ECE 131A	HW #5	Jason Chapman

① a) $X = A-1$, $A \sim \text{geometric}(p)$

$$P(A=a) = (1-p)^{a-1} \cdot p$$

$$P(X=x) = (1-p)^x \cdot p \qquad x \geq 0$$

$$\Phi_X(w) = \mathbb{E}[e^{jwx}]$$

$$= \sum_{x=0}^{\infty} e^{jwx}(1-p)^x p = p \sum_{k=0}^{\infty} [e^{jw}(1-p)]^x \quad // \ \sum ar^k = \frac{a}{1-r}$$

$$= \frac{p}{1-e^{jw}(1-p)} \quad \Rightarrow \quad \boxed{\Phi_X(w) = \frac{p}{1-e^{jw}(1-p)}}$$

b) $Y \sim U[a,b]$

$$\Phi_Y(w) = \mathbb{E}[e^{jwy}]$$

$$= \int_a^b e^{jwy} f_Y(y)\,dy = \int_a^b e^{jwy} \frac{1}{b-a}\,dy = \frac{1}{b-a}\int_a^b e^{jwy}\,dy$$

$$= \frac{1}{b-a}\left[\frac{1}{jw}e^{jwy}\right]_a^b = \frac{1}{b-a}\left[\frac{1}{jw}\left(e^{jwb} - e^{jwa}\right)\right]$$

$$\Rightarrow \quad \boxed{\Phi_Y(w) = \frac{e^{jwb} - e^{jwa}}{(b-a)\,jw}}$$

c.) $Z \sim U[c,d]$ // discrete $P(Z=z) = \frac{1}{d-c+1}$

$$\Phi_Z(w) = \mathbb{E}[e^{jwz}]$$

$$= \sum_{z=c}^{d} e^{jwz}\frac{1}{d-c+1} = \frac{1}{d-c+1}\sum_{z=c}^{d} e^{jwz} = \frac{1}{d-c}\left[e^{jwc} + e^{jwc+1} + e^{jwz} - e^{jw}\right.$$

$$= \frac{1}{d-c+1}\left[e^{jwc} + e^{jwc}e + e^{jwc}e^2 + \dots e^{jwd}\right] \quad // \ e^{jwd} = e^{jwc}e^{jwd}c^{-1}$$

$$\Rightarrow \quad \boxed{\Phi_Z(w) = \frac{e^{jwc} - e^{jw(d+1)}}{(d-c+1)(1-e^{jw})}}$$

(2) $Y = aX + b$, $a>0$, cts. RV

a) $\Phi_X(\omega) = \mathbb{E}[e^{j\omega X}]$

$\Phi_Y(\omega) = \mathbb{E}[e^{j\omega Y}] = \mathbb{E}[e^{j\omega(aX+b)}] = \mathbb{E}[e^{j\omega aX + j\omega b}] = e^{j\omega b} \cdot \mathbb{E}[e^{aj\omega X}]$

$\Rightarrow \boxed{\Phi_Y(\omega) = e^{j\omega b} \cdot \Phi_X(a\omega)}$

b) $X \sim \exp(\lambda)$, $b=0$

$\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega} \Rightarrow \Phi_Y(\omega) = e^{j\omega b} \cdot \frac{\lambda}{\lambda - j\omega a}$

$\frac{d\Phi_Y}{d\omega} = \frac{d}{d\omega} e^{j\omega b}\left(\frac{\lambda}{\lambda - j\omega a}\right) = \frac{ja e^{j\omega b}\lambda}{(\lambda - j\omega a)^2} + \frac{jb e^{j\omega b}\lambda}{(\lambda - j\omega a)}\Big|_{\omega=0} = \frac{ja\lambda}{\lambda^2} + \frac{jb\cancel{\lambda}}{\cancel{\lambda}} = \frac{ja}{\lambda} + jb$

$\mathbb{E}[Y] = \frac{1}{j} \circ \left[\frac{ja}{\lambda} + jb\right] = \frac{a}{\lambda} + b$

$Y \sim \exp\left(\frac{\lambda}{a}\right)$ with a scalar addition of b

c.) $X \sim N(\mu, \sigma^2)$

// Let $Z \sim N(0,1)$

$\Phi_X(\omega) = \mathbb{E}[e^{j\omega X}]$

$= \mathbb{E}[e^{j\omega(\mu + \sigma Z)}] = \mathbb{E}[e^{j\omega\mu} e^{j\omega\sigma Z}]$

$= e^{j\omega\mu} \Phi_Z(\sigma\omega) = e^{j\omega\mu} e^{-\frac{1}{2}(\sigma\omega)^2}$

$\Phi_Y(\omega) = \mathbb{E}[e^{j\omega Y}] = \mathbb{E}[e^{j\omega(aX+b)}] = \mathbb{E}[e^{j\omega aX} e^{j\omega b}]$

$= e^{j\omega b} \Phi_X(a\omega)$

$\Rightarrow \boxed{\Phi_Y(\omega) = e^{j\omega b} e^{j\omega a\mu} e^{-\frac{1}{2}(\sigma a\omega)^2}}$

- For a RV that is a linear transformation of a Gaussian RV, you will get another Gaussian RV w/ Mean $= a\mu + b$ \& Variance $= (\sigma a)^2$

③ Chernoff bound for $X \sim \exp(\lambda)$

$$P(X \geq a) \leq e^{-as} \mathbb{E}[e^{sx}]$$

// we know $\mathbb{E}[e^{jwx}] = \frac{\lambda}{\lambda - jw}$ for $\exp(x)$

$$\Rightarrow \mathbb{E}[e^{sx}] = \frac{\lambda}{\lambda - s} \quad \Rightarrow \quad \boxed{P(X \geq a) \leq e^{-as} \frac{\lambda}{\lambda - s}}$$

$$P(X \geq a) = 1 - P(X \leq a)$$

$$= 1 - F_X(a)$$

$$= 1 - (1 - e^{-\lambda a}) = e^{-\lambda a} \quad \Rightarrow \quad \boxed{P(X \geq a) = e^{-\lambda a}}$$

4.) $Y = \frac{X}{n}$, $X \sim \text{binomial}(p, n)$

$$\{|Y - p| \geq a\} \Rightarrow \left|\frac{X}{n} - p\right| \geq a \Rightarrow \frac{X}{n} - p \leq -a \text{ or } \frac{X}{n} - p \geq a$$

$$\Rightarrow X - np \geq na \quad \Rightarrow X \geq n(a+p)$$

$$P(|X - np| \geq na) \leq \frac{VAR(X)}{(na)^2} \qquad // \; np = \mathbb{E}[X]$$

$$\leq \frac{\cancel{n}p(1-p)}{n^{\cancel{2}}a^2}$$

$$\Rightarrow P(|X - np| \geq na) \leq \frac{p(1-p)}{na^2} \qquad P(|Y - p| \geq a) \leq \frac{p(1-p)}{na^2}$$

→ As $n \to \infty$ the probability that the event $\{|Y - p| \geq a\}$ approaches zero

This implies that the distribution of Y as $n \to \infty$ is that of a bernoulli RV.

⑤ a) $Q(-x) = 1 - Q(x)$

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du$$

$$Q(-x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^\infty e^{-\frac{u^2}{2}} du$$

// gaussian RV are symmetric about mean

$$\Rightarrow Q(-x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^\infty e^{-\frac{u^2}{2}} du = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du + \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{u^2}{2}} du = 1$$ // integral of pdf from $-\infty$ to $\infty = 1$ ✓

$1 = Q(x) + Q(-x) \Rightarrow Q(-x) = 1 - Q(x)$ ✓

b.) $Q(a) = P(X \geq a) = \int_a^\infty f_X(x)\,dx \leq \int_{-\infty}^\infty f_X(x) e^{sx} e^{-sa} dx$

$$\leq e^{-sa} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} e^{sx} dx$$

$$\leq e^{-sa} \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{sx - \frac{x^2}{2}} dx$$

$$\leq \frac{e^{-sa}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^\infty e^{sx - \frac{x^2}{2}} dx}_{\sqrt{2\pi}\, e^{s^2/2}}$$

$$\Rightarrow Q(a) \leq e^{-sa} e^{s^2/2} \leq e^{-sa + \frac{s^2}{2}}$$

$$h(s) = e^{\frac{s^2}{2} - sa}$$

$$h'(s) = e^{\frac{s^2}{2} - sa} \cdot (s - a) = 0 \Rightarrow s = a$$

$$Q(a) \leq e^{-a^2 + \frac{a^2}{2}} \leq e^{-\frac{a^2}{2}}$$

$$\Rightarrow Q(x) \leq e^{-\frac{x^2}{2}}$$ ✓