

[§1.6] 3. (a) Addition where p = “Alice is a mathematics major” and q = “Alice is a computer science major”

$$\therefore \frac{p}{p \vee q}$$

(e) Modus ponens where p = “it is rainy” and q = “the pool is closed”

$$\therefore \frac{\begin{array}{c} p \\ p \rightarrow q \end{array}}{q}$$

6. Let

p = “it rains,”

q = “it is foggy,”

r = “the sailing race is held,”

s = “the lifesaving demonstration will go on,” and

t = “the trophy is awarded.”

Then

1	$(\neg p \vee \neg q) \rightarrow (r \wedge s)$	premise
2	$r \rightarrow t$	premise
3	$\neg t$	premise
4	$\neg r$	modus tollens with 2 and 3
5	$\neg(r \wedge s) \rightarrow \neg(\neg p \vee \neg q)$	contrapositive from 1
6	$\neg(r \wedge s) \rightarrow \neg(\neg(p \wedge q))$	De Morgan’s law
7	$\neg(r \wedge s) \rightarrow (p \wedge q)$	double negation
8	$\neg r \vee \neg s$	addition from 4
9	$\neg(r \wedge s)$	De Morgan’s law
10	$p \wedge q$	modus ponens with 7 and 9
11	p	simplification

9. (a) Let

p = “I take Tuesday off,”

q = “I take Thursday off,”

r = “it rains on Tuesday,”

s = “it rains on Thursday,”

t = “it snows on Tuesday,” and

u = “it snows on Thursday.”

Then

	1	$p \rightarrow (r \vee t)$	premise
	2	$q \rightarrow (s \vee u)$	premise
	3	$p \vee q$	premise
	4	$\neg r \wedge \neg t$	premise
	5	$\neg u$	premise
	6	$\neg(r \vee t)$	De Morgan's law from 4
★	7	$\neg p$	modus tollens with 1 and 6
★	8	q	disjunctive syllogism with 3 and 7
	9	$s \vee u$	modus ponens with 2 and 8
★	10	s	disjunctive syllogism with 5 and 9

so the conclusions are

- 7. I did not take Tuesday off,
- 8. I took Thursday off, and
- 10. it rained Thursday.

(d) Let the domain consist of people, and

$P(x)$ = “ x is a computer science major” and

$Q(x)$ = “ x has a personal computer.”

Then

	1	$\forall x (P(x) \rightarrow Q(x))$	premise
	2	$\neg Q(\text{Ralph})$	premise
	3	$Q(\text{Ann})$	premise
★	4	$\neg P(\text{Ralph})$	contrapositive from 1 and 2

so the conclusion is

- 4. Ralph is not a computer science major.

10c. Let the domain consist of bugs, and

$P(x)$ = “ x are insects,”

$Q(x)$ = “ x have six legs,” and

$R(x, y)$ = “ x eat y .”

Then

	1	$\forall x (P(x) \rightarrow Q(x))$	premise
	2	$P(\text{dragonflies})$	premise
	3	$\neg Q(\text{spiders})$	premise
	4	$R(\text{spiders}, \text{dragonflies})$	premise
	5	$(P(a) \rightarrow Q(a))$ for any a	universal instantiation from 1
	6	$(\neg Q(a) \rightarrow \neg P(a))$ for any a	contrapositive
★	7	$\forall x (\neg Q(x) \rightarrow \neg P(x))$	universal generalization
	8	$P(\text{spiders}) \rightarrow Q(\text{spiders})$	set $a = \text{spiders}$
	9	$P(\text{dragonflies}) \rightarrow Q(\text{dragonflies})$	set $a = \text{dragonflies}$ from 5
★	10	$Q(\text{dragonflies})$	modus ponens with 2 and 9
★	11	$\neg P(\text{spiders})$	modus tollens with 3 and 8
	12	$P(\text{dragonflies}) \wedge (\neg Q(\text{spiders})) \wedge R(\text{spiders}, \text{dragonflies})$	conjunction with 2, 3, and 4
★	13	$\exists x \exists y (P(x) \wedge (\neg Q(y)) \wedge R(y, x))$	existential generalization

so the conclusions are

- 7. Any bug that does not have six legs is not an insect;
 - 10. dragonflies have six legs;
 - 11. spiders are not insects;
 - 13. there exists a non-insect that eats an insect.
13. (a) Let the domain consist of students, and

$P(x) = \text{"}x \text{ is in this class,"}$

$Q(x) = \text{"}x \text{ knows how to write programs in JAVA,"}$ and

$R(x) = \text{"}x \text{ can get a high-paying job."}$

The conclusion, $\exists x (P(x) \wedge R(x))$, is arrived from

1	$P(\text{Doug})$	premise
2	$Q(\text{Doug})$	premise
3	$\forall x (Q(x) \rightarrow R(x))$	premise
4	$Q(\text{Doug}) \rightarrow R(\text{Doug})$	universal instantiation
5	$R(\text{Doug})$	modus ponens with 2 and 4
6	$P(\text{Doug}) \wedge R(\text{Doug})$	conjunction with 1 and 5
7	$\exists x (P(x) \wedge R(x))$	existential generalization

- (b) Let the domain consist of people, and

$P(y) = \text{"}y \text{ is in this class,"}$

$S(y) = \text{"}y \text{ enjoys whale watching,"}$ and

$T(y) = \text{"}y \text{ cares about ocean pollution."}$

The conclusion, $\exists y (P(y) \wedge T(y))$, is arrived from

1	$\exists y (P(y) \wedge S(y))$	premise
2	$\forall y (S(y) \rightarrow T(y))$	premise
3	$P(a) \wedge S(a)$ for some person a	existential instantiation from 1
4	$P(a)$ for some person a	simplification
5	$S(a)$ for some person a	simplification from 3
6	$(S(a) \rightarrow T(a))$ for some person a	universal instantiation from 2
7	$T(a)$ for some person a	modus ponens with 5 and 6
8	$(P(a) \wedge T(a))$ for some person a	conjunction with 4 and 7
9	$\exists y (P(y) \wedge T(y))$	existential generalization

15. (a) Correct argument. Let the domain consist of people, and

$P(x)$ = “ x is a student in this class” and

$Q(x)$ = “ x understands logic.”

The conclusion, $Q(\text{Xavier})$, is arrived from

1	$\forall x (P(x) \rightarrow Q(x))$	premise
2	$P(\text{Xavier})$	premise
3	$(P(a) \rightarrow Q(a))$ for some person a	universal instantiation from 1
4	$P(\text{Xavier}) \rightarrow Q(\text{Xavier})$	set $a = \text{Xavier}$
5	$Q(\text{Xavier})$	modus ponens with 2 and 4

- (c) Incorrect argument. Let the domain consist of animals, and

$R(y)$ = “ y is a parrot” and

$S(y)$ = “ y likes fruit.”

The conclusion, $\neg S(\text{my pet bird})$, is invalid;

1	$\forall y (R(y) \rightarrow S(y))$	premise
2	$\neg R(\text{my pet bird})$	premise
3	$(R(b) \rightarrow S(b))$ for some animal b	universal instantiation from 1
4	$R(\text{my pet bird}) \rightarrow S(\text{my pet bird})$	set $b = \text{my pet bird}$

from here it would be a fallacy of denying the hypothesis to say $(\neg R(\text{my pet bird})) \rightarrow \neg S(\text{my pet bird})$. It is indeterminate whether “my pet bird” likes fruit.

17. The argument misuses existential instantiation. The correct rule is

$$\therefore \frac{\exists x H(x)}{H(c) \text{ for some element } c \text{ in the domain}}$$

where c cannot be set to any *specific* element, such as Lola.

23. The variable c used in (5) cannot be assumed equal to the variable of the same name in (3). A different variable name, perhaps d , would be required in (5).

[§1.7] 3. Proof. Let n be an even number. By definition,

$$n = 2k$$

where k is an integer. Squaring both sides gives

$$\begin{aligned} n^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2(2k^2). \end{aligned}$$

Because $2k^2$ is also an integer, the equation above shows that n^2 equals 2 times an integer, that is, an even number. \square

5. Proof. By the definition of an even integer,

$$m + n = 2a$$

and

$$n + p = 2b$$

where a and b are integers. Using these equalities, the sum of m and p is

$$\begin{aligned} m + p &= (m + n) + (n + p) - 2n \\ &= 2a + 2b - 2n \\ &= 2(a + b - n). \end{aligned}$$

Because a , b , and n are integers, $a + b - n$ is also an integer, so the above shows $m + p$ equals 2 times an integer. Thus $m + p$ is even. \square This was a direct proof.

9. Proof. (By contradiction.) Let x be an irrational number and y be a rational number. Assume for the purposes of contradiction that the sum of x and y is rational, that is,

$$x + y = \frac{p}{q}$$

where p and q are integers. Then

$$x = \frac{p}{q} - y$$

is equivalent to

$$x = \frac{p}{q} - \frac{r}{s}$$

where r and s are integers, since y is rational. By cross-multiplication on the right hand side,

$$x = \frac{ps - rq}{qs};$$

since $ps - rq$ and qs are integers, x is rational. This contradicts the premise that x is irrational, so the assumption that $x + y$ is rational is false. That is, $x + y$ is irrational. \square

10. Proof. Let x and y be rational numbers. By definition, x and y can be expressed as

$$x = \frac{a}{b}$$

and

$$y = \frac{c}{d}$$

where a , b , c , and d are integers. Then the product of x and y is

$$xy = \frac{ac}{bd}$$

where ac and bd are integers because the product of any two integers is an integer. Since xy can be expressed as the ratio of two integers, it is rational. \square

15. Proof. (By contraposition.) Assume $((x \geq 1) \vee (y \geq 1))$ is false. Then, by De Morgan's law,

$$(x < 1) \wedge (y < 1),$$

which implies

$$x + y < 2.$$

This is the negation of $(x + y \geq 2)$. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true: $((x \geq 1) \vee (y \geq 1))$. \square

26. The biconditional statement to be proven in both directions is that for any positive integer n ,

$$n \text{ is even} \Leftrightarrow 7n + 4 \text{ is even.}$$

Proof (\Rightarrow). Suppose n is even. Then

$$n = 2k$$

where k is an integer. Then

$$\begin{aligned} 7n + 4 &= 7(2k) + 4 \\ &= 2(7k + 2). \end{aligned}$$

Since $7k + 2$ is also an integer, the above is an expression of $7n + 4$ as 2 times an integer, which implies that $7n + 4$ is even. \square

Proof (\Leftarrow). Suppose $7n + 4$ is even. Then

$$7n + 4 = 2p$$

where p is an integer. Dividing both sides by 2 gives

$$p = \frac{7}{2}n + 2.$$

Since p is an integer, n is a multiple of 2 so that the fractional term reduces to an integer. Because n is a multiple of 2, n is even. \square

33. It suffices to show that for all real numbers x ,

$$x \text{ irrational} \xrightarrow{\textcircled{a}} 3x + 2 \text{ irrational} \xrightarrow{\textcircled{b}} \frac{x}{2} \text{ irrational} \xrightarrow{\textcircled{c}} x \text{ irrational}.$$

Proof \textcircled{a} . (By contraposition.) Assume $3x + 2$ is rational, that is,

$$3x + 2 = \frac{p}{q}$$

where p and q are integers. Then, by subtracting 2 from both sides and dividing by 3,

$$\begin{aligned} x &= \frac{1}{3} \left(\frac{p}{q} - 2 \right) \\ &= \frac{p - 2q}{3q}. \end{aligned}$$

Since both $p - 2q$ and $3q$ are integers, x is rational. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. \square

Proof \textcircled{b} . (By contraposition.) Assume $x/2$ is rational, that is,

$$\frac{x}{2} = \frac{r}{s}$$

where r and s are integers. Multiplying both sides by 6 gives

$$3x = \frac{6r}{s},$$

whereupon adding 2 to both sides gives

$$\begin{aligned} 3x + 2 &= \frac{6r}{s} + 2 \\ &= \frac{6r + 2s}{s}. \end{aligned}$$

Since both $6r + 2s$ and s are integers, $3x + 2$ is rational. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. \square

Proof \textcircled{c} . (By contraposition.) Assume x is rational, that is,

$$x = \frac{u}{v}$$

where u and v are integers. Dividing both sides by 2 gives

$$\frac{x}{2} = \frac{u}{2v}.$$

Since both u and $2v$ are integers, $x/2$ is rational. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. \square

41. It suffices to show that for all integers n ,

$$n \text{ even} \xrightarrow{\textcircled{a}} n + 1 \text{ odd} \xrightarrow{\textcircled{b}} 3n + 1 \text{ odd} \xrightarrow{\textcircled{c}} 3n \text{ even} \xrightarrow{\textcircled{d}} n \text{ even}.$$

Proof Ⓐ. Suppose n is even. Then

$$n = 2k$$

where k is an integer. Then

$$n + 1 = 2k + 1$$

where $2k + 1$ is an integer that is not divisible by 2, so by definition, $n + 1$ is odd. \square

Proof Ⓑ. Suppose $n + 1$ is odd. Then

$$n + 1 = 2k + 1$$

where k is an integer. Then

$$\begin{aligned} 3n + 1 &= 2n + (n + 1) \\ &= 2n + (2k + 1) \\ &= 2(n + k) + 1, \end{aligned}$$

and since $(n + k)$ is an integer, $2(n + k) + 1$ is not divisible by 2. Therefore, $3n + 1$ is odd. \square

Proof Ⓒ. Suppose $3n + 1$ is odd. Then

$$3n + 1 = 2k + 1$$

where k is an integer. Subtracting 1 from both sides gives

$$3n = 2k,$$

which means $3n$ is even since it equals 2 times an integer. \square

Proof Ⓓ. (By contraposition.) Assume n is odd, that is,

$$n = 2k + 1$$

where k is an integer. Then

$$\begin{aligned} 3n &= 3(2k + 1) \\ &= 6k + 3 \\ &= 6k + 2 + 1 \\ &= 2(3k + 1) + 1, \end{aligned}$$

and since $(3k + 1)$ is an integer, $2(3k + 1) + 1$ is not divisible by 2, so $3n$ is odd. Because the negation of the conclusion of the conditional statement implies that the hypothesis is false, the original conditional statement is true. \square

[§1.8] 1. Proof. (By exhaustion.) If $n = 1$,

$$(1^1 + 1 = 2) \geq (2^1 = 2);$$

if $n = 2$,

$$(2^2 + 1 = 5) \geq (2^2 = 4);$$

if $n = 3$,

$$(3^2 + 1 = 10) \geq (2^3 = 8);$$

if $n = 4$,

$$(4^2 + 1 = 17) \geq (2^4 = 16).$$

□

3. Proof. There are two cases to consider.

Case 1. Suppose $x \geq y$. Then

$$\max(x, y) = x$$

and

$$\min(x, y) = y,$$

so

$$\max(x, y) + \min(x, y) = x + y.$$

Case 2. Suppose $x < y$. Then

$$\max(x, y) = y$$

and

$$\min(x, y) = x,$$

so

$$\max(x, y) + \min(x, y) = y + x,$$

which, by the commutative property of addition, is equivalent to $x + y$. □