EE 131A | HW #4 | Jason Chapman

① $f_X(x) = \begin{cases} cx(1+2x-x^2) & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

a) We know that $\int_{-\infty}^{\infty} f_X(x) = 1$, can use this property to solve for c.

$\Rightarrow \int_{-\infty}^{0} 0\,dx + \int_0^1 cx(1+2x-x^2)\,dx + \int_1^{\infty} 0\,dx = 1$

$\Rightarrow \int_0^1 cx + 2cx^2 - cx^3\,dx = \frac{1}{2}cx^2\Big|_0^1 + \frac{2}{3}cx^3\Big|_0^1 - \frac{1}{4}cx^4\Big|_0^1 = 1$

$\Rightarrow c\left(\frac{1}{2} + \frac{2}{3} - \frac{1}{4}\right) = 1 \Rightarrow \boxed{c = \frac{12}{11}}$

See Attached for Matlab plot

b) $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \left(\frac{12}{11}t + \frac{24}{11}t^2 - \frac{12}{11}t^3\right)dt$

$= \frac{6}{11}t^2 + \frac{8}{11}t^3 - \frac{3}{11}t^4\Big|_0^x = \frac{1}{11}\left(6x^2 + 8x^3 - 3x^4\right)$

$$\boxed{F_X(x) = \begin{cases} 0 & -\infty < x \le 0 \\ \frac{1}{11}(6x^2 + 8x^3 - 3x^4) & 0 \le x \le 1 \\ 1 & 1 \le x < \infty \end{cases}}$$

See Matlab for Plot.

c) $P(0.25 < X < 0.5) = \int_{0.25}^{0.5} f_X(x)\,dx = \int_{0.25}^{0.5} \frac{12}{11}\left(x + 2x^2 - x^3\right)dx$

$= \frac{12}{11}\left[\frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{0.25}^{0.5}$

$= \frac{12}{11}\left[\left(\frac{1}{2}(.5)^2 - \frac{1}{2}(.25)^2\right) + \left(\frac{2}{3}(.5)^3 - \frac{2}{3}(.25)^3\right) - \left(\frac{1}{4}(.5)^4 - \frac{1}{4}(.25)^4\right)\right]$

$= 0.16583808 \Rightarrow \boxed{P(0.25 < X < 0.5) = 0.1658}$

② $R = A\cos(\Theta)$, $A \ge 0$ & Θ Uniform on $(0, \pi)$.

PDF of $\Theta = \begin{cases} \frac{1}{\pi - 0} & \Theta \in [0, \pi] \\ 0 & \Theta \notin [0, \pi] \end{cases}$

since $\Theta \in [0, \pi]$, $\cos(\Theta) \in [1, -1]$

$\cancel{F_R(r) = P(R \le r) = P(A\cos\Theta \le r) = P(\Theta \ge \cos^{-1}(\frac{r}{A})) = \frac{\cos^{-1}(\frac{r}{A})}{\pi}}$

$$F_R(r) = P[R \le r]$$
$$= P[A\cos\theta \le r]$$
$$= P\left[\theta \le \cos^{-1}\left(\tfrac{r}{A}\right)\right]$$
$$= P\left[\theta \le \cos^{-1}\left(\tfrac{-r}{A}\right)\right] \quad // \text{ from CDF of } \theta \sim U[0,\pi),\ P[\theta \le \theta] = \frac{\theta}{\pi}$$
$$= \frac{\cos^{-1}\left(-\frac{r}{A}\right)}{\pi}$$

$$f_R(r) = \frac{\partial}{\partial r} F_R(r) = \frac{\partial}{\partial r}\left[\frac{\cos^{-1}\left(\frac{-r}{A}\right)}{\pi}\right] = \frac{1}{\pi}\frac{\partial}{\partial r}\cos^{-1}\left(\frac{-r}{A}\right)$$
$$= \frac{1}{\pi}\left[\frac{-1}{\sqrt{1-\left(\frac{-r}{A}\right)^2}} \cdot \frac{-1}{A}\right] = \frac{1}{\pi A} \cdot \frac{1}{\sqrt{1-\left(\frac{r}{A}\right)^2}}, \quad -A \le r \le A$$

$$\boxed{f_R(r) = \begin{cases} \dfrac{1}{\pi A\sqrt{1-(r/A)^2}}, & -A \le r \le A \\ 0 & \text{elsewhere} \end{cases}}$$

// check $\int_{-1}^{1} f_R(r) = 1$ on Wolfram Alpha ✓

③ $X = -\ln(8U)$, $U \sim U[0,1]$

$X \in [-\ln 8, \infty)$

$$F_X(x) = P(X \le x)$$
$$= P(-\ln 8U \le x)$$
$$= P\left(U \ge \tfrac{1}{8}e^{-x}\right)$$
$$= 1 - P\left(U \le \tfrac{1}{8}e^{-x}\right) = 1 - \tfrac{1}{8}e^{-x} \quad // \ F_U(u) = \frac{u}{1}$$

$$f_X(x) = \frac{\partial}{\partial x}F_X(x)$$
$$= \frac{\partial}{\partial x}\left(1 - \tfrac{1}{8}e^{-x}\right)$$
$$= \tfrac{1}{8}e^{-x} \quad // \ \int_{-\ln 8}^{\infty} \tfrac{1}{8}e^{-x}\,\partial x = 1 \ ✓$$

$$\boxed{f_X(x) = \begin{cases} \tfrac{1}{8}e^{-x}, & x \in [-\ln 8, \infty) \\ 0 & \text{elsewhere} \end{cases}}$$

④ $X = U^n \quad U \sim U[0,1]$

$F_X(x) = P(X \le x)$

$= P(U^n \le x)$

$= P(U \le x^{\frac{1}{n}})$

$= F_U(x^{1/n}) \quad // \ F_U(u) = u$

$= x^{1/n}$

$\Rightarrow \boxed{F_X(x) = x^{1/n} \quad \text{for } 0 \le x \le 1}$

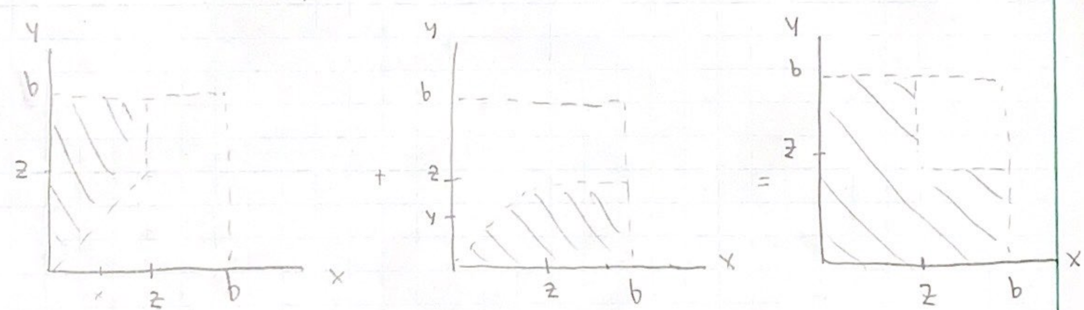
$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \frac{\partial}{\partial x}(x^{1/n})$

$= x^{(\frac{1}{n}-1)} \cdot \frac{1}{n} = \frac{x^{\frac{1-n}{n}}}{n}$

$\Rightarrow \boxed{f_X(x) = \frac{x^{\frac{1-n}{n}}}{n} \quad \text{for } 0 \le x \le 1}$

⑤ $\{(x,y): 0 \le x \le b,\ 0 \le y \le b\}, \quad Z = \min(x,y)$

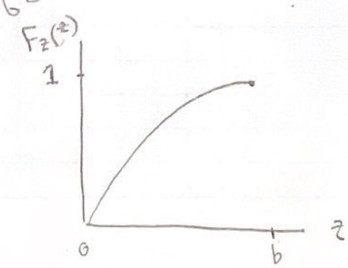
a.)



b.) $F_Z(z) = P(Z \le z)$

$= \frac{b^2 - (b-z)^2}{b^2} = \frac{b^2 - [b^2 - 2bz + z^2]}{b^2} = \frac{2bz - z^2}{b^2}$

$\Rightarrow \boxed{F_Z(z) = \frac{2bz - z^2}{b^2}, \quad 0 < z < b}$



CDF of Z

c) $P[Z>0] = 1 - P[Z\le 0] = 1 - F_Z(0) = 1 - \frac{2b(0)-(0)^2}{b^2} = 1 - 0 = 1$

$P[Z>b] = 1 - P[Z\le b] = 1 - F_Z(b) = 1 - \frac{2b(b)-b^2}{b^2} = 1-1 = 0$

$P[Z\le b/2] = F_Z(b/2) = \frac{2b(b/2)-(b/2)^2}{b^2} = \frac{b^2 - \frac{1}{4}b^2}{b^2} = 1 - \frac{1}{4} = 3/4$

$P[Z>b/4] = 1 - P[Z\le b/4] = 1 - F_Z(b/4) = 1 - \frac{2b(b/4)-(b/4)^2}{b^2} = 1 - \frac{b^2/2 - b^2/16}{b^2} = 1 - \frac{7}{16} = \frac{9}{16}$

$$\Rightarrow \boxed{\begin{aligned} P[Z>0] &= 1 \\ P[Z>b] &= 0 \\ P[Z\le b/2] &= 3/4 \\ P[Z>b/4] &= 9/16 \end{aligned}}$$

d.) $f_Z(z) = \frac{\partial}{\partial z} F(z) = \frac{\partial}{\partial z}\left(\frac{2bz - z^2}{b^2}\right) = \frac{2b-2z}{b^2}, \ 0<z<b$

$$\Rightarrow \boxed{f_Z(z) = \frac{2b-2z}{b^2}, \quad 0<z<b}$$