ECE 131A	HW #6	Jason Chapman

(1)

$$f_{X,Y}(x,y) = \begin{cases} ce^{-x}e^{-y}, & 0 \le x \le y \le \infty \\ 0, & \text{otherwise} \end{cases}$$

a)

$$\int_0^\infty \int_0^y ce^{-x}e^{-y}\,dx\,dy = c\int_0^\infty e^{-y}\left[-e^{-x}\Big|_0^y\right]dy = c\int_0^\infty -e^{-2y}+e^{-y}\,dy$$

$$\Rightarrow c\left[\tfrac{1}{2}e^{-2y}-e^{-y}\right]_0^\infty = c\left[0-\left(\tfrac{1}{2}-1\right)\right] = c\left[\tfrac{1}{2}\right] \Rightarrow \boxed{c=2}$$

b)

$$f_X(x) = \int_x^\infty f_{X,Y}(x,y)\,dy = \int_x^\infty 2e^{-x}e^{-y}\,dy = -2e^{-2x} \Rightarrow \boxed{f_X(x) = 2e^{-2x} \quad x \ge 0}$$

$$f_Y(y) = \int_0^y f_{X,Y}(x,y)\,dx = \int_0^y 2e^{-x}e^{-y}\,dx = 2e^{-y}\left(-e^{-x}\Big|_0^y\right) \Rightarrow \boxed{f_Y(y) = 2e^{-y}-2e^{-2y} \quad y \ge 0}$$

c) $P[X+Y \le 1]$

X & Y nonnegative

$Y \in [0,1] \Rightarrow X \in [0, 1-Y]$

$$\int_0^1 \int_0^{1-y} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^{1-y} e^{-x}e^{-y}\,dx\,dy = \int_0^1 \left[-e^{-x}\right]\Big|_0^{1-y} e^{-y}\,dy$$

$$\Rightarrow \int_0^1 (-e^{y-1}+1)e^{-y}\,dy = \int_0^1 -e^{y-1}e^{-y}+e^{-y}\,dy = \int_0^1 (-e^{-1}+e^{-y})\,dy$$

$$\Rightarrow \int_0^1 \left(-\tfrac{1}{e}\right)dy + \int_0^1 e^{-y}\,dy = -\tfrac{1}{e} + \left[-e^{-y}\right]\Big|_0^1 = -\tfrac{1}{e} + (-e^{-1}) + 1 = -\tfrac{2}{e}+1$$

$$\Rightarrow \boxed{P[X+Y \le 1] = -\tfrac{2}{e}+1 \approx 0.264}$$

(2)

$$f_{X,Y}(x,y) = \begin{cases} \frac{x}{10}+\frac{y}{20}, & 0 \le x \le 2,\ 1 \le y \le 5 \\ 0, & \text{otherwise} \end{cases}$$

a)

$$f_X(x) = \int_1^5 f_{X,Y}(x,y)\,dy = \int_1^5 \frac{x}{10}+\frac{y}{20}\,dy = \frac{x}{10}\int_1^5 dy + \int_1^5 \frac{y}{20}\,dy = \frac{x}{10}[5-1] + \left[\frac{y^2}{40}\right]\Big|_1^5$$

$$= \frac{4x}{10} + \frac{25-1}{40} = \frac{2x+3}{5}$$

$$f_Y(y) = \int_0^2 f_{X,Y}(x,y)\,dx = \int_0^2 \frac{x}{10}+\frac{y}{20}\,dx = \int_0^2 \frac{x}{10}\,dx + \frac{y}{20}\int_0^2 dx = \left[\frac{x^2}{20}\right]\Big|_0^2 + \frac{y}{20}[2]$$

$$= \frac{4}{20} + \frac{2y}{20} = \frac{2+y}{10}$$

$$f_X(x)\cdot f_Y(y) = \left(\frac{2x+3}{5}\right)\cdot\left(\frac{2+y}{10}\right) = \frac{(2x+3)(2+y)}{50} = \frac{4x+2xy+6+3y}{50}$$

$f_X(x)f_Y(y) \neq f_{XY}(x,y) \Rightarrow$ <u>Not Independent</u>

b) $Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$$\mathbb{E}[XY] = \iint x\cdot y\, f_{x,y}(x,y)\,dx\,dy$$

$$= \int_1^5\int_0^2 x\cdot y\left(\frac{x}{10}+\frac{y}{20}\right)dx\,dy$$

$$= \int_1^5\int_0^2 \left(\frac{x^2y}{10}+\frac{xy^2}{20}\right)dx\,dy$$

$$= \int_1^5\left[\frac{x^3y}{30}\Big|_0^2 + \frac{x^2y^2}{40}\Big|_0^2\right]dy = \int_1^5\left(\frac{8y}{30}+\frac{4y^2}{40}\right)dy$$

$$= \frac{8y^2}{60}\Big|_1^5 + \frac{4y^3}{120}\Big|_1^5 = \frac{10}{3} - \frac{2}{15} + \frac{25}{6} - \frac{1}{30} = \frac{22}{3}$$

$$\mathbb{E}[X] = \iint x\cdot f_{x,y}(x,y)\,dx\,dy$$

$$= \int_1^5\int_0^2 x\left(\frac{x}{10}+\frac{y}{20}\right)dx\,dy = \int_1^5\int_0^2\left(\frac{x^2}{10}+\frac{xy}{20}\right)dx\,dy$$

$$= \int_1^5\left[\frac{x^3}{30}\Big|_0^2 + \frac{x^2y}{40}\Big|_0^2\right]dy = \int_1^5\left(\frac{8}{30}+\frac{4y}{40}\right)dy$$

$$= \frac{8}{30}(5-1) + \frac{4y^2}{80}\Big|_1^5 = \frac{16}{15} + \frac{5}{4} - \frac{1}{20} = \frac{34}{15}$$

$$\mathbb{E}[Y] = \iint y\cdot f_{x,y}(x,y)\,dx\,dy = \int_1^5\int_0^2\left(\frac{xy}{10}+\frac{y^2}{20}\right)dx\,dy$$

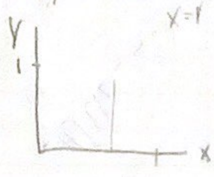
$$= \int_1^5\left[\frac{x^2y}{20}\Big|_0^2 + \frac{xy^2}{20}\right]dy = \int_1^5\left(\frac{4y}{20}+\frac{2y^2}{20}\right)dy$$

$$= \frac{4y^2}{40}\Big|_1^5 + \frac{2y^3}{60}\Big|_1^5 = \frac{5}{2} - \frac{1}{10} + \frac{25}{6} - \frac{1}{30} = \frac{98}{15}$$

$$Cov(X,Y) = \frac{22}{3} - \left(\frac{34}{15}\right)\left(\frac{98}{15}\right) = \frac{-1682}{225} = \boxed{-7.4756 = Cov(X,Y)}$$

③ A point (X,Y) selected inside a triangle defined by: $\{(x,y): 0 \le y \le x \le 1\}$

a) $0 \le y \le x \le 1$



$$\int_0^1\int_0^x k\,dy\,dx = 1 \Rightarrow \int_0^1 kx\,dx = \frac{1}{2}kx^2\Big|_0^1 = \frac{k}{2} = 1$$

$\Rightarrow k = 1$ $\quad f_{X,Y}(x,y) = 2$

$$F_{X,Y}(x,y) = \int_0^x\int_0^y f_{X,Y}(x',y')\,dx'dy' = \int_0^y\int_u^x 2\,dw\,du = 2\int_0^y (x-u)\,du = 2\left[xu - \tfrac{1}{2}u^2\right]_0^y$$

$$\Rightarrow F_{X,Y} = 2xy - y^2$$

$$\boxed{f_{X,Y}(x,y) = \begin{cases} 2 & 0\le y\le x\le 1 \\ 0 & \text{else}\end{cases} \qquad F_{X,Y}(x,y) = \begin{cases} 0 & y\le 0,\ x\le 0 \\ 2xy-y^2 & 0\le y\le x\le 1 \\ 1 & y\ge 1,\ x\ge 1\end{cases}}$$

b.)

$$f_X(x) = \int_0^x 2\,dy = 2x \quad 0\le x\le 1$$

$$f_Y(y) = \int_y^1 2\,dx = 2(1-y) = 2-2y \quad 0\le y\le 1$$

$$F_X(x) = F_{XY}(x,\infty) = \lim_{y\to\infty} F_{X,Y} \quad // \quad y\to\infty \Rightarrow y = x, \Rightarrow F_{XY}(x,x) = 2x^2 - x^2 = x^2$$

$$F_Y(y) = F_{XY}(\infty,y) = \lim_{x\to\infty} F_{X,Y} \quad // \quad x\to\infty \Rightarrow x=1 \Rightarrow F_{XY}(1,y) = 2y-y^2$$

$$\Rightarrow \boxed{\begin{array}{ll} f_X(x) = 2x \quad 0\le x\le 1 & F_X(x) = x^2 \quad 0\le x\le 1,\ x\le y \\ f_Y(y) = 2-2y \quad 0\le y\le 1 & F_Y(y) = 2y-y^2 \quad 0\le y\le 1,\ 1\le x \end{array}}$$

c) $A:\left\{X\le \frac{1}{2},\ Y\le\frac{3}{4}\right\}$; $B:\left\{\frac{1}{4}\le X\le\frac{3}{4},\ \frac{1}{4}< Y\le\frac{3}{4}\right\}$

$$P(A) = F_{X,Y}\left(\tfrac{1}{2},\tfrac{3}{4}\right) = 2\left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right) - \left(\tfrac{3}{4}\right)^2 = \tfrac{3}{16}$$

$$P(B) = F_{X,Y}\left(\tfrac{3}{4},\tfrac{3}{4}\right) - F_{X,Y}\left(\tfrac{1}{4},\tfrac{3}{4}\right) - F_{X,Y}\left(\tfrac{3}{4},\tfrac{1}{4}\right) + F_{X,Y}\left(\tfrac{1}{4},\tfrac{1}{4}\right) \Rightarrow$$

$$= \tfrac{9}{16} - \left(\tfrac{-3}{16}\right) - \tfrac{5}{16} + \tfrac{1}{16} = \tfrac{8}{16} = \tfrac{1}{2}$$

$$\boxed{\begin{array}{l} P(A) = \frac{3}{16} \\ P(B) = \frac{1}{2}\end{array}}$$

d.) $P(Y < X^2)$

$$x\in[0,1] \Rightarrow y\in[0,x^2)$$

$$\int_0^1\int_0^{x^2} f_{XY}(x,y)\,dxdy = 2\int_0^1 x^2\,dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

$$\boxed{P[Y<X^2] = \tfrac{2}{3}}$$

④ $Z = X + Y$, $f_{X,Y}(x,y) = 8xy$ for $0 \le y \le x \le 1$

$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y')\,dy'$

$= \int_0^x 8xu\,du = 4xu^2\Big|_0^x = 4x^3$

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x', y)\,dx'$

$= \int_y^1 8uy\,du = 4u^2y\Big|_y^1 = 4y - 4y^3$

$f_X(x) \cdot f_Y(y) = (4x^3)(4y - 4y^3) = 16x^3y - 16x^3y^3 \neq f_{X,Y}(x,y)$

$\Rightarrow$ X & Y are <u>not</u> independent

$f_Z(x,y) = f_X(x) + f_Y(y) = 4x^3 + 4y - 4y^3$

$\Rightarrow$ $\boxed{f_Z(x,y) = 4\left[x^3 + y - y^3\right] \quad 0 \le y \le x \le 1}$

⑤ $U \sim \text{Uniform}[0,1]$

a.) $X \sim \exp(\lambda)$, $\lambda = 0.5$

$F_X(x) = 1 - e^{-\lambda x}$

$1 - e^{-\lambda x} = u \Rightarrow 1 - u = e^{-\lambda x} \Rightarrow x = -\frac{1}{\lambda}\ln(1-u)$

$\boxed{F_X^{-1}(u) = -2\ln(1-u)}$

b.) See Attached

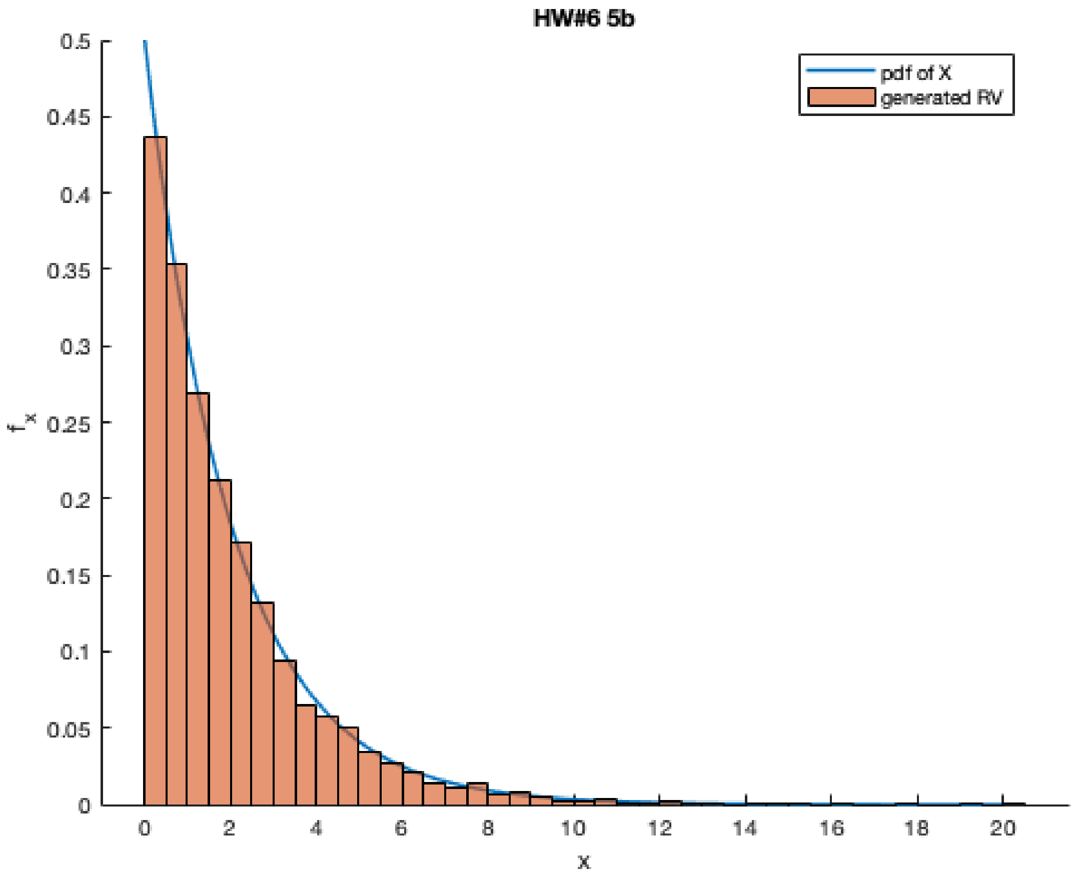
c.) $X = \{1, 2, 3, 4\}$ Uniform RV $p_i = 1/4$

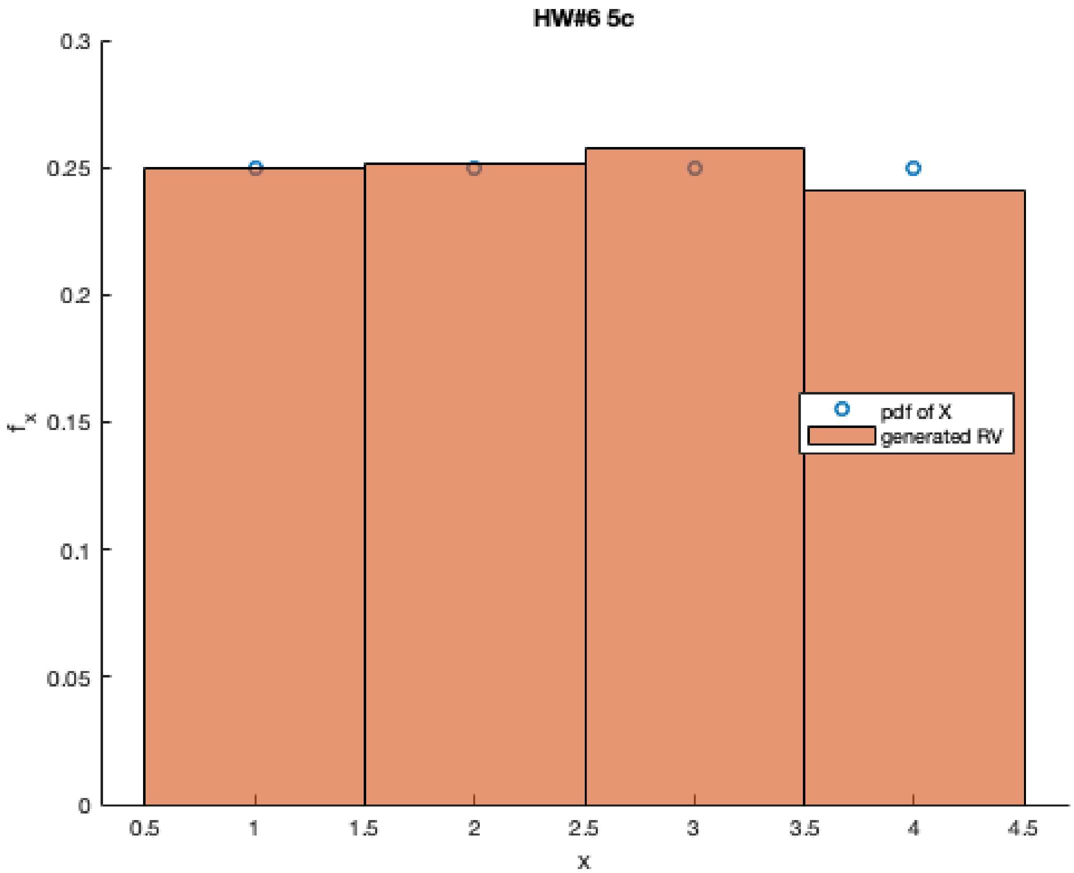
$X = \begin{cases} x_1 & \text{if } U \le p_1 \\ x_2 & \text{if } \sum_{i=1}^{k-1} p_i < U \le \sum_{i=1}^{k} p_i \end{cases} \Rightarrow \boxed{F_X^{-1}(u) = \begin{cases} 1 & \text{if } u \le 1/4 \\ 2 & \text{if } 1/4 < u \le \frac{1}{2} \\ 3 & \text{if } 1/2 < u \le 3/4 \\ 4 & \text{if } 3/4 < u \le 1 \end{cases}}$

ECE 131a HW6

Jason Chapman

```
close all; clear; clc;

% 5b
n = 5000;
x = linspace(0,20);
lambda = 0.5;
fx = lambda*exp(-lambda*x);
u = rand([1 n]);
Fx_inv = -(1/lambda)*log(1-u);

figure(1)
hold on
plot(x,fx,'Linewidth',1.5)
histogram(Fx_inv,'normalization','pdf')
xlabel('x')
ylabel('f_{x}')
title('HW#6 5b')
legend('pdf of X','generated RV','Location','Best')

% 5c
x = [1 2 3 4];
fx = [.25 .25 .25 .25];
for i = 1:n
    if u(i) <= 1/4
        Fx_inv(i) = 1;
    elseif u(i) > 1/4 && u(i) <= 1/2
        Fx_inv(i) = 2;
    elseif u(i) > 1/2 && u(i) <= 3/4
        Fx_inv(i) = 3;
    elseif u(i) > 3/4
        Fx_inv(i) = 4;
    end
end

figure(2)
hold on
plot(x,fx,'o','Linewidth',1.5)
histogram(Fx_inv,'normalization','pdf')
xlabel('x')
ylabel('f_{x}')
title('HW#6 5c')
legend('pdf of X','generated RV','Location','Best')
```


HW#6 5b
pdf of X
generated RV
f_X
x
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0
2
4
6
8
10
12
14
16
18
20



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