[§5.1] 3. (a) $P(1) =$ “$1^2 = \dfrac{1(1+1)(2\cdot 1+1)}{6}$”

(b) $1^2 = 1$ and $\dfrac{1(1+1)(2\cdot 1+1)}{6} = \dfrac{6}{6} = 1$. Thus $P(1)$ is true.

(c) $P(k) =$ “$k^2 = \dfrac{k(k+1)(2k+1)}{6}$” for some positive integer k.

(d) $P(k) \to P(k+1)$ is true for all positive integers k.

(e) Assume $P(k)$ is true for some positive integer k. It will be shown that $P(k+1)$ is true, namely, that

$$1^2 + 2^2 \cdots + (k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

is also true. Adding $(k+1)^2$ to both sides of the equation in $P(k)$ gives

$$\begin{aligned}
1^2 + \cdots + k^2 + (k+1)^2 &\overset{\text{IH}}{=} \frac{k(k+1)(2k+1)}{6} + (k+1)^2 && \text{by the inductive hypothesis}\\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}\\
&= \frac{(k+1)(k(2k+1)+6(k+1))}{6}\\
&= \frac{(k+1)(2k^2+7k+6)}{6}\\
&= \frac{(k+1)(k+2)(2k+3)}{6}.
\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$. This completes the inductive step.

(f) Part (b) shows $P(1)$ is true and part (e) shows $\forall k \in \mathbb{Z}^+ \ (P(k) \to P(k+1))$ is true. By the principle of mathematical induction, $\forall n \in \mathbb{Z}^+ \ P(n)$ is true.

6. *Proof by induction.* Let $P(n) =$ “$1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$” for all $n \in \mathbb{Z}^+$.

Basis step. $P(1)$ is true: $1\cdot 1! = 1$ and $(1+1)! - 1 = 1$.

Inductive step. Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$. It will be shown that $P(k+1)$ is true, namely, that

$$1\cdot 1! + 2\cdot 2! + \cdots + (k+1)\cdot(k+1)! = (k+2)! - 1$$

is also true. Adding $(k+1)\cdot(k+1)!$ to both sides of the equation in $P(k)$ gives

$$\begin{aligned}
1\cdot 1! + \cdots + k\cdot k! + (k+1)\cdot(k+1)! &\overset{\text{IH}}{=} (k+1)! - 1 + (k+1)\cdot(k+1)! && \text{by the ind. hyp.}\\
&= \underbrace{(k+1)!\cdot(1+(k+1))}_{1\cdots(k+1)(k+2)} - 1\\
&= (k+2)! - 1
\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$. This completes the inductive step.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$. □

10. (a) The formula is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

n	expression value
1	1/2
2	2/3
3	3/4
4	4/5

(b) *Proof by induction.* Let $P(n) =$ "$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$" for all $n \in \mathbb{Z}^+$.

Basis step. $P(1)$ is true: $\dfrac{1}{1(1+1)}=\dfrac{1}{2}$ and $\dfrac{1}{1+1}=\dfrac{1}{2}$.

Inductive step. Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$. It will be shown that $P(k+1)$ is true, namely, that

$$\frac{1}{1\cdot 2}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

is also true. Adding $\frac{1}{(k+1)(k+2)}$ to both sides of the equation in $P(k)$ gives

$$\begin{aligned}\frac{1}{1\cdot 2}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} &\overset{\text{IH}}{=} \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} && \text{by the ind. hyp.}\\ &= \frac{k(k+2)+1}{(k+1)(k+2)}\\ &= \frac{k^2+2k+1}{(k+1)(k+2)}\\ &= \frac{(k+1)^2}{(k+1)(k+2)}\\ &= \frac{k+1}{k+2}\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$. This completes the inductive step.
By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$

18. (a) $P(2) =$ "$2! < 2^2$"
(b) $2! = 2$, which is less than $2^2 = 4$.
(c) $P(k) =$ "$k! < k^k$" for some integer $k > 1$.
(d) $P(k) \to P(k+1)$ is true for all positive integers $k > 1$.
(e) Assume $P(k)$ is true for some integer $k > 1$. It will be shown that $P(k+1)$ is true, namely, that

$$(k+1)! < (k+1)^{k+1}$$

is also true. Multiplying both sides of the equation in $P(k)$ by $(k+1)$ gives

$$\begin{aligned}k!(k+1) &\overset{\text{IH}}{<} k^k(k+1) && \text{by the ind. hyp.}\\ &< (k+1)^k(k+1)\\ &= (k+1)^{k+1}\end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$. This completes the inductive step.

(f) Part (b) shows $P(2)$ is true and part (e) shows $(P(k) \to P(k+1))$ is true for all integers $k > 1$.

By the principle of mathematical induction, $P(n)$ is true for all integers $n > 1$.

21. *Proof by induction.* Let $P(n) =$ "$2^n > n^2$" for all integers $n > 4$.

Basis step. $P(5)$ is true: $2^5 = 32$ is greater than $5^2 = 25$.

Inductive step. Assume $P(k)$ is true for some integer $k > 4$. It will be shown that $P(k+1)$ is true, namely, that

$$2^{k+1} > (k+1)^2$$

is also true. Multiplying both sides of the equation in $P(k)$ by 2 gives

$$2^k 2 \overset{\text{IH}}{>} 2k^2$$

by the inductive hypothesis. Now, observe that (since $k > 4$)

$$\begin{aligned}(k+1)^2 &= k^2 + 2k + 1 \\ &< k^2 + 3k \\ &< k^2 + k^2\end{aligned}$$

where the final term on the right hand side is equal to $2k^2$. Hence

$$2^k 2 = 2^{k+1} > 2k^2 > (k+1)^2$$

and so $P(k+1)$ follows from $P(k)$. This completes the inductive step.

By the principle of mathematical induction, $P(n)$ is true for all integers $n > 4$. $\square$

32. *Proof by induction.* Let $P(n) =$ "$3 \mid (n^3 + 2n)$" for all $n \in \mathbb{Z}^+$.

Basis step. $P(1)$ is true: $1^3 + 2 \cdot 1 = 3$, which is divisible by 3.

Inductive step. Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$. It will be shown that $P(k+1)$ is true, namely, that

$$3 \mid ((k+1)^3 + 2(k+1))$$

is also true. Algebraic manipulation gives

$$\begin{aligned}(k+1)^3 + 2(k+1) &= k^3 + 3k^2 + 5k + 3 \\ &= \underbrace{k^3 + 2k}_{\text{dividend in } P(k)} + 3(k^2 + k + 1),\end{aligned}$$

where the bracketed term is divisible by 3 by the inductive hypothesis, and the remaining term is divisible by 3 because it is an integer multiple of 3.

Thus the entire sum is divisible by 3, and $P(k+1)$ follows from $P(k)$. This completes the inductive step.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$

38. *Proof by induction.* Let $P(n) =$ "$A_j \subseteq B_j$ for $j = 1, \ldots, n \Rightarrow \displaystyle\bigcup_{j=1}^{n} A_j \subseteq \bigcup_{j=1}^{n} B_j$" for all $n \in \mathbb{Z}^+$.

Basis step. $P(1)$ is true: $A_1 \subseteq B_1 \Rightarrow \bigcup_{j=1}^{1} A_j \subseteq \bigcup_{j=1}^{1} B_j$ because the union of one set is itself.

Inductive step. Assume $P(k)$ is true for some $k \in \mathbb{Z}^+$. It will be shown that $P(k+1)$ is true, namely, that

$$A_{k+1} \subseteq B_{k+1} \Rightarrow \bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j$$

is also true. It suffices to show $\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j$ is true.

First, observe that

$$\bigcup_{j=1}^{k+1} A_j = \left(\bigcup_{j=1}^{k} A_j\right) \cup A_{k+1} \qquad \text{and} \qquad \bigcup_{j=1}^{k+1} B_j = \left(\bigcup_{j=1}^{k} B_j\right) \cup B_{k+1}.$$

Next, denote an arbitrary element $x \in \bigcup_{j=1}^{k+1} A_j$; equivalently,

$$\left(x \in \bigcup_{j=1}^{k} A_j\right) \vee \left(x \in A_{k+1}\right).$$

Case 1. Suppose $x \in \bigcup_{j=1}^{k} A_j$.

By the inductive hypothesis and by the definition of a subset, $x \in \bigcup_{j=1}^{k} B_j$.

Case 2. Suppose $x \in A_{k+1}$.

By the premise in $P(k+1)$ and by the definition of a subset, $x \in B_{k+1}$.

Hence

$$\left(x \in \bigcup_{j=1}^{k} B_j\right) \vee \left(x \in B_{k+1}\right);$$

equivalently, $x \in \bigcup_{j=1}^{k+1} B_j$. Now since

$$x \in \bigcup_{j=1}^{k+1} A_j \Rightarrow x \in \bigcup_{j=1}^{k+1} B_j$$

is true, it must also be true (by the definition of a subset) that

$$\bigcup_{j=1}^{k+1} A_j \subseteq \bigcup_{j=1}^{k+1} B_j,$$

so $P(k+1)$ follows from $P(k)$. This completes the inductive step.

By the principle of mathematical induction, $P(n)$ is true for all integers $n \in \mathbb{Z}^+$. $\square$

49. The assertion that the first k and last k horses overlap,

$$\underbrace{1, 2, 3, \ldots, k}_{\text{same color}}, k+1 \qquad \text{and} \qquad 1, \underbrace{2, 3, \ldots, k, k+1}_{\text{same color}},$$

is false when $k = 1$. Then no horses exist between 1 and $k+1$, which may be different colors.

[§5.2] 3. (a) $P(8) = 1 \cdot 3¢ + 1 \cdot 5¢$
$P(9) = 3 \cdot 3¢ + 0 \cdot 5¢$
$P(10) = 0 \cdot 3¢ + 2 \cdot 5¢$

(b) $P(n)$ is true for $8 \leq n \leq k$ where $k \geq 10$ and n, k are integers.

(c) $[P(8) \wedge P(9) \wedge P(10) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$ for every integer $k \geq 10$.

(d) If $k \geq 10$, then

$$k + 1 = \underbrace{(k-2)}_{\geq 8} + 3.$$

By the inductive hypothesis, $P(k-2)$ is true. The above equation shows that by adding one 3¢ stamp, $P(k+1)$ is true.

(e) The basis step in part (a) and the inductive step in part (d) have been completed, so by strong induction, $P(n)$ is true for all positive integers $n \geq 8$.

7. Any dollar amount in $\mathbb{N}$ except \$1 and \$3 can be formed.

\$2 is one \$2 bill. \$4 is two \$2 bills. Let $P(n)$ be the statement that any integer $n \geq 5$ dollars can be formed using just \$2 and \$5 bills.

Proof (by strong induction).

Basis step. $P(5) = 0 \cdot \$2 + 1 \cdot \$5 \qquad P(6) = 3 \cdot \$2 + 0 \cdot \$5$.

Inductive step. The inductive hypothesis is the statement that $P(j)$ is true for all $5 \leq j \leq k$ where $k \geq 6$ and j, k are integers.

Since $k - 1 \geq 5$, $P(k-1)$ is true by the inductive hypothesis. Since $k-1$ dollars can be formed, $k+1$ dollars can be formed by adding one \$2 bill.

Thus $P(k+1)$ is true. This completes the inductive step.

By strong induction, $P(n)$ is true for all integers $n \geq 5$. □