[§1.6] 3. (a) $\boxed{\text{Addition}}$ where $p =$ "Alice is a mathematics major" and $q =$ "Alice is a computer science major"

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

(e) $\boxed{\text{Modus ponens}}$ where $p =$ "it is rainy" and $q =$ "the pool is closed"

$$\begin{array}{rl} & p \\ & p \to q \\ \hline \therefore & q \end{array}$$

6. Let

$$\begin{aligned} p &= \text{"it rains,"} \\ q &= \text{"it is foggy,"} \\ r &= \text{"the sailing race is held,"} \\ s &= \text{"the lifesaving demonstration will go on," and} \\ t &= \text{"the trophy is awarded."} \end{aligned}$$

Then

1	$(\neg p \vee \neg q) \to (r \wedge s)$	premise
2	$r \to t$	premise
3	$\neg t$	premise
4	$\neg r$	modus tollens with 2 and 3
5	$\neg(r \wedge s) \to \neg(\neg p \vee \neg q)$	contrapositive from 1
6	$\neg(r \wedge s) \to \neg(\neg(p \wedge q))$	De Morgan's law
7	$\neg(r \wedge s) \to (p \wedge q)$	double negation
8	$\neg r \vee \neg s$	addition from 4
9	$\neg(r \wedge s)$	De Morgan's law
10	$p \wedge q$	modus ponens with 7 and 9
11	p	simplification

9. (a) Let

$$\begin{aligned} p &= \text{"I take Tuesday off,"} \\ q &= \text{"I take Thursday off,"} \\ r &= \text{"it rains on Tuesday,"} \\ s &= \text{"it rains on Thursday,"} \\ t &= \text{"it snows on Tuesday," and} \\ u &= \text{"it snows on Thursday."} \end{aligned}$$

Then

	1	$p \to (r \lor t)$	premise
	2	$q \to (s \lor u)$	premise
	3	$p \lor q$	premise
	4	$\neg r \land \neg t$	premise
	5	$\neg u$	premise
	6	$\neg(r \lor t)$	De Morgan's law from 4
★	7	$\neg p$	modus tollens with 1 and 6
★	8	q	disjunctive syllogism with 3 and 7
	9	$s \lor u$	modus ponens with 2 and 8
★	10	s	disjunctive syllogism with 5 and 9

so the conclusions are

7. I did not take Tuesday off,
8. I took Thursday off, and
10. it rained Thursday.

(d) Let the domain consist of people, and

$$P(x) = \text{“}x \text{ is a computer science major” and}$$
$$Q(x) = \text{“}x \text{ has a personal computer.”}$$

Then

	1	$\forall x\,(P(x) \to Q(x))$	premise
	2	$\neg Q(\text{Ralph})$	premise
	3	$Q(\text{Ann})$	premise
★	4	$\neg P(\text{Ralph})$	contrapositive from 1 and 2

so the conclusion is

4. Ralph is not a computer science major.

10c. Let the domain consist of bugs, and

$$P(x) = \text{“}x \text{ are insects,”}$$
$$Q(x) = \text{“}x \text{ have six legs,” and}$$
$$R(x, y) = \text{“}x \text{ eat } y\text{.”}$$

Then

	1	$\forall x\,(P(x) \to Q(x))$	premise
	2	$P(\text{dragonflies})$	premise
	3	$\neg Q(\text{spiders})$	premise
	4	$R(\text{spiders}, \text{dragonflies})$	premise
	5	$(P(a) \to Q(a))$ for any a	universal instantiation from 1
	6	$(\neg Q(a) \to \neg P(a))$ for any a	contrapositive
★	7	$\forall x\,(\neg Q(x) \to \neg P(x))$	universal generalization
	8	$P(\text{spiders}) \to Q(\text{spiders})$	set $a = \text{spiders}$
	9	$P(\text{dragonflies}) \to Q(\text{dragonflies})$	set $a = \text{dragonflies}$ from 5
★	10	$Q(\text{dragonflies})$	modus ponens with 2 and 9
★	11	$\neg P(\text{spiders})$	modus tollens with 3 and 8
	12	$P(\text{dragonflies}) \wedge (\neg Q(\text{spiders})) \wedge R(\text{spiders}, \text{dragonflies})$	conjunction with 2, 3, and 4
★	13	$\exists x\,\exists y\,(P(x) \wedge (\neg Q(y)) \wedge R(y, x))$	existential generalization

so the conclusions are

7. Any bug that does not have six legs is not an insect;
10. dragonflies have six legs;
11. spiders are not insects;
13. there exists a non-insect that eats an insect.

13. (a) Let the domain consist of students, and

$$\begin{aligned} P(x) &= \text{“}x \text{ is in this class,”} \\ Q(x) &= \text{“}x \text{ knows how to write programs in JAVA,” and} \\ R(x) &= \text{“}x \text{ can get a high-paying job.”} \end{aligned}$$

The conclusion, $\exists x\,(P(x) \wedge R(x))$, is arrived from

1	$P(\text{Doug})$	premise
2	$Q(\text{Doug})$	premise
3	$\forall x\,(Q(x) \to R(x))$	premise
4	$Q(\text{Doug}) \to R(\text{Doug})$	universal instantiation
5	$R(\text{Doug})$	modus ponens with 2 and 4
6	$P(\text{Doug}) \wedge R(\text{Doug})$	conjunction with 1 and 5
7	$\exists x\,(P(x) \wedge R(x))$	existential generalization

(b) Let the domain consist of people, and

$$\begin{aligned} P(y) &= \text{“}y \text{ is in this class,”} \\ S(y) &= \text{“}y \text{ enjoys whale watching,” and} \\ T(y) &= \text{“}y \text{ cares about ocean pollution.”} \end{aligned}$$

The conclusion, $\exists y\,(P(y) \wedge T(y))$, is arrived from

1	$\exists y\,(P(y) \wedge S(y))$	premise
2	$\forall y\,(S(y) \rightarrow T(y))$	premise
3	$P(a) \wedge S(a)$ for some person a	existential instantiation from 1
4	$P(a)$ for some person a	simplification
5	$S(a)$ for some person a	simplification from 3
6	$(S(a) \rightarrow T(a))$ for some person a	universal instantiation from 2
7	$T(a)$ for some person a	modus ponens with 5 and 6
8	$(P(a) \wedge T(a))$ for some person a	conjunction with 4 and 7
9	$\exists y\,(P(y) \wedge T(y))$	existential generalization

15. (a) $\boxed{\text{Correct argument.}}$ Let the domain consist of people, and

$$P(x) = \text{“}x \text{ is a student in this class” and}$$
$$Q(x) = \text{“}x \text{ understands logic.”}$$

The conclusion, $Q(\text{Xavier})$, is arrived from

1	$\forall x\,(P(x) \rightarrow Q(x))$	premise
2	$P(\text{Xavier})$	premise
3	$(P(a) \rightarrow Q(a))$ for some person a	universal instantiation from 1
4	$P(\text{Xavier}) \rightarrow Q(\text{Xavier})$	set $a = \text{Xavier}$
5	$Q(\text{Xavier})$	modus ponens with 2 and 4

(c) $\boxed{\text{Incorrect argument.}}$ Let the domain consist of animals, and

$$R(y) = \text{“}y \text{ is a parrot” and}$$
$$S(y) = \text{“}y \text{ likes fruit.”}$$

The conclusion, $\neg S(\text{my pet bird})$, is invalid;

1	$\forall y\,(R(y) \rightarrow S(y))$	premise
2	$\neg R(\text{my pet bird})$	premise
3	$(R(b) \rightarrow S(b))$ for some animal b	universal instantiation from 1
4	$R(\text{my pet bird}) \rightarrow S(\text{my pet bird})$	set $b = \text{my pet bird}$

from here it would be a fallacy of denying the hypothesis to say $(\neg R(\text{my pet bird})) \rightarrow \neg S(\text{my pet bird})$. It is indeterminate whether “my pet bird” likes fruit.

17. The argument misuses existential instantiation. The correct rule is

$$\begin{array}{ll} & \exists x\, H(x) \\ \hline \therefore & H(c) \text{ for some element } c \text{ in the domain} \end{array}$$

where c cannot be set to any *specific* element, such as Lola.

23. The variable c used in (5) cannot be assumed equal to the variable of the same name in (3). A different variable name, perhaps d, would be required in (5).

[§1.7] 3. Proof. Let n be an even number. By definition,

$$n = 2k$$

where k is an integer. Squaring both sides gives

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

Because $2k^2$ is also an integer, the equation above shows that n^2 equals 2 times an integer, that is, an even number. □

5. Proof. By the definition of an even integer,

$$m + n = 2a$$

and

$$n + p = 2b$$

where a and b are integers. Using these equalities, the sum of m and p is

$$\begin{aligned} m + p &= (m + n) + (n + p) - 2n \\ &= 2a + 2b - 2n \\ &= 2(a + b - n). \end{aligned}$$

Because a, b, and n are integers, $a + b - n$ is also an integer, so the above shows $m + p$ equals 2 times an integer. Thus $m + p$ is even. □ $\boxed{\text{This was a direct proof.}}$

9. Proof. (By contradiction.) Let x be an irrational number and y be a rational number. Assume for the purposes of contradiction that the sum of x and y is rational, that is,

$$x + y = \frac{p}{q}$$

where p and q are integers. Then

$$x = \frac{p}{q} - y$$

is equivalent to

$$x = \frac{p}{q} - \frac{r}{s}$$

where r and s are integers, since y is rational. By cross-multiplication on the right hand side,

$$x = \frac{ps - rq}{qs};$$

since $ps - rq$ and qs are integers, x is rational. This contradicts the premise that x is irrational, so the assumption that $x + y$ is rational is false. That is, $x + y$ is irrational. □

10. <u>Proof.</u> Let x and y be rational numbers. By definition, x and y can be expressed as

$$x = \frac{a}{b}$$

and

$$y = \frac{c}{d}$$

where a, b, c, and d are integers. Then the product of x and y is

$$xy = \frac{ac}{bd}$$

where ac and bd are integers because the product of any two integers is an integer. Since xy can be expressed as the ratio of two integers, it is rational. □

15. <u>Proof.</u> (By contraposition.) Assume $((x \geq 1) \vee (y \geq 1))$ is false. Then, by De Morgan's law,

$$(x < 1) \wedge (y < 1),$$

which implies

$$x + y < 2.$$

This is the negation of $(x + y \geq 2)$. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true: $((x \geq 1) \vee (y \geq 1))$. □

26. The biconditional statement to be proven in both directions is that for any positive integer n,

$$n \text{ is even} \Leftrightarrow 7n + 4 \text{ is even.}$$

<u>Proof (⇒).</u> Suppose n is even. Then

$$n = 2k$$

where k is an integer. Then

$$\begin{aligned} 7n + 4 &= 7(2k) + 4 \\ &= 2(7k + 2) \end{aligned}$$

Since $7k + 2$ is also an integer, the above is an expression of $7n + 4$ as 2 times an integer, which implies that $7n + 4$ is even. □

<u>Proof (⇐).</u> Suppose $7n + 4$ is even. Then

$$7n + 4 = 2p$$

where p is an integer. Dividing both sides by 2 gives

$$p = \frac{7}{2}n + 2.$$

Since p is an integer, n is a multiple of 2 so that the fractional term reduces to an integer. Because n is a multiple of 2, n is even. □

33. It suffices to show that for all real numbers x,

$$x \text{ irrational} \xrightarrow{\text{ⓐ}} 3x+2 \text{ irrational} \xrightarrow{\text{ⓑ}} \frac{x}{2} \text{ irrational} \xrightarrow{\text{ⓒ}} x \text{ irrational}.$$

Proof ⓐ. (By contraposition.) Assume $3x+2$ is rational, that is,

$$3x+2 = \frac{p}{q}$$

where p and q are integers. Then, by subtracting 2 from both sides and dividing by 3,

$$\begin{aligned} x &= \frac{1}{3}\left(\frac{p}{q} - 2\right) \\ &= \frac{p-2q}{3q}. \end{aligned}$$

Since both $p - 2q$ and $3q$ are integers, x is rational. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. □

Proof ⓑ. (By contraposition.) Assume $x/2$ is rational, that is,

$$\frac{x}{2} = \frac{r}{s}$$

where r and s are integers. Multiplying both sides by 6 gives

$$3x = \frac{6r}{s},$$

whereupon adding 2 to both sides gives

$$\begin{aligned} 3x+2 &= \frac{6r}{s} + 2 \\ &= \frac{6r+2s}{s}. \end{aligned}$$

Since both $6r+2s$ and s are integers, $3x+2$ is rational. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. □

Proof ⓒ. (By contraposition.) Assume x is rational, that is,

$$x = \frac{u}{v}$$

where u and v are integers. Dividing both sides by 2 gives

$$\frac{x}{2} = \frac{u}{2v}.$$

Since both u and $2v$ are integers, $x/2$ is rational. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. □

41. It suffices to show that for all integers n,

$$n \text{ even} \xrightarrow{ⓐ} n+1 \text{ odd} \xrightarrow{ⓑ} 3n+1 \text{ odd} \xrightarrow{ⓒ} 3n \text{ even} \xrightarrow{ⓓ} n \text{ even}.$$

Proof ⓐ. Suppose n is even. Then

$$n = 2k$$

where k is an integer. Then

$$n+1 = 2k+1$$

where $2k+1$ is an integer that is not divisible by 2, so by definition, $n+1$ is odd. $\square$

Proof ⓑ. Suppose $n+1$ is odd. Then

$$n+1 = 2k+1$$

where k is an integer. Then

$$\begin{aligned} 3n+1 &= 2n + (n+1) \\ &= 2n + (2k+1) \\ &= 2(n+k) + 1, \end{aligned}$$

and since $(n+k)$ is an integer, $2(n+k)+1$ is not divisible by 2. Therefore, $3n+1$ is odd. $\square$

Proof ⓒ. Suppose $3n+1$ is odd. Then

$$3n+1 = 2k+1$$

where k is an integer. Subtracting 1 from both sides gives

$$3n = 2k,$$

which means $3n$ is even since it equals 2 times an integer. $\square$

Proof ⓓ. (By contraposition.) Assume n is odd, that is,

$$n = 2k+1$$

where k is an integer. Then

$$\begin{aligned} 3n &= 3(2k+1) \\ &= 6k+3 \\ &= 6k+2+1 \\ &= 2(3k+1)+1, \end{aligned}$$

and since $(3k+1)$ is an integer, $2(3k+1)+1$ is not divisible by 2, so $3n$ is odd. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. $\square$

[§1.8] 1. Proof. (By exhaustion.) If $n = 1$,

$$(1^1 + 1 = 2) \geq (2^1 = 2);$$

if $n = 2$,

$$(2^2 + 1 = 5) \geq (2^2 = 4);$$

if $n = 3$,

$$(3^2 + 1 = 10) \geq (2^3 = 8);$$

if $n = 4$,

$$(4^2 + 1 = 17) \geq (2^4 = 16).$$

□

3. Proof. There are two cases to consider.

Case 1. Suppose $x \geq y$. Then

$$\max(x, y) = x$$

and

$$\min(x, y) = y,$$

so

$$\max(x, y) + \min(x, y) = x + y.$$

Case 2. Suppose $x < y$. Then

$$\max(x, y) = y$$

and

$$\min(x, y) = x,$$

so

$$\max(x, y) + \min(x, y) = y + x,$$

which, by the commutative property of addition, is equivalent to $x + y$. □