ECE 131A | HW #8 | Jason Chapman

① X & Y iid $\exp(\lambda)$

$Q = X + R; \quad R = Y$

$$f_{Q,R}(z,r) = \frac{f_{X,Y}(A^{-1}z)}{|\det(A)|}$$

$$\begin{bmatrix} Q \\ R \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad // \quad \begin{aligned} X &= Q - R \\ Y &= R \\ \det(A) &= 1 \end{aligned}$$

$f_X(x) = \lambda e^{-\lambda x}$

$f_Y(y) = \lambda e^{-\lambda y}$

$$f_{X,Y}(x,y) = (\lambda e^{-\lambda x})(\lambda e^{-\lambda y}) = \lambda^2 e^{-\lambda x - \lambda y} = \lambda^2 e^{-\lambda(x+y)}$$

$$\Rightarrow \boxed{f_{Q,R}(z,r) = \lambda^2 e^{-\lambda(z-r+r)} = \lambda^2 e^{-\lambda z} \quad z \geq 0}$$

② Prove $-1 \leq \rho_{X,Y} \leq 1$

$$\rho_{X,Y} = \frac{\operatorname{cov}(X,Y)}{\sigma_X \cdot \sigma_Y} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\left[\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \cdot \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2\right]^{1/2}}$$

$$\rho_{X,Y}^2 = \frac{\left(\sum_{i=1}^{n} a_i b_i\right)^2}{\left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2} \quad \text{w/} \quad \begin{aligned} a_i &= x_i - \bar{x} \\ b_i &= y_i - \bar{y} \end{aligned}$$

$$\Rightarrow \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i\right)^2 \left(\sum_{i=1}^{n} b_i\right)^2$$

$$\Rightarrow \rho_{X,Y}^2 \leq 1 \Rightarrow |\rho_{X,Y}| \leq 1 \Rightarrow -1 \leq \rho_{X,Y} \leq 1 \quad \checkmark$$

③ a.) 100 Transistors, Find probability # broken between 35 $ 65 $ between 50 $ 55

$X_i \sim$ Bernoulli (0.05), $n=100$

$E[X] = 0.5$, $VAR[X] = 0.25$

$S_n = X_1 + X_2 + \dots X_{100}$

$E[S_n] = np = 100(.5) = 50$, $VAR(S_n) = n(1-p)p = 25$

$$Z_n = \frac{S_n - E[S_n]}{\sqrt{VAR(S_n)}} = \frac{S_n - 50}{5}$$

A: # broken between 35 $ 65

$$P[35 \le N \le 65] = P\left[\frac{35-50}{5} \le \frac{N-50}{5} \le \frac{65-50}{5}\right]$$

$$\Rightarrow P[-3 \le Z_n \le 3] \quad // \quad Z_n \sim N(0,1), \text{ call it } Y$$

$$\approx 1 - P(Y \ge 3) - P(Y \le -3) = 1 - Q(3) - Q(-3) = 0.9973$$

B: # broken between 50 $ 55

$$P[50 \le N \le 55] = P\left[\frac{50-50}{5} \le \frac{N-50}{5} \le \frac{55-50}{5}\right]$$

$$\Rightarrow P[0 \le Z_n \le 1] \quad // \quad Z_n \sim N(0,1), \text{ call it } Y$$

$$\approx 1 - P(Y \ge 1) - P(Y \le 0) = 1 - Q(1) - Q(0) = 0.3413$$

$$\boxed{\begin{array}{l} P[\#\text{ broken between } 35 \$ 65] = 0.9973 \\ P[\#\text{ broken between } 50 \$ 55] = 0.3413 \end{array}}$$

b.) 1000 Transistors, $P[350 - 650]$, $P[500, 550]$

$S_n = X_1 + X_2 + \dots + X_{1000}$

$E[S_n] = 500$, $VAR(S_n) = 250$

$$P[350 \le N \le 650] = P\left[\frac{350-500}{\sqrt{250}} \le \frac{N-500}{\sqrt{250}} \le \frac{650-500}{\sqrt{250}}\right] = P[-9.4868 \le Z_n \le 9.4868]$$

$$\approx 1 - Q(9.4868) - Q(-9.4868) = 1$$

$$P[500 \le N \le 550] = P\left[\frac{500-500}{\sqrt{250}} \le \frac{N-500}{\sqrt{250}} \le \frac{550-500}{\sqrt{250}}\right] = P[0 \le Z_n \le \sqrt{10}]$$

$$\approx 1 - Q(\sqrt{10}) - Q(0) = 0.4992$$

$$\boxed{\begin{array}{l} P[\#\text{ broken between } 350 \$ 650] = 1 \\ P[\#\text{ broken between } 500 \$ 550] = 0.4992 \end{array}}$$

c) $X_i \sim \text{bernoulli}(0.2)$

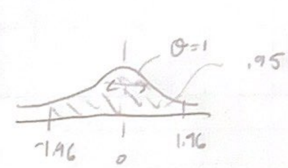
$P\left[|f_A(n) - 0.2| < 0.02\right] \geq 0.95$

$\mathbb{E}[f_A(n)] = 0.2, \quad VAR[f_A(n)] = \frac{0.16}{n}$

$P\left[\left|\frac{f_A(n) - 0.2}{\sqrt{\frac{0.16}{n}}}\right| < \frac{0.02}{\sqrt{\frac{0.16}{n}}}\right] \geq 0.95 \qquad // \quad \frac{f_A(n) - 0.2}{.4/\sqrt{n}} = Z_n \sim [0,1]$

$P\left[|Z_n| < \frac{0.02\sqrt{n}}{0.4}\right] \geq 0.95$

$\Rightarrow P\left[-0.05\sqrt{n} \leq Z < 0.05\sqrt{n}\right] \geq 0.95 \qquad //$



$\Rightarrow 0.05\sqrt{n} = 1.96$

$\Rightarrow n = \left(\frac{1.96}{.05}\right)^2 = 1536.64 \Rightarrow \boxed{n = 1537}$

④ X & Y jointly gaussian. $\mathbb{E}[X] = 0, \mathbb{E}[Y] = 0, \sigma_x = 2, \sigma_y = 4, \mathbb{E}[X|Y] = \frac{Y}{4}$

a) $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho xy}{\sigma_x\sigma_y}\right)\right]$

$\mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\, f_{X|Y}(x,y)\, dx$

$// \quad f_{X|Y}(x,y) = \frac{1}{\sqrt{2\pi}\, b} e^{-\frac{1}{2}\left(\frac{x-a}{b}\right)^2} \quad$ w/ $\quad a = \rho\frac{\sigma_x}{\sigma_y} y$

$b = \sigma_x\sqrt{1-\rho^2}$

$\mathbb{E}[X|Y] = \frac{1}{\sqrt{2\pi}\, b}\int_{-\infty}^{\infty} x e^{-\frac{(x-a)^2}{2b^2}} dx = \int\left((x-a)e^{-\frac{(x-a)^2}{2b^2}}\right)dx + a\int e^{-\frac{(x-a)^2}{2b^2}} dx =$

$= a = \rho\frac{\sigma_x}{\sigma_y} y \quad ✓$

b) $\rho_{xy} = \frac{COV(X,Y)}{\sqrt{VAR(X)}\cdot\sqrt{VAR(Y)}} = \frac{\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]}{\sigma_x \cdot \sigma_y} = \frac{\mathbb{E}[XY]}{8} = 0$

$= \mathbb{E}[X]\cdot\mathbb{E}[Y]$

$\Rightarrow \boxed{f_{X,Y}(x,y) = \frac{1}{16\pi} \exp\left[-\frac{1}{2}\left(\frac{x^2}{4} + \frac{y^2}{16}\right)\right]}$

c) $\boxed{f_{X|Y}(x|y) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{2}\right)^2}}$

(5) $X_1, X_2, \ldots$ independent RVs w/ mean μ & var θ^2

$Y_n = X_n + X_{n+1} + X_{n+2}$. For $j \geq 0$, find $\mathrm{cov}(Y_n, Y_{n+j})$

$$
\begin{aligned}
\mathrm{cov}(Y_n, Y_{n+1}) &= \mathrm{cov}(X_n + X_{n+1} + X_{n+2}, X_{n+1} + Y_{n+2} + X_{n+3}) \\
&= \mathrm{cov}(X_n, X_{n+1} + X_{n+2} + X_{n+3}) + \mathrm{cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \mathrm{cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2} + X_{n+3}) \\
&= \mathrm{cov}(X_{n+1} + X_{n+2}, Y_{n+1} + X_{n+2}) + \mathrm{cov}(X_{n+1} + X_{n+2}, X_{n+3}) \\
&= \mathrm{cov}(X_{n+1} + X_{n+2}, X_{n+1} + X_{n+2}) \\
&= \mathrm{VAR}(X_{n+1} + X_{n+2}) = 2\theta^2
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{cov}(Y_n, Y_{n+2}) &= \mathrm{cov}(X_n + X_{n+1} + X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \\
&= \mathrm{cov}(X_n + X_{n+1}, X_{n+2} + X_{n+3} + X_{n+4}) + \mathrm{cov}(X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \\
&= \mathrm{cov}(X_{n+2}, X_{n+2} + X_{n+3} + X_{n+4}) \\
&= \mathrm{cov}(X_{n+2}, X_{n+2}) + \mathrm{cov}(X_{n+2}, X_{n+3} + X_{n+4}) \\
&= \mathrm{cov}(X_{n+2}, X_{n+2}) = \mathrm{VAR}(X_{n+2}) = \theta^2
\end{aligned}
$$

For $j \geq 3$, $\mathrm{cov}(Y_n, Y_{n+j}) = 0$

$\Rightarrow \boxed{\mathrm{cov}(Y_n, Y_{n+j}) = 3\theta^2}$