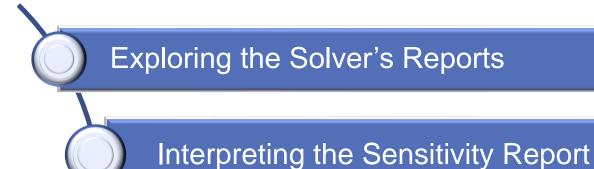
Topics covered:



Exploring the Sensitivity Assistant – Spider Plot & Solver Tables

A brief look at the Simplex Method



Introduction

- When solving an LP problem we assume that values of all model coefficients are known with certainty.
- Such certainty rarely exists.
- Sensitivity analysis helps answer questions about how sensitive the optimal solution is to changes in various coefficients in a model.

General Form of a Linear Programming (LP) Problem

MAX (or MIN): $c_1X_1 + c_2X_2 + ... + c_nX_n$

c_i's are costs or prices

Subject to:

$$a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \le b_1$$

a_{ij}'s are resource requirements

$$a_{k1}X_1 + a_{k2}X_2 + ... + a_{kn}X_n \le b_k$$

:

$$a_{m1}X_1 + a_{m2}X_2 + ... + a_{mn}X_n = b_m$$

• How sensitive is a solution to changes in the c_i , a_{ij} ,

and b_i ?

b_i's indicate resource availability



Approaches to Sensitivity Analysis

- Change the data and re-solve the model!
 - Sometimes this is the only practical approach.
- Solver also produces sensitivity reports that can answer various questions...



Solver's Sensitivity Report

- Answers questions about:
 - Amounts by which objective function coefficients can change without changing the optimal solution.
 - The impact on the optimal objective function value of changes in constrained resources.
 - The impact on the optimal objective function value of forced changes in decision variables.
 - The impact changes in constraint coefficients will have on the optimal solution.



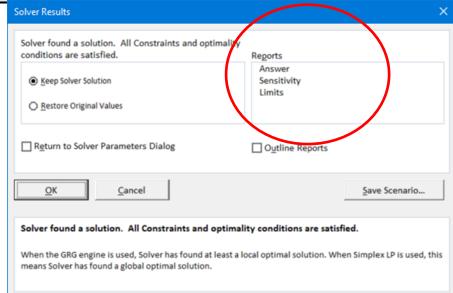
Once Again, We'll Use The Blue Ridge Hot Tubs Example...

MAX:
$$350X_1 + 300X_2$$
 } profit
S.T.: $1X_1 + 1X_2 <= 200$ } pumps
 $9X_1 + 6X_2 <= 1566$ } labor
 $12X_1 + 16X_2 <= 2880$ } tubing
 $X_1, X_2 >= 0$ } nonnegativity



Solver Results

| | Blue Ridge | e Hot Tubs | | |
|-------------------|------------|-------------|--------------|-----------|
| | Aqua-Spas | Hydro-Luxes | | |
| Number to Make | 122 | 78 | Total Profit | |
| Unit Profits | \$350 | \$300 | \$66,100 | |
| Constraints | | | Used | Available |
| - Pumps Required | 1 | 1 | 200 | 200 |
| - Labor Required | 9 | 6 | 1,566 | 1566 |
| - Tubing Required | 12 | 16 | 2,712 | 2880 |





Solver Reports

- Answer Report
- Sensitivity Report
- Limits Report

See file Lecture 3.xlsm



Answer Report

3 sections:

- Section 1: Final value of optimal solution obj.function.
- Section 2: Final values of decision variables.
- Section 3: How much of each resource is used.
- Names are determined by the worksheet names we have used.
- Note that the Answer Report gives us the same information as the worksheet output but in a more useful format for reporting purposes.

Answer Report

Value of Objective

| Objective | Cell (Max) | Function | | | | | _ | | |
|------------|----------------|-----------------|--------|----------|----------|------------------|-------------|----------------|-----|
| Cell | | Name | Origin | al Value | Fina | Value | • | | |
| \$D\$6 | Unit Profits T | otal Profit | | \$66,100 | | \$66,100 |) ~ | | |
| | | | | | | | | Onti | mal |
| Variable (| Cells | Decision | Vari | ables | | | | Opti Solu - | |
| Cell | | Name | Origin | al Value | Fina | Value | Integer | - G OIG | |
| \$B\$5 | Number to M | ake Aqua-Spas | | 122 | | 122 | Contin | | |
| \$C\$5 | Number to M | ake Hydro-Luxes | | 78 | | 78 | Contin | | |
| | | Resource | 9 | | | | | | |
| Constrain | its | constrair | nts | | | | | | |
| Cell | | Name | | Value | For | rmula | Status | Slack |) |
| \$D\$9 | Pumps Re | q'd Used | | 200 | \$D\$9< | =\$E\$9 | Binding | 0 | |
| \$D\$10 | Labor Req | 'd Used | | 1,566 | \$D\$10 | <=\$E\$10 | Binding | 0 | |
| \$D\$11 | Tubing Re | g'd Used | | 2,712 | \$D\$11 | <=\$E\$11 | Not Binding | 168 | |
| \$B\$5 | Number to M | ake Aqua-Spas | | 122 | \$B\$5>: | =0 | Not Binding | 122 | |
| \$C\$5 | Number to M | ake Hvdro-Luxes | | 78 | \$C\$5>: | =0 | Not Bindina | 78 | |

Non-negativity conditions

Answer report notes

- Section 1 and 2 together tell us the optimal values for the decision variables and the optimal value of the objective function (e.g. profit or return).
- Section 3 gives info. on the constraints:
 - Resource constraints:
 - Which are binding i.e. which resources are all used up
 - Which are non-binding i.e. some of the resource is not used
 - Non-negativity constraints
 - Which decision variables have values greater than their lower bound of zero



Answer Report for Hot Tubs

- Sections 1 and 2 tell us the optimal values for the decision variables
 - $-X_1$ = Aqua-spas = 122
 - $-X_2 = Hydro-Luxe = 78$
 - The optimal value of the objective function = \$66,100
- Section 3 gives info on the constraints:
 - Resource constraints- Which are binding/non-binding (i.e. which resources are all used up and which are not.)
 - Pumps and labour are binding.
 - Tubing is non-binding.
 - Non-negativity constraints: Both decision variables (Aqua-spas and Hydro-Luxes) are greater than their lower bound of zero.

The Sensitivity Report

The sensitivity report consists of 2 sections:

- The variable cells section
 - What happens when we change the values of the coefficients in the objective function
 - What happens if we include variables which are not part of the optimal solution
- The constraints section
 - What happens to the value of the objective function if we increase (or decrease) the amount of available resources



The Sensitivity Report

Microsoft Excel 16.0 Sensitivity Report

Worksheet: [Lecture 3.xlsm]Production Report (Solution)

Report Created: 4/08/2018 12:24:15 AM

Variable Cells

| | | Final | Reduced | Objective | Allowable | Allowable |
|--------|----------------------------|-------|---------|-------------|-----------|-----------|
| Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| \$B\$5 | Number to Make Aqua-Spas | 122 | 0 | 350 | 100 | 50 |
| \$C\$5 | Number to Make Hydro-Luxes | 78 | 0 | 300 | 50 | 66.6667 |

Constraints

| | | Final | Shadow | Constraint | Allowable | Allowable |
|---------|-------------------|-------|---------|------------|-----------|-----------|
| Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| \$D\$9 | Pumps Req'd Used | 200 | 200 | 200 | 7 | 26 |
| \$D\$10 | Labor Req'd Used | 1566 | 16.6667 | 1566 | 234 | 126 |
| \$D\$11 | Tubing Req'd Used | 2712 | 0 | 2880 | 1E+30 | 168 |

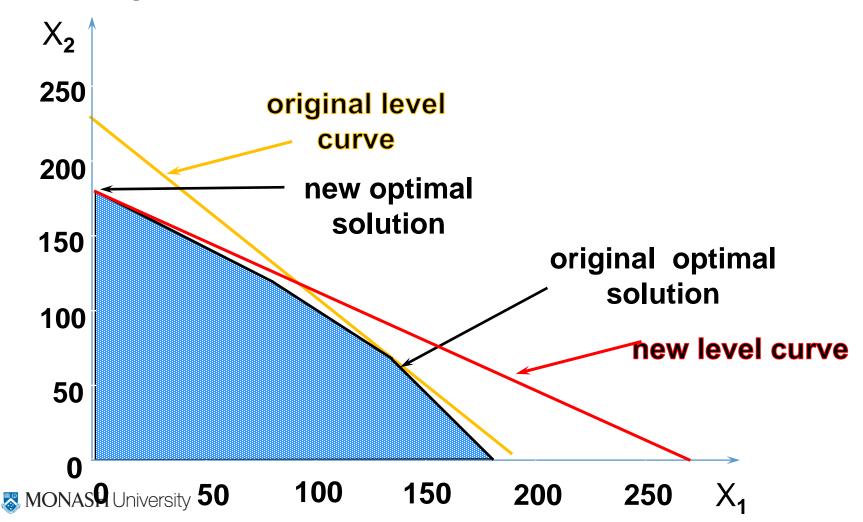


Changing the objective function coefficients

- The objective function is: MAX: $350 X_1 + 300 X_2$.
- Suppose we are not sure of the profit for X_1 and X_2 .
- I.e. what happens if we change one of the coefficients?
- We can see this diagrammatically and by looking at the sensitivity report.



How Changes in Objective Coefficients Change the Slope of the Level Curve



Sensitivity Report: Variable Cells

- Tells the amount by which each objective function coefficient can change without affecting the optimal solution – assuming all other coefficients remain constant.
- Max. increase in profit for Aqua-Spas is \$100 Max. decrease in profit for Aqua-Spas is \$50
- In the absence of degeneracy, any zero value in Allowable Increase or Decrease for any objective function coefficient indicates alternate optimal solutions exist.

Variable Cells

| | | Final | Reduced | Objective | Mowable | Allowable |
|--------|----------------------------|-------|---------|------------------|----------|-----------|
| Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| \$B\$5 | Number to Make Aqua-Spas | 122 | 0 | 350 | 100 | 50 |
| \$C\$5 | Number to Make Hydro-Luxes | 78 | 0 | 300 | 50 | 66.6667 |
| | | | | | | |



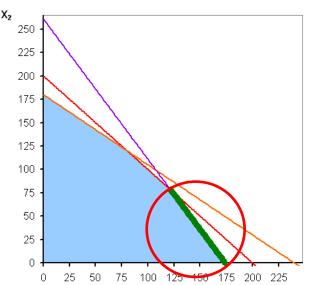
Changes in Objective Function Coefficients

- Values in the "Allowable Increase" and Allowable Decrease" columns for the Decision Cells indicate the amounts by which an objective function coefficient can change without changing the optimal solution (i.e. the optimal values of the decision variables), assuming all other coefficients remain constant.
- The value of the optimal solution however, does change.

Alternate Optimal Solutions

Note: Values of zero (0) in "Allowable Increase" or "Allowable Decrease" columns for the Changing Cells indicate that an alternate optimal solution exists

| Α | djustabl | e Cells | | | | | |
|---|----------|----------------------------|-------|---------|-------------|-----------|-----------|
| | | | Final | Reduced | Objective | Allowable | Allowable |
| | Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| | \$B\$5 | Number to make Aqua-Spas | 174 | 0 | 450 | 1E+28 | (0) |
| | \$C\$5 | Number to make Hydro-Luxes | 0 | 0 | 300 | 0 | 1E+30 |



e.g. (122,78) and (174,0) are both optimal



Changes in Constraint RHS Values

- The <u>shadow price</u> of a constraint indicates the amount by which the objective function value changes given a unit *increase* in the RHS value of the constraint, <u>assuming all other coefficients</u> remain constant.
- Shadow prices holds if RHS value of constraint falls within Allowable Increase or Decrease values.
- Shadow prices for non-binding constraints are always zero.

So what is shadow price...

- The shadow prices of resources equate the marginal value of the resources consumed with the marginal benefit of the goods being produced.
- The shadow prices of a resource is the amount over and above what you are currently paying for and which you would be willing to pay to acquire an additional unit of that resource.
- Resources in excess supply have a shadow price (or marginal value) of zero.

Sensitivity Report for Hot Tubs

- 1 unit increase in labour increases OF by \$16.67.
- 10 units increase in labour increases by \$16,67*10 = \$166.67.
- 1 unit decrease in labour decreases OF by \$16.67.
- Tubing is a non-binding constraint as there is a slack of 168 units (2880 2712) and therefore, has zero shadow price.

Constraints

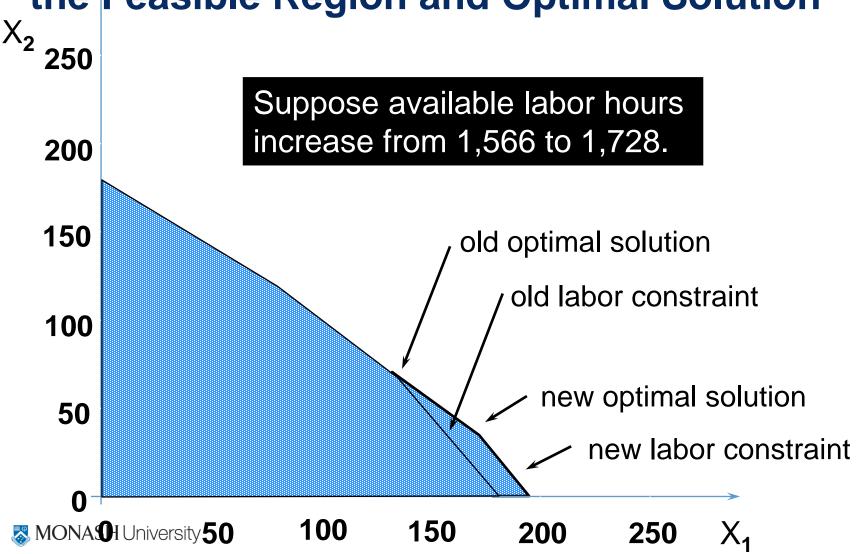
| | | Final | Shadow | Constraint | Allowable | Allowable |
|---------|------------------------|-------|--------|------------|------------------|------------------|
| Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| \$D\$9 | - Pumps Required Used | 200 | 200.00 | 200 | 7 | 26 |
| \$D\$10 | - Labor Required Used | 1,566 | 16.67 | 1566 | 234 | 126 |
| \$D\$11 | - Tubing Required Used | 2,712 | 0.00 | 2880 | 1E+30 | 168 |



Comments About Changes in Constraint RHS Values

- Shadow prices only indicate the changes that occur in the objective function value as RHS values change.
- Changing a RHS value for a binding constraint also changes the feasible region and the optimal solution (see graph on following slide).
- To find the optimal solution after changing a binding RHS value, you must re-solve the problem.

How Changing an RHS Value Can Change the Feasible Region and Optimal Solution



Sensitivity Report - Constraints

- Binding constraint:
 - A change in RHS value changes the feasible solutions & optimal solution. You have to re-solve model to find new optimal solution (the report does not tell you this!)
- Non-binding constraint:
 - The shadow price is always zero (because adding more of a resource that has not been used up is no help!)

Constraints

| | | Final | Shadow | Constraint | Allowable | Allowable |
|---------|------------------------|-------|--------|------------|-----------|-----------|
| Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| \$D\$9 | - Pumps Required Used | 200 | 200.00 | 200 | 7 | 26 |
| \$D\$10 | - Labor Required Used | 1,566 | 16.67 | 1566 | 234 | 126 |
| \$D\$11 | - Tubing Required Used | 2,712 | 0.00 | 2880 | 1E+30 | 168 |



Other Uses of Shadow Prices

- Suppose a new Hot Tub (the Typhoon-Lagoon) is being considered. It generates a marginal profit of \$320 and requires:
 - -1 pump (shadow price = \$200)
 - 8 hours of labor (shadow price = \$16.67)
 - 13 feet of tubing (shadow price = \$0)
- Q: Would it be profitable to produce any?

A:
$$\$320 - \$200*1 - \$16.67*8 - \$0*13 = -\$13.33 = No!$$

Reduced cost = Profit - \sum (shadow price x units of resources required)

Negative

What solver says...

Microsoft Excel 12.0 Sensitivity Report

Worksheet: [Fig4-8.xls]Production Report Created: 6/03/2009 4:24:12 Aqua spas = 350 - 200*1 - 16.67*9 - 0*12 = 0 Hydro-luxes = 300 - 200*1 - 16.67*6 - 0*16 = 0 Typhoon-lagoons = 320 - 200*1 - 16.67*8 - 0*13 = -13.33

Adjustable Cells

| | | Final | Re | educed | Objective | Allowable | Allowable |
|--------|--------------------------------|-------|----|--------|-------------|-----------|-----------|
| Cell | Name | Value | | Cost | Coefficient | Increase | Decrease |
| \$B\$5 | Number to make Aqua-Spas | 122 | | 0 | 350 | 100 | 20 |
| \$C\$5 | Number to make Hydro-Luxes | 78 | | 0 | 300 | 50 | 40 |
| \$D\$5 | Number to make Typhoon-Lagoons | 0 | | -13 | 320 | 13.33333 | 1E+30 |

Constraints

| | | Final | Shadow | Constraint | Allowable | Allowable |
|---------|-------------------|-------|--------|------------|-----------|-----------|
| Cell | Name | Value | Price | R.H. Side | Increase | Decrease |
| \$E\$9 | Pumps Req'd Used | 200 | 200 | 200 | 7 | 26 |
| \$E\$10 | Labor Req'd Used | 1566 | 17 | 1566 | 234 | 126 |
| \$E\$11 | Tubing Req'd Used | 2712 | 0 | 2880 | 1E+30 | 168 |



The Meaning of Reduced Costs

- The <u>reduced cost</u> for each product equals its per-unit marginal profit minus the per-unit value of the resources it consumes (priced at their shadow prices).
- Products whose marginal profits are less than the marginal value of the goods required for their production will not be produced in an optimal solution (unless a constraint forces the product to be produced)
- Reduced costs tell us by how much we would have to increase the profitability of an item before it would be included in the optimal solution (for a maximisation problem). And vice-versa for a minimisation problem.

Interpretation of Reduced Costs

- Current solution will remain optimal provided marginal profit on Typhoon-Lagoons is less than \$320 + \$13.33 = \$333.33.
- If marginal profit for Typhoon-Lagoons is greater than \$333.33, producing this product would be profitable and optimal solution would change.

Adjustable Cells

| | | Final | Reduced | Objective | Allowable | Allowable |
|--------|-------------------------------|-------|---------|-------------|-----------|-----------|
| Cell | Name | Value | Cost | Coefficient | Increase | Decrease |
| \$B\$5 | Number to make Aqua-Spa | 122 | 0.00 | 350 | 100 | 20 |
| \$C\$5 | Number to make Hydro-Luxe | 78 | 0.00 | 300 | 50 | 40 |
| \$D\$5 | Number to make Typhoon-Lagoon | 0 | -13.33 | 320 | 13.333333 | 1E+30 |



Analyzing Changes in Constraint Coefficients

Q: Suppose a Typhoon-Lagoon required only 7 labor hours rather than 8. Is it now profitable to produce any?

Q: What is the maximum amount of labor Typhoon-Lagoons could require and still be profitable?

A: We need \$320 - \$200*1 - \$16.67* L_3 - \$0*13 >=0

The above is true if $L_3 \le 120/16.67 = 7.20$ hrs



Simultaneous Changes in Objective Function Coefficients

- The <u>100% Rule</u> can be used to determine if the optimal solutions changes when more than one objective function coefficient changes.
- Two cases can occur:
 - Case 1: All variables with changed obj. coefficients have non-zero reduced costs.
 - Case 2: At least one variable with changed obj. coefficient has a reduced cost of zero.

Simultaneous Changes in Objective Function Coefficients: Case 1

 The current solution remains optimal provided the obj. coefficient changes are all within their Allowable Increase or Decrease.

(All variables with changed obj. coefficients have non-zero reduced costs.)

Simultaneous Changes in Objective Function Coefficients: Case 2

(At least one variable with changed obj. coefficient has a reduced cost of zero.)

For each variable compute:

$$\mathbf{r}_{j} = \begin{cases} \frac{\Delta c_{j}}{I_{j}}, & \text{if } \Delta c_{j} \ge 0\\ \frac{-\Delta c_{j}}{D_{j}}, & \text{if } \Delta c_{j} < 0 \end{cases}$$

- If more than one objective function coefficient changes, the current solution remains optimal provided the r_j sum to <= 1.
- If the r_j sum to > 1, the current solution, might remain optimal, but this is not guaranteed.

A Warning About Degeneracy

- The solution to an LP problem is degenerate if the Allowable Increase or Decrease on any constraint is zero (0).
- When the solution is degenerate:
 - 1. The methods mentioned earlier for detecting alternate optimal solutions cannot be relied upon.
 - 2. The reduced costs for the changing cells may not be unique. Also, the objective function coefficients for changing cells must change by at least as much as (and possibly more than) their respective reduced costs before the optimal solution would change.

- When the solution is degenerate (cont'd):
 - 3. The allowable increases and decreases for the objective function coefficients still hold and, in fact, the coefficients may have to be changed beyond the allowable increase and decrease limits before the optimal solution changes.
 - 4. The given shadow prices and their ranges may still be interpreted in the usual way but they may not be unique. That is, a different set of shadow prices and ranges may also apply to the problem (even if the optimal solution is unique!).

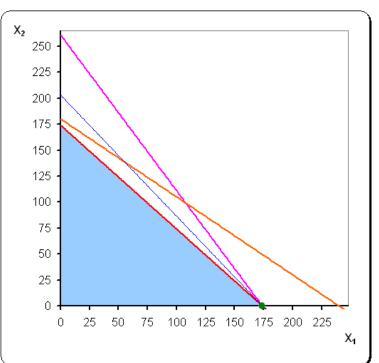
Degenerate case

Setting # pumps to 174.

| | | | _ | |
|-------------------|------------|-------------|--------------|-----------|
| | Blue Ridge | Hot Tubs | | |
| | Aqua-Spas | Hydro-Luxes | | |
| Number to Make | 174 | 0 | Total Profit | |
| Unit Profits | \$350 | \$300 | \$60,900 | |
| | | | | |
| Constraints | | _ | Used | Available |
| - Pumps Required | 1 | 1 | \$174 | 174 |
| - Labor Required | 9 | 6 | \$1,566 | 1566 |
| - Tubing Required | 12 | 16 | \$2,088 | 2880 |

Adjustable Cells

| | | Final | Reduced | |
|--------|----------------------------|-------|---------|--|
| Cell | Name | Value | Cost | |
| \$B\$5 | Number to Make Aqua-Spas | 174 | 0 | |
| \$C\$5 | Number to Make Hydro-Luxes | 0 | 0 | |



Constraints

| | | Final | Shadow | Constraint | Allowable | Allowable | |
|---------|---|---------|--------|------------|-----------|-----------|--|
| Cell | Name | Value | Price | R.H. Side | Increase | Decrease | |
| \$D\$9 | - Pumps Required Used | \$174 | \$200 | 174 | 33 | 0 | |
| \$D\$10 | Labor Required Used | \$1,566 | \$17 | 1566 | 0 | 522 | |
| \$D\$11 | - Tubing Required Used | \$2,088 | \$0 | 2880 | 1E+30 | 792 | |



Limits report

 Summarises the value of the objective function as each variable cell ranges through its limits (and the others remain constant).

See file Hot Tub Analysis.xlsm

| Target | | | | |
|--------|---------------------------|----------|--|--|
| Cell | Name | Value | | |
| \$D\$6 | Unit Profits Total Profit | \$66,100 | | |

| Adjustable Cell Name Value | | | | Target Result | • • | Target Result |
|-------------------------------|----------------------------|-----|---|------------------|-----|------------------|
| \$B\$5 | Number to Make Aqua-Spas | 122 | 0 | 23400 | 122 | 66100 |
| \$C\$5 | Number to Make Hydro-Luxes | 78 | 0 | 42700 | 78 | 66100 |

The Sensitivity Assistant

- Sensitivity.xla is an add-in that allows you to create:
 - Spider Tables & Plots
 - Summarize the optimal value for one output cell as individual changes are made to various input cells.
 - Solver Tables
 - Summarize the optimal value of multiple output cells as changes are made to a single input cell.

The Sensitivity Assistant

See files:

Spider plot and Solver Table.xlsx

(Also refer to file: "How to install sensitivity assistant.pdf" for instructions)



Tutorial 2 this week:

- LP with Spreadsheet Modelling
 - Goals for Good Spreadsheet Design
 - Implementing LP Problems in Spreadsheet
 - Solution Analysis

