Chapter 6

Integer Linear Programming

Learning Objectives

After reading this chapter students should be able to:

1. Discuss how integer variables impact the solution of integer linear programming (ILP) problems.
2. Describe the operation of the branch-and-bound algorithm.
3. Explain the meaning of an linear programming (LP) relaxation to an ILP problem.
4. Explain the concept of a bound on the optimal objective function value of an ILP problem.
5. Explain the meaning of the integer tolerance setting in ILP software.
6. Use general integer and binary variables to formulate ILP models.
7. Explain and use linking constraints.
8. Run a parameterized optimization.
9. Explain and use the following functions: SUM( ), MIN( ), SUMPRODUCT( ), VLOOKUP( ), PsiOptParam( ).

6-0 Introduction

When some or all of the decision variables in an LP problem are restricted to assuming only integer values, the resulting problem is referred to as an **integer linear programming** (ILP) problem. Many practical business problems need integer solutions. For example, when scheduling workers, a company needs to determine the optimal number of employees to assign to each shift. If we formulate this problem as an LP problem, its optimal solution could involve allocating fractional numbers of workers (e.g., 7.33 workers) to different shifts; but this is not an integer feasible solution. Similarly, if an airline is trying to decide how many 767s, 757s, and A-300s to purchase for its fleet, it must obtain an integer solution because the airline cannot buy fractions of planes.

This chapter discusses how to solve optimization problems in which certain decision variables must assume only integer values. This chapter also shows how the use of integer variables allows us to build more accurate models for a number of business problems.

6-1 Integrality Conditions

To illustrate some of the issues involved in an ILP problem, let's consider again the decision problem faced by Howie Jones, the owner of Blue Ridge Hot Tubs, described in Chapters 2, 3, and 4. This company sells two models of hot tubs, the Aqua-Spa and the Hydro-Lux, which it produces by purchasing prefabricated fiberglass hot tub shells and installing a common water pump and an appropriate amount of tubing. Each Aqua-Spa produced requires 1 pump, 9 hours of labor, and 12 feet of tubing, and contributes \$350 to profits. Each Hydro-Lux produced requires 1 pump, 6 hours of labor, and 16 feet of tubing, and contributes \$300 to profits. Assuming the company has 200 pumps, 1,566 labor hours, and 2,880 feet of tubing available, we created the following LP formulation for this problem where X_1 and X_2 represent the number of Aqua-Spas and Hydro-Luxes to produce:

$$
\begin{array}{llrl}
\text{MAX:} & 350X_1 + 300X_2 & & \} \text{ profit} \\
\text{Subject to:} & 1X_1 + 1X_2 & \leq 200 & \} \text{ pump constraint} \\
& 9X_1 + 6X_2 & \leq 1{,}566 & \} \text{ labor constraint} \\
& 12X_1 + 16X_2 & \leq 2{,}880 & \} \text{ tubing constraint} \\
& X_1, X_2 & \geq 0 & \} \text{ nonnegativity conditions}
\end{array}
$$

Blue Ridge Hot Tubs is undoubtedly interested in obtaining the best possible *integer solution* to this problem because hot tubs can be sold only as discrete units. Thus, we can be sure the company wants to find the *optimal integer solution* to this problem. So, in addition to the constraints stated previously, we add the following integrality condition to the formulation of the problem:

$$X_1 \text{ and } X_2 \text{ must be integers}$$

An **integrality condition** indicates that some (or all) of the variables in the formulation must assume only **integer values**. We refer to such variables as the integer variables in a problem. In contrast, variables that are not required to assume strictly integer values are referred to as **continuous variables**. Although it is easy to state integrality conditions for a problem, such conditions often make a problem more difficult (and sometimes impossible) to solve.

6-2 Relaxation

One approach to finding the optimal integer solution to a problem is to relax, or ignore, the integrality conditions and solve the problem as if it were a standard LP problem where all the variables are assumed to be continuous. This model is sometimes referred to as the **LP relaxation** of the original ILP problem. Consider the following ILP problem:

$$
\begin{array}{llrl}
\text{MAX:} & 2X_1 + 3X_2 & & \\
\text{Subject to:} & X_1 + 3X_2 & \leq 8.25 \\
& 2.5X_1 + X_2 & \leq 8.75 \\
& X_1, X_2 & \geq 0 \\
& \multicolumn{2}{l}{X_1, X_2 \text{ must be integers}}
\end{array}
$$

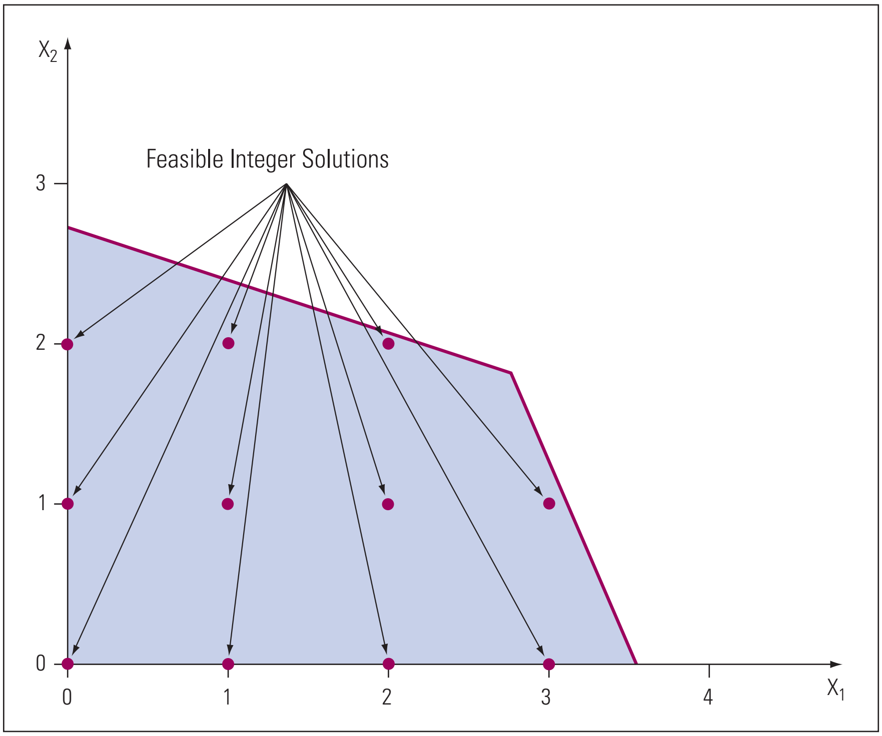
The LP relaxation for this problem is represented by:

$$\begin{aligned}
\text{MAX:} \quad & 2X_1 + 3X_2 \\
\text{Subject to:} \quad & X_1 + 3X_2 \leq 8.25 \\
& 2.5X_1 + X_2 \leq 8.75 \\
& X_1, X_2 \geq 0
\end{aligned}$$

The only difference between the ILP and its LP relaxation is that all integrality conditions imposed by the ILP are dropped in the relaxation. However, as illustrated in Figure 6.1, this change has a significant impact on the feasible regions for the two problems.

FIGURE 6.1

Integer feasible region vs. LP feasible region

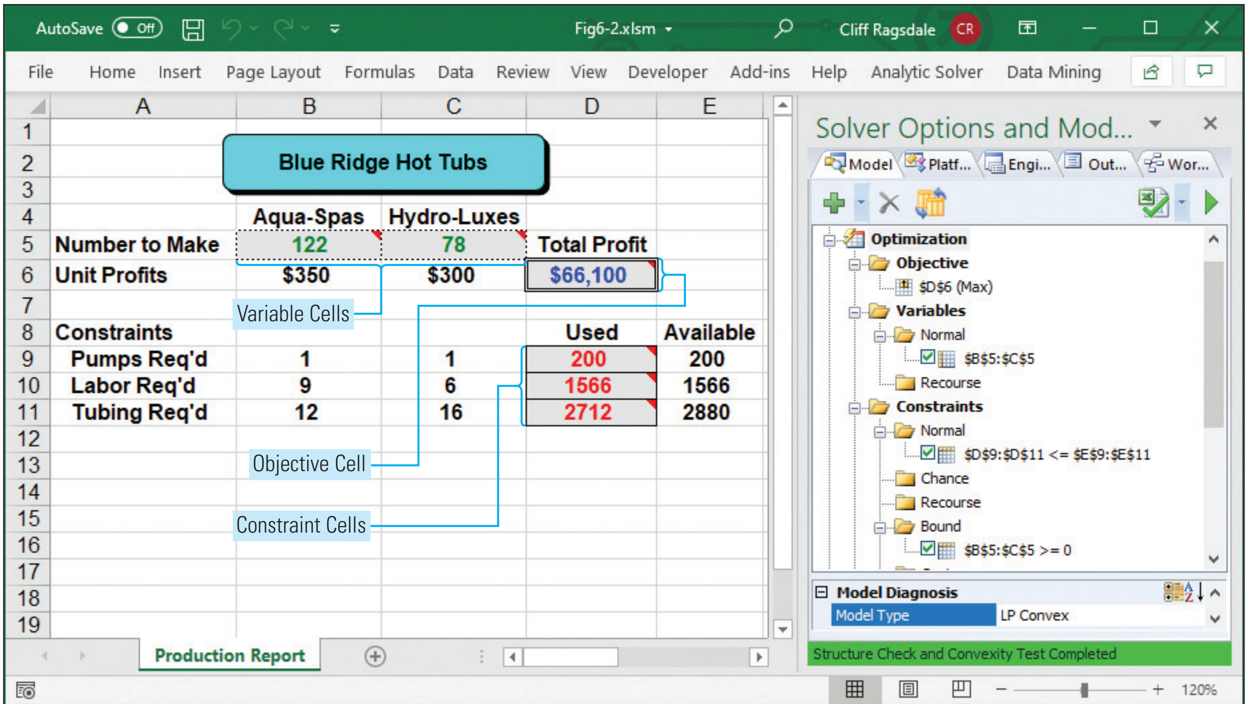


As shown in Figure 6.1, the feasible region for the ILP consists of only 11 discrete points. On the other hand, the feasible region for its LP relaxation consists of an infinite number of points represented by the shaded area. This figure illustrates an important point about the relationship between the feasible region of an ILP and its LP relaxation. The feasible region of the LP relaxation of an ILP problem *always* encompasses *all* the feasible integer solutions to the original ILP problem. Although the relaxed feasible region might include additional noninteger solutions, it will *not* include any integer solutions that are not feasible solutions to the original ILP.

6-3 Solving the Relaxed Problem

The LP relaxation of an ILP problem is often easy to solve using the simplex method. As explained in Chapter 2, an optimal solution to an LP problem occurs at one of the corner points of its feasible region (assuming that the problem has a bounded optimal solution). Thus, if we are extremely lucky, the optimal solution to the LP relaxation of an ILP problem might occur at an integer corner point of the relaxed feasible region. In this case, we find the optimal integer solution to the ILP problem simply by solving its LP relaxation. This is exactly what happened in Chapters 2 and 3 when we originally solved the relaxed LP model for the hot tub problem. Figure 6.2 (and the file Fig6-2.xlsm that accompanies this book) shows the solution to this problem.

FIGURE 6.2 *Integer solution obtained as optimal solution to the Blue Ridge Hot Tubs LP problem*



Key Cell Formulas

Cell	Formula	Copied to
D6	=B6*$B$5+C6*$C$5	D9:D11

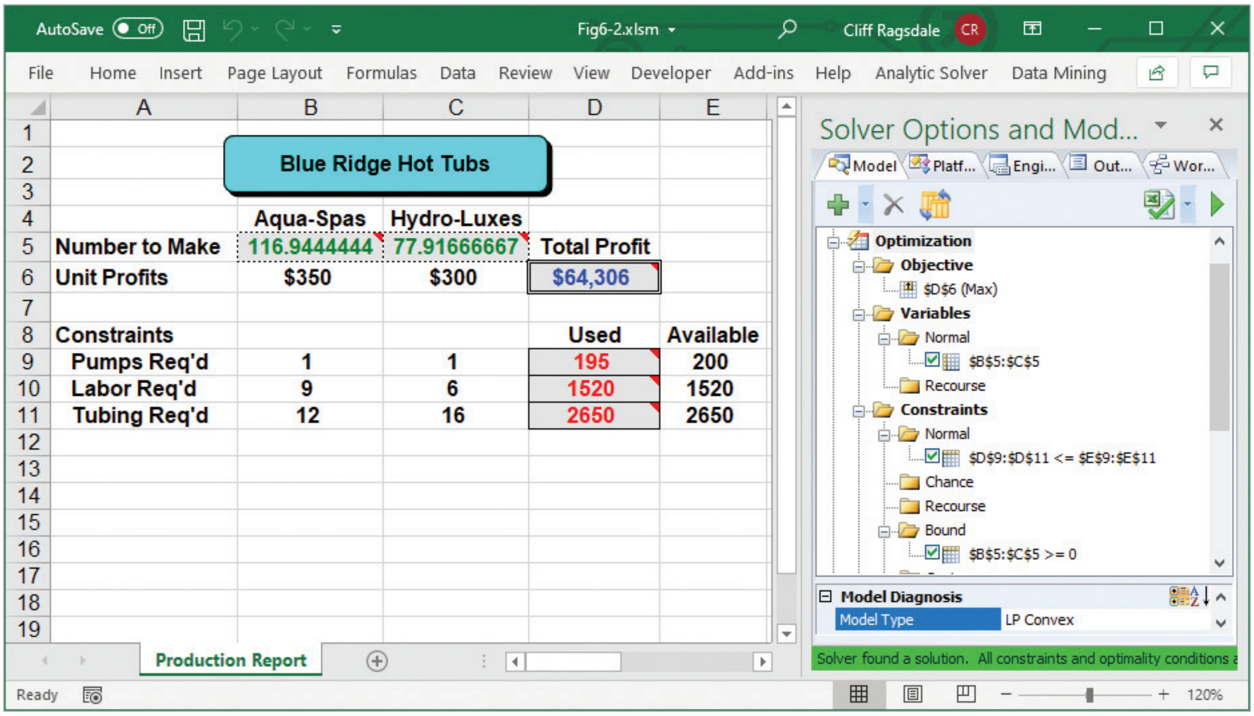
The optimal solution to the relaxed LP formulation of the hot tub problem assigns integer values to the decision variables ($X_1 = 122$ and $X_2 = 78$). So in this case, the relaxed LP problem happens to have an integer-valued optimal solution. However, as you might expect, this will not always be the case.

Suppose, for example, that Blue Ridge Hot Tubs has only 1,520 hours of labor and 2,650 feet of tubing available during its next production cycle. The company might be interested in solving the following ILP problem:

$$
\begin{array}{llrcl}
\text{MAX:} & 350X_1 + 300X_2 & & & \} \text{ profit} \\
\text{Subject to:} & 1X_1 + 1X_2 & \leq & 200 & \} \text{ pump constraint} \\
 & 9X_1 + 6X_2 & \leq & 1{,}520 & \} \text{ labor constraint} \\
 & 12X_1 + 16X_2 & \leq & 2{,}650 & \} \text{ tubing constraint} \\
 & X_1, X_2 & \geq & 0 & \} \text{ nonnegativity conditions} \\
 & X_1, X_2 \text{ must be integers} & & & \} \text{ integrality conditions}
\end{array}
$$

If we relax the integrality conditions and solve the resulting LP problem, we obtain the solution shown in Figure 6.3. This solution indicates that producing 116.9444 Aqua-Spas and 77.9167 Hydro-Luxes will generate a maximum profit of $64,306. But this solution violates the integrality conditions stated in the original problem. As a general rule, the optimal solution to the LP relaxation of an ILP problem is not guaranteed to produce an integer solution. In such cases, other techniques must be applied to find the optimal integer solution for the problem being solved. (There are some exceptions to this rule. In particular, the network flow problems discussed in Chapter 5 often can be viewed as ILP problems. For reasons that go beyond the scope of this text, the LP relaxation of network flow problems will always have integer solutions if the supplies and/or demands at each node are integers and the problem is solved using the simplex method.)

FIGURE 6.3 *Noninteger solution obtained as optimal solution to the revised Blue Ridge Hot Tubs LP problem*



6-4 Bounds

Before discussing how to solve ILP problems, an important point must be made about the relationship between the optimal solution to an ILP problem and the optimal solution to its LP relaxation: *The objective function value for the optimal solution to the ILP problem can never be better than the objective function value for the optimal solution to its LP relaxation.*

For example, the solution shown in Figure 6.3 indicates that if the company could produce (and sell) fractional numbers of hot tubs, it could make a maximum profit of $64,306 by producing 116.9444 Aqua-Spas and 77.9167 Hydro-Luxes. No other feasible solution (integer or otherwise) could result in a better value of the objective function. If a better feasible solution existed, the optimization procedure would have identified this better solution as optimal because our aim was to maximize the value of the objective function.

Although solving the LP relaxation of the revised hot tub problem might not provide the optimal integer solution to our original ILP problem, it does indicate that the objective function value of the optimal integer solution cannot possibly be greater than $64,306. This information can be important in helping us evaluate the quality of integer solutions we might discover during our search for the optimal solution.

Key Concept

For *maximization* problems, the objective function value at the optimal solution to the LP relaxation represents an *upper bound* on the optimal objective function value of the original ILP problem. For *minimization* problems, the objective function value at the optimal solution to the LP relaxation represents a *lower bound* on the optimal objective function value of the original ILP problem.

6-5 Rounding

As mentioned earlier, the solution to the LP relaxation of an ILP problem might satisfy the ILP problem's integrality conditions and, therefore, represent the optimal integer solution to the problem. But what should we do if this is not the case (as usually happens)? One frequently used technique involves rounding the relaxed LP solution.

When the solution to the LP relaxation of an ILP problem does not result in an integer solution, it is tempting to think that simply rounding this solution will generate the optimal integer solution. Unfortunately, this is not the case. For example, if the values for the decision variables shown in Figure 6.3 are manually rounded up to their closest integer values, as shown in Figure 6.4, the resulting solution is infeasible. The company cannot manufacture 117 Aqua-Spas and 78 Hydro-Luxes because this would involve using more labor and tubing than are available.

Because rounding up does not always work, perhaps we should round down, or truncate, the values for the decision variables identified in the LP relaxation. As shown in Figure 6.5, this results in a feasible solution where 116 Aqua-Spas and 77 Hydro-Luxes are manufactured for a total profit of $63,700. However, this approach presents two possible problems. First, rounding down could also result in an infeasible solution, as shown in Figure 6.6.

FIGURE 6.4

Infeasible integer solution obtained by rounding up

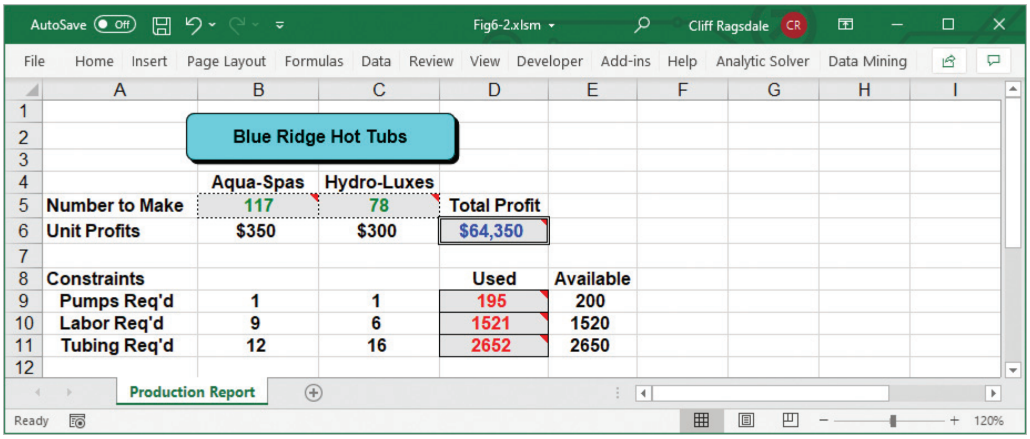
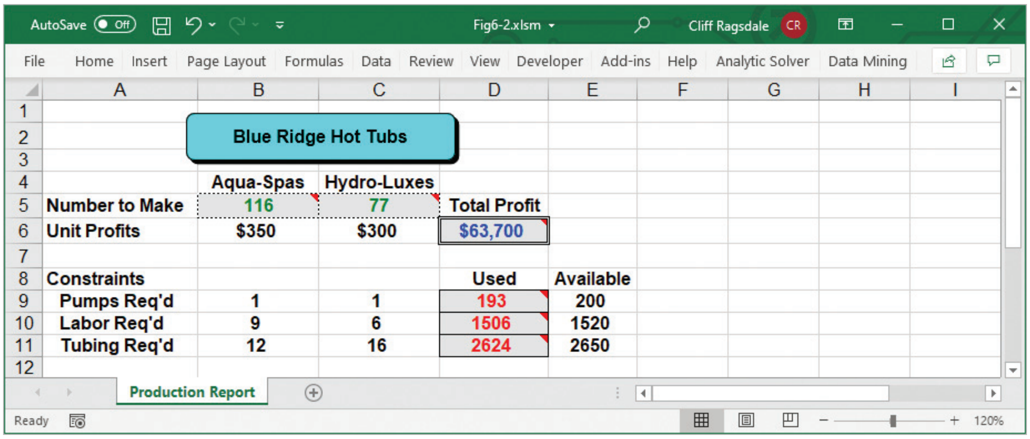


FIGURE 6.5

Feasible integer solution obtained by rounding down



Another problem with rounding down is that even if it results in a feasible integer solution to the problem, there is no guarantee that it is the *optimal* integer solution. For example, the integer solution obtained by rounding down shown in Figure 6.5 produced a total profit of $63,700. However, as shown in Figure 6.7, a better integer solution exists for this problem. If the company produces 118 Aqua-Spas and 76 Hydro-Luxes, it can achieve a total profit of $64,100 (which is the optimal integer solution to this problem). Simply rounding the solution to the LP relaxation of an ILP problem is not guaranteed to provide the optimal integer solution. Although the integer solution obtained in this problem by rounding is very close to the optimal integer solution, rounding does not always work this well.

As we have seen, the solution to the LP relaxation of an ILP is not guaranteed to produce an integer solution, and rounding the solution to the LP relaxation is not guaranteed to produce the optimal integer solution. Therefore, we need another way to find the optimal integer solution to an ILP problem. Various procedures have been developed for this purpose. The most effective and widely used of these procedures is the **branch-and-bound (B&B) algorithm**. The B&B algorithm theoretically allows us to solve any ILP problem by solving a series of LP problems called candidate problems. For those who are interested, a discussion of how the B&B algorithm works is given at the end of this chapter.

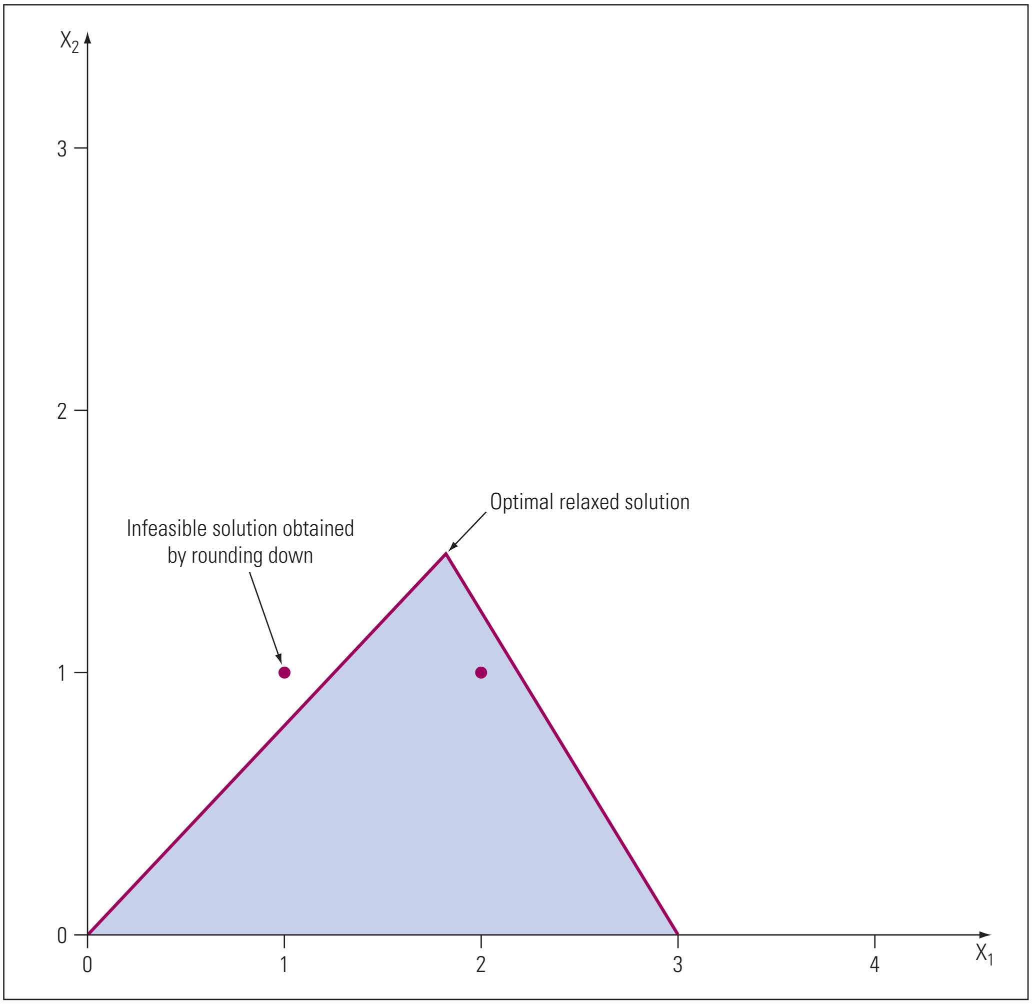
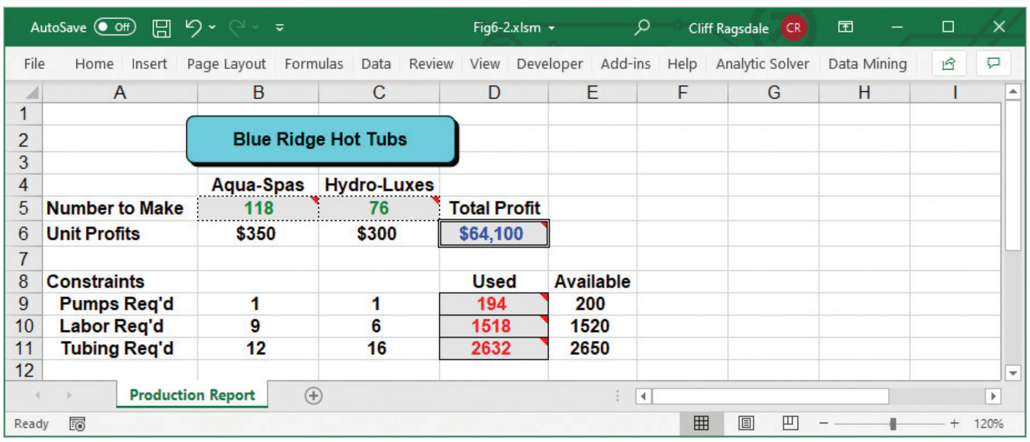


FIGURE 6.6 *How rounding down can result in an infeasible integer solution*

FIGURE 6.7 *Optimal integer solution to the revised Blue Ridge Hot Tubs problem*



	A	B	C	D	E
1					
2		Blue Ridge Hot Tubs			
3					
4		Aqua-Spas	Hydro-Luxes		
5	Number to Make	118	76	Total Profit	
6	Unit Profits	$350	$300	$64,100	
7					
8	Constraints			Used	Available
9	Pumps Req'd	1	1	194	200
10	Labor Req'd	9	6	1518	1520
11	Tubing Req'd	12	16	2632	2650
12					

6-6 Stopping Rules

Finding the optimal solution for simple ILP problems can sometimes require the evaluation of hundreds of candidate problems. More complex problems can require the evaluation of thousands of candidate problems, which can be a very time-consuming task even for the fastest computers. For this reason, many ILP packages allow you to specify a suboptimality tolerance of X% (where X is some numeric value), which tells the B&B algorithm to stop when it finds an integer solution that is no more than X% worse than the optimal integer solution. This is another area where obtaining upper or lower bounds on the optimal integer solution can be helpful.

As noted earlier, if we relax all the integrality conditions in an ILP with a maximization objective and solve the resulting LP problem, the objective function value at the optimal solution to the relaxed problem provides an upper bound on the optimal integer solution. For example, when we relaxed the integrality conditions for the revised Blue Ridge Hot Tubs problem and solved it as an LP, we obtained the solution shown earlier in Figure 6.3, which has an objective function value of $64,306. Thus, we know that the optimal integer solution to this problem cannot have an objective function value greater than $64,306. Now, suppose the owner of Blue Ridge Hot Tubs is willing to settle for any integer solution to its problem that is no more than 5% below the optimal integer solution. It is easy to determine that 95% of $64,306 is $61,090 ($0.95 \times \$64{,}306 = \$61{,}090$). Therefore, any integer solution with an objective function value of at least $61,090 can be no worse than 5% below the optimal integer solution.

Specifying suboptimality tolerances can be helpful if you are willing to settle for a good but suboptimal solution to a difficult ILP problem. However, most B&B packages employ some sort of default suboptimality tolerance and, therefore, might produce a suboptimal solution to the ILP problem without indicating that a better solution might exist. (We will look at an example where this occurs shortly.) It is important to be aware of suboptimality tolerances because they can determine whether or not the true optimal solution to an ILP problem is found.

6-7 Solving ILP Problems Using Solver

Now that you have some understanding of the effort required to solve ILP problems, you can appreciate how using Solver simplifies this process. This section shows how to use Solver with the revised Blue Ridge Hot Tubs problem.

Figure 6.8 shows the Solver settings required to solve the revised Blue Ridge Hot Tubs problem as a standard LP problem. However, none of these parameters indicate that the cells representing the decision variables (cells B5 and C5) must assume integer values. To communicate this to Solver, we need to add constraints to the problem as shown in Figure 6.9.

In Figure 6.9, cells B5 through C5 are specified as the cell references for the additional constraints. Because we want these cells to assume only integer values, we need to select the “int” option from the drop-down menu, as shown in Figure 6.9 and click OK.

Figure 6.10 shows the Solver setting and optimal solution with cells B5 and C5 constrained to assume only integer values. The message at the bottom of the Analytic Solver Task Pane indicates that Solver found a solution “within tolerance” that satisfies all constraints. Thus, we might suspect that the optimal integer solution to this problem involves producing 117 Aqua-Spas and 77 Hydro-Luxes for a total profit of $64,050. However, if you refer back to Figure 6.7, you will recall that an even better integer solution to this problem can be obtained by producing 118 Aqua-Spas and 76 Hydro-Luxes for a total profit of $64,100. So why did Solver select an integer solution with a total profit of $64,050 when a better integer solution exists? The answer lies in Solver’s suboptimality tolerance factor.

FIGURE 6.8 *Solver parameters for the relaxed Blue Ridge Hot Tubs problem*

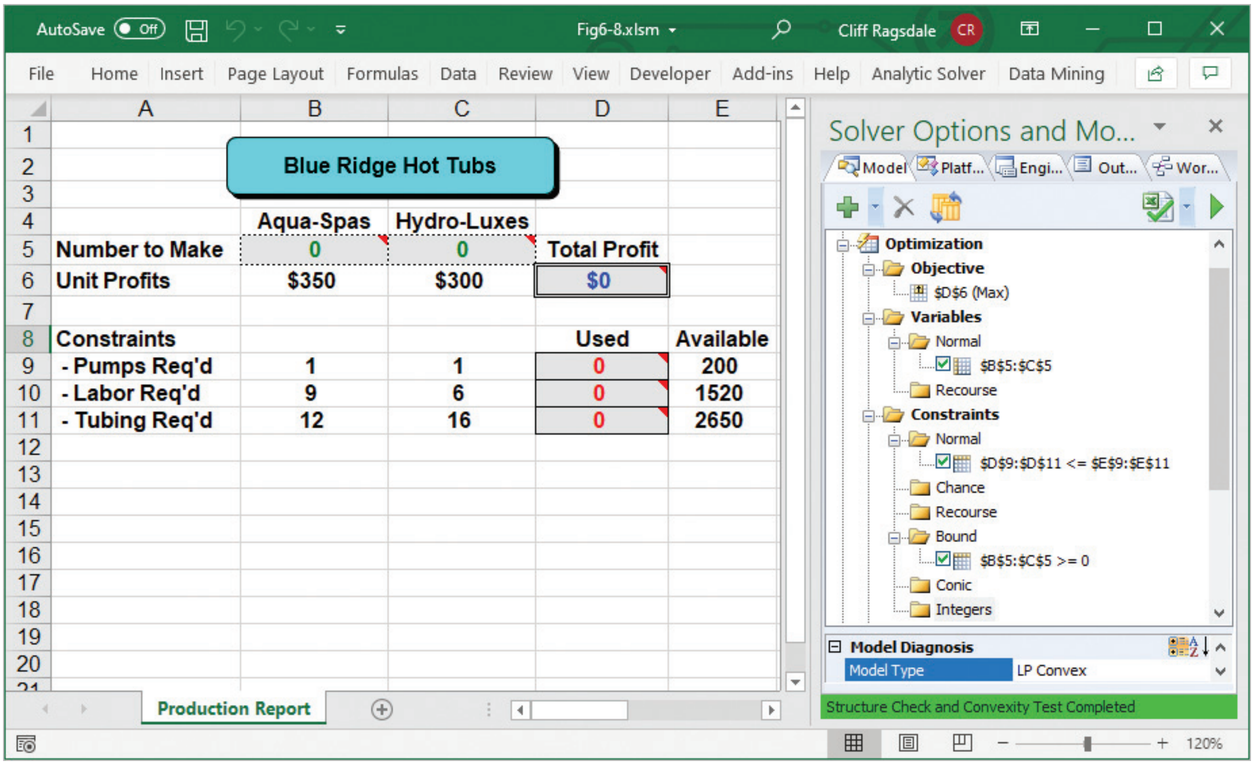


FIGURE 6.9 *Selecting integer constraints*

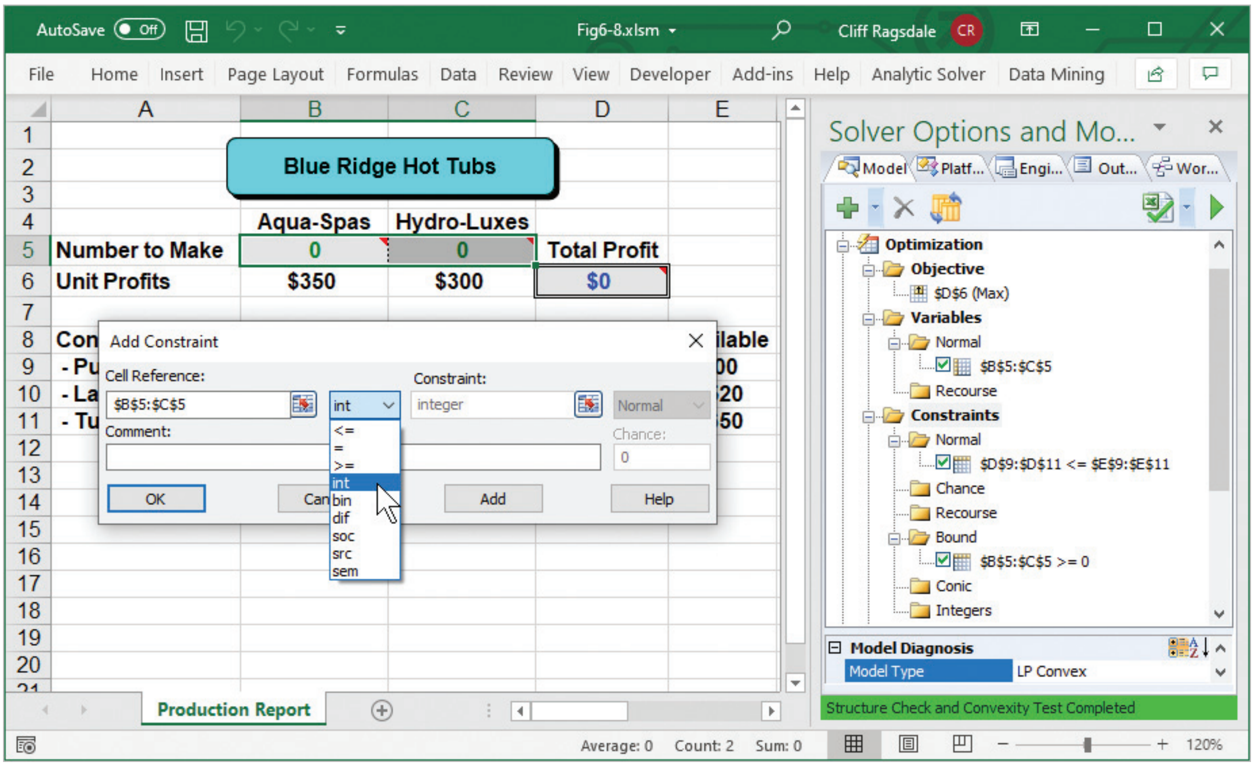
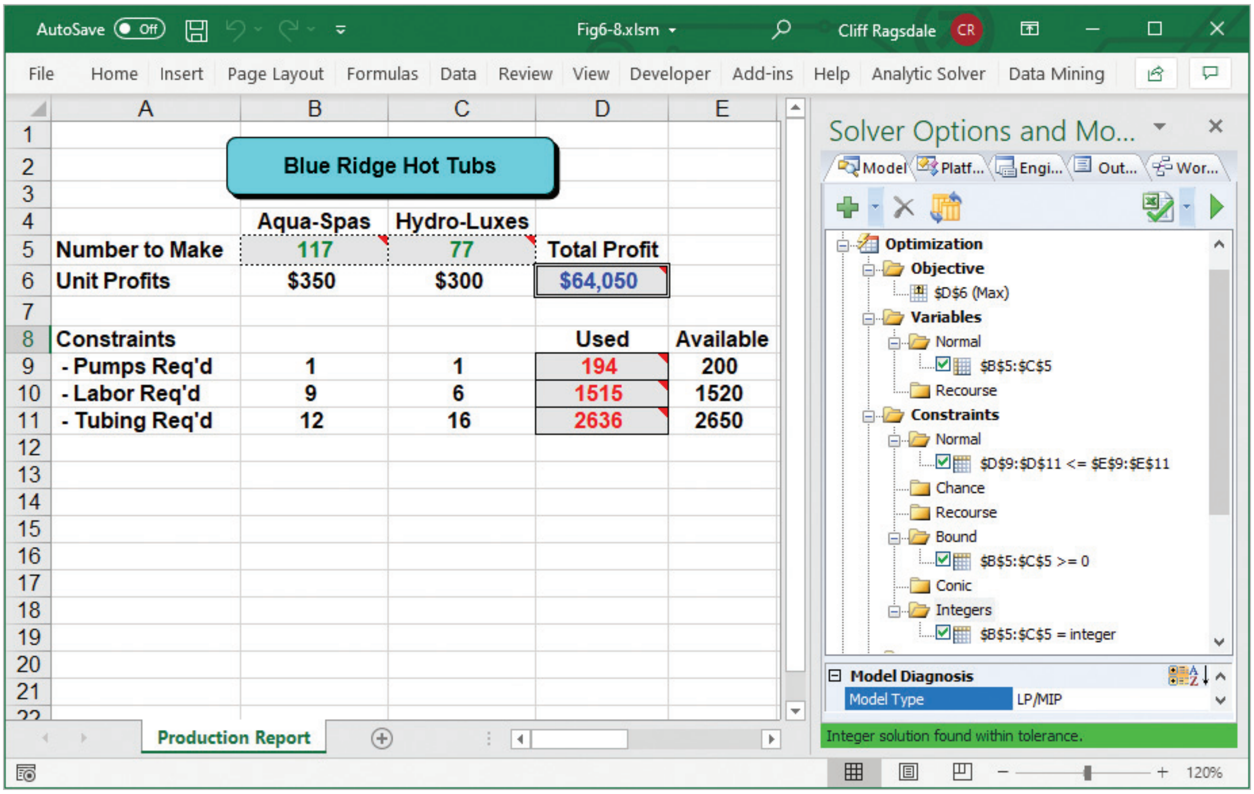


FIGURE 6.10 *Solver parameters and optimal solution to the revised Blue Ridge Hot Tubs problem with integer constraints*



By default, Solver uses a suboptimality tolerance factor of 5%. So, when Solver found the integer solution with the objective function value of $64,050 shown in Figure 6.10, it determined that this solution was within 5% of the optimal integer solution and abandoned its search. (Again, note the message in Figure 6.10, "Integer solution found within tolerance.") To ensure that Solver finds the best possible solution to an ILP problem, we must change its suboptimality tolerance factor by clicking the Engine tab in the Analytic Solver Task Pane and then changing the Integer Tolerance value as shown in Figure 6.11.

As shown in Figure 6.11, you can set a number of options to control Solver's operations. The Integer Tolerance option represents Solver's suboptimality tolerance value. To make sure Solver finds the best possible solution to an ILP problem, we must change this setting from its default value of 0.05 to 0. If we do this and re-solve the current problem, we obtain the solution shown in Figure 6.12. This solution is the best possible integer solution to the problem.

6-8 Other ILP Problems

Many decision problems encountered in business can be modeled as ILPs. As we have seen from the Blue Ridge Hot Tubs example, some problems that are initially formulated as LP problems might turn into ILP formulations if they require integer solutions. However, the importance of ILP extends beyond simply allowing us to obtain integer solutions to LP problems.

The ability to constrain certain variables to assume only integer values enables us to model a number of important conditions more accurately. For example, up to this point, we have not

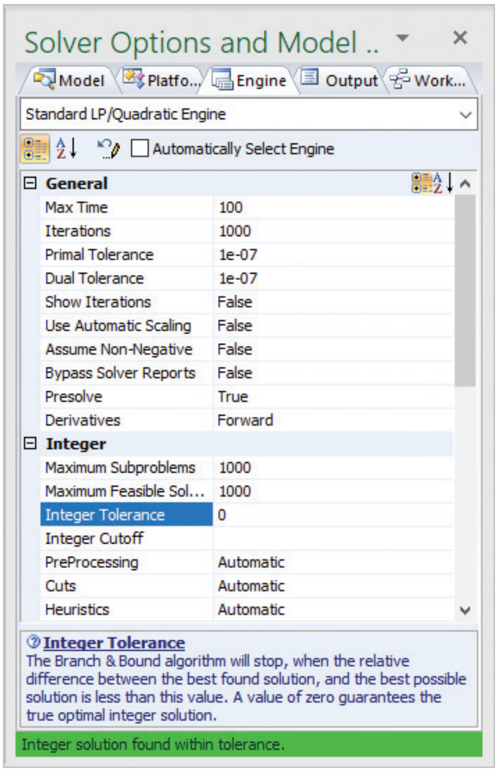
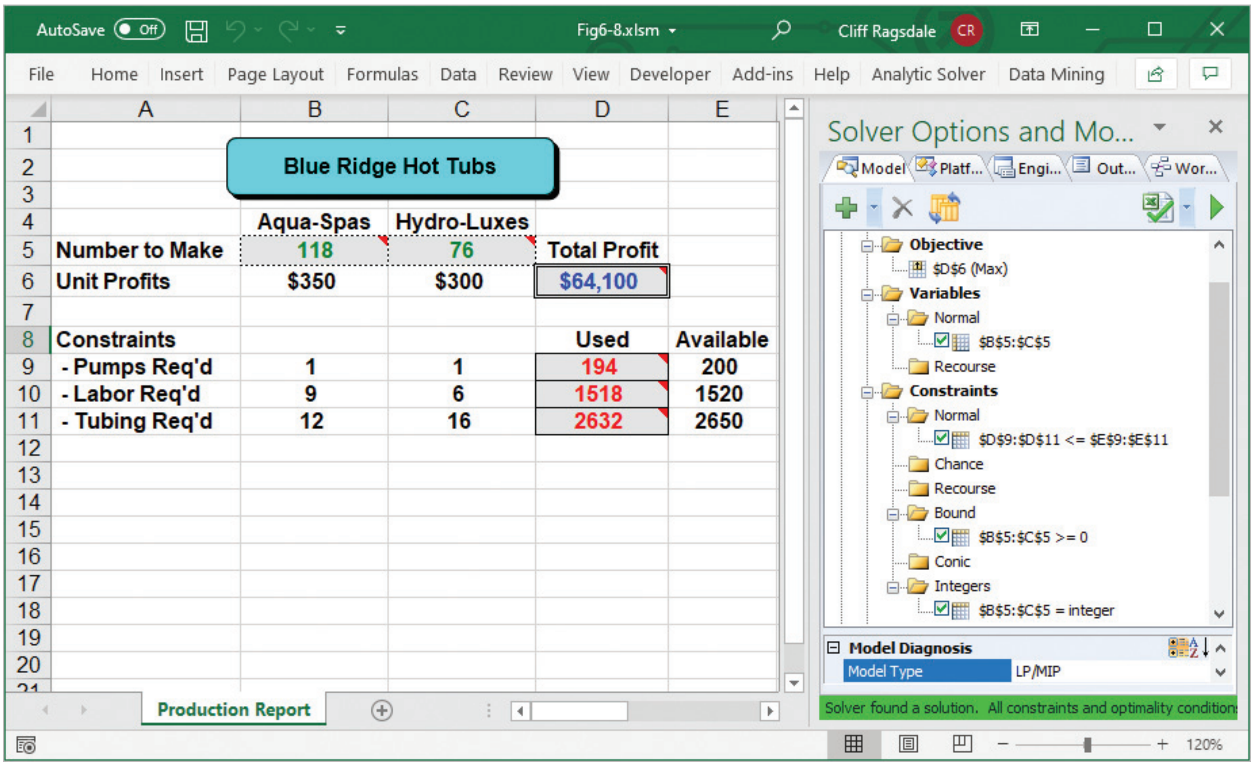


FIGURE 6.11

Changing the suboptimality tolerance factor

FIGURE 6.12 *Optimal integer solution to the revised Blue Ridge Hot Tubs problem*



considered the impact of quantity discounts, setup or lump-sum costs, or batch size restrictions on a given decision problem. Without ILP techniques, we could not model these decision issues. We now consider several examples that illustrate the expanded modeling capabilities available through the use of integer variables.

6-9 An Employee Scheduling Problem

Anyone responsible for creating work schedules for a number of employees can appreciate the difficulties in this task. It can be very difficult to develop a feasible schedule, much less an optimal schedule. Trying to ensure that a sufficient number of workers is available when needed is a complicated task when you must consider multiple shifts, rest breaks, and lunch or dinner breaks. However, some sophisticated ILP models have been devised to solve these problems. Although a discussion of these models is beyond the scope of this text, we will consider a simple example of an employee scheduling problem to give you an idea of how ILP models are applied in this area.

Air-Express is an express shipping service that guarantees overnight delivery of packages anywhere in the continental United States. The company has various operations centers, called hubs, at airports in major cities across the country. Packages are received at hubs from other locations and then shipped to intermediate hubs or to their final destinations.

The manager of the Air-Express hub in Baltimore, Maryland, is concerned about labor costs at the hub and is interested in determining the most effective way to schedule workers. The hub operates 7 days a week, and the number of packages it handles each day varies from one day to the next. Using historical data on the average number of packages received each day, the manager estimates the number of workers needed to handle the packages as shown in the following table:

Day of Week	Workers Required
Sunday	18
Monday	27
Tuesday	22
Wednesday	26
Thursday	25
Friday	21
Saturday	19

The package handlers working for Air-Express are unionized and are guaranteed a 5-day work week with two consecutive days off. The base wage for the handlers is $655 per week. Because most workers prefer to have Saturday or Sunday off, the union has negotiated bonuses of $25 per day for its members who work on these days. The possible shifts and salaries for package handlers are given in the following table:

Shift	Days Off	Wage
1	Sunday and Monday	$680
2	Monday and Tuesday	$705
3	Tuesday and Wednesday	$705
4	Wednesday and Thursday	$705

5	Thursday and Friday	\$705
6	Friday and Saturday	\$680
7	Saturday and Sunday	\$655

The manager wants to keep the total wage expense for the hub as low as possible. With this in mind, how many package handlers should be assigned to each shift if the manager wants to have a sufficient number of workers available each day?

6-9a DEFINING THE DECISION VARIABLES

In this problem, the manager must decide how many workers to assign to each shift. Because there are seven possible shifts, we need the following seven decision variables:

X_1 = the number of workers assigned to shift 1
X_2 = the number of workers assigned to shift 2
X_3 = the number of workers assigned to shift 3
X_4 = the number of workers assigned to shift 4
X_5 = the number of workers assigned to shift 5
X_6 = the number of workers assigned to shift 6
X_7 = the number of workers assigned to shift 7

6-9b DEFINING THE OBJECTIVE FUNCTION

The objective in this problem is to minimize the total wages paid. Each worker on shift 1 and 6 is paid \$680 per week, and each worker on shift 7 is paid \$655. All other workers are paid \$705 per week. Thus, the objective of minimizing the total wage expense is expressed as:

$$\text{MIN:} \quad 680X_1 + 705X_2 + 705X_3 + 705X_4 + 705X_5 + 680X_6 + 655X_7 \text{ \} total wage expense}$$

6-9c DEFINING THE CONSTRAINTS

The constraints for this problem must ensure that at least 18 workers are scheduled for Sunday, at least 27 are scheduled for Monday, and so on. We need one constraint for each day of the week.

To make sure that at least 18 workers are available on Sunday, we must determine which decision variables represent shifts that are scheduled to work on Sunday. Because shifts 1 and 7 are the only shifts that have Sunday scheduled as a day off, the remaining shifts, 2 through 6, all are scheduled to work on Sunday. The following constraint ensures that at least 18 workers are available on Sunday:

$$0X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 0X_7 \geq 18 \qquad \text{\} workers required on Sunday}$$

Because workers on shifts 1 and 2 have Monday off, the constraint for Monday should ensure that the sum of the variables representing the number of workers on the remaining shifts, 3 through 7, is at least 27. This constraint is expressed as:

$$0X_1 + 0X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 \geq 27 \qquad \text{\} workers required on Monday}$$

Constraints for the remaining days of the week are generated easily by applying the same logic used in generating the previous two constraints. The resulting constraints are stated as:

$$1X_1 + 0X_2 + 0X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 \geq 22 \quad \} \text{ workers required on Tuesday}$$
$$1X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 + 1X_7 \geq 26 \quad \} \text{ workers required on Wednesday}$$
$$1X_1 + 1X_2 + 1X_3 + 0X_4 + 0X_5 + 1X_6 + 1X_7 \geq 25 \quad \} \text{ workers required on Thursday}$$
$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 0X_5 + 0X_6 + 1X_7 \geq 21 \quad \} \text{ workers required on Friday}$$
$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 0X_6 + 0X_7 \geq 19 \quad \} \text{ workers required on Saturday}$$

Finally, all our decision variables must assume nonnegative integer values. These conditions are stated as:

$$X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0$$
$$\text{All } X_i \text{ must be integers}$$

6-9d A NOTE ABOUT THE CONSTRAINTS

At this point, you might wonder why the constraints for each day are greater than or equal to rather than equal to constraints. For example, if Air-Express needs only 19 people on Saturday, why do we have a constraint that allows *more* than 19 people to be scheduled? The answer to this question relates to feasibility. Suppose we restate the problem so that all the constraints are equal to constraints. There are two possible outcomes for this problem: (1) it might have a feasible optimal solution, or (2) it might not have a feasible solution.

In the first case, if the formulation using equal to constraints has a feasible optimal solution, this same solution also must be a feasible solution to our formulation using greater than or equal to constraints. Because both formulations have the same objective function, the solution to our original formulation could not be worse (in terms of the optimal objective function value) than a formulation using equal to constraints.

In the second case, if the formulation using equal to constraints has no feasible solution, there is no schedule where the *exact* number of employees required can be scheduled each day. To find a feasible solution in this case, we would need to make the constraints less restrictive by allowing for more than the required number of employees to be scheduled (that is, using greater than or equal to constraints).

Therefore, using greater than or equal to constraints does not preclude a solution where the exact number of workers needed is scheduled for each shift, if such a schedule is feasible and optimal. If such a schedule is not feasible or not optimal, the formulation using greater than or equal to constraints also guarantees that a feasible optimal solution to the problem will be obtained.

6-9e IMPLEMENTING THE MODEL

The ILP model for the Air-Express scheduling problem is summarized as:

$$\text{MIN:} \quad 680X_1 + 705X_2 + 705X_3 + 705X_4 + 705X_5 + 680X_6 + 655X_7 \quad \} \text{ total wage expense}$$

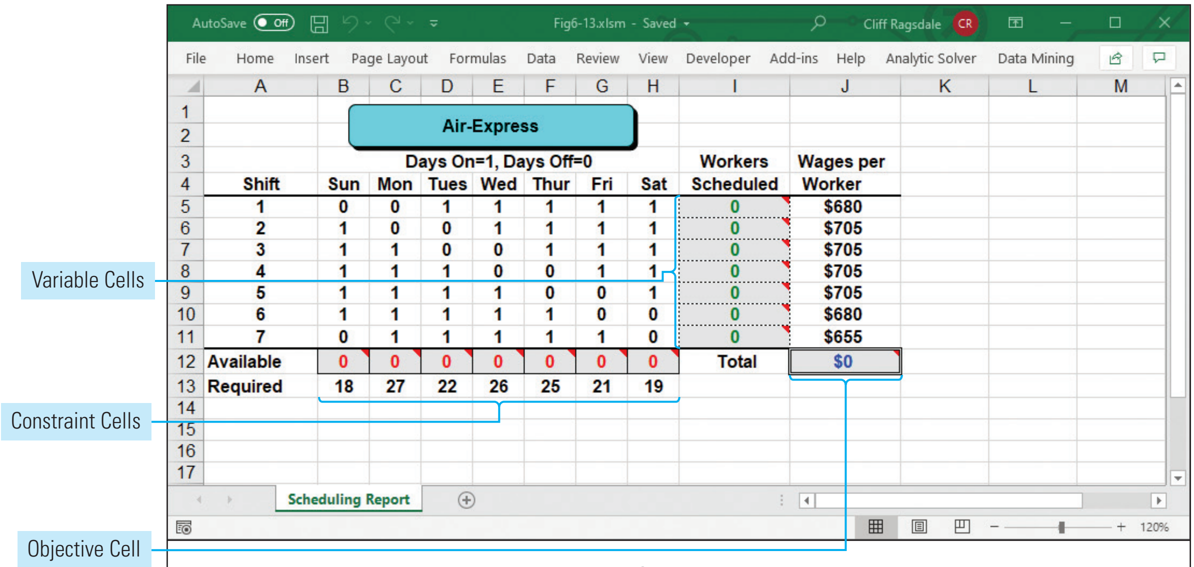
Subject to:

$$0X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 0X_7 \geq 18 \quad \} \text{ workers required on Sunday}$$
$$0X_1 + 0X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 \geq 27 \quad \} \text{ workers required on Monday}$$
$$1X_1 + 0X_2 + 0X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 \geq 22 \quad \} \text{ workers required on Tuesday}$$
$$1X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 + 1X_7 \geq 26 \quad \} \text{ workers required on Wednesday}$$

$$1X_1 + 1X_2 + 1X_3 + 0X_4 + 0X_5 + 1X_6 + 1X_7 \geq 25 \quad \} \text{ workers required on Thursday}$$
$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 0X_5 + 0X_6 + 1X_7 \geq 21 \quad \} \text{ workers required on Friday}$$
$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 0X_6 + 0X_7 \geq 19 \quad \} \text{ workers required on Saturday}$$
$$X_1, X_2, X_3, X_4, X_5, X_6, X_7 \geq 0$$
All X_i must be integers

A convenient way of implementing this model is shown in Figure 6.13 (and in the file Fig6-13.xlsm that accompanies this book). Each row in the table shown in this spreadsheet corresponds to one of the seven shifts in the problem. For each day of the week, entries have been made to indicate which shifts are scheduled to be on or off. For example, shift 1 is scheduled off Sunday and Monday, and works on the remaining days of the week. Notice that the values for each day of the week in Figure 6.13 correspond directly to the coefficients in the constraint in our ILP model for the same day of the week. The required number of workers for each day is listed in cells B13 through H13 and corresponds to the RHS values of each constraint. The wages to be paid to each worker on the various shifts are listed in cells J5 through J11 and correspond to the objective function coefficients in our model.

FIGURE 6.13 *Spreadsheet model for the Air-Express employee scheduling problem*



	Days On=1, Days Off=0							Workers	Wages per
Shift	Sun	Mon	Tues	Wed	Thur	Fri	Sat	Scheduled	Worker
1	0	0	1	1	1	1	1	0	$680
2	1	0	0	1	1	1	1	0	$705
3	1	1	0	0	1	1	1	0	$705
4	1	1	1	0	0	1	1	0	$705
5	1	1	1	1	0	0	1	0	$705
6	1	1	1	1	1	0	0	0	$680
7	0	1	1	1	1	1	0	0	$655
Available	0	0	0	0	0	0	0	Total	$0
Required	18	27	22	26	25	21	19		

Key Cell Formulas

Cell	Formula	Copied to
B12	=SUMPRODUCT(B5:B11,$I$5:$I$11)	C12:H12 and J12

Cells I5 through I11 indicate the number of workers assigned to each shift, and correspond to the decision variables X_1 through X_7 in our algebraic formulation of the LP model. The LHS formula for each constraint is implemented easily using the SUMPRODUCT( ) function. For example, the formula in cell B12 implements the LHS of the constraint for the number of workers needed on Sunday as:

Formula for cell B12: =SUMPRODUCT(B5:B11,$I$5:$I$11)
(Copy to C12 through H12 and J12.)

This formula is then copied to cells C12 through H12 to implement the LHS formulas of the remaining constraints. With the coefficients for the objective function entered in cells J5 through J11, the previous formula is also copied to cell J12 to implement the objective function for this model.

6-9f SOLVING THE MODEL

Figure 6.14 shows the Solver parameters required to solve this problem. The optimal solution is shown in Figure 6.15.

FIGURE 6.14 *Solver settings and options for the Air-Express scheduling problem*

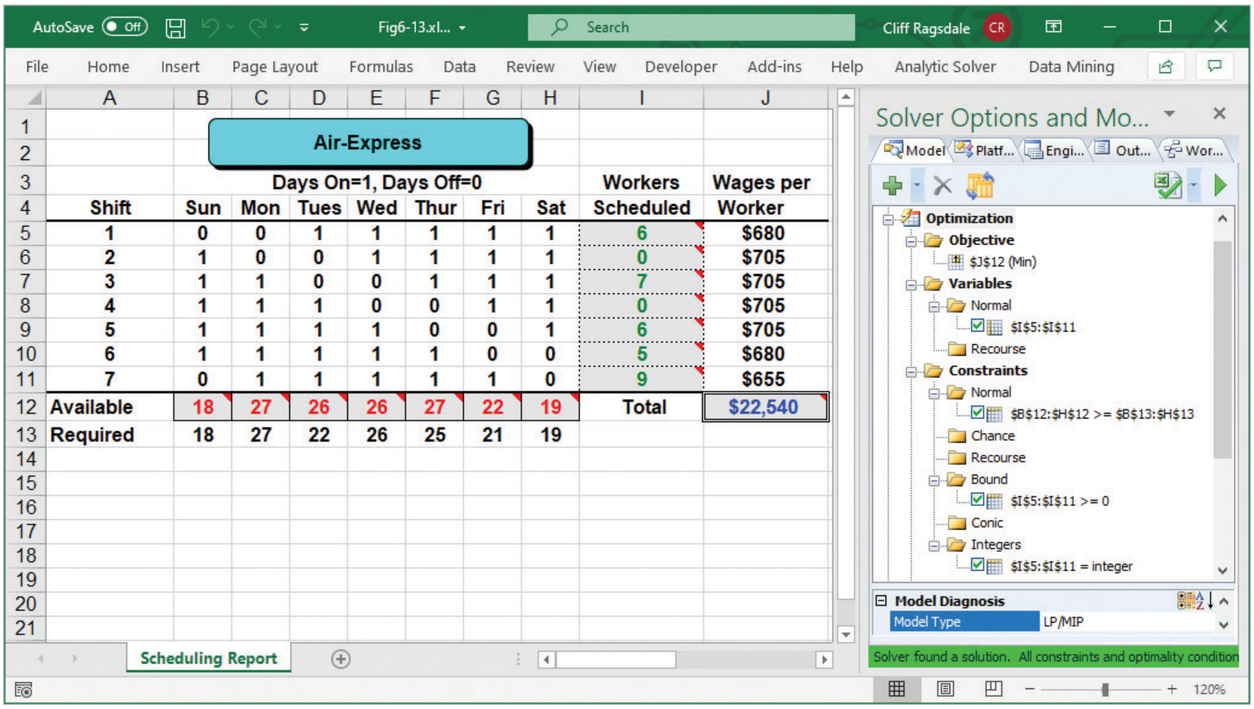
Solver Settings:

Objective: J12 (Min)
Variable cells: I5:I11
Constraints:
    B12:H12 >= B13:H13
    I5:I11 = integer
    I5:I11 >= 0

Solver Options:

    Standard LP/Quadratic Engine (Simplex LP)
    Integer Tolerance = 0

FIGURE 6.15 *Optimal solution to the Air-Express employee scheduling problem*



6-9g ANALYZING THE SOLUTION

The solution shown in Figure 6.15 ensures that the available number of employees is at least as great as the required number of employees for each day. The minimum total wage expense associated with this solution is $22,540. (There are alternate optimal solutions to this problem.)

6-10 Binary Variables

As mentioned earlier, some LP problems naturally evolve into ILP problems when we realize that we need to obtain integer solutions. For example, in the Air-Express problem discussed in the previous section, we needed to determine the number of workers to assign to each of seven shifts. Because workers are discrete units, we needed to impose integrality conditions on the decision variables in this model representing the number of workers scheduled for each shift. To do so, we changed the continuous variables in the model into **general integer variables**, or variables that could assume any integer value (provided that the constraints of the problem are not violated). In many other situations, we might want to use **binary integer variables** (or binary variables), which can assume *only two* integer values: 0 and 1. Binary variables can be useful in a number of practical modeling situations, as illustrated in the following examples.

6-11 A Capital Budgeting Problem

In a capital budgeting problem, a decision-maker is presented with several potential projects or investment alternatives and must determine which projects or investments to choose. The projects or investments typically require different amounts of various resources (e.g., money, equipment, personnel) and generate different cash flows to the company. The cash flows for each project or investment are converted to a net present value (NPV). The problem is to determine which set of projects or investments to select in order to achieve the maximum possible NPV. Consider the following example:

> In his position as vice president of research and development (R&D) for CRT Technologies, Mark Schwartz is responsible for evaluating and choosing which R&D projects to support. The company received 18 R&D proposals from its scientists and engineers, and identified 6 projects as being consistent with the company's mission. However, the company does not have the funds available to undertake all six projects. Mark must determine which of the projects to select. The funding requirements for each project are summarized in the following table along with the NPV the company expects each project to generate.

		Capital (in $1,000s) Required in				
Project	**Expected NPV (in $1,000s)**	**Year 1**	**Year 2**	**Year 3**	**Year 4**	**Year 5**
1	$141	$ 75	$25	$20	$15	$10
2	$187	$ 90	$35	$ 0	$ 0	$30
3	$121	$ 60	$15	$15	$15	$15
4	$ 83	$ 30	$20	$10	$ 5	$ 5
5	$265	$100	$25	$20	$20	$20
6	$127	$ 50	$20	$10	$30	$40

The company currently has \$250,000 available to invest in new projects. It has budgeted \$75,000 for continued support for these projects in year 2 and \$50,000 per year for years 3, 4, and 5. Surplus funds in any year are reappropriated for other uses within the company and may not be carried over to future years.

6-11a DEFINING THE DECISION VARIABLES

Mark must decide which of the six projects to select. Thus, we need six variables to represent the alternatives under consideration. We will let $X_1, X_2, \ldots, X_6$ represent the six decision variables for this problem and assume they operate as:

$$X_i = \begin{cases} 1, \text{ if project } i \text{ is selected} \\ 0, \text{ otherwise} \end{cases} \quad i = 1, 2, \ldots, 6$$

Each decision variable in this problem is a binary variable that assumes the value 1 if the associated project is selected, or the value 0 if the associated project is not selected. In essence, each variable acts like an "on/off switch" to indicate whether or not a given project has been selected.

6-11b DEFINING THE OBJECTIVE FUNCTION

The objective in this problem is to maximize the total NPV of the selected projects. This is stated mathematically as:

$$\text{MAX:} \quad 141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6$$

Notice that this objective function simply sums the NPV figures for the selected projects.

6-11c DEFINING THE CONSTRAINTS

We need one capital constraint for each year to ensure that the selected projects do not require more capital than is available. This set of constraints is represented by:

$$\begin{aligned}
75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 &\le 250 && \}\text{ year 1 capital constraint} \\
25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 &\le 75 && \}\text{ year 2 capital constraint} \\
20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 &\le 50 && \}\text{ year 3 capital constraint} \\
15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 &\le 50 && \}\text{ year 4 capital constraint} \\
10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 &\le 50 && \}\text{ year 5 capital constraint}
\end{aligned}$$

6-11d SETTING UP THE BINARY VARIABLES

In our formulation of this problem, we assume that each decision variable is a binary variable. We must include this assumption in the formal statement of our model by adding the constraints:

$$\text{All } X_i \text{ must be binary}$$

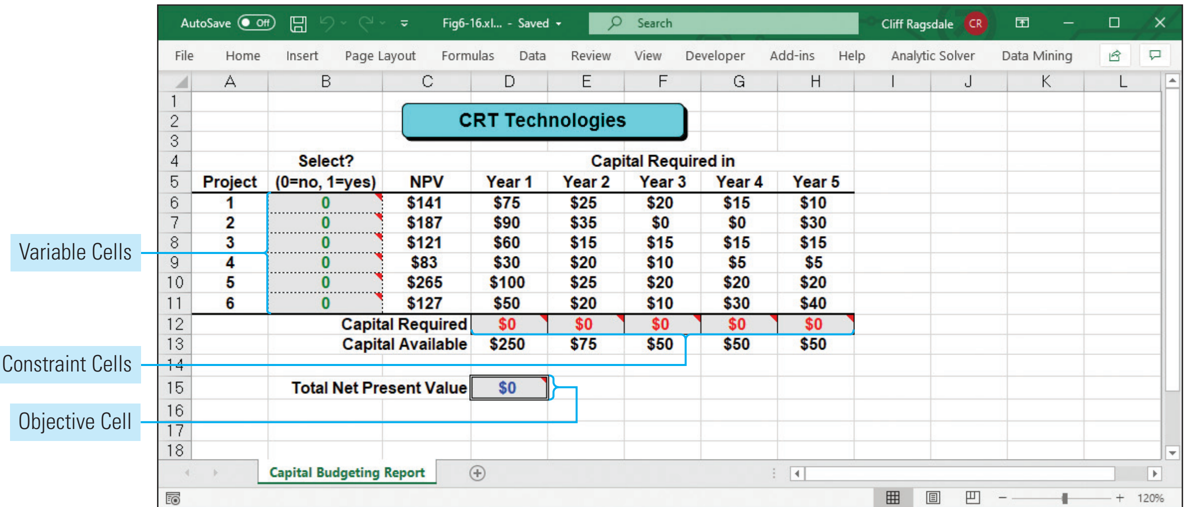
6-11e IMPLEMENTING THE MODEL

The ILP model for the CRT Technologies project selection problem is summarized as:

$$
\begin{array}{ll}
\text{MAX:} & 141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6 \\
\text{Subject to:} & 75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 \leq 250 \\
& 25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 \leq 75 \\
& 20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 \leq 50 \\
& 15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 \leq 50 \\
& 10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 \leq 50 \\
& \text{All } X_i \text{ must be binary}
\end{array}
$$

This model is implemented in the spreadsheet shown in Figure 6.16 (and in the file Fig6-16.xlsm that accompanies this book). In this spreadsheet, the data for each project are listed in separate rows.

FIGURE 6.16 *Spreadsheet model for the CRT Technologies project selection problem*



Project	Select? (0=no, 1=yes)	NPV	Year 1	Year 2	Year 3	Year 4	Year 5
1	0	$141	$75	$25	$20	$15	$10
2	0	$187	$90	$35	$0	$0	$30
3	0	$121	$60	$15	$15	$15	$15
4	0	$83	$30	$20	$10	$5	$5
5	0	$265	$100	$25	$20	$20	$20
6	0	$127	$50	$20	$10	$30	$40
		Capital Required	$0	$0	$0	$0	$0
		Capital Available	$250	$75	$50	$50	$50
		Total Net Present Value	$0				

Key Cell Formulas

Cell	Formula	Copied to
D12	=SUMPRODUCT(D6:D11,$B$6:$B$11)	E12:H12
D15	=SUMPRODUCT(C6:C11,$B$6:$B$11)	--

Cells B6 through B11 contain values of 0 to indicate that they are reserved for representing the six variables in our algebraic model. The LHS formula for the capital constraint is entered in cell D12 and then copied to cells E12 through H12 as:

Formula for cell D12: =SUMPRODUCT(D6:D11,$B$6:$B$11)
(Copy to E12 through H12.)

The RHS values for the constraints are listed in cells D13 through H13. Finally, the objective function of the model is implemented in cell D15 as:

Formula for cell D15: =SUMPRODUCT(C6:C11,$B$6:$B$11)

6-11f SOLVING THE MODEL

To solve this model, we must tell Solver where we have implemented our objective function, decision variables, and constraints. The Solver settings and options shown in Figure 6.17 indicate that the objective function is implemented in cell D15 and that the decision variables are represented by cells B6 through B11. Also, notice that only two sets of constraints are specified for this problem.

FIGURE 6.17

Solver settings and options for the CRT Technologies project selection problem

Solver Settings:
Objective: D15 (Max) Variable cells: B6:B11 Constraints: B6:B11 = binary D12:H12 <= D13:H13
Solver Options: Standard LP/Quadratic Engine (Simplex LP) Integer Tolerance = 0

The first set of constraints ensures that cells B6 through B11 will operate as binary variables. We implemented these constraints by referring to the cells in the spreadsheet that represent our decision variables and selecting the "bin" (for binary) option in the Add Constraint dialog (see Figure 6.9). The last set of constraints shown indicates that the values in cells D12 through H12 must be less than or equal to the values in cells D13 through H13 when the problem is solved. These conditions correspond to the capital constraints in the problem.

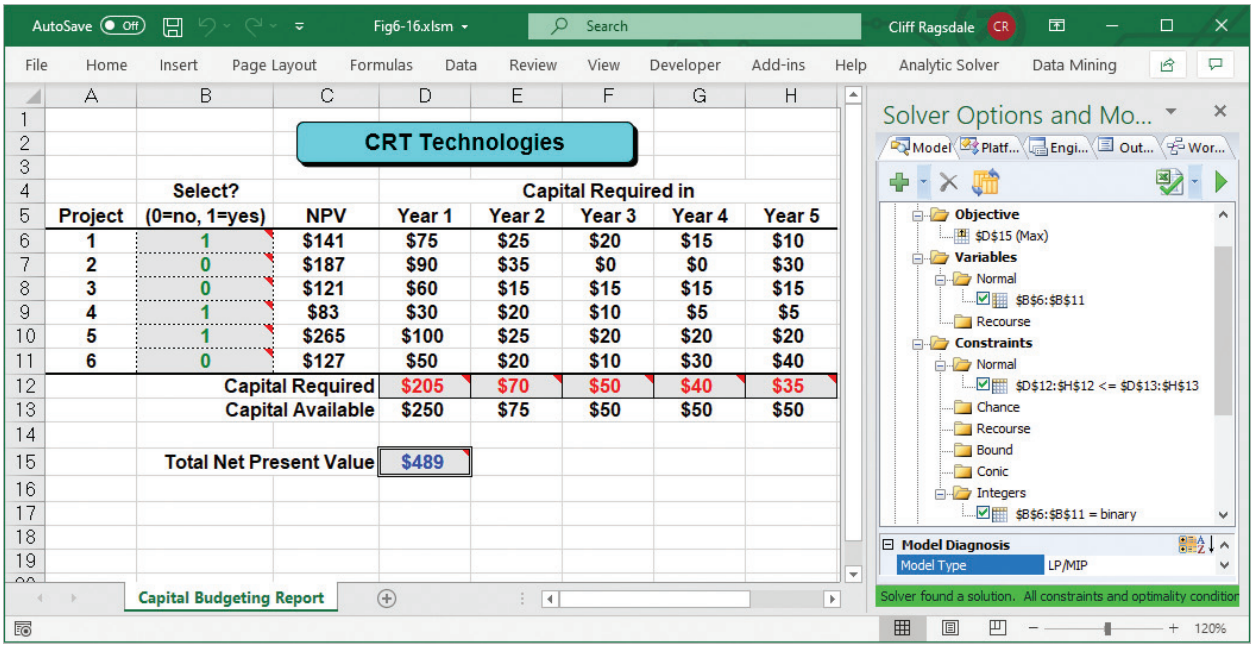
Because this model contains six decision variables and each variable can assume only one of two values, at most $2^6 = 64$ possible integer solutions exist for this problem. Some of these integer solutions will not fall in the feasible region, so we might suspect that this problem will not be too difficult to solve optimally. If we set the Integer Tolerance factor to 0 and solve the problem, we obtain the solution shown in Figure 6.18.

6-11g COMPARING THE OPTIMAL SOLUTION TO A HEURISTIC SOLUTION

The optimal solution shown in Figure 6.18 indicates that if CRT Technologies selects projects 1, 4, and 5, it can achieve a total NPV of $489,000. Although this solution does not use all of the capital available in each year, it is still the best possible integer solution to the problem.

Another approach to solving this problem is to create a ranked list of the projects in decreasing order by NPV and then select projects from this list, in order, until the capital is depleted. As shown in Figure 6.19, if we apply this heuristic to the current problem, we would select projects 5 and 2, but we could not select any more projects due to a lack of capital in year 5. This solution would generate a total NPV of $452,000. Again, we can see the potential benefit of optimization techniques over heuristic solution techniques.

FIGURE 6.18 *Optimal integer solution to the CRT Technologies project selection problem*



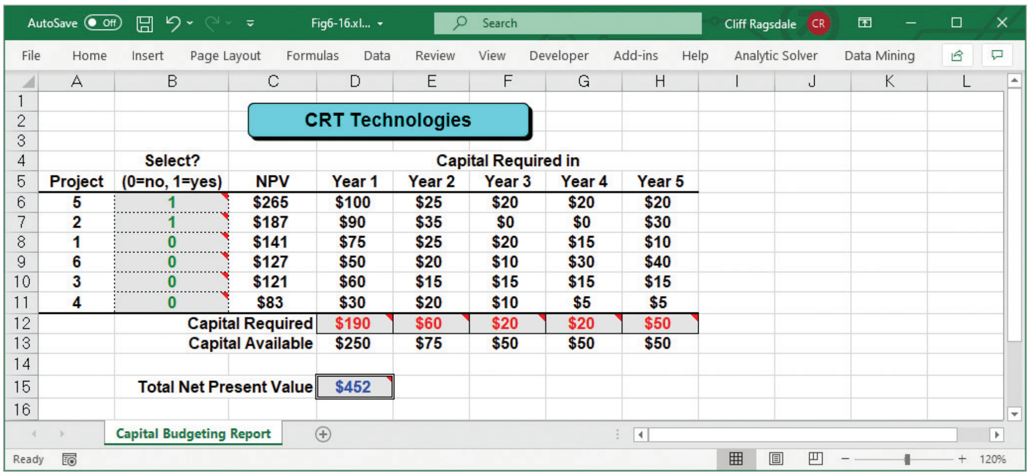


FIGURE 6.19 *A suboptimal heuristic solution to the CRT Technologies project selection problem*

6-12 Binary Variables and Logical Conditions

Binary variables can be used to model a number of logical conditions that might apply in a variety of problems. For example, in the CRT Technologies problem, several of the projects under consideration (e.g., projects 1, 3, and 6) might represent alternative approaches for producing a certain part for a product. The company might want to limit the solution to include *no more than one* of these three alternatives. The following type of constraint accomplishes this restriction:

$$X_1 + X_3 + X_6 \leq 1$$

Because X_1, X_3, and X_6 represent binary variables, no more than one of them can assume the value 1 and still satisfy the previous constraint. If we want to ensure that the solution includes *exactly one* of these alternatives, we could include the following constraint in our model:

$$X_1 + X_3 + X_6 = 1$$

As an example of another type of logical condition, suppose that project 4 involves a cellular communications technology that will not be available to the company unless it undertakes project 5. In other words, the company cannot select project 4 unless it also selects project 5. This type of relationship can be imposed on the solution with the constraint:

$$X_4 - X_5 \leq 0$$

The four possible combinations of values for X_4 and X_5 and their relationships to the previous constraint are summarized as in the following table:

Value of			
X_4	X_5	Meaning	Feasible?
0	0	Do not select either project	Yes
1	1	Select both projects	Yes
0	1	Select 5, but not 4	Yes
1	0	Select 4, but not 5	No

As indicated in this table, the previous constraint prohibits any solution in which project 4 is selected and project 5 is not selected.

As these examples illustrate, you can model certain several different conditions using binary variables. Several problems at the end of this chapter allow you to use binary variables (and your own creativity) to formulate models for decision problems that involve these types of logical conditions.

6-13 The Line Balancing Problem

Assembling a product or delivering a service is often a multi-step process where several tasks are required to complete the product or service. To keep operations flowing smoothly and efficiently, the tasks are usually grouped into packages of work that can be completed by a workstation within a set amount of time, known as the **cycle time**. During each cycle, the work package assigned to each workstation is completed and made ready for the next station in the process. Because some tasks must be completed before others can be performed, the creation of work packages must carefully consider the task precedence requirements.

> Colpitts Control Devices manufactures hand-operated steering mechanisms for powered wheelchairs used by people who are unable to walk. Creating one steering mechanism requires eight assembly tasks. Figure 6.20 summarizes the precedence relations among these tasks as well as the time required to perform each task (in minutes). For instance, task A must be completed before task B, task C must be completed before task F, and so on. The company would like to group the tasks into the minimum number of workstations required to achieve a cycle time of 0.5 minutes.

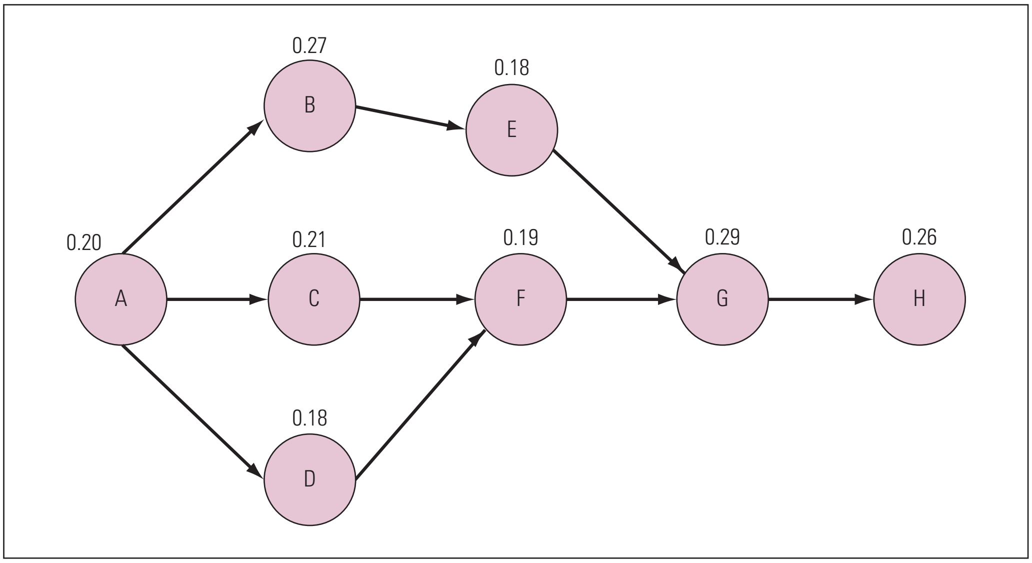


FIGURE 6.20

Task precedence and times for the line balancing problem

6-13a DEFINING THE DECISION VARIABLES

There are eight tasks in this problem so, in the worst case, each task might need to be assigned to its own unique workstation. Thus, we should allow for up to eight workstations. Ignoring the precedence relations (that we will enforce with constraints), any task may be assigned to any of the workstations. This leads to the following set of binary decision variables for the problem:

$$X_{ij} = \begin{cases} 1, \text{ if task } i \text{ is assigned to wrokstation } j \\ 0, \text{ otherwise} \end{cases}, \; i = \text{A, B}, \ldots, \text{H}, j = 1, 2, \ldots, 8$$

One additional decision variable is required for this problem and will be introduced below in section 6-13c.

6-13b DEFINING THE CONSTRAINTS

A number of different constraints apply to this problem. First, we must ensure that each of the eight tasks is assigned to exactly one workstation. This is accomplished as follows:

$$X_{A1} + X_{A2} + X_{A3} + X_{A4} + X_{A5} + X_{A6} + X_{A7} + X_{A8} = 1 \quad \} \text{ Task A assignment constraint}$$
$$X_{B1} + X_{B2} + X_{B3} + X_{B4} + X_{B5} + X_{B6} + X_{B7} + X_{B8} = 1 \quad \} \text{ Task B assignment constraint}$$

and so on to…

$$X_{H1} + X_{H2} + X_{H3} + X_{H4} + X_{H5} + X_{H6} + X_{H7} + X_{H8} = 1 \quad \} \text{ Task H assignment constraint}$$

Next, we must ensure that the time required to complete the tasks assigned to each workstation does not exceed the desired cycle time of 0.5 minutes. This is accomplished via:

$$\left.\begin{aligned} 0.2X_{A1} + 0.27X_{B1} + 0.21X_{C1} + 0.18X_{D1} + 0.18X_{E1} + 0.19X_{F1} + 0.29X_{G1} \\ + 0.26X_{H1} \leq 0.5 \end{aligned}\right\} \begin{array}{l}\text{Task time for} \\ \text{workstation 1}\end{array}$$

$$\left.\begin{aligned} 0.2X_{A2} + 0.27X_{B2} + 0.21X_{C2} + 0.18X_{D2} + 0.18X_{E2} + 0.19X_{F2} + 0.29X_{G2} \\ + 0.26X_{H2} \leq 0.5 \end{aligned}\right\} \begin{array}{l}\text{Task time for} \\ \text{workstation 2}\end{array}$$

and so on to…

$$\left.\begin{aligned}0.2X_{A8} + 0.27X_{B8} + 0.21X_{C8} + 0.18X_{D8} + 0.18X_{E8} + 0.19X_{F8} + 0.29X_{G8}\\ + 0.26X_{H8} \le 0.5\end{aligned}\right\}\text{Task time for workstation 8}$$

The arrows in Figure 6.20 summarize the required precedence conditions for this problem. For instance, task A must be completed before task B. This may be accomplished by task A being assigned to a workstation that precedes the workstation to which task B is assigned. Alternatively, this may also be accomplished by assigning tasks A and B to the same workstation (assuming both tasks can be accomplished within the specified cycle time). In general, the workstation (1, 2, … , 8) that task i is assigned to may be computed from the decision variables as:

$$WS_i = 1X_{i1} + 2X_{i2} + 3X_{i3} + 4X_{i4} + 5X_{i5} + 6X_{i6} + 7X_{i7} + 8X_{i8}$$

For example, if task C is assigned to workstation 3 (i.e., $X_{C3}=1$) then $WS_C = 3X_{C3} = 3$. Using this definition of WS_i the precedence constraints may be stated as follows:

$WS_A \le WS_B$ (implement as $WS_A - WS_B \le 0$) } Task B is stationed with or after A
$WS_A \le WS_C$ (implement as $WS_A - WS_C \le 0$) } Task C is stationed with or after A
$WS_A \le WS_D$ (implement as $WS_A - WS_D \le 0$) } Task D is stationed with or after A
$WS_B \le WS_E$ (implement as $WS_B - WS_E \le 0$) } Task E is stationed with or after B
$WS_C \le WS_F$ (implement as $WS_C - WS_F \le 0$) } Task F is stationed with or after C
$WS_D \le WS_F$ (implement as $WS_D - WS_F \le 0$) } Task F is stationed with or after D
$WS_E \le WS_G$ (implement as $WS_E - WS_G \le 0$) } Task G is stationed with or after E
$WS_F \le WS_G$ (implement as $WS_F - WS_G \le 0$) } Task G is stationed with or after F
$WS_G \le WS_H$ (implement as $WS_G - WS_H \le 0$) } Task H is stationed with or after G

Notice that the first of these precedence constraints ensures that task B is either assigned to the same workstation as task A (if $WS_A = WS_B$) or one following it (if $WS_A < WS_B$). Similar interpretations apply to the other precedence constraints.

6-13c DEFINING THE OBJECTIVE

Recall that the objective in this problem, which is to determine the minimum number of workstations required to achieve a cycle time of 0.5 minutes. Because WS_i represents the workstation number to which task i is assigned and we want to use as few workstations as possible, we would like to minimize the maximum assigned workstation number. That is, we would like to use the objective:

$$\text{MIN: } \text{MAX}(WS_A, WS_B, WS_C, WS_D, WS_E, WS_F, WS_G, WS_H)$$

Unfortunately, this objective function is not a linear combination of the decision variables. However, we can express the same objective in a linear fashion by introducing an additional variable (Q) and eight additional constraints as follows:

$$\text{MIN: } Q$$

$$WS_A \le Q$$
$$WS_B \le Q$$
and so on to…
$$WS_H \le Q$$

Because the variable Q must be greater than or equal to the values of all the assigned workstation numbers, and because we are trying to minimize it, Q will always be set equal to the maximum assigned workstation number. At the same time, this objective function tries to find a solution where the maximum assigned workstation number (and the value of Q) is as small as possible. Therefore, this technique allows us to minimize the maximum assigned workstation number (and also minimizes the number of workstations used). (Note that this technique for minimizing the maximum of several computed values proves useful in a number of optimization modeling situations.)

6-13d IMPLEMENTING THE MODEL

The ILP model for the Colpitts Control Devices workload balancing problem is summarized as:

MIN: Q

Subject to:

$X_{A1} + X_{A2} + X_{A3} + X_{A4} + X_{A5} + X_{A6} + X_{A7} + X_{A8} = 1$ } Assignment constraint for task A

$X_{B1} + X_{B2} + X_{B3} + X_{B4} + X_{B5} + X_{B6} + X_{B7} + X_{B8} = 1$ } Assignment constraint for task B

and so on to…

$X_{H1} + X_{H2} + X_{H3} + X_{H4} + X_{H5} + X_{H6} + X_{H7} + X_{H8} = 1$ } Assignment constraint for task H

$0.2X_{A1} + 0.27X_{B1} + 0.21X_{C1} + 0.18X_{D1} + 0.18X_{E1} + 0.19X_{F1} + 0.29X_{G1} + 0.26X_{H1} \leq 0.5$ } Task time for workstation 1

$0.2X_{A2} + 0.27X_{B2} + 0.21X_{C2} + 0.18X_{D2} + 0.18X_{E2} + 0.19X_{F2} + 0.29X_{G2} + 0.26X_{H2} \leq 0.5$ } TaskTask time for workstation 2

and so on to…

$0.2X_{A8} + 0.27X_{B8} + 0.21X_{C8} + 0.18X_{D8} + 0.18X_{E8} + 0.19X_{F8} + 0.29X_{G8} + 0.26X_{H8} \leq 0.5$ } Task time for workstation 8

$WS_A \leq Q$ } Objective constraint for task A's workstation number

$WS_B \leq Q$ } Objective constraint for task B's workstation number

and so on to…

$WS_H \leq Q$ } Objective constraint for task H's workstation number

$WS_A - WS_B \leq 0$ } Task B is stationed with or after A

$WS_A - WS_C \leq 0$ } Task C is stationed with or after A

$WS_A - WS_D \leq 0$ } Task D is stationed with or after A

$WS_B - WS_E \leq 0$ } Task E is stationed with or after B

$WS_C - WS_F \leq 0$ } Task F is stationed with or after C

$WS_D - WS_F \leq 0$ } Task F is stationed with or after D

$WS_E - WS_G \leq 0$ } Task G is stationed with or after E

$WS_F - WS_G \leq 0$ } Task G is stationed with or after F

$WS_G - WS_H \leq 0$ } Task H is stationed with or after G

Where:

$$WS_A = 1X_{A1} + 2X_{A2} + 3X_{A3} + 4X_{A4} + 5X_{A5} + 6X_{A6} + 7X_{A7} + 8X_{A8} \left.\right\} \text{Task A's workstation number}$$

$$WS_B = 1X_{B1} + 2X_{B2} + 3X_{B3} + 4X_{B4} + 5X_{B5} + 6X_{B6} + 7X_{A7} + 8X_{B8} \left.\right\} \text{Task B's workstation number}$$

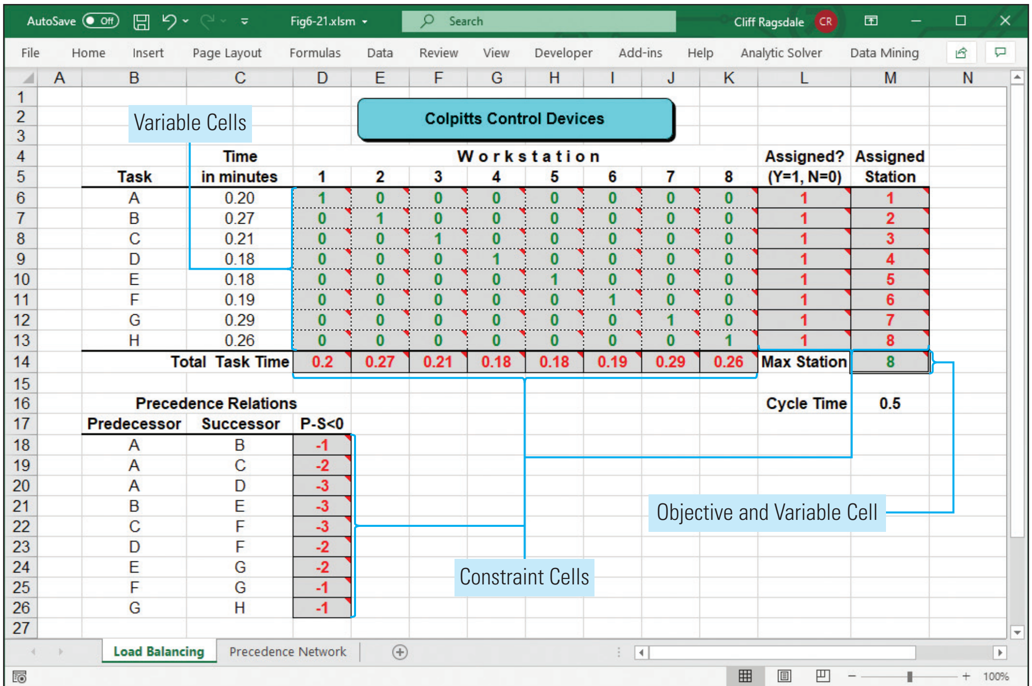
and so on to…

$$WS_H = 1X_{H1} + 2X_{H2} + 3X_{H3} + 4X_{H4} + 5X_{H5} + 6X_{H6} + 7X_{H7} + 8X_{H8} \left.\right\} \text{Task H's workstation number}$$

All X_{ij} are binary

A convenient way of implementing this model is shown in Figure 6.21 (and in the file Fig6-21.xlsm that accompanies this book). Cells D6 through K13 in this workbook represent the binary decision variables indicating to what workstation each task is assigned. An arbitrary starting solution is shown in Figure 6.21 with each task being assigned to a unique workstation.

FIGURE 6.21 *Spreadsheet model for the line balancing problem*



Key Cell Formulas

Cell	Formula	Copied to
D14	=SUMPRODUCT(D6:D13,\$C\$6:\$C\$13)	E14:K14
L6	=SUM(D6:K6)	L7:L13
M6	=SUMPRODUCT(D6:K6,\$D\$5:\$K\$5)	M7:M13
D18	=VLOOKUP(B18,\$B\$6:\$M\$13,12)– VLOOKUP(C18,\$B\$6:\$M\$13,12)	D19:D26

The LHS formulas for the task assignment constraints are implemented in cells L6 through L13, containing formulas summing the decision variables in their respective rows. Each of these cells will be constrained to equal 1.

Formula for cell L6: =SUM(D6:K6)
(Copy to L7 through L13.)

Formulas computing the total task time assigned to each workstation are implemented in cells D14 through K14 and will be constrained to not exceed the desired cycle time specified in cell M16.

Formula for cell D14: =SUMPRODUCT(D6:D13,$C$6:$C$13)
(Copy to E14 through K14.)

The workstation number to which each task is assigned is computed in cells M6 through M13 as:

Formula for cell M6: =SUMPRODUCT(D6:K6,$D$5:$K$5)
(Copy to M7 through M13.)

The predecessor and successor tasks for each require precedence relation are listed in cells B18 through C26. Recall that each successor task must be stationed with or after its associated predecessor task. The LHS formula for each of these constraints is implemented in cells D18 through D26 and will be constrained to be less than or equal to zero.

Formula for cell D18: =VLOOKUP(B18,$B$6:$M$13,12)-VLOOKUP(C18,$B$6:$M$13,12)
(Copy to D19 through D26.)

This first VLOOKUP( ) function in this formula "looks up" the value in cell B18 in the first column of the range B6 through M13 and, when it finds the matching value, returns the value in the 12th column of the matching row (as specified by the value 12 as the third argument in the VLOOKUP( ) function). So, for cell D18, the first VLOOKUP( ) function looks for the letter A (from B18) in the first column of the range B6 through M13, and locates this value in the first row of the range. It then returns the value 1 found in the 12th column on that same row (cell M6) in the range B6 through M13. The second VLOOK( ) performs the identical operation for the value in cell C18 (i.e., the letter B) and returns the value 2 found cell M7. Copying this formula to cells D19 through D26 computes the differences between the assigned workstation number for each predecessor and successor pairing. (By default, the VLOOKUP( ) function assumes the values in the first column of the given range appear in ascending order. If that's not the case, an optional fourth argument should be passed to the VLOOKUP( ) function with a Boolean value of False.)

Finally, the objective function for this problem is implemented in cell M14. While it seems intuitive and tempting to use the formula MAX(M6:M13) in this cell, recall that this is not a linear function of the decision variables. Instead, cell M14 should be defined to be both a decision variable cell *and* the objective cell. As with any other variable cell, Solver will determine the optimal value for the cell so no formula should be placed in the cell. (The value 8 shown in cell M14 in Figure 6.21 was entered manually.) We will instruct Solver to choose and minimize the value in cell M14 while keeping its value greater than or equal to the values in cells M6 through M13. This, in turn, will minimize the number of workstations required.

6-13e ANALYZING THE SOLUTION

The Solver parameters and settings used to solve this problem are shown in Figure 6.22. Again, note that cell M14 is both a variable cell and the objective cell we wish to minimize. The optimal solution to the problem is shown in Figure 6.23.

FIGURE 6.22

Solver settings and options for the line balancing problem

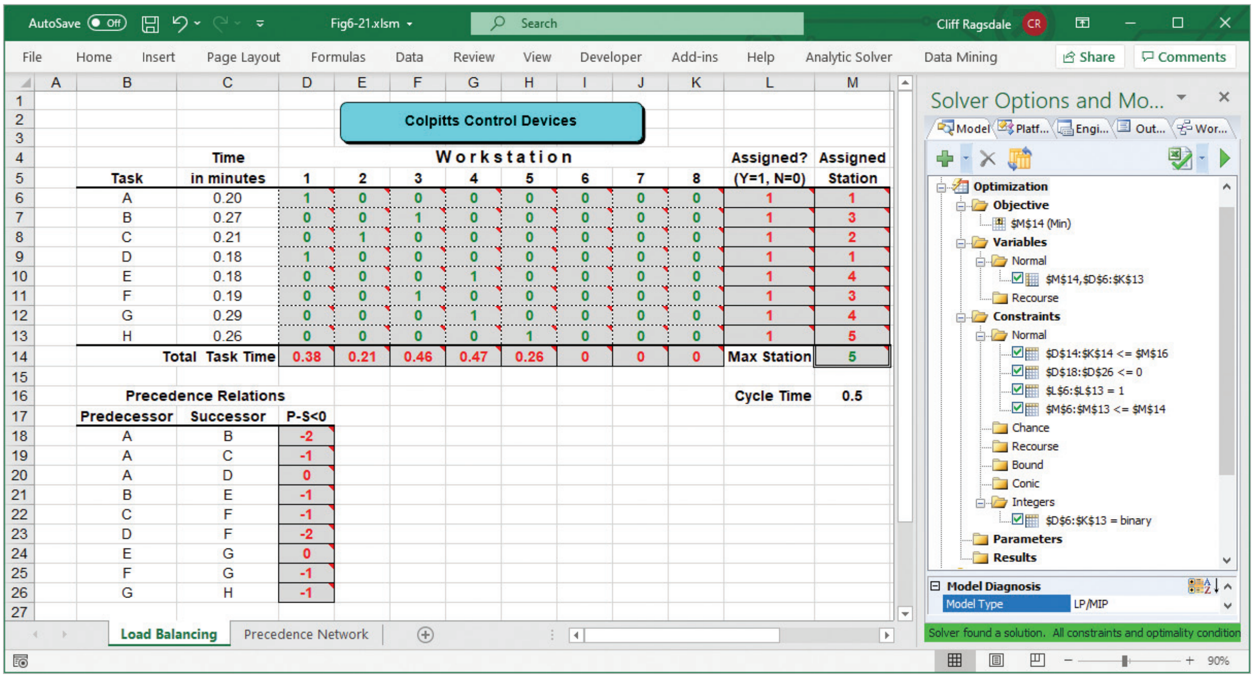
Solver Settings:

Objective: M14 (Min)
Variable cells: D6:K13, M14
Constraints:
  L6:L13 = 1
  D14:K14 <= M16
  D18:D26 <= 0
  M6:M13 <= M14
  D6:K13 = binary

Solver Options:

  Standard LP/Quadratic Engine (Simplex LP)
  Integer Tolerance = 0

FIGURE 6.23 *Optimal solution to the line balancing problem*



Colpitts Control Devices

Task	Time in minutes	Workstation 1	2	3	4	5	6	7	8	Assigned? (Y=1, N=0)	Assigned Station
A	0.20	1	0	0	0	0	0	0	0	1	1
B	0.27	0	0	1	0	0	0	0	0	1	3
C	0.21	0	1	0	0	0	0	0	0	1	2
D	0.18	1	0	0	0	0	0	0	0	1	1
E	0.18	0	0	0	1	0	0	0	0	1	4
F	0.19	0	0	1	0	0	0	0	0	1	3
G	0.29	0	0	0	1	0	0	0	0	1	4
H	0.26	0	0	0	0	1	0	0	0	1	5
	Total Task Time	0.38	0.21	0.46	0.47	0.26	0	0	0	Max Station	5
										Cycle Time	0.5

Precedence Relations

Predecessor	Successor	P-S<0
A	B	-2
A	C	-1
A	D	0
B	E	-1
C	F	-1
D	F	-2
E	G	0
F	G	-1
G	H	-1

This solution indicates that five workstations are required. Tasks A and D are assigned to workstation 1, task C to workstation 2, tasks B and F to workstation 3, tasks E and G to workstation 4, and task H to workstation 5. Note that there are alternate optimal solutions to this problem. Also note that the task times assigned to all workstations vary and are all less than the required cycle time of 0.5 minutes.

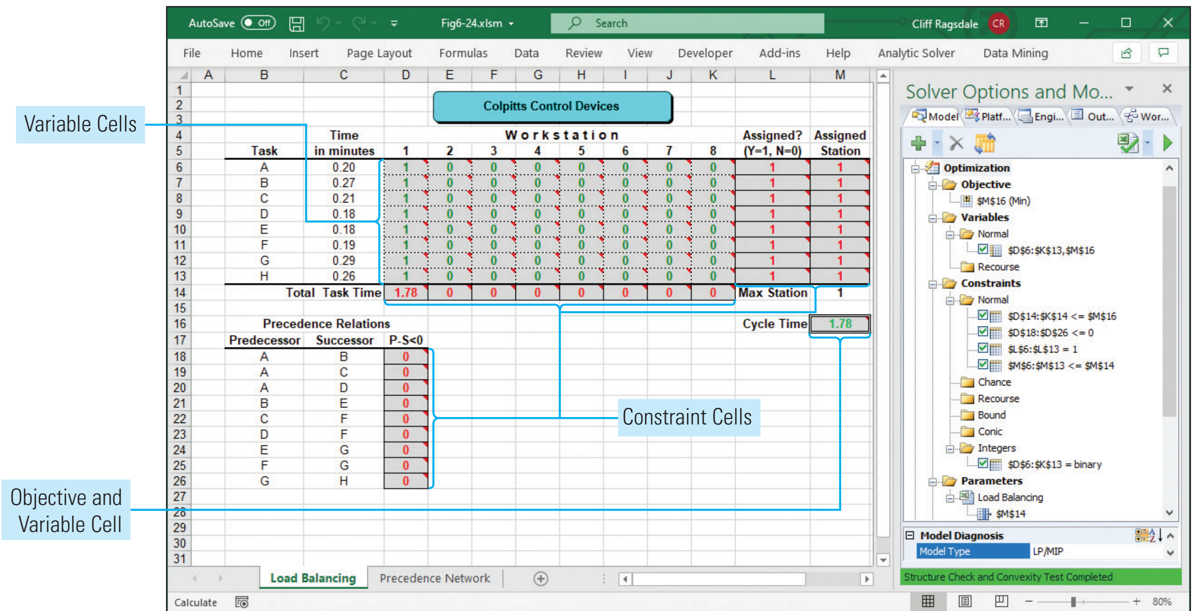
6-13f EXTENSION

Up to this point, we have approached the line balancing problem from the perspective of determining the minimum number of workstations required to achieve a specified cycle time. Another approach to the problem involves determining the minimum cycle time associated with a particular number of workstations. In the present example, if there is a single workstation (with all tasks assigned to that workstation), the minimum cycle time is 1.78 minutes—or the sum of the task times for all of the tasks. Similarly, if there are eight workstations (with each job assigned to a unique workstation), the minimum cycle time is 0.29 minutes—or the maximum individual task time. But what if there were 2, 3, 4, 5, 6, or 7 workstations? What is the minimum cycle time associated with each of those configurations? Fortunately, with a few easy changes, we can easily run a parameterized optimization on our existing model to answer these questions.

Figure 6.24 (and the file Fig6-24.xlsm that accompanies this book) illustrates the required changes to the model. First, notice that cell M16 (representing the cycle time) is now a decision variable cell *and* the objective cell we wish to minimize. Also, cell M14 now simply represents the RHS value for the constraints in cells M6 through M13 which compute the assigned workstation number for each task. Figure 6.24 shows the solution we would obtain if only one workstation is allowed. However, the formula in cell M14 allows us to "parameterize" or specify several values for this cell.

Formula for cell M14: =PsiOptParam(1, 8)

FIGURE 6.24 *Spreadsheet model for the parameterized line balancing problem*



Key Cell Formulas

Cell	Formula	Copied to
M14	=PsiOptParam(1,8)	--
D14	=SUMPRODUCT(D6:D13,$C$6:$C$13)	E14:K14
L6	=SUM(D6:K6)	L7:L13
M6	=SUMPRODUCT(D6:K6,$D$5:$K$5)	M7:M13
D18	=VLOOKUP(B18,$B$6:$M$13,12)–VLOOKUP(C18,$B$6:$M$13,12)	D19:D26

The PsiOptParam( ) function in cell M14 tells Analytic Solver that we want to consider values between 1 and 8 for this cell. (Though not needed in this example, a third argument for the PsiOptParam( ) function can be specified to indicate a default value for the cell and is useful when using more than one PsiOptParam( ) functions in the same model.) As shown in Figure 6.25, the Platform tab in Analytic Solver task pane allows us to indicate that we want to run 8 optimizations. When we run the model using the settings shown in Figure 6.26, Analytic Solver will run eight optimizations, changing the value in cell M14 from 1 to 8 in equal increments.

FIGURE 6.25

Analytic Solver Platform setting for running multiple optimizations

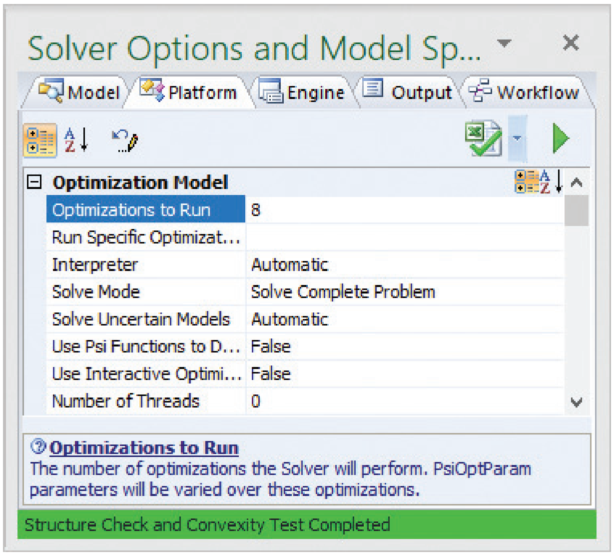


FIGURE 6.26

Solver settings and options for the parameterized line balancing problem

Solver Settings:

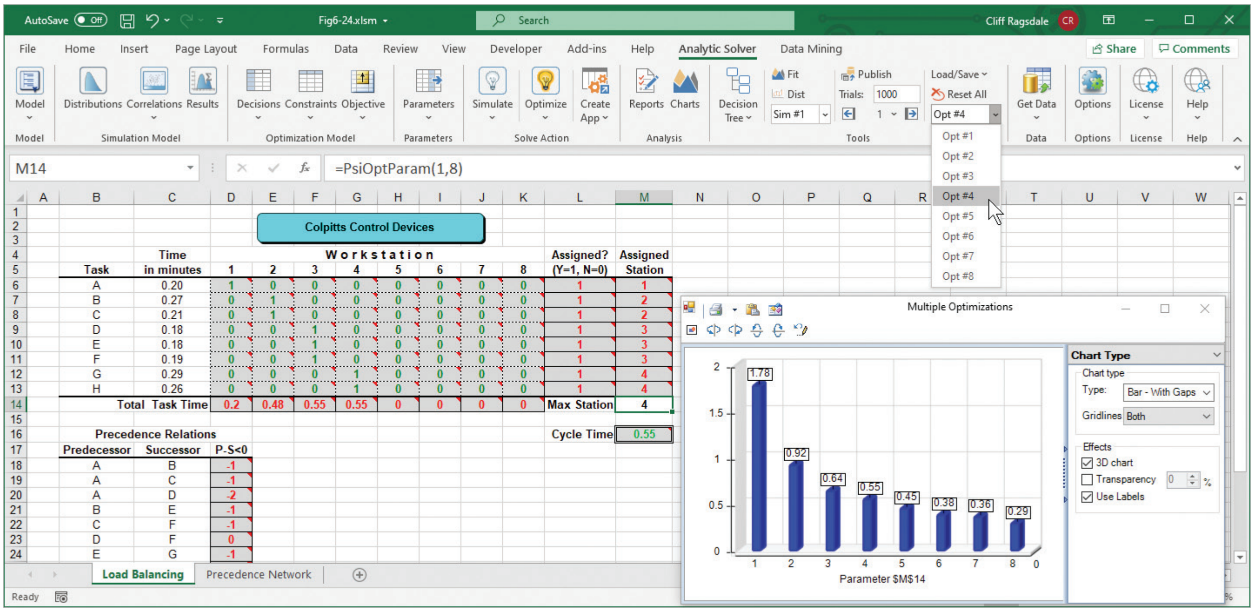
Objective: M16 (Min)
Variable cells: D6:K13, M16
Constraints:
L6:L13 = 1
D14:K14 <= M16
D18:D26 <= 0
M6:M13 <= M14
D6:K13 = binary

Solver Options:

Standard LP/Quadratic Engine (Simplex LP)
Integer Tolerance = 0
Optimizations to Run = 8

As shown in Figure 6.27, we may view the results for any of the eight optimizations using the displayed dropdown on the Analytic Solver tab. Note that when we allow four workstations the minimum cycle time is 0.55 minutes. Assuming that more workstations require more staffing, there is a trade-off between labor costs and cycle time; where lower cycle times involve higher labor costs and vice versa. After running a parameterized optimization, Analytic Solver provides options of graphing key results. As an example, the graph in Figure 6.27 shows the minimum cycle time (optimal objective value) for each of the eight parameterized optimizations. This sort of graph is very helpful in assisting management determining where the benefit of additional reductions in cycle time is not worth the cost of manning additional workstations.

FIGURE 6.27 *Exploring results for the parameterized line balancing problem*



To create the graph shown in Figure 6.27, first run the eight parameterized optimizations. After Analytic Solver performs the optimizations, you can easily construct a graph like the one shown in Figure 6.27 by following these steps:

1. Click the Charts icon on the Analytic Solver tab.
2. Select Multiple Optimizations, Monitored Cells.
3. Expand the Objective option, select $M$16, and click the ">" button.
4. Click OK.

Analytic Solver then produces a basic graph of the optimization results and offers a variety of options that allow you to edit and customize its appearance.

6-14 The Fixed-Charge Problem

In most of the LP problems discussed in earlier chapters, we formulated objective functions to maximize profits or minimize costs. In each of these cases, we associated a per-unit cost or per-unit profit with each decision variable to create the objective function. However, in some situations, the decision to produce a product results in a lump-sum cost, or fixed-charge, in addition to a per-unit cost or profit. These types of problems are known as **fixed-charge** or fixed-cost problems. The following are some examples of fixed-costs:

- the cost to lease, rent, or purchase a piece of equipment or a vehicle that will be required if a particular action is taken
- the setup cost required to prepare a machine or production line to produce a different type of product
- the cost to construct a new production line or facility that will be required if a particular decision is made
- the cost of hiring additional personnel that will be required if a particular decision is made

In each of these examples, the fixed costs are *new* costs that will be incurred if a particular action or decision is made. In this respect, fixed costs are different from **sunk costs**, which are costs that will be incurred regardless of what decision is made. Sunk costs are irrelevant for decision-making purposes because, by definition, decisions do not influence these costs. On the other hand, fixed costs are important factors in decision-making because the decision determines whether or not these costs will be incurred. The following example illustrates the formulation and solution of a fixed-charge problem.

Remington Manufacturing is planning its next production cycle. The company can produce three products, each of which must undergo machining, grinding, and assembly operations. The following table summarizes the hours of machining, grinding, and assembly required by each unit of each product, and the total hours of capacity available for each operation.

	Hours Required By			
Operation	**Product 1**	**Product 2**	**Product 3**	**Total Hours Available**
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400

The cost accounting department has estimated that each unit of product 1 manufactured and sold will contribute \$48 to profit, and each unit of products 2 and 3 contributes \$55 and \$50, respectively. However, manufacturing a unit of product 1 requires a setup operation on the production line that costs \$1,000. Similar setups are required for products 2 and 3 at costs of \$800 and \$900, respectively. The marketing department believes it can sell all the products produced. Therefore, the management of Remington wants to determine the most profitable mix of products to produce.

6-14a DEFINING THE DECISION VARIABLES

Although only three products are under consideration in this problem, we need six variables to formulate the problem accurately. We can define these variables as:

$$X_i = \text{the number of units of product } i \text{ to be produced, } i = 1, 2, 3$$

$$Y_i = \begin{cases} 1, \text{ if } X_i > 0 \\ 0, \text{ if } X_i = 0 \end{cases}, \ i = 1, 2, 3$$

We need three variables, X_1, X_2, and X_3, to correspond to the units of products 1, 2, and 3 produced. Each of the X_i variables has a corresponding binary variable, Y_i, that will equal 1 if X_i assumes any positive value, or will equal 0 if X_i is 0. For now, do not be concerned about how this relationship between the X_i and Y_i is enforced. We will explore that soon.

6-14b DEFINING THE OBJECTIVE FUNCTION

Given our definition of the decision variables, the objective function for our model is stated as:

$$\text{MAX:} \quad 48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$$

The first three terms in this function calculate the marginal profit generated by the number of products 1, 2, and 3 sold. The last three terms in this function subtract the fixed costs for the products produced. For example, if X_1 assumes a positive value, we know from our definition of the Y_i variables that Y_1 should equal 1. And if $Y_1 = 1$, the value of the objective function will be reduced by $1,000 to reflect payment of the setup cost. On the other hand, if $X_1 = 0$, we know that $Y_1 = 0$. Therefore, if no units of X_1 are produced, the setup cost for product 1 will not be incurred in the objective. Similar relationships exist between X_2 and Y_2 and between X_3 and Y_3.

6-14c DEFINING THE CONSTRAINTS

Several sets of constraints apply to this problem. Capacity constraints are needed to ensure that the number of machining, grinding, and assembly hours used does not exceed the number of hours available for each of these resources. These constraints are stated as:

$$2X_1 + 3X_2 + 6X_3 \le 600 \quad \}\text{ machining constraint}$$
$$6X_1 + 3X_2 + 4X_3 \le 300 \quad \}\text{ grinding constraint}$$
$$5X_1 + 6X_2 + 2X_3 \le 400 \quad \}\text{ assembly constraint}$$

We also need to include integer and nonnegativity conditions on the X_i variables as:

$$X_i \ge 0 \text{ and integer, } i = 1, 2, 3$$

The following constraint on the Y_i variables is needed to ensure that they operate as binary variables:

$$\text{All } Y_i \text{ must be binary}$$

As mentioned earlier, we must ensure that the required relationship between the X_i and Y_i variables is enforced. In particular, the value of the Y_i variables can be determined from the X_i variables. Therefore, we need constraints to establish this *link* between the value of the Y_i variables and the X_i variables. These linking constraints are represented by:

$$X_1 \le M_1Y_1$$
$$X_2 \le M_2Y_2$$
$$X_3 \le M_3Y_3$$

In each of these constraints, the M_i is a numeric constant that represents an upper bound on the optimal value of the X_i. Let's assume that all the M_i are arbitrarily large numbers; for example, $M_i = 10{,}000$. Then each constraint sets up a link between the value of the X_i and the Y_i. For example, if any X_i variables in the previous constraints assume a value greater than 0, the corresponding Y_i variable must assume the value 1 or the constraint will be violated. On the other hand, if any of the X_i variables are equal to 0, the corresponding Y_i variables could equal 0 or 1 and still satisfy the constraint. However, if we consider the objective function to this problem, we know that when given a choice, Solver will always set the Y_i equal to 0 (rather than 1) because this results in a better objective function value. Therefore, we can conclude that if any X_i variables are equal to 0, Solver will set the corresponding Y_i variable equal to 0 because this is feasible and results in a better objective function value.

6-14d DETERMINING VALUES FOR "BIG M"

The M_i values used in the linking constraints are sometimes referred to as "Big M" values because they can be assigned arbitrarily large values. However, for reasons that go beyond the scope of this text, these types of problems are easier to solve if the M_i values are kept as small as possible.

As indicated earlier, the M_i values impose upper bounds on the values of the X_i. So, if a problem indicates that a company could manufacture and sell no more than 60 units of X_1, for example, we could let $M_1 = 60$. However, even if upper bounds for the X_i are not explicitly indicated, it is sometimes easy to derive implicit upper bounds for these variables.

Let's consider the variable X_1 in the Remington problem. What is the maximum number of units of X_1 that can be produced in this problem? Referring back to our capacity constraints, if the company produces 0 units of X_2 and X_3, it would run out of machining capacity after producing $600/2 = 300$ units of X_1. Similarly, it would run out of grinding capacity after producing $300/6 = 50$ units of X_1, and it would run out of assembly capacity after producing $400/5 = 80$ units of X_1. Therefore, the maximum number of units of X_1 the company can produce is 50. Using similar logic, we can determine that the maximum units of X_2 the company can produce is $\text{MIN}(600/3, 300/3, 400/6) = 66.67$, and the maximum units of X_3 is $\text{MIN}(600/6, 300/4, 400/2) = 75$. Thus, for this problem reasonable upper bounds for X_1, X_2, and X_3 are represented by $M_1 = 50$, $M_2 = 66.67$, and $M_3 = 75$, respectively. (Note that the method illustrated here for obtaining reasonable values for the M_i does not apply if any of the coefficients in the machining, grinding, or assembly constraints are negative. Why is this?) When possible, you should determine reasonable values for the M_i in this type of problem. However, if this is not possible, you can assign arbitrarily large values to the M_i.

6-14e IMPLEMENTING THE MODEL

Using the values for the M_i calculated earlier, our ILP formulation of Remington's production planning model is summarized as:

$$
\begin{array}{lll}
\text{MAX:} & 48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3 & \\
\text{Subject to:} & 2X_1 + 3X_2 + 6X_3 \le 600 & \} \text{ machining constraint} \\
 & 6X_1 + 3X_2 + 4X_3 \le 300 & \} \text{ grinding constraint} \\
 & 5X_1 + 6X_2 + 2X_3 \le 400 & \} \text{ assembly constraint} \\
 & X_1 - 50Y_1 \le 0 & \} \text{ linking constraint} \\
 & X_2 - 67Y_2 \le 0 & \} \text{ linking constraint} \\
 & X_3 - 75Y_3 \le 0 & \} \text{ linking constraint} \\
 & \text{All } Y_i \text{ must be binary} & \} \text{ binary constraints} \\
 & \text{All } X_i \text{ must be integer} & \} \text{ integrality conditions} \\
 & X_i \ge 0,\ i = 1, 2, 3 & \} \text{ nonnegativity conditions}
\end{array}
$$

This model expresses the linking constraints in a slightly different (but algebraically equivalent) manner in order to follow our convention of having all the variables on the LHS of the inequality and a constant on the RHS. This model is implemented in the spreadsheet shown in Figure 6.28 (and in the file Fig6-28.xlsm that accompanies this book).

In the spreadsheet in Figure 6.28, cells B5, C5, and D5 represent the variables X_1, X_2, and X_3, and cells B15, C15, and D15 represent Y_1, Y_2, and Y_3. The coefficients for the objective function are in cells B7 through D8. The objective function is implemented in cell F8 with the formula:

Formula for cell F8: = SUMPRODUCT(B7:D7,B5:D5) –
SUMPRODUCT(B8:D8,B15:D15)

Cells B11 through D13 contain the coefficients for the machining, grinding, and assembly constraints. The LHS formulas for these constraints are implemented in cells E11 through E13,

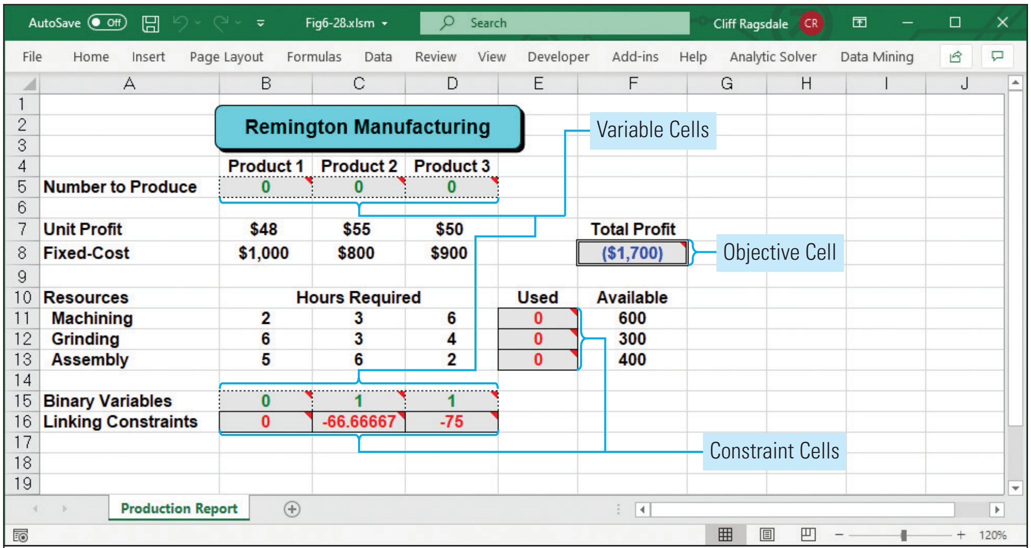


FIGURE 6.28

Spreadsheet model for Remington's fixed-charge problem

Key Cell Formulas

Cell	Formula	Copied to
B16	=B5-MIN($F$11/B11,$F$12/B12,$F$13/B13)*B15	C16:D16
E11	=SUMPRODUCT(B11:D11,$B$5:$D$5)	E12:E13
F8	=SUMPRODUCT(B7:D7,B5:D5) −SUMPRODUCT(B8:D8,B15:D15)	--

and cells F11 through F13 contain the RHS values for these constraints. Finally, the LHS formulas for the linking constraints are entered in cells B16 through D16 as:

Formula for cell B16: = B5-MIN($F$11/B11,$F$12/B12,$F$13/B13)*B15
(Copy to cells C16 through D16.)

Instead of entering the values for M_i in these constraints, we implemented formulas that would automatically calculate correct M_i values if the user of this spreadsheet changed any of the coefficients or RHS values in the capacity constraints.

6-14f SOLVING THE MODEL

The required Solver settings and options for this problem are shown in Figure 6.29. Notice that the ranges B5 through D5 and B15 through D15, which correspond to the X_i and Y_i variables, are both listed as ranges of cells that Solver can change. Also, notice the necessary binary constraint is imposed on cells B15 through D15.

Because so few integer variables exist in this problem, we should be able to obtain an optimal integer solution easily. If we set the Integer Tolerance to 0, we obtain the optimal solution to this problem shown in Figure 6.30.

6-14g ANALYZING THE SOLUTION

The solution shown in Figure 6.30 indicates that the company should produce 0 units of product 1, 56 units of product 2, and 32 units of product 3 ($X_1 = 0$, $X_2 = 56$, and $X_3 = 32$). Solver assigned values of 0, 1, and 1, respectively, to cells B15, C15, and D15 ($Y_1 = 0$, $Y_2 = 1$, and $Y_3 = 1$).

FIGURE 6.29 *Solver settings and options for Remington's fixed-charge problem*

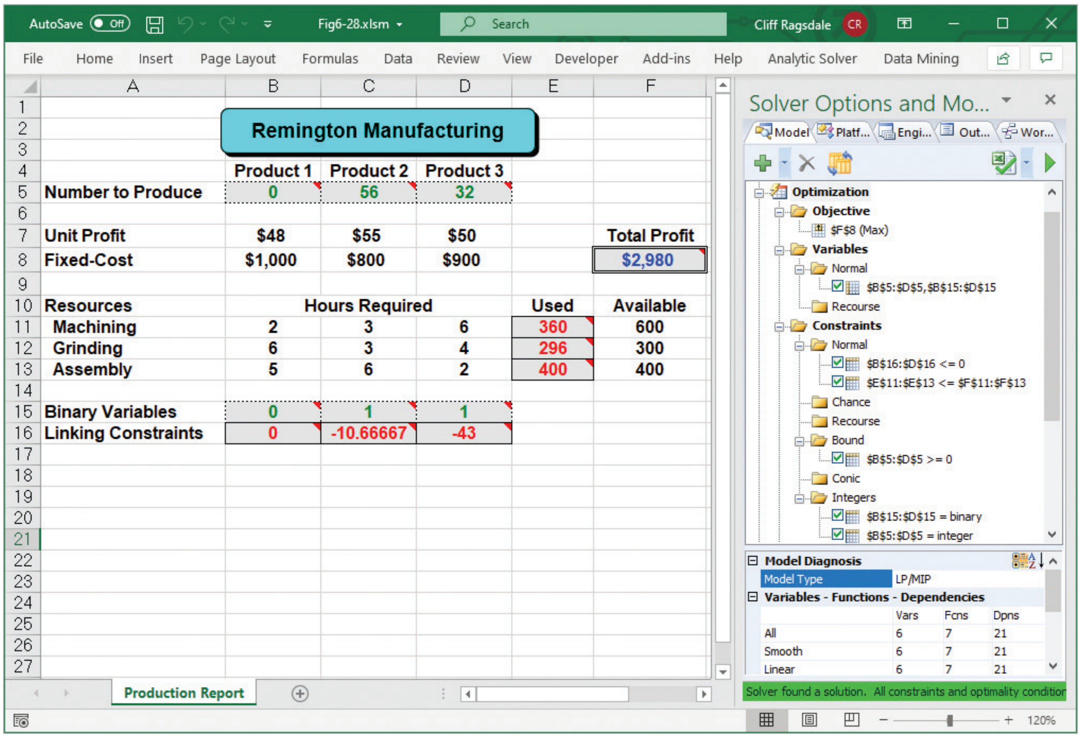
Solver Settings:

Objective: F8 (Max)
Variable cells: B5:D5, B15:D15
Constraints:
E11:E13 <= F11:F13
B16:D16 <= 0
B5:D5 >= 0
B5:D5 = integer
B15:D15 = binary

Solver Options:

Standard LP/Quadratic Engine (Simplex LP)
Integer Tolerance = 0

FIGURE 6.30 *Optimal integer solution to Remington's fixed-charge problem*



Thus, Solver maintained the proper relationship between the X_i and Y_i because the linking constraints were specified for this problem.

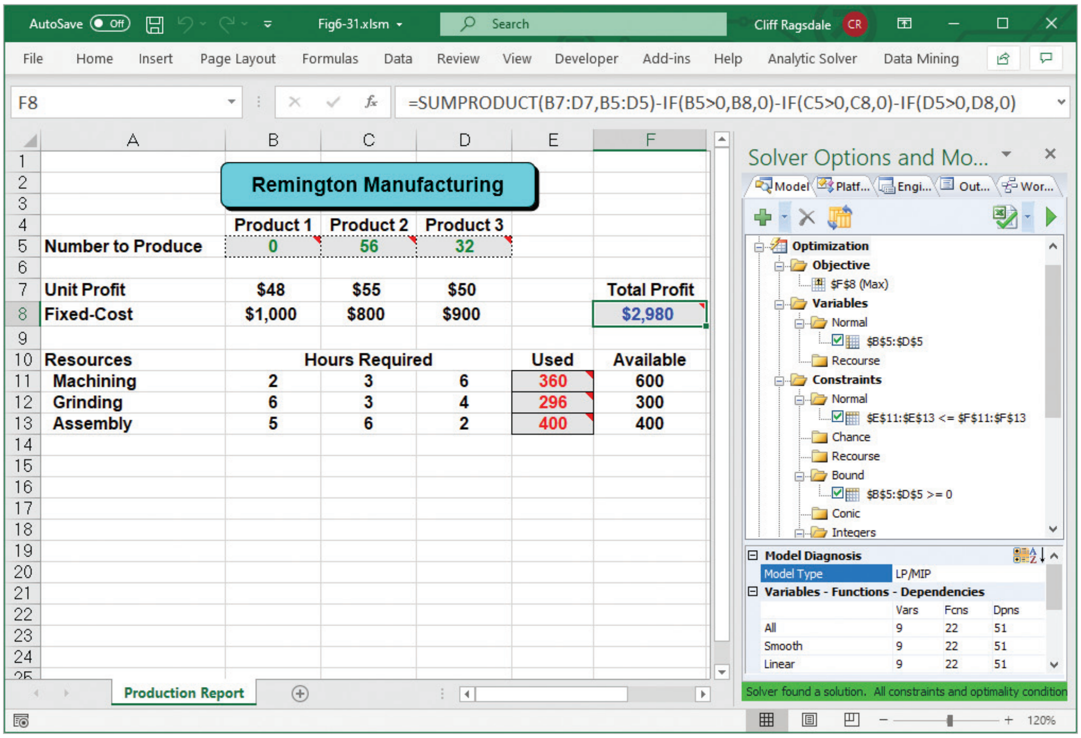
The values in cells B16, C16, and D16 indicate the amounts by which the values for X_1, X_2, and X_3 (in cells B5, C5, and D5) fall below the upper bounds imposed by their respective linking constraints. Thus, the optimal value of X_2 is approximately 10.67 units below its upper bound of 66.67 and the optimal value of X_3 is 43 units below its upper bound of 75. Because the optimal value of Y_1 is zero, the linking constraint for X_1 and Y_1 imposes an upper bound of 0 on X_1. Thus, the value in cell B16 indicates that the optimal value of X_1 is 0 units below its upper bound of 0.

6-14h A COMMENT ON IF( ) FUNCTIONS

In Figure 6.30 it is important to note that we are treating cells B15, C15, and D15, which represent the binary variables Y_1, Y_2, and Y_3, just like any other cells representing decision variables. We simply entered values of 0 into these cells to indicate that they represent decision variables. We then let Solver determine what values should be placed in these cells so that all the constraints are satisfied and the objective function is maximized. Some people try to make Solver's job (or their own life) "easier" by using an alternate approach with IF( ) functions in the objective to turn on or off the fixed costs depending on the values of cells B5, C5, and D5, which correspond to the variables X_1, X_2, and X_3. For example, consider the model in Figure 6.31 (and Fig6-31.xlsm in the files accompanying this book) where we have eliminated the binary variables and linking constraints and replaced them with IF( ) functions in the objective (cell F8) to model the fixed costs in this problem as follows:

Formula for cell F8: ==SUMPRODUCT(B7:D7,B5:D5)-IF(B5>0,B8,0)-
IF(C5>0,C8,0)-IF(D5>0,D8,0)

FIGURE 6.31 *An alternate implementation of Remington's fixed-charge problem with IF( ) functions replacing the binary variables and linking constraints*



Although this approach seems to make sense, it can produce unwanted results. Using IF( ) functions in this way introduces discontinuities in the spreadsheet model that makes it more difficult for Solver (particularly Excel's built-in Solver) to find the optimal solution. One of the rather amazing features of Analytic Solver is its ability to automatically transform a model containing certain types of IF( ) functions into an equivalent integer programming model without IF( ) functions.

When you solve this problem, the diagnostic information on the Output tab in the Analytic Solver task pane (not shown) indicates that this model is diagnosed as a nonsmooth problem and Solver automatically transforms it into a "LP Convex" problem. Note that this transformation is not made on your worksheet but, instead, refers to how Analytic Solver is handling the model internally. Now, although the solution shown in Figure 6.31 matches the optimal solution shown in Figure 6.30, note that the numbers of variables (Vars), functions (Fcns), and dependencies (Dpns) listed in the bottom of the Analytic Solver task pane in Figure 6.31 are significantly higher than those listed in Figure 6.30. That is, Analytic Solver's automatic transformation of the model in Figure 6.31 (with IF( ) functions) resulted in a problem with 9 variables, 22 functions, and 51 dependencies, whereas our original model in Figure 6.30 (using binary variables and linking constraints) has only 6 variables, 7 functions, and 21 dependencies. Thus, while the same solution was obtained using IF( ) functions, it required Analytic Solver to formulate and solve a significantly more complicated model. In this case, the added complexity was not an issue. However, it is easy to see how the complications caused by IF( ) functions could become problematic as problem size increases. Additionally, Analytic Solver cannot always successfully transform a model containing IF( ) functions. Thus, for a variety of reasons, it is best to avoid IF( ) functions when possible and not rely on Analytic Solver's ability to automatically transform some models containing them. (If desired, you can disable this type of automatic transformation in Analytic Solver by setting the "Nonsmooth Model Transformation" property to "Never" on the Platform tab in the task pane.)

6-15 Minimum Order/Purchase Size

Many investment, production, and distribution problems have minimum purchase amounts or minimum production lot size requirements that must be met. For example, a particular investment opportunity might require a minimum investment of $25,000. Or, a supplier of a given part used in a production process might require a minimum order of 10 units. Similarly, many manufacturing companies have a policy of not producing any units of a given item unless a certain minimum lot size will be produced.

To see how these types of minimum order/purchase requirements can be modeled, suppose that in the previous problem, Remington Manufacturing did not want to produce any units of product 3 (X_3) unless it produced at least 40 units of this product. This type of restriction is modeled as:

$$X_3 \leq M_3Y_3$$

$$X_3 \geq 40Y_3$$

The first constraint is the same type of linking constraint described earlier, in which M_3 represents an upper bound on X_3 (or an arbitrarily large number) and Y_3 represents a binary variable. If X_3 assumes any positive value, Y_3 must equal 1 (if $X_3 > 0$, then $Y_3 = 1$). However, according to the second constraint, if Y_3 equals 1, then X_3 must be greater than or equal to 40 (if $Y_3 = 1$, then $X_3 \geq 40$). On the other hand, if X_3 equals 0, Y_3 must also equal 0 in order to satisfy both constraints. Together, these two constraints ensure that if X_3 assumes any positive value, that value must be at least 40. This example illustrates how binary variables can be used to model a practical condition that is likely to occur in a variety of decision problems.

6-16 Quantity Discounts

In all the LP problems considered to this point, we have assumed that the profit or cost coefficients in the objective function were constant. For example, consider our revised Blue Ridge Hot Tubs problem, which is represented by:

$$
\begin{array}{llrcrl}
\text{MAX:} & 350X_1 + 300X_2 & & & & \} \text{ profit} \\
\text{Subject to:} & 1X_1 + 1X_2 & \leq & 200 & & \} \text{ pump constraint} \\
 & 9X_1 + 6X_2 & \leq & 1{,}520 & & \} \text{ labor constraint} \\
 & 12X_1 + 16X_2 & \leq & 2{,}650 & & \} \text{ tubing constraint} \\
 & X_1, X_2 & \geq & 0 & & \} \text{ nonnegativity conditions} \\
 & X_1, X_2 \text{ must be integers} & & & & \} \text{ integrality conditions}
\end{array}
$$

This model assumes that *every* additional Aqua-Spa (X_1) manufactured and sold results in a \$350 increase in profit. It also assumes that every additional Hydro-Lux (X_2) manufactured and sold results in a \$300 increase in profit. However, as the production of these products increases, quantity discounts might be obtained on component parts that would cause the profit margin on these items to increase.

For example, suppose that if the company produces more than 75 Aqua-Spas, it will be able to obtain quantity discounts and other economies of scale that would increase the profit margin to \$375 per unit for each unit produced in excess of 75. Similarly, suppose that if the company produces more than 50 Hydro-Luxes, it will be able to increase its profit margin to \$325 for each unit produced in excess of 50. That is, each of the first 75 units of X_1 and the first 50 units of X_2 would produce profits of \$350 and \$300 per unit, respectively, and each additional unit of X_1 and X_2 would produce profits of \$375 and \$325 per unit, respectively. How do we model this type of problem?

6-16a FORMULATING THE MODEL

To accommodate the different profit rates that can be generated by producing Aqua-Spas and Hydro-Luxes, we need to define new variables for the problem, where

X_{11} = the number of Aqua-Spas produced at \$350 profit per unit
X_{12} = the number of Aqua-Spas produced at \$375 profit per unit
X_{21} = the number of Hydro-Luxes produced at \$300 profit per unit
X_{22} = the number of Hydro-Luxes produced at \$325 profit per unit

Using these variables, we can begin to reformulate our problem as:

$$
\begin{array}{llrl}
\text{MAX:} & 350X_{11} + 375X_{12} + 300X_{21} + 325X_{22} & & \\
\text{Subject to:} & 1X_{11} + 1X_{12} + 1X_{21} + 1X_{22} \leq & 200 & \} \text{ pump constraint} \\
 & 9X_{11} + 9X_{12} + 6X_{21} + 6X_{22} \leq & 1{,}520 & \} \text{ labor constraint} \\
 & 12X_{11} + 12X_{12} + 16X_{21} + 16X_{22} \leq & 2{,}650 & \} \text{ tubing constraint} \\
 & \text{All } X_{ij} \geq 0 & & \} \text{ simple lower bounds} \\
 & \text{All } X_{ij} \text{ must be integers} & & \} \text{ integrality conditions}
\end{array}
$$

This formulation is not complete. Notice that the variable X_{12} would always be preferred over X_{11} because X_{12} requires exactly the same resources as X_{11} and generates a larger per-unit profit. The same relationship holds between X_{22} and X_{21}. Thus, the optimal solution to the problem is

$X_{11} = 0$, $X_{12} = 118$, $X_{21} = 0$, and $X_{22} = 76$. However, this solution is not allowable because we cannot produce any units of X_{12} until we have produced 75 units of X_{11}; and we cannot produce any units of X_{22} until we have produced 50 units of X_{21}. Therefore, we must identify some additional constraints to ensure that these conditions are met.

6-16b THE MISSING CONSTRAINTS

To ensure that the model does not allow any units of X_{12} to be produced unless we have produced 75 units of X_{11}, consider the constraints:

$$X_{12} \leq M_{12}Y_1$$
$$X_{11} \geq 75Y_1$$

In the first constraint, M_{12} represents some arbitrarily large numeric constant and Y_1 represents a binary variable. The first constraint requires that $Y_1 = 1$ if any units of X_{12} are produced (if $X_{12} > 0$, then $Y_1 = 1$). However, if $Y_1 = 1$, then the second constraint would require X_{11} to be at least 75. According to the second constraint, the only way that fewer than 75 units of X_{11} can be produced is if $Y_1 = 0$, which, by the first constraint, implies $X_{12} = 0$. These two constraints do not allow any units of X_{12} to be produced unless at least 75 units of X_{11} have been produced. The following constraints ensure that the model does not allow any units of X_{22} to be produced unless we have produced 50 units of X_{21}:

$$X_{22} \leq M_{22}Y_2$$
$$X_{21} \geq 50Y_2$$

If we include these new constraints in our previous formulation (along with the constraints necessary to make Y_1 and Y_2 operate as binary variables), we would have an accurate formulation of the decision problem. The optimal solution to this problem is $X_{11} = 75$, $X_{12} = 43$, $X_{21} = 50$, $X_{22} = 26$.

6-17 A Contract Award Problem

Other conditions often arise in decision problems that can be modeled effectively using binary variables. The following example, which involves awarding contracts, illustrates some of these conditions.

B&G Construction is a commercial building company located in Tampa, Florida. The company has recently signed contracts to construct four buildings in different locations in southern Florida. Each building project requires large amounts of cement to be delivered to the building sites. At B&G's request, three cement companies have submitted bids for supplying the cement for these jobs. The following table summarizes the prices the three companies charge per delivered ton of cement and the maximum amount of cement that each company can provide.

	Cost per Delivered Ton of Cement				
	Project 1	**Project 2**	**Project 3**	**Project 4**	**Max. Supply**
Company 1	$120	$115	$130	$125	525
Company 2	$100	$150	$110	$105	450
Company 3	$140	$ 95	$145	$165	550
Total Tons Needed	450	275	300	350	

For example, company 1 can supply a maximum of 525 tons of cement, and each ton delivered to projects 1, 2, 3, and 4 will cost \$120, \$115, \$130, and \$125, respectively. The costs vary primarily because of the different distances between the cement plants and the construction sites. The numbers in the last row of the table indicate the total amount of cement (in tons) required for each project.

In addition to the maximum supplies listed, each cement company placed special conditions on its bid. Specifically, company 1 indicated that it will not supply orders of less than 150 tons for any of the construction projects. Company 2 indicated that it can supply more than 200 tons to no more than one of the projects. Company 3 indicated that it will accept only orders that total 200 tons, 400 tons, or 550 tons.

B&G can contract with more than one supplier to meet the cement requirements for a given project. The problem is to determine what amounts to purchase from each supplier to meet the demands for each project at the least total cost.

This problem seems like a transportation problem in which we want to determine how much cement should be shipped from each cement company to each construction project in order to meet the demands of the projects at a minimum cost. However, the special conditions imposed by each supplier require side constraints, which are not usually found in a standard transportation problem. First, we'll discuss the formulation of the objective function and the transportation constraints. Then, we'll consider how to implement the side constraints required by the special conditions in the problem.

6-17a FORMULATING THE MODEL: THE OBJECTIVE FUNCTION AND TRANSPORTATION CONSTRAINTS

To begin formulating this problem, we need to define our decision variables as:

$$X_{ij} = \text{tons of cement purchased from company } i \text{ for construction project } j$$

The objective function to minimize total cost is represented by:

$$\begin{aligned} \text{MIN:} \quad & 120X_{11} + 115X_{12} + 130X_{13} + 125X_{14} \\ & + 100X_{21} + 150X_{22} + 110X_{23} + 105X_{24} \\ & + 140X_{31} + 95X_{32} + 145X_{33} + 165X_{34} \end{aligned}$$

To ensure that the maximum supply of cement from each company is not exceeded, we need the following constraints:

$$\begin{aligned} & X_{11} + X_{12} + X_{13} + X_{14} \leq 525 && \} \text{ supply from company 1} \\ & X_{21} + X_{22} + X_{23} + X_{24} \leq 450 && \} \text{ supply from company 2} \\ & X_{31} + X_{32} + X_{33} + X_{34} \leq 550 && \} \text{ supply from company 3} \end{aligned}$$

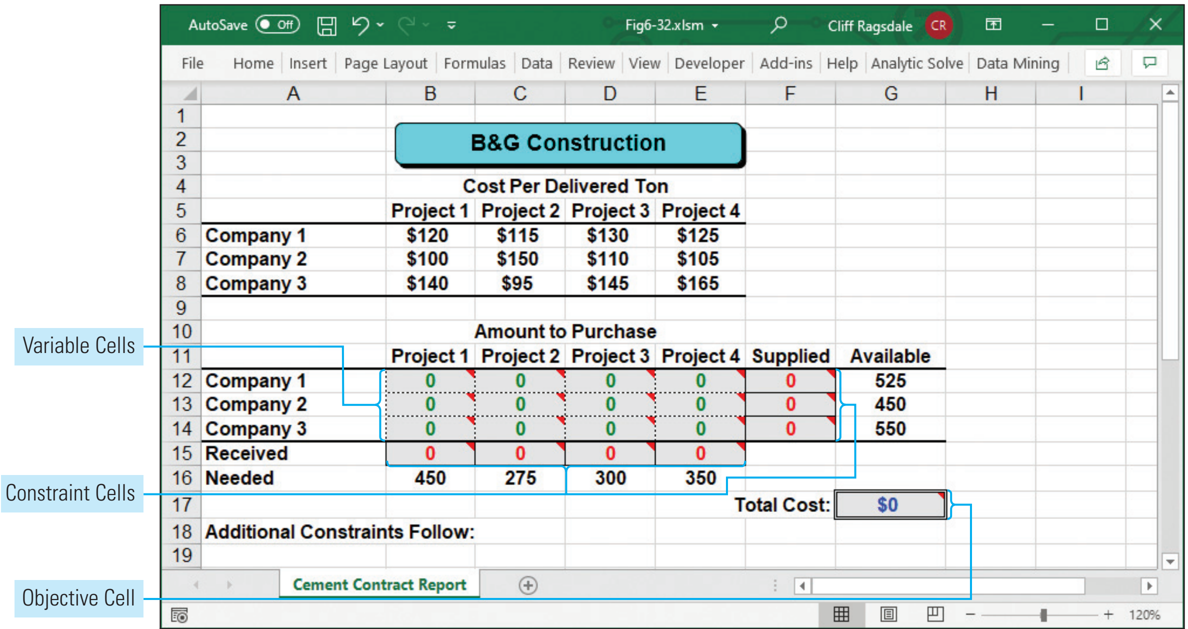
To ensure that the requirement for cement at each construction project is met, we need the following constraints:

$$\begin{aligned} & X_{11} + X_{21} + X_{31} = 450 && \} \text{ demand for cement at project 1} \\ & X_{12} + X_{22} + X_{32} = 275 && \} \text{ demand for cement at project 2} \\ & X_{13} + X_{23} + X_{33} = 300 && \} \text{ demand for cement at project 3} \\ & X_{14} + X_{24} + X_{34} = 350 && \} \text{ demand for cement at project 4} \end{aligned}$$

6-17b IMPLEMENTING THE TRANSPORTATION CONSTRAINTS

The objective function and the constraints of this problem are implemented in the spreadsheet model shown in Figure 6.32 (and in the file Fig6-32.xlsm that accompanies this book).

FIGURE 6.32 *Spreadsheet model for the transportation portion of B&G's contract award problem*



Key Cell Formulas

Cell	Formula	Copied to
B15	=SUM(B12:B14)	C15:E15
F12	=SUM(B12:E12)	F13:F14
G17	=SUMPRODUCT(B6:E8,B12:E14)	--

In this spreadsheet, the costs per delivered ton of cement are shown in cells B6 through E8. Cells B12 through E14 represent the decision variables in the model. The objective function is entered in cell G17 as:

Formula for cell G17: =SUMPRODUCT(B6:E8,B12:E14)

The LHS formulas of the supply constraints are entered in cells F12 through F14 as:

Formula for cell F12: =SUM(B12:E12)
(Copy to F13 through F14.)

Cells G12 through G14 contain the RHS values for these constraints. The LHS formulas for the demand constraints are entered in cells B15 through E15 as:

Formula for cell B15: =SUM(B12:B14)
(Copy to C15 through E15.)

Cells B16 through E16 contain the RHS values for these constraints.

6-17c FORMULATING THE MODEL: THE SIDE CONSTRAINTS

Company 1 indicated that it will not accept orders for less than 150 tons for any of the construction projects. This minimum-size order restriction is modeled by the following eight constraints, where the Y_{ij} represent binary variables:

$$
\begin{array}{ll}
X_{11} \leq 525Y_{11} & \text{(implement as } X_{11} - 525Y_{11} \leq 0) \\
X_{12} \leq 525Y_{12} & \text{(implement as } X_{12} - 525Y_{12} \leq 0) \\
X_{13} \leq 525Y_{13} & \text{(implement as } X_{13} - 525Y_{13} \leq 0) \\
X_{14} \leq 525Y_{14} & \text{(implement as } X_{14} - 525Y_{14} \leq 0) \\
X_{11} \geq 150Y_{11} & \text{(implement as } X_{11} - 150Y_{11} \geq 0) \\
X_{12} \geq 150Y_{12} & \text{(implement as } X_{12} - 150Y_{12} \geq 0) \\
X_{13} \geq 150Y_{13} & \text{(implement as } X_{13} - 150Y_{13} \geq 0) \\
X_{14} \geq 150Y_{14} & \text{(implement as } X_{14} - 150Y_{14} \geq 0)
\end{array}
$$

Each constraint has an algebraically equivalent constraint, which will ultimately be used in implementing the constraint in the spreadsheet. The first four constraints represent linking constraints that ensure if X_{11}, X_{12}, X_{13}, or X_{14} is greater than 0, then its associated binary variable (Y_{11}, Y_{12}, Y_{13}, or Y_{14}) must equal 1. (These constraints also indicate that 525 is the maximum value that can be assumed by X_{11}, X_{12}, X_{13}, and X_{14}.) The next four constraints ensure that if X_{11}, X_{12}, X_{13}, or X_{14} is greater than 0, it must be at least 150. We include these constraints in the formulation of this model to ensure that any order given to company 1 is for at least 150 tons of cement.

Company 2 indicated that it can supply more than 200 tons to no more than one of the projects. This type of restriction is represented by the following set of constraints where, again, the Y_{ij} represent binary variables:

$$
\begin{array}{ll}
X_{21} \leq 200 + 250Y_{21} & \text{(implement as } X_{21} - 200 - 250Y_{21} \leq 0) \\
X_{22} \leq 200 + 250Y_{22} & \text{(implement as } X_{22} - 200 - 250Y_{22} \leq 0) \\
X_{23} \leq 200 + 250Y_{23} & \text{(implement as } X_{23} - 200 - 250Y_{23} \leq 0) \\
X_{24} \leq 200 + 250Y_{24} & \text{(implement as } X_{24} - 200 - 250Y_{24} \leq 0) \\
Y_{21} + Y_{22} + Y_{23} + Y_{24} \leq 1 & \text{(implement as is)}
\end{array}
$$

The first constraint indicates that the amount supplied from company 2 for project 1 must be less than 200 if $Y_{21} = 0$, or less than 450 (the maximum supply from company 2) if $Y_{21} = 1$. The next three constraints have similar interpretations for the amount supplied from company 2 to projects 2, 3, and 4, respectively. The last constraint indicates that at most one of Y_{21}, Y_{22}, Y_{23}, and Y_{24} can equal 1. Therefore, only one of the projects can receive more than 200 tons of cement from company 2.

The final set of constraints for this problem addresses company 3's stipulation that it will accept only orders totaling 200, 400, or 550 tons. This type of condition is modeled using binary Y_{ij} variables as:

$$X_{31} + X_{32} + X_{33} + X_{34} = 200Y_{31} + 400Y_{32} + 550Y_{33}$$
$$\text{(implement as } X_{31} + X_{32} + X_{33} + X_{34} - 200Y_{31} - 400Y_{32} - 550Y_{33} = 0)$$
$$Y_{31} + Y_{32} + Y_{33} \leq 1 \text{ (implement as is)}$$

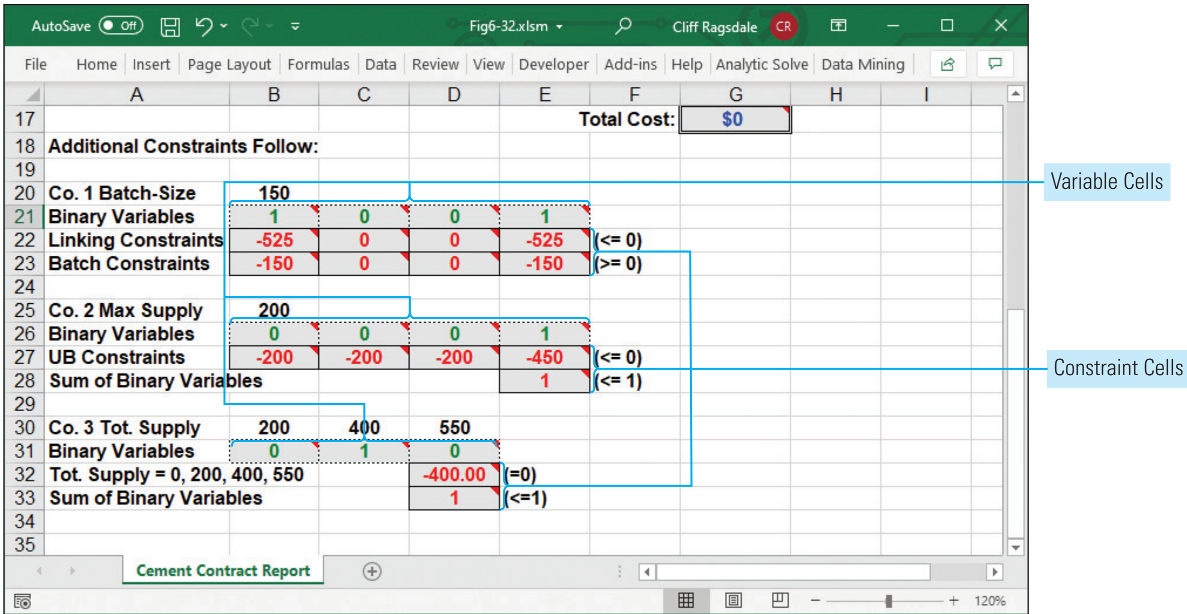
These constraints allow for the total amount ordered from company 3 to assume four distinct values. If $Y_{31} = Y_{32} = Y_{33} = 0$, then no cement will be ordered from company 3. If $Y_{31} = 1$, then 200 tons must be ordered. If $Y_{32} = 1$, then 400 tons must be ordered. Finally, if $Y_{33} = 1$, then

550 tons must be ordered from company 3. These two constraints enforce the special condition imposed by company 3.

6-17d IMPLEMENTING THE SIDE CONSTRAINTS

Although the side constraints in this problem allow us to impose important restrictions on the feasible solutions that can be considered, these constraints serve more of a "mechanical" purpose—to make the model work—but are not of primary interest to management. Thus, it is often convenient to implement side constraints in an out-of-the-way area of the spreadsheet so that they do not detract from the primary purpose of the spreadsheet, in this case, to determine how much cement to order from each potential supplier. Figure 6.33 shows how the side constraints for the current problem can be implemented in a spreadsheet.

FIGURE 6.33 *Spreadsheet model for the side constraints in B&G's contract award problem*



Key Cell Formulas

Cell	Formula	Copied to
B22	=B12−$G$12*B21	C22:E22
B23	=B12−$B$20*B21	C23:E23
B27	=B13−$B$25−($G$13−$B$25)*B26	C27:E27
E28	=SUM(B26:E26)	--
D32	=SUM(B14:E14)−SUMPRODUCT(B30:D30,B31:D31)	--
D33	=SUM(B31:D31)	--

To implement the side constraints for company 1, we enter the batch-size restriction of 150 in cell B20 and reserve cells B21 through E21 to represent the binary variables Y_{11}, Y_{12}, Y_{13}, and Y_{14}. The LHS formulas for the linking constraints for company 1 are implemented in cells B22 through E22 as:

Formula for cell B22: =B12−$G$12*B21
(Copy to C22 through E22.)

Cell F22 contains a reminder for us to tell Solver that these cells must be less than or equal to 0. The LHS formulas for the batch-size constraints for company 1 are implemented in cells B23 through E23 as:

Formula for cell B23: =B12−$B$20*B21
(Copy to C23 through E23.)

Cell F23 contains a reminder for us to tell Solver that these cells must be greater than or equal to 0. To implement the side constraints for company 2, we enter the maximum supply value of 200 in cell B25 and reserve cells B26 through E26 to represent the binary variables Y_{21}, Y_{22}, Y_{23}, and Y_{24}. The LHS formulas for the maximum supply constraints are implemented in cells B27 through E27 as:

Formula for cell B27: =B13−$B$25−($G$13−$B$25)*B26
(Copy to C27 through E27.)

Cell F27 reminds us to tell Solver that these cells must be less than or equal to 0. As discussed earlier, to ensure that no more than one order from company 2 exceeds 200 tons, the sum of the binary variables for company 2 cannot exceed 1. The LHS formula for this constraint is entered in cell E28 as:

Formula for cell E28: =SUM(B26:E26)

Cell F28 reminds us to tell Solver that this cell must be less than or equal to 1.
To implement the side constraints for company 3, the three possible total order amounts are entered in cells B30 through D30. Cells B31 through D31 are reserved to represent the binary variables Y_{31}, Y_{32}, and Y_{33}. The LHS formula for company 3's total supply side constraint is entered in cell D32 as:

Formula for cell D32: =SUM(B14:E14)−SUMPRODUCT(B30:D30,B31:D31)

Cell E32 reminds us to tell Solver that cell D32 must equal 0. Finally, to ensure that no more than one of the binary variables for company 3 is set equal to 1, we enter the sum of these variables in cell D33 as:

Formula for cell D33: =SUM(B31:D31)

Cell E33 reminds us to tell Solver that this cell must be less than or equal to 1.

6-17e SOLVING THE MODEL

The Solver parameters required for this problem are shown in Figure 6.34. Note that all of the cells representing binary variables must be identified as variable cells and must be constrained to assume only integer values of 0 or 1.

6-17f ANALYZING THE SOLUTION

An optimal solution to this problem is shown in Figure 6.35 (there are alternate optimal solutions to this problem). In this solution, the amounts of cement required by each construction project are met exactly. Also, each condition imposed by the side constraints for each company is met. Specifically, the orders awarded to company 1 are for at least 150 tons; a maximum of one of the orders awarded to company 2 exceeds 200 tons; and the sum of the orders awarded to company 3 is exactly equal to 400 tons.

FIGURE 6.34

Solver settings and options for B&G's contract award problem

Solver Settings:

Objective: G17 (Min)
Variable cells: B12:E14, B21:E21, B26:E26, B31:D31
Constraints:

- $B12:E14 \geq 0$
- B21:E21 = binary
- B26:E26 = binary
- B31:D31 = binary
- $F12:F14 \leq G12:G14$
- $B15:E15 = B16:E16$
- $B22:E22 \leq 0$
- $B23:E23 \geq 0$
- $B27:E27 \leq 0$
- $E28 \leq 1$
- $D32 = 0$
- $D33 \leq 1$

Solver Options:

- Standard LP/Quadratic Engine (Simplex LP)
- Integer Tolerance = 0

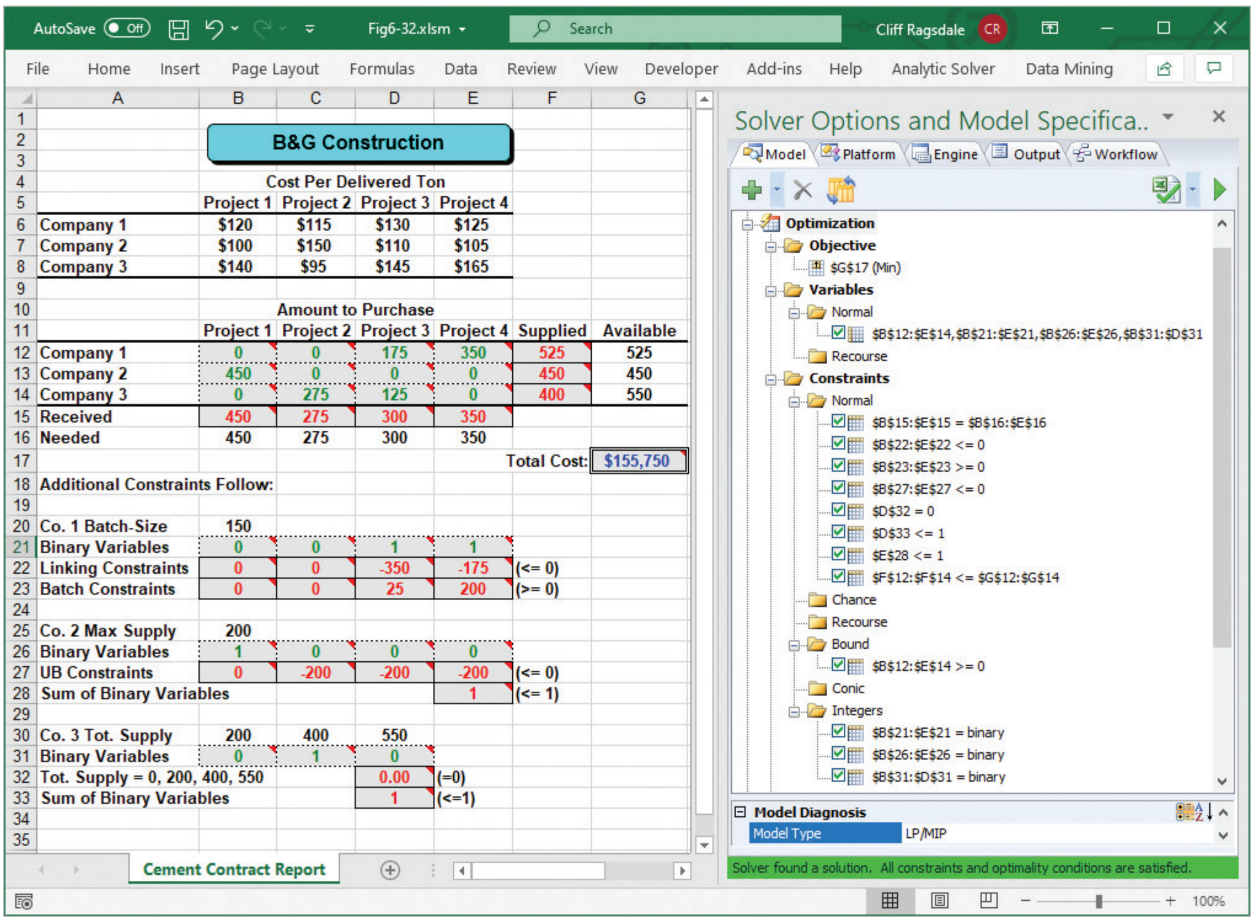


FIGURE 6.35 *Optimal solution to B&G's contract award problem*

6-18 The Branch-and-Bound Algorithm (Optional)

As mentioned earlier, a special procedure, known as the branch-and-bound (B&B) algorithm, is required to solve ILPs. Although we can easily indicate the presence of integer variables in a model, it usually requires quite a bit of effort on Solver's part to actually solve an ILP problem using the B&B algorithm. To better appreciate and understand what is involved in the B&B algorithm, let's consider how it works.

The B&B algorithm starts by relaxing all the integrality conditions in an ILP and solving the resulting LP problem. As noted earlier, if we are lucky, the optimal solution to the relaxed LP problem might happen to satisfy the original integrality conditions. If this occurs, then we are done—the optimal solution to the LP relaxation is also the optimal solution to the ILP. However, it is more likely that the optimal solution to the LP will violate one or more of the original integrality conditions. For example, consider the problem whose integer and relaxed feasible regions were shown in Figure 6.1 and are repeated in Figure 6.36:

$$
\begin{aligned}
\text{MAX:} \quad & 2X_1 + 3X_2 \\
\text{Subject to:} \quad & X_1 + 3X_2 \le 8.25 \\
& 2.5X_1 + X_2 \le 8.75 \\
& X_1, X_2 \ge 0 \\
& X_1, X_2 \text{ must be integers}
\end{aligned}
$$

If we relax the integrality conditions in this problem and solve the resulting LP problem, we obtain the solution $X_1 = 2.769$, $X_2 = 1.826$ shown in Figure 6.36. This solution clearly violates the integrality conditions stated in the original problem. Part of the difficulty here is that none of the corner points of the relaxed feasible region are integer feasible (other than the origin). We know that the optimal solution to an LP problem will occur at a corner point of its feasible region but, in this case, none of those corner points (except the origin) correspond to integer solutions. Thus, we need to modify the problem so that the integer feasible solutions to the problem occur at corner points of the relaxed feasible region. This is accomplished by branching.

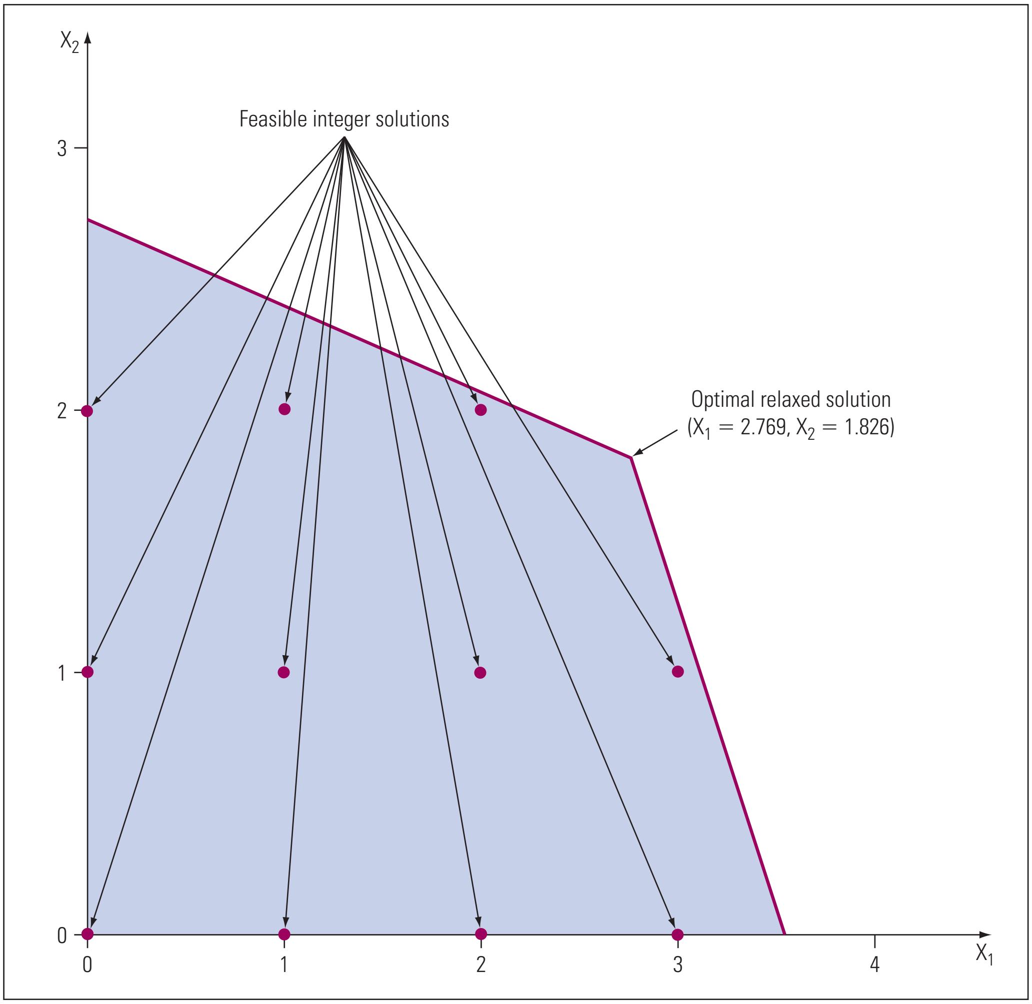
6-18a BRANCHING

Any integer variable in an ILP that assumes a fractional value in the optimal solution to the relaxed problem can be designated as a **branching variable**. For example, the variables X_1 and X_2 in the previous problem should assume only integer values but were assigned the values $X_1 = 2.769$ and $X_2 = 1.826$ in the optimal solution to the LP relaxation of the problem. Either of these variables could be selected as branching variables.

Let's arbitrarily choose X_1 as our branching variable. Because the current value of X_1 is not integer feasible, we want to eliminate this solution from further consideration. Many other solutions in this same vicinity of the relaxed feasible region can be eliminated as well. That is, X_1 must assume a value less than or equal to 2 ($X_1 \le 2$) or greater than or equal to 3 ($X_1 \ge 3$) in the optimal integer solution to the ILP. Therefore, all other possible solutions where X_1 assumes values between 2 and 3 (such as the current solution where $X_1 = 2.769$) can be eliminated

FIGURE 6.36

Solution to LP relaxation at noninteger corner point



from consideration. By branching on X_1, our original ILP problem can be subdivided into the following two candidate problems:

Problem I: MAX: $2X_1 + 3X_2$

Subject to:

$$X_1 + 3X_2 \le 8.25$$
$$2.5X_1 + X_2 \le 8.75$$
$$X_1 \le 2$$
$$X_1, X_2 \ge 0$$

X_1, X_2 must be integers

Problem II: MAX: $2X_1 + 3X_2$

Subject to:

$$X_1 + 3X_2 \le 8.25$$
$$2.5X_1 + X_2 \le 8.75$$
$$X_1 \ge 3$$
$$X_1, X_2 \ge 0$$

X_1, X_2 must be integers

The integer and relaxed feasible regions for each candidate problem are shown in Figure 6.37. Notice that a portion of the relaxed feasible region shown in Figure 6.36 has been eliminated in Figure 6.37, but none of the feasible integer solutions shown in Figure 6.36 have been eliminated. This is a general property of the branching operation in the B&B algorithm. Also notice that several feasible integer solutions now occur on the boundary lines of the feasible regions shown in Figure 6.37. More importantly, one of these feasible integer solutions occurs at an extreme point of the relaxed feasible region for problem I (at the point $X_1 = 2, X_2 = 0$). If we relax the integrality conditions in problem I and solve the resulting LP, we could obtain an integer solution because one of the corner points of the relaxed feasible region corresponds to such a point. (However, this integer feasible extreme point still might not be the optimal solution to the relaxed LP problem.)

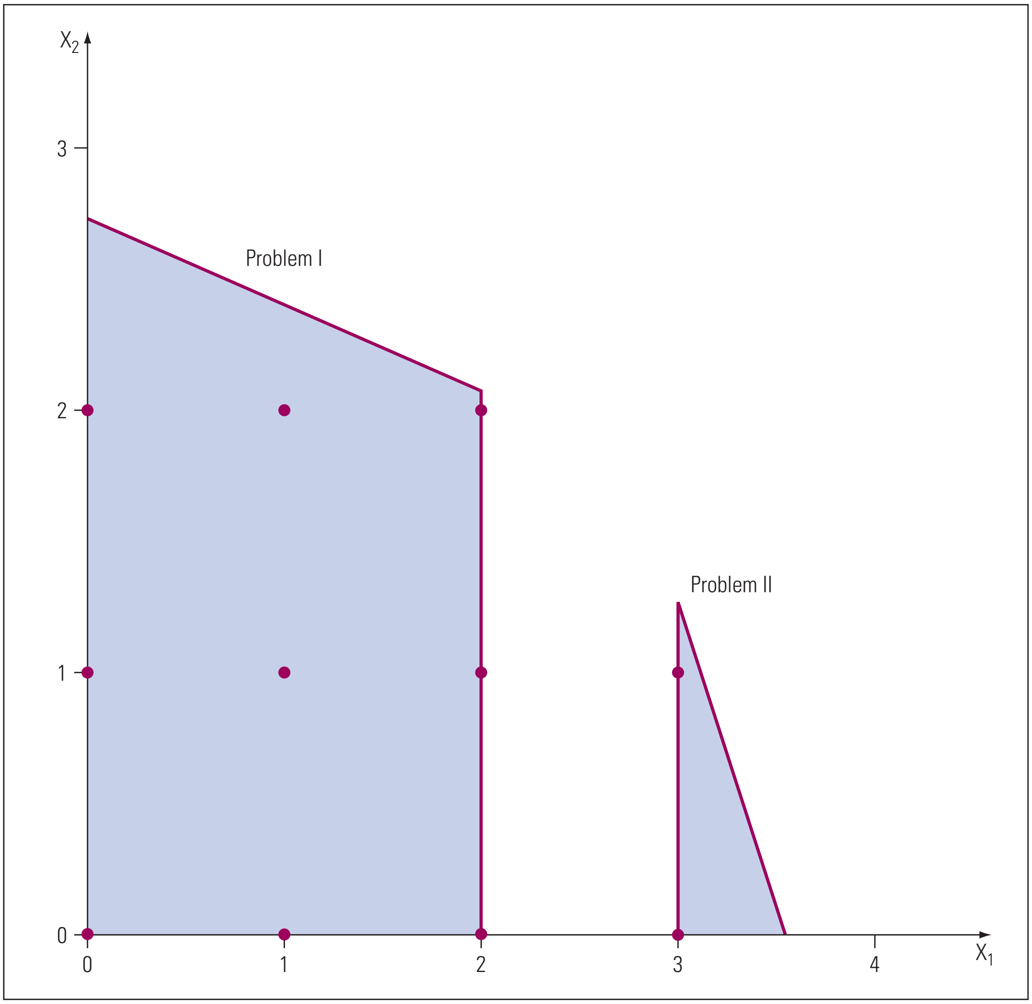


FIGURE 6.37

Feasible solutions to the candidate problems after the first branch

6-18b BOUNDING

The next step in the B&B algorithm is to select one of the existing candidate problems for further analysis. Let's arbitrarily select problem I. If we relax the integrality conditions in problem I and solve the resulting LP, we obtain the solution $X_1 = 2, X_2 = 2.083$ and an objective function value of 10.25. This value represents an upper bound on the best possible integer solution that can be obtained from problem I. That is, because the relaxed solution to problem I is not integer feasible, we have not yet found the best possible integer solution for this problem. However, we do know that the objective function value of the best possible integer solution that can be obtained from problem I can be no greater than 10.25. As you will see, this information can be useful in reducing the amount of work required to locate the optimal integer solution to an ILP problem.

6-18c BRANCHING AGAIN

Because the relaxed solution to problem I is not entirely integer feasible, the B&B algorithm proceeds by selecting X_2 as a branching variable and creating two additional candidate problems from problem I. These problems are represented as:

$$\begin{array}{lll} \text{Problem III:} & \text{MAX:} & 2X_1 + 3X_2 \\ & \text{Subject to:} & X_1 + 3X_2 \leq 8.25 \\ & & 2.5X_1 + X_2 \leq 8.75 \\ & & X_1 \leq 2 \\ & & X_2 \leq 2 \\ & & X_1, X_2 \geq 0 \\ & & X_1, X_2 \text{ must be integers} \end{array}$$

$$\begin{array}{lll} \text{Problem IV:} & \text{MAX:} & 2X_1 + 3X_2 \\ & \text{Subject to:} & X_1 + 3X_2 \leq 8.25 \\ & & 2.5X_1 + X_2 \leq 8.75 \\ & & X_2 \leq 2 \\ & & X_1 \geq 3 \\ & & X_1, X_2 \geq 0 \\ & & X_1, X_2 \text{ must be integers} \end{array}$$

Problem III is created by adding the constraint $X_2 \leq 2$ to problem I. Problem IV is created by adding the constraint $X_2 \geq 3$ to problem I. Thus, our previous solution to problem I (where $X_2 = 2.083$) will be eliminated from consideration as a possible solution to the LP relaxations of problems III and IV.

Problem IV is infeasible because there are no feasible solutions where $X_2 \geq 3$. The integer and relaxed feasible regions for problems II and III are summarized in Figure 6.38.

All of the corner points to the relaxed feasible region of problem III correspond to integer feasible solutions. Thus, if we relax the integrality conditions in problem III and solve the resulting LP problem, we must obtain an integer feasible solution. The solution to problem III is represented by $X_1 = 2$, $X_2 = 2$ and has an objective function value of 10.

6-18d BOUNDING AGAIN

Although we have obtained an integer feasible solution to our problem, we won't know if it is the *optimal* integer solution until we evaluate the remaining candidate problem (i.e., problem II). If we relax the integrality conditions in problem II and solve the resulting LP problem, we obtain the solution $X_1 = 3$, $X_2 = 1.25$ with an objective function value of 9.75.

Because the solution to problem II is not integer feasible, we might be inclined to branch on X_2 in a further attempt to determine the best possible integer solution for problem II. However, this is not necessary. Earlier we noted that for *maximization* ILP problems, the objective function value at the optimal solution to the LP relaxation of the problem represents an *upper bound* on the optimal objective function value of the original ILP problem. This means that even though we do not yet know the optimal integer solution to problem II, we do know that its objective function value cannot be greater than 9.75. And because 9.75 is worse than the objective function value for the integer solution obtained from problem III, we cannot find a better integer solution by continuing

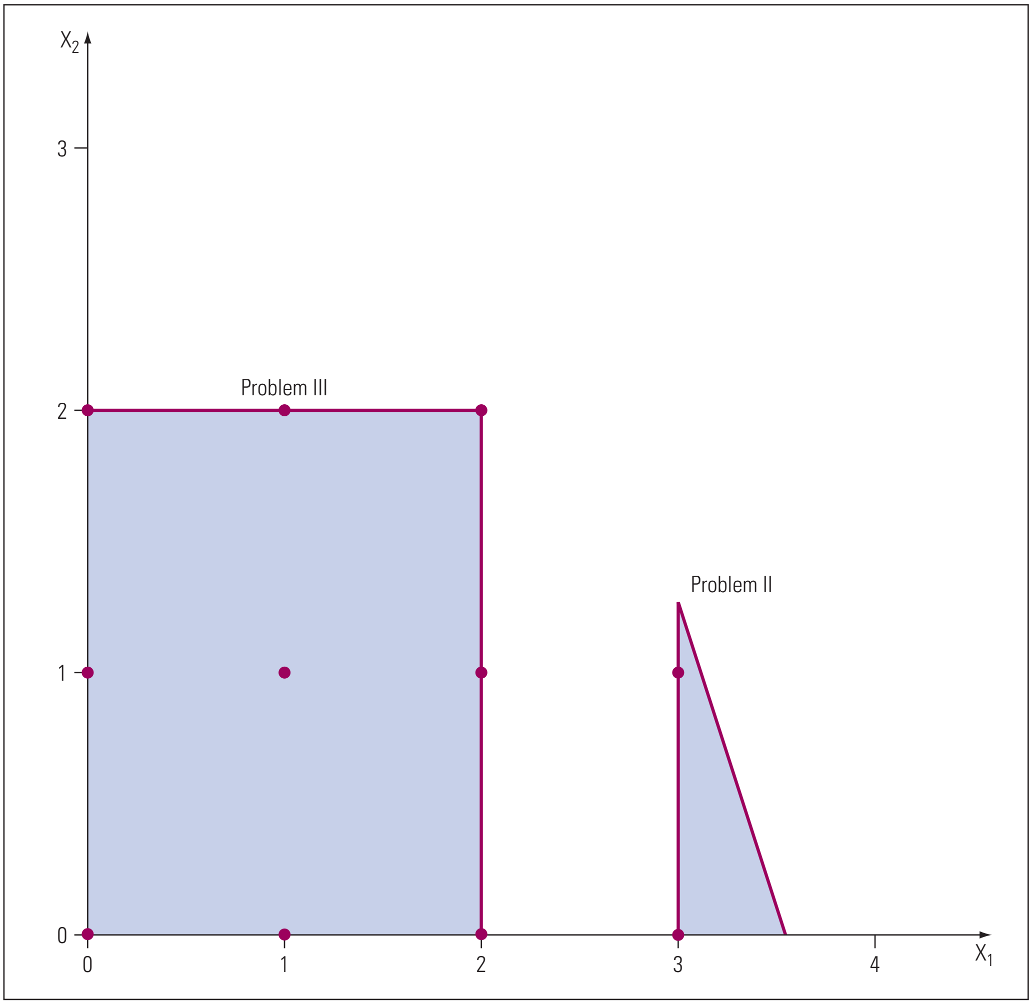


FIGURE 6.38

Feasible solutions to the candidate problems after the second branch

to branch problem II. Therefore, problem II can be eliminated from further consideration. Because we have no more candidate problems to consider, we can conclude that the optimal integer solution to our problem is $X_1 = 2$, $X_2 = 2$ with an optimal objective function value of 10.

6-18e SUMMARY OF B&B EXAMPLE

The steps involved in the solution to our example problem can be represented graphically in the form of a *branch-and-bound tree*, as shown in Figure 6.39. Although Figure 6.36 indicates that 11 integer solutions exist for this problem, we do not have to locate all of them in order to prove that the integer solution we found is the optimal solution. The bounding operation of the B&B algorithm eliminated the need to explicitly enumerate all the integer feasible solutions and select the best of those as the optimal solution.

If the relaxed solution to problem II was greater than 10 (say 12.5), then the B&B algorithm would have continued branching from this problem in an attempt to find a better integer solution (an integer solution with an objective function value greater than 10). Similarly, if problem IV had a feasible noninteger solution, we would have needed to perform further branching from that problem if its relaxed objective value was better than that of the best known integer feasible solution. Thus, the first integer solution obtained using B&B will not always be the optimal integer solution. A more detailed description of the operations of the B&B algorithm is given in Figure 6.40.

FIGURE 6.39

Branch-and-bound tree for the example problem

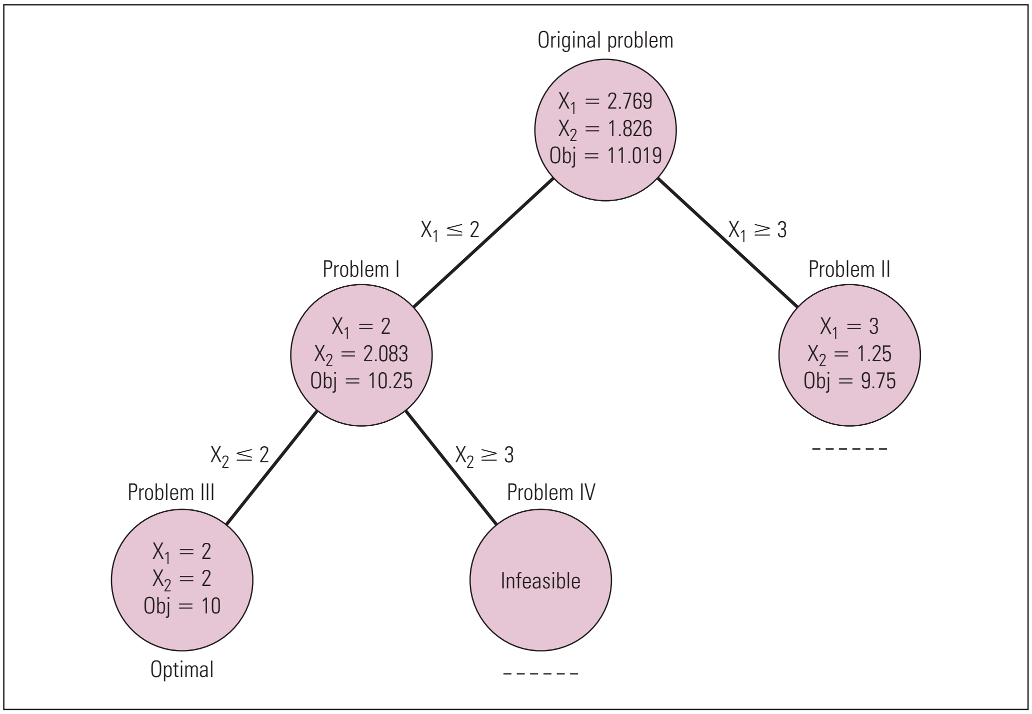


FIGURE 6.40

Detailed description of the B&B algorithm for solving ILP problems

THE BRANCH-AND-BOUND ALGORITHM

1. Relax all the integrality conditions in ILP and solve the resulting LP problem. If the optimal solution to the relaxed LP problem happens to satisfy the original integrality conditions, stop—this is the optimal integer solution. Otherwise, proceed to step 2.
2. If the problem being solved is a maximization problem let $Z_{best} = -$infinity. If it is a minimization problem, let $Z_{best} = +$infinity. (In general Z_{best} represents the objective function value of the best known integer solution as the algorithm proceeds.)
3. Let X_j represent one of the variables that violated the integrality conditions in the solution to the problem that was solved most recently and let b_j represent its noninteger value. Let $\text{INT}(b_j)$ represent the largest integer that is less than b_j. Create two new candidate problems: one by appending the constraint $X_j \le \text{INT}(b_j)$ to the most recently solved LP problem, and the other by appending the constraint $X_j \ge \text{INT}(b_j) + 1$ to the most recently solved LP problem. Place both of these new LP problems in a list of candidate problems to be solved.
4. If the list of candidate problems is empty, proceed to step 9. Otherwise, remove a candidate problem from the list, relax any integrality conditions in the problem, and solve it.
5. If there is not solution to the current candidate problem (i.e., it is infeasible), proceed to step 4. Otherwise, let Z_{cp} denote the optimal objective function value for the current candidate problem.
6. If Z_{cp} is not better than Z_{best} (for a maximization problem $Z_{cp} \le Z_{best}$ or for a minimization problem $Z_{cp} \ge Z_{best}$), proceed to step 4.
7. If the solution to the current candidate problem *does not* satisfy the original integrality conditions, proceed to step 3.
8. If the solution to the current candidate problem *does* satisfy the original integrality conditions, a better integer solution has been found. Thus, let $Z_{best} = Z_{cp}$ and save the solution obtained for this candidate problem. Then go back to step 4.
9. Stop. The optimal solution has been found and has an objective function value given by the current value of Z_{best}.

6-19 Summary

This chapter discussed the issues involved in formulating and solving ILP problems. In some cases, acceptable integer solutions to ILP problems can be obtained by rounding the solution to the LP relaxation of the problem. However, this procedure can lead to suboptimal solutions, which might still be viable if you can show that the solution obtained by rounding is within an acceptable distance from the optimal integer solution. This approach might be the only practical way to obtain integer solutions for some ILP problems.

The B&B algorithm is a powerful technique for solving ILP problems. A great deal of skill and creativity are involved in formulating ILPs so that they can be solved efficiently using the B&B technique. Binary variables can be useful in overcoming a number of the simplifying assumptions often made in the formulation of LP models. Here again, quite a bit of creativity might be required on the part of the model builder to identify the constraints to implement various logical conditions in a given problem.

6-20 References

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THE WORLD OF BUSINESS ANALYTICS

Who Eats the Float?—Maryland National Improves Check Clearing Operations and Cuts Costs

Maryland National Bank (MNB) of Baltimore typically processes about 500,000 checks worth over $250,000,000 each day. Those checks not drawn on MNB or a local bank must be cleared via the Federal Reserve System, a private clearing bank, or a "direct send" by courier service to the bank on which they were drawn.

Because funds are not available until the check clears, banks try to maximize the availability of current funds by reducing the float—the time interval required for a check to clear. Banks publish an availability schedule listing the number of days before funds from a deposited check are available to the customer. If clearing time is longer than the schedule, the bank must "eat the float." If the check is cleared through

(Continued)

the Federal Reserve and clearing takes longer than the Federal Reserve availability schedule, then the Federal Reserve "eats the float." If clearing time is actually less than the local bank's availability schedule, the customer "eats the float." The cost of float is related to the daily cost of capital.

MNB uses a system based on binary integer LP to decide the timing and method to be used for each bundle of checks of a certain type (called a cash letter). Total clearing costs (the objective function) include float costs, clearing charges from the Federal Reserve or private clearing banks, and transportation costs for direct sends. Constraints ensure that exactly one method is chosen for each check type and that a method can be used only at a time that method is available. Use of this system saves the bank $100,000 annually.

Source: Markland, Robert E., and Robert M. Nauss. "Improving Transit Check Clearing Operations at Maryland National Bank." *Interfaces*, vol. 13, no. 1, February 1983, pp. 1–9.

Questions and Problems

1. As shown in Figure 6.1, the feasible region for an ILP consists of a relatively small, *finite* number of points, whereas the feasible region of its LP relaxation consists of an *infinite* number of points. Why, then, are ILPs so much harder to solve than LPs?
2. Identify reasonable values for M_{12} and M_{22} in the example on quantity discounts presented in section 6-16b of this chapter.
3. Consider the following optimization problem:

$$\begin{array}{ll} \text{MIN:} & X_1 + X_2 \\ \text{Subject to:} & \\ & -4X1 + 4X2 \leq 1 \\ & -8X1 + 10X2 \geq 15 \\ & X_1, X_2 \geq 0 \end{array}$$

   a. What is the optimal solution to this LP problem?
   b. Now suppose that X_1 and X_2 must be integers. What is the optimal solution?
   c. What general principle of integer programming is illustrated by this question?
4. The following questions refer to the CRT Technologies project selection example presented in this chapter. Formulate a constraint to implement the conditions described in each of the following statements.
   a. Out of projects 1, 2, 4, and 6, CRT's management wants to select exactly two projects.
   b. Project 2 can be selected only if project 3 is selected and vice-versa.
   c. Project 5 cannot be undertaken unless both projects 3 and 4 are also undertaken.
   d. If projects 2 and 4 are undertaken, then project 5 must also be undertaken.
5. In the CRT Technologies project selection example in this chapter, the problem indicates that surplus funds in any year are reappropriated and cannot be carried over to the next year. Suppose this is no longer the case and surplus funds may be carried over to future years.
   a. Modify the spreadsheet model given for this problem to reflect this change in assumptions.
   b. What is the optimal solution to the revised problem?
6. The following questions refer to the Blue Ridge Hot Tubs example discussed in this chapter.
   a. Suppose Howie Jones has to purchase a single piece of equipment for $1,000 in order to produce any Aqua-Spas or Hydro-Luxes. How will this affect the formulation of the model of his decision problem?

b. Suppose Howie must buy one piece of equipment that costs $900 in order to produce any Aqua-Spas and a different piece of equipment that costs $800 in order to produce any Hydro-Luxes. How will this affect the formulation of the model for his problem?

7. In the Colpitts Control Devices workload balancing problem presented in the chapter a successor task could be assigned to the same workstation as an immediate predecessor task. Suppose we change that assumption so that a successor task cannot be assigned to a workstation containing an immediate predecessor task.
   a. What change is required to enforce this new restriction?
   b. With that new restriction in force, solve Colpitts' workload balancing problem to determine the minimum cycle time with 1, 2, 3, 4, 5, 6, 7, and 8 workstations and produce a graph of the results like the one shown in Figure 6.27.
   c. Explain how and why your results differ from those shown in the graph in Figure 6.27.

8. Bowden Transport provides dispatching services for independent truckers who specialize in transporting cars purchased online from the seller to the buyer. At present, there are four cars needing to be picked up and delivered and five trucks in the vicinity of these cars. The following table summarizes the marginal cost of each truck picking up and delivering each of the cars along with the current number of available car carrying spots available on each truck.

	Marginal Cost to Pick Up and Deliver				
	Car 1	**Car 2**	**Car 3**	**Car 4**	**Capacity**
Truck 1	$276	$497	$251	$364	2 cars
Truck 2	$179	$375	$298	$190	1 car
Truck 3	$150	$475	$344	$492	1 car
Truck 4	$ 97	$163	$285	$185	1 car
Truck 5	$305	$150	$225	$165	2 cars

Bowden charges the car buyer a flat fee of $600 to pick up and deliver each car and keeps 50% of the profit earned.
   a. Formulate an ILP model for this problem.
   b. Implement your ILP model in a spreadsheet and solve it.
   c. What is the optimal solution?

9. Enrique Brava is responsible for upgrading the wireless network for his employer. He has identified seven possible locations to install new nodes for the network. Each node can provide service to different regions within his employer's corporate campus. The cost of installing each node and the regions that can be served by each node are summarized below.

   Node 1: Regions 1, 2, 5; Cost $700
   Node 2: Regions 3, 6, 7; Cost $600
   Node 3: Regions 2, 3, 7, 9; Cost $900
   Node 4: Regions 1, 3, 6, 10; Cost $1,250
   Node 5: Regions 2, 4, 6, 8; Cost $850
   Node 6: Regions 4, 5, 8, 10; Cost $1,000
   Node 7: Regions 1, 5, 7, 8, 9; Cost $100

   a. Formulate an ILP for this problem.
   b. Implement your model in a spreadsheet and solve it.
   c. What is the optimal solution?

10. Ken Stark is an operations analyst for an insurance company in Muncie, Indiana. Over the next 6 weeks the company needs to send 2,028,415 pieces of marketing literature to customers in the following 16 states:

State	Mailing Pieces
AZ	82,380
CA	212,954
CT	63,796
GA	136,562
IL	296,479
MA	99,070
ME	38,848
MN	86,207
MT	33,309
NC	170,997
NJ	104,974
NV	29,608
OH	260,858
OR	63,605
TX	214,076
VA	134,692
TOTAL	**2,028,415**

In order to coordinate with other marketing efforts, all the mailings for a given state must go out the same week (i.e., if Ken decides to schedule mailings for Georgia in week 2, then all of the 136,562 pieces of mail for Georgia must be sent that week). Ken would like to balance the work load in each week as much as possible and, in particular, would like to minimize the maximum amount of mail to be processed in any given week during the 6-week period.

a. Create a spreadsheet model to determine which states should be processed each week in order to achieve Ken's objective.
b. What is the optimal solution?

11. Garden City Beach is a popular summer vacation destination for thousands of people. Each summer, the city hires temporary lifeguards to ensure the safety of the vacationing public. Garden City's lifeguards are assigned to work five consecutive days each week and then have two days off. However, the city's insurance company requires them to have at least the following number of lifeguards on duty each day of the week:

	Minimum Number of Lifeguards Required Each Day						
	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
Lifeguards	18	17	16	16	16	14	19

The city manager would like to determine the minimum number of lifeguards that will have to be hired.

a. Formulate an ILP for this problem.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?
d. Several lifeguards have expressed a preference to be off on Saturdays and Sundays. What is the maximum number of lifeguards that can be off on the weekend without increasing the total number of life guards required?

12. Joni Wu manages the GoldRush Casino in New Orleans. She would like to adjust the assortment of gaming machines in the casino to ensure they are operating in the most profitable manner. The following table summarizes the current assortment of gaming machines in the casino. Joni is willing to increase or decrease the number of each type of gaming machine by as much as 10% (rounded to the closest integer). However, due to space limitations the total number of gaming machines must remain the same.

Machine Type	Units on Floor	Average Daily Profit per Unit
$0.01 Reel Slots	243	$123
$0.05 Reel Slots	9	$ 46
$0.25 Reel Slots	45	$ 82
$0.50 Reel Slots	16	$ 76
$1.00 Reel Slots	40	$ 89
$5.00 Reel Slots	12	$205
$0.01 Video Slots	658	$316
$0.05 Video Slots	8	$108
$0.01 Video Poker	8	$207
$0.05 Video Slots	67	$137
$0.25 Video Slots	84	$133
$1.00 Video Slots	6	$115
$0.01 Multi-Game	75	$117
$0.05 Multi-Game	257	$ 70
$0.25 Multi-Game	232	$ 90
$1.00 Multi-Game	18	$266
$5.00 Multi-Game	8	$114
$10.00 Multi-Game	6	$776
$0.05 Video Keno	30	$ 47

a. How much does the casino currently make in profit on average each day?
b. Create an optimization model in a spreadsheet to solve Joni's problem.
c. What is the optimal solution and how much profit should the casino expect to make on average each day under the optimal assortment of gaming machines?

13. Snookers Restaurant is open from 8:00 am to 10:00 pm daily. Besides the hours they are open for business, workers are needed an hour before opening and an hour after closing for setup and clean-up activities. The restaurant operates with both full-time and part-time workers on the following shifts:

Shift	Daily Pay Rate
7:00 a.m.–11:00 a.m.	$32
7:00 a.m.– 3:00 p.m.	$80
11:00 a.m.– 3:00 p.m.	$32
11:00 a.m.– 7:00 p.m.	$80
3:00 p.m.– 7:00 p.m.	$32
3:00 p.m.–11:00 p.m.	$80
7:00 p.m.–11:00 p.m.	$32

The following numbers of workers are needed during each of the indicated time blocks.

Hours	Workers Needed
7:00 a.m.–11:00 a.m.	11
11:00 a.m.– 1:00 p.m.	24
1:00 p.m.– 3:00 p.m.	16
3:00 p.m.– 5:00 p.m.	10
5:00 p.m.– 7:00 p.m.	22
7:00 p.m.– 9:00 p.m.	17
9:00 p.m.–11:00 p.m.	6

At least one full-time worker must be available during the hour before opening and after closing. Additionally, at least 30% of the employees should be full-time (8-hour) workers during the restaurant's busy periods from 11:00 a.m.–1:00 p.m. and 5:00 p.m.–7:00 p.m.

a. Formulate an ILP for this problem with the objective of minimizing total daily labor costs.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?

14. A manufacturer of industrial motors has identified ten new prospective customers for its products with estimated each customer's annual sales potential as follows:

Customer	1	2	3	4	5	6	7	8	9	10
Sales Potential (in $1,000,000s)	$113	$106	$84	$52	$155	$103	$87	$91	$128	$131

The company would like to allocate these ten prospective customers to five of its current salespeople in the most equitable way possible. (Each customer may be assigned to only one sales person.) To do this, *ideally*, the customers assigned to each of the five salespeople would have exactly the same sales potential. If such a solution is not possible, the company would like to minimize the total amount by which the actual sales potentials for the customers assigned to each salesperson deviate from the ideal allocation.

a. Ideally, what sales potential should be assigned to each salesperson?
b. Formulate a mathematical programming model for this problem. (*Hint:* For each salesperson, create two decision variables to represent the amount by which his or her assigned sales potential is, respectively, under or over the ideal sales potential.)
c. Implement your model in a spreadsheet and solve it.
d. What is the optimal solution?

15. A power company is considering how to increase its generating capacity to meet expected demand in its growing service area. Currently, the company has 750 megawatts (MW) of generating capacity but projects it will need the following minimum generating capacities in each of the next five years:

	Year				
	1	2	3	4	5
Minimum Capacity in Megawatts (MW)	780	860	950	1060	1180

The company can increase its generating capacity by purchasing four different types of generators: 10 MW, 25 MW, 50 MW, and/or 100 MW. The cost of acquiring and installing each of the four types of generators in each of the next five years is summarized in the following table:

Generator Size	Cost of Generator (in $1,000s) in Year 1	2	3	4	5
10 MW	$300	$250	$200	$170	$145
25 MW	$460	$375	$350	$280	$235
50 MW	$670	$558	$465	$380	$320
100 MW	$950	$790	$670	$550	$460

a. Formulate a mathematical programming model to determine the least costly way of expanding the company's generating assets to the minimum required levels.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?

16. Health Care Systems of Florida (HCSF) is planning to build a number of new emergency-care clinics in central Florida. HCSF management has divided a map of the area into seven regions. They want to locate the emergency centers so that all seven regions will be conveniently served by at least one facility. Five possible sites are available for constructing the new facilities. The regions that can be served conveniently by each site are indicated by X in the following table:

Region	Possible Building Sites: Sanford	Altamonte	Apopka	Casselberry	Maitland
1	X		X		
2	X	X		X	X
3		X		X	
4			X		X
5	X	X			
6			X		X
7				X	X
Cost ($1,000s)	$450	$650	$550	$500	$525

a. Formulate an ILP problem to determine which sites should be selected in order to provide convenient service to all locations in the least costly manner.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?

17. Charles McKeown is an acquisitions editor for a college textbook publisher. The file Books.xlsx that accompanies this book contains a list of 151 textbooks that Charles has an opportunity to acquire from another publisher. For each title, the file lists the price (acquisition cost) and net present value of expected future sales. Assume that Charles may select up to 20 tiles from this list and spend $12 million on these acquisitions.
a. Create an optimization model in a spreadsheet to solver Charles' problem.
b. Which titles should Charles acquire, how much of the budget would be used, and what is the expected NPV of these titles?

18. Radford Castings can produce brake shoes on six different machines. The following table summarizes the manufacturing costs associated with producing the brake shoes on each

machine along with the available capacity on each machine. If the company has received an order for 1,800 brake shoes, how should it schedule these machines?

Machine	Fixed Cost	Variable Cost	Capacity
1	$1000	$21	500
2	$ 950	$23	600
3	$ 875	$25	750
4	$ 850	$24	400
5	$ 800	$20	600
6	$ 700	$26	800

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it.
c. What is the optimal solution?

19. The teenage daughter of a recently deceased movie star inherited a number of items from her famous father's estate. Rather than convert these assets to cash immediately, her financial advisor has recommended that she let some of these assets appreciate in value before disposing of them. An appraiser has given the following estimates of the assets' worth (in $1,000s) for each of the next five years.

	Year 1	Year 2	Year 3	Year 4	Year 5
Car	$ 35	$ 37	$ 39	$ 42	$ 45
Piano	$ 16	$ 17	$ 18	$ 19	$ 20
Necklace	$125	$130	$136	$139	$144
Desk	$ 25	$ 27	$ 29	$ 30	$ 33
Golf Clubs	$ 40	$ 43	$ 46	$ 50	$ 52
Humidor	$ 5	$ 7	$ 8	$ 10	$ 11

Knowing this teenager's propensity to spend money, her financial advisor would like to develop a plan to dispose of these assets that will maximize the amount of money received and ensure that at least $30,000 of new funds become available each year to pay her college tuition.
a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it.
c. What is the optimal solution?

20. A developer of video game software has seven proposals for new games. Unfortunately, the company cannot develop all the proposals because its budget for new projects is limited to $950,000 and it has only 20 programmers to assign to new projects. The financial requirements, returns, and the number of programmers required by each project are summarized in the following table. Projects 2 and 6 require specialized programming knowledge that only one of the programmers has. Both of these projects cannot be selected because the programmer with the necessary skills can be assigned to only one of the projects. (Note: All dollar amounts represent thousands.)

Project	Programmers Required	Capital Required	Estimated NPV
1	7	$250	$650
2	6	$175	$550
3	9	$300	$600
4	5	$150	$450
5	6	$145	$375
6	4	$160	$525
7	8	$325	$750

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it.
c. What is the optimal solution?

21. Tropicsun is a leading grower and distributor of fresh citrus products with three large citrus groves scattered around central Florida in the cities of Mt. Dora, Eustis, and Clermont. Tropicsun currently has 275,000 bushels of citrus at the grove in Mt. Dora, 400,000 bushels at the grove in Eustis, and 300,000 at the grove in Clermont. Tropicsun has citrus processing plants in Ocala, Orlando, and Leesburg with processing capacities to handle 200,000, 600,000, and 225,000 bushels, respectively. Tropicsun contracts with a local trucking company to transport its fruit from the groves to the processing plants. The trucking company charges a flat rate of $8 per mile regardless of how many bushels of fruit are transported. The following table summarizes the distances (in miles) between each grove and processing plant:

Distances (in Miles) Between Groves and Plants

	Processing Plant		
Grove	**Ocala**	**Orlando**	**Leesburg**
Mt. Dora	21	50	40
Eustis	35	30	22
Clermont	55	20	25

Tropicsun wants to determine how many bushels to ship from each grove to each processing plant in order to minimize the total transportation cost.
a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it.
c. What is the optimal solution?

22. A real estate developer is planning to build an apartment building specifically for graduate students on a parcel of land adjacent to a major university. Four types of apartments can be included in the building: efficiencies, and one-, two-, or three-bedroom units. Each efficiency requires 500 square feet; each one-bedroom apartment requires 700 square feet; each two-bedroom apartment requires 800 square feet; and each three-bedroom unit requires 1,000 square feet. The developer believes that the building should include no more than 15 one-bedroom units, 22 two-bedroom units, and 10 three-bedroom units. Local zoning ordinances do not allow the developer to build more than 40 units in this particular building location, and restrict the building to a maximum of 40,000 square feet. The developer has already agreed to lease 5 one-bedroom units and 8 two-bedroom units to a local rental agency that is a "silent partner" in this endeavor. Market studies indicate that efficiencies can be rented for $350 per month, one-bedroom units for $450 per month, two-bedroom units for $550 per month, and three-bedroom units for $750 per month. How many rental units of each type should the developer include in the building plans in order to maximize the potential rental income from the building?
a. Formulate an LP model for this problem.
b. Create a spreadsheet model for this problem and solve it using Solver.
c. What is the optimal solution?
d. Which constraint in this model limits the builder's potential rental income from increasing any further?

23. Bellows Lumber Yard, Inc. stocks standard length, 25-foot boards, which it cuts to custom lengths to fill individual customer orders. An order has just come in for 5,000 7-foot boards, 1,200 9-foot boards, and 300 11-foot boards. The lumber yard manager has identified six ways to cut the 25-foot boards to fill this order. The six cutting patterns are summarized in the following table.

Number of Boards Produced			
Cutting Pattern	7 ft	9 ft	11 ft
1	3	0	0
2	2	1	0
3	2	0	1
4	1	2	0
5	0	1	1
6	0	0	2

One possibility (cutting pattern 1) is to cut a 25-foot board into three 7-foot boards, and not to cut any 9- or 11-foot boards. Note that cutting pattern 1 uses a total of 21 feet of board and leaves a 4-foot piece of scrap. Another possibility (cutting pattern 4) is to cut a 25-foot board into one 7-foot board and two 9-foot boards (using all 25 feet of the board). The remaining cutting patterns have similar interpretations. The lumber yard manager wants to fill this order using the fewest number of 25-foot boards as possible. To do this, the manager needs to determine how many 25-foot boards to run through each cutting pattern.

a. Formulate an LP model for this problem.
b. Create a spreadsheet model for this problem and solve it using Solver.
c. What is the optimal solution?
d. Suppose the manager wants to minimize waste. Would the solution change?

24. Howie's Carpet World has just received an order for carpets for a new office building. The order is for 4,000 yards of carpet 4-feet wide, 20,000 yards of carpet 9-feet wide, and 9,000 yards of carpet 12-feet wide. Howie can order two kinds of carpet rolls, which he will then have to cut to fill this order. One type of roll is 14-feet wide, 100-yards long, and costs \$1,000 per roll; the other is 18-feet wide, 100-yards long, and costs \$1,400 per roll. Howie needs to determine how many of the two types of carpet rolls to order and how they should be cut. He wants to do this in the least costly way possible.
    a. Formulate an LP model for this problem.
    b. Create a spreadsheet model for this problem and solve it using Solver.
    c. What is the optimal solution?
    d. Suppose Howie wants to minimize waste. Would the solution change?

25. A manufacturer is considering alternatives for building new plants in order to be located closer to three of its primary customers with whom it intends to develop long-term relationships. The net cost of manufacturing and transporting each unit of the product to its customers will vary depending on where the plant is built and the production capacity of the plant. These costs are summarized in the following table:

	Net Cost per Unit to Supply Customer		
Plant	X	Y	Z
1	35	30	45
2	45	40	50
3	70	65	50
4	20	45	25
5	65	45	45

The annual demand for products from customers X, Y, and Z is expected to be 40,000, 25,000, and 35,000 units, respectively. The annual production capacity and construction costs for each plant are:

Plant	Production Capacity	Construction Cost (in $1000s)
1	40,000	$1,325
2	30,000	$1,100
3	50,000	$1,500
4	20,000	$1,200
5	40,000	$1,400

The company wants to determine which plants to build in order to satisfy customer demand at a minimum total cost.

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it.
c. What is the optimal solution?

26. Refer to the previous question. Suppose plants 1 and 2 represent different building alternatives for the same site (i.e., only one of these plants can be built). Similarly, suppose plants 4 and 5 represent different building alternatives for another site.
    a. What additional constraints are required to model these new conditions?
    b. Revise the spreadsheet to reflect these additional constraints and solve the resulting problem.
    c. What is the optimal solution?
27. GLMH Shipping is a start-up company that plans to offer same-day shipping services between 20 major cities in the United States. In order to provide this service, GLMH needs to build hubs at airports in several of these cities. GLMH wants to select hub locations in a way that ensures each of the 20 cities is within 500 miles of at least one of the hub locations. The file Airports.xlsx contains data describing the estimated cost of establishing a hub in each city as well as a matrix summarizing the distances in miles between each of the cities.
    a. Create a spreadsheet model to determine where hubs should be located in order to achieve GLMH's objectives in the most cost-effective manner.
    b. In what cities should GLMH create hubs and what is the total cost of this plan?
28. A company manufactures three products: A, B, and C. The company currently has an order for 3 units of product A, 7 units of product B, and 4 units of product C. There is no inventory for any of these products. All three products require special processing that can be done on one of two machines. The cost of producing each product on each machine is summarized in the following table:

	Cost of Producing a Unit of Product		
Machine	A	B	C
1	$13	$ 9	$10
2	$11	$12	$ 8

The time required to produce each product on each machine is summarized in the following table:

	Time (Hours) Needed to Produce a Unit of Product		
Machine	A	B	C
1	0.4	1.1	0.9
2	0.5	1.2	1.3

Assume machine 1 can be used for 8 hours and machine 2 can be used for 6 hours. Each machine must undergo a special setup operation to prepare it to produce each product. After completing this setup for a product, any number of that product type can be produced. The setup costs for producing each product on each machine are summarized in the following table:

	Setup Costs for Producing		
Machine	**A**	**B**	**C**
1	\$55	\$93	\$60
2	\$65	\$58	\$75

a. Formulate an ILP model to determine how many units of each product to produce on each machine in order to meet demand at a minimum cost.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?

29. Clampett Oil purchases crude oil products from suppliers in Texas (TX), Oklahoma (OK), Pennsylvania (PA), and Alabama (AL), from which it refines four end-products: gasoline, kerosene, heating oil, and asphalt. Because of differences in the quality and chemical characteristics of the oil from the different suppliers, the amount of each end-product that can be refined from a barrel of crude oil varies depending on the source of the crude. Additionally, the amount of crude available from each source varies, as does the cost of a barrel of crude from each supplier. These values are summarized in the following table. For example, the first line of this table indicates that a barrel of crude oil from Texas can be refined into 2 barrels of gasoline, 2.8 barrels of kerosene, 1.7 barrels of heating oil, or 2.4 barrels of asphalt. Each supplier requires a minimum purchase of at least 500 barrels.

Raw Material Characteristics

		Possible Production per Barrel					
Crude Oils	**Barrels Available**	**Gas**	**Kerosene**	**Heat**	**Asphalt**	**Cost per Barrel**	**Trucking Cost**
TX	1,500	2.00	2.80	1.70	2.40	\$22	\$1,500
OK	2,000	1.80	2.30	1.75	1.90	\$21	\$1,700
PA	1,500	2.30	2.20	1.60	2.60	\$22	\$1,500
AL	1,800	2.10	2.60	1.90	2.40	\$23	\$1,400

The company owns a tanker truck that picks up whatever crude oil it purchases. This truck can hold 2,000 barrels of crude. The cost of sending the truck to pick up oil from the various locations is shown in the column labeled "Trucking Cost." The company's plans for its next production cycle specify 750 barrels of gasoline, 800 barrels of kerosene, 1,000 barrels of heating oil, and 300 barrels of asphalt to be produced.

a. Formulate an ILP model that can be solved to determine the purchasing plan that will allow the company to implement its production plan at the least cost.
b. Implement this model in a spreadsheet and solve it.
c. What is the optimal solution?

30. The Clampett Oil Company has a tanker truck that it uses to deliver fuel to customers. The tanker has five different storage compartments with capacities to hold 2,500, 2,000, 1,500, 1,800 and 2,300 gallons, respectively. The company has an order to deliver 2,700 gallons of diesel fuel; 3,500 gallons of regular unleaded gasoline; and 4,200 gallons of premium unleaded gasoline. If each storage compartment can hold only one type of fuel, how should Clampett Oil load the tanker? If it is impossible to load the truck with the full order, the company wants to minimize the total number of gallons by which the order is short. (*Hint:* Consider using slack variables to represent shortage amounts.)

a. Formulate an ILP model for this problem.
b. Implement this model in a spreadsheet and solve it.
c. What is the optimal solution?

31. Dan Boyd is a financial planner trying to determine how to invest $100,000 for one of his clients. The cash flows for the five investments under consideration are summarized in the following table:

Summary of Cash In-Flows and Out-Flows (at Beginning of Year)

	A	B	C	D	E
Year 1	−1.00	0.00	−1.00	0.00	−1.00
Year 2	+0.45	−1.00	0.00	0.00	0.00
Year 3	+1.05	0.00	0.00	−1.00	1.25
Year 4	0.00	+1.30	+1.65	+1.30	0.00

For example, if Dan invests $1 in investment A at the beginning of year 1, he will receive $0.45 at the beginning of year 2 and another $1.05 at the beginning of year 3. Alternatively, he can invest $1 in investment B at the beginning of year 2 and receive $1.30 at the beginning of year 4. Entries of "0.00" in the preceding table indicate times when no cash in-flows or out-flows can occur. The minimum required investment for each of the possible investments is $50,000. Also, at the beginning of each year, Dan may also place any or all of the available money in a money market account that is expected to yield 5% per year. How should Dan plan his investments if he wants to maximize the amount of money available to his client at the end of year 4?

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it using Solver.
c. What is the optimal solution?

32. Bavarian Motor Company (BMC) manufacturers cars and SUVs in Europe and ships them to distributors in the United States. Presently, BMC has an inventory of 200 cars and 140 SUVs in Newark, NJ and 300 cars and 180 SUVs in Jacksonville, FL. These vehicles need to be transported by rail to meet the demand for BMC distributors in the cities summarized in the following table:

	Vehicles Needed	
City	**Cars**	**SUVs**
Boston	100	75
Columbus	60	40
Richmond	80	55
Atlanta	170	95
Mobile	70	50

BMC rents rail cars to move its inventory of vehicles between these cities. Each rail car can hold up to 12 vehicles and are readily available in any quantity needed. The cost of renting a rail car and having it moved among these cities is summarized in Figure 6.41.

a. Formulate an ILP model for this problem.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?

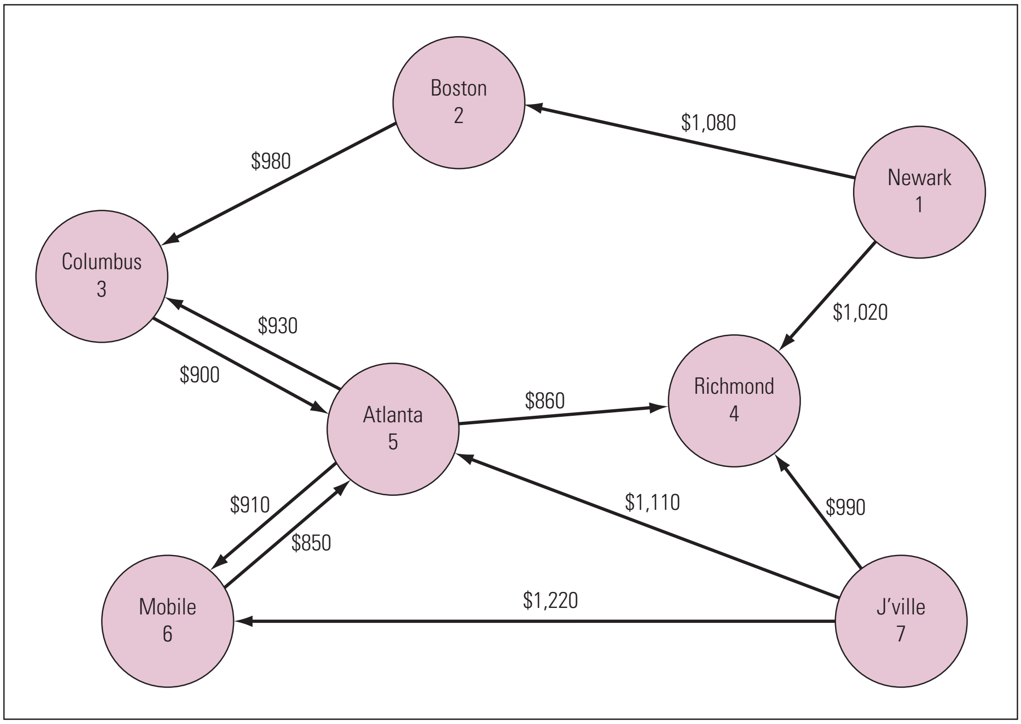
33. The Mega-Bucks Corporation is planning its production schedule for the next four weeks and is forecasting the following demand for compound X—a key raw material used in its production process:

Forecasted Demand of Compound X

Week	1	2	3	4
Demand	400 lbs.	150 lbs.	200 lbs.	350 lbs.

FIGURE 6.41

Costs per rail car rental for BMC's vehicle distribution problem



The company currently has no compound X on hand. The supplier of this product delivers only in batch sizes that are multiples of 100 pounds (0, 100, 200, 300, and so on). The price of this material is $125 per 100 pounds. Deliveries can be arranged weekly, but there is a delivery charge of $50. Mega-Bucks estimates that it costs $15 for each 100 pounds of compound X held in inventory from one week to the next. Assuming Mega-Bucks does not want more than 50 pounds of compound X in inventory at the end of week 4, how much should it order each week so that the demand for this product will be met in the least costly manner?

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it using Solver.
c. What is the optimal solution?

34. An automobile manufacturer is considering mechanical design changes in one of its top-selling cars to reduce the weight of the car by at least 400 pounds to improve its fuel efficiency. Design engineers have identified 10 changes that could be made in the car to make it lighter (e.g., using composite body pieces rather than metal). The weight saved by each design change and the estimated costs of implementing each change are summarized in the following table:

	Design Change									
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
Weight Saved (lbs)	50	75	25	150	60	95	200	40	80	30
Cost (in $1,000s)	$150	$350	$50	$450	$90	$35	$650	$75	$110	$30

Changes 4 and 7 represent alternate ways of modifying the engine block and, therefore, only one of these options could be selected. The company wants to determine which changes to make in order to reduce the total weight of the car by at least 400 pounds in the least costly manner.

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it.
c. What is the optimal solution?

35. Darten Restaurants owns and operates several different restaurant chains including Red Snapper and the Olive Grove. The company is considering opening a number of new units in Ohio. There are 10 different sites available for the company to build new restaurants and the company can build either type of restaurant at a given site. The following table summarizes the estimated net present value (NPV) of the cash flows (in millions) resulting from locating each type of restaurant at each of the sites and also indicates which sites are within 15 miles of each other.

Site	Red Snapper NPV	Olive Grove NPV	Other Sites within 15 miles
1	$11.8	$16.2	2, 3, 4
2	13.3	13.8	1, 3, 5
3	19.0	14.6	1, 2, 4, 5
4	17.8	12.4	1, 3
5	10.0	13.7	2, 3, 9
6	16.1	19.0	7
7	13.3	10.8	6, 8
8	18.8	15.2	7
9	17.2	15.9	5, 10
10	14.4	16.8	9

a. Suppose the company does not want to build two units from the same chain within 15 miles of each other (e.g., it does not want to build two Red Snappers within 15 miles of each other nor is it willing to build two Olive Groves within 15 miles of each other). Create a spreadsheet model to determine which (if any) restaurant it should build at each site in order to maximize total NPV.
b. What is the optimal solution?

36. Paul Bergey is in charge of loading cargo ships for International Cargo Company (ICC) at the port in Newport News, Virginia. Paul is preparing a loading plan for an ICC freighter destined for Ghana. An agricultural commodities dealer would like to transport the following products aboard this ship.

Commodity	Amount Available (tons)	Volume per Ton (cubic feet)	Profit per Ton ($)
1	4,800	40	70
2	2,500	25	50
3	1,200	60	60
4	1,700	55	80

Paul can elect to load any and/or all of the available commodities. However, the ship has three cargo holds with the following capacity restrictions:

Cargo Hold	Weight Capacity (tons)	Volume Capacity (cubic feet)
Forward	3,000	145,000
Center	6,000	180,000
Rear	4,000	155,000

Only one type of commodity can be placed into any cargo hold. However, because of balance considerations, the weight in the forward cargo hold must be within 10% of the weight in the rear cargo hold and the center cargo hold must be between 40% and 60% of the total weight on board.

a. Formulate an ILP model for this problem.
b. Create a spreadsheet model for this problem and solve it using Solver.
c. What is the optimal solution?

37. KPS Communications is planning to bring wireless internet access to the town of Ames, Iowa. Using a geographic information system, KPS has divided Ames into the following 5 by 5 grid. The values in each block of the grid indicate the expected annual revenue (in $1,000s) KPS will receive if wireless Internet service is provided to the geographic area represented by each block.

Expected Annual Revenue by Area (in $1,000s)

$34	$43	$62	$42	$34
$64	$43	$71	$48	$65
$57	$57	$51	$61	$30
$32	$38	$70	$56	$40
$68	$73	$30	$56	$44

KPS can build wireless towers in any block in the grid at a cost of $150,000 per tower. Each tower can provide wireless service to the block it is in and to all adjacent blocks. (Blocks are considered to be adjacent if they share a side. Blocks touching only at cornerpoint are not considered adjacent.) KPS would like to determine how many towers to build and where to build them in order to maximize profits in the first year of operations. (Note: If a block can receive wireless service from two different towers, the revenue for that block should only be counted once.)

a. Create a spreadsheet model for this problem and solve it.
b. What is the optimal solution and how much money will KPS make in the first year?
c. Suppose KPS is required to provide wireless service to all of the blocks. What is the optimal solution and how much money will KPS make in the first year?

38. The emergency services coordinator for Dade County, Tallys DeCampinas, is interested in locating the county's two ambulances to maximize the number of residents that can be reached within 4 minutes in emergency situations. The county is divided into six regions, and the average times required to travel from one region to the next are summarized in the following table:

	To Region					
From Region	1	2	3	4	5	6
1	0	4	3	6	6	5
2	4	0	7	5	5	6
3	3	7	0	4	3	5
4	6	5	4	0	7	5
5	6	5	3	7	0	2
6	5	6	5	5	2	0

The population (in 1,000s) in regions 1 through 6 are estimated, respectively, as 21, 35, 15, 60, 20, and 37. In which two regions should the ambulances be placed?

a. Formulate an ILP model for this problem.
b. Implement your model in a spreadsheet and solve it.
c. What is the optimal solution?
d. How many ambulances would be required to provide coverage within 4 minutes to all residents?
e. Suppose the county wants to locate three ambulances in such a way to provide coverage to all residents within 4 minutes and maximize the redundancy in the system. (Assume redundancy means being able to provide service by one or more ambulances within 4 minutes.) Where should the ambulances be located?

39. The CoolAire Company manufactures air conditioners that are sold to five different retail customers across the United States. The company is evaluating its manufacturing and logistics strategy to ensure that it is operating in the most efficient manner possible. The company can produce air conditioners at six plants across the country and stock these units in any of four different warehouses. The cost of manufacturing and shipping a unit between each plant and warehouse is summarized in the following table along with the monthly capacity and fixed cost of operating each plant.

	Warehouse 1	Warehouse 2	Warehouse 3	Warehouse 4	Fixed Cost	Capacity
Plant 1	$700	$1,000	$900	$1,200	$55,000	300
Plant 2	$800	$ 500	$600	$ 700	$40,000	200
Plant 3	$850	$ 600	$700	$ 500	$45,000	300
Plant 4	$600	$ 800	$500	$ 600	$50,000	250
Plant 5	$500	$ 600	$450	$ 700	$42,000	350
Plant 6	$700	$ 600	$750	$ 500	$40,000	400

Similarly, the per-unit cost of shipping units from each warehouse to each customer is given in the following table, along with the monthly fixed cost of operating each warehouse.

	Customer 1	Customer 2	Customer 3	Customer 4	Customer 5	Fixed Cost
Warehouse 1	$40	$80	$60	$90	$50	$40,000
Warehouse 2	$60	$50	$75	$40	$35	$50,000
Warehouse 3	$55	$40	$65	$60	$80	$35,000
Warehouse 4	$80	$30	$80	$50	$60	$60,000

The monthly demand from each customer is summarized next:

	Customer 1	Customer 2	Customer 3	Customer 4	Customer 5
Demand	200	300	200	150	250

CoolAire would like to determine which plants and warehouses it should operate to meet demand in the most cost-effective manner.

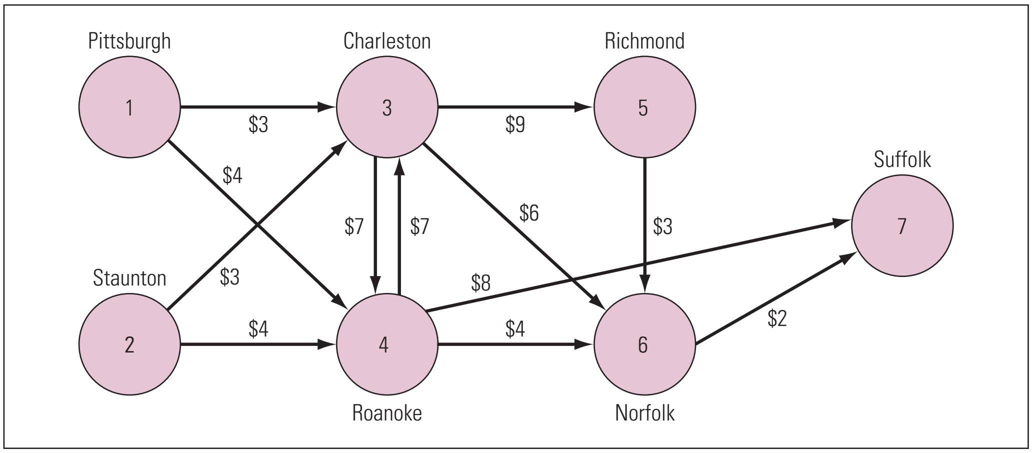
a. Create a spreadsheet model for this problem and solve it.
b. Which plants and warehouses should CoolAire operate?
c. What is the optimal shipping plan?

40. A blood bank wants to determine the least expensive way to transport available blood donations from Pittsburgh and Staunton to hospitals in Charleston, Roanoke, Richmond, Norfolk, and Suffolk. Figure 6.42 shows the possible shipping paths between cities along with the per unit cost of shipping along each possible arc. Additionally, the courier service used by the blood bank charges a flat rate of $125 any time it makes a trip across any of these arcs, regardless of how many units of blood are transported. Also assume that each arc may be

traversed only once. The van used by the courier service can carry a maximum of 800 units of blood.

FIGURE 6.42

Possible shipping routes for the blood bank problem



Assume Pittsburgh has 600 units of blood type O positive (O+) and 800 units of blood type AB available. Assume Staunton has 500 units of O+ and 600 units of AB available. The following table summarizes the number of units of each blood type needed at the various hospitals:

	Units Needed	
Hospital	**O+**	**AB**
Charleston	100	200
Roanoke	100	100
Richmond	500	300
Norfolk	200	500
Suffolk	150	250

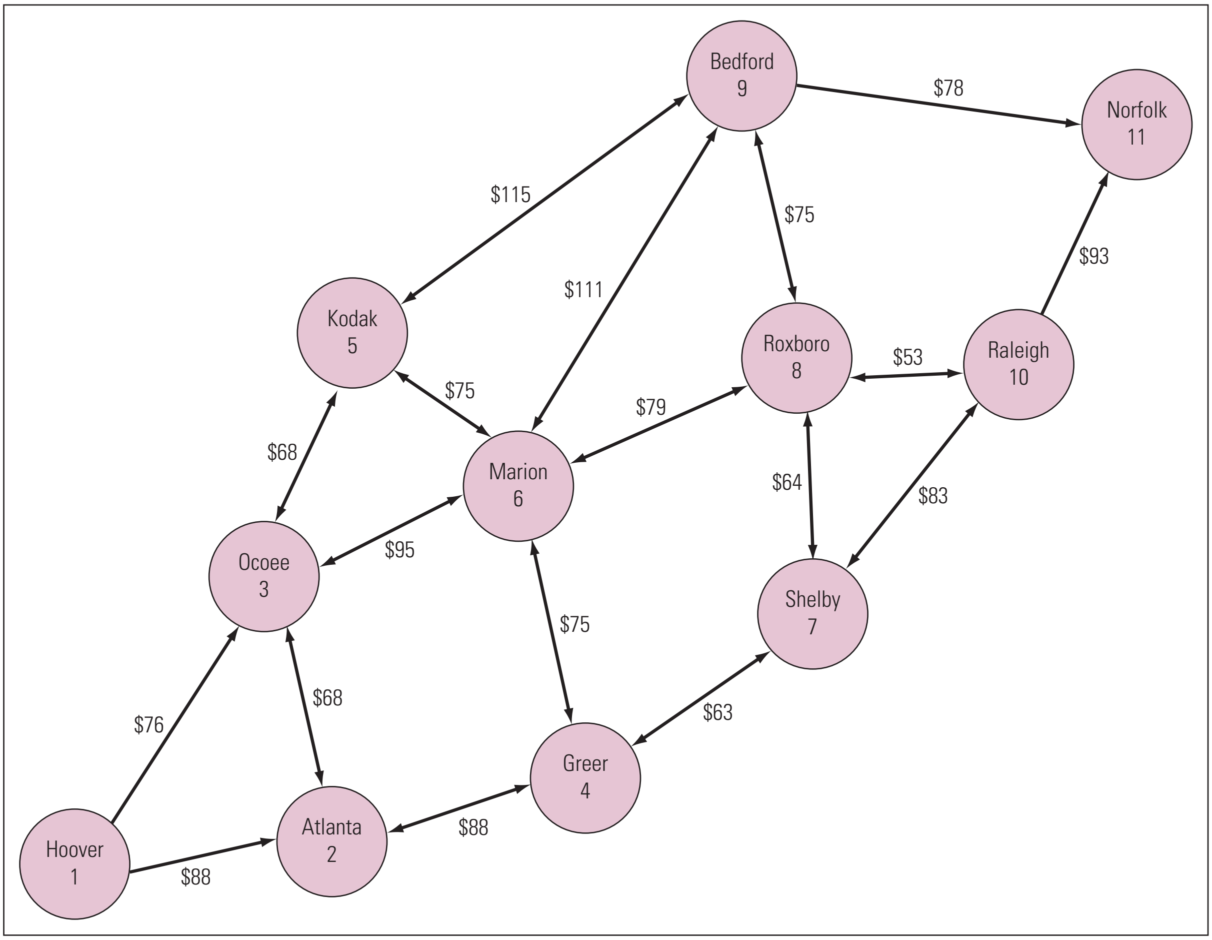
a. Create a spreadsheet model for this problem.
b. What is the optimal solution?
c. Suppose that the courier services switch to a new type of van that can carry no more than 1,000 units of blood between any two cities. What is the optimal solution to this revised problem?

41. Home Sweet Home Appliances manufactures specialty kitchen appliances at its factory in Hoover, Alabama. Presently, the company is preparing to deliver custom refrigerators and dishwashers to distributors in a number of different cities. It has 20 refrigerators in stock in its main warehouse in Hoover, AL and another 15 in inventory in Greer, SC. It also has 10 dishwashers at the warehouse in Hoover and another in inventory 10 in Ocoee, TN. Distributors in Roxboro, NC, Bedford, VA, and Norfolk, VA are requesting, respectively, 5, 10, and 15 refrigerators. Distributors in Marion, NC and Norfolk, VA are each requesting 10 dishwashers. The company would like to dispatch a single delivery truck from its warehouse in Hoover to make the necessary pickups and deliveries. The truck can hold a maximum of 45 total appliances. The graph in Figure 6.43 shows the different routes the

truck can take and the cost associated with each arc. Note that each bidirectional arc in the graph could be replaced by two unidirectional arcs going in opposite directions. For instance, the bidirectional arc from node 4 to node 6 could be replaced by a unidirectional arc from node 4 to node 6 and another unidirectional arc from node 6 to node 4.

a. Create a spreadsheet model for this problem and solver it.
b. What is the optimal route for the delivery truck?
c. What is the total cost associated with this route?

FIGURE 6.43 *Possible shipping routes for Home Sweet Home*



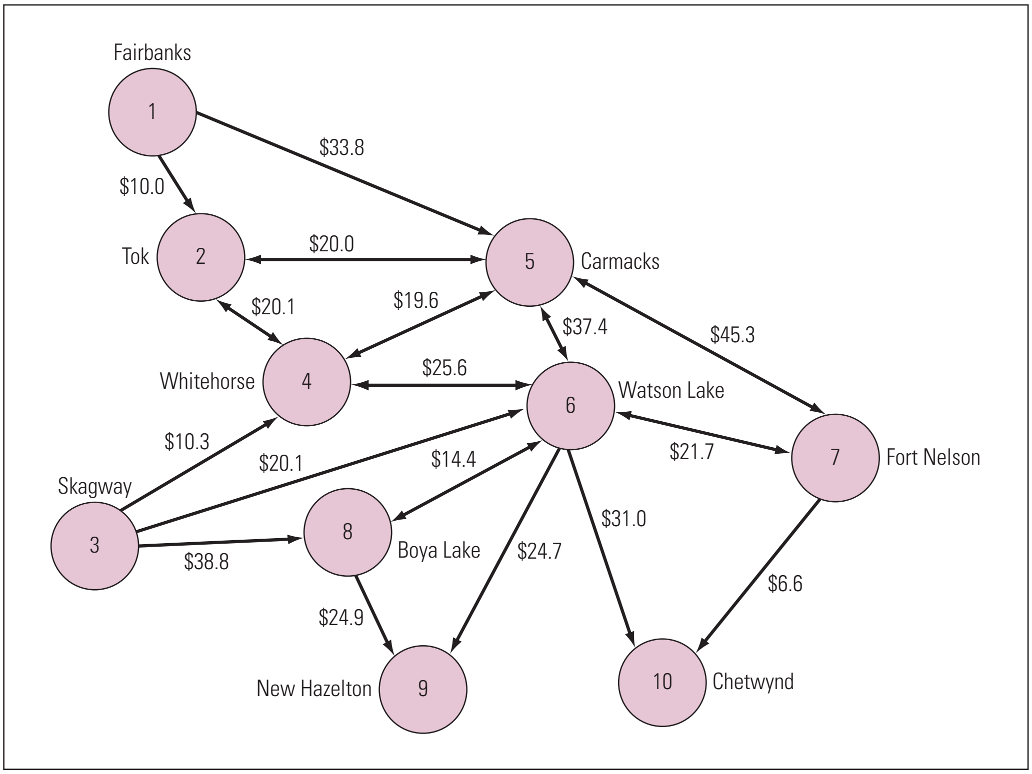
42. Alaskan Railroad is an independent, stand-alone railroad operation not connected to any other rail service in North America. As a result, rail shipments between Alaska and the rest of North America must be shipped by truck for thousands of miles or loaded onto ocean-going cargo vessels and transported by sea. Alaskan Railroad recently began talks with the nation of Canada about expanding its railroad lines to connect with the North American railway system. Figure 6.44 summarizes the various rail segments (and associated costs in

millions of U.S. dollars) that could be built. The North American railroad system currently provides service to New Hazelton and Chetwynd. Alaskan Railroad would like to expand its railway so as to be able to reach both of these cities from Skagway and Fairbanks.

a. Implement an optimization model to determine the least expensive way to connect the city of Skagway to New Hazelton and Chetwynd and also connect Fairbanks to these same cities.
b. What is the optimal solution?

FIGURE 6.44

Possible rail lines for Alaskan Railroads



43. CaroliNet is a satellite TV service provider for residential customers in the state of North Carolina. The company is planning to expand and offer satellite TV service in South Carolina as well. The company wants to establish a set of service hubs throughout the state in such a way to ensure that all residents of the state have a service hub either in their own country or in an adjacent county. Figure 6.45 (and the file CaroliNet.xlsm that accompanies this book) shows an Excel spreadsheet with a matrix indicating county adjacencies throughout the state. That is, values of 1 in the matrix indicate counties that are adjacent to one another while values of 0 indicate counties that are not adjacent to one another. (Note that a county is also considered to be adjacent to itself.)

a. Assume CaroliNet wants to minimize the number of hubs they must install. In what counties should the hubs be installed?
b. Suppose CaroliNet is willing to install hubs in exactly 10 different counties. In what counties should the hubs be installed if the company wants to maximize its service coverage?

FIGURE 6.45 *County adjacency matrix for the CaroliNet ISP location problem*

County Adjacency Matrix (1=adjacent counties)

	Abbeville	Aiken	Allendale	Anderson	Bamberg	Barnwell	Beaufort	Berkeley	Calhoun	Charleston	Cherokee	Chester	Chesterfield
Abbeville	1	0	0	1	0	0	0	0	0	0	0	0	0
Aiken	0	1	0	0	0	1	0	0	0	0	0	0	0
Allendale	0	0	1	0	1	1	0	0	0	0	0	0	0
Anderson	1	0	0	1	0	0	0	0	0	0	0	0	0
Bamberg	0	0	1	0	1	1	0	0	0	0	0	0	0
Barnwell	0	1	1	0	1	1	0	0	0	0	0	0	0
Beaufort	0	0	0	0	0	0	1	0	0	0	0	0	0
Berkeley	0	0	0	0	0	0	0	1	0	1	0	0	0
Calhoun	0	0	0	0	0	0	0	0	1	0	0	0	0
Charleston	0	0	0	0	0	0	0	1	0	1	0	0	0
Cherokee	0	0	0	0	0	0	0	0	0	0	1	0	0
Chester	0	0	0	0	0	0	0	0	0	0	0	1	0
Chesterfield	0	0	0	0	0	0	0	0	0	0	0	0	1
Claredon	0	0	0	0	0	0	0	1	1	0	0	0	0
Colleton	0	0	1	0	1	0	1	0	0	1	0	0	0
Darlington	0	0	0	0	0	0	0	0	0	0	0	0	1
Dillon	0	0	0	0	0	0	0	0	0	0	0	0	0
Dorchester	0	0	0	0	1	0	0	1	0	1	0	0	0
Edgefield	0	1	0	0	0	0	0	0	0	0	0	0	0
Fairfield	0	0	0	0	0	0	0	0	0	0	0	1	0
Florence	0	0	0	0	0	0	0	0	0	0	0	0	0
Georgetown	0	0	0	0	0	0	0	1	0	0	0	0	0
Greenville	1	0	0	1	0	0	0	0	0	0	0	0	0
Greenwood	1	0	0	0	0	0	0	0	0	0	0	0	0
Hampton	0	0	1	0	1	0	1	0	0	0	0	0	0
Horry	0	0	0	0	0	0	0	0	0	0	0	0	0
Jasper	0	0	0	0	0	0	1	0	0	0	0	0	0
Kershaw	0	0	0	0	0	0	0	0	0	0	0	0	1
Lancaster	0	0	0	0	0	0	0	0	0	0	0	1	1
Laurens	1	0	0	1	0	0	0	0	0	0	0	0	0

44. Solve the following problem manually using the B&B algorithm. You can use the computer to solve the individual problems generated. Create a branch-and-bound tree to display the steps you complete.

$$
\begin{aligned}
\text{MAX:} \quad & 6X_1 + 8X_2 \\
\text{Subject to:} \quad & 6X_1 + 3X_2 \leq 18 \\
& 2X_1 + 3X_2 \leq 9 \\
& X_1, X_2 \geq 0 \\
& X_1, X_2 \text{ must be integers}
\end{aligned}
$$

45. During the execution of the B&B algorithm, many candidate problems are likely to be generated and awaiting further analysis. In the B&B example in this chapter, we chose the next candidate problem to analyze in a rather arbitrary way. What other, more structured ways might we use to select the next candidate problem? What are the pros and cons of these techniques?

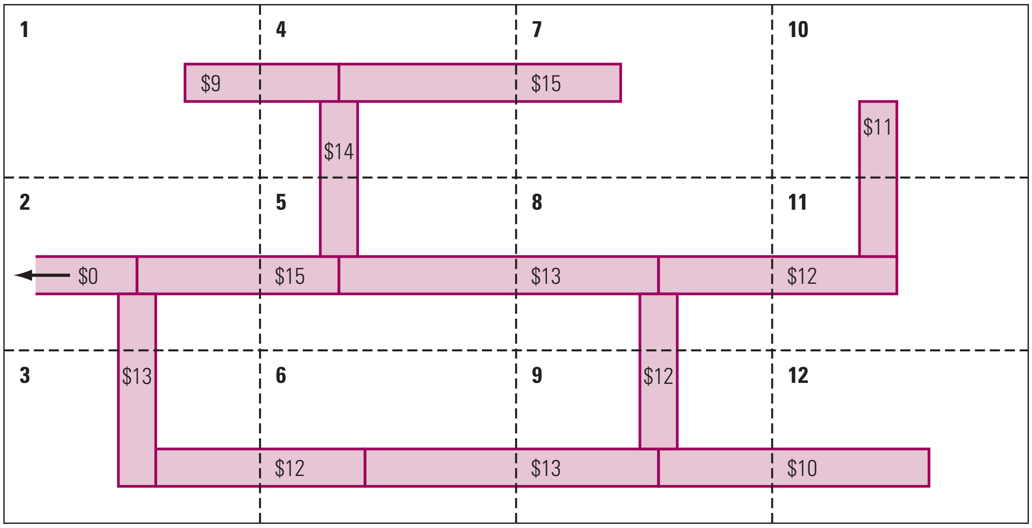
CASE 6-1 Optimizing a Timber Harvest

The state of Virginia is one of the largest producers of wood furniture in the United States, with the furniture industry accounting for 50% of value added to wood materials. Over the past 40 years the inventory volume of wood in Virginia's forests has increased by 81%. Today, 15.4 million acres, which is well over half of the state, are covered in forest. Private owners hold 77% of this land. When making decisions about which trees to harvest, forestry professionals consider many factors and must follow numerous laws and regulations.

Figure 6.46 depicts a tract of forested land that has been section off into 12 harvestable areas, indicated by dashed lines. Area 2 provides the only access to the forest via a paved road, so any timber cut must ultimately be transported out of the forest through area 2. Currently, there are no roads through this forest. So to harvest the timber, forest roads will need to be built. The allowable routes for these roads are also shown in Figure 6.45 and are determined largely on the geography of the land and location of streams and wildlife habitats.

FIGURE 6.46

Forest diagram for the timber harvesting problem



Not all areas of the forest have to be harvested. However, to harvest any area, a forest road must be built to that area. The cost of building each section of forest road (in $1,000s) is indicated in the figure. Finally, the net value of the harvestable timber in each area is estimated as follows:

	Harvested Value (in $1000s)											
Area	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
Value	$15	$7	$10	$12	$8	$17	$14	$18	$13	$12	$10	$11

Which areas should be harvested and what roads should be built to make the most profitable use of this forest?

a. Create a spreadsheet model for this problem.
b. What is the optimal solution?
c. Suppose the cost of building the road connecting areas 4 and 5 dropped to $12,000. What impact does this have on the optimal solution?

Power Dispatching at Old Dominion

CASE 6-2

The demand for electricity varies greatly during the day. Because large amounts of electricity cannot be stored economically, electric power companies cannot manufacture electricity and hold it in inventory until it is needed. Instead, power companies must balance the production of power with the demand for power in real time. One of the greatest uncertainties in forecasting the demand for electricity is the weather. Most power companies employ meteorologists who constantly monitor weather patterns and update computer models that predict the demand for power over a rolling, 7-day planning horizon. This forecasted 7-day window of demand is referred to as the company's load profile and is typically updated every hour basis.

Every power company has a base-load demand that is relatively constant. To satisfy this base-load demand, a power company uses its most economical, low-cost power generating assets and keeps them running continuously. To meet additional demands for power above the base-load, a power company must dispatch (or turn on) other generators. These other generators are sometimes called "peakers" as they help the power company meet the highest demands or peak-loads. It costs different amounts of money to bring different types of peakers online. And because different peakers use different types of fuel (e.g., coal, gas, biomass), their operating costs per megawatt (MW) generated also differ. Thus, dispatchers for a power company continually have to decide which generator to bring online or turn off to meet their load profile in the least costly manner.

The Old Dominion Power (ODP) Company provides electrical power throughout Virginia and the Carolinas. Suppose ODP's peak-load profile (that is the estimated load above base load) in MWs is currently estimated as follows:

	Day						
	1	**2**	**3**	**4**	**5**	**6**	**7**
Load (in MWs)	4,300	3,700	3,900	4,000	4,700	4,800	3,600

ODP currently has three peaking generators offline that are available to help meet this load. The generators have the following operating characteristics:

Generator Location	Startup Cost	Cost per Day	Maximum MW Capacity per Day
New River	$800	$200 + $5 per MW	2,100
Galax	$1,000	$300 + $4 per MW	1,900
James River	$700	$250 + $7 per MW	3,000

To get an offline generator up and running, a startup cost must be paid. After a generator is running, it can continue to run indefinitely without having to pay this startup cost again. However, if the generator is turned off at any point, the setup cost must be paid again to get it back up and running. Each day that a generator runs, there is both a fixed and variable cost that must be paid. For example, any day that the New River generator is online, it incurs a fixed cost of $200 plus $5 per MW generated. So even if this generator is not producing any MWs, it still costs $200 per day to keep it running (so as to avoid a restart). When they are running, each generator can supply up to the maximum daily MWs listed in the final column of the table.

a. Formulate a mathematical programming model for ODP's power dispatching problem.
b. Implement your model in a spreadsheet and solve it.

c. What is the optimal solution?
d. Suppose ODP can sometimes buy power from a competitor. How much should ODP be willing to pay to acquire 300 MWs of power on day 1? Explain your answer.
e. What concerns, if any, would you have about implementing this plan?

CASE 6-3 The MasterDebt Lockbox Problem

MasterDebt is a national credit card company with thousands of card holders located across the United States. Every day throughout the month, MasterDebt sends out statements to different customers summarizing their charges for the previous month. Customers then have 30 days to remit a payment for their bills. MasterDebt includes a pre-addressed envelope with each statement for customers to use in making their payments.

One of the critical problems facing MasterDebt involves determining what address to put on the pre-addressed envelopes sent to various parts of the country. The amount of time that elapses between when a customer writes his check and when MasterDebt receives the cash for the check is referred to as *float*. Checks can spend several days floating in the mail and in processing before being cashed. This float time represents lost revenue to MasterDebt because if they could receive and cash these checks immediately, they could earn additional interest on these funds.

To reduce the interest being lost from floating checks, MasterDebt would like to implement a lockbox system to speed the processing of checks. Under such a system, MasterDebt might have all its customers on the West Coast send their payments to a bank in Sacramento which, for a fee, processes the checks and deposits the proceeds in a MasterDebt account. Similarly, MasterDebt might arrange for a similar service with a bank on the East Coast for its customers there. Such lockbox systems are a common method companies use to improve their cash flows.

MasterDebt has identified six different cities as possible lockbox sites. The annual fixed cost of operating a lockbox in each of the possible locations is given in the following table:

Annual Lockbox Operating Costs (in $1,000s)

Sacramento	Denver	Chicago	Dallas	New York	Atlanta
$25	$30	$35	$35	$30	$35

An analysis was done to determine the average number of days a check floats when sent from seven different regions of the country to each of these six cities. The results of this analysis are summarized in the following table. This table indicates, for instance, that a check sent from the central region of the country to New York spends an average of 3 days in the mail and in processing before MasterDebt actually receives the cash for the check.

Average Days of Float Between Regions and Possible Lockbox Locations

	Sacramento	Denver	Chicago	Dallas	New York	Atlanta
Central	4	2	2	2	3	3
Mid-Atlantic	6	4	3	4	2	2
Midwest	3	2	3	2	5	4
Northeast	6	4	2	5	2	3
Northwest	2	3	5	4	6	7
Southeast	7	4	3	2	4	2
Southwest	2	3	6	2	7	6

Further analysis was done to determine the average amount of payments being sent from each region of the country. These results are given next:

Average Daily Payments (in $1,000s) by Region	
	Payments
Central	$45
Mid-Atlantic	$65
Midwest	$50
Northeast	$90
Northwest	$70
Southeast	$80
Southwest	$60

Thus, if payments from the Central Region are sent to New York, on any given day, there is an average of $135,000 in undeposited checks from the Central Region. Because MasterDebt can earn 15% on cash deposits, it would be losing $20,250 per year in potential interest on these checks alone.

a. Which of the six potential lockbox locations should MasterDebt use and to which lockbox location should each region be assigned?
b. How would your solution change if a maximum of four regions could be assigned to any lockbox location?

Removing Snow in Montreal

CASE 6-4

Snow removal and disposal are important and expensive activities in Montreal and many northern cities. While snow can be cleared from streets and sidewalks by plowing and shoveling, in prolonged subfreezing temperatures, the resulting banks of accumulated snow can impede pedestrian and vehicular traffic and must be removed.

To allow timely removal and disposal of snow, a city is divided up into several sectors and snow removal operations are carried out concurrently in each sector. In Montreal, accumulated snow is loaded onto trucks and hauled away to disposal sites (e.g., rivers, quarries, sewer chutes, surface holding areas). For contractual reasons, each sector may be assigned to only a *single* disposal site. (However, each disposal site may receive snow from multiple sectors.) The different types of disposal sites can accommodate different amounts of snow due to either the physical size of the disposal facility or environmental restrictions on the amount of snow (often contaminated by salt and de-icing chemicals) that can be dumped into rivers. The annual capacities for five different snow disposal sites are given in the following table (in 1,000s of cubic meters).

	Disposal Site				
	1	**2**	**3**	**4**	**5**
Capacity	350	250	500	400	200

The cost of removing and disposing of snow depends mainly on the distance it must be trucked. For planning purposes, the city of Montreal uses the straight-line distance between the center of each sector to each of the various disposal sites as an approximation of the cost involved in transporting snow between these locations. The following table summarizes these distances (in kilometers) for ten sectors in the city.

	Disposal Site				
Sector	1	2	3	4	5
1	3.4	1.4	4.9	7.4	9.3
2	2.4	2.1	8.3	9.1	8.8
3	1.4	2.9	3.7	9.4	8.6
4	2.6	3.6	4.5	8.2	8.9
5	1.5	3.1	2.1	7.9	8.8
6	4.2	4.9	6.5	7.7	6.1
7	4.8	6.2	9.9	6.2	5.7
8	5.4	6.0	5.2	7.6	4.9
9	3.1	4.1	6.6	7.5	7.2
10	3.2	6.5	7.1	6.0	8.3

Using historical snowfall data, the city is able to estimate the annual volume of snow requiring removal in each sector as four times the length of streets in the sectors in meters (i.e., it is assumed each linear meter of street generates 4 cubic meters of snow to remove over an entire year). The following table estimates the snow removal requirements (in 1,000s of cubic meters) for each sector in the coming year.

Estimated Annual Snow Removal Requirements									
1	2	3	4	5	6	7	8	9	10
153	152	154	138	127	129	111	110	130	135

a. Create a spreadsheet that Montreal could use to determine the most efficient snow removal plan for the coming year. Assume it costs \$0.10 to transport 1 cubic meter of snow 1 kilometer.
b. What is the optimal solution?
c. How much will it cost Montreal to implement your snow disposal plan?
d. Ignoring the capacity restrictions at the disposal sites, how many different assignments of sectors to disposal sites are possible?
e. Suppose Montreal can increase the capacity of a single disposal site by 100,000 cubic meters. Which disposal site's capacity (if any) should be increased and how much should the city be willing to pay to obtain this extra disposal capacity?

Source: Based on Campbell, J. and A. Langevin. "The Snow Disposal Assignment Problem." *Journal of the Operational Research Society*, 1995, pp. 919–929.