FIT3158
Business Decision Modelling

SEMESTER 2, 2022

Lecture 11

Simulation and Queuing Theory

Topics Covered:

- Introduction to Simulation – simulating a waiting room queue
- Understanding the basics of queuing theory
- Review of probability theory – Poisson and Exponential distributions
- Exploring the steady state behaviour of queues and Kendall notation

Student Evaluation of Teaching and Units (SETU)

- SETU is open for Semester 2 2022.
- All students are encouraged to participate.
- Your feedback on unit content and lecture/tutorials is very important.
- You will see a block in Moodle linking you to the survey.

Assignment Due on Friday of Week 11

- Use consultations for any final clarifications.
- Refer to delivery instructions in the assignment specifications.
- Submit three separate Excel Workbooks for each case in the assignment.
- Make sure your solver settings are visible when solver is opened in the submitted files.
- Include team member contribution on each part of the assignment as specified.

(We expect all team members to have equally contributed to the assignment.)

Introduction to Simulation

- In many spreadsheets, the value for one or more cells representing independent variables is unknown or uncertain.

e.g., cost

- As a result, there is uncertainty about the value the dependent variable will assume:

$$Y = f(X_1, X_2, \ldots, X_k)$$

- Simulation can be used to analyze these types of models.
- Simulation is a technique that lets us estimate and describe various characteristics of the performance measure of a model when one or more independent variables are uncertain.
- The objective in simulation is to describe the distribution and characteristics of the ***possible values*** of the performance measure *Y*, given the ***possible values*** and behavior of the independent variables *X(1),…,X(k)*.

we mean here what we are optimizing for it could be like a total cost or total profit or total revenue

Simulation

analyze the change when only a single variable is changing at a time when everything else is remaining the same

- The idea behind simulation is similar to the notion of playing out many "what-if" scenarios.
- The difference is that the process of assigning values to the cells in the spreadsheet that represent random variables is automated so that:
  - The values are assigned in a nonbiased way.
  - The spreadsheet user is relieved of the burden of determining these values.
- With simulation we repeatedly and randomly generate sample values for each uncertain input variable $X(1),\ldots,X(t)$ in our model and then compute the resulting value of our performance measure Y.
- We can then use the sample values of Y to estimate the true distribution and other characteristics of the performance measure Y (e.g. frequency distribution, mean, variance etc.)

Types of Simulation Models

- A static simulation model is representation of a system at a particular point in time.
- A dynamic simulation is a representation of a system as it evolves over time – can be discrete-event or continuous-event simulation (e.g. modelling a production line over time).

Monte Carlo Simulation

- Monte Carlo simulation – generates random variables from pre-defined probability distributions to simulate future events.
- Named after the Roulette wheel at Monte Carlo. Named by John von Neumann at Los Alamos Scientific Laboratory.
- Simulation driven by random events: Dice, random numbers or coins.
- A range of probability distributions can be applied.
- Some random events:
  - Whether a customer chooses item A, B or C.
  - The time between successive customers entering a shop.
  - The time each customer takes to be served.
  - The number of hamburgers sold in one hour.

Random Variables & Risk

- A *random variable* is any variable whose value cannot be predicted or set with certainty.
- Many “input cells” in spreadsheet models are actually random variables.
  - the future cost of raw materials
  - future interest rates.
  - future number of employees in a firm
  - expected product demand.
- Decisions made on the basis of uncertain information often involve risk.
- “Risk” implies the potential for loss.

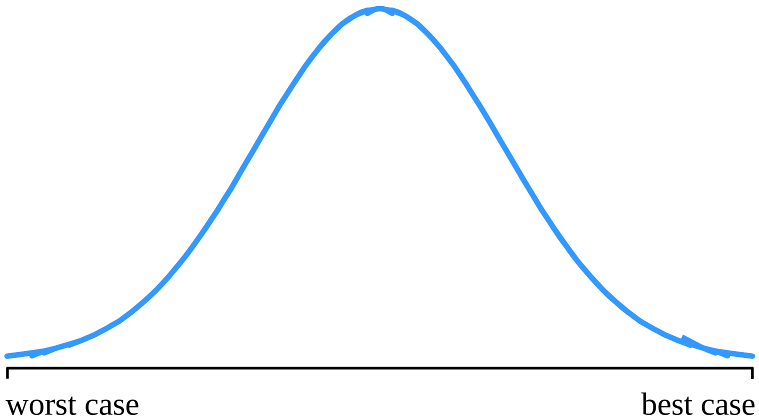
Why Analyse Risk?

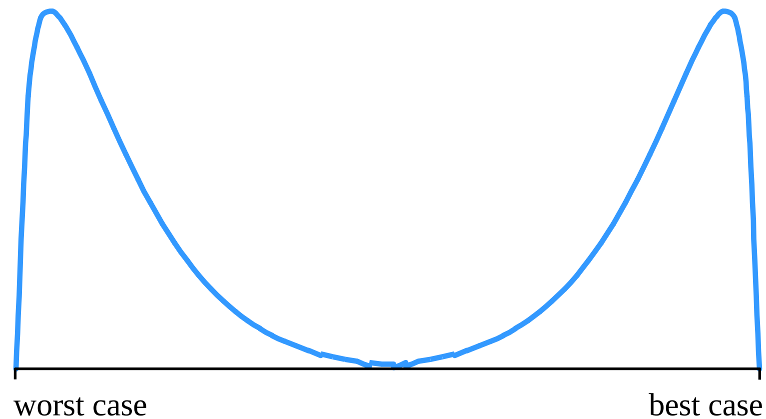
- Plugging in expected values for uncertain cells tells us nothing about the variability of the performance measure we base decisions on.
- Suppose a $5,000 investment is expected to return $10,000 in two years. Would you invest if...
  - the outcomes could range from $9,000 to $11,000?
  - the outcomes could range from -$30,000 to $50,000?
- Alternatives with the same expected value may result in different levels of risk.

Methods of Risk Analysis

- Best-Case/Worst-Case Analysis
  - Plug in most optimistic or pessimistic values for each uncertain cells
- What-if Analysis
  - Plug in different values for uncertain cells and see what happens
- Simulation
  - Resembles automated what-if analysis
  - Values are selected in an unbiased manner
  - Computer generates hundreds (or thousands) of scenarios

Possible Performance Measure Distributions Within a Range

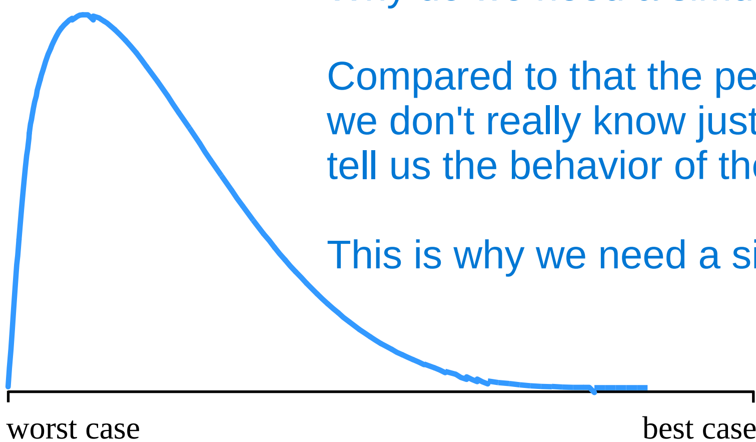


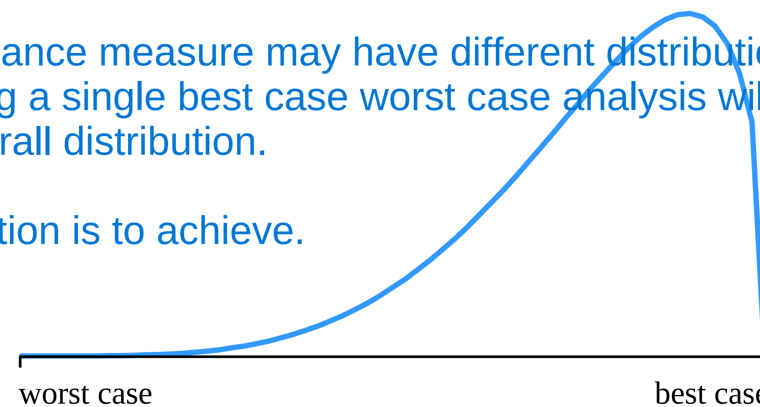


Why do we need a simulation rather than Best-Worst case?

Compared to that the performance measure may have different distributions, we don't really know just doing a single best case worst case analysis will not tell us the behavior of the overall distribution.

This is why we need a simulation is to achieve.





What-If Analysis

- Plug in different values for the uncertain cells and see what happens.
- This is easy to do with spreadsheets.
- Problems:
  - Values may be chosen in a biased way.
  - Hundreds or thousands of scenarios may be required to generate a representative distribution.
  - Does not supply the tangible evidence (facts and figures) needed to justify decisions to management.

Simulation

- Resembles automated what-if analysis.
- Values for uncertain cells are selected in an unbiased manner.
- The computer generates hundreds (or thousands) of scenarios.
- We analyze the results of these scenarios to better understand the behavior of the performance measure.
- This allows us to make decisions using solid empirical evidence.
- Simulation in Excel: - Four-step process:

1) Identify uncertain cells in model.
2) Implement appropriate random number generators (RNGs) for each uncertain cell.
3) Replicate model *n* times, and record value of bottom-line performance measure.
4) Analyse sample values collected on performance measure.

Limitations in Simulation

- **Simulation Models**
  - can be costly / time-consuming to develop
  - are often problem-specific
    - model for one problem unlikely to be suitable for another
- **Simulation**
  - does not guarantee optimal results
  - is not a method of optimisation
    - parameters can be tested over a range of values to find the best combination, but if there are many parameters, the process can be time consuming
- **Variability introduced by random events**
  - needs to be statistically controlled (by running multiple – ideally independent - trials)

Random Number Generators (RNGs)

- RNG – a Mathematical function that randomly generates (returns) a value from a particular probability distribution
- Implement RNGs for uncertain cells to allow us to sample from distribution of values expected for different cells
- <u>Discrete vs Continuous RVs</u>:
  - A discrete random variable may assume one of a fixed set of (usually integer) values.
    - Example: The number of defective tyres on a new car can be 0, 1, 2, 3, or 4.
  - A continuous random variable may assume one of an infinite number of values in a specified range.
    - Example: The amount of gasoline in a new car can be any value between 0 and the maximum capacity of the fuel tank.

How RNGs Work

- The RAND( ) function returns **uniformly** distributed random numbers between 0.0 and 0.9999999.
- Suppose we want to simulate the act of tossing a fair coin.
- Let 0 represent "heads" and 1 represent "tails".
- Consider the following RNG:
- =IF(RAND( )<0.5,0,1)

Refer: Lecture 11.xlsm (RNG)

Random Number	Head or Tail
0.82366351	1
0.14460405	0
0.55567153	1
0.66598144	1
0.22892131	0

Simulating the Roll of a Die

- We want the values 1, 2, 3, 4, 5 & 6 to occur randomly with equal probability of occurrence.
- Consider the following RNG:
- =INT(6*RAND())+1

If 6*RAND( ) falls in the interval:	**INT(6*RAND( ))+1 returns the value:**
0.0 to 0.999	1
1.0 to 1.999	2
2.0 to 2.999	3
3.0 to 3.999	4
4.0 to 4.999	5
5.0 to 5.999	6

Refer: Lecture 11.xlsm (RNG)

Example – The Newsboy Problem

A newsboy sells papers on the corner of the street.

Demand is uncertain and each morning he has to decide how many to order from his supplier.

If he buys too many papers he will have unsold papers left at the end of the day. If he buys too few papers he has lost the opportunity of making a higher profit.

The table on the next slide shows an empirical distribution for the possible number of customers per day with associated probability.

Source: The Newsboy problem (from Computer Models for Business Decisions, R.J.Willis).

Example – The Newsboy Problem

The purchase price of papers is 30 cents and he sells them for 40 cents.

He gets 5c back for each unsold paper and loses $1 in goodwill each time someone asks for a paper and he is out of stock.

How many papers should he stock each day to maximise profit in the long term?

customers per day	probability
25	0.1
26	0.1
27	0.2
28	0.25
29	0.2
30	0.15

Refer: Lecture 11.xlsm (Newsboy)

empirical probability distribution
for what for the expected number of customers per
day or the demand

Newsboy Problem

- Note, An *empirical* distribution is one that has been observed or recorded, usually from historical data.
- We will use Monte Carlo simulation to model 100 days of sales and calculate the average daily sales for each stock level.
- We first calculate the cumulative probability for each customer level:

customers per day	probability	Cumulative probability
25	0.1	0.1
26	0.1	0.2
27	0.2	0.4
28	0.25	0.65
29	0.2	0.85
30	0.15	1

Newsboy Problem cont…

- We need to simulate demand for 100 days.
- To generate a random number between 0 and 1 enter: **=**RAND()
- This function returns an **evenly distributed random number** greater than or equal to 0 and less than 1. A new random number is returned every time the worksheet is calculated.
- We can use the Rand() function and the cumulative probability distribution to generate demand.
- Put x=Rand(), then:

➢ if x < .1 this corresponds to a demand of 25

➢ If .1<=x<.2 this corresponds to a demand of 26

➢ If .2<=x<.4 this corresponds to a demand of 27

… and so on

customers per day	probability	Cumulative probability
25	0.1	0.1
26	0.1	0.2
27	0.2	0.4
28	0.25	0.65
29	0.2	0.85
30	0.15	1

Newsboy Problem

We will generate the demand probability for 100 days, then use Vlookup() to find the associated demand, e.g.:

demand			customers per day
0.30477	27	0	25
		0.1	26
		0.2	27
		0.4	28
		0.65	29
		0.85	30

cumulative_prob

Combine into 1 equation:
=VLOOKUP(RAND(),cumulative_prob,2,TRUE)

Note: F9 recalculates the value of RAND()

Newsboy Problem

Next derive a formula that will provide the profit/loss for each combination of stock level and demand

If demand>= stock_level,

profit=(selling_price-cost_price)*stock_level – (demand-stock_level)*(opportunity_loss)

Otherwise

profit=(selling_price-cost_price)*demand-(stock_level-demand)*return_price

C18 | =IF(B18>=B2,(B4-B3)*B2-(B18-B2)*B5,(B4-B3)*B18-(B2-B18)*(B3-B6))

	A	B	C	D	E	F	G	H	I	J	K
1											
2	stock level	27									
3	cost price	0.3		customers per day	probability	Cumulative probability			customers per day		
4	selling price	0.4		25	0.1	0.1		0	25		
5	opportunity loss	1		26	0.1	0.2		0.1	26		
6	return price	0.05		27	0.2	0.4		0.2	27		
7				28	0.25	0.65		0.4	28		
8				29	0.2	0.85		0.65	29		
9				30	0.15	1		0.85	30		
10											
17	Day	Demand	Profit								
18	1	27	2.7								

Newsboy Problem

- We can now simulate 1 day of sales.
- Copying this 99 more times, we can simulate 100 days of sales and find the average profit for a stock level of 27:

stock level	27
cost price	0.3
selling price	0.4
opportunity loss	1
return price	0.05

customers per day	probability	Cumulative probability
25	0.1	0.1
26	0.1	0.2
27	0.2	0.4
28	0.25	0.65
29	0.2	0.85
30	0.15	1

	customers per day
0	25
0.1	26
0.2	27
0.4	28
0.65	29
0.85	30

average daily profit
1.463

100 days

	demand	profit
1	30	-0.3
2	30	-0.3
3	29	0.7
4	29	0.7
5	29	0.7
6	25	2
7	26	2.35
8	28	1.7
9	28	1.7
10	27	2.7
11	29	0.7

Newsboy Problem

- To do this for every possible stock level (25-30) – use a one variable data table:

	A	B	C	D	E	F	G	H	I	J
1										
2	stock level	27								
3	cost price	0.3		customers per day	probability	Cumulative probability			customers per day	
4	selling price	0.4		25	0.1	0.1		0	25	
5	opportunity loss	1		26	0.1	0.2		0.1	26	
6	return price	0.05		27	0.2	0.4		0.2	27	
7				28	0.25	0.65		0.4	28	
8				29	0.2	0.85		0.65	29	
9				30	0.15	1		0.85	30	
10										
11	day	demand	profit							
12	1	29	0.7		average daily profit					
13	2	27	2.7		1.488			**stock level**	**ave. daily profit**	
14	3	25	2					27	1.488	
								25	-0.4	
								26	0.8485	
								27	1.546	
								28	2.1235	
								29	2.3185	
								30	2.2685	

Note: the table indicates that a stock level of 29 is optimal. However if the simulation is repeated, this may alter. We need to simulate for a longer time period to get a stable result.

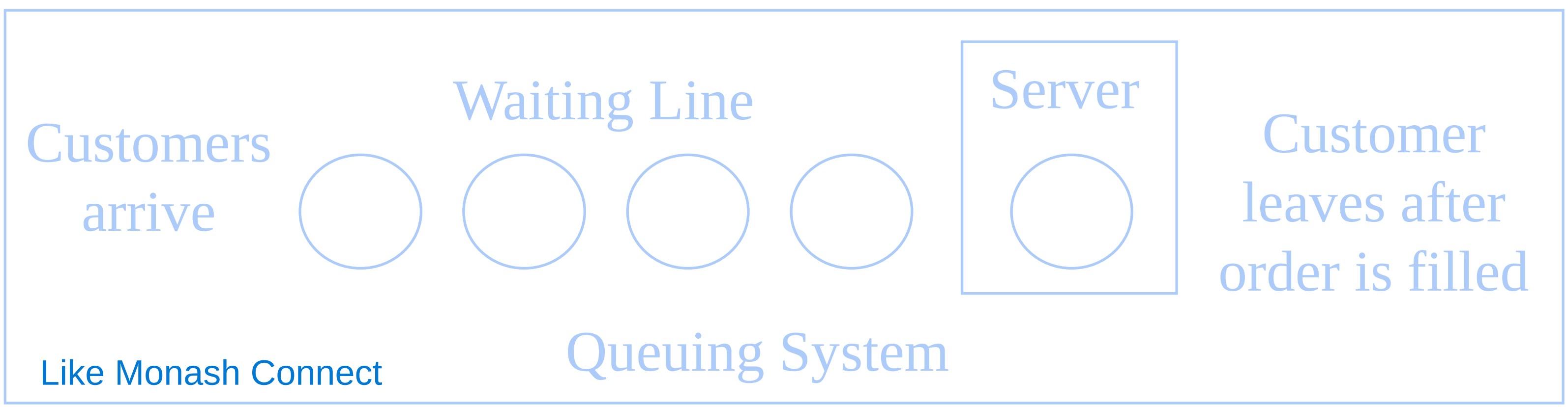
The result can be verified using expected values.

NEXT…

Simulating a Waiting Line and Introduction to Queuing Theory

Structure of a Waiting Line System

- Queuing theory is the study of waiting lines.
- Four characteristics of a queuing system are:
  - the manner in which customers arrive
  - the time required for service
  - the priority determining the order of service
  - the number and configuration of servers in the system.



Structure of a Waiting Line System

- In general, the arrival of customers into the system is a random event. Frequently the arrival pattern is modelled as a Poisson process whereby the time between customers has an exponential probability distribution.
- Service time is also usually a random variable. Again, the exponential distribution is commonly used to model service time.
- The most common queue discipline is first come, first served (FCFS). Another queuing discipline is last come, first served (last in, first out: LIFO) which is the discipline of people entering and exiting a lift.

Simulating a Waiting Room Queue

- Patients arrive at a Doctor’s surgery at various times according to a probability distribution, based on historical information, given below.
- Similarly, the Doctor’s consultation times have the distribution:

Dist of the arrival time

Time Between Patients	Probability
1	0.20
2	0.20
3	0.20
4	0.20
5	0.20

Dist of the service time

Doctor's Consultation Time	Probability
2	0.25
5	0.50
10	0.25

- Simulate the arrival and service of the first 10 patients for the day.

Refer: Lecture 11.xlsm (Simulation)

Waiting Room Queue

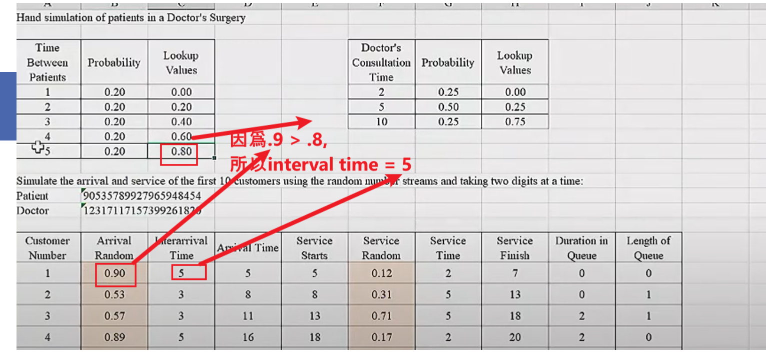
Arrival time

- Set up the following table:

Customer Number	Arrival Random	Interarrival Time	Arrival Time	Service Starts	Service Random	Service Time	Service Finish	Duration in Queue	Length of Queue
1	0.90	5			0.12	2			
2	0.53	3			0.31	5			
3	0.57	3			0.71	5			
4	0.89	5			0.17	2			
5	0.92	5			0.15	2			
6	0.79	4			0.73	5			
7	0.65	4			0.99	10			
8	0.94	5			0.26	5			
9	0.84	5			0.18	2			
10	0.54	3			0.20	2			

- We use random number data (already generated) to determine the times between arrivals, and the consultation times.

Waiting Room Queue



Hand simulation of patients in a Doctor's Surgery

Time Between Patients	Probability	Lookup Values
1	0.20	0.00
2	0.20	0.20
3	0.20	0.40
4	0.20	0.60
5	0.20	0.80

Doctor's Consultation Time	Probability	Lookup Values
2	0.25	0.00
5	0.50	0.25
10	0.25	0.75

因爲.9 > .8,
所以interval time = 5

Simulate the arrival and service of the first 10 customers using the random number streams and taking two digits at a time:

Patient 90535789927965948454

Doctor 1231711715739926182

Customer Number	Arrival Random	Interarrival Time	Arrival Time	Service Starts	Service Random	Service Time	Service Finish	Duration in Queue	Length of Queue
1	0.90	5	5	5	0.12	2	7	0	0
2	0.53	3	8	8	0.31	5	13	0	1
3	0.57	3	11	13	0.71	5	18	2	1
4	0.89	5	16	18	0.17	2	20	2	0

- The completed table: Max(A, B)

Customer Number	Arrival Random	Interarrival Time	Arrival Time	Service Starts	Service Random	Service Time	Service Finish	Duration in Queue	Length of Queue
1	0.90	5	5	5	0.12	2	7 B	0	0
2	0.53	3	8 A	8	0.31	5	13	0	1
3	0.57	3	11	13	0.71	5	18	13-11 2	1
4	0.89	5	16	18	0.17	2	20	18-16 2	0
5	0.92	5	21	21	0.15	2	23	21-21 0	0
6	0.79	4	25	25	0.73	5	30	0	1
7	0.65	4	29	30	0.99	10	40	1	2
8	0.94	5	34	40	0.26	5	45	6	2
9	0.84	5	39	45	0.18	2	47	6	1
10	0.54	3	42	47	0.20	2	49	5	0

- The issues at stake here, are the **service level** of patients seen through waiting time, and **server utilisation**, seen as the Doctor's idle time. These issues will be investigated more thoroughly.
- *But first, probability theory and queuing systems more generally...*

Review of Probability Theory

- Because queues typically form as the consequence of random events, they are modelled using probability theory. Some basic concepts are:
- Random Variables:
  - A random variable is a numerical description of the outcome of an experiment.
  - A random variable can be classified as being either discrete or continuous depending on the numerical values it assumes.
  - A discrete random variable may assume either a finite number of values or an infinite sequence of values.
  - A continuous random variable may assume any numerical value in an interval or collection of intervals.

Poisson Probability Distribution

- Properties of a Poisson distributed event:
- The number of occurrences of an event randomly distributed in space or over time.
- The occurrence of each event is independent of any other event occurring.
- The number of events occurring is theoretically infinite.
- Poisson Probability Function:

$$f(x) = \frac{\theta^x e^{-\theta}}{x!}$$

- where
- $f(x)$ = probability of x occurrences in an interval
- θ = mean number of occurrences in an interval
- $e \approx 2.71828$

Example: Poisson Distribution

The number of customers entering a post office each hour is a Poisson distributed random variable with a mean of 5.

a) What is the probability that exactly two customers arrive in a one hour period?

$$\theta = 5, \quad f(2) = \frac{5^2 (2.71828)^{-5}}{2!} = 0.0842$$

$$\boxed{f(x) = \frac{\theta^x e^{-\theta}}{x!}}$$

b) What is the probability of more than 7 customers arriving in a one hour period? *(Using cumulative probability tables).*

$$P(x > 7) = 1 - P(x \leq 7) = 1 - 0.8666 = 0.1334$$

Poisson Probability Tables

Probabilities for the Poisson Distribution

θ

x	0.5	1.0	2.0	3.0	4.0	5.0
0	0.6065	0.3679	0.1353	0.0498	0.0183	0.0067
1	0.3033	0.3679	0.2707	0.1494	0.0733	0.0337
2	0.0758	0.1839	0.2707	0.2240	0.1465	0.0842
3	0.0126	0.0613	0.1804	0.2240	0.1954	0.1404
4	0.0016	0.0153	0.0902	0.1680	0.1954	0.1755
5	0.0002	0.0031	0.0361	0.1008	0.1563	0.1755
6	0.0000	0.0005	0.0120	0.0504	0.1042	0.1462
7	0.0000	0.0001	0.0034	0.0216	0.0595	0.1044
8	0.0000	0.0000	0.0009	0.0081	0.0298	0.0653
9	0.0000	0.0000	0.0002	0.0027	0.0132	0.0363
10	0.0000	0.0000	0.0000	0.0008	0.0053	0.0181
11	0.0000	0.0000	0.0000	0.0002	0.0019	0.0082
12	0.0000	0.0000	0.0000	0.0001	0.0006	0.0034
13	0.0000	0.0000	0.0000	0.0000	0.0002	0.0013

Cumulative probabilities for the Poisson Distribution

θ

x	0.5	1.0	2.0	3.0	4.0	5.0
0	0.6065	0.3679	0.1353	0.0498	0.0183	0.0067
1	0.9098	0.7358	0.4060	0.1991	0.0916	0.0404
2	0.9856	0.9197	0.6767	0.4232	0.2381	0.1247
3	0.9982	0.9810	0.8571	0.6472	0.4335	0.2650
4	0.9998	0.9963	0.9473	0.8153	0.6288	0.4405
5	1.0000	0.9994	0.9834	0.9161	0.7851	0.6160
6	1.0000	0.9999	0.9955	0.9665	0.8893	0.7622
7	1.0000	1.0000	0.9989	0.9881	0.9489	0.8666
8	1.0000	1.0000	0.9998	0.9962	0.9786	0.9319
9	1.0000	1.0000	1.0000	0.9989	0.9919	0.9682
10	1.0000	1.0000	1.0000	0.9997	0.9972	0.9863
11	1.0000	1.0000	1.0000	0.9999	0.9991	0.9945
12	1.0000	1.0000	1.0000	1.0000	0.9997	0.9980
13	1.0000	1.0000	1.0000	1.0000	0.9999	0.9993

Refer: Lecture 11.xlsm

Continuous Probability Distributions

- A continuous random variable can assume any value in an interval on the real line or in a collection of intervals.
- It is not possible to talk about the probability of the random variable assuming a particular value.
- Instead, we talk about the probability of the random variable assuming a value within a given interval.
- The probability of the random variable assuming a value within some given interval from x_1 to x_2 is defined to be the area under the graph of the probability density function between x_1 and x_2.

Exponential Probability Distribution

- Exponential Probability Density Function

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \qquad \text{for } x > 0,\ \theta > 0$$

where θ = mean, $e \approx 2.71828$

- Cumulative Exponential Distribution Function

$$P(x \le x_0) = 1 - e^{-x_0/\theta}$$

$$P(x \ge x_0) = e^{-x_0/\theta}$$

where x_0 = some specific value of x

Example: Exponential Distribution

The time that an insurance company's clerk takes to process claim applications is an exponentially distributed random variable with a mean of 5 minutes.

a) What is the probability that a claim will take more than 10 minutes to process? We need x_0 and theta

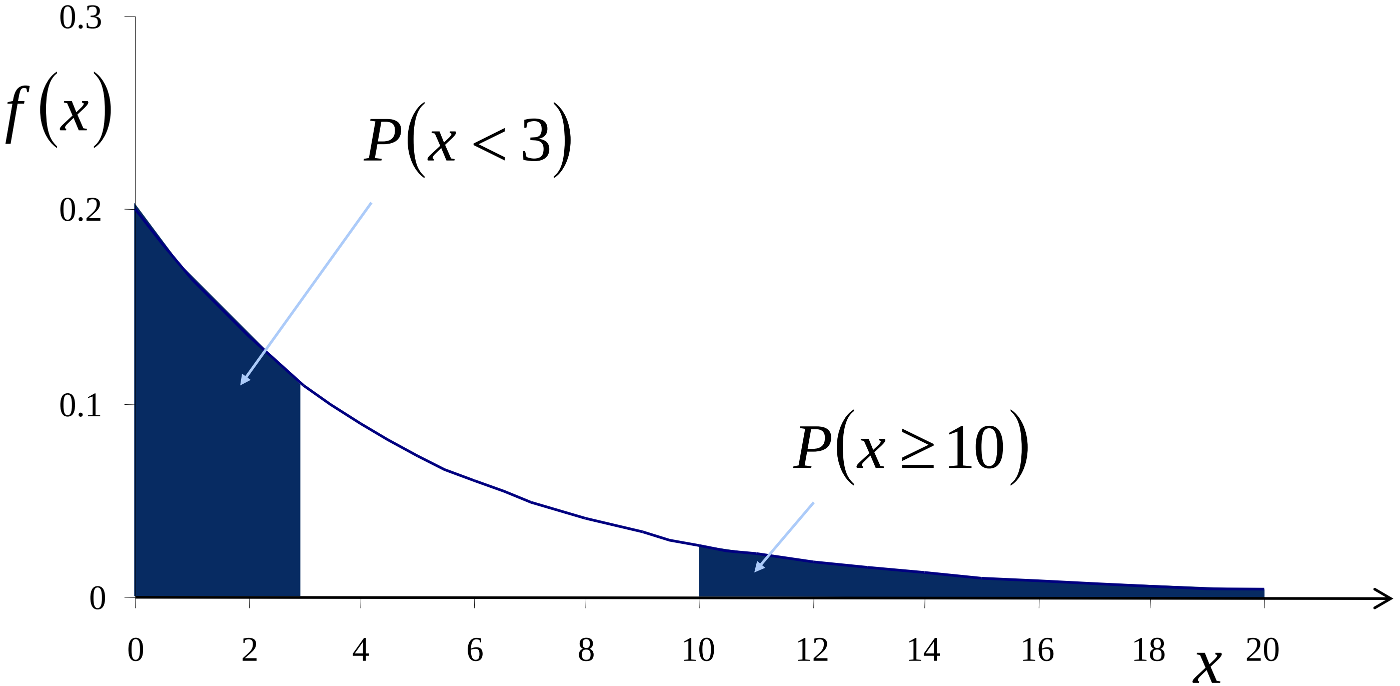
$$P(x \geq 10) = e^{-10/5} = e^{-2} = 0.1353$$

b) What is the probability that a claim will take less than 3 minutes to process?

$$P(x \leq 3) = 1 - e^{-3/5} = 1 - 0.5488 = 0.4512$$

Example: Exponential Distribution

Graph of the Probability Density Function when θ = 5 showing the calculated probabilities.



Relationship Between the Poisson and Exponential Distributions

The continuous exponential probability distribution is related to the discrete Poisson distribution in the following way:

The Poisson distribution provides an appropriate description of the number of occurrences over a certain period of time when the time interval between each occurrence is has an exponential distribution.

the number of occurrences = poisson distribution
the time interval between the occurrences = exponential distribution

e.g., Previous example.
number of arrivals = poisson distribution
when the time between the arrivals = an exponential distribution

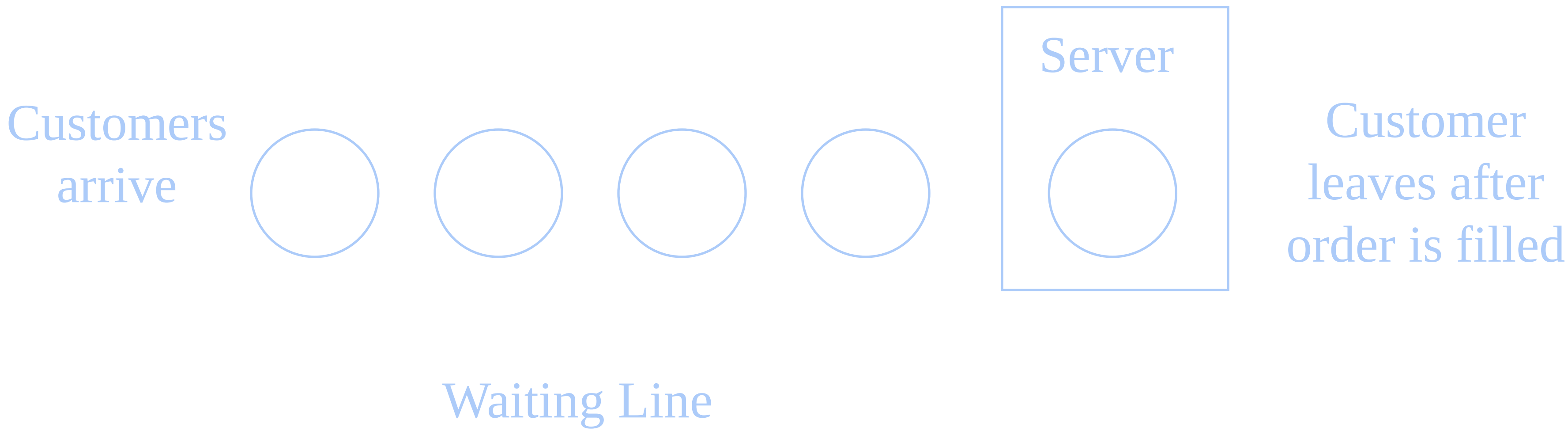
so, poisson distribution provides a description of the number of occurrences in a time interval
when the time interval between each occurrence is having a exponential
probability distribution

Queuing Systems

- Kendall notation is a three part code of the form *A*/*B*/*s,* used to describe various queuing systems.
- *A* identifies the arrival distribution, *B* the service distribution and *s* the number of servers in the system.
- Frequently used symbols for the arrival and service processes are: *M* - Markov distributions (Poisson/exponential), *D* - Deterministic (constant) and *G* - General distribution (with a known mean and variance).
- For example, *M*/*M*/*k* refers to a system in which arrivals occur according to a Poisson distribution, service times follow an exponential distribution and there are *k* servers working at identical service rates.
- The doctor's waiting room example is a G/G/1 queue.

M/M/1 Queuing System

- This is a queuing system with exponentially distributed times between customers and exponentially distributed service times. A single queue is served by a single server.



Queuing System Input Characteristics

λ = the average arrival rate.

$1/\lambda$ = the average time between arrivals.

μ = the average service rate for each server.

$1/\mu$ = the average service time.

σ = the standard deviation of the service time.

- For example, if 10 customers arrive every minute on average, then the average time between arrivals is 0.1 minutes.
- If a server takes an average of 5 minutes, with a standard deviation of 1 minute to serve customers then:

$$1/\mu = 5, \mu = 0.2, \sigma = 1.$$

Operating Characteristics of M/M/1 Queues

- Some steady-state (long-term average) characteristics of an M/M/1 Queue are:

$$\lambda = \text{the mean number of arrival per time period}$$

$$\mu = \text{the mean number of services per time period}$$

$$P_0 = 1 - \frac{\lambda}{\mu} \text{ the probability that no units are in the system}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} \text{ the average number of units in the waiting line}$$

$$L = L_q + \frac{\lambda}{\mu} \text{ the average number of units in the system}$$

Operating Characteristics of M/M/1 Queues

$$W_q = \frac{L_q}{\lambda}$$ the average time a unit spends in the waiting line

$$W = W_q + \frac{1}{\mu}$$ the average time a unit spends in the system

$$P_w = \frac{\lambda}{\mu}$$ the probability that an arriving unit has to wait for service

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$ the probability of n units in the system

M/M/1 Example

Students arrive at a service counter at the rate of 8 students per hour. Each student takes, on average, 5 minutes to serve. Assume that the time between arrivals, and the service time are exponentially distributed.

a) What is the probability that exactly 4 students arrive in a certain hour?

Student arrivals have a Poisson distribution with mean of 8:

$$f(4) = \frac{8^4 e^{-8}}{4!} = 0.0573$$

b) What is the probability that a particular student will take more than 7 minutes to serve?

Service time is exponentially distributed with a mean of 5 minutes:

$$P(x \geq 7) = e^{-7/5} = 0.2466$$

M/M/1 Example Cont…

c) What is the average number of students waiting at any time?

Service rate is 12 students per hour:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{8^2}{12(12-8)} = 1.333\dot{3}$$

d) What is the probability of 5 students waiting at any one time?

5 students waiting means there are 6 students in the system:

$$P_6 = \left(\frac{\lambda}{\mu}\right)^6 \left(1 - \frac{\lambda}{\mu}\right) = \left(\frac{8}{12}\right)^6 \left(1 - \frac{8}{12}\right) = 0.0293$$

M/M/1 Example Cont…

e) Examine the sensitivity of the steady state performance characteristics of the waiting line to the rate of student enquiries.

The mean arrival rate for the system (lambda)	5	6	7	***8***	9	10	11
The mean service rate for each channel (mu)	12	12	12	***12***	12	12	12
Number of units in the system (n)	6	6	6	***6***	6	6	6
Operating Characteristics							
The probability that no units are in the system (Po)	0.58	0.50	0.42	***0.33***	0.25	0.17	0.08
The average number of units in the waiting line (Lq)	0.30	0.50	0.82	***1.33***	2.25	4.17	10.08
The average number of units in the system (L)	0.71	1.00	1.40	***2.00***	3.00	5.00	11.00
The average time a unit is in the waiting line (Wq)	0.06	0.08	0.12	***0.17***	0.25	0.42	0.92
The average time a unit spends in the system (W)	0.14	0.17	0.20	***0.25***	0.33	0.50	1.00
The probability that a unit has to wait (Pw)	0.42	0.50	0.58	***0.67***	0.75	0.83	0.92
Probability of n units in the system (Pn)	0.00	0.01	0.02	***0.03***	0.04	0.06	0.05
Server utilisation factor (rho)	0.42	0.50	0.58	***0.67***	0.75	0.83	0.92

Refer: Lecture 11.xlsm

Summary

- Introduction to Simulation and Queues
- Simulation of patients in a doctor’s waiting room
- Basic Queuing Theory
- Review of probability theory
  - Poisson distribution
  - Exponential distribution
- Steady state behaviour of queues.
  - M/M/1,
  - M/G/1, M/M/2 queuing systems (next week)

End of Lecture 11

References:

Ragsdale, C.. (8th Ed 2017/9th Ed 2021). Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e) Cengage Learning: Chapters 12&13

Lapin, L. and Whisler, W., Quantitative Decision Making with Spreadsheet Applications 7th Ed., Wadsworth (Thomson Learning) Belmont, 2002: Chapter 17

Tutorial 10 this week:

- Time series forecasting
- Regression with Seasonal and Non-seasonal data
- Additive and Multiplicative models

Homework

- Go through the lecture:
  - ✓ Work through the doctor's waiting room simulation exercise
  - ✓ Familiarise yourself with the Queuing Theory
  - ✓ Understand the probability theory and work through the exercises:
    - Poisson distribution
    - Exponential distribution
  - ✓ Work through the examples for M/M/1 queuing systems.

☞ ***Preparation for exams:***

- Work on the review/sample questions for each topic

- Poisson Probability Function:

$$f(x) = \frac{\theta^x e^{-\theta}}{x!}$$

- where
- $f(x)$ = probability of x occurrences in an interval
- θ = mean number of occurrences in an interval
- $e \approx 2.71828$

- Exponential Probability Density Function

$$f(x) = \frac{1}{\theta} e^{-x/\theta} \qquad \text{for } x > 0,\ \theta > 0$$

where θ = mean, $e \approx 2.71828$

- Cumulative Exponential Distribution Function

$$P(x \le x_0) = 1 - e^{-x_0/\theta}$$
$$P(x \ge x_0) = e^{-x_0/\theta}$$

where x_0 = some specific value of x

The number of customers entering a post office each hour is a Poisson distributed random variable with a mean of 5.

a) What is the probability that exactly two customers arrive in a one hour period?

$$\theta = 5,\ f(2) = \frac{5^2 (2.71828)^{-5}}{2!} = 0.0842 \qquad \boxed{f(x) = \frac{\theta^x e^{-\theta}}{x!}}$$

The time that an insurance company's clerk takes to process claim applications is an exponentially distributed random variable with a mean of 5 minutes.

a) What is the probability that a claim will take more than 10 minutes to process? We need x_0 and theta

$$P(x \ge 10) = e^{-10/5} = e^{-2} = 0.1353$$

b) What is the probability that a claim will take less than 3 minutes to process?

$$P(x \le 3) = 1 - e^{-3/5} = 1 - 0.5488 = 0.4512$$

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