CHAPTER 10

Distribution and Network Models

CONTENTS

The models discussed in this chapter belong to a special class of linear programming problems called *network flow* problems. We begin by discussing models commonly encountered when dealing with problems related to supply chains, specifically transportation and transshipment problems. We then consider three other types of network problems: assignment problems, shortest-route problems, and maximal flow problems.

In each case, we present a graphical representation of the problem in the form of a *network*. We then show how the problem can be formulated and solved as a linear program. In the last section of the chapter we present a production and inventory problem that is an interesting application of the transshipment problem.

10.1 Supply Chain Models

A **supply chain** describes the set of all interconnected resources involved in producing and distributing a product. For instance, a supply chain for automobiles could include raw material producers, automotive-parts suppliers, distribution centers for storing automotive parts, assembly plants, and car dealerships. All the materials needed to produce a finished automobile must flow through the supply chain. In general, supply chains are designed to satisfy customer demand for a product at minimum cost. Those that control the supply chain must make decisions such as where to produce the product, how much should be produced, and where it should be sent. We will look at two specific types of problems common in supply chain models that can be solved using linear programing: transportation problems and transshipment problems.

Transportation Problem

The **transportation problem** arises frequently in planning for the distribution of goods and services from several supply locations to several demand locations. Typically, the quantity of goods available at each supply location (origin) is limited, and the quantity of goods needed at each of several demand locations (destinations) is known. The usual objective in a transportation problem is to minimize the cost of shipping goods from the origins to the destinations.

Let us illustrate by considering a transportation problem faced by Foster Generators. This problem involves the transportation of a product from three plants to four distribution centers. Foster Generators operates plants in Cleveland, Ohio; Bedford, Indiana; and York, Pennsylvania. Production capacities over the next three-month planning period for one particular type of generator are as follows:

Origin	Plant		Three-Month Production Capacity (units)
1	Cleveland		5,000
2	Bedford		6,000
3	York		2,500
		Total	13,500

The firm distributes its generators through four regional distribution centers located in Boston, Chicago, St. Louis, and Lexington; the three-month forecast of demand for the distribution centers is as follows:

Destination	Distribution Center	Three-Month Demand Forecast (units)
1	Boston	6,000
2	Chicago	4,000
3	St. Louis	2,000
4	Lexington	1,500
	Total	13,500

Management would like to determine how much of its production should be shipped from each plant to each distribution center. Figure 10.1 shows graphically the 12 distribution routes Foster can use. Such a graph is called a **network**; the circles are referred to as

FIGURE 10.1 THE NETWORK REPRESENTATION OF THE FOSTER GENERATORS TRANSPORTATION PROBLEM

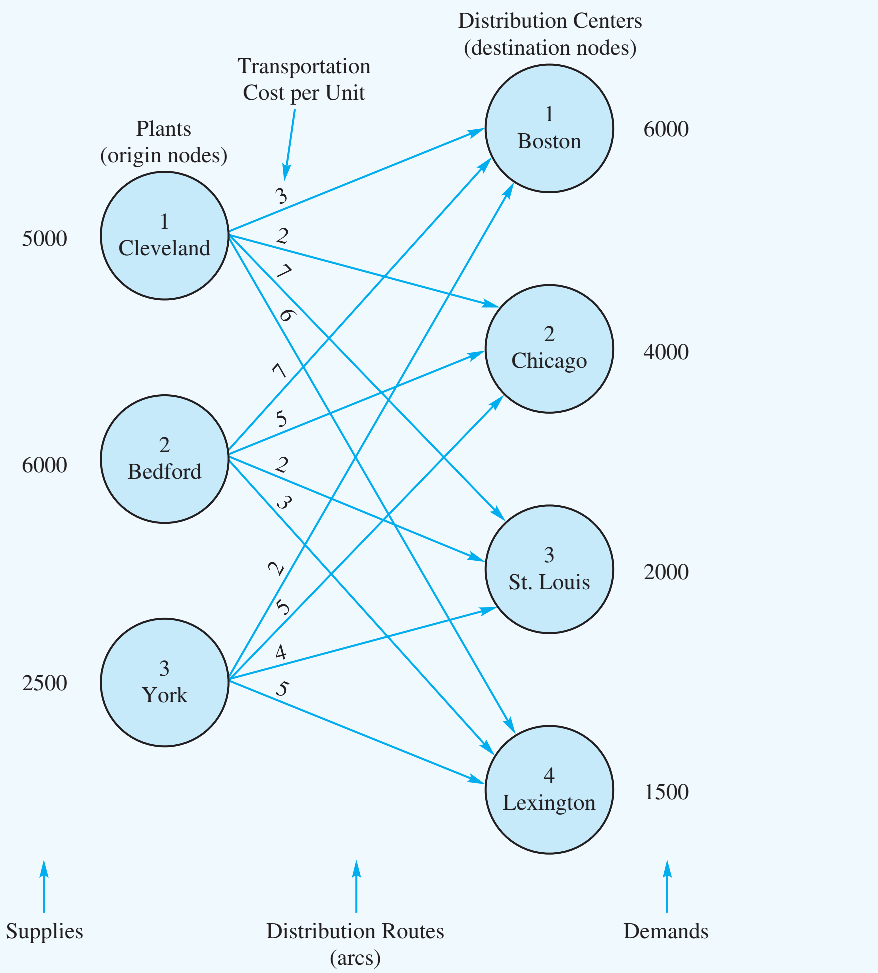

TABLE 10.1 TRANSPORTATION COST PER UNIT FOR THE FOSTER GENERATORS TRANSPORTATION PROBLEM

	Destination			
Origin	**Boston**	**Chicago**	**St. Louis**	**Lexington**
Cleveland	3	2	7	6
Bedford	7	5	2	3
York	2	5	4	5

nodes and the lines connecting the nodes as **arcs**. Each origin and destination is represented by a node, and each possible shipping route is represented by an arc. The amount of the supply is written next to each origin node, and the amount of the demand is written next to each destination node. The goods shipped from the origins to the destinations represent the flow in the network. Note that the direction of flow (from origin to destination) is indicated by the arrows.

Try Problem 1 for practice in developing a network model of a transportation problem.

For Foster's transportation problem, the objective is to determine the routes to be used and the quantity to be shipped via each route that will provide the minimum total transportation cost. The cost for each unit shipped on each route is given in Table 10.1 and is shown on each arc in Figure 10.1.

A linear programming model can be used to solve this transportation problem. We use double-subscripted decision variables, with x_{11} denoting the number of units shipped from origin 1 (Cleveland) to destination 1 (Boston), x_{12} denoting the number of units shipped from origin 1 (Cleveland) to destination 2 (Chicago), and so on. In general, the decision variables for a transportation problem having m origins and n destinations are written as follows:

The first subscript identifies the "from" node of the corresponding arc and the second subscript identifies the "to" node of the arc.

$$x_{ij} = \text{number of units shipped from origin } i \text{ to destination } j$$
$$\text{where } i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n$$

Because the objective of the transportation problem is to minimize the total transportation cost, we can use the cost data in Table 10.1 or on the arcs in Figure 10.1 to develop the following cost expressions:

$$\begin{aligned}
&\text{Transportation costs for}\\
&\text{units shipped from Cleveland} = 3x_{11} + 2x_{12} + 7x_{13} + 6x_{14}\\
&\text{Transportation costs for}\\
&\text{units shipped from Bedford} \quad = 7x_{21} + 5x_{22} + 2x_{23} + 3x_{24}\\
&\text{Transportation costs for}\\
&\text{units shipped from York} \qquad = 2x_{31} + 5x_{32} + 4x_{33} + 5x_{34}
\end{aligned}$$

The sum of these expressions provides the objective function showing the total transportation cost for Foster Generators.

Transportation problems need constraints because each origin has a limited supply and each destination has a demand requirement. We consider the supply constraints first. The capacity at the Cleveland plant is 5000 units. With the total number of units shipped from the Cleveland plant expressed as $x_{11} + x_{12} + x_{13} + x_{14}$, the supply constraint for the Cleveland plant is

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 5000 \quad \text{Cleveland supply}$$

With three origins (plants), the Foster transportation problem has three supply constraints. Given the capacity of 6000 units at the Bedford plant and 2500 units at the York plant, the two additional supply constraints are

$$x_{21} + x_{22} + x_{23} + x_{24} \leq 6000 \quad \text{Bedford supply}$$
$$x_{31} + x_{32} + x_{33} + x_{34} \leq 2500 \quad \text{York supply}$$

With the four distribution centers as the destinations, four demand constraints are needed to ensure that destination demands will be satisfied:

To obtain a feasible solution, the total supply must be greater than or equal to the total demand.

$$x_{11} + x_{21} + x_{31} = 6000 \quad \text{Boston demand}$$
$$x_{12} + x_{22} + x_{32} = 4000 \quad \text{Chicago demand}$$
$$x_{13} + x_{23} + x_{33} = 2000 \quad \text{St. Louis demand}$$
$$x_{14} + x_{24} + x_{34} = 1500 \quad \text{Lexington demand}$$

Combining the objective function and constraints into one model provides a 12-variable, 7-constraint linear programming formulation of the Foster Generators transportation problem:

$$\begin{array}{ll}
\text{Min} & 3x_{11} + 2x_{12} + 7x_{13} + 6x_{14} + 7x_{21} + 5x_{22} + 2x_{23} + 3x_{24} + 2x_{31} + 5x_{32} + 4x_{33} + 5x_{34} \\
\text{s.t.} & \\
& x_{11} + x_{12} + x_{13} + x_{14} \leq 5000 \\
& x_{21} + x_{22} + x_{23} + x_{24} \leq 6000 \\
& x_{31} + x_{32} + x_{33} + x_{34} \leq 2500 \\
& x_{11} + x_{21} + x_{31} = 6000 \\
& x_{12} + x_{22} + x_{32} = 4000 \\
& x_{13} + x_{23} + x_{33} = 2000 \\
& x_{14} + x_{24} + x_{34} = 1500 \\
& x_{ij} \geq 0 \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4
\end{array}$$

Comparing the linear programming formulation to the network in Figure 10.1 leads to several observations: All the information needed for the linear programming formulation is on the network. Each node has one constraint and each arc has one variable. The sum of the variables corresponding to arcs from an origin node must be less than or equal to the origin's supply, and the sum of the variables corresponding to the arcs into a destination node must be equal to the destination's demand.

Can you now use Excel to solve a linear programming model of a transportation problem? Try Problem 2.

We solved the Foster Generators problem using Excel Solver. The optimal objective function values and optimal decision variable values are shown in Figure 10.2, which indicates that the minimum total transportation cost is $39,500. The values for the decision variables show the optimal amounts to ship over each route. For example, 3500 units should be shipped from Cleveland to Boston, and 1500 units should be shipped from Cleveland to Chicago. Other values of the decision variables indicate the remaining shipping quantities and routes. Table 10.2 shows the minimum cost transportation schedule, and Figure 10.3 summarizes the optimal solution on the network.

Problem Variations

The Foster Generators problem illustrates use of the basic transportation model. Variations of the basic transportation model may involve one or more of the following situations:

1. Total supply not equal to total demand
2. Maximization objective function
3. Route capacities or route minimums
4. Unacceptable routes

FIGURE 10.2 OPTIMAL SOLUTION FOR THE FOSTER GENERATORS TRANSPORTATION PROBLEM

Objective Cell (Min)

Name	Original Value	Final Value
Minimize Total Cost	0.000	39500.000

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
X11	Cleveland to Boston	0.000	3500.000	Contin
X12	Cleveland to Chicago	0.000	1500.000	Contin
X13	Cleveland to St. Louis	0.000	0.000	Contin
X14	Cleveland to Lexington	0.000	0.000	Contin
X21	Bedford to Boston	0.000	0.000	Contin
X22	Bedford to Chicago	0.000	2500.000	Contin
X23	Bedford to St. Louis	0.000	2000.000	Contin
X24	Bedford to Lexington	0.000	1500.000	Contin
X31	York to Boston	0.000	2500.000	Contin
X32	York to Chicago	0.000	0.000	Contin
X33	York to St. Louis	0.000	0.000	Contin
X34	York to Lexington	0.000	0.000	Contin

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TABLE 10.2 OPTIMAL SOLUTION TO THE FOSTER GENERATORS TRANSPORTATION PROBLEM

Route				
From	**To**	**Units Shipped**	**Cost per Unit**	**Total Cost**
Cleveland	Boston	3500	\$3	\$10,500
Cleveland	Chicago	1500	\$2	\$ 3,000
Bedford	Chicago	2500	\$5	\$12,500
Bedford	St. Louis	2000	\$2	\$ 4,000
Bedford	Lexington	1500	\$3	\$ 4,500
York	Boston	2500	\$2	\$ 5,000
				\$39,500

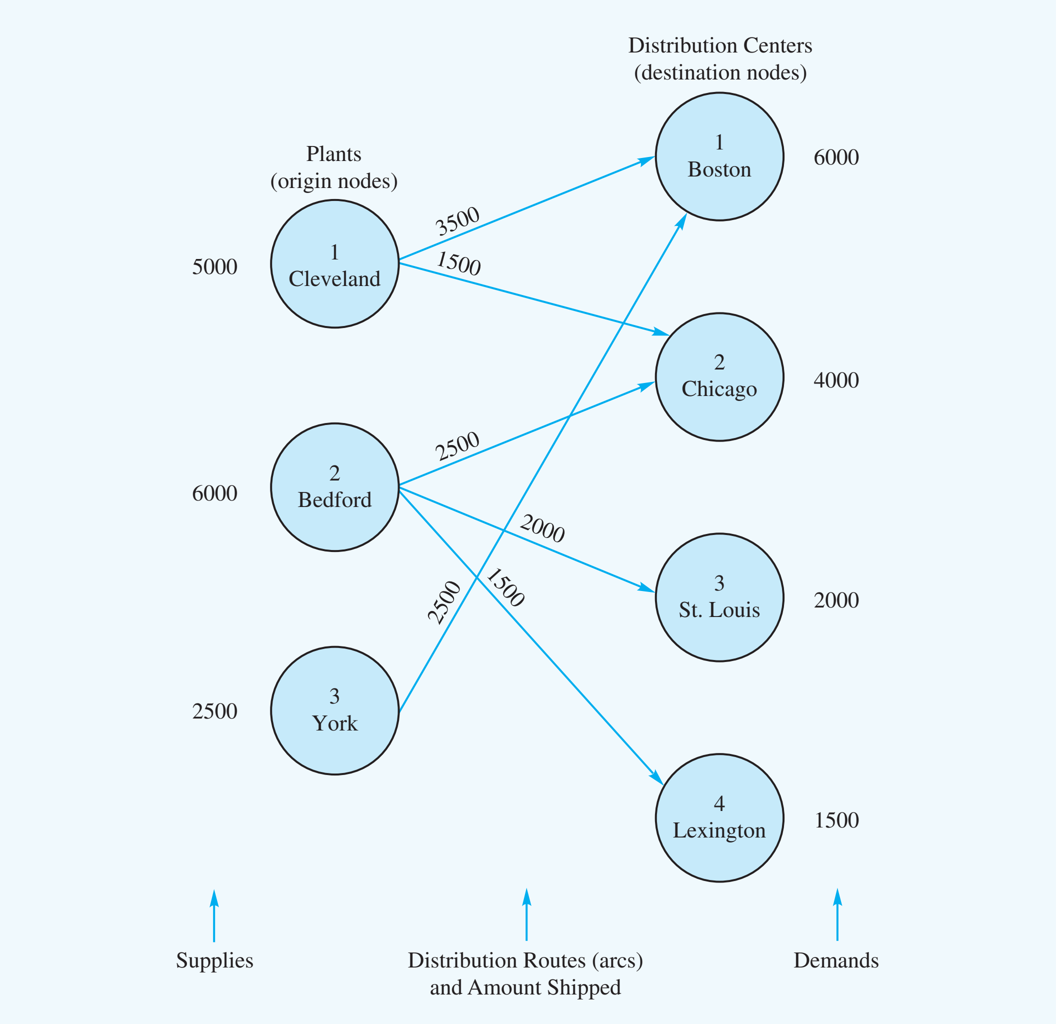
With slight modifications in the linear programming model, we can easily accommodate these situations.

Total Supply Not Equal to Total Demand Often *the total supply is not equal to the total demand.* If total supply exceeds total demand, no modification in the linear programming formulation is necessary. Excess supply will appear as slack in the linear programming solution. Slack for any particular origin can be interpreted as the unused supply or amount not shipped from the origin.

Whenever total supply is less than total demand, the model does not determine how the unsatisfied demand is handled (e.g., backorders). The manager must handle this aspect of the problem.

If total supply is less than total demand, the linear programming model of a transportation problem will not have a feasible solution. In this case, we modify the network representation by adding a **dummy origin** with a supply equal to the difference between the total demand and the total supply. With the addition of the dummy origin and an arc from

FIGURE 10.3 NETWORK DIAGRAM FOR THE OPTIMAL SOLUTION TO THE FOSTER GENERATORS TRANSPORTATION PROBLEM



the dummy origin to each destination, the linear programming model will have a feasible solution. A zero cost per unit is assigned to each arc leaving the dummy origin so that the value of the optimal solution for the revised problem will represent the shipping cost for the units actually shipped (no shipments actually will be made from the dummy origin). When the optimal solution is implemented, the destinations showing shipments being received from the dummy origin will be the destinations experiencing a shortfall or unsatisfied demand.

Try Problem 6 for practice with a case in which demand is greater than supply with a maximization objective.

Maximization Objective Function In some transportation problems, the objective is to find a solution that maximizes profit or revenue. Using the values for profit or revenue per unit as coefficients in the objective function, we simply solve a maximization rather than a minimization linear program. This change does not affect the constraints.

Route Capacities or Route Minimums The linear programming formulation of the transportation problem also can accommodate capacities or minimum quantities for one or more of the routes. For example, suppose that in the Foster Generators problem the York–Boston route (origin 3 to destination 1) had a capacity of 1000 units because of limited space availability on its normal mode of transportation. With x_{31} denoting the amount shipped from York to Boston, the route capacity constraint for the York–Boston route would be

$$x_{31} \leq 1000$$

Similarly, route minimums can be specified. For example,

$$x_{22} \geq 2000$$

would guarantee that a previously committed order for a Bedford–Chicago delivery of at least 2000 units would be maintained in the optimal solution.

Unacceptable Routes Finally, establishing a route from every origin to every destination may not be possible. To handle this situation, we simply drop the corresponding arc from the network and remove the corresponding variable from the linear programming formulation. For example, if the Cleveland–St. Louis route were unacceptable or unusable, the arc from Cleveland to St. Louis could be dropped in Figure 10.1, and x_{13} could be removed from the linear programming formulation. Solving the resulting 11-variable, 7-constraint model would provide the optimal solution while guaranteeing that the Cleveland–St. Louis route is not used.

A General Linear Programming Model

To show the general linear programming model for a transportation problem with m origins and n destinations, we use the following notation:

$$
\begin{aligned}
x_{ij} &= \text{number of units shipped from origin } i \text{ to destination } j \\
c_{ij} &= \text{cost per unit of shipping from origin } i \text{ to destination } j \\
s_i &= \text{supply or capacity in units at origin } i \\
d_j &= \text{demand in units at destination } j
\end{aligned}
$$

The general linear programming model is as follows:

$$
\begin{aligned}
&\text{Min} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\
&\text{s.t.} \\
&\qquad \sum_{j=1}^{n} x_{ij} \leq s_i \qquad i = 1, 2, \ldots, m \quad \text{Supply} \\
&\qquad \sum_{i=1}^{m} x_{ij} = d_j \qquad j = 1, 2, \ldots, n \quad \text{Demand} \\
&\qquad x_{ij} \geq 0 \qquad \text{for all } i \text{ and } j
\end{aligned}
$$

As mentioned previously, we can add constraints of the form $x_{ij} \leq L_{ij}$ if the route from origin i to destination j has capacity L_{ij}. A transportation problem that includes constraints of this type is called a **capacitated transportation problem**. Similarly, we can add route minimum constraints of the form $x_{ij} \geq M_{ij}$ if the route from origin i to destination j must handle at least M_{ij} units.

The Q.M. in Action, Optimizing Freight Car Assignments at Union Pacific, describes how Union Pacific railroad used an optimization model to solve a transportation problem of assigning empty freight cars to customer requests.

Q.M. in ACTION

*OPTIMIZING FREIGHT CAR ASSIGNMENTS AT UNION PACIFIC**

Union Pacific (UP) is one of the largest railroads in North America. It owns over 100,000 freight cars, which it uses to service its customers via a network of over 30,000 miles of railroad track. In response to customer demand, UP moves empty freight cars to its customer locations, where the cars are loaded. UP then transports the loaded cars to destinations designated by the customers.

At any point in time, Union Pacific may have hundreds of customer requests for empty freight cars to transport their products. Empty freight cars are typically scattered throughout UP's rail network at previous delivery destinations. A day-to-day decision faced by UP operations managers is how to assign these empty freight cars to current freight car requests from its customers. The assignments need to be cost effective but also must meet the customers' needs in terms of service time.

UP partnered with researchers from Purdue University to develop an optimization model to assist with the empty freight car assignment problem. In order to be useful, the model had to be simple enough to be solved quickly and had to run within UP's existing information systems. A transportation model was developed, with supply being the empty freight cars at their current locations and demand being the current and forecasted requests at the customer locations. The objective function includes not just the cost of transporting the cars, but other factors such as early and late delivery penalties and customer priority. This allows the managers to trade off a variety of factors with the cost of assignments to ensure that the proper level of service is achieved. The model outputs the number of empty cars to move from each current location to the locations of customers requesting cars. The model is used on a daily basis for operations planning and is also used to study the potential impact of changes in operational policies.

*Based on A. Narisetty et al., "An Optimization Model for Empty Freight Car Assignment at Union Pacific Railroad," *Interfaces* 38, no. 2 (March/April 2008): 89–102.

Transshipment Problem

The **transshipment problem** is an extension of the transportation problem in which intermediate nodes, referred to as *transshipment nodes*, are added to account for locations such as warehouses. In this more general type of distribution problem, shipments may be made between any pair of the three general types of nodes: origin nodes, transshipment nodes, and destination nodes. For example, the transshipment problem permits shipments of goods from origins to intermediate nodes and on to destinations, from one origin to another origin, from one intermediate location to another, from one destination location to another, and directly from origins to destinations.

As was true for the transportation problem, the supply available at each origin is limited, and the demand at each destination is specified. The objective in the transshipment problem is to determine how many units should be shipped over each arc in the network so that all destination demands are satisfied with the minimum possible transportation cost.

Try Problem 11, part (a), for practice in developing a network representation of a transshipment problem.

Let us consider the transshipment problem faced by Ryan Electronics. Ryan is an electronics company with production facilities in Denver and Atlanta. Components produced at either facility may be shipped to either of the firm's regional warehouses, which are located in Kansas City and Louisville. From the regional warehouses, the firm supplies retail outlets in Detroit, Miami, Dallas, and New Orleans. The key features of the problem are shown in the network model depicted in Figure 10.4. Note that the supply at each origin and demand at each destination are shown in the left and right margins, respectively. Nodes 1 and 2 are the origin nodes; nodes 3 and 4 are the transshipment nodes; and nodes 5, 6, 7, and 8 are the destination nodes. The transportation cost per unit for each distribution route is shown in Table 10.3 and on the arcs of the network model in Figure 10.4.

FIGURE 10.4 NETWORK REPRESENTATION OF THE RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

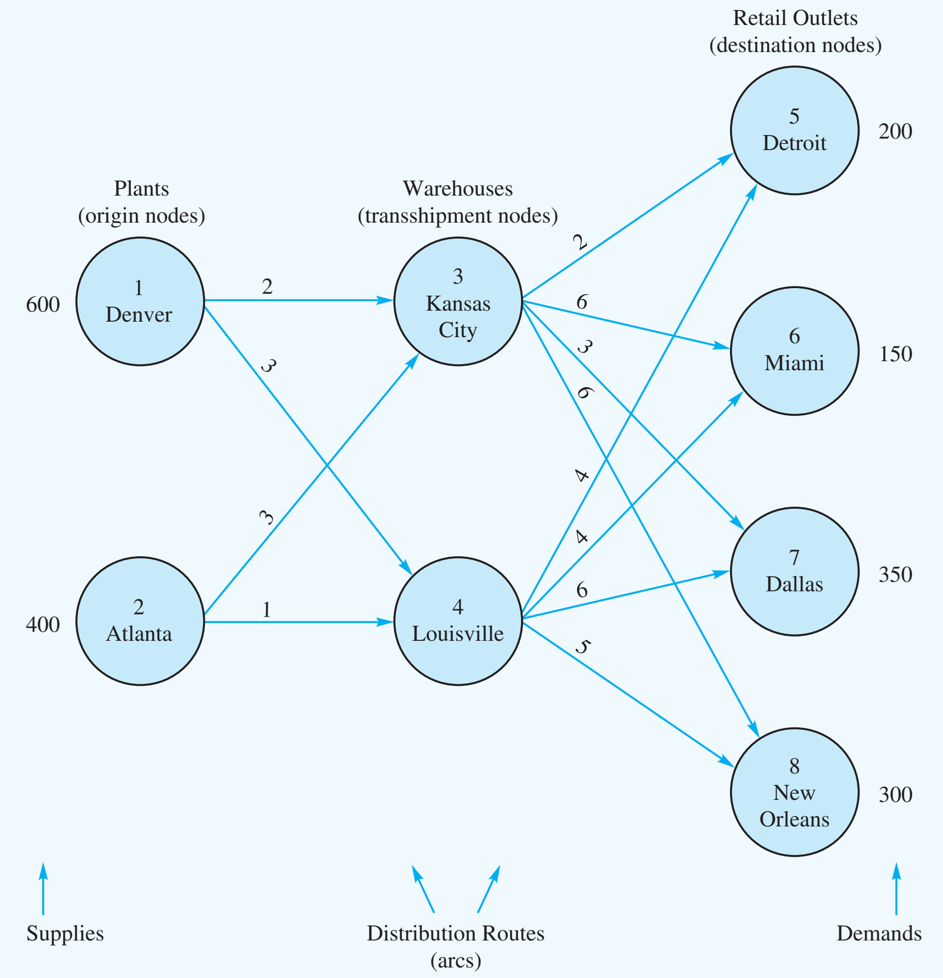


TABLE 10.3 TRANSPORTATION COST PER UNIT FOR THE RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

	Warehouse	
Plant	**Kansas City**	**Louisville**
Denver	2	3
Atlanta	3	1

	Retail Outlet			
Warehouse	**Detroit**	**Miami**	**Dallas**	**New Orleans**
Kansas City	2	6	3	6
Louisville	4	4	6	5

As with the transportation problem, we can formulate a linear programming model of the transshipment problem from a network representation. Again, we need a constraint for each node and a variable for each arc. Let x_{ij} denote the number of units shipped from node i to node j. For example, x_{13} denotes the number of units shipped from the Denver plant to the Kansas City warehouse, x_{14} denotes the number of units shipped from the Denver plant to the Louisville warehouse, and so on. Because the supply at the Denver plant is 600 units, the amount shipped from the Denver plant must be less than or equal to 600. Mathematically, we write this supply constraint as

$$x_{13} + x_{14} \leq 600$$

Similarly, for the Atlanta plant we have

$$x_{23} + x_{24} \leq 400$$

We now consider how to write the constraints corresponding to the two transshipment nodes. For node 3 (the Kansas City warehouse), we must guarantee that the number of units shipped out must equal the number of units shipped into the warehouse. If

$$\begin{array}{l}\text{Number of units}\\ \text{shipped out of node } 3 = x_{35} + x_{36} + x_{37} + x_{38}\end{array}$$

and

$$\begin{array}{l}\text{Number of units}\\ \text{shipped into node } 3 = x_{13} + x_{23}\end{array}$$

we obtain

$$x_{35} + x_{36} + x_{37} + x_{38} = x_{13} + x_{23}$$

Placing all the variables on the left-hand side provides the constraint corresponding to node 3 as

$$-x_{13} - x_{23} + x_{35} + x_{36} + x_{37} + x_{38} = 0$$

Similarly, the constraint corresponding to node 4 is

$$-x_{14} - x_{24} + x_{45} + x_{46} + x_{47} + x_{48} = 0$$

To develop the constraints associated with the destination nodes, we recognize that for each node the amount shipped to the destination must equal the demand. For example, to satisfy the demand for 200 units at node 5 (the Detroit retail outlet), we write

$$x_{35} + x_{45} = 200$$

Similarly, for nodes 6, 7, and 8, we have

$$\begin{aligned} x_{36} + x_{46} &= 150 \\ x_{37} + x_{47} &= 350 \\ x_{38} + x_{48} &= 300 \end{aligned}$$

Try Problem 11, parts (b) and (c), for practice in developing the linear programming model and in solving a transshipment problem on the computer.

As usual, the objective function reflects the total shipping cost over the 12 shipping routes. Combining the objective function and constraints leads to a 12-variable, 8-constraint linear programming model of the Ryan Electronics transshipment problem (see Figure 10.5). Figure 10.6 shows the optimal solution from the answer report, and Table 10.4 summarizes the optimal solution.

As mentioned at the beginning of this section, in the transshipment problem, arcs may connect any pair of nodes. All such shipping patterns are possible in a transshipment

FIGURE 10.5 LINEAR PROGRAMMING FORMULATION OF THE RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

$$\text{Min } 2x_{13} + 3x_{14} + 3x_{23} + 1x_{24} + 2x_{35} + 6x_{36} + 3x_{37} + 6x_{38} + 4x_{45} + 4x_{46} + 6x_{47} + 5x_{48}$$

s.t.

$$\begin{array}{l}
\left.\begin{array}{ll}
x_{13} + x_{14} & \leq 600 \\
x_{23} + x_{24} & \leq 400
\end{array}\right\} \text{Origin node constraints} \\
\left.\begin{array}{ll}
-x_{13} - x_{23} + x_{35} + x_{36} + x_{37} + x_{38} & = 0 \\
-x_{14} - x_{24} + x_{45} + x_{46} + x_{47} + x_{48} & = 0
\end{array}\right\} \text{Transshipment node constraints} \\
\left.\begin{array}{ll}
x_{35} + x_{45} & = 200 \\
x_{36} + x_{46} & = 150 \\
x_{37} + x_{47} & = 350 \\
x_{38} + x_{48} & = 300
\end{array}\right\} \text{Destination node constraints} \\
x_{ij} \geq 0 \text{ for all } i \text{ and } j
\end{array}$$

FIGURE 10.6 OPTIMAL SOLUTION FOR THE RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

Objective Cell (Min)

Name	Original Value	Final Value
Minimize Total Cost	0.000	5200.000

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
X13	Denver–Kansas City	0.000	550.000	Contin
X14	Denver–Louisville	0.000	50.000	Contin
X23	Atlanta–Kansas City	0.000	0.000	Contin
X24	Atlanta–Louisville	0.000	400.000	Contin
X35	Kansas City–Detroit	0.000	200.000	Contin
X36	Kansas City–Miami	0.000	0.000	Contin
X37	Kansas City–Dallas	0.000	350.000	Contin
X38	Kansas City–New Orleans	0.000	0.000	Contin
X45	Louisville–Detroit	0.000	0.000	Contin
X46	Louisville–Miami	0.000	150.000	Contin
X47	Louisville–Dallas	0.000	0.000	Contin
X48	Louisville–New Orleans	0.000	300.000	Contin

WEB file

Ryan

problem. We still require only one constraint per node, but the constraint must include a variable for every arc entering or leaving the node. For origin nodes, the sum of the shipments out minus the sum of the shipments in must be less than or equal to the origin supply. For destination nodes, the sum of the shipments in minus the sum of the shipments out must equal demand. For transshipment nodes, the sum of the shipments out must equal the sum of the shipments in, as before.

For an illustration of this more general type of transshipment problem, let us modify the Ryan Electronics problem. Suppose that it is possible to ship directly from Atlanta to New Orleans at \$4 per unit and from Dallas to New Orleans at \$1 per unit. The network model corresponding to this modified Ryan Electronics problem is shown in Figure 10.7,

TABLE 10.4 OPTIMAL SOLUTION TO THE RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

Route			Cost	
From	**To**	**Units Shipped**	**per Unit**	**Total Cost**
Denver	Kansas City	550	\$2	\$1100
Denver	Louisville	50	\$3	\$ 150
Atlanta	Louisville	400	\$1	\$ 400
Kansas City	Detroit	200	\$2	\$ 400
Kansas City	Dallas	350	\$3	\$1050
Louisville	Miami	150	\$4	\$ 600
Louisville	New Orleans	300	\$5	\$1500
				\$5200

FIGURE 10.7 NETWORK REPRESENTATION OF THE MODIFIED RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

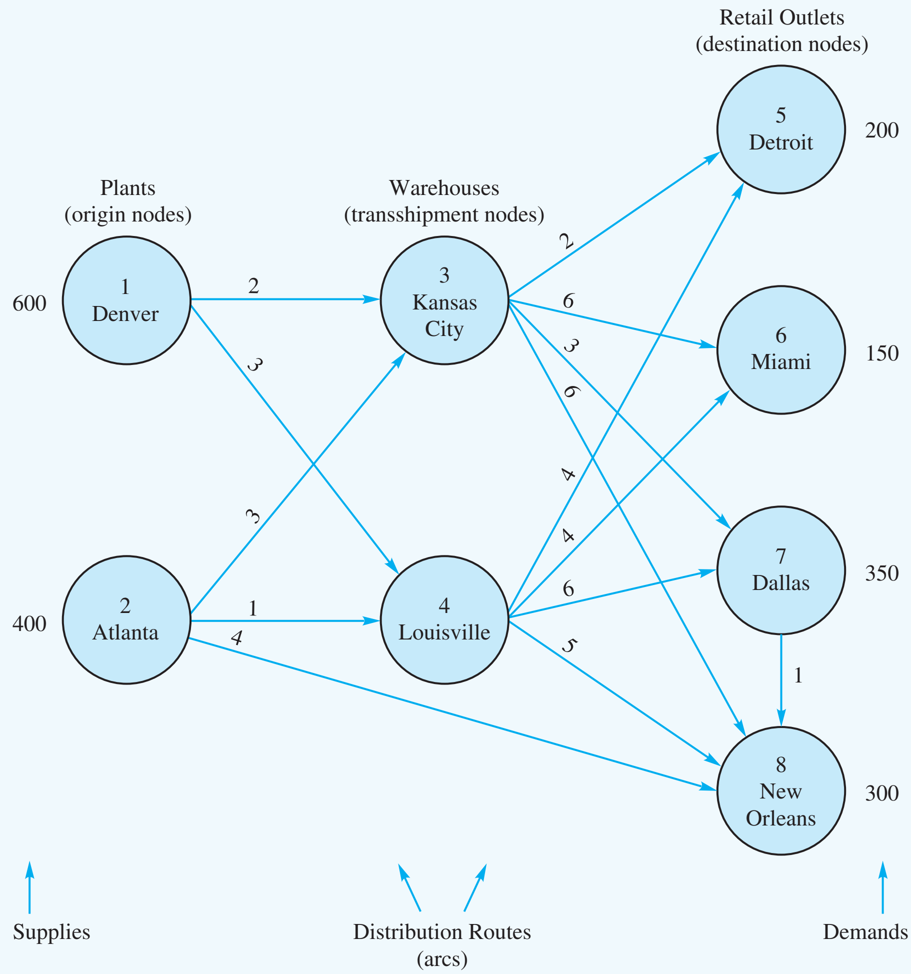


FIGURE 10.8 LINEAR PROGRAMMING FORMULATION OF THE MODIFIED RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

$$
\begin{array}{ll}
\text{Min} & 2x_{13} + 3x_{14} + 3x_{23} + 1x_{24} + 2x_{35} + 6x_{36} + 3x_{37} + 6x_{38} + 4x_{45} + 4x_{46} + 6x_{47} + 5x_{48} + 4x_{28} + 1x_{78} \\
\text{s.t.} & \\
& \left.\begin{array}{l}
x_{13} + x_{14} \leq 600 \\
x_{23} + x_{24} + x_{28} \leq 400
\end{array}\right\} \text{Origin node constraints} \\
& \left.\begin{array}{l}
-x_{13} - x_{23} + x_{35} + x_{36} + x_{37} + x_{38} = 0 \\
-x_{14} - x_{24} + x_{45} + x_{46} + x_{47} + x_{48} = 0
\end{array}\right\} \text{Transshipment node constraints} \\
& \left.\begin{array}{l}
x_{35} + x_{45} = 200 \\
x_{36} + x_{46} = 150 \\
x_{37} + x_{47} - x_{78} = 350 \\
x_{38} + x_{48} + x_{28} + x_{78} = 300
\end{array}\right\} \text{Destination node constraints} \\
& x_{ij} \geq 0 \text{ for all } i \text{ and } j
\end{array}
$$

FIGURE 10.9 OPTIMAL SOLUTION FOR THE MODIFIED RYAN ELECTRONICS TRANSSHIPMENT PROBLEM

Objective Cell (Min)

Name	Original Value	Final Value
Total Cost	0.000	4600.000

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
X13	Denver–Kansas City	0.000	550.000	Contin
X14	Denver–Louisville	0.000	50.000	Contin
X23	Atlanta–Kansas City	0.000	0.000	Contin
X24	Atlanta–Louisville	0.000	100.000	Contin
X35	Kansas City–Detroit	0.000	200.000	Contin
X36	Kansas City–Miami	0.000	0.000	Contin
X37	Kansas City–Dallas	0.000	350.000	Contin
X38	Kansas City–New Orleans	0.000	0.000	Contin
X45	Louisville–Detroit	0.000	0.000	Contin
X46	Louisville–Miami	0.000	150.000	Contin
X47	Louisville–Dallas	0.000	0.000	Contin
X48	Louisville–New Orleans	0.000	0.000	Contin
X28	Atlanta–New Orleans	0.000	300.000	Contin
X78	Dallas–New Orleans	0.000	0.000	Contin

WEB file
Modified Ryan

the linear programming formulation is shown in Figure 10.8, and the optimal solution from the answer report is shown in Figure 10.9.

Try Problem 12 for practice working with transshipment problems with this more general structure.

In Figure 10.7 we added two new arcs to the network model. Thus, two new variables are necessary in the linear programming formulation. Figure 10.8 shows that the new variables x_{28} and x_{78} appear in the objective function and in the constraints corresponding to the nodes to which the new arcs are connected. Figure 10.9 shows that the value of the optimal solution has been reduced \$600 by allowing these additional shipping routes. The value of $x_{28} = 300$ indicates that 300 units are being shipped directly from Atlanta to New Orleans.

The value of $x_{78} = 0$ indicates that no units are shipped from Dallas to New Orleans in this solution.[1]

Problem Variations

As with transportation problems, transshipment problems may be formulated with several variations, including

1. Total supply not equal to total demand
2. Maximization objective function
3. Route capacities or route minimums
4. Unacceptable routes

The linear programming model modifications required to accommodate these variations are identical to the modifications required for the transportation problem. When we add one or more constraints of the form $x_{ij} \leq L_{ij}$ to show that the route from node i to node j has capacity L_{ij}, we refer to the transshipment problem as a **capacitated transshipment problem**.

A General Linear Programming Model

To show the general linear programming model for the transshipment problem, we use the following notation:

$$
\begin{aligned}
x_{ij} &= \text{number of units shipped from node } i \text{ to node } j \\
c_{ij} &= \text{cost per unit of shipping from node } i \text{ to node } j \\
s_i &= \text{supply at origin node } i \\
d_j &= \text{demand at destination node } j
\end{aligned}
$$

The general linear programming model for the transshipment problem is as follows:

$$
\begin{aligned}
&\text{Min} \quad \sum_{\text{all arcs}} c_{ij} x_{ij} \\
&\text{s.t.} \\
&\qquad \sum_{\text{arcs out}} x_{ij} - \sum_{\text{arcs in}} x_{ij} \leq s_i \qquad \text{Origin nodes } i \\
&\qquad \sum_{\text{arcs out}} x_{ij} - \sum_{\text{arcs in}} x_{ij} = 0 \qquad \text{Transshipment nodes} \\
&\qquad \sum_{\text{arcs in}} x_{ij} - \sum_{\text{arcs out}} x_{ij} = d_j \qquad \text{Destination nodes } j \\
&\qquad\quad x_{ij} \geq 0 \text{ for all } i \text{ and } j
\end{aligned}
$$

The Q.M. in Action, Product Sourcing Heuristic at Procter & Gamble, describes a transshipment model used by Procter & Gamble to help make strategic decisions related to sourcing and distribution.

---

[1]This is an example of a linear programming with alternate optimal solutions. The solution $x_{13} = 600$, $x_{14} = 0$, $x_{23} = 0$, $x_{24} = 150$, $x_{28} = 250$, $x_{35} = 200$, $x_{36} = 0$, $x_{37} = 400$, $x_{38} = 0$, $x_{45} = 0$, $x_{46} = 150$, $x_{47} = 0$, $x_{48} = 0$, $x_{78} = 50$ is also optimal. Thus, in this solution both new routes are used: $x_{28} = 250$ units are shipped from Atlanta to New Orleans and $x_{78} = 50$ units are shipped from Dallas to New Orleans.

NOTES AND COMMENTS

1. Supply chain models used in practice usually lead to large linear programs. Problems with 100 origins and 100 destinations are not unusual. Such a problem would involve (100)(100) = 10,000 variables.
2. To handle a situation in which some routes may be unacceptable, we stated that you could drop the corresponding arc from the network and remove the corresponding variable from the linear programming formulation. Another approach often used is to assign an extremely large objective function cost coefficient to any unacceptable arc. If the problem has already been formulated, another option is to add a constraint to the formulation that sets the variable you want to remove equal to zero.
3. The optimal solution to a transportation model will consist of integer values for the decision variables as long as all supply and demand values are integers. The reason is the special mathematical structure of the linear programming model. Each variable appears in exactly one supply and one demand constraint, and all coefficients in the constraint equations are 1 or 0.
4. In the general linear programming formulation of the transshipment problem, the constraints for the destination nodes are often written as

$$\sum_{\text{arcs out}} x_{ij} - \sum_{\text{arcs in}} x_{ij} = -d_j$$

The advantage of writing the constraints this way is that the left-hand side of each constraint then represents the flow out of the node minus the flow in.

Q.M. in ACTION

*PRODUCT SOURCING HEURISTIC AT PROCTER & GAMBLE**

A few years ago Procter & Gamble (P&G) embarked on a major strategic planning initiative called the North American Product Sourcing Study. P&G wanted to consolidate its product sources and optimize its distribution system design throughout North America. A decision support system used to aid in this project was called the Product Sourcing Heuristic (PSH) and was based on a transshipment model much like the ones described in this chapter.

In a preprocessing phase, the many P&G products were aggregated into groups that shared the same technology and could be made at the same plant. The PSH employing the transshipment model was then used by product strategy teams responsible for developing product sourcing options for these product groups. The various plants that could produce the product group were the source nodes, the company's regional distribution centers were the transshipment nodes, and P&G's customer zones were the destinations. Direct shipments to customer zones as well as shipments through distribution centers were employed.

The product strategy teams used the heuristic interactively to explore a variety of questions concerning product sourcing and distribution. For instance, the team might be interested in the impact of closing two of five plants and consolidating production in the three remaining plants. The product sourcing heuristic would then delete the source nodes corresponding to the two closed plants, make any capacity modifications necessary to the sources corresponding to the remaining three plants, and re-solve the transshipment problem. The product strategy team could then examine the new solution, make some more modifications, solve again, and so on.

The Product Sourcing Heuristic was viewed as a valuable decision support system by all who used it. When P&G implemented the results of the study, it realized annual savings in the $200 million range. The PSH proved so successful in North America that P&G used it in other markets around the world.

*Based on information provided by Franz Dill and Tom Chorman of Procter & Gamble.

10.2 Assignment Problem

The **assignment problem** arises in a variety of decision-making situations; typical assignment problems involve assigning jobs to machines, agents to tasks, sales personnel to sales territories, contracts to bidders, and so on. A distinguishing feature of the assignment problem is that *one* agent is assigned to *one and only one* task. Specifically, we look for the set of assignments that will optimize a stated objective, such as minimize cost, minimize time, or maximize profits.

To illustrate the assignment problem, let us consider the case of Fowle Marketing Research, which has just received requests for market research studies from three new clients. The company faces the task of assigning a project leader (agent) to each client (task). Currently, three individuals have no other commitments and are available for the project leader assignments. Fowle's management realizes, however, that the time required to complete each study will depend on the experience and ability of the project leader assigned. The three projects have approximately the same priority, and management wants to assign project leaders to minimize the total number of days required to complete all three projects. If a project leader is to be assigned to one client only, which assignments should be made?

To answer the assignment question, Fowle's management must first consider all possible project leader–client assignments and then estimate the corresponding project completion times. With three project leaders and three clients, nine assignment alternatives are possible. The alternatives and the estimated project completion times in days are summarized in Table 10.5.

Try Problem 17, part (a), for practice in developing a network model for an assignment problem.

Figure 10.10 shows the network representation of Fowle's assignment problem. The nodes correspond to the project leaders and clients, and the arcs represent the possible assignments of project leaders to clients. The supply at each origin node and the demand at each destination node are 1; the cost of assigning a project leader to a client is the time it takes that project leader to complete the client's task. Note the similarity between the network models of the assignment problem (Figure 10.10) and the transportation problem (Figure 10.1). The assignment problem is a special case of the transportation problem in which all supply and demand values equal 1, and the amount shipped over each arc is either 0 or 1.

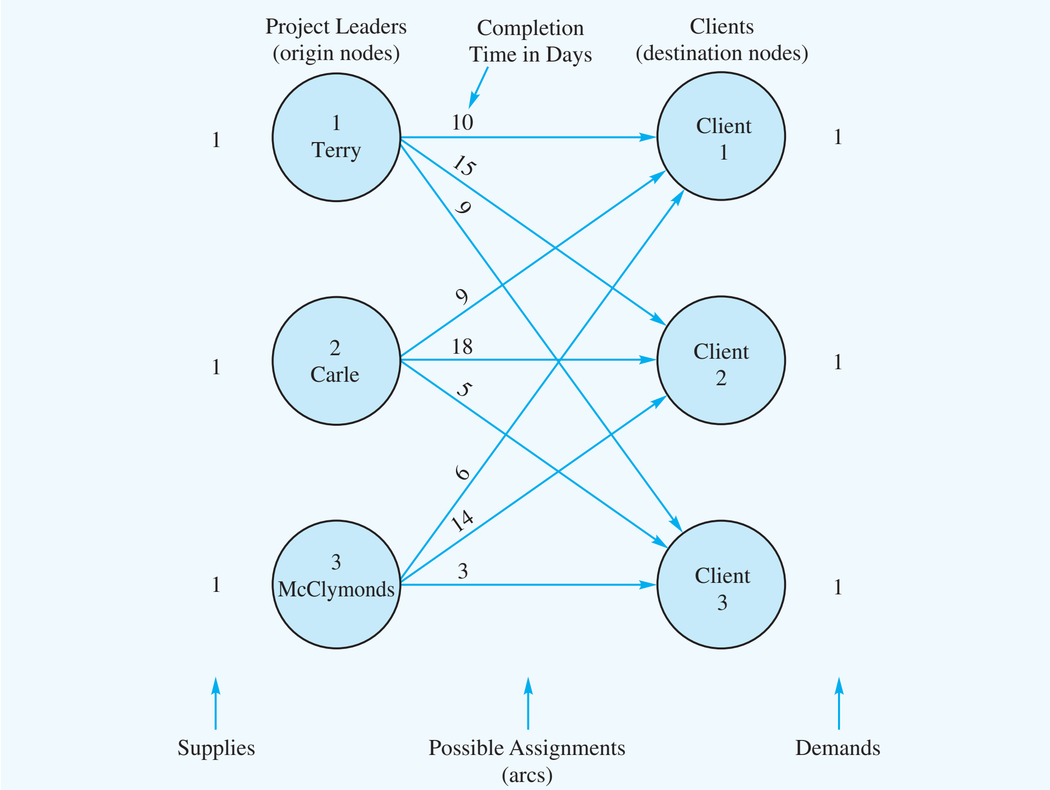
The assignment problem is a special case of the transportation problem.

Because the assignment problem is a special case of the transportation problem, a linear programming formulation can be developed. Again, we need a constraint for each node and a variable for each arc. As in the transportation problem, we use double-subscripted decision variables, with x_{11} denoting the assignment of project leader 1 (Terry) to client 1, x_{12}

TABLE 10.5 ESTIMATED PROJECT COMPLETION TIMES (DAYS) FOR THE FOWLE MARKETING RESEARCH ASSIGNMENT PROBLEM

	Client		
Project Leader	**1**	**2**	**3**
1. Terry	10	15	9
2. Carle	9	18	5
3. McClymonds	6	14	3

FIGURE 10.10 A NETWORK MODEL OF THE FOWLE MARKETING RESEARCH ASSIGNMENT PROBLEM



denoting the assignment of project leader 1 (Terry) to client 2, and so on. Thus, we define the decision variables for Fowle's assignment problem as

$$x_{ij} = \begin{cases} 1 & \text{if project leader } i \text{ is assigned to client } j \\ 0 & \text{otherwise} \end{cases}$$

where $i = 1, 2, 3$, and $j = 1, 2, 3$

Using this notation and the completion time data in Table 10.5, we develop completion time expressions:

$$\begin{aligned} &\text{Days required for Terry's assignment} &&= 10x_{11} + 15x_{12} + 9x_{13} \\ &\text{Days required for Carle's assignment} &&= 9x_{21} + 18x_{22} + 5x_{23} \\ &\text{Days required for McClymonds's assignment} &&= 6x_{31} + 14x_{32} + 3x_{33} \end{aligned}$$

The sum of the completion times for the three project leaders will provide the total days required to complete the three assignments. Thus, the objective function is

$$\text{Min } 10x_{11} + 15x_{12} + 9x_{13} + 9x_{21} + 18x_{22} + 5x_{23} + 6x_{31} + 14x_{32} + 3x_{33}$$

Because the number of project leaders equals the number of clients, all the constraints could be written as equalities. But when the number of project leaders exceeds the number of clients, less-than-or-equal-to constraints must be used for the project leader constraints.

The constraints for the assignment problem reflect the conditions that each project leader can be assigned to at most one client and that each client must have one assigned project leader. These constraints are written as follows:

$$
\begin{array}{ll}
x_{11} + x_{12} + x_{13} \leq 1 & \text{Terry's assignment} \\
x_{21} + x_{22} + x_{23} \leq 1 & \text{Carle's assignment} \\
x_{31} + x_{32} + x_{33} \leq 1 & \text{McClymonds's assignment} \\
x_{11} + x_{21} + x_{31} = 1 & \text{Client 1} \\
x_{12} + x_{22} + x_{32} = 1 & \text{Client 2} \\
x_{13} + x_{23} + x_{33} = 1 & \text{Client 3}
\end{array}
$$

Note that each node in Figure 10.10 has one constraint.

Try Problem 17, part (b), for practice in formulating and solving a linear programming model for an assignment problem on the computer.

Combining the objective function and constraints into one model provides the following nine-variable, six-constraint linear programming model of the Fowle Marketing Research assignment problem:

$$
\begin{array}{ll}
\text{Min} & 10x_{11} + 15x_{12} + 9x_{13} + 9x_{21} + 18x_{22} + 5x_{23} + 6x_{31} + 14x_{32} + 3x_{33} \\
\text{s.t.} & \\
& \begin{array}{lllllllllr}
x_{11} + & x_{12} + & x_{13} & & & & & & & \leq 1 \\
& & & x_{21} + & x_{22} + & x_{23} & & & & \leq 1 \\
& & & & & & x_{31} + & x_{32} + & x_{33} & \leq 1 \\
x_{11} & & & + x_{21} & & & + x_{31} & & & = 1 \\
& x_{12} & & & + x_{22} & & & + x_{32} & & = 1 \\
& & x_{13} & & & + x_{23} & & & + x_{33} & = 1
\end{array} \\
& x_{ij} \geq 0 \quad \text{for } i = 1, 2, 3; j = 1, 2, 3
\end{array}
$$

Figure 10.11 shows the optimal solution from the answer report for this model. Terry is assigned to client 2 ($x_{12} = 1$), Carle is assigned to client 3 ($x_{23} = 1$), and McClymonds

FIGURE 10.11 OPTIMAL SOLUTION FOR THE FOWLE MARKETING RESEARCH ASSIGNMENT PROBLEM

WEB file

Fowle

Objective Cell (Min)

Name	Original Value	Final Value
Minimize Completion Time	0.000	26.000

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
X11	Terry to Client 1	0.000	0.000	Contin
X12	Terry to Client 2	0.000	1.000	Contin
X13	Terry to Client 3	0.000	0.000	Contin
X21	Carle to Client 1	0.000	0.000	Contin
X22	Carle to Client 2	0.000	0.000	Contin
X23	Carle to Client 3	0.000	1.000	Contin
X31	McClymonds to Client 1	0.000	1.000	Contin
X32	McClymonds to Client 2	0.000	0.000	Contin
X33	McClymonds to Client 3	0.000	0.000	Contin

TABLE 10.6 OPTIMAL PROJECT LEADER ASSIGNMENTS FOR THE FOWLE MARKETING RESEARCH ASSIGNMENT PROBLEM

Project Leader	Assigned Client		Days
Terry	2		15
Carle	3		5
McClymonds	1		6
		Total	26

is assigned to client 1 ($x_{31} = 1$). The total completion time required is 26 days. This solution is summarized in Table 10.6.

Problem Variations

Because the assignment problem can be viewed as a special case of the transportation problem, the problem variations that may arise in an assignment problem parallel those for the transportation problem. Specifically, we can handle

1. Total number of agents (supply) not equal to the total number of tasks (demand)
2. A maximization objective function
3. Unacceptable assignments

The situation in which the number of agents does not equal the number of tasks is analogous to total supply not equaling total demand in a transportation problem. If the number of agents exceeds the number of tasks, the extra agents simply remain unassigned in the linear programming solution. If the number of tasks exceeds the number of agents, the linear programming model will not have a feasible solution. In this situation, a simple modification is to add enough dummy agents to equalize the number of agents and the number of tasks. For instance, in the Fowle problem we might have had five clients (tasks) and only three project leaders (agents). By adding two dummy project leaders, we can create a new assignment problem with the number of project leaders equal to the number of clients. The objective function coefficients for the assignment of dummy project leaders would be zero so that the value of the optimal solution would represent the total number of days required by the assignments actually made (no assignments will actually be made to the clients receiving dummy project leaders).

If the assignment alternatives are evaluated in terms of revenue or profit rather than time or cost, the linear programming formulation can be solved as a maximization rather than a minimization problem. In addition, if one or more assignments are unacceptable, the corresponding decision variable can be removed from the linear programming formulation. This situation could happen, for example, if an agent did not have the experience necessary for one or more of the tasks.

A General Linear Programming Model

To show the general linear programming model for an assignment problem with m agents and n tasks, we use the following notation:

$$x_{ij} = \begin{cases} 1 & \text{if agent } i \text{ is assigned to task } j \\ 0 & \text{otherwise} \end{cases}$$

$$c_{ij} = \text{the cost of assigning agent } i \text{ to task } j$$

The general linear programming model is as follows:

$$\text{Min} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.

$$\sum_{j=1}^{n} x_{ij} \leq 1 \qquad i = 1, 2, \ldots, m \quad \text{Agents}$$

$$\sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1, 2, \ldots, n \quad \text{Tasks}$$

$$x_{ij} \geq 0 \qquad \text{for all } i \text{ and } j$$

At the beginning of this section, we indicated that a distinguishing feature of the assignment problem is that *one* agent is assigned to *one and only one* task. In generalizations of the assignment problem where one agent can be assigned to two or more tasks, the linear programming formulation of the problem can be easily modified. For example, let us assume that in the Fowle Marketing Research problem Terry could be assigned up to two clients; in this case, the constraint representing Terry's assignment would be $x_{11} + x_{12} + x_{13} \leq 2$. In general, if a_i denotes the upper limit for the number of tasks to which agent i can be assigned, we write the agent constraints as

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad i = 1, 2, \ldots, m$$

If some tasks require more than one agent, the linear programming formulation can also accommodate the situation. Use the number of agents required as the right-hand side of the appropriate task constraint.

NOTES AND COMMENTS

1. As noted, the assignment model is a special case of the transportation model. We stated in the Notes and Comments at the end of the preceding section that the optimal solution to the transportation problem will consist of integer values for the decision variables as long as the supplies and demands are integers. For the assignment problem, all supplies and demands equal 1; thus, the optimal solution must be integer valued and the integer values must be 0 or 1.
2. Combining the method for handling multiple assignments with the notion of a dummy agent provides another means of dealing with situations when the number of tasks exceeds the number of agents. That is, we add one dummy agent but provide the dummy agent with the capability to handle multiple tasks. The number of tasks the dummy agent can handle is equal to the difference between the number of tasks and the number of agents.
3. The Q.M. in Action, Assigning Project Managers at Heery International, describes how managers are assigned to construction projects. The application involves multiple assignments.

Q.M. *in* ACTION

*ASSIGNING PROJECT MANAGERS AT HEERY INTERNATIONAL**

Heery International contracts with the State of Tennessee and others for a variety of construction projects, including higher education facilities, hotels, and park facilities. At any particular time, Heery typically has more than 100 ongoing projects. Each of these projects must be assigned a single manager. With seven managers available, more than 700 = 7(100) assignments are possible. Assisted by an outside consultant, Heery International

(*continued*)

*Based on Larry J. LeBlanc, Dale Randels, Jr., and T. K. Swann, "Heery International's Spreadsheet Optimization Model for Assigning Managers to Construction Projects," *Interfaces* (November/December 2000): 95–106.

developed a mathematical model for assigning construction managers to projects.

The assignment problem developed by Heery uses 0–1 decision variables for each manager/project pair, just as in the assignment problem discussed previously. The goal in assigning managers is to balance the workload among managers and, at the same time, to minimize travel cost from the manager's home to the construction site. Thus, an objective function coefficient for each possible assignment was developed that combined project intensity (a function of the size of the project budget) with the travel distance from the manager's home to the construction site. The objective function calls for minimizing the sum over all possible assignments of the product of these coefficients with the assignment variables.

With more construction projects than managers, it was necessary to consider a variation of the standard assignment problem involving multiple assignments. Of the two sets of constraints, one set enforces the requirement that each project receive one and only one manager. The other set of constraints limits the number of assignments each manager can accept by placing an upper bound on the total intensity that is acceptable over all projects assigned.

Heery International implemented this assignment model with considerable success. According to Emory F. Redden, a Heery vice president, "The optimization model . . . has been very helpful for assigning managers to projects. . . . We have been satisfied with the assignments chosen at the Nashville office. . . . We look forward to using the model in our Atlanta office and elsewhere in the Heery organization."

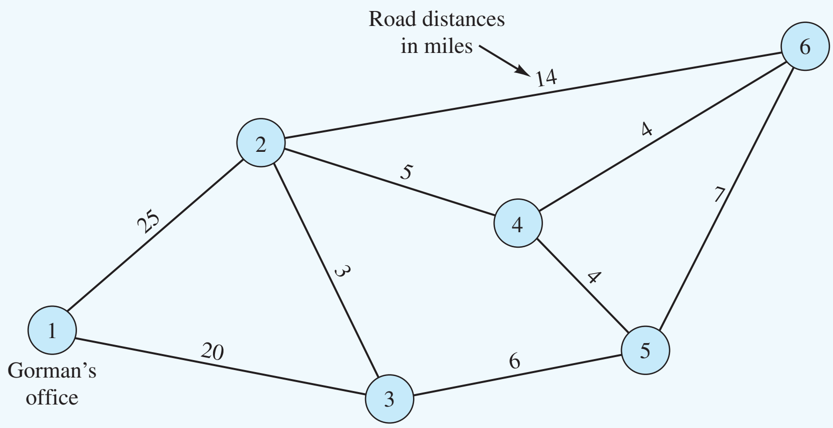
10.3 Shortest-Route Problem

In this section we consider a problem in which the objective is to determine the **shortest route**, or *path*, between two nodes in a network. We will demonstrate the shortest-route problem by considering the situation facing the Gorman Construction Company. Gorman has several construction sites located throughout a three-county area. With multiple daily trips carrying personnel, equipment, and supplies from Gorman's office to the construction sites, the costs associated with transportation activities are substantial. The travel alternatives between Gorman's office and each construction site can be described by the road network shown in Figure 10.12. The road distances in miles between the nodes are shown above the corresponding arcs. In this application, Gorman would like to determine the route that will minimize the total travel distance between Gorman's office (located at node 1) and the construction site located at node 6.

A key to developing a model for the shortest-route problem is to understand that the problem is a special case of the transshipment problem. Specifically, the Gorman shortest-route problem can be viewed as a transshipment problem with one origin node (node 1), one destination node (node 6), and four transshipment nodes (nodes 2, 3, 4 and 5). The transshipment network for the Gorman shortest-route problem is shown in Figure 10.13. Arrows added to the arcs show the direction of flow, which is always *out* of the origin node and *into* the destination node. Note also that two directed arcs are shown between the pairs of transshipment nodes. For example, one arc going from node 2 to node 3 indicates that the shortest route may go from node 2 to node 3, and one arc going from node 3 to node 2 indicates that the shortest route may go from node 3 to node 2. The distance between two transshipment nodes is the same in either direction.

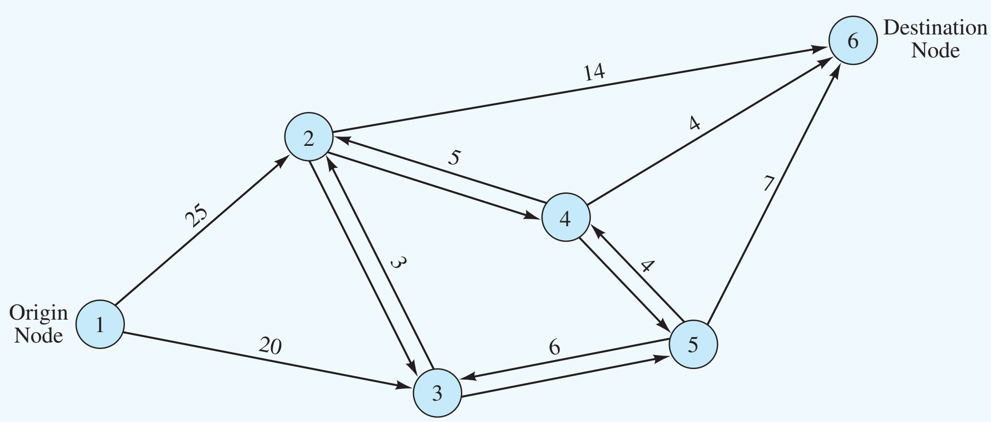
To find the shortest route between node 1 and node 6, think of node 1 as having a supply of 1 unit and node 6 as having a demand of 1 unit. Let x_{ij} denote the number of units that flow or are shipped from node i to node j. Because only 1 unit will be shipped from node 1 to node 6, the value of x_{ij} will be either 1 or 0. Thus, if $x_{ij} = 1$, the arc from node i to node j

FIGURE 10.12 ROAD NETWORK FOR THE GORMAN COMPANY SHORTEST-ROUTE PROBLEM



Note: (1) The length of each arc is not necessarily proportional to the travel distance it represents.
(2) All roads are two-way; thus, flow may be in either direction.

FIGURE 10.13 TRANSSHIPMENT NETWORK FOR THE GORMAN SHORTEST-ROUTE PROBLEM



Note: Nodes 2, 3, 4, and 5 are transshipment nodes.

is on the shortest route from node 1 to node 6; if $x_{ij} = 0$, the arc from node i to node j is not on the shortest route. Because we are looking for the shortest route between node 1 and node 6, the objective function for the Gorman problem is

$$\begin{aligned}\text{Min} \quad & 25x_{12} + 20x_{13} + 3x_{23} + 3x_{32} + 5x_{24} + 5x_{42} + 14x_{26} + 6x_{35} + 6x_{53} \\ & + 4x_{45} + 4x_{54} + 4x_{46} + 7x_{56}\end{aligned}$$

To develop the constraints for the model, we begin with node 1. Because the supply at node 1 is 1 unit, the flow out of node 1 must equal 1. Thus, the constraint for node 1 is written

$$x_{12} + x_{13} = 1$$

For transshipment nodes 2, 3, 4, and 5, the flow out of each node must equal the flow into each node; thus, the flow out minus the flow in must be 0. The constraints for the four transshipment nodes are as follows:

	Flow Out	**Flow In**
Node 2	$x_{23} + x_{24} + x_{26}$	$-x_{12} - x_{32} - x_{42} = 0$
Node 3	$x_{32} + x_{35}$	$-x_{13} - x_{23} - x_{53} = 0$
Node 4	$x_{42} + x_{45} + x_{46}$	$-x_{24} - x_{54} = 0$
Node 5	$x_{53} + x_{54} + x_{56}$	$-x_{35} - x_{45} = 0$

Because node 6 is the destination node with a demand of 1 unit, the flow into node 6 must equal 1. Thus, the constraint for node 6 is written as

$$x_{26} + x_{46} + x_{56} = 1$$

Including the negative constraints $x_{ij} \geq 0$ for all i and j, the linear programming model for the Gorman shortest-route problem is shown in Figure 10.14.

The optimal solution from the answer report for the Gorman shortest-route problem is shown in Figure 10.15. The objective function value of 32 indicates that the shortest route between Gorman's office located at node 1 to the construction site located at node 6 is 32 miles. With $x_{13} = 1$, $x_{32} = 1$, $x_{24} = 1$, and $x_{46} = 1$, the shortest route from node 1 to node 6

FIGURE 10.14 LINEAR PROGRAMMING FORMULATION OF THE GORMAN SHORTEST-ROUTE PROBLEM

$$\text{Min } 25x_{12} + 20x_{13} + 3x_{23} + 3x_{32} + 5x_{24} + 5x_{42} + 14x_{26} + 6x_{35} + 6x_{53} + 4x_{45} + 4x_{54} + 4x_{46} + 7x_{56}$$

s.t.

$$\begin{array}{ll}
x_{12} + x_{13} = 1 & \text{Origin node} \\
\left.\begin{array}{l}
-x_{12} + x_{23} - x_{32} + x_{24} - x_{42} + x_{26} = 0 \\
-x_{13} - x_{23} + x_{32} + x_{35} - x_{53} = 0 \\
-x_{24} + x_{42} + x_{45} - x_{54} + x_{46} = 0 \\
-x_{35} + x_{53} - x_{45} + x_{54} + x_{56} = 0
\end{array}\right\} & \text{Transshipment nodes} \\
x_{26} + x_{46} + x_{56} = 1 & \text{Destination node}
\end{array}$$

$x_{ij} \geq 0$ for all i and j

FIGURE 10.15 OPTIMAL SOLUTION FOR THE GORMAN SHORTEST-ROUTE PROBLEM

Objective Cell (Min)

Name	Original Value	Final Value
Total Distance	0.000	32.000

Variable Cells

Cell	Name	Original Value	Final Value	Integer
X12	Flow from Node 1 to 2	0.000	0.000	Contin
X13	Flow from Node 1 to 3	0.000	1.000	Contin
X23	Flow from Node 2 to 3	0.000	0.000	Contin
X32	Flow from Node 3 to 2	0.000	1.000	Contin
X24	Flow from Node 2 to 4	0.000	1.000	Contin
X42	Flow from Node 4 to 2	0.000	0.000	Contin
X26	Flow from Node 2 to 6	0.000	0.000	Contin
X35	Flow from Node 3 to 5	0.000	0.000	Contin
X53	Flow from Node 5 to 3	0.000	0.000	Contin
X45	Flow from Node 4 to 5	0.000	0.000	Contin
X54	Flow from Node 5 to 4	0.000	0.000	Contin
X46	Flow from Node 4 to 6	0.000	1.000	Contin
X56	Flow from Node 5 to 6	0.000	0.000	Contin

WEB file
Gorman

Try Problem 23 to practice solving a shortest-route problem.

is 1–3–2–4–6; in other words, the shortest route takes us from node 1 to node 3; then from node 3 to node 2; then from node 2 to node 4; and finally from node 4 to node 6.

A General Linear Programming Model

To show the general linear programming model for the shortest-route problem, we use the following notation:

$$x_{ij} = \begin{cases} 1 & \text{if the arc from node } i \text{ to node } j \text{ is on the shortest route} \\ 0 & \text{otherwise} \end{cases}$$

$$c_{ij} = \text{the distance, time, or cost associated with the arc from node } i \text{ to node } j$$

The general linear programming model for the shortest-route problem is as follows:

$$\text{Min} \quad \sum_{\text{all arcs}} c_{ij}x_{ij}$$

s.t.

$$\begin{aligned} \sum_{\text{arcs out}} x_{ij} \qquad\qquad\quad &= 1 \quad \text{Origin node } i \\ \sum_{\text{arcs out}} x_{ij} - \sum_{\text{arcs in}} x_{ij} &= 0 \quad \text{Transshipment nodes} \\ \sum_{\text{arcs in}} x_{ij} &= 1 \quad \text{Destination node } j \end{aligned}$$

NOTES AND COMMENTS

1. In the Gorman problem we assumed that all roads in the network are two-way. As a result, the road connecting nodes 2 and 3 in the road network resulted in the creation of two corresponding arcs in the transshipment network. Two decision variables, x_{23} and x_{32}, were required to show that the shortest route might go from node 2 to node 3 or from node 3 to node 2. If the road connecting nodes 2 and 3 had been a one-way road allowing flow only from node 2 to node 3, decision variable x_{32} would not have been included in the model.

10.4 Maximal Flow Problem

The objective in a **maximal flow** problem is to determine the maximum amount of flow (vehicles, messages, fluid, etc.) that can enter and exit a network system in a given period of time. In this problem, we attempt to transmit flow through all arcs of the network as efficiently as possible. The amount of flow is limited due to capacity restrictions on the various arcs of the network. For example, highway types limit vehicle flow in a transportation system, while pipe sizes limit oil flow in an oil distribution system. The maximum or upper limit on the flow in an arc is referred to as the **flow capacity** of the arc. Even though we do not specify capacities for the nodes, we do assume that the flow out of a node is equal to the flow into the node.

As an example of the maximal flow problem, consider the north–south interstate highway system passing through Cincinnati, Ohio. The north–south vehicle flow reaches a level of 15,000 vehicles per hour at peak times. Due to a summer highway maintenance program, which calls for the temporary closing of lanes and lower speed limits, a network of alternate routes through Cincinnati has been proposed by a transportation planning committee. The alternate routes include other highways as well as city streets. Because of differences in speed limits and traffic patterns, flow capacities vary depending on the particular streets and roads used. The proposed network with arc flow capacities is shown in Figure 10.16.

FIGURE 10.16 NETWORK OF HIGHWAY SYSTEM AND FLOW CAPACITIES (1000S/HOUR) FOR CINCINNATI

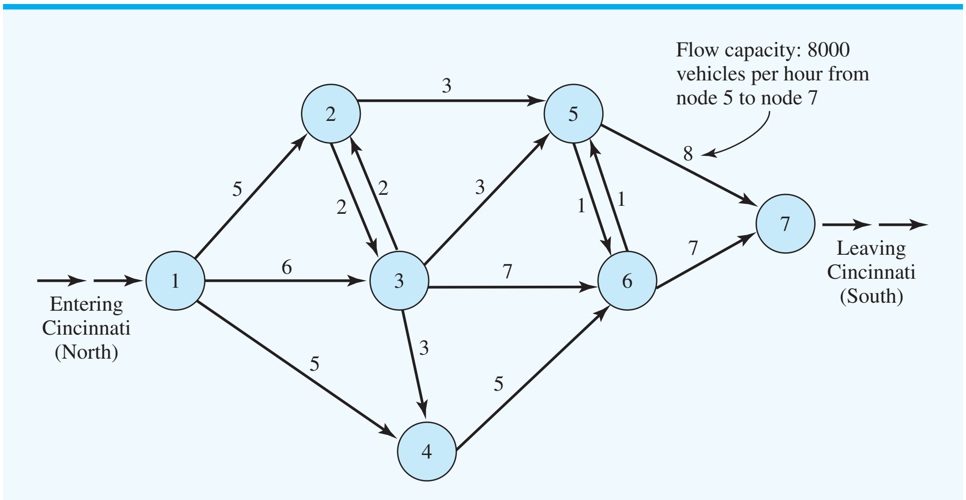
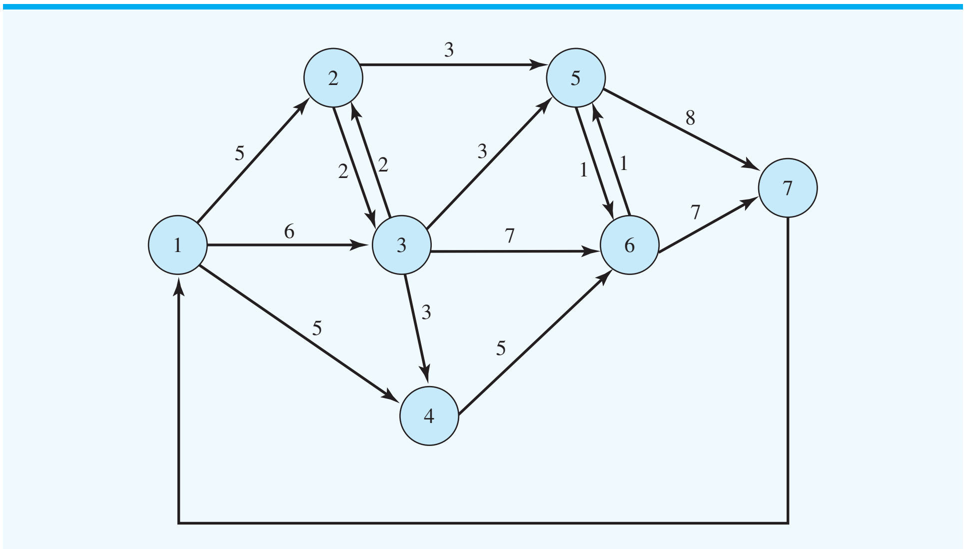


FIGURE 10.17 FLOW OVER ARC FROM NODE 7 TO NODE 1 TO REPRESENT TOTAL FLOW THROUGH THE CINCINNATI HIGHWAY SYSTEM



The direction of flow for each arc is indicated, and the arc capacity is shown next to each arc. Note that most of the streets are one-way. However, a two-way street can be found between nodes 2 and 3 and between nodes 5 and 6. In both cases, the capacity is the same in each direction.

We will show how to develop a capacitated transshipment model for the maximal flow problem. First, we will add an arc from node 7 back to node 1 to represent the total flow through the highway system. Figure 10.17 shows the modified network. The newly added arc shows no capacity; indeed, we will want to maximize the flow over that arc. Maximizing the flow over the arc from node 7 to node 1 is equivalent to maximizing the number of cars that can get through the north–south highway system passing through Cincinnati.

The decision variables are as follows:

$$x_{ij} = \text{amount of traffic flow from node } i \text{ to node } j$$

The objective function that maximizes the flow over the highway system is

$$\text{Max } x_{71}$$

As with all transshipment problems, each arc generates a variable and each node generates a constraint. For each node, a conservation of flow constraint represents the requirement that the flow out must equal the flow in. Or, stated another way, the flow out minus the flow in must equal zero. For node 1, the flow out is $x_{12} + x_{13} + x_{14}$, and the flow in is x_{71}. Therefore, the constraint for node 1 is

$$x_{12} + x_{13} + x_{14} - x_{71} = 0$$

The conservation of flow constraints for the other six nodes are developed in a similar fashion.

	Flow Out	Flow In	
Node 2	$x_{23} + x_{25}$	$-x_{12} - x_{32}$	$= 0$
Node 3	$x_{32} + x_{34} + x_{35} + x_{36}$	$-x_{13} - x_{23}$	$= 0$
Node 4	x_{46}	$-x_{14} - x_{34}$	$= 0$
Node 5	$x_{56} + x_{57}$	$-x_{25} - x_{35} - x_{65}$	$= 0$
Node 6	$x_{65} + x_{67}$	$-x_{36} - x_{46} - x_{56}$	$= 0$
Node 7	x_{71}	$-x_{57} - x_{67}$	$= 0$

Additional constraints are needed to enforce the capacities on the arcs. These 14 simple upper-bound constraints are given.

$$
\begin{array}{llll}
x_{12} \le 5 & x_{13} \le 6 & x_{14} \le 5 & \\
x_{23} \le 2 & x_{25} \le 3 & & \\
x_{32} \le 2 & x_{34} \le 3 & x_{35} \le 5 & x_{36} \le 7 \\
x_{46} \le 5 & & & \\
x_{56} \le 1 & x_{57} \le 8 & & \\
x_{65} \le 1 & x_{67} \le 7 & &
\end{array}
$$

Note that the only arc without a capacity is the one we added from node 7 to node 1.

The optimal solution from the answer report for this 15-variable, 21-constraint linear programming problem is shown in Figure 10.18. We note that the value of the optimal

FIGURE 10.18 OPTIMAL SOLUTION FOR THE CINCINNATI HIGHWAY SYSTEM MAXIMAL FLOW PROBLEM

Objective Cell (Max)

Name	Original Value	Final Value
Max Flow	0.000	14.000

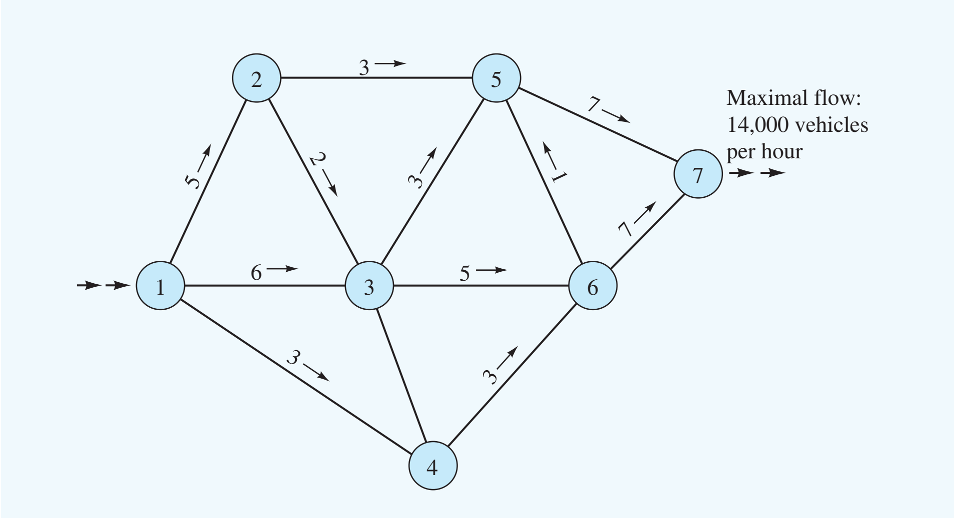
Variable Cells

Cell	Name	Original Value	Final Value	Integer
X12	Flow from 1 to 2	0.000	3.000	Contin
X13	Flow from 1 to 3	0.000	6.000	Contin
X14	Flow from 1 to 4	0.000	5.000	Contin
X23	Flow from 2 to 3	0.000	0.000	Contin
X25	Flow from 2 to 5	0.000	3.000	Contin
X34	Flow from 3 to 4	0.000	0.000	Contin
X35	Flow from 3 to 5	0.000	3.000	Contin
X36	Flow from 3 to 6	0.000	3.000	Contin
X32	Flow from 3 to 2	0.000	0.000	Contin
X46	Flow from 4 to 6	0.000	5.000	Contin
X56	Flow from 5 to 6	0.000	0.000	Contin
X57	Flow from 5 to 7	0.000	7.000	Contin
X65	Flow from 6 to 5	0.000	1.000	Contin
X67	Flow from 6 to 7	0.000	7.000	Contin
X71	Flow from 7 to 1	0.000	14.000	Contin

WEB file

Cincinnati

FIGURE 10.19 MAXIMAL FLOW PATTERN FOR THE CINCINNATI HIGHWAY SYSTEM NETWORK



Try Problem 29 for practice in solving a maximal flow problem.

solution is 14. This result implies that the maximal flow over the highway system is 14,000 vehicles. Figure 10.19 shows how the vehicle flow is routed through the original highway network. We note, for instance, that 3000 vehicles per hour are routed between nodes 1 and 2, 6000 vehicles per hour are routed between nodes 1 and 3, 0 vehicles are routed between nodes 2 and 3, and so on.

The results of the maximal flow analysis indicate that the planned highway network system will not handle the peak flow of 15,000 vehicles per hour. The transportation planners will have to expand the highway network, increase current arc flow capacities, or be prepared for serious traffic problems. If the network is extended or modified, another maximal flow analysis will determine the extent of any improved flow. The Q.M. in Action, Optimizing Restoration Capacity at AT&T, notes that AT&T solved shortest-route and maximal flow problems in designing a transmission network.

NOTES AND COMMENTS

1. The maximal flow problem of this section can also be solved with a slightly different formulation if the extra arc between nodes 7 and 1 is not used. The alternate approach is to maximize the flow into node 7 ($x_{57} + x_{67}$) and drop the conservation of flow constraints for nodes 1 and 7. However, the formulation used in this section is most common in practice.
2. Network models can be used to describe a variety of management science problems. Unfortunately, no one network solution algorithm can be used to solve every network problem. It is important to recognize the specific type of problem being modeled in order to select the correct specialized solution algorithm.

Q.M. *in* ACTION

*OPTIMIZING RESTORATION CAPACITY AT AT&T**

AT&T is a global telecommunications company that provides long-distance voice and data, video, wireless, satellite, and Internet services. The company uses state-of-the-art switching and transmission equipment to provide service to more than 80 million customers. In the continental United States, AT&T's transmission network consists of more than 40,000 miles of fiber-optic cable. On peak days AT&T handles as many as 290 million calls of various types.

Power outages, natural disasters, cable cuts, and other events can disable a portion of the transmission network. When such events occur, spare capacity comprising the restoration network must be immediately employed so that service is not disrupted. Critical issues with respect to the restoration network are as follows: How much capacity is necessary? and Where should it be located? In 1997, AT&T assembled a RestNet team to address these issues.

To optimize restoration capacity, the RestNet team developed a large-scale linear programming model. One subproblem in the model involves determining the shortest route connecting an origin and destination whenever a failure occurs in a span of the transmission network. Another subproblem solves a maximal flow problem to find the best restoration paths from each switch to a disaster recovery switch.

The RestNet team was successful, and its work is an example of how valuable management science methodology is to companies. According to C. Michael Armstrong, chair and CEO, "Last year the work of the RestNet team allowed us to reduce capital spending by tens of millions of dollars."

*Based on Ken Ambs, Sebastian Cwilich, Mei Deng, David J. Houck, David F. Lynch, and Dicky Yan, "Optimizing Restoration Capacity in the AT&T Network," *Interfaces* (January/February 2000): 26–44.

10.5 A Production and Inventory Application

The introduction to supply chain models in Section 10.1 involved applications for the shipment of goods from several supply locations or origins to several demand sites or destinations. Although the shipment of goods is the subject of many supply chain problems, supply chain models can be developed for applications that have nothing to do with the physical shipment of goods from origins to destinations. In this section we show how to use a transshipment model to solve a production and inventory problem.

Contois Carpets is a small manufacturer of carpeting for home and office installations. Production capacity, demand, production cost per square yard, and inventory holding cost per square yard for the next four quarters are shown in Table 10.7. Note that production capacity, demand, and production costs vary by quarter, whereas the cost of carrying inventory from one quarter to the next is constant at $0.25 per yard. Contois wants to

TABLE 10.7 PRODUCTION, DEMAND, AND COST ESTIMATES FOR CONTOIS CARPETS

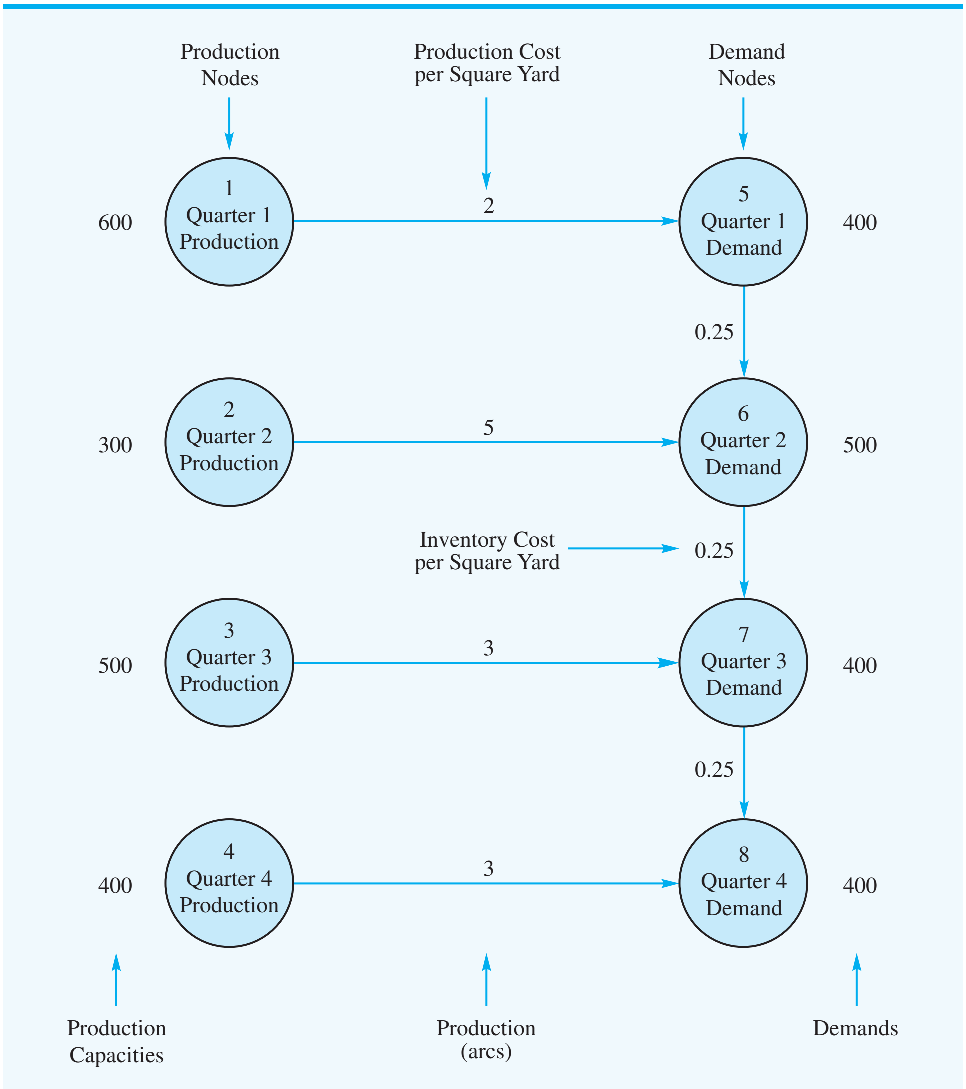
Quarter	Production Capacity (square yards)	Demand (square yards)	Production Cost ($/square yard)	Inventory Cost ($/square yard)
1	600	400	2	0.25
2	300	500	5	0.25
3	500	400	3	0.25
4	400	400	3	0.25

determine how many yards of carpeting to manufacture each quarter to minimize the total production and inventory cost for the four-quarter period.

The network flows into and out of demand nodes are what make the model a transshipment model.

We begin by developing a network representation of the problem. First, we create four nodes corresponding to the production in each quarter and four nodes corresponding to the demand in each quarter. Each production node is connected by an outgoing arc to the demand node for the same period. The flow on the arc represents the number of square yards of carpet manufactured for the period. For each demand node, an outgoing arc represents the amount of inventory (square yards of carpet) carried over to the demand node for the next period. Figure 10.20 shows the network model. Note that nodes 1–4 represent the production for each quarter and that nodes 5–8 represent the demand for each quarter. The quarterly production capacities are shown in the left margin, and the quarterly demands are shown in the right margin.

FIGURE 10.20 NETWORK REPRESENTATION OF THE CONTOIS CARPETS PROBLEM



The objective is to determine a production scheduling and inventory policy that will minimize the total production and inventory cost for the four quarters. Constraints involve production capacity and demand in each quarter. As usual, a linear programming model can be developed from the network by establishing a constraint for each node and a variable for each arc.

Let x_{15} denote the number of square yards of carpet manufactured in quarter 1. The capacity of the facility is 600 square yards in quarter 1, so the production capacity constraint is

$$x_{15} \leq 300$$

Using similar decision variables, we obtain the production capacities for quarters 2–4:

$$\begin{aligned} x_{26} &\leq 300 \\ x_{37} &\leq 500 \\ x_{48} &\leq 400 \end{aligned}$$

We now consider the development of the constraints for each of the demand nodes. For node 5, one arc enters the node, which represents the number of square yards of carpet produced in quarter 1, and one arc leaves the node, which represents the number of square yards of carpet that will not be sold in quarter 1 and will be carried over for possible sale in quarter 2. In general, for each quarter the beginning inventory plus the production minus the ending inventory must equal demand. However, because quarter 1 has no beginning inventory, the constraint for node 5 is

$$x_{15} - x_{56} = 400$$

The constraints associated with the demand nodes in quarters 2, 3, and 4 are

$$\begin{aligned} x_{56} + x_{26} - x_{67} &= 500 \\ x_{67} + x_{37} - x_{78} &= 400 \\ x_{78} + x_{48} &= 400 \end{aligned}$$

Note that the constraint for node 8 (fourth-quarter demand) involves only two variables because no provision is made for holding inventory for a fifth quarter.

The objective is to minimize total production and inventory cost, so we write the objective function as

$$\text{Min} \quad 2x_{15} + 5x_{26} + 3x_{37} + 3x_{48} + 0.25x_{56} + 0.25x_{67} + 0.25x_{78}$$

The complete linear programming formulation of the Contois Carpets problem is

$$\begin{array}{ll} \text{Min} & 2x_{15} + 5x_{26} + 3x_{37} + 3x_{48} + 0.25x_{56} + 0.25x_{67} + 0.25x_{78} \\ \text{s.t.} & \end{array}$$

$$\begin{array}{ccccccccccc} x_{15} & & & & & & & & & & \leq 600 \\ & x_{26} & & & & & & & & & \leq 300 \\ & & x_{37} & & & & & & & & \leq 500 \\ & & & x_{48} & & & & & & & \leq 400 \\ x_{15} & & & & - & x_{56} & & & & & = 400 \\ & x_{26} & & & + & x_{56} & - & x_{67} & & & = 500 \\ & & x_{37} & & & & + & x_{67} & - & x_{78} & = 400 \\ & & & x_{48} & & & & & + & x_{78} & = 400 \end{array}$$

$$x_{ij} \geq 0 \quad \text{for all } i \text{ and } j$$

FIGURE 10.21 OPTIMAL SOLUTION FOR THE CONTOIS CARPETS PROBLEM

Objective Cell (Min)

Name	Original Value	Final Value
Total Cost	0.000	5150.000

WEB file

Contois

Variable Cells

Model Variable	Name	Original Value	Final Value	Integer
X15	Flow from Node 1 to 5	0.000	600.000	Contin
X26	Flow from Node 2 to 6	0.000	300.000	Contin
X37	Flow from Node 3 to 7	0.000	400.000	Contin
X48	Flow from Node 4 to 8	0.000	400.000	Contin
X56	Flow from Node 5 to 6	0.000	200.000	Contin
X67	Flow from Node 6 to 7	0.000	0.000	Contin
X78	Flow from Node 7 to 8	0.000	0.000	Contin

Figure 10.21 shows the optimal solution from the answer report for this problem. Contois Carpets should manufacture 600 square yards of carpet in quarter 1, 300 square yards in quarter 2, 400 square yards in quarter 3, and 400 square yards in quarter 4. Note also that 200 square yards will be carried over from quarter 1 to quarter 2. The total production and inventory cost is $5150.

NOTES AND COMMENTS

1. For the network models presented in this chapter, the amount leaving the starting node for an arc is always equal to the amount entering the ending node for that arc. An extension of such a network model is the case where a gain or a loss occurs as an arc is traversed. The amount entering the destination node may be greater or smaller than the amount leaving the origin node. For instance, if cash is the commodity flowing across an arc, the cash earns interest from one period to the next. Thus, the amount of cash entering the next period is greater than the amount leaving the previous period by the amount of interest earned. Networks with gains or losses are treated in more advanced texts on network flow programming.

Summary

In this chapter we introduced models related to supply chain problems—specifically, transportation and transshipment problems—as well as assignment, shortest-route, and maximal flow problems. All of these types of problems belong to the special category of linear programs called *network flow problems*. In general, the network model for these problems consists of nodes representing origins, destinations, and, if necessary, transshipment points in the network system. Arcs are used to represent the routes for shipment, travel, or flow between the various nodes.

Transportation problems and transshipment problems are commonly encountered when dealing with supply chains. The general transportation problem has m origins and n destinations. Given the supply at each origin, the demand at each destination, and unit shipping cost between each origin and each destination, the transportation model determines the optimal amounts to ship from each origin to each destination. The transshipment problem is an extension of the transportation problem involving transfer points referred to as transshipment nodes. In this more general model, we allow arcs between any pair of nodes in the network.

The assignment problem is a special case of the transportation problem in which all supply and all demand values are 1. We represent each agent as an origin node and each task as a destination node. The assignment model determines the minimum cost or maximum profit assignment of agents to tasks.

The shortest-route problem finds the shortest route or path between two nodes of a network. Distance, time, and cost are often the criteria used for this model. The shortest-route problem can be expressed as a transshipment problem with one origin and one destination. By shipping one unit from the origin to the destination, the solution will determine the shortest route through the network.

The maximal flow problem can be used to allocate flow to the arcs of the network so that flow through the network system is maximized. Arc capacities determine the maximum amount of flow for each arc. With these flow capacity constraints, the maximal flow problem is expressed as a capacitated transshipment problem.

In the last section of the chapter, we showed how a variation of the transshipment problem could be used to solve a production and inventory problem. In the chapter appendix we show how to use Excel to solve three of the distribution and network problems presented in the chapter.

Glossary

Supply chain The set of all interconnected resources involved in producing and distributing a product.

Transportation problem A network flow problem that often involves minimizing the cost of shipping goods from a set of origins to a set of destinations; it can be formulated and solved as a linear program by including a variable for each arc and a constraint for each node.

Network A graphical representation of a problem consisting of numbered circles (nodes) interconnected by a series of lines (arcs); arrowheads on the arcs show the direction of flow. Transportation, assignment, and transshipment problems are network flow problems.

Nodes The intersection or junction points of a network.

Arcs The lines connecting the nodes in a network.

Dummy origin An origin added to a transportation problem to make the total supply equal to the total demand. The supply assigned to the dummy origin is the difference between the total demand and the total supply.

Capacitated transportation problem A variation of the basic transportation problem in which some or all of the arcs are subject to capacity restrictions.

Transshipment problem An extension of the transportation problem to distribution problems involving transfer points and possible shipments between any pair of nodes.

Capacitated transshipment problem A variation of the transshipment problem in which some or all of the arcs are subject to capacity restrictions.

Assignment problem A network flow problem that often involves the assignment of agents to tasks; it can be formulated as a linear program and is a special case of the transportation problem.

Shortest route Shortest path between two nodes in a network.

Maximal flow The maximum amount of flow that can enter and exit a network system during a given period of time.

Flow capacity The maximum flow for an arc of the network. The flow capacity in one direction may not equal the flow capacity in the reverse direction.

Problems

SELF test

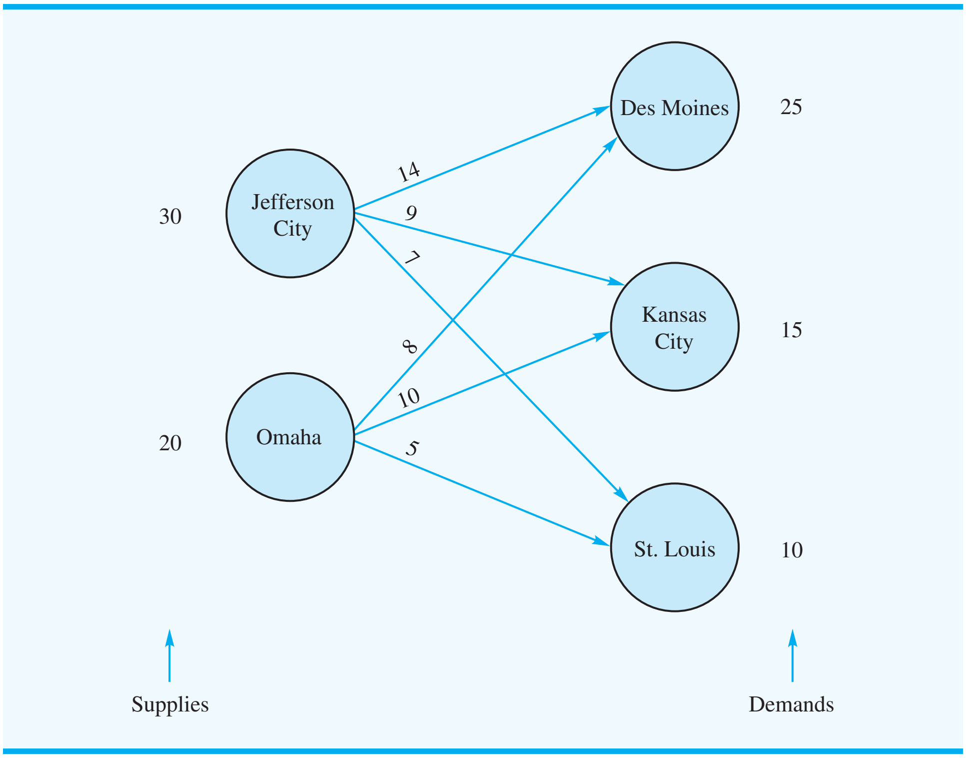
1. A company imports goods at two ports: Philadelphia and New Orleans. Shipments of one product are made to customers in Atlanta, Dallas, Columbus, and Boston. For the next planning period, the supplies at each port, customer demands, and shipping costs per case from each port to each customer are as follows:

	Customers				
Port	**Atlanta**	**Dallas**	**Columbus**	**Boston**	**Port Supply**
Philadelphia	2	6	6	2	5000
New Orleans	1	2	5	7	3000
Demand	1400	3200	2000	1400	

Develop a network representation of the distribution system (transportation problem).

SELF test

2. Consider the following network representation of a transportation problem:



The supplies, demands, and transportation costs per unit are shown on the network.

a. Develop a linear programming model for this problem; be sure to define the variables in your model.
b. Solve the linear program to determine the optimal solution.

3. Tri-County Utilities, Inc., supplies natural gas to customers in a three-county area. The company purchases natural gas from two companies: Southern Gas and Northwest Gas. Demand forecasts for the coming winter season are as follows: Hamilton County, 400 units; Butler County, 200 units; and Clermont County, 300 units. Contracts to provide the following quantities have been written: Southern Gas, 500 units; and Northwest Gas, 400 units. Distribution costs for the counties vary, depending upon the location of the suppliers. The distribution costs per unit (in thousands of dollars) are as follows:

		To	
From	**Hamilton**	**Butler**	**Clermont**
Southern Gas	10	20	15
Northwest Gas	12	15	18

a. Develop a network representation of this problem.
b. Develop a linear programming model that can be used to determine the plan that will minimize total distribution costs.
c. Describe the distribution plan and show the total distribution cost.
d. Recent residential and industrial growth in Butler County has the potential for increasing demand by as much as 100 units. Which supplier should Tri-County contract with to supply the additional capacity?

4. GloFish, Inc. has genetically engineered a species of fish that glows in normal lighting conditions. The company believes the new fish will be a huge success as a new pet option for children and adults alike. GloFish, Inc. has developed two varieties of its glowing fish: one that glows red and one that glows blue. GloFish currently "grows" its fish at two different fish farms in the United States: one in Michigan and one in Texas. The Michigan farm can produce up to 1 million red and 1 million blue GloFish per year; the Texas farm can produce up to 600,000 GloFish, but only in the blue variety. GloFish ships its fish between the fish farms and its three retail stores using a third-party shipper. The shipment rates between origins and destinations are shown in the following table. These costs are per fish and do not depend on the color of the fish being shipped.

		Cost of Shipping GloFish	
	Retailer 1	**Retailer 2**	**Retailer 3**
Michigan	$1.00	$2.50	$0.50
Texas	$2.00	$1.50	$2.80

Estimated demands by each retailer for each color of fish are shown in the following table.

		Demand for GloFish	
	Retailer 1	**Retailer 2**	**Retailer 3**
Red	320,000	300,000	160,000
Blue	380,000	450,000	290,000

a. What is the optimal policy for the fish farms? How many red and blue fish should be produced in Michigan and shipped to each retailer? How many blue fish should be produced in Texas and shipped to each retailer?
b. What is the minimum shipping cost that can be incurred and still meet demand requirements at retailers 1, 2, and 3?
c. How much should GloFish be willing to invest to enable the Texas farm to produce both red and blue GloFish while maintaining the maximum of 600,000 total fish produced at the Texas farm?

5. Premier Consulting's two consultants, Avery and Baker, can be scheduled to work for clients up to a maximum of 160 hours each over the next four weeks. A third consultant, Campbell, has some administrative assignments already planned and is available for clients up to a maximum of 140 hours over the next four weeks. The company has four clients with projects in process. The estimated hourly requirements for each of the clients over the four-week period are as follows:

Client	Hours
A	180
B	75
C	100
D	85

Hourly rates vary for the consultant–client combination and are based on several factors, including project type and the consultant's experience. The rates (dollars per hour) for each consultant–client combination are as follows:

	Client			
Consultant	A	B	C	D
Avery	100	125	115	100
Baker	120	135	115	120
Campbell	155	150	140	130

a. Develop a network representation of the problem.
b. Formulate the problem as a linear program, with the optimal solution providing the hours each consultant should be scheduled for each client to maximize the consulting firm's billings. What is the schedule and what is the total billing?
c. New information shows that Avery doesn't have the experience to be scheduled for client B. If this consulting assignment is not permitted, what impact does it have on total billings? What is the revised schedule?

SELF test

6. Klein Chemicals, Inc., produces a special oil-based material that is currently in short supply. Four of Klein's customers have already placed orders that together exceed the combined capacity of Klein's two plants. Klein's management faces the problem of deciding how many units it should supply to each customer. Because the four customers are in different industries, different prices can be charged because of the various industry pricing structures. However, slightly different production costs at the two plants and varying transportation costs between the plants and customers make a "sell to the highest bidder"

strategy unacceptable. After considering price, production costs, and transportation costs, Klein established the following profit per unit for each plant–customer alternative:

	Customer			
Plant	D_1	D_2	D_3	D_4
Clifton Springs	$32	$34	$32	$40
Danville	$34	$30	$28	$38

The plant capacities and customer orders are as follows:

Plant	Capacity (units)	Distributor Orders (units)
Clifton Springs	5000	D_1 2000
		D_2 5000
Danville	3000	D_3 3000
		D_4 2000

How many units should each plant produce for each customer to maximize profits? Which customer demands will not be met? Show your network model and linear programming formulation.

7. Aggie Power Generation supplies electrical power to residential customers for many U.S. cities. Its main power generation plants are located in Los Angeles, Tulsa, and Seattle. The following table shows Aggie Power Generation's major residential markets, the annual demand in each market (in megawatts or MWs), and the cost to supply electricity to each market from each power generation plant (prices are in $/MW).

	Distribution Costs			
City	**Los Angeles**	**Tulsa**	**Seattle**	**Demand (MWs)**
Seattle	$356.25	$593.75	$59.38	950.00
Portland	$356.25	$593.75	$178.13	831.25
San Francisco	$178.13	$475.00	$296.88	2375.00
Boise	$356.25	$475.00	$296.88	593.75
Reno	$237.50	$475.00	$356.25	950.00
Bozeman	$415.63	$415.63	$296.88	593.75
Laramie	$356.25	$415.63	$356.25	1187.50
Park City	$356.25	$356.25	$475.00	712.50
Flagstaff	$178.13	$475.00	$593.75	1187.50
Durango	$356.25	$296.88	$593.75	1543.75

a. If there are no restrictions on the amount of power that can be supplied by any of the power plants, what is the optimal solution to this problem? Which cities should be supplied by which power plants? What is the total annual power distribution cost for this solution?

b. If at most 4000 MWs of power can be supplied by any one of the power plants, what is the optimal solution? What is the annual increase in power distribution cost that results from adding these constraints to the original formulation?

8. Forbelt Corporation has a one-year contract to supply motors for all refrigerators produced by the Ice Age Corporation. Ice Age manufactures the refrigerators at four locations around the country: Boston, Dallas, Los Angeles, and St. Paul. Plans call for the following number (in thousands) of refrigerators to be produced at each location:

Boston	50
Dallas	70
Los Angeles	60
St. Paul	80

Forbelt's three plants are capable of producing the motors. The plants and production capacities (in thousands) are as follows:

Denver	100
Atlanta	100
Chicago	150

Because of varying production and transportation costs, the profit that Forbelt earns on each lot of 1000 units depends on which plant produced the lot and which destination it was shipped to. The following table gives the accounting department estimates of the profit per unit (shipments will be made in lots of 1000 units):

	Shipped To			
Produced At	**Boston**	**Dallas**	**Los Angeles**	**St. Paul**
Denver	7	11	8	13
Atlanta	20	17	12	10
Chicago	8	18	13	16

With profit maximization as a criterion, Forbelt's management wants to determine how many motors should be produced at each plant and how many motors should be shipped from each plant to each destination.

a. Develop a network representation of this problem.
b. Find the optimal solution.

9. The Ace Manufacturing Company has orders for three similar products:

Product	**Orders (units)**
A	2000
B	500
C	1200

Three machines are available for the manufacturing operations. All three machines can produce all the products at the same production rate. However, due to varying defect percentages of each product on each machine, the unit costs of the products vary depending on the machine used. Machine capacities for the next week and the unit costs are as follows:

Machine	Capacity (units)
1	1500
2	1500
3	1000

	Product		
Machine	A	B	C
1	\$1.00	\$1.20	\$0.90
2	\$1.30	\$1.40	\$1.20
3	\$1.10	\$1.00	\$1.20

Use the transportation model to develop the minimum cost production schedule for the products and machines. Show the linear programming formulation.

10. Hatcher Enterprises uses a chemical called Rbase in production operations at five divisions. Only six suppliers of Rbase meet Hatcher's quality control standards. All six suppliers can produce Rbase in sufficient quantities to accommodate the needs of each division. The quantity of Rbase needed by each Hatcher division and the price per gallon charged by each supplier are as follows:

Division	Demand (1000s of gallons)
1	40
2	45
3	50
4	35
5	45

Supplier	Price per gallon (\$)
1	12.60
2	14.00
3	10.20
4	14.20
5	12.00
6	13.00

The cost per gallon (\$) for shipping from each supplier to each division is provided in the following table:

	Supplier					
Division	1	2	3	4	5	6
1	2.75	2.50	3.15	2.80	2.75	2.75
2	0.80	0.20	5.40	1.20	3.40	1.00
3	4.70	2.60	5.30	2.80	6.00	5.60
4	2.60	1.80	4.40	2.40	5.00	2.80
5	3.40	0.40	5.00	1.20	2.60	3.60

Hatcher believes in spreading its business among suppliers so that the company will be less affected by supplier problems (e.g., labor strikes or resource availability). Company policy requires that each division have a separate supplier.

a. For each supplier–division combination, compute the total cost of supplying the division's demand.
b. Determine the optimal assignment of suppliers to divisions.

SELF test

11. The distribution system for the Herman Company consists of three plants, two warehouses, and four customers. Plant capacities and shipping costs per unit (in $) from each plant to each warehouse are as follows:

	Warehouse		
Plant	**1**	**2**	**Capacity**
1	4	7	450
2	8	5	600
3	5	6	380

Customer demand and shipping costs per unit (in $) from each warehouse to each customer are as follows:

a. Develop a network representation of this problem.
b. Formulate a linear programming model of the problem.
c. Solve the linear program to determine the optimal shipping plan.

	Customer			
Warehouse	**1**	**2**	**3**	**4**
1	6	4	8	4
2	3	6	7	7
Demand	300	300	300	400

12. Refer to Problem 11. Suppose that shipments between the two warehouses are permitted at $2 per unit and that direct shipments can be made from plant 3 to customer 4 at a cost of $7 per unit.
a. Develop a network representation of this problem.
b. Formulate a linear programming model of this problem.
c. Solve the linear program to determine the optimal shipping plan.

13. Sports of All Sorts produces, distributes, and sells high-quality skateboards. Its supply chain consists of three factories (located in Detroit, Los Angeles, and Austin) that produce skateboards. The Detroit and Los Angeles facilities can produce 350 skateboards per week, but the Austin plant is larger and can produce up to 700 skateboards per week. Skateboards must be shipped from the factories to one of four distribution centers, or DCs (located in Iowa, Maryland, Idaho, and Arkansas). Each distribution center can process (repackage, mark for sale, and ship) at most 500 skateboards per week.

Skateboards are then shipped from the distribution centers to retailers. Sports of All Sorts supplies three major U.S. retailers: Just Sports, Sports 'N Stuff, and The Sports Dude. The weekly demands are 200 skateboards at Just Sports, 500 skateboards at Sports 'N Stuff, and 650 skateboards at The Sports Dude. The following tables display the per-unit costs for shipping skateboards between the factories and DCs and for shipping between the DCs and the retailers.

	Shipping Costs ($ per skateboard)			
Factory/DCs	**Iowa**	**Maryland**	**Idaho**	**Arkansas**
Detroit	$25.00	$25.00	$35.00	$40.00
Los Angeles	$35.00	$45.00	$35.00	$42.50
Austin	$40.00	$40.00	$42.50	$32.50
Retailers/DCs	**Iowa**	**Maryland**	**Idaho**	**Arkansas**
Just Sports	$30.00	$20.00	$35.00	$27.50
Sports 'N Stuff	$27.50	$32.50	$40.00	$25.00
The Sports Dude	$30.00	$40.00	$32.50	$42.50

a. Draw the network representation for this problem.
b. Build a model to minimize the transportation cost of a logistics system that will deliver skateboards from the factories to the distribution centers and from the distribution centers to the retailers. What is the optimal production strategy and shipping pattern for Sports of All Sorts? What is the minimum attainable transportation cost?
c. Sports of All Sorts is considering expansion of the Iowa DC capacity to 800 units per week. The annual amortized cost of expansion is $40,000. Should the company expand the Iowa DC capacity so that it can process 800 skateboards per week? (Assume 50 operating weeks per year.)

14. The Moore & Harman Company is in the business of buying and selling grain. An important aspect of the company's business is arranging for the purchased grain to be shipped to customers. If the company can keep freight costs low, profitability will improve.

    The company recently purchased three rail cars of grain at Muncie, Indiana; six rail cars at Brazil, Indiana; and five rail cars at Xenia, Ohio. Twelve carloads of grain have been sold. The locations and the amount sold at each location are as follows:

Location	Number of Rail Car Loads
Macon, GA	2
Greenwood, SC	4
Concord, SC	3
Chatham, NC	3

All shipments must be routed through either Louisville or Cincinnati. Shown are the shipping costs per bushel (in cents) from the origins to Louisville and Cincinnati and the costs per bushel to ship from Louisville and Cincinnati to the destinations.

	To		
From	**Louisville**	**Cincinnati**	
Muncie	8	6	← Cost per bushel from Muncie to Cincinnati is 6¢
Brazil	3	8	
Xenia	9	3	

	To			
From	**Macon**	**Greenwood**	**Concord**	**Chatham**
Louisville	44	34	34	32
Cincinnati	57	35	28	24

Cost per bushel from Cincinnati to Greenwood is 35¢

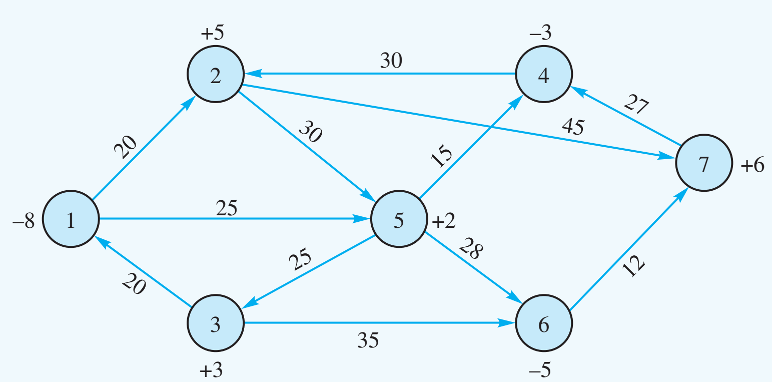
Determine a shipping schedule that will minimize the freight costs necessary to satisfy demand. Which (if any) rail cars of grain must be held at the origin until buyers can be found?

15. The following linear programming formulation is for a transshipment problem:

$$
\begin{array}{ll}
\text{Min} & 11x_{13} + 12x_{14} + 10x_{21} + 8x_{34} + 10x_{35} + 11x_{42} + 9x_{45} + 12x_{52} \\
\text{s.t.} & \\
& \begin{array}{rrrrrrrrl}
x_{13} + & x_{14} - & x_{21} & & & & & & \leq 5 \\
& & x_{21} & & & - \; x_{42} & & - \; x_{52} & \leq 3 \\
x_{13} & & & - \; x_{34} - & x_{35} & & & & = 6 \\
& - \; x_{14} & & - \; x_{34} & & + \; x_{42} + & x_{45} & & \leq 2 \\
& & & & x_{35} & & + \; x_{45} - & x_{52} & = 4
\end{array} \\
& x_{ij} \geq 0 \quad \text{for all } i, j
\end{array}
$$

Show the network representation of this problem.

16. A rental car company has an imbalance of cars at seven of its locations. The following network shows the locations of concern (the nodes) and the cost to move a car between locations. A positive number by a node indicates an excess supply at the node, and a negative number indicates an excess demand.



a. Develop a linear programming model of this problem.
b. Solve the model formulated in part (a) to determine how the cars should be redistributed among the locations.

SELF test

17. Scott and Associates, Inc., is an accounting firm that has three new clients. Project leaders will be assigned to the three clients. Based on the different backgrounds and experiences of the leaders, the various leader–client assignments differ in terms of projected completion times. The possible assignments and the estimated completion times in days are as follows:

	Client		
Project Leader	**1**	**2**	**3**
Jackson	10	16	32
Ellis	14	22	40
Smith	22	24	34

a. Develop a network representation of this problem.
b. Formulate the problem as a linear program, and solve. What is the total time required?

18. CarpetPlus sells and installs floor covering for commercial buildings. Brad Sweeney, a CarpetPlus account executive, was just awarded the contract for five jobs. Brad must now assign a CarpetPlus installation crew to each of the five jobs. Because the commission Brad will earn depends on the profit CarpetPlus makes, Brad would like to determine an assignment that will minimize total installation costs. Currently, five installation crews are available for assignment. Each crew is identified by a color code, which aids in tracking of job progress on a large white board. The following table shows the costs (in hundreds of dollars) for each crew to complete each of the five jobs:

	Job				
Crew	**1**	**2**	**3**	**4**	**5**
Red	30	44	38	47	31
White	25	32	45	44	25
Blue	23	40	37	39	29
Green	26	38	37	45	28
Brown	26	34	44	43	28

a. Develop a network representation of the problem.
b. Formulate and solve a linear programming model to determine the minimum cost assignment.

19. A local television station plans to drop four Friday evening programs at the end of the season. Steve Botuchis, the station manager, developed a list of six potential replacement programs. Estimates of the advertising revenue ($) that can be expected for each of the new programs in the four vacated time slots are as follows. Mr. Botuchis asked you to find the assignment of programs to time slots that will maximize total advertising revenue.

	5:00–5:30 P.M.	**5:30–6:00 P.M.**	**7:00–7:30 P.M.**	**8:00–8:30 P.M.**
Home Improvement	5000	3000	6000	4000
World News	7500	8000	7000	5500
NASCAR Live	8500	5000	6500	8000
Wall Street Today	7000	6000	6500	5000
Hollywood Briefings	7000	8000	3000	6000
Ramundo & Son	6000	4000	4500	7000

20. The U.S. Cable Company uses a distribution system with five distribution centers and eight customer zones. Each customer zone is assigned a sole source supplier; each customer zone receives all of its cable products from the same distribution center. In an effort to balance demand and workload at the distribution centers, the company's vice president of logistics specified that distribution centers may not be assigned more than three customer zones. The following table shows the five distribution centers and cost of supplying each customer zone (in thousands of dollars):

	Customer Zones							
Distribution Centers	**Los Angeles**	**Chicago**	**Columbus**	**Atlanta**	**Newark**	**Kansas City**	**Denver**	**Dallas**
Plano	70	47	22	53	98	21	27	13
Nashville	75	38	19	58	90	34	40	26
Flagstaff	15	78	37	82	111	40	29	32
Springfield	60	23	8	39	82	36	32	45
Boulder	45	40	29	75	86	25	11	37

a. Determine the assignment of customer zones to distribution centers that will minimize cost.
b. Which distribution centers, if any, are not used?
c. Suppose that each distribution center is limited to a maximum of two customer zones. How does this constraint change the assignment and the cost of supplying customer zones?

21. United Express Service (UES) uses large quantities of packaging materials at its four distribution hubs. After screening potential suppliers, UES identified six vendors that can provide packaging materials that will satisfy its quality standards. UES asked each of the six vendors to submit bids to satisfy annual demand at each of its four distribution hubs over the next year. The following table lists the bids received (in thousands of dollars). UES wants to ensure that each of the distribution hubs is serviced by a different vendor. Which bids should UES accept, and which vendors should UES select to supply each distribution hub?

	Distribution Hub			
Bidder	**1**	**2**	**3**	**4**
Martin Products	190	175	125	230
Schmidt Materials	150	235	155	220
Miller Containers	210	225	135	260
D&J Burns	170	185	190	280
Larbes Furnishings	220	190	140	240
Lawler Depot	270	200	130	260

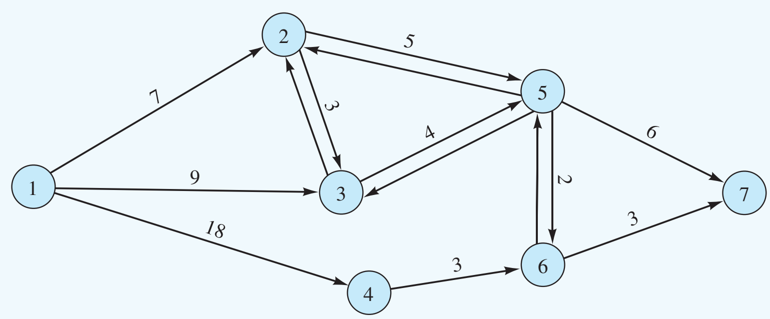
22. The quantitative methods department head at a major midwestern university will be scheduling faculty to teach courses during the coming autumn term. Four core courses need to be covered. The four courses are at the undergraduate (UG), master of business administration (MBA), master of science (MS), and doctor of philosophy (Ph.D.) levels. Four professors will be assigned to the courses, with each professor receiving one of the courses. Student evaluations of professors are available from previous terms. Based on a rating scale of 4 (excellent), 3 (very good), 2 (average), 1 (fair), and 0 (poor), the average student evaluations for each professor are shown. Professor D does not have a Ph.D. and cannot

be assigned to teach the Ph.D. level course. If the department head makes teaching assignments based on maximizing the student evaluation ratings over all four courses, what staffing assignments should be made?

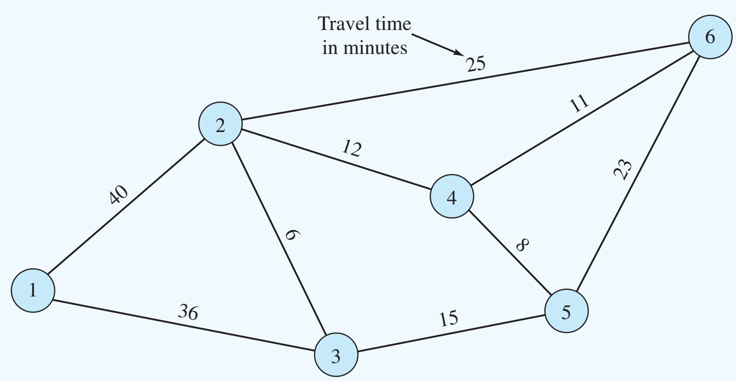
	Course			
Professor	**UG**	**MBA**	**MS**	**Ph.D.**
A	2.8	2.2	3.3	3.0
B	3.2	3.0	3.6	3.6
C	3.3	3.2	3.5	3.5
D	3.2	2.8	2.5	—

SELF test

23. Find the shortest route from node 1 to node 7 in the network shown.

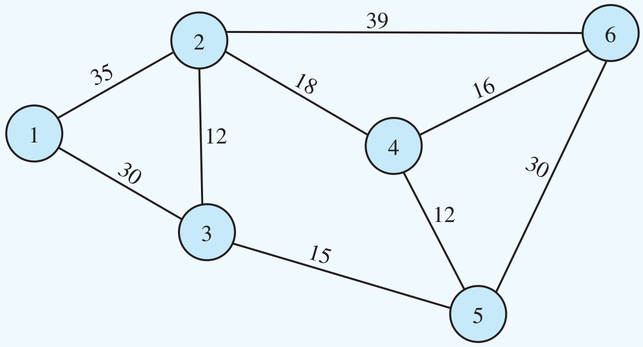


24. In the original Gorman Construction Company problem, we found the shortest distance from the office (node 1) to the construction site located at node 6. Because some of the roads are highways and others are city streets, the shortest-distance routes between the office and the construction site may not necessarily provide the quickest or shortest-time route. Shown here is the Gorman road network with travel time rather than distance. Find the shortest route from Gorman's office to the construction site at node 6 if the objective is to minimize travel time rather than distance.



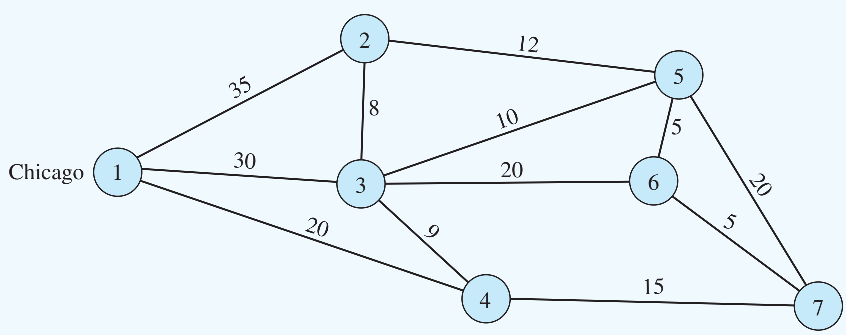
25. Cleveland Area Rapid Delivery (CARD) operates a delivery service in the Cleveland metropolitan area. Most of CARD's business involves rapid delivery of documents and parcels

between offices during the business day. CARD promotes its ability to make fast and on-time deliveries anywhere in the metropolitan area. When a customer calls with a delivery request, CARD quotes a guaranteed delivery time. The following network shows the street routes available. The numbers above each arc indicate the travel time in minutes between the two locations.

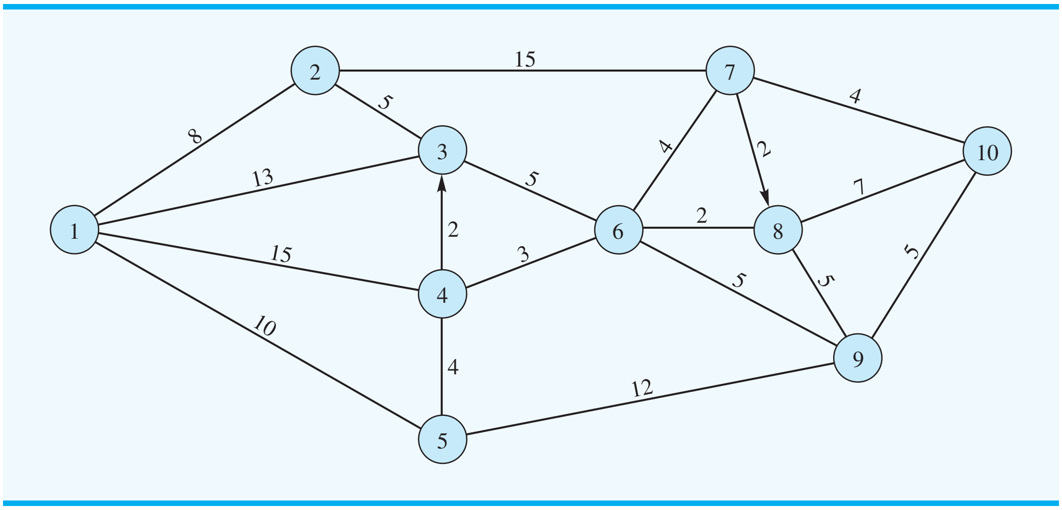


a. Develop a linear programming model that can be used to find the minimum time required to make a delivery from location 1 to location 6.
b. How long does it take to make a delivery from location 1 to location 6?
c. Assume that it is now 1:00 P.M. and that CARD just received a request for a pickup at location 1. The closest CARD courier is 8 minutes away from location 1. If CARD provides a 20% safety margin in guaranteeing a delivery time, what is the guaranteed delivery time if the package picked up at location 1 is to be delivered to location 6?

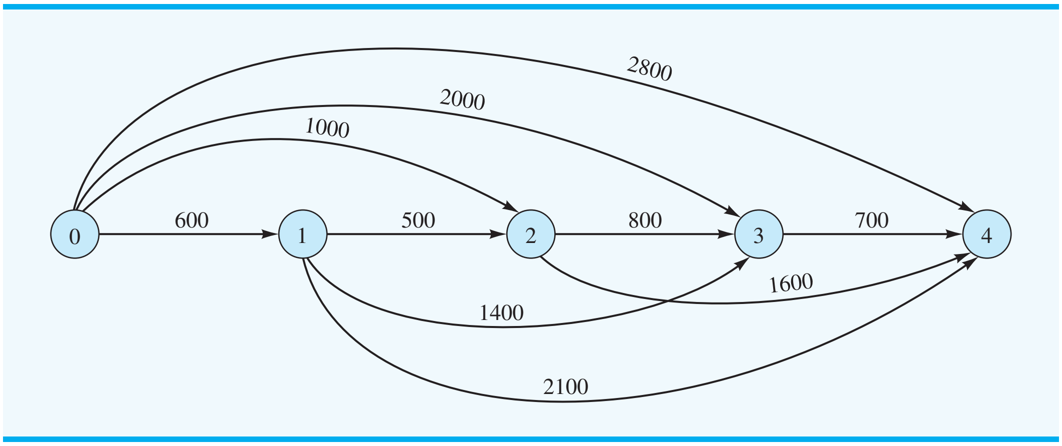
26. Morgan Trucking Company operates a special pickup and delivery service between Chicago and six other cities located in a four-state area. When Morgan receives a request for service, it dispatches a truck from Chicago to the city requesting service as soon as possible. With both fast service and minimum travel costs as objectives for Morgan, it is important that the dispatched truck take the shortest route from Chicago to the specified city. Assume that the following network (not drawn to scale) with distances given in miles represents the highway network for this problem. Find the shortest-route distances from Chicago to node 6.



27. City Cab Company identified 10 primary pickup and drop locations for cab riders in New York City. In an effort to minimize travel time and improve customer service and the utilization of the company's fleet of cabs, management would like the cab drivers to take the shortest route between locations whenever possible. Using the following network of roads and streets, what is the route a driver beginning at location 1 should take to reach location 10? The travel times in minutes are shown on the arcs of the network. Note that there are two one-way streets and that the direction is shown by the arrows.

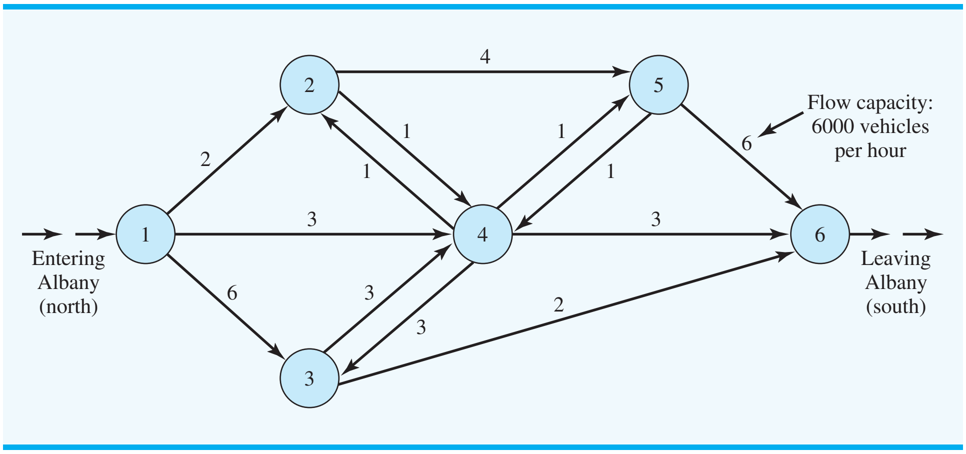


28. The five nodes in the following network represent points one year apart over a four-year period. Each node indicates a time when a decision is made to keep or replace a firm's computer equipment. If a decision is made to replace the equipment, a decision must also be made as to how long the new equipment will be used. The arc from node 0 to node 1 represents the decision to keep the current equipment one year and replace it at the end of the year. The arc from node 0 to node 2 represents the decision to keep the current equipment two years and replace it at the end of year 2. The numbers above the arcs indicate the total cost associated with the equipment replacement decisions. These costs include discounted purchase price, trade-in value, operating costs, and maintenance costs. Use a shortest-route model to determine the minimum cost equipment replacement policy for the four-year period.



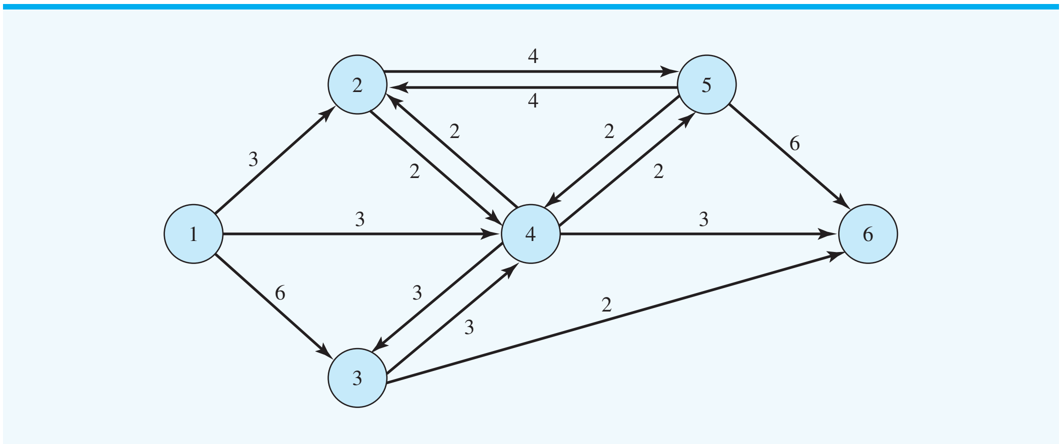
SELF test

29. The north–south highway system passing through Albany, New York, can accommodate the capacities shown.



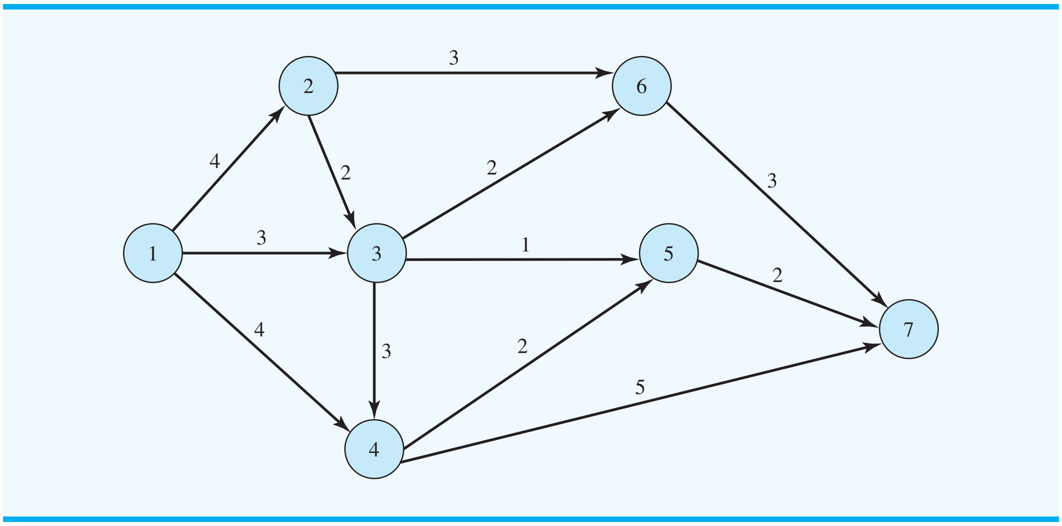
Can the highway system accommodate a north–south flow of 10,000 vehicles per hour?

30. If the Albany highway system described in Problem 29 has revised flow capacities as shown in the following network, what is the maximal flow in vehicles per hour through the system? How many vehicles per hour must travel over each road (arc) to obtain this maximal flow?

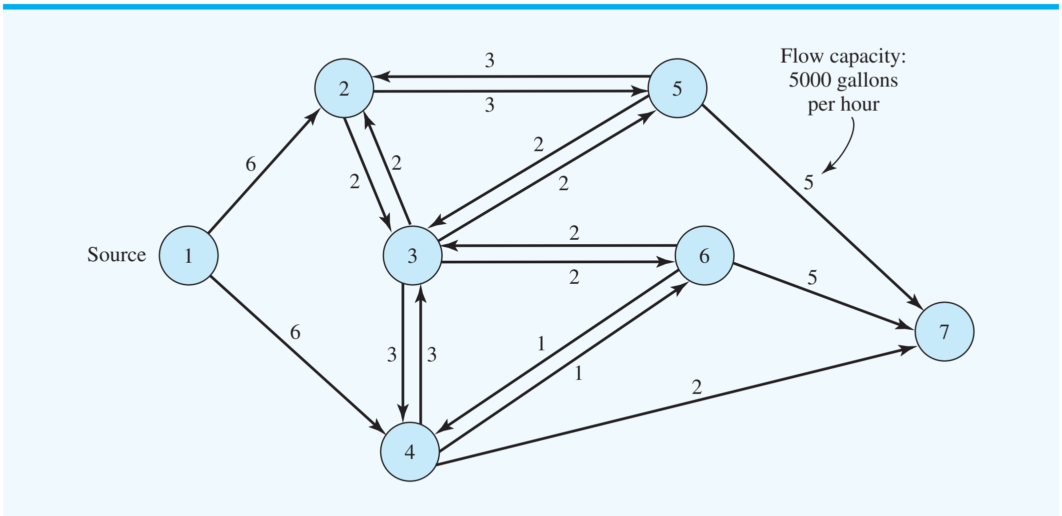


31. A long-distance telephone company uses a fiber-optic network to transmit phone calls and other information between locations. Calls are carried through cable lines and switching nodes. A portion of the company's transmission network is shown here. The numbers above each arc show the capacity in thousands of messages that can be transmitted over that branch of the network.

To keep up with the volume of information transmitted between origin and destination points, use the network to determine the maximum number of messages that may be sent from a city located at node 1 to a city located at node 7.



32. The High-Price Oil Company owns a pipeline network that is used to convey oil from its source to several storage locations. A portion of the network is as follows:

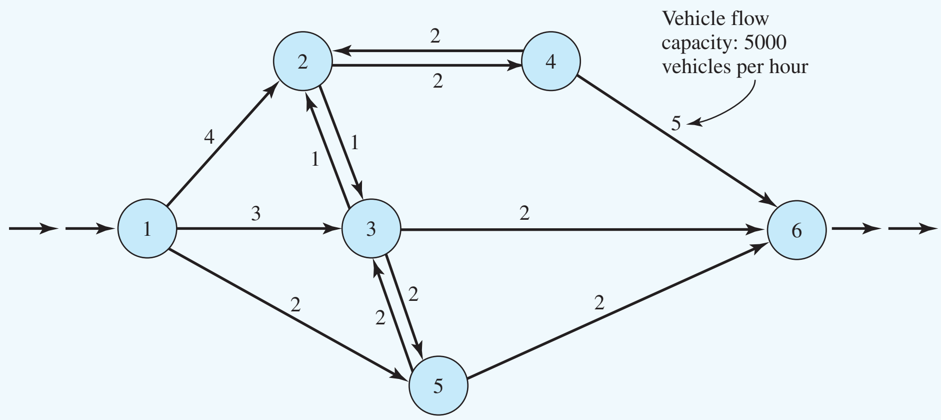


Due to the varying pipe sizes, the flow capacities vary. By selectively opening and closing sections of the pipeline network, the firm can supply any of the storage locations.

a. If the firm wants to fully utilize the system capacity to supply storage location 7, how long will it take to satisfy a location 7 demand of 100,000 gallons? What is the maximal flow for this pipeline system?

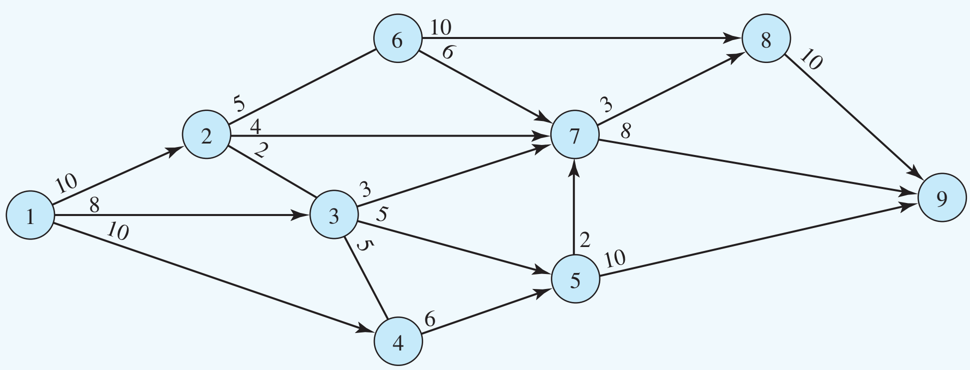
b. If a break occurs on line 2–3 and that line is closed down, what is the maximal flow for the system? How long will it take to transmit 100,000 gallons to location 7?

33. For the following highway network system, determine the maximal flow in vehicles per hour:



The highway commission is considering adding highway section 3–4 to permit a flow of 2000 vehicles per hour or, at an additional cost, a flow of 3000 vehicles per hour. What is your recommendation for the 3–4 arc of the network?

34. A chemical processing plant has a network of pipes that are used to transfer liquid chemical products from one part of the plant to another. The following pipe network has pipe flow capacities in gallons per minute as shown. What is the maximum flow capacity for the system if the company wishes to transfer as much liquid chemical as possible from location 1 to location 9? How much of the chemical will flow through the section of pipe from node 3 to node 5?



35. Refer to the Contois Carpets problem, for which the network representation is shown in Figure 10.20. Suppose that Contois has a beginning inventory of 50 yards of carpet and requires an inventory of 100 yards at the end of quarter 4.
    a. Develop a network representation of this modified problem.
    b. Develop a linear programming model and solve for the optimal solution.

36. Sanders Fishing Supply of Naples, Florida, manufactures a variety of fishing equipment that it sells throughout the United States. For the next three months, Sanders estimates demand for a particular product at 150, 250, and 300 units, respectively. Sanders can

supply this demand by producing on regular time or overtime. Because of other commitments and anticipated cost increases in month 3, the production capacities in units and the production costs per unit are as follows:

Production	Capacity (units)	Cost per Unit
Month 1—Regular	275	\$ 50
Month 1—Overtime	100	\$ 80
Month 2—Regular	200	\$ 50
Month 2—Overtime	50	\$ 80
Month 3—Regular	100	\$ 60
Month 3—Overtime	50	\$100

Inventory may be carried from one month to the next, but the cost is \$20 per unit per month. For example, regular production from month 1 used to meet demand in month 2 would cost Sanders \$50 + \$20 = \$70 per unit. This same month 1 production used to meet demand in month 3 would cost Sanders \$50 + 2(\$20) = \$90 per unit.

a. Develop a network representation of this production scheduling problem as a transportation problem. (*Hint*: Use six origin nodes; the supply for origin node 1 is the maximum that can be produced in month 1 on regular time, and so on.)
b. Develop a linear programming model that can be used to schedule regular and overtime production for each of the three months.
c. What is the production schedule, how many units are carried in inventory each month, and what is the total cost?
d. Is there any unused production capacity? If so, where?

Case Problem 1 Solutions Plus

Solutions Plus is an industrial chemicals company that produces specialized cleaning fluids and solvents for a wide variety of applications. Solutions Plus just received an invitation to submit a bid to supply Great North American railroad with a cleaning fluid for locomotives. Great North American needs the cleaning fluid at 11 locations (railway stations); it provided the following information to Solutions Plus regarding the number of gallons of cleaning fluid required at each location (see Table 10.8).

Solutions Plus can produce the cleaning fluid at its Cincinnati plant for \$1.20 per gallon. Even though the Cincinnati location is its only plant, Solutions Plus has negotiated with

TABLE 10.8 GALLONS OF CLEANING FLUID REQUIRED AT EACH LOCATION

Location	Gallons Required	Location	Gallons Required
Santa Ana	22,418	Glendale	33,689
El Paso	6,800	Jacksonville	68,486
Pendleton	80,290	Little Rock	148,586
Houston	100,447	Bridgeport	111,475
Kansas City	241,570	Sacramento	112,000
Los Angeles	64,761		

TABLE 10.9 FREIGHT COST ($ PER GALLON)

	Cincinnati	Oakland
Santa Ana	—	0.22
El Paso	0.84	0.74
Pendleton	0.83	0.49
Houston	0.45	—
Kansas City	0.36	—
Los Angeles	—	0.22
Glendale	—	0.22
Jacksonville	0.34	—
Little Rock	0.34	—
Bridgeport	0.34	—
Sacramento	—	0.15

an industrial chemicals company located in Oakland, California, to produce and ship up to 500,000 gallons of the locomotive cleaning fluid to selected Solutions Plus customer locations. The Oakland company will charge Solutions Plus $1.65 per gallon to produce the cleaning fluid, but Solutions Plus thinks that the lower shipping costs from Oakland to some customer locations may offset the added cost to produce the product.

The president of Solutions Plus, Charlie Weaver, contacted several trucking companies to negotiate shipping rates between the two production facilities (Cincinnati and Oakland) and the locations where the railroad locomotives are cleaned. Table 10.9 shows the quotes received in terms of dollars per gallon. The "—" entries in Table 10.9 identify shipping routes that will not be considered because of the large distances involved. These quotes for shipping rates are guaranteed for one year.

To submit a bid to the railroad company, Solutions Plus must determine the price per gallon it will charge. Solutions Plus usually sells its cleaning fluids for 15% more than its cost to produce and deliver the product. For this big contract, however, Fred Roedel, the director of marketing, suggested that maybe the company should consider a smaller profit margin. In addition, to ensure that if Solutions Plus wins the bid, it will have adequate capacity to satisfy existing orders as well as accept orders for other new business, the management team decided to limit the number of gallons of the locomotive cleaning fluid produced in the Cincinnati plant to 500,000 gallons at most.

Managerial Report

You are asked to make recommendations that will help Solutions Plus prepare a bid. Your report should address, but not be limited to, the following issues:

1. If Solutions Plus wins the bid, which production facility (Cincinnati or Oakland) should supply the cleaning fluid to the locations where the railroad locomotives are cleaned? How much should be shipped from each facility to each location?
2. What is the breakeven point for Solutions Plus? That is, how low can the company go on its bid without losing money?
3. If Solutions Plus wants to use its standard 15% markup, how much should it bid?
4. Freight costs are significantly affected by the price of oil. The contract on which Solutions Plus is bidding is for two years. Discuss how fluctuation in freight costs might affect the bid Solutions Plus submits.

Case Problem 2 Supply Chain Design

The Darby Company manufactures and distributes meters used to measure electric power consumption. The company started with a small production plant in El Paso and gradually built a customer base throughout Texas. A distribution center was established in Fort Worth, Texas, and later, as business expanded, a second distribution center was established in Santa Fe, New Mexico.

The El Paso plant was expanded when the company began marketing its meters in Arizona, California, Nevada, and Utah. With the growth of the West Coast business, the Darby Company opened a third distribution center in Las Vegas and just two years ago opened a second production plant in San Bernardino, California.

Manufacturing costs differ between the company's production plants. The cost of each meter produced at the El Paso plant is $10.50. The San Bernardino plant utilizes newer and more efficient equipment; as a result, manufacturing costs are $0.50 per meter less than at the El Paso plant.

Due to the company's rapid growth, not much attention had been paid to the efficiency of its supply chain, but Darby's management decided that it is time to address this issue. The cost of shipping a meter from each of the two plants to each of the three distribution centers is shown in Table 10.10.

The quarterly production capacity is 30,000 meters at the older El Paso plant and 20,000 meters at the San Bernardino plant. Note that no shipments are allowed from the San Bernardino plant to the Fort Worth distribution center.

The company serves nine customer zones from the three distribution centers. The forecast of the number of meters needed in each customer zone for the next quarter is shown in Table 10.11.

TABLE 10.10 SHIPPING COST PER UNIT FROM PRODUCTION PLANTS TO DISTRIBUTION CENTERS (IN $)

	Distribution Center		
Plant	**Fort Worth**	**Santa Fe**	**Las Vegas**
El Paso	3.20	2.20	4.20
San Bernardino	—	3.90	1.20

TABLE 10.11 QUARTERLY DEMAND FORECAST

Customer Zone	Demand (meters)
Dallas	6300
San Antonio	4880
Wichita	2130
Kansas City	1210
Denver	6120
Salt Lake City	4830
Phoenix	2750
Los Angeles	8580
San Diego	4460

TABLE 10.12 SHIPPING COST FROM THE DISTRIBUTION CENTERS TO THE CUSTOMER ZONES

	Customer Zone								
Distribution Center	**Dallas**	**San Antonio**	**Wichita**	**Kansas City**	**Denver**	**Salt Lake City**	**Phoenix**	**Los Angeles**	**San Diego**
Fort Worth	0.3	2.1	3.1	4.4	6.0	—	—	—	—
Santa Fe	5.2	5.4	4.5	6.0	2.7	4.7	3.4	3.3	2.7
Las Vegas	—	—	—	—	5.4	3.3	2.4	2.1	2.5

The cost per unit of shipping from each distribution center to each customer zone is given in Table 10.12; note that some distribution centers cannot serve certain customer zones. These are indicated by a dash, "—".

In its current supply chain, demand at the Dallas, San Antonio, Wichita, and Kansas City customer zones is satisfied by shipments from the Fort Worth distribution center. In a similar manner, the Denver, Salt Lake City, and Phoenix customer zones are served by the Santa Fe distribution center, and the Los Angeles and San Diego customer zones are served by the Las Vegas distribution center. To determine how many units to ship from each plant, the quarterly customer demand forecasts are aggregated at the distribution centers, and a transportation model is used to minimize the cost of shipping from the production plants to the distribution centers.

Managerial Report

You are asked to make recommendations for improving Darby Company's supply chain. Your report should address, but not be limited to, the following issues:

1. If the company does not change its current supply chain, what will its distribution costs be for the following quarter?
2. Suppose that the company is willing to consider dropping the distribution center limitations; that is, customers could be served by any of the distribution centers for which costs are available. Can costs be reduced? If so, by how much?
3. The company wants to explore the possibility of satisfying some of the customer demand directly from the production plants. In particular, the shipping cost is $0.30 per unit from San Bernardino to Los Angeles and $0.70 from San Bernardino to San Diego. The cost for direct shipments from El Paso to San Antonio is $3.50 per unit. Can distribution costs be further reduced by considering these direct plant-to-customer shipments?
4. Over the next five years, Darby is anticipating moderate growth (5000 meters) to the north and west. Would you recommend that Darby consider plant expansion at this time?

Appendix 10.1 Excel Solution of Transportation, Transshipment, and Assignment Problems

In this appendix we will use an Excel worksheet to solve transportation, transshipment, and assignment problems. We start with the Foster Generators transportation problem (see Section 10.1).

Transportation Problem

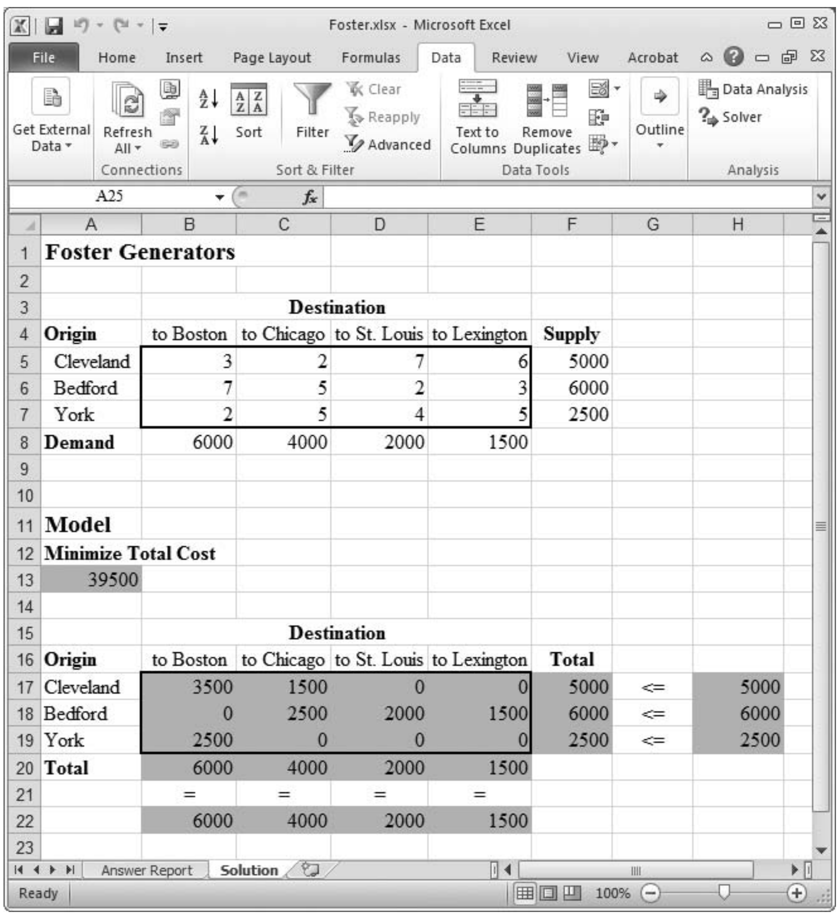
The first step is to enter the data for the transportation costs, the origin supplies, and the destination demands in the top portion of the worksheet. Then the linear programming model is developed in the bottom portion of the worksheet. As with all linear programs, the worksheet model has four key elements: the decision variables, the objective function, the constraint left-hand sides, and the constraint right-hand sides. For a transportation problem, the decision variables are the amounts shipped from each origin to each destination; the objective function is the total transportation cost; the left-hand sides are the number of units shipped from each origin and the number of units shipped into each destination; and the right-hand sides are the origin supplies and the destination demands.

The formulation and solution of the Foster Generators problem are shown in Figure 10.22. The data are in the top portion of the worksheet. The model appears in the bottom portion of the worksheet.

FIGURE 10.22 EXCEL SOLUTION OF THE FOSTER GENERATORS PROBLEM

WEB file

Foster



	A	B	C	D	E	F	G	H
1	Foster Generators							
2								
3				Destination				
4	Origin	to Boston	to Chicago	to St. Louis	to Lexington	Supply		
5	Cleveland	3	2	7	6	5000		
6	Bedford	7	5	2	3	6000		
7	York	2	5	4	5	2500		
8	Demand	6000	4000	2000	1500			
9								
10								
11	Model							
12	Minimize Total Cost							
13	39500							
14								
15				Destination				
16	Origin	to Boston	to Chicago	to St. Louis	to Lexington	Total		
17	Cleveland	3500	1500	0	0	5000	<=	5000
18	Bedford	0	2500	2000	1500	6000	<=	6000
19	York	2500	0	0	0	2500	<=	2500
20	Total	6000	4000	2000	1500			
21		=	=	=	=			
22		6000	4000	2000	1500			
23								

Formulation

The data and descriptive labels are contained in cells A1:F8. The transportation costs are in cells B5:E7. The origin supplies are in cells F5:F7, and the destination demands are in cells B8:E8. The key elements of the model required by the Excel Solver are the decision variables, the objective function, the constraint left-hand sides, and the constraint right-hand sides.

Decision Variables Cells B17:E19 are reserved for the decision variables. The optimal values are shown to be $x_{11} = 3500$, $x_{12} = 1500$, $x_{22} = 2500$, $x_{23} = 2000$, $x_{24} = 1500$, and $x_{41} = 2500$. All other decision variables equal zero, indicating that nothing will be shipped over the corresponding routes.

Objective Function The formula SUMPRODUCT(B5:E7,B17:E19) has been placed into cell A13 to compute the cost of the solution. The minimum cost solution is shown to have a value of $39,500.

Left-Hand Sides Cells F17:F19 contain the left-hand sides for the supply constraints, and cells B20:E20 contain the left-hand sides for the demand constraints.

Cell F17 = SUM(B17:E17) (Copy to F18:F19)
Cell B20 = SUM(B17:B19) (Copy to C20:E20)

Right-Hand Sides Cells H17:H19 contain the right-hand sides for the supply constraints, and cells B22:E22 contain the right-hand sides for the demand constraints.

Cell H17 = F5 (Copy to H18:H19)
Cell B22 = B8 (Copy to C22:E22)

Excel Solution

The solution shown in Figure 10.22 can be obtained by selecting **Solver** from the **Analysis Group** in the **Data Ribbon**. The Data Ribbon is displayed at the top of the worksheet in Figure 10.22.When the **Solver Parameters** dialog box appears, enter the proper values for the constraints and the objective function, select **Simplex LP,** and click the checkbox for **Make Unconstrained Variables Non-negative.** Then click **Solve.** The information entered into the **Solver Parameters** dialog box is shown in Figure 10.23.

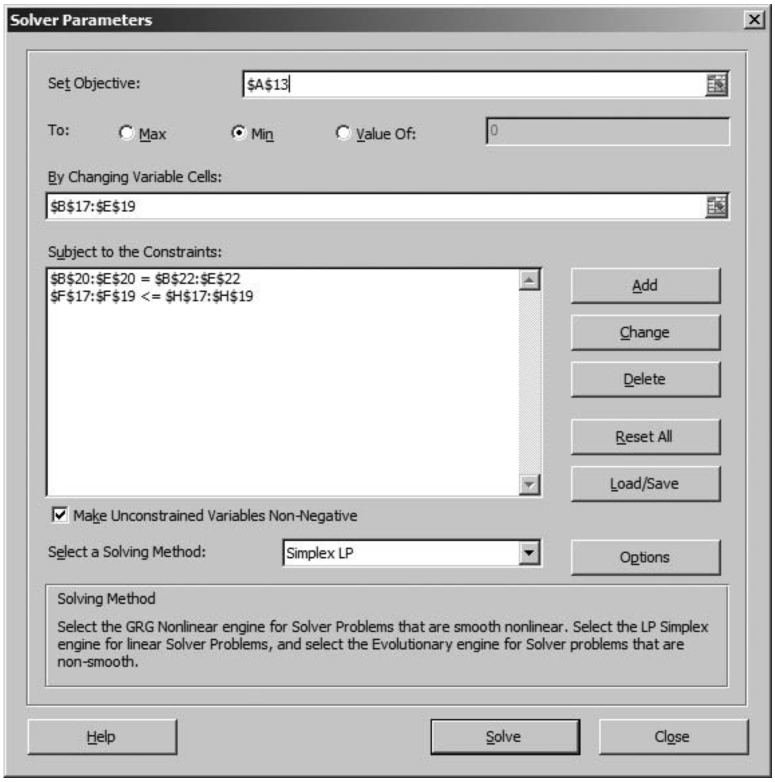
Transshipment Problem

The worksheet model we present for the transshipment problem can be used for all the network flow problems (transportation, transshipment, and assignment) in this chapter. We organize the worksheet into two sections: an arc section and a node section. Let us illustrate by showing the worksheet formulation and solution of the Ryan Electronics transshipment problem. Refer to Figure 10.24 as we describe the steps involved.

Formulation

The arc section uses cells A4:C16. Each arc is identified in cells A5:A16. The arc costs are identified in cells B5:B16, and cells C5:C16 are reserved for the values of the decision variables (the amount shipped over the arcs).

FIGURE 10.23 EXCEL SOLVER PARAMETERS DIALOG BOX FOR THE FOSTER GENERATORS PROBLEM



The node section uses cells F5:K14. Each of the nodes is identified in cells F7:F14. The following formulas are entered into cells G7:H14 to represent the flow out and the flow in for each node:

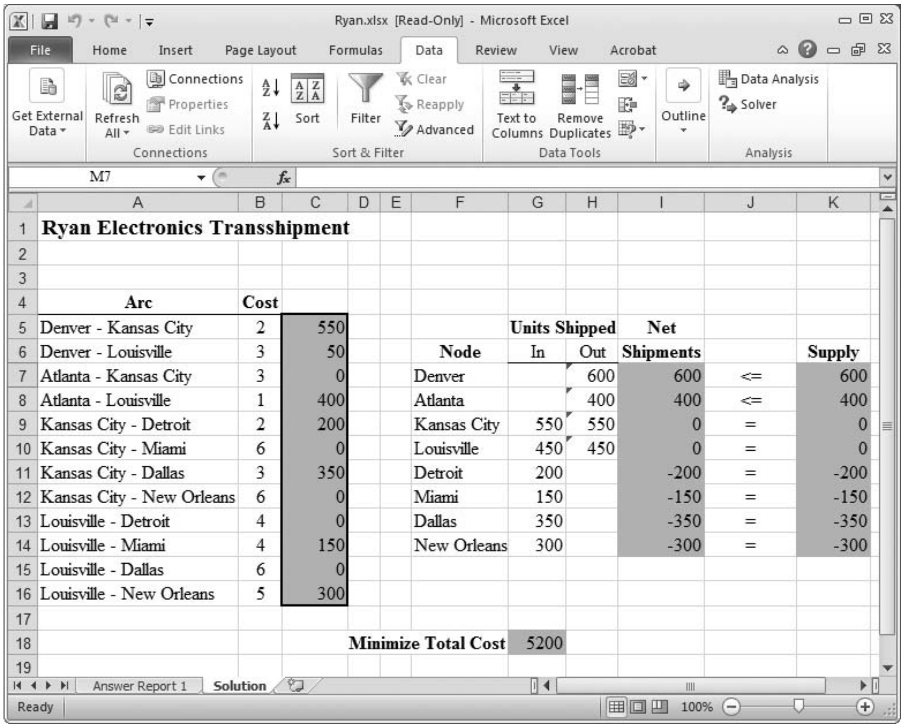
Units shipped in:	Cell G9 = C5+C7
	Cell G10 = C6+C8
	Cell G11 = C9+C13
	Cell G12 = C10+C14
	Cell G13 = C11+C15
	Cell G14 = C12+C16
Units shipped out:	Cell H7 = SUM(C5:C6)
	Cell H8 = SUM(C7:C8)
	Cell H9 = SUM(C9:C12)
	Cell H10 = SUM(C13:C16)

The net shipments in cells I7:I14 are the flows out minus the flows in for each node. For supply nodes, the flow out will exceed the flow in, resulting in positive net shipments. For demand nodes, the flow out will be less than the flow in, resulting in negative net

FIGURE 10.24 EXCEL SOLUTION FOR THE RYAN ELECTRONICS PROBLEM

WEB file

Ryan



	A	B	C	D	E	F	G	H	I	J	K
1	**Ryan Electronics Transshipment**										
2											
3											
4	**Arc**	**Cost**									
5	Denver - Kansas City	2	550				**Units Shipped**		**Net**		
6	Denver - Louisville	3	50			**Node**	In	Out	**Shipments**		**Supply**
7	Atlanta - Kansas City	3	0			Denver		600	600	<=	600
8	Atlanta - Louisville	1	400			Atlanta		400	400	<=	400
9	Kansas City - Detroit	2	200			Kansas City	550	550	0	=	0
10	Kansas City - Miami	6	0			Louisville	450	450	0	=	0
11	Kansas City - Dallas	3	350			Detroit	200		-200	=	-200
12	Kansas City - New Orleans	6	0			Miami	150		-150	=	-150
13	Louisville - Detroit	4	0			Dallas	350		-350	=	-350
14	Louisville - Miami	4	150			New Orleans	300		-300	=	-300
15	Louisville - Dallas	6	0								
16	Louisville - New Orleans	5	300								
17											
18				**Minimize Total Cost**			5200				
19											

shipments. The "net" supply appears in cells K7:K14. Note that the net supply is negative for demand nodes.

Decision Variables Cells C5:C16 are reserved for the decision variables. The optimal number of units to ship over each arc is shown.

Objective Function The formula =SUMPRODUCT(B5:B16,C5:C16) is placed into cell G18 to show the total cost associated with the solution. As shown, the minimum total cost is $5200.

Left-Hand Sides The left-hand sides of the constraints represent the net shipments for each node. Cells I7:I14 are reserved for these constraints.

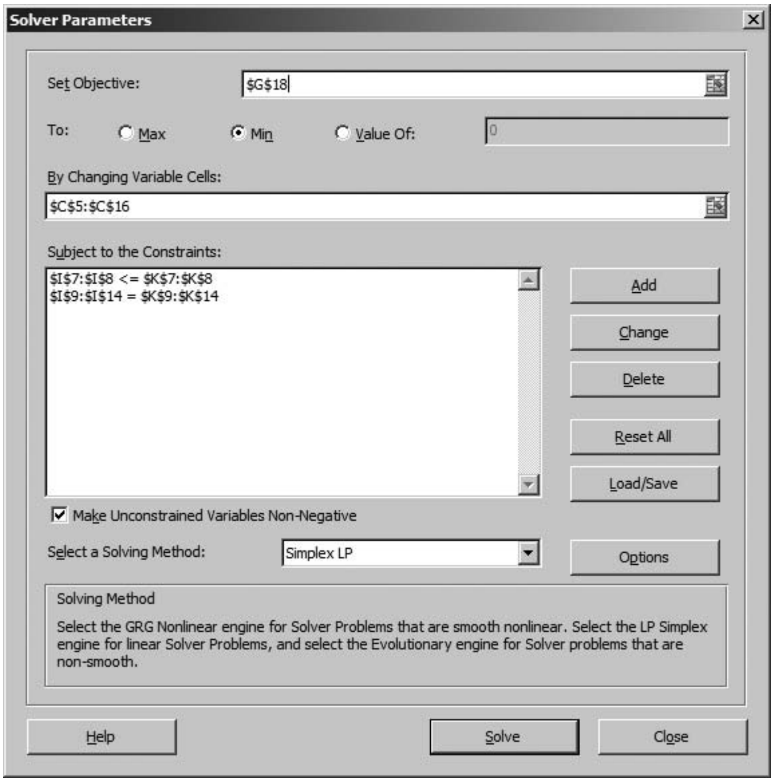
Cell I7 = H7-G7 (Copy to I8:I14)

Right-Hand Sides The right-hand sides of the constraints represent the supply at each node. Cells K7:K14 are reserved for these values. (Note the negative supply at the four demand nodes.)

Excel Solution

The solution can be obtained by selecting **Solver** from the **Analysis Group** in the **Data Ribbon**. The Data Ribbon is displayed at the top of the worksheet in Figure 10.24. When

FIGURE 10.25 EXCEL SOLVER PARAMETERS DIALOG BOX FOR THE RYAN ELECTRONICS PROBLEM



the **Solver Parameters** dialog box appears, enter the proper values for the constraints and the objective function, select **Simplex LP**, and click the checkbox for **Make Unconstrained Variables Non-negative.** Then click **Solve.** The information entered into the **Solver Parameters** dialog box is shown in Figure 10.25.

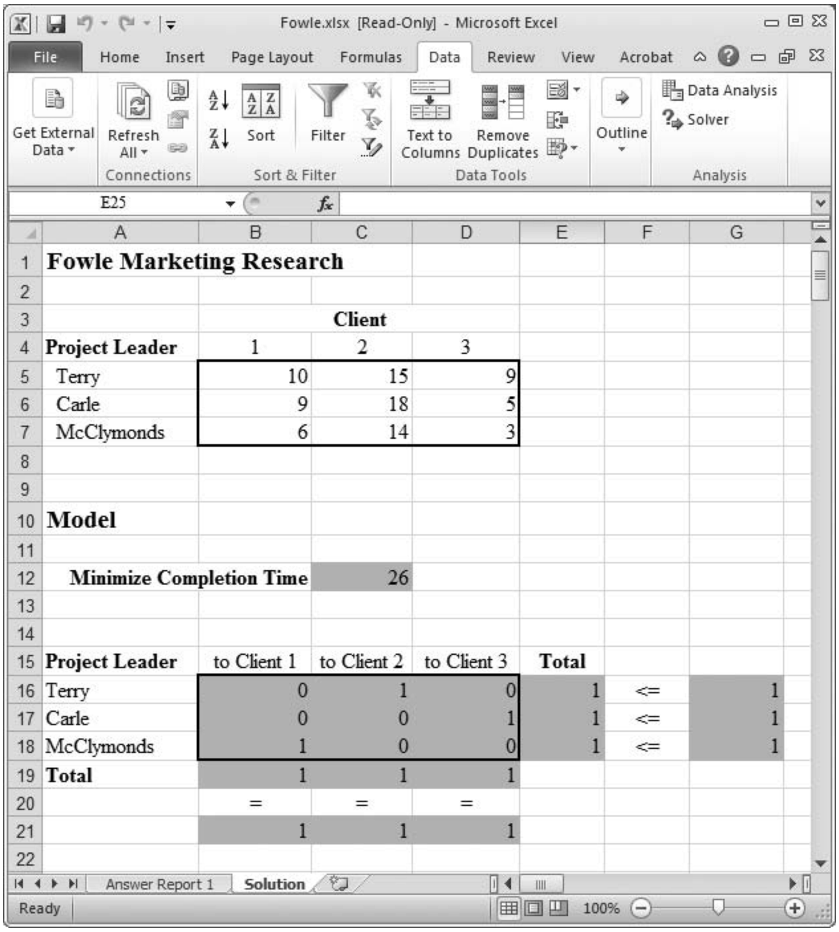
Assignment Problem

The first step is to enter the data for the assignment costs in the top portion of the worksheet. Even though the assignment model is a special case of the transportation model, it is not necessary to enter values for origin supplies and destination demands because they are always equal to 1.

The linear programming model is developed in the bottom portion of the worksheet. As with all linear programs, the model has four key elements: the decision variables, the objective function, the constraint left-hand sides, and the constraint right-hand sides. For an assignment problem the decision variables indicate whether an agent is assigned to a task (with a 1 for yes or 0 for no); the objective function is the total cost of all assignments; the constraint left-hand sides are the number of tasks that are assigned to each agent and the number of agents that are assigned to each task; and the right-hand sides are the number of tasks each agent can handle (1) and the number of agents each task requires (1). The worksheet formulation and solution for the Fowle marketing research problem are shown in Figure 10.26.

FIGURE 10.26 EXCEL SOLUTION OF THE FOWLE MARKETING RESEARCH PROBLEM

WEB file
Fowle



	A	B	C	D	E	F	G
1	**Fowle Marketing Research**						
2							
3			**Client**				
4	**Project Leader**	1	2	3			
5	Terry	10	15	9			
6	Carle	9	18	5			
7	McClymonds	6	14	3			
8							
9							
10	**Model**						
11							
12	**Minimize Completion Time**		26				
13							
14							
15	**Project Leader**	to Client 1	to Client 2	to Client 3	**Total**		
16	Terry	0	1	0	1	<=	1
17	Carle	0	0	1	1	<=	1
18	McClymonds	1	0	0	1	<=	1
19	**Total**	1	1	1			
20		=	=	=			
21		1	1	1			
22							

Formulation

The data and descriptive labels are contained in cells A3:D7. Note that we have not inserted supply and demand values because they are always equal to 1 in an assignment problem. The model appears in the bottom portion of the worksheet.

Decision Variables — Cells B16:D18 are reserved for the decision variables. The optimal values are shown to be $x_{12} = 1$, $x_{23} = 1$, and $x_{31} = 1$, with all other variables $= 0$.

Objective Function — The formula =SUMPRODUCT(B5:D7,B16:D18) has been placed into cell C12 to compute the number of days required to complete all the jobs. The minimum time solution has a value of 26 days.

Left-Hand Sides — Cells E16:E18 contain the left-hand sides of the constraints for the number of clients each project leader can handle. Cells

B19:D19 contain the left-hand sides of the constraints requiring that each client must be assigned a project leader.

Cell E16 = SUM(B16:D16) (Copy to E17:E18)

Cell B19 = SUM(B16:B18) (Copy to C19:D19)

Right-Hand Sides — Cells G16:G18 contain the right-hand sides for the project leader constraints, and cells B21:D21 contain the right-hand sides for the client constraints. All right-hand-side cell values are 1.

Excel Solution

The solution shown in Figure 10.26 can be obtained by selecting **Solver** from the **Analysis Group** in the **Data Ribbon**. The Data Ribbon is displayed at the top of the worksheet in Figure 10.26. When the **Solver Parameters** dialog box appears, enter the proper values for the constraints and the objective function, select **Simplex LP,** and click the checkbox for **Make Unconstrained Variables Non-negative.** Then click **Solve.** The information entered into the **Solver Parameters** dialog box is shown in Figure 10.27.

FIGURE 10.27 EXCEL SOLVER PARAMETERS DIALOG BOX FOR THE FOWLE MARKETING RESEARCH PROBLEM