FIT3158
Business Decision Modelling

SEMESTER 2, 2022

Lecture 4

- Integer Linear Programming (ILP)

Topics Covered:

- Discuss the problems faced with integrality constraints
- Formulating Integer Linear Programming Problems
- Goal Programming (GP) and MOLP - Non-Examinable

Introduction

- Integer Linear Programming (ILP)
    - When one or more variables in an LP problem must assume an integer value
- ILPs occur frequently…
    - Scheduling workers
    - Manufacturing products
- Integer variables also allow us to build more accurate models for a number of common business problems.
    - Quantity discounts
    - Setup and lump sum costs
    - Batch size restrictions

Integrality Conditions

MAX:	$350X_1 + 300X_2$	} profit
S.T.:	$1X_1 + 1X_2 <= 200$	} pumps
	$9X_1 + 6X_2 <= 1566$	} labor
	$12X_1 + 16X_2 <= 2880$	} tubing
	$X_1, X_2 >= 0$	} non-negativity
	X_1, X_2 must be integers	} integrality

Integrality conditions are easy to state but make the problem much more difficult (and sometimes impossible) to solve.

Relaxation

- **Original ILP** **MAX:** $\mathbf{2X_1 + 3X_2}$

S.T.: $X_1 + 3X_2 <= 8.25$

$2.5X_1 + X_2 <= 8.75$

$X_1, X_2 >= 0$

X_1, X_2 must be integers

This constraint is dropped in LP Relaxation

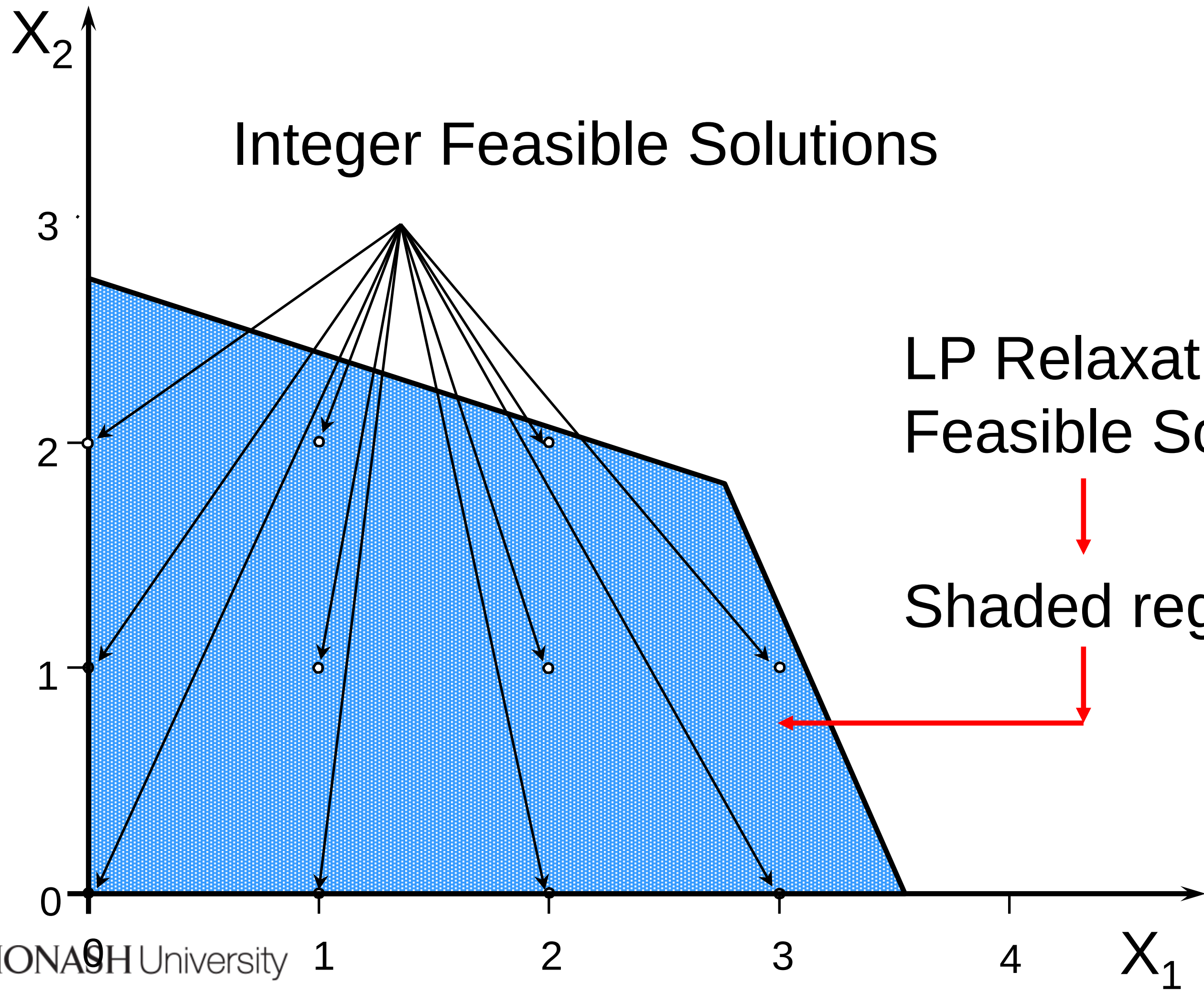
- LP Relaxation MAX: $2X_1 + 3X_2$

S.T.: $X_1 + 3X_2 <= 8.25$

$2.5X_1 + X_2 <= 8.75$

$X_1, X_2 >= 0$

Integer Feasible vs. LP Feasible Region



Solving ILP Problems

- When solving an LP relaxation, sometimes you "get lucky" and obtain an integer feasible solution.
- Example: Blue Ridge Hot Tubs

MAX: $350X_1 + 300X_2$ } profit

S.T.: $1X_1 + 1X_2 <= 200$ } pumps

$9X_1 + 6X_2 <= 1566$ } labor

$12X_1 + 16X_2 <= 2880$ } tubing

$X_1, X_2 >= 0$ } non-negativity

Optimal solution: $X_1 = 122$ and $X_2 = 78$

Integer solution

Solving ILP Problems

- But what if we reduce the amount of labor available to 1520 hours and the amount of tubing to 2650 feet?
- See file Lecture 4.xlsm *(Blue Ridge)*

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

Bounds

Blue Ridge Hot Tubs

	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

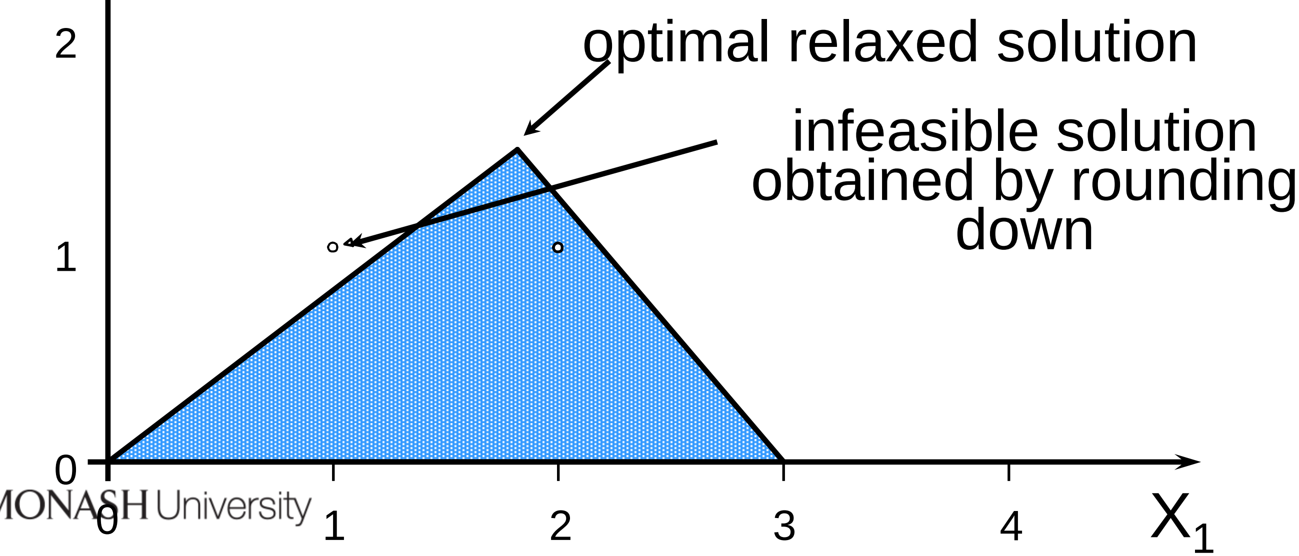
- The optimal solution to an LP relaxation of an ILP problem gives us a *bound* on the optimal objective function value.
- For maximization problems, the optimal relaxed objective function values is an upper bound on the optimal integer value.
- For minimization problems, the optimal relaxed objective function values is a lower bound on the optimal integer value.

Rounding

- It is tempting to simply round a fractional solution to the closest integer solution.

- In general, this does not work reliably:
  - The rounded solution may be infeasible.
  - The rounded solution may be suboptimal.

How Rounding Down Can Result in an Infeasible Solution



Optimal relaxed solution

X_1=1.91

X_2=1.45

Rounding Up

- LP solution:

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

- Round up - Infeasible

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	117	78	Total Profit	
Unit Profits	$350	$300	$64,350	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1521	1520
Tubing Req'd	12	16	2652	2650

Rounding Down

- LP

solution

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

- Round down – Feasible …..but

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	116	77	Total Profit	
Unit Profits	$350	$300	$63,700	
Constraints			Used	Available
Pumps Req'd	1	1	193	200
Labor Req'd	9	6	1506	1520
Tubing Req'd	12	16	2624	2650

Rounding Down Causes Sub-Optimality

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	116	77	Total Profit	
Unit Profits	$350	$300	$63,700	
Constraints			Used	Available
Pumps Req'd	1	1	193	200
Labor Req'd	9	6	1506	1520
Tubing Req'd	12	16	2624	2650

- A better integer solution exists (i.e. better than the above sub-optimal solution):

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	118	76	Total Profit	
Unit Profits	$350	$300	$64,100	
Constraints			Used	Available
Pumps Req'd	1	1	194	200
Labor Req'd	9	6	1518	1520
Tubing Req'd	12	16	2632	2650

Branch-and-Bound

- The Branch-and-Bound (B&B) algorithm can be used to solve ILP problems.
- Requires the solution of a series of LP problems termed “candidate problems”.
- *Theoretically*, this can solve any ILP.
- *Practically*, it often takes *LOTS* of computational effort (and time).

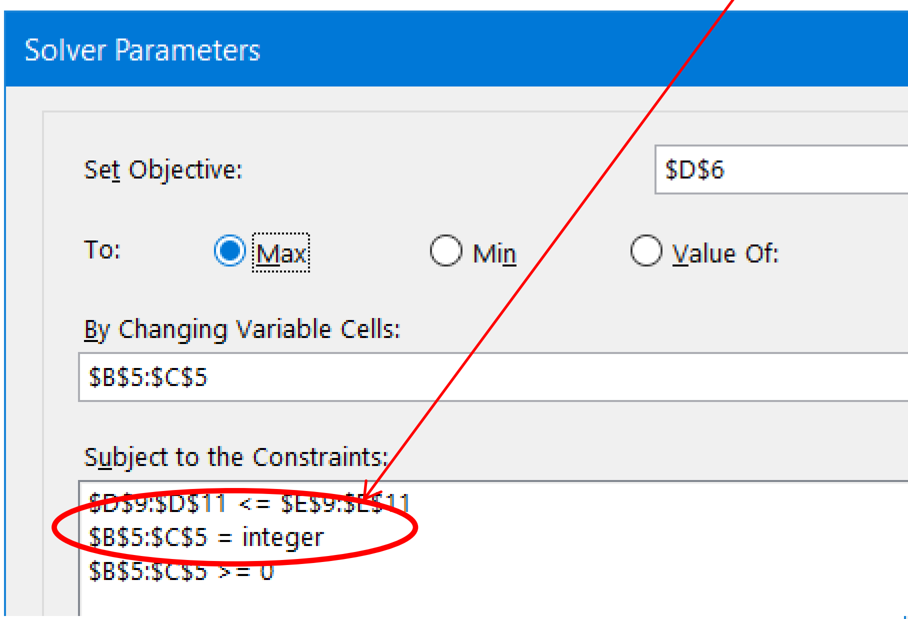
Stopping Rules

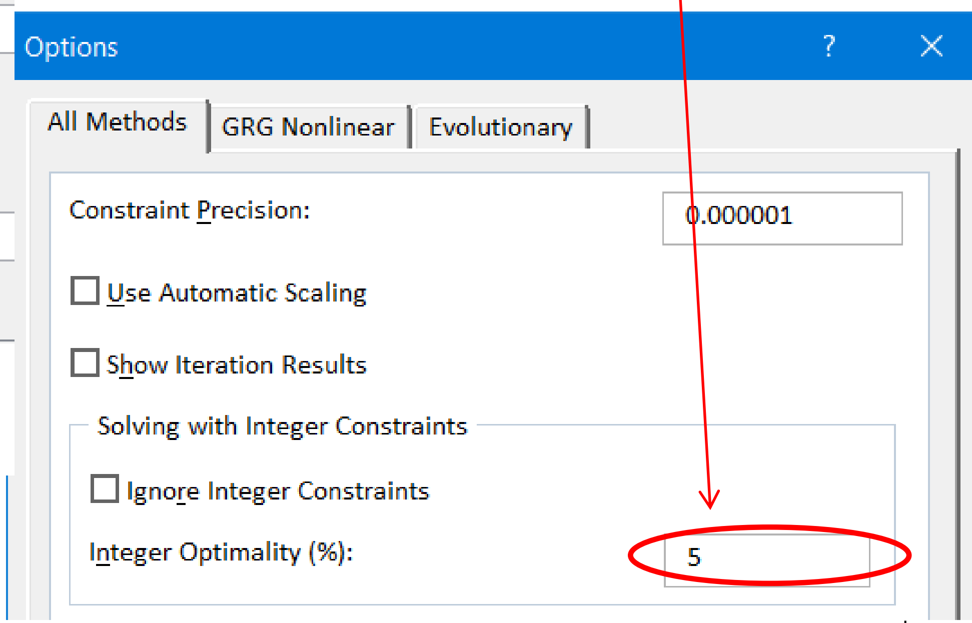
- Because B&B can take so long, most ILP packages allow you to specify a sub-optimality tolerance factor.
- This allows you to stop once an integer solution is found that is within some % of the global optimal solution.
- Bounds obtained from LP relaxations are helpful here.
  - Example
    - LP relaxation has an optimal obj. value of $64,306.
    - 95% of $64,306 is $61,090.
    - Thus, an integer solution with obj. value of $61,090 or better must be within 5% of the optimal solution.

Using Solver

Let's see how to specify integrality conditions and sub-optimality tolerances using Solver…

See file Lecture 4.xlsm (*Blue Ridge – ILP*)





An Employee Scheduling Problem: Air-Express

- An express shipping company – guarantees o/night delivery
- Various hubs across the country – shipments go to hubs, then on to their destination
- Manager of Baltimore hub is concerned about labour costs and wants to investigate the most effective way of scheduling of workers
- Hub open 7 days per week
- # packages varies from 1 day to the next
- An estimate of the number of workers needed on each day of the week has been calculated using historical data

An Employee Scheduling Problem: Air-Express

Day of Week	Workers Needed
Sunday	18
Monday	27
Tuesday	22
Wednesday	26
Thursday	25
Friday	21
Saturday	19

Shift	Days Off	Wage
1	Sun & Mon	$680
2	Mon & Tue	$705
3	Tue & Wed	$705
4	Wed & Thr	$705
5	Thr & Fri	$705
6	Fri & Sat	$680
7	Sat & Sun	$655

Defining the Decision Variables

X_1 = the number of workers assigned to shift 1

X_2 = the number of workers assigned to shift 2

X_3 = the number of workers assigned to shift 3

X_4 = the number of workers assigned to shift 4

X_5 = the number of workers assigned to shift 5

X_6 = the number of workers assigned to shift 6

X_7 = the number of workers assigned to shift 7

Defining the Objective Function

Minimize the total wage expense.

$$\text{MIN: } 680X_1 + 705X_2 + 705X_3 + 705X_4 + 705X_5 + 680X_6 + 655X_7$$

Wage per shift

Shift	Days Off	Wage
1	Sun & Mon	$680
2	Mon & Tue	$705
3	Tue & Wed	$705
4	Wed & Thr	$705
5	Thr & Fri	$705
6	Fri & Sat	$680
7	Sat & Sun	$655

Defining the Constraints

- Workers required each day

$$0X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 0X_7 >= 18 \text{ \} Sunday}$$

$$0X_1 + 0X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 >= 27 \text{ \} Monday}$$

$$1X_1 + 0X_2 + 0X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 >= 22 \text{ \}Tuesday}$$

$$1X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 + 1X_7 >= 26 \text{ \} Wednesday}$$

$$1X_1 + 1X_2 + 1X_3 + 0X_4 + 0X_5 + 1X_6 + 1X_7 >= 25 \text{ \} Thursday}$$

$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 0X_5 + 0X_6 + 1X_7 >= 21 \text{ \} Friday}$$

$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 0X_6 + 0X_7 >= 19 \text{ \} Saturday}$$

- Non-negativity & integrality conditions

$X_i >= 0$ and integer for all i

Implementing the Model

See file Lecture 4.xlsm (*AirExpress*)

	A	B	C	D	E	F	G	H	I	J
1		Air-Express								
2										
3		Days On=1, Days Off=0							Workers	Wages per
4	Shift	Sun	Mon	Tues	Wed	Thur	Fri	Sat	Scheduled	Worker
5	1	0	0	1	1	1	1	1	6	$680
6	2	1	0	0	1	1	1	1	0	$705
7	3	1	1	0	0	1	1	1	5	$705
8	4	1	1	1	0	0	1	1	1	$705
9	5	1	1	1	1	0	0	1	7	$705
10	6	1	1	1	1	1	0	0	5	$680
11	7	0	1	1	1	1	1	0	9	$655
12	Available	18	27	28	27	25	21	19	Total	$22,540
13	Required	18	27	22	26	25	21	19		
14										

At least as many as required

Binary Variables

- Binary variables are integer variables that can assume only two values: 0 or 1.
- These variables can be useful in a number of practical modeling situations…

A Capital Budgeting Problem: CRT Technologies

Project	Expected NPV (in $000s)	Capital (in $000s) Required in Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

- The company has $250,000 available to invest in new projects. It has budgeted $75,000 for continued support for these projects in year 2 and $50,000 per year for years 3, 4, and 5.
- Unused funds in any year cannot be carried over.

Defining Decision Variables & Objective Function

$$X_i = \begin{cases} 1, \text{if project } i \text{ is selected} \\ 0, \text{otherwise} \end{cases} \quad i = 1,2,\ldots,6$$

Maximize total NPV of selected projects

MAX: $141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6$

Expected NPV ($000s)

Project yr	Expected NPV (in $000s)	Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

Defining Constraints

- Capital Constraints

  must ensure for each year that the selected projects do not require more capital than is available

e.g. year 2, $75,000 is available, so:

$$25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75$$

Project yr	Expected NPV (in $000s)	Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

Defining the Constraints

- Capital Constraints

$$75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 <= 250 \quad \} \text{ year 1}$$

$$25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75 \quad \} \text{ year 2}$$

$$20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 <= 50 \quad \} \text{ year 3}$$

$$15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 <= 50 \quad \} \text{ year 4}$$

$$10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 <= 50 \quad \} \text{ year 5}$$

- Binary Constraints

All X_i must be binary

Implementing the Model

See file Lecture 4.xlsm(*CRT*)

CRT Technologies

	Select?		Capital Required in				
Project	(0=no, 1=yes)	NPV	Year 1	Year 2	Year 3	Year 4	Year 5
1	1	$141	$75	$25	$20	$15	$10
2	0	$187	$90	$35	$0	$0	$30
3	0	$121	$60	$15	$15	$15	$15
4	1	$83	$30	$20	$10	$5	$5
5	1	$265	$100	$25	$20	$20	$20
6	0	$127	$50	$20	$10	$30	$40
		Capital Required	$205	$70	$50	$40	$35
		Capital Available	$250	$75	$50	$50	$50
		Total Net Present Value	$489				

Binary Variables & Logical Conditions

- Binary variables are also useful in modeling a number of logical conditions.

  Can choose to select none of these projects, i.e., 0

  – Of projects 1, 3 & 6, no more than one may be selected: $X_1 + X_3 + X_6 <= 1$

  CANNOT choose to select none of these projects, i.e., 1

  – Of projects 1, 3 & 6, exactly one must be selected: $X_1 + X_3 + X_6 = 1$

  – Project 4 cannot be selected unless project 5 is also selected: $X_4 – X_5 <= 0$

The Fixed-Charge Problem

- Many decisions result in a fixed or lump-sum cost being incurred:
  - The cost to lease, rent, or purchase a piece of equipment or a vehicle that will be required if a particular action is taken.
  - The setup cost required to prepare a machine or to produce a different type of product.
  - The cost to construct a new production line that will be required if a particular decision is made.
  - The cost of hiring additional personnel that will be required if a particular decision is made.

Example Fixed-Charge Problem: Remington Manufacturing

Hours Required By:

Operation	Prod. 1	Prod. 2	Prod. 3	Hours Available
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400
Unit Profit	$48	$55	$50	
Setup Cost	$1000	$800	$900	

Fixed charge for making any quantity of prod 1, prod 2 or prod 3

Defining Decision Variables

X_i = the amount of product *i* to be produced, *i* = 1, 2, 3

$$Y_i = \begin{cases} 1, \text{if } X_i > 0 \\ 0, \text{if } X_i = 0 \end{cases} \quad i = 1, 2, 3$$

Y_i are binary variables that will be used to include the fixed charges

Defining the Objective Function

Maximize total profit.

Binary variable

$$\text{MAX:}\quad 48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$$

Unit Profit

Setup cost

Defining the Constraints

- Resource Constraints

  $2X_1 + 3X_2 + 6X_3 <= 600$ } machining

  $6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

  $5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- Non-negativity & integer conditions

  $X_i >= 0, i = 1, 2,...3$

  X_i integer, i=1,..3

- Binary Constraints

  All Y_i must be binary

- Is there a missing link?
- Yes - we need to ensure that $Y_i =1$ if $X_i >0$

1:03:00

Linking Constraints

- Linking Constraints (with "Big M")

$X_1 <= M_1Y_1$ or $X_1 - M_1Y_1 <= 0$

$X_2 <= M_2Y_2$ or $X_2 - M_2Y_2 <= 0$

$X_3 <= M_3Y_3$ or $X_3 - M_3Y_3 <= 0$

- If $X_i > 0$ these constraints force the associated Y_i to equal 1.
- If $X_i = 0$ these constraints allow Y_i to equal 0 or 1, but the objective will cause Solver to choose 0.
- Note that M_i imposes an upper bounds on X_i.
- It helps to find reasonable values for the M_i.

Maximum value of Xi = Mi

But we don't want to constrain X_i any further

Finding Reasonable Values for M1

- Consider the resource constraints

$2X_1 + 3X_2 + 6X_3 <= 600$ } machining

$6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

$5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- What is the maximum value X_1 can assume?

Let $X_2 = X_3 = 0$

$X_1 = \text{MIN}(600/2, 300/6, 400/5)$

$= \text{MIN}(300, 50, 80)$

$= 50$

So we can put $M_1 = 50$

- Maximum values for X_2 & X_3 can be found similarly.

Summary of the Model

MAX: $48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$

S.T.:

$2X_1 + 3X_2 + 6X_3 <= 600$ } machining

$6X_1 + 3X_2 + 4X_3 <= 30$ } grinding

$5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

$$\left.\begin{aligned} X_1 - 50Y_1 &<= 0 \\ X_2 - 67Y_2 &<= 0 \\ X_3 - 75Y_3 &<= 0 \end{aligned}\right\} \textbf{linking constraints}$$

All Y_i must be binary

$X_i >= 0, i = 1, 2, 3$ (= integer)

Implementing Model

See file Lecture 4.xlsm (Remington)

linking constraints:

$X_1 - 50Y_1 <= 0$

$X_2 - 67Y_2 <= 0$

$X_3 - 75Y_3 <= 0$

	A	B	C	D	E	F	G
1							
2		Remington Manufacturin					
3							
4		Product 1	Product 2	Product 3			
5	Number to Produce	0	56	32			
6							
7	Unit Profit	$48	$55	$50		Total Profit	
8	Fixed-Cost	$1,000	$800	$900		$2,980	
9							
10	Resources		Hours Required		Used	Available	
11	Machining	2	3	6	360	600	
12	Grinding	6	3	4	296	300	
13	Assembly	5	6	2	400	400	
14							
15	Binary Variables	0	1	1			
16	Linking Constraints	0	-10.66667	-43			
17							

Potential Pitfall

- Do not use IF( ) functions to model the relationship between the X_i and Y_i.
  - Suppose cell B5 represents X_1
  - Suppose cell B15 represents Y_1
  - You'll want to let B15 = IF(B5>0,1,0)
  - This will not work with Solver!
- Treat the Y_i just like any other variable.
  - Make them changing cells.
  - Use the linking constraints to enforce the proper relationship between the X_i and Y_i.

Minimum Order Size Restrictions

Suppose Remington doesn't want to manufacture any units of product 3 unless it produces at least 40 units...

Use M3 = min(600/6,300/4,400/2)=75

Consider,

$$X_3 <= M_3 Y_3$$

$$X_3 >= 40\ Y_3$$

See Lecture 4.xlsm (*Remington – Min order*)

B&G – A Contract Award Problem

- B&G Construction has 4 building projects and can purchase cement from 3 companies for the following costs:

	Cost per Delivered Ton of Cement				Max.
	Project 1	Project 2	Project 3	Project 4	Supply
Co. 1	$120	$115	$130	$125	525
Co. 2	$100	$150	$110	$105	450
Co. 3	$140	$95	$145	$165	550
Needs (tons)	450	275	300	350	

Defining the Decision Variables

X_{ij} = tons of cement purchased from company *i* for project *j*

Defining the Objective Function

Minimize total cost

$$\text{MIN:} \quad 120X_{11} + 115X_{12} + 130X_{13} + 125X_{14}$$
$$+ 100X_{21} + 150X_{22} + 110X_{23} + 105X_{24}$$
$$+ 140X_{31} + 95X_{32} + 145X_{33} + 165X_{34}$$

A Contract Award Problem

- Side constraints:

    Co. 1 will not supply orders of less than 150 tons for any project

    Co. 2 can supply more than 200 tons to no more than one of the projects

    Co. 3 will accept only orders that total 200, 400, or 550 tons

Defining the Constraints

- Supply Constraints

$X_{11} + X_{12} + X_{13} + X_{14} <= 525$ } company 1

$X_{21} + X_{22} + X_{23} + X_{24} <= 450$ } company 2

$X_{31} + X_{32} + X_{33} + X_{34} <= 550$ } company 3

- Demand Constraints

$X_{11} + X_{21} + X_{31} = 450$ } project 1

$X_{12} + X_{22} + X_{32} = 275$ } project 2

$X_{13} + X_{23} + X_{33} = 300$ } project 3

$X_{14} + X_{24} + X_{34} = 350$ } project 4

Defining the Constraints - I

- Company 1 Side Constraints

$$X_{11} <= 525Y_{11}$$

$$X_{12} <= 525Y_{12}$$

$$X_{13} <= 525Y_{13}$$

$$X_{14} <= 525Y_{14}$$

$$X_{11} >= 150Y_{11}$$

$$X_{12} >= 150Y_{12}$$

$$X_{13} >= 150Y_{13}$$

$$X_{14} >= 150Y_{14}$$

Y_{ij} binary

Defining the Constraints- II & III

- Company 2 Side Constraints

  $X_{21} <= 200 + 250Y_{21}$

  $X_{22} <= 200 + 250Y_{22}$

  $X_{23} <= 200 + 250Y_{23}$

  $X_{24} <= 200 + 250Y_{24}$

  $Y_{21} + Y_{22} + Y_{23} + Y_{24} <= 1$

  Y_{ij} binary

- Company 3 Side Constraints

  $X_{31} + X_{32} + X_{33} + X_{34} = 200Y_{31} + 400Y_{32} + 550Y_{33}$

  $Y_{31} + Y_{32} + Y_{33} <= 1$

Implementing the Transportation Constraints

See file Lecture 4.xlsm(*B&G*)

These are non-examinable but good to know …

- ***Goal Programming***
- ***Multiple Objective LP (MOLP)***

Multiple Objectives

- Most optimisation problems considered to this point have had a single objective.
- Often, more than one objective can be identified for a given problem.
  - Maximize Return or Minimize Risk
  - Maximize Profit or Minimize Pollution
- These objectives often conflict with one another.
- How can such problems be dealt with?

Goal Programming (GP)

- Most LP problems have *hard constraints* that cannot be violated...
  - There are 1,566 labor hours available.
  - There is $850,000 available for projects.
- In some cases, hard constraints are too restrictive...
  - You have a maximum price in mind when buying a car (this is your "goal" or target price).
  - If you can't buy the car for this price you'll likely find a way to spend more.
- We use *soft constraints* to represent such goals or targets we'd like to achieve.

GP Example: Beach Hotel Expansion

- Davis McKeown wants to expand the convention center at his hotel in Myrtle Beach, SC.
- The types of conference rooms being considered are:

	Size (sq ft)	Unit Cost
Small	400	$18,000
Medium	750	$33,000
Large	1,050	$45,150

- Davis would like to add 5 small, 10 medium and 15 large conference rooms.
- He also wants the total expansion to be 25,000 square feet and to limit the cost to $1,000,000.

Defining Goals

Goal 1: The expansion should include *approximately* 5 small conference rooms.

Goal 2: The expansion should include *approximately* 10 medium conference rooms.

Goal 3: The expansion should include *approximately* 15 large conference rooms.

Goal 4: The expansion should consist of *approximately* 25,000 square feet.

Goal 5: The expansion should cost *approximately* $1,000,000.

Defining Decision Variables

X_1 = number of small rooms to add

X_2 = number of medium rooms to add

X_3 = number of large rooms to add

Deviational Variables:

Amounts by which each goal deviates from its target value

$$d_i^- \ and \ d_i^+$$

(-) represents amount of underachievement of each goal's target value

(+) represents amount of overachievement of each goal's target value

Defining the Goal Constraints- I

- Small Rooms

$$X_1 + d_1^- - d_1^+ = 5$$

- Medium Rooms

$$X_2 + d_2^- - d_2^+ = 10$$

- Large Rooms

$$X_3 + d_3^- - d_3^+ = 15$$

where

$$d_i^-, d_i^+ \geq 0$$

e.g.

If $X_1 = 3$,

$d_1(-) = 2$

$d_1(+) = 0$

If $X_2 = 13$,

$d_2(-) = 0$

$d_2(+) = 3$

Defining the Goal Constraints- II

- Total Expansion

$$400X_1 + 750X_2 + 1{,}050X_3 + d_4^- - d_4^+ = 25{,}000$$

- **Total Cost (in $1,000s)**

$$18X_1 + 33X_2 + 45.15X_3 + d_5^- - d_5^+ = 1{,}000$$

where

$$d_i^-, d_i^+ \geq 0$$

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GP Objective Functions – Option 1

- There are numerous objective functions we could formulate for a GP problem.
- Minimize the sum of the deviations:

$$\text{MIN} \sum_i \left(d_i^- + d_i^+ \right)$$

- Problem: The deviations measure different things, so what does this objective represent?
- e.g. 7 rooms + 1500 \$ = 1507 of ?

GP Objective Functions – Option 2

- Minimize the sum of percentage deviations
  - MIN $$\sum_i \frac{1}{t_i}\left(d_i^- + d_i^+\right)$$
  - where ti represents the target value of goal i
- Problem: Suppose the first goal is underachieved by 1 small room and the fifth goal is overachieved by $20,000.
  - We underachieve goal 1 by 1/5=20%
  - We overachieve goal 5 by 20,000/1,000,000= 2%
  - This implies being $20,000 over budget is just as undesirable as having one too few small rooms.

Is this true? Only the decision maker can say for sure.

GP Objective Functions – Option 3

- Weights can be used in the previous objectives to allow the decision maker indicate
  - desirable vs. undesirable deviations
  - the relative importance of various goals
- Minimize the weighted sum of deviations

MIN $$\sum_i \left(w_i^- d_i^- + w_i^+ d_i^+ \right)$$

- Minimize the weighted sum of % deviations

MIN $$\sum_i \frac{1}{t_i} \left(w_i^- d_i^- + w_i^+ d_i^+ \right)$$

Defining the Objective

- Assume
  - It is undesirable to underachieve (-) any of the first three room goals
  - It is undesirable to overachieve (-, +) or underachieve the 25,000 sq ft expansion goal
  - It is undesirable to overachieve (+) the $1,000,000 total cost goal

$$\text{MIN: } \frac{w_1^-}{5}d_1^- + \frac{w_2^-}{5}d_2^- + \frac{w_3^-}{5}d_3^- + \frac{w_4^-}{25{,}000}d_4^- + \frac{w_4^+}{25{,}000}d_4^+ + \frac{w_5^+}{1{,}000{,}000}d_5^+$$

Initially, we will assume all the above weights equal 1.

Implementation - Lecture 4_GP.xlsm

About GP

- GP involves making trade-offs among the goals until the most satisfying solution is found.
- GP objective function values should not be compared because the weights are changed in each iteration. Compare the solutions!
- An arbitrarily large weight will effectively change a soft constraint to a hard constraint.
- Hard constraints can be place on deviational variables.

Summary of GP

1. Identify the decision variables in the problem.
2. Identify any hard constraints in the problem and formulate them in the usual way.
3. State the goals of the problem along with their target values.
4. Create constraints using the decision variables that would achieve the goals exactly.
5. Transform the above constraints into goal constraints by including deviational variables.
6. Determine which deviational variables represent undesirable deviations from the goals.
7. Formulate an objective that penalizes the undesirable deviations.
8. Identify appropriate weights for the objective.
9. Solve the problem.
10. Inspect the solution to the problem. If the solution is unacceptable, return to step 8 and revise the weights as needed.

Multiple Objective Linear Programming (MOLP)

- An MOLP problem is an LP problem with more than one objective function.
- MOLP problems can be viewed as special types of GP problems where we must also determine target values for each goal or objective.
- Analyzing these problems effectively also requires that we use the MiniMax objective

Summary of MOLP

1. Identify the decision variables in the problem.
2. Identify the objectives in the problem and formulate them as usual.
3. Identify the constraints in the problem and formulate them as usual.
4. Solve the problem once for each of the objectives identified in step 2 to determine the optimal value of each objective.
5. Restate the objectives as goals using the optimal objective values identified in step 4 as the target values.
6. For each goal, create a deviation function that measures the amount by which any given solution fails to meet the goal (either as an absolute or a percentage).
7. For each of the functions identified in step 6, assign a weight to the function and create a constraint that requires the value of the weighted deviation function to be less than the MINIMAX variable Q.
8. Solve the resulting problem with the objective of minimizing Q.
9. Inspect the solution to the problem. If the solution is unacceptable, adjust the weights in step 7 and return to step 8.

End of Lecture 4

Content References:

Ragsdale, C. Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e/9e) Cengage Learning: Chapter 6 & 7

Homework

- Go through today's lecture examples and Ragsdale Chapter 6, to:
  - ✓ Familiarise yourself with the ILP formulation and models
  - ✓ Understand the use of "Big M" in linking constraints
- Concepts and modeling techniques used in GP and MOLP problems are non-examinable and there only for your reference with Ragsdale chapter 7.

☞ ***Readings for next week Lecture:***

Ragsdale, C. Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e/9e) Cengage Learning: Chapter 5

Tutorial 3 this week:

- Interpreting Solver reports
  - Answer Report
  - Sensitivity Report
  - Limits report

- Spider plot
- Solver tables