Topics Covered:

- Discuss the problems faced with integrality constraints
- Formulating Integer Linear Programming Problems
- Goal Programming (GP) and MOLP - Non-Examinable

Introduction

- Integer Linear Programming (ILP)
  - When one or more variables in an LP problem must assume an integer value
- ILPs occur frequently…
  - Scheduling workers
  - Manufacturing products
- Integer variables also allow us to build more accurate models for a number of common business problems.
  - Quantity discounts
  - Setup and lump sum costs
  - Batch size restrictions

Integrality Conditions

MAX: $350X_1 + 300X_2$ } profit

S.T.: $1X_1 + 1X_2 <= 200$ } pumps

$9X_1 + 6X_2 <= 1566$ } labor

$12X_1 + 16X_2 <= 2880$ } tubing

$X_1, X_2 >= 0$ } non-negativity

X_1, X_2 must be integers } integrality

Integrality conditions are easy to state but make the problem much more difficult (and sometimes impossible) to solve.

Relaxation

- **Original ILP**

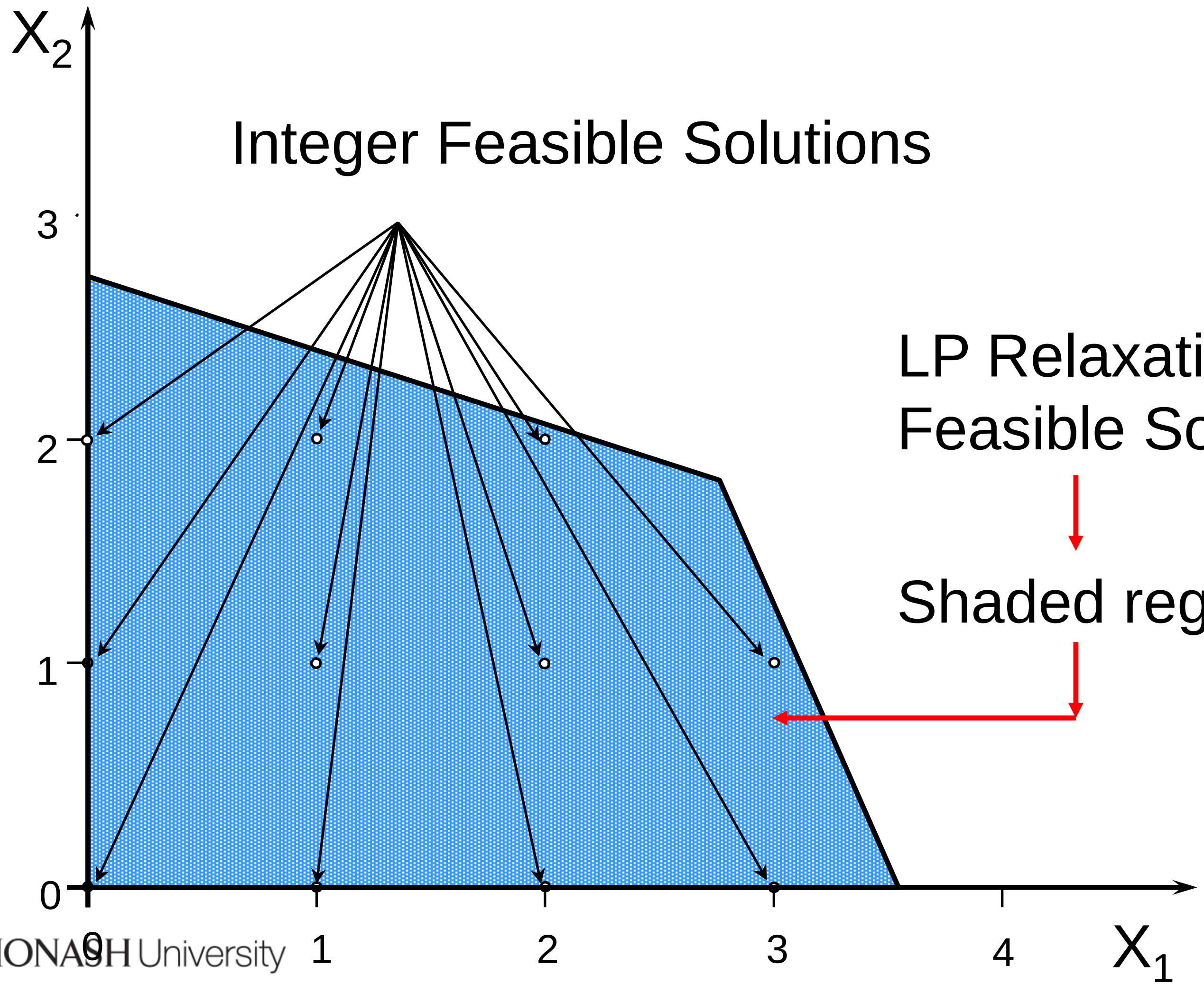
$$\begin{aligned} \textbf{MAX:}\quad & \mathbf{2X_1 + 3X_2} \\ \text{S.T.:}\quad & X_1 + 3X_2 <= 8.25 \\ & 2.5X_1 + X_2 <= 8.75 \\ & X_1, X_2 >= 0 \\ & X_1, X_2 \text{ must be integers} \end{aligned}$$

This constraint is dropped in LP Relaxation

- LP Relaxation

$$\begin{aligned} \text{MAX:}\quad & 2X_1 + 3X_2 \\ \text{S.T.:}\quad & X_1 + 3X_2 <= 8.25 \\ & 2.5X_1 + X_2 <= 8.75 \\ & X_1, X_2 >= 0 \end{aligned}$$

Integer Feasible vs. LP Feasible Region



Solving ILP Problems

- When solving an LP relaxation, sometimes you "get lucky" and obtain an integer feasible solution.
- Example: Blue Ridge Hot Tubs

MAX: $350X_1 + 300X_2$ } profit

S.T.: $1X_1 + 1X_2 <= 200$ } pumps

$9X_1 + 6X_2 <= 1566$ } labor

$12X_1 + 16X_2 <= 2880$ } tubing

$X_1, X_2 >= 0$ } non-negativity

Optimal solution: $X_1 = 122$ and $X_2 = 78$

Integer solution

Solving ILP Problems

- But what if we reduce the amount of labor available to 1520 hours and the amount of tubing to 2650 feet?
- See file Lecture 4.xlsm *(Blue Ridge)*

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

Bounds

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

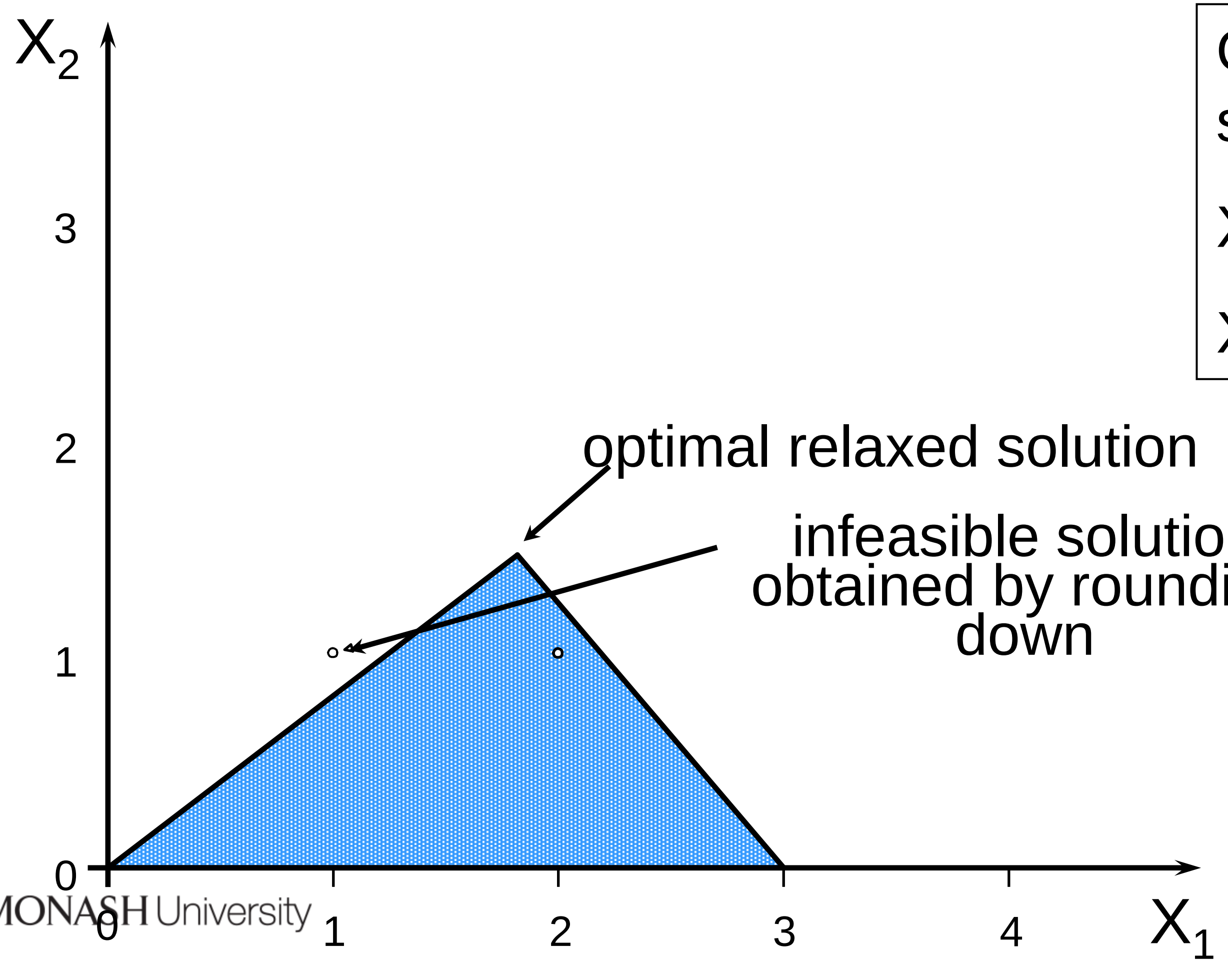
- The optimal solution to an LP relaxation of an ILP problem gives us a *bound* on the optimal objective function value.
- For maximization problems, the optimal relaxed objective function values is an <u>upper bound</u> on the optimal integer value.
- For minimization problems, the optimal relaxed objective function values is a <u>lower bound</u> on the optimal integer value.

Rounding

- It is tempting to simply round a fractional solution to the closest integer solution.

- In general, this does not work reliably:
  - The rounded solution may be infeasible.
  - The rounded solution may be suboptimal.

How Rounding Down Can Result in an Infeasible Solution



Optimal relaxed solution

X_1=1.91

X_2=1.45

Rounding Up

- LP solution:

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

- Round up - Infeasible

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	117	78	Total Profit	
Unit Profits	$350	$300	$64,350	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1521	1520
Tubing Req'd	12	16	2652	2650

Rounding Down

- LP solution

	Aqua-Spas	Hydro-Luxes		
Blue Ridge Hot Tubs				
Number to Make	116.9444444	77.91666667	Total Profit	
Unit Profits	$350	$300	$64,306	
Constraints			Used	Available
Pumps Req'd	1	1	195	200
Labor Req'd	9	6	1520	1520
Tubing Req'd	12	16	2650	2650

- Round down – Feasible …..but

	Aqua-Spas	Hydro-Luxes		
Blue Ridge Hot Tubs				
Number to Make	116	77	Total Profit	
Unit Profits	$350	$300	$63,700	
Constraints			Used	Available
Pumps Req'd	1	1	193	200
Labor Req'd	9	6	1506	1520
Tubing Req'd	12	16	2624	2650

Rounding Down Causes Sub-Optimality

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	116	77	Total Profit	
Unit Profits	$350	$300	$63,700	
Constraints			Used	Available
Pumps Req'd	1	1	193	200
Labor Req'd	9	6	1506	1520
Tubing Req'd	12	16	2624	2650

- A better integer solution exists (i.e. better than the above sub-optimal solution):

Blue Ridge Hot Tubs				
	Aqua-Spas	Hydro-Luxes		
Number to Make	118	76	Total Profit	
Unit Profits	$350	$300	$64,100	
Constraints			Used	Available
Pumps Req'd	1	1	194	200
Labor Req'd	9	6	1518	1520
Tubing Req'd	12	16	2632	2650

Branch-and-Bound

- The Branch-and-Bound (B&B) algorithm can be used to solve ILP problems.
- Requires the solution of a series of LP problems termed "candidate problems".
- *Theoretically*, this can solve any ILP.
- *Practically*, it often takes *LOTS* of computational effort (and time).

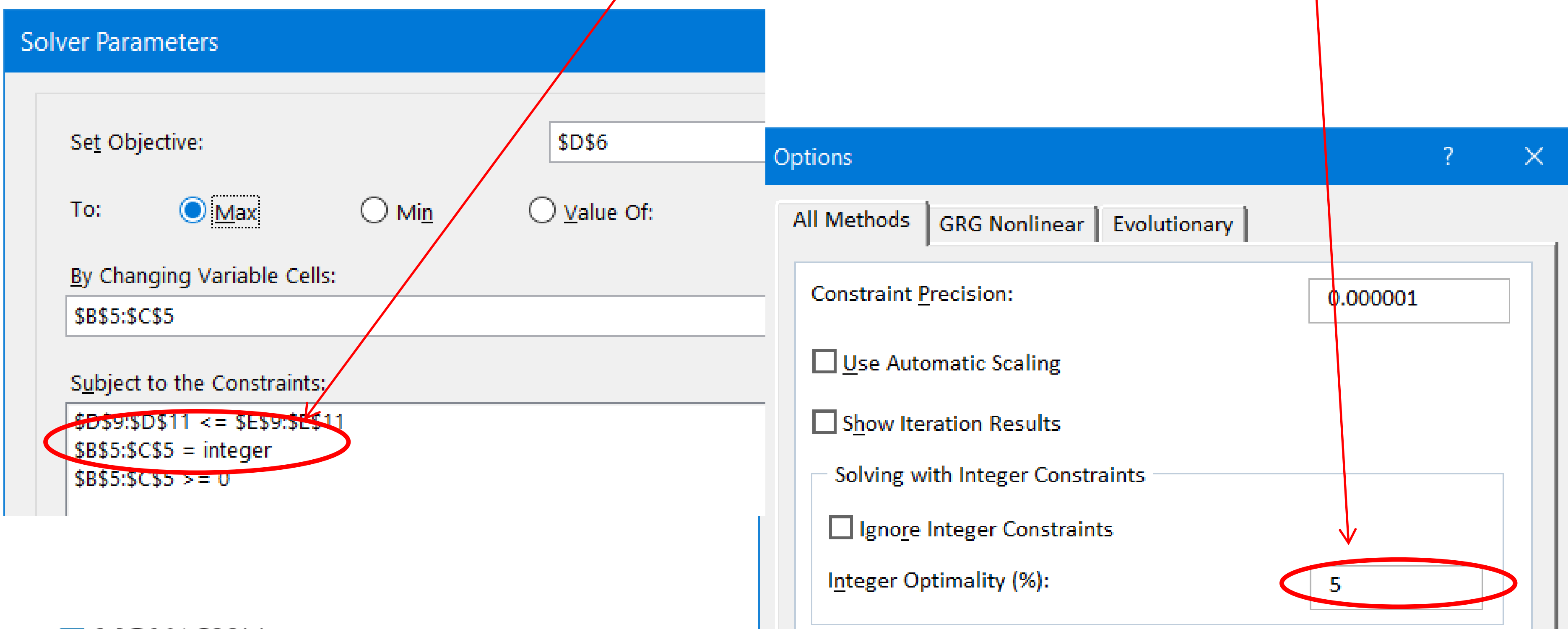
Stopping Rules

- Because B&B can take so long, most ILP packages allow you to specify a sub-optimality tolerance factor.
- This allows you to stop once an integer solution is found that is within some % of the global optimal solution.
- Bounds obtained from LP relaxations are helpful here.
  - Example
    - LP relaxation has an optimal obj. value of $64,306.
    - 95% of $64,306 is $61,090.
    - Thus, an integer solution with obj. value of $61,090 or better must be within 5% of the optimal solution.

Using Solver

Let’s see how to specify integrality conditions and sub-optimality tolerances using Solver…

See file Lecture 4.xlsm (*Blue Ridge – ILP*)



An Employee Scheduling Problem: Air-Express

- An express shipping company – guarantees o/night delivery
- Various hubs across the country – shipments go to hubs, then on to their destination
- Manager of Baltimore hub is concerned about labour costs and wants to investigate the most effective way of scheduling of workers
- Hub open 7 days per week
- # packages varies from 1 day to the next
- An estimate of the number of workers needed on each day of the week has been calculated using historical data

An Employee Scheduling Problem: Air-Express

Day of Week	Workers Needed
Sunday	18
Monday	27
Tuesday	22
Wednesday	26
Thursday	25
Friday	21
Saturday	19

Shift	Days Off	Wage
1	Sun & Mon	$680
2	Mon & Tue	$705
3	Tue & Wed	$705
4	Wed & Thr	$705
5	Thr & Fri	$705
6	Fri & Sat	$680
7	Sat & Sun	$655

Defining the Decision Variables

X_1 = the number of workers assigned to shift 1

X_2 = the number of workers assigned to shift 2

X_3 = the number of workers assigned to shift 3

X_4 = the number of workers assigned to shift 4

X_5 = the number of workers assigned to shift 5

X_6 = the number of workers assigned to shift 6

X_7 = the number of workers assigned to shift 7

Defining the Objective Function

Minimize the total wage expense.

$$\text{MIN: } 680X_1 + 705X_2 + 705X_3 + 705X_4 + 705X_5 + 680X_6 + 655X_7$$

Wage per shift

Shift	Days Off	Wage
1	Sun & Mon	$680
2	Mon & Tue	$705
3	Tue & Wed	$705
4	Wed & Thr	$705
5	Thr & Fri	$705
6	Fri & Sat	$680
7	Sat & Sun	$655

Defining the Constraints

- Workers required each day

$$0X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 0X_7 >= 18 \text{ \} Sunday}$$

$$0X_1 + 0X_2 + 1X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 >= 27 \text{ \} Monday}$$

$$1X_1 + 0X_2 + 0X_3 + 1X_4 + 1X_5 + 1X_6 + 1X_7 >= 22 \text{ \}Tuesday}$$

$$1X_1 + 1X_2 + 0X_3 + 0X_4 + 1X_5 + 1X_6 + 1X_7 >= 26 \text{ \} Wednesday}$$

$$1X_1 + 1X_2 + 1X_3 + 0X_4 + 0X_5 + 1X_6 + 1X_7 >= 25 \text{ \} Thursday}$$

$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 0X_5 + 0X_6 + 1X_7 >= 21 \text{ \} Friday}$$

$$1X_1 + 1X_2 + 1X_3 + 1X_4 + 1X_5 + 0X_6 + 0X_7 >= 19 \text{ \} Saturday}$$

- Non-negativity & integrality conditions

$X_i >= 0$ and integer for all i

Implementing the Model

See file Lecture 4.xlsm (*AirExpress*)

	A	B	C	D	E	F	G	H	I	J
1		Air-Express								
2										
3		Days On=1, Days Off=0							Workers	Wages per
4	Shift	Sun	Mon	Tues	Wed	Thur	Fri	Sat	Scheduled	Worker
5	1	0	0	1	1	1	1	1	6	$680
6	2	1	0	0	1	1	1	1	0	$705
7	3	1	1	0	0	1	1	1	5	$705
8	4	1	1	1	0	0	1	1	1	$705
9	5	1	1	1	1	0	0	1	7	$705
10	6	1	1	1	1	1	0	0	5	$680
11	7	0	1	1	1	1	1	0	9	$655
12	Available	18	27	28	27	25	21	19	Total	$22,540
13	Required	18	27	22	26	25	21	19		
14										

At least as many as required

Binary Variables

- Binary variables are integer variables that can assume only two values: 0 or 1.
- These variables can be useful in a number of practical modeling situations….

A Capital Budgeting Problem: CRT Technologies

Project	Expected NPV (in $000s)	Capital (in $000s) Required in Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

- The company has $250,000 available to invest in new projects. It has budgeted $75,000 for continued support for these projects in year 2 and $50,000 per year for years 3, 4, and 5.
- Unused funds in any year cannot be carried over.

Defining Decision Variables & Objective Function

$$X_i = \begin{cases} 1, \text{if project } i \text{ is selected} \\ 0, \text{otherwise} \end{cases} \quad i = 1,2,\ldots,6$$

Maximize total NPV of selected projects

MAX: $141X_1 + 187X_2 + 121X_3 + 83X_4 + 265X_5 + 127X_6$

Expected NPV ($000s)

Project yr	Expected NPV (in $000s)	Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

Defining Constraints

- Capital Constraints

  must ensure for each year that the selected projects do not require more capital than is available

e.g. year 2, $75,000 is available, so:

$$25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75$$

Project yr	Expected NPV (in $000s)	Year 1	Year 2	Year 3	Year 4	Year 5
1	$141	$75	$25	$20	$15	$10
2	$187	$90	$35	$0	$0	$30
3	$121	$60	$15	$15	$15	$15
4	$83	$30	$20	$10	$5	$5
5	$265	$100	$25	$20	$20	$20
6	$127	$50	$20	$10	$30	$40

Defining the Constraints

- Capital Constraints

$$75X_1 + 90X_2 + 60X_3 + 30X_4 + 100X_5 + 50X_6 <= 250 \quad \text{\} year 1}$$

$$25X_1 + 35X_2 + 15X_3 + 20X_4 + 25X_5 + 20X_6 <= 75 \quad \text{\} year 2}$$

$$20X_1 + 0X_2 + 15X_3 + 10X_4 + 20X_5 + 10X_6 <= 50 \quad \text{\} year 3}$$

$$15X_1 + 0X_2 + 15X_3 + 5X_4 + 20X_5 + 30X_6 <= 50 \quad \text{\} year 4}$$

$$10X_1 + 30X_2 + 15X_3 + 5X_4 + 20X_5 + 40X_6 <= 50 \quad \text{\} year 5}$$

- Binary Constraints

All X_i must be binary

Implementing the Model

See file Lecture 4.xlsm(*CRT*)

	A	B	C	D	E	F	G	H
1								
2			CRT Technologies					
3								
4		Select?			Capital Required in			
5	Project	(0=no, 1=yes)	NPV	Year 1	Year 2	Year 3	Year 4	Year 5
6	1	1	$141	$75	$25	$20	$15	$10
7	2	0	$187	$90	$35	$0	$0	$30
8	3	0	$121	$60	$15	$15	$15	$15
9	4	1	$83	$30	$20	$10	$5	$5
10	5	1	$265	$100	$25	$20	$20	$20
11	6	0	$127	$50	$20	$10	$30	$40
12			Capital Required	$205	$70	$50	$40	$35
13			Capital Available	$250	$75	$50	$50	$50
14								
15			Total Net Present Value	$489				
16								

Binary Variables & Logical Conditions

- Binary variables are also useful in modeling a number of logical conditions.
  - Of projects 1, 3 & 6, no more than one may be selected: $X_1 + X_3 + X_6 <= 1$
  - Of projects 1, 3 & 6, exactly one must be selected: $X_1 + X_3 + X_6 = 1$
  - Project 4 cannot be selected unless project 5 is also selected: $X_4 – X_5 <= 0$

The Fixed-Charge Problem

- Many decisions result in a fixed or lump-sum cost being incurred:
  - The cost to lease, rent, or purchase a piece of equipment or a vehicle that will be required if a particular action is taken.
  - The setup cost required to prepare a machine or to produce a different type of product.
  - The cost to construct a new production line that will be required if a particular decision is made.
  - The cost of hiring additional personnel that will be required if a particular decision is made.

Example Fixed-Charge Problem: Remington Manufacturing

Hours Required By:

Operation	Prod. 1	Prod. 2	Prod. 3	Hours Available
Machining	2	3	6	600
Grinding	6	3	4	300
Assembly	5	6	2	400
Unit Profit	$48	$55	$50	
Setup Cost	$1000	$800	$900	

Fixed charge for making any quantity of prod 1, prod 2 or prod 3

Defining Decision Variables

X_i = the amount of product *i* to be produced, *i* = 1, 2, 3

$$Y_i = \begin{cases} 1, \text{ if } X_i > 0 \\ 0, \text{ if } X_i = 0 \end{cases} \quad i = 1, 2, 3$$

Y_i are binary variables that will be used to include the fixed charges

Defining the Objective Function

Maximize total profit.

Binary variable

$$\text{MAX:}\quad \underbrace{48X_1 + 55X_2 + 50X_3}_{\text{Unit Profit}} - \underbrace{1000Y_1 - 800Y_2 - 900Y_3}_{\text{Setup cost}}$$

Unit Profit

Setup cost

Defining the Constraints

- Resource Constraints

  $2X_1 + 3X_2 + 6X_3 <= 600$ } machining

  $6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

  $5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- Non-negativity & integer conditions

  $X_i >= 0$, $i = 1, 2,...3$

  X_i integer, i=1,..3

- Binary Constraints

  All Y_i must be binary

- Is there a missing link?
- Yes - we need to ensure that $Y_i = 1$ if $X_i > 0$

Linking Constraints

- Linking Constraints (with "Big M")

$$X_1 <= M_1 Y_1 \qquad \text{or} \qquad X_1 - M_1 Y_1 <= 0$$

$$X_2 <= M_2 Y_2 \qquad \text{or} \qquad X_2 - M_2 Y_2 <= 0$$

$$X_3 <= M_3 Y_3 \qquad \text{or} \qquad X_3 - M_3 Y_3 <= 0$$

- If $X_i > 0$ these constraints force the associated Y_i to equal 1.
- If $X_i = 0$ these constraints allow Y_i to equal 0 or 1, but the objective will cause Solver to choose 0.
- Note that M_i imposes an upper bounds on X_i.
- It helps to find reasonable values for the M_i.

But we don't want to constrain X_i any further

Finding Reasonable Values for M1

- Consider the resource constraints

  $2X_1 + 3X_2 + 6X_3 <= 600$ } machining

  $6X_1 + 3X_2 + 4X_3 <= 300$ } grinding

  $5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

- What is the maximum value X_1 can assume?

  Let $X_2 = X_3 = 0$

  $X_1 = MIN(600/2, 300/6, 400/5)$

  $= MIN(300, 50, 80)$

  $= 50$

  So we can put M_1 =50

- Maximum values for X_2 & X_3 can be found similarly.

Summary of the Model

MAX: $48X_1 + 55X_2 + 50X_3 - 1000Y_1 - 800Y_2 - 900Y_3$

S.T.:

$2X_1 + 3X_2 + 6X_3 <= 600$ } machining

$6X_1 + 3X_2 + 4X_3 <= 30$ } grinding

$5X_1 + 6X_2 + 2X_3 <= 400$ } assembly

$X_1 - 50Y_1 <= 0$

$X_2 - 67Y_2 <= 0$

$X_3 - 75Y_3 <= 0$

} **linking constraints**

All Y_i must be binary

$X_i >= 0, i = 1, 2, 3$ (= integer)

Implementing Model

See file Lecture 4.xlsm (Remington)

linking constraints:

$X_1 - 50Y_1 <= 0$

$X_2 - 67Y_2 <= 0$

$X_3 - 75Y_3 <= 0$

	A	B	C	D	E	F	G
1							
2		Remington Manufacturin					
3							
4		Product 1	Product 2	Product 3			
5	Number to Produce	0	56	32			
6							
7	Unit Profit	$48	$55	$50		Total Profit	
8	Fixed-Cost	$1,000	$800	$900		$2,980	
9							
10	Resources		Hours Required		Used	Available	
11	Machining	2	3	6	360	600	
12	Grinding	6	3	4	296	300	
13	Assembly	5	6	2	400	400	
14							
15	Binary Variables	0	1	1			
16	Linking Constraints	0	-10.66667	-43			
17							

Potential Pitfall

- Do not use IF( ) functions to model the relationship between the X_i and Y_i.
  - Suppose cell B5 represents X_1
  - Suppose cell B15 represents Y_1
  - You'll want to let B15 = IF(B5>0,1,0)
  - This will not work with Solver!
- Treat the Y_i just like any other variable.
  - Make them changing cells.
  - Use the linking constraints to enforce the proper relationship between the X_i and Y_i.

Minimum Order Size Restrictions

Suppose Remington doesn't want to manufacture any units of product 3 unless it produces at least 40 units...

Use M3 = min(600/6,300/4,400/2)=75

Consider,

$$X_3 <= M_3 Y_3$$

$$X_3 >= 40\ Y_3$$

See [Lecture 4.xlsm]() (*Remington – Min order*)

B&G – A Contract Award Problem

- B&G Construction has 4 building projects and can purchase cement from 3 companies for the following costs:

	Cost per Delivered Ton of Cement				Max.
	Project 1	Project 2	Project 3	Project 4	Supply
Co. 1	$120	$115	$130	$125	525
Co. 2	$100	$150	$110	$105	450
Co. 3	$140	$95	$145	$165	550
Needs (tons)	450	275	300	350	

Defining the Decision Variables

X_{ij} = tons of cement purchased from company i for project j

Defining the Objective Function

Minimize total cost

$$
\begin{aligned}
\text{MIN:}\quad & 120X_{11} + 115X_{12} + 130X_{13} + 125X_{14} \\
& + 100X_{21} + 150X_{22} + 110X_{23} + 105X_{24} \\
& + 140X_{31} + 95X_{32} + 145X_{33} + 165X_{34}
\end{aligned}
$$

A Contract Award Problem

- Side constraints:

  Co. 1 will not supply orders of less than 150 tons for any project

  Co. 2 can supply more than 200 tons to no more than one of the projects

  Co. 3 will accept only orders that total 200, 400, or 550 tons

Defining the Constraints

- Supply Constraints

$$X_{11} + X_{12} + X_{13} + X_{14} <= 525 \quad \} \text{ company } 1$$

$$X_{21} + X_{22} + X_{23} + X_{24} <= 450 \quad \} \text{ company } 2$$

$$X_{31} + X_{32} + X_{33} + X_{34} <= 550 \quad \} \text{ company } 3$$

- Demand Constraints

$$X_{11} + X_{21} + X_{31} = 450 \quad \} \text{ project } 1$$

$$X_{12} + X_{22} + X_{32} = 275 \quad \} \text{ project } 2$$

$$X_{13} + X_{23} + X_{33} = 300 \quad \} \text{ project } 3$$

$$X_{14} + X_{24} + X_{34} = 350 \quad \} \text{ project } 4$$

Defining the Constraints - I

- Company 1 Side Constraints

$$X_{11} <= 525Y_{11}$$

$$X_{12} <= 525Y_{12}$$

$$X_{13} <= 525Y_{13}$$

$$X_{14} <= 525Y_{14}$$

$$X_{11} >= 150Y_{11}$$

$$X_{12} >= 150Y_{12}$$

$$X_{13} >= 150Y_{13}$$

$$X_{14} >= 150Y_{14}$$

Y_{ij} binary

Defining the Constraints- II & III

- Company 2 Side Constraints

$$X_{21} <= 200 + 250Y_{21}$$

$$X_{22} <= 200 + 250Y_{22}$$

$$X_{23} <= 200 + 250Y_{23}$$

$$X_{24} <= 200 + 250Y_{24}$$

$$Y_{21} + Y_{22} + Y_{23} + Y_{24} <= 1$$

Y_{ij} binary

- Company 3 Side Constraints

$$X_{31} + X_{32} + X_{33} + X_{34} = 200Y_{31} + 400Y_{32} + 550Y_{33}$$

$$Y_{31} + Y_{32} + Y_{33} <= 1$$

Implementing the Transportation Constraints

See file Lecture 4.xlsm(*B&G*)

References:

Ragsdale, C. Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e/9e) Cengage Learning: Chapter 6 & 7