EEE 131A | HW # 7 | Jason Chapman

(1)

$$X \rightarrow \oplus \xleftarrow{} N \rightarrow y \; , \quad Y \sim N(0,\theta_x), \quad N \sim N(0,\theta_y) \; ; \quad Y = X + N \, , \text{ independent}$$

a.) $\rho_{x,y} = \dfrac{\text{cov}(X,Y)}{\sqrt{\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}}$ // $\mathbb{E}[X] = 0, \; \text{VAR}[X] = \theta_x^2$

$\mathbb{E}[Y] = \mathbb{E}[X] + \mathbb{E}[N] = 0 + 0 = 0$

$\text{Cov}(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] = \mathbb{E}[X(X+N)] = \mathbb{E}[X^2] + \mathbb{E}[XN]$

$= \text{Var}[X] + (\mathbb{E}[X])^2 + \mathbb{E}[X]\mathbb{E}[N]$ // $X \perp N$

$= \theta_x^2$

$\text{Var}[Y] = \text{Var}[X] + \text{Var}[N] + 2\text{Cov}(X,N) = \theta_x^2 + \theta_N^2 + 2(0)$ // $X \perp N$

corellation coefficient $\rho = \dfrac{\text{cov}(X,Y)}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \dfrac{\theta_x^2}{\sqrt{\theta_x^2(\theta_x^2+\theta_y^2)}} \Rightarrow \boxed{\rho = \dfrac{\theta_x}{\sqrt{\theta_x^2+\theta_y^2}}}$

b.) $f(a) = \mathbb{E}[(X-aY)^2]$

/ minimize error when $\dfrac{\partial f}{\partial a} = 0$ & $\dfrac{\partial^2 f}{\partial a^2} > 0$

$\Rightarrow \dfrac{\partial f}{\partial a} = \dfrac{\partial}{\partial a}\left[\mathbb{E}[(X-aY)^2]\right] = \mathbb{E}\left[2(X-aY)(-Y)\right] = 0$

$\Rightarrow \mathbb{E}[XY] = a\,\mathbb{E}[Y^2]$

$\text{Cov}(X,Y) = a\left(\text{Var}(Y) + (\mathbb{E}[Y])^2\right)$

$\theta_x^2 = a(\theta_x^2 + \theta_N^2 + 0) \Rightarrow \boxed{a = \dfrac{\theta_x^2}{\theta_x^2+\theta_N^2}}$

c.) $\text{MSE} = \mathbb{E}[(X-aY)^2]$

$= \mathbb{E}[X^2] + a^2\,\mathbb{E}[Y^2] - 2a\,\mathbb{E}[XY]$

$= \theta_x^2 + a^2(\theta_x^2+\theta_N^2) - 2a\theta_x^2$

$= \theta_x^2 + \dfrac{\theta_x^4}{\theta_x^2+\theta_N^2} - \dfrac{2\theta_x^4}{\theta_x^2+\theta_N^2} = \theta_x^2 - \dfrac{\theta_x^4}{\theta_x^2+\theta_N^2} = \dfrac{\theta_x^4 + \theta_x^2\theta_N^2 - \theta_x^4}{\theta_x^2+\theta_N^2}$

$\Rightarrow \boxed{\text{MSE} = \dfrac{\theta_x^2\theta_N^2}{\theta_x^2+\theta_N^2}}$

② $f_{XY}(x,y) = \begin{cases} \frac{1}{36}xy, & 0 \le x \le 6,\ 0 \le y \le \sqrt{x} \\ 0, & \text{else} \end{cases}$

a.) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y')\,dy' = \int_0^{\sqrt{x}} \frac{1}{36}xy'\,dy' = \frac{1}{36}x\int_0^{\sqrt{x}} y'\,dy' = \frac{x}{36}\left[\frac{1}{2}y'^2\right]_0^{\sqrt{x}} = \frac{x^2}{72}$

$\boxed{f_X(x) = \frac{x^2}{72}}$

b.) $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\frac{1}{36}xy}{\frac{1}{2}y} = \frac{x}{18} \Rightarrow \boxed{f_{X|Y}(x|y) = \frac{x}{18} \quad 0 \le y \le \sqrt{x}}$

c.) $\mathbb{E}[X|Y=y]$, for $0 \le y \le 1$

$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|Y=y)\,dx = \int_0^6 x\frac{x}{18}\,dx = \frac{1}{18}\left[\frac{1}{3}x^3\right]_0^6 = 4$

$\Rightarrow \boxed{\mathbb{E}[X|Y=y] = 4}$

③ $Z = \frac{X}{X+Y}$; $X = \exp(4)$, $Y = \exp(4)$, $X \perp\!\!\!\perp Y$. $x, y > 0$

$Z = \frac{x}{x+y} \Rightarrow y = \frac{1-z}{z}x$

$\Rightarrow f_Z(z) = \int_0^{\infty}\int_0^{\frac{zy}{1-z}} 16e^{-4y}e^{-4x}\,dx\,dy = \int_0^{\infty} -4e^{-4y}\left[e^{-4x}\right]_0^{\frac{zy}{1-z}}\,dy$

$= \int_0^{\infty} -4e^{-4y}\left(e^{-\frac{4zy}{1-z}} - 1\right)dy = \int_0^{\infty} -4e^{-4y}e^{-\frac{4zy}{1-z}} - \int_0^{\infty} 4e^{-4y}\,dy$

$= z - 1 - 1 = z$

$\Rightarrow \boxed{f_z(z) = z}$

④

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho xy}{\sigma_x\sigma_y}\right)\right]$$

a.)

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x',y)\,dx' = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}Q(x,y)}\,dx$$

where

$$Q(x,y) = \frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{1-\rho^2}$$

$$= \left(\frac{x-a}{b}\right)^2 + c = \frac{x^2}{b^2} - 2\frac{a}{b^2}x + \frac{a^2}{b^2} + c$$

$$= \frac{x^2}{\sigma_x^2(1-\rho^2)} - 2\left[\frac{\mu_x}{\sigma_x^2(1-\rho^2)} + \rho\frac{y-\mu_y}{\sigma_x\sigma_y(1-\rho^2)}\right]x + \frac{\mu_x^2}{\sigma_x^2(1-\rho^2)} + 2\rho\frac{(y-\mu_y)}{\sigma_x\sigma_y(1-\rho^2)}\mu_x + \frac{(y-\mu_y)^2}{\sigma_y^2(1-\rho^2)}$$

$$\Rightarrow b^2 = \sigma_x^2(1-\rho^2), \quad a = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$$

$$c = \frac{\mu_x^2}{\sigma_x^2(1-\rho^2)} + 2\rho\frac{(y-\mu_y)}{\sigma_x\sigma_y(1-\rho^2)}\mu_x + \frac{(y-\mu_y)^2}{\sigma_y^2(1-\rho^2)} - \frac{\left[\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right]^2}{\sigma_x^2(1-\rho^2)}$$

$$= \frac{1}{\sigma_x^2(1-\rho^2)}\left[\frac{\sigma_x^2}{\sigma_y^2}(1-\rho^2)(y-\mu_y)^2\right] = \left(\frac{y-\mu_y}{\sigma_y}\right)^2$$

$$\Rightarrow f_Y(y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[\left(\frac{x-a}{b}\right)^2 + c\right]}dx = \frac{\sqrt{2\pi}\,b e^{-c/2}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\,b}e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2} \Rightarrow \boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{1}{2}\left(\frac{y}{\sigma_y}\right)^2} \quad \text{Mean: } 0 \quad \text{Variance: } \sigma_y}$$

b.)

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f(y)} = \frac{e^{-\frac{1}{2}Q(x,y)}}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \Big/ \frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$= \frac{e^{-\frac{1}{2}\left[\left(\frac{x-a}{b}\right)^2 + c\right] + \frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}}$$

$$\Rightarrow \boxed{f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\,b}e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2}}$$

Mean: $\mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$

Variance: $\sigma_x^2(1-\rho^2)$

$a = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$

$b = \sigma_x\sqrt{1-\rho^2}$

(5) $Y = \begin{cases} X & \text{if } X \geq 0 \\ X^2 & \text{if } X < 0 \end{cases}$ $\quad X \sim N(0,9)$

$\underline{Y=X}$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{3\sqrt{2\pi}} e^{-\frac{x^2}{18}}$

$\underline{Y=X^2}$ $\quad X = \pm\sqrt{y} = -\sqrt{y}$ as $X < 0$

$\Rightarrow f_Y(y) = f_X(\sqrt{y}) \cdot \left|\frac{dx}{dy}\right| = \frac{1}{3\sqrt{2\pi}} e^{\frac{-y}{18}}$.

$$F_Y(y) = P(Y \leq y) = P(X \leq y) + P(X^2 \leq y)$$
$$= P(X \leq y \mid X > 0) + P(X^2 \leq y \mid X < 0)$$
$$= P(X \leq y) + P(X \leq -\sqrt{y})$$
$$= F_X(y) + F_X(-\sqrt{y})$$

$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}\left[F_X(y) + F_X(-\sqrt{y})\right] = f_X(y) \cdot 1 + f_X(-\sqrt{y}) \cdot \frac{d}{dy}[-\sqrt{}$

$$= \frac{1}{3\sqrt{2\pi}} e^{\frac{-y^2}{18}} + \frac{1}{3\sqrt{2\pi}} e^{\frac{-y}{18}} \cdot -\frac{1}{2} y^{-1/2}$$

$\Rightarrow$ $\boxed{f_Y(y) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{y^2}{18}} - \frac{1}{6\sqrt{2\pi y}} e^{-\frac{y}{18}}}$