FIT3158
Business Decision Modelling

SEMESTER 2, 2022

Lecture 12

Queuing Theory Cont.

Review and Exam Preparation

Queuing Systems

- Kendall notation is a three part code of the form *A*/*B*/*s,* used to describe various queuing systems.
- *A* identifies the arrival distribution, *B* the service distribution and *s* the number of servers in the system.
- Frequently used symbols for the arrival and service processes are: *M* - Markov distributions (Poisson/exponential), *D* - Deterministic (constant) and *G* - General distribution (with a known mean and variance).
- For example, *M*/*M*/*k* refers to a system in which arrivals occur according to a Poisson distribution, service times follow an exponential distribution and there are *k* servers working at identical service rates.
- The doctor's waiting room example is a G/G/1 queue.

Structure of a Queue

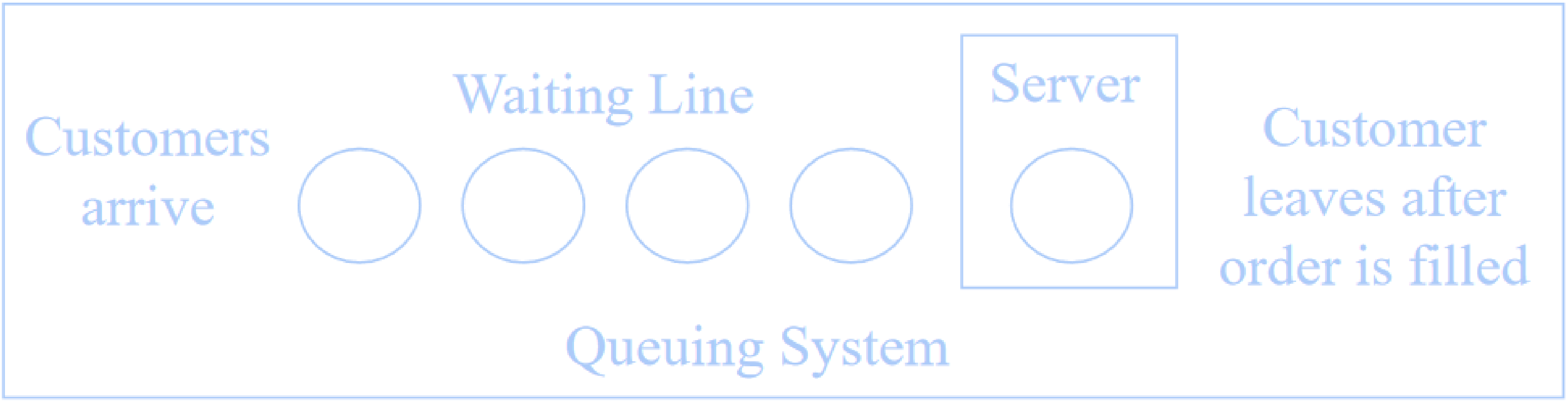
Four characteristics of a queuing system are:

the manner in which customers arrive

the time required for service

the priority determining the order of service

the number and configuration of servers in the system.



M/G/1 Queuing System

- This is a queuing system with exponentially distributed times between customers, however, customer service times have a distribution characterised by a mean and standard deviation.

$$\lambda = \text{the average number of arrivals per time period}$$

$$\mu = \text{the average number of services per time period}$$

$$\frac{1}{\mu} = \text{the average service time}$$

$$\sigma = \text{the standard deviation of service time}$$

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda/\mu)^2}{2(1-\lambda/\mu)}, \; L = L_q + \frac{\lambda}{\mu}$$

Other results as for M/M/1 case.

M/G/1 Example

Students arrive at a service counter at the rate of 8 students per hour. Each student takes, on average, 5 minutes to serve. Assume that the time between arrivals is exponentially distributed, however, the standard deviation of service time is 3 minutes (0.05 hours)

a) What is the average length of the queue?

$$L_q = \frac{\lambda^2\sigma^2 + (\lambda/\mu)^2}{2(1-\lambda/\mu)} = \frac{8^2 \times 0.05^2 + (8/12)^2}{2(1-8/12)} = 0.907$$

b) What is the average number of students waiting at any time?

$$W_q = \frac{L_q}{\lambda} = \frac{0.907}{8} = 0.113$$

M/G/1 Example Cont…

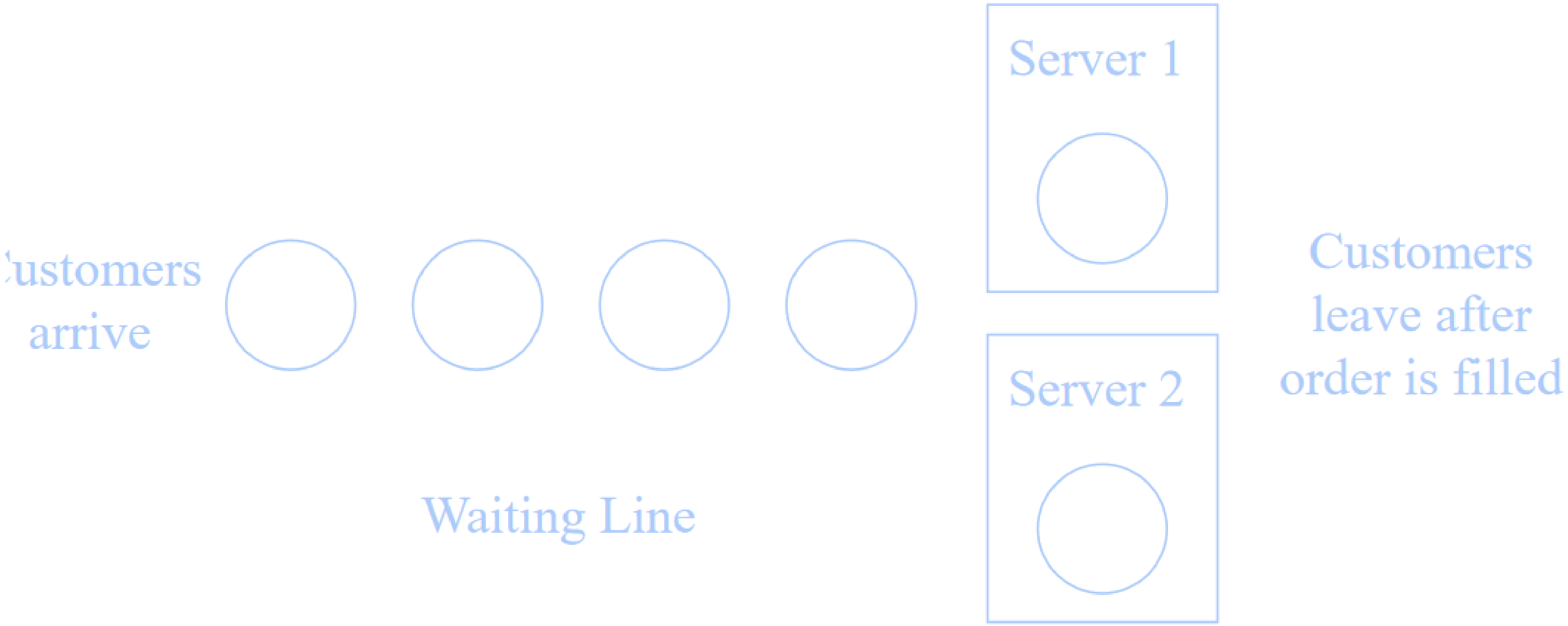
- Examine the sensitivity of the steady state performance characteristics of the waiting line to the standard deviation of student service time (varying σ by one minute at a time).

The mean arrival rate for the system (lambda)	8	8	8	***8***	8	8	8
The mean service rate for each channel (mu)	12	12	12	***12***	12	12	12
The standard deviation of service time (sigma)	0	0.017	0.033	**0.05**	0.067	0.083	0.1
Number of units in the system (n)	6	6	6	***6***	6	6	6
Operating Characteristics							
The probability that no units are in the system (Po)	0.33	0.33	0.33	***0.33***	0.33	0.33	0.33
The average number of units in the waiting line (Lq)	0.67	0.69	0.77	***0.91***	1.09	1.33	1.63
The average number of units in the system (L)	1.33	1.36	1.44	***1.57***	1.76	2.00	2.29
The average time a unit is in the waiting line (Wq)	0.08	0.09	0.10	***0.11***	0.14	0.17	0.20
The average time a unit spends in the system (W)	0.17	0.17	0.18	***0.20***	0.22	0.25	0.29
The probability that a unit has to wait (Pw)	0.67	0.67	0.67	***0.67***	0.67	0.67	0.67
Probability of n units in the system (Pn)	0.03	0.03	0.03	***0.03***	0.03	0.03	0.03
Server utilisation factor (rho)	0.67	0.67	0.67	***0.67***	0.67	0.67	0.67

- Refer Lecture 12.xlsm

M/M/S Queuing System

- This is a queuing system with exponentially distributed times between customers and exponentially distributed service times. A single queue is served by a multiple servers. For example, the M/M/2 case is:



Operating Characteristics of M/M/S Queues

- Some steady-state (long-term average) characteristics of an M/M/S Queue are:

$$P_0 = 1 \Bigg/ \left[\sum_{n=0}^{S-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^S}{S!} \left(\frac{1}{1-\lambda/S\mu} \right) \right]$$

$$L = L_q + \frac{\lambda}{\mu}$$

$$P_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} P_0 & \text{if } 0 \le n \le S \\ \dfrac{(\lambda/\mu)^n}{S!S^{n-S}} P_0 & \text{if } n \ge S \end{cases}$$

$$W_q = \frac{L_q}{\lambda}$$

$$W = W_q + \frac{1}{\mu}$$

$$L_q = \frac{(\lambda/\mu)^S (\lambda/S\mu)}{S!(1-\lambda/S\mu)^2} P_0$$

$$\rho = \frac{\lambda}{S\mu}$$

M/M/S Example

During the summer enrolment period, students arrive at a service counter at the rate of 20 students per hour. Each student takes, on average, 5 minutes to serve. Assume that the time between arrivals, and the service time are exponentially distributed.

a) When the counter is staffed by 2 servers, what is the average probability of no students waiting or being served.

$$P_0 = 1 \Big/ \left[\sum_{n=0}^{S-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^S}{S!} \left(\frac{1}{1-\lambda/S\mu} \right) \right]$$

$$= 1 \Big/ \left[\frac{(20/12)^0}{0!} + \frac{(20/12)^1}{1!} + \frac{(20/12)^2}{2!} \left(\frac{1}{1-20/24} \right) \right] = 0.9091$$

M/M/S Example Cont…

- Examine the sensitivity of the steady state performance characteristics of the waiting line to the number of servers.

The mean arrival rate for the system (lambda)	20	20	20	20	20
The mean service rate for each channel (mu)	12	12	12	12	12
The number of channels (S)	2	3	4	5	6
Number of units in the system (n)	6	6	6	6	6
Operating Characteristics					
The probability that no units are in the system (Po)	0.091	0.173	0.186	0.188	0.189
The average number of units in the waiting line (Lq)	3.788	0.375	0.073	0.015	0.003
The average number of units in the system (L)	5.455	2.041	1.740	1.682	1.670
The average time a unit spends in the waiting line (Wq)	0.189	0.019	0.004	0.001	0.000
The average time a unit spends in the system (W)	0.273	0.102	0.087	0.084	0.083
The probability that a unit arriving has to wait for service (Pw)	0.758	0.300	0.102	0.030	0.008
Probability of n units in the system (Pn)	0.061	0.023	0.010	0.007	0.006
Server utilisation factor (rho)	0.833	0.556	0.417	0.333	0.278

- Refer Lecture 12.xlsm

Economic Analysis of Queuing Systems

- We may want to determine the total cost of a queuing system in order to determine the service model that minimises the total cost of the system.
- This approach requires that the cost of service is known (usually $/hour per server or machine) and that the cost of waiting is known.
- When a company is serving its staff, for example, the cost of waiting is easily determined (for example an in-house call centre), when customers are external to the company, a cost of waiting must be estimated. This could include penalty for customer dissatisfaction.

Economic Analysis Cont…

- Determine the optimal number of servers in a call centre when staff calls are received at the rate of 25 per hour. Servers are able to cope with, on average, 15 enquiries per hour. Call centre employees cost $20 per hour and the waiting time for staff is calculated at $10 per hour.
- Revise your estimate of the number of servers for management, whose waiting time is calculated at $80 per hour.

$$\text{Waiting Cost} = L_q \times \text{Wait Cost/Person/Time}$$

$$\text{Service Cost} = \text{Number of Servers} \times \text{Service Cost/Server/Time}$$

Economic Analysis Cont…

Economic Analysis of an M/M/S Queue

The mean arrival rate for the system (lambda)	25	25	25	25	25
The mean service rate for each channel (mu)	15	15	15	15	15
The number of channels (S)	2	3	4	5	6
Cost $/hour Server	20	20	20	20	20
Cost $/hour Customer	10	10	10	10	10
Operating Characteristics					
The probability that no units are in the system (Po)	0.091	0.173	0.186	0.188	0.189
The average number of units in the waiting line (Lq)	3.788	0.375	0.073	0.015	0.003
The average time a unit spends in the waiting line (Wq)	0.152	0.015	0.003	0.001	0.000
The average time a unit spends in the system (W)	0.218	0.082	0.070	0.067	0.067
Server utilisation factor (rho)	0.833	0.556	0.417	0.333	0.278
Economic Analysis					
Cost of Service $/hour	40.00	60.00	80.00	100.00	120.00
Cost of Waiting $/hour	37.88	3.75	0.73	0.15	0.03
Total Cost of System $/hour	77.88	63.75	80.73	100.15	120.03

Review Questions & Topic Summary

Week 1

- Course Overview.
- Introduction to modelling and decision analysis
  - What is Business Analytics and why it is useful for businesses today?
  - Characteristics and benefits of modelling
  - Discuss the modelling approach to decision making
  - Discuss the problem-solving framework for leveraging business opportunities

Week 1 Cont…

- Introduction to Optimisation and Linear Programming
  - Explore the use of linear programming (LP) in solving optimisation problems
  - Formulating linear problems;
  - Solving two dimensional linear problems:
- Solving LP problems using graphical approach
  - Graphically using level curves,
  - Enumerating corner points.

Question 1 – LP Problem

- The Quality Desk Company makes two types of computer desks from laminated particle board.
- The Presidential model requires 30 square feet of particle board, 1 keyboard sliding mechanism, and 5 hours of labour to fabricate. It sells for $149.
- The Senator model requires 24 square feet of particle board, 1 keyboard sliding mechanism, and 3 hours of labour to fabricate. It sells for $135.
- In the coming week, the company can buy up to 15,000 square feet of particle board at $1.35 per square foot and up to 600 keyboard sliding mechanisms at a cost of $4.75 each. The company views manufacturing labour as fixed cost and has 3000 labour hours available in the coming week for the fabrication of these desks.

Formulating the LP Model

Let:

P = number of Presidential desks produced,

S = number of Senator desks produced

MAX: $103.75\,P + 97.85\,S$

ST:

$$30\,P + 24\,S \le 15{,}000$$

$$1\,P + 1\,S \le 600$$

$$5\,P + 3\,S \le 3000$$

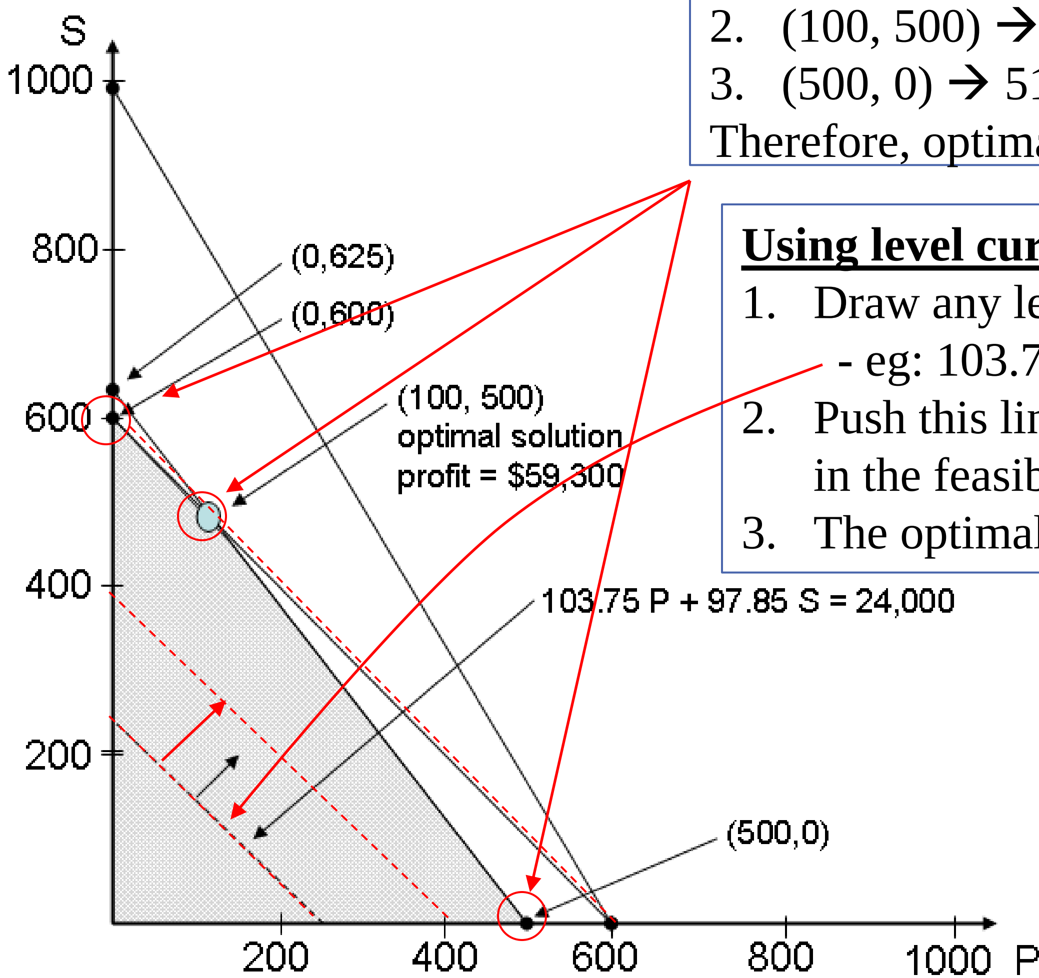
$$P, S \ge 0$$

Solving the Problem

By enumerating corner points:
1. (0, 600) → 103.75 * 0 + 97.85 * 600 = 58,710
2. (100, 500) → 59,300
3. (500, 0) → 51,875

Therefore, optimal solution =$59,300

Using level curve:
1. Draw any level curve
   - eg: 103.75P + 97.85S = 24,000
2. Push this line up till it touches the last point in the feasible region.
3. The optimal solution =$59,300



Week 2

- Modelling and Solving LP Problems in a Spreadsheet
  - Characteristics of optimization problems:
    - decisions, constraints, objective, maximise or minimise;
  - Linear programming: the general form of a linear program;
- Anomalies in LP solutions:
  - Alternate Optimal Solutions;
  - Redundant Constraints, Unbounded Solutions, Infeasibility;
- LP problems with more than two variables;
  - Using the Excel Solver to find solutions to Linear Programming problems;
  - Scaling variables to achieve the best solution.

Week 3

- Linear Programming – Sensitivity Analysis:
  - The Solver’s reports:
    - The Answer report: optimal solution, final value of decision variables, resource usage;
    - The Limits report: how the objective function varies as each variable ranges between its limits;
    - The Sensitivity report: The most important. Adjustable cells – final value, reduced cost, allowable increase/decrease. Constraints – final value, shadow price, allowable increase/decrease;
  - Degeneracy: allowable increase/decrease of any constraint is zero. Sometimes caused by redundant constraints.

Question 2 – Sensitivity Analysis

- Given the following sensitivity report:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B4	Number to make: X1	9.49	0	5	1.54	1
C4	Number to make: X2	1.74	0	6	1.5	1.47

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
D8	Resource 1 used	42	0	48	1E+30	6
D9	Resource 2 used	132	0.24	132	12	12
D10	Resource 3 used	24	1.24	24	1.33	2

Question 2 Cont…

a) What range of values can the objective function coefficient for variable X1 assume without changing the optimal solution?

   Answer: 4 – 6.54

b) Which resources are binding?

   Answer: Resource 2 & 3

c) If you're able to increase Resource 1, how much would you be willing to pay and why?

   Answer: Will not want to acquire additional unit of this resource as there's a slack of 6 units.

d) If you're able to increase Resource 2, how much would you be willing to pay?

   Answer: $0.24 on top of what they are currently paying up to 12 units.

Week 4

- Linear Programming – Integer Linear Programming:
  - One or more variable in an LP must assume a linear value;
  - Relaxation of the solution (ie assume non-integer first);
  - Branch and bound algorithm (mentioned only);
  - Optimal solution – stopping rules;
  - Employee scheduling problem;
  - Fixed charge problems using ‘Big M’;
  - Quantity discounts.

Question 3 – ILP Problem

- A shipping company wants to build two new warehouses to cover 4 regions. The company wants to minimize the cost of building the two warehouses while strategically trying to locate the warehouse to cover most regions. The company has identified the regions that can be covered by each warehouse site and are indicated by a 1 in the following table:
- Formulate the ILP for this problem.

	Warehouse Sites			
Region	1	2	3	4
A		1		1
B	1		1	1
C	1	1	1	
D	1			1
COST ($000s)	210	160	200	260

Question 3 Cont…

Answer:

MIN: $210\ X_1 + 160\ X_2 + 200\ X_3 + 260\ X_4$

Subject to:

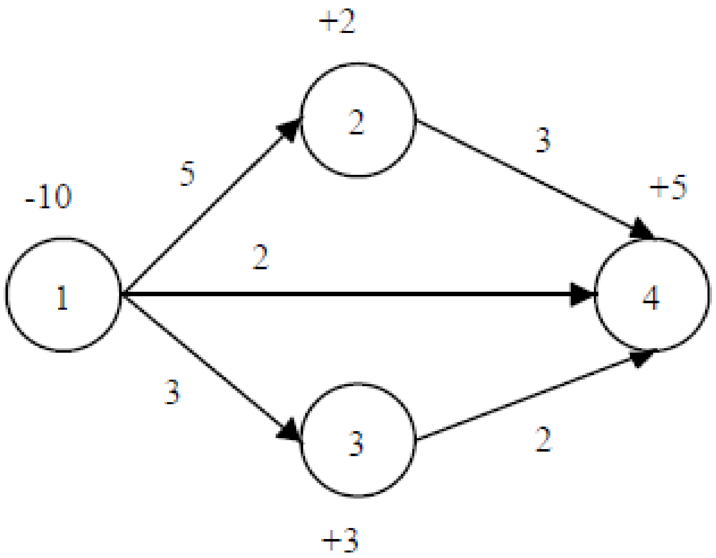
$$X_2 + X_4 \geq 1$$
$$X_1 + X_3 + X_4 \geq 1$$
$$X_1 + X_2 + X_3 \geq 1$$
$$X_1 + X_4 \geq 1$$
$$X_1 + X_2 + X_3 + X_4 = 2$$
$$X_i = 0, 1$$

Week 5

- Formulating and solving trans-shipment problems
- Modelling the shortest path problems
- Understanding generalised network flow problems
- Modelling Maximal Flow Problems
- Exploring the algorithm for minimal spanning tree problem

Question 4 – Network modelling I

- Formulate the LP formulation for this network (trans-shipment) model:



Answer:

MIN: $5X_{12} + 3X_{13} + 2X_{14} + 3X_{24} + 2X_{34}$

Subject to:

$-X_{12} - X_{13} - X_{14} = -10$

$X_{12} - X_{24} = 2$

$X_{13} - X_{34} = 3$

$X_{14} + X_{24} + X_{34} = 5$

$X_{ij} \geq 0$ for all i and j

Week 6

- Solving transportation problems using North-West Corner method
- Solving transportation problems with Vogel Approximation Method (VAM)
- Solving transportation problems with MODI (closed loop) Method
- Modelling Assignment Problems
- The Travelling Salesman Problem (TSP)
- Solving TSP non-examinable

Question 5 – Network Modelling II

Consider the following distribution problem for Ace Widgets:

	Shipping Costs to Warehouses				
Plants	**W1**	**W2**	**W3**	**W4**	**Capacity**
P1	2	6	4	12	100
P2	7	3	10	11	250
P3	5	8	9	13	300
Demand	50	150	200	250	

a) Apply the North-west corner method to determine a starting solution. Compute the total cost.

b) Using the solution obtained in part (a), determine the new shipping schedule according to the closed-loop path (MODI method). Compute the total cost.

c) Draw a network diagram to depict Ace Widgets Shipping Distribution.

d) Formulate an LP formulation for Ace Widgets.

Try this at home

Week 7

- Deterministic Inventory Models:
  - Economic Order Quantity (EOQ) Model;
  - Inventory Model with Planned Shortages;
  - Quantity Discounts for the EOQ Model – for which we have to evaluate the EOQ under the different cost regimes and choose the option with greatest profit (least annual costs);
  - Inventory costs: holding, order, backorder, item;
  - Sensitivity total costs to changes in Q^*;
  - Number of orders per year;
  - Time between orders, when to order;
  - Total annual cost.

Question 6 – Deterministic Inventory Models

A baseball card dealer must determine how many 1955 reproduced Willie Mays cards to stock. He experiences an annual demand of 100 cards. Each card is acquired from a big dealer for $2. Each shipment must be sent by registered mail at a cost of $4 regardless of quantity. Inventory is financed through a 16% bank loan.

Suppose a shortage penalty applies in the amount of $0.04 per card short (on an annual basis).

a) What is the economic order quantity?

b) What is the optimal order/inventory level?

c) If the optimal policy is used, determine the number of cards on backorder when a shipment arrives.

Answer:

a) $$Q^* = \sqrt{\frac{2(100)4}{.16(2)}}\sqrt{\frac{.04+.16(2)}{.04}} = 150$$

b) $$S^* = \sqrt{\frac{2(100)4}{.16(2)}}\sqrt{\frac{.04}{.04+.16}} = 50/3 = 16.667$$

c) $Q^* – S^* = 150 – 16.667 = 133.33$

Optimal order quantity, $Q^* = \sqrt{\frac{2Ak}{ch}\left(\frac{p+ch}{p}\right)}$

Quantity at the beginning of each cycle, $S^* = \sqrt{\frac{2Ak}{ch}\left(\frac{p}{p+ch}\right)}$

Maximum number of backorders $= Q^* - S^*$

Number of orders per year $= \frac{A}{Q^*}$

Total annual cost = setup + holding + backorder

$$= \frac{Ak}{Q} + \frac{chS^2}{2Q} + \frac{p(Q-S)^2}{2Q}$$

Week 8

- Inventory modelling under uncertainty (stochastic demand)
  - Probabilistic Model
    - Review of the Normal distribution
  - Single-period order quantity model
  - Reorder-point quantity model
  - Periodic-review order quantity model

Question 7 – Stochastic Inventory Model

The demand for Halloween pumpkins at the Black Cat's Patch is normally distributed with a mean of 1,000 and a standard deviation of 200. Each pumpkin costs $0.50 and sells for $0.90. Unsold pumpkins are disposed of at a cost of $0.10 each.

a) How many pumpkins should be ordered?

b) For the quantity in (a), determine the probability that there will be a shortage?

Answer:

(a) c = \$.50 $p_S = 0$ $p_R = \$.90$ $h_E = \$.10$

$$\frac{p_S + p_R - c}{(p_S + p_R - c) + (h_E + c)} = \frac{0 + .90 - .50}{(0 + .90 - .50) + (-.10 + .50)} = .40$$

$\Pr[D \leq Q^*] = 0.40$ $z = -0.25$

$Q^* = m + zs = 1{,}000 - 0.25(200) = 950$ pumpkins

(b) Pr[shortage] = Pr[D > 950]

=1 – .40 = .60

Optimal probability of no shortage:

$$P(\text{demand} < Q^*) = \frac{C_u}{C_u + C_v} = \frac{(P_S + P_R - c)}{(P_S + P_R - c) + (h_E + c)}$$

Week 9

- Decision making under risk and uncertainty:
  - Problem Formulation;
  - Non-probabilistic decision rules:
    - Maximax, Maximin, Minimax regret;
  - Probabilistic decision rules:
    - Expected Monetary Value, Expected Regret or Opportunity Loss;
  - Expected Value of Perfect Information;
  - Sensitivity Analysis;
  - Decision Trees.

Week 9 Cont…

- Decision Making with Sample Information:
  - Evaluating (rolling back) Decision Trees;
  - Expected Value of Sample Information;
  - Bayes' Theorem: prior and posterior probabilities, incorporating information given as conditional probabilities.

Question 8 – Bayes Theorem

- Using the Oil Wildcatting example in Lecture 9
- Lucky Luke is an Oil Wildcatter (a person who searches for oil). Based on 20 years of experience he estimates the probability of oil beneath Crockpot Dome. Let:

$$\left.\begin{array}{l} A_1 = \text{oil below Crockpot Dome} \rightarrow \mathrm{P}(A_1) = 0.2 \\ A_2 = \text{no oil below Crockpot Dome} \rightarrow \mathrm{P}(A_2) = 0.8 \end{array}\right\} \text{Prior Probabilities}$$

- Lucky Luke orders a seismic survey. The petroleum engineering consultant is 90% reliable in confirming oil when there actually is oil, but only 70% reliable in predicting that there is no oil when there actually is no oil.

$\rightarrow \quad \mathrm{P}(B|A_1) = 0.9 \qquad \mathrm{P}(B^c|A_2) = 0.7$

Example: Oil Wildcatting Probability Tree

Prior Probability	Conditional Probability	Joint Probability
$P(A_1) = .2$	$P(B \mid A_1) = .9$	$P(A_1 \cap B) = .18$
	$P(B^c \mid A_1) = .1$	$P(A_1 \cap B^c) = .02$
$P(A_2) = .8$	$P(B \mid A_2) = .3$	$P(A_2 \cap B) = .24$
	$P(B^c \mid A_2) = .7$	$P(A_2 \cap B^c) = .56$

Calculating Posterior Probability & EVSI

- What is the value of the petrol consultant's information?
- Let's assume that if Lucky Luke is successful in finding oil then he will earn $100,000. If he drills and does not find oil then he loses $30,000.
- Assuming that he will not proceed if the test result is negative, we see the expected returns after a positive test result. In this case, the expected value of sample information is $29,714.

States of Nature	Prior Probabilities	Conditional Probabilities	Joint Probabilities	Posterior Probabilities
A1	0.2	0.9	0.18	0.43
A2	0.8	0.3	0.24	0.57
			0.42	

If oil found	$ 100,000			$ 100,000	EVSI
If no oil found	-$ 30,000			-$ 30,000	
Payoff	-$ 4,000			$ 25,714	$ 29,714

Week 10

- Forecasting I:
  - The Components of a Time Series – trend, cyclic, seasonal and random;
  - Moving Average Forecasts;
  - Simple Exponential Smoothing;
  - Measures of Forecast Accuracy;
- Forecasting II:
  - Forecasting using linear regression based methods;
  - Seasonal forecasting using a Multiplicative Model;
  - Seasonal forecasting using an Additive Model;

Week 11/12

- Monte Carlo Simulation:
  - Random number generation in a spreadsheet;
  - Simulating the roll of a die, toss of a coin, empirical distributions using a lookup table;
  - The Newsboy Problem;
  - Simulating a waiting line
- Queuing Theory
  - Understand the basics of queuing theory
  - Review of probability theory – Poisson and Exponential distribution
  - Exploring the steady state behaviour of queues and Kendall notation
  - Economic analysis of queuing systems

The Exam

- Closed book, invigilated eExam consisting of two parts;
- Part A: Multiple choice.
  - 10 Questions worth a total of 10 marks.
- Part B: Structured questions.
  - 6 Structured topical Questions, each with multiple parts requiring short answers worth a total of 60 marks.
- Manage your Time. Do the easy questions first – and get them right!
- Calculator and two working sheets permitted.
- You do not need to separately upload anything - all answers to be typed in the e-Assessment portal

Preparing for the Exam

- Lecture review questions for each topic available under Scheduled Final Assessments in Moodle.
- Start with Lecture and Tutorial materials then move on to review questions provided.
- Attempt the Sample Exam that will be provided on the mock e-Exam platform
  - Time it!
- The formula sheet is also provided and also as in the sample exam.
- Exam Consultation times: Will be published on Moodle.

Your feedback is valuable

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At Monash University, we are always seeking ways to improve your learning experience. One way we do this is by asking for your feedback on the content, structure, assessment tasks and learning technologies used in this unit. Please take the time to complete this survey and help us to improve your learning experience over the lectures/tutorials and content covered for this unit.

Good luck and all the best in your exam!