Chapter 4

Sensitivity Analysis and the Simplex Method

4.0 Introduction

In chapters 2 and 3, we studied how to formulate and solve LP models for a variety of decision problems. However, formulating and solving an LP model does not necessarily mean that the original decision problem has been solved. After solving an LP model, a number of questions often arise about the optimal solution. In particular, we might be interested in how sensitive the optimal solution is to changes in various coefficients of the LP model.

Businesses rarely know with certainty what costs will be incurred or the exact amount of resources that will be consumed or available in a given situation or time period. Thus, optimal solutions obtained using models that assume all relevant factors are known with certainty might be viewed with skepticism by management. Sensitivity analysis can help overcome this skepticism and provide a better picture of how the solution to a problem will change if different factors in the model change. Sensitivity analysis also can help answer a number of practical managerial questions that might arise about the solution to an LP problem.

4.1 The Purpose of Sensitivity Analysis

As noted in chapter 2, any problem that can be stated in the following form is an LP problem:

$$
\begin{array}{ll}
\text{MAX (or MIN):} & c_1X_1 + c_2X_2 + \cdots + c_nX_n \\
\text{Subject to:} & a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n \leq b_1 \\
& \vdots \\
& a_{k1}X_1 + a_{k2}X_2 + \cdots + a_{kn}X_n \geq b_k \\
& \vdots \\
& a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = b_m
\end{array}
$$

All the coefficients in this model (the c_j, a_{ij}, and b_i) represent numeric constants. So, when we formulate and solve an LP problem, we implicitly assume that we can specify the exact values for these coefficients. However, in the real world, these coefficients might change from day to day or minute to minute. For example, the price a company charges for its products can change on a daily, weekly, or monthly basis. Similarly, if a skilled machinist calls in sick, a manufacturer might have less capacity to produce items on a given machine than was originally planned.

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Realizing that such uncertainties exist, a manager should consider how sensitive an LP model's solution is to changes or estimation errors that might occur in: (1) the objective function coefficients (the c_j), (2) the constraint coefficients (the a_{ij}), and (3) the RHS values for the constraints (the b_i). A manager also might ask a number of "What if?" questions about these values. For example, what if the cost of a product increases by 7%? What if a reduction in setup time allows for additional capacity on a given machine? What if a worker's suggestion results in a product requiring only 2 hours of labor rather than three? Sensitivity analysis addresses these issues by assessing the sensitivity of the solution to uncertainty or estimation errors in the model coefficients, as well as the solution's sensitivity to changes in model coefficients that might occur because of human intervention.

4.2 Approaches to Sensitivity Analysis

You can perform sensitivity analysis on an LP model in a number of ways. If you want to determine the effect of some change in the model, the most direct approach is simply to change the model and re-solve it. This approach is suitable if the model does not take an excessive amount of time to change or solve. In addition, if you are interested in studying the consequences of *simultaneously* changing several coefficients in the model, this might be the only practical approach to sensitivity analysis.

Solver also provides some sensitivity information after solving an LP problem. As mentioned in chapter 3, one of the benefits of using the simplex method to solve LP problems is its speed—it is considerably faster than the other optimization techniques. However, the simplex method also provides more sensitivity analysis information than the other techniques. In particular, the simplex method provides us with information about the following:

- The range of values the objective function coefficients can assume without changing the optimal solution
- The impact on the optimal objective function value of increases or decreases in the availability of various constrained resources
- The impact on the optimal objective function value of forcing changes in the values of certain decision variables away from their optimal values
- The impact that changes in constraint coefficients will have on the optimal solution to the problem

4.3 An Example Problem

We will again use the Blue Ridge Hot Tubs problem to illustrate the types of sensitivity analysis information available using Solver. The LP formulation of the problem is repeated here, where X_1 represents the number of Aqua-Spas and X_2 represents the number of Hydro-Luxes to be produced:

$$\begin{array}{lrcl}
\text{MAX:} & 350X_1 + 300X_2 & & \}\ \text{profit} \\
\text{Subject to:} & 1X_1 + 1X_2 \leq 200 & & \}\ \text{pump constraint} \\
 & 9X_1 + 6X_2 \leq 1{,}566 & & \}\ \text{labor constraint} \\
 & 12X_1 + 16X_2 \leq 2{,}880 & & \}\ \text{tubing constraint} \\
 & X_1, X_2 \geq 0 & & \}\ \text{nonnegativity conditions}
\end{array}$$

This model is implemented in the spreadsheet shown in Figure 4.1 (and file Fig4-1.xlsm that accompanies this book). (See chapter 3 for details on the procedure used

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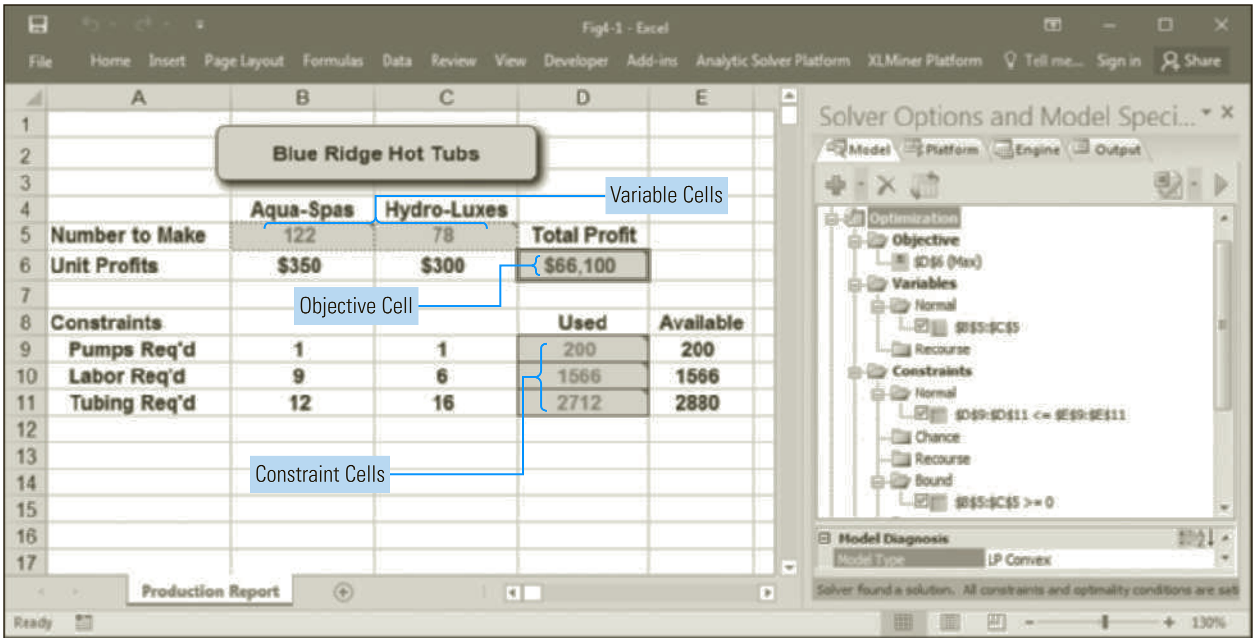
FIGURE 4.1 *Spreadsheet model for the Blue Ridge Hot Tubs product mix problem*



Key Cell Formulas

Cell	Formula	Copied to
D6	=SUMPRODUCT(B6:C6,$B$5:$C$5)	D9:D11

to create and solve this spreadsheet model.) After solving the LP problem, a number of reports are available about its solution via the Reports icon on the Analytic Solver Platform tab on the ribbon.

Software Note

When solving an LP problem, be sure to use Solver's Standard LP/Quadratic Engine. This allows for the maximum amount of sensitivity analysis information in the reports discussed throughout this chapter.

4.4 The Answer Report

Figure 4.2 shows the Answer Report for the Blue Ridge Hot Tubs problem. To create this report, first solve the LP problem in the usual way and then click Reports, Optimization, Answer on the Analytic Solver Platform tab on the ribbon. This report summarizes the solution to the problem, and is fairly self-explanatory. The first section of the report summarizes the original and final (optimal) value of the objective cell. The next section summarizes the original and final (optimal) values of the decision variable cells.

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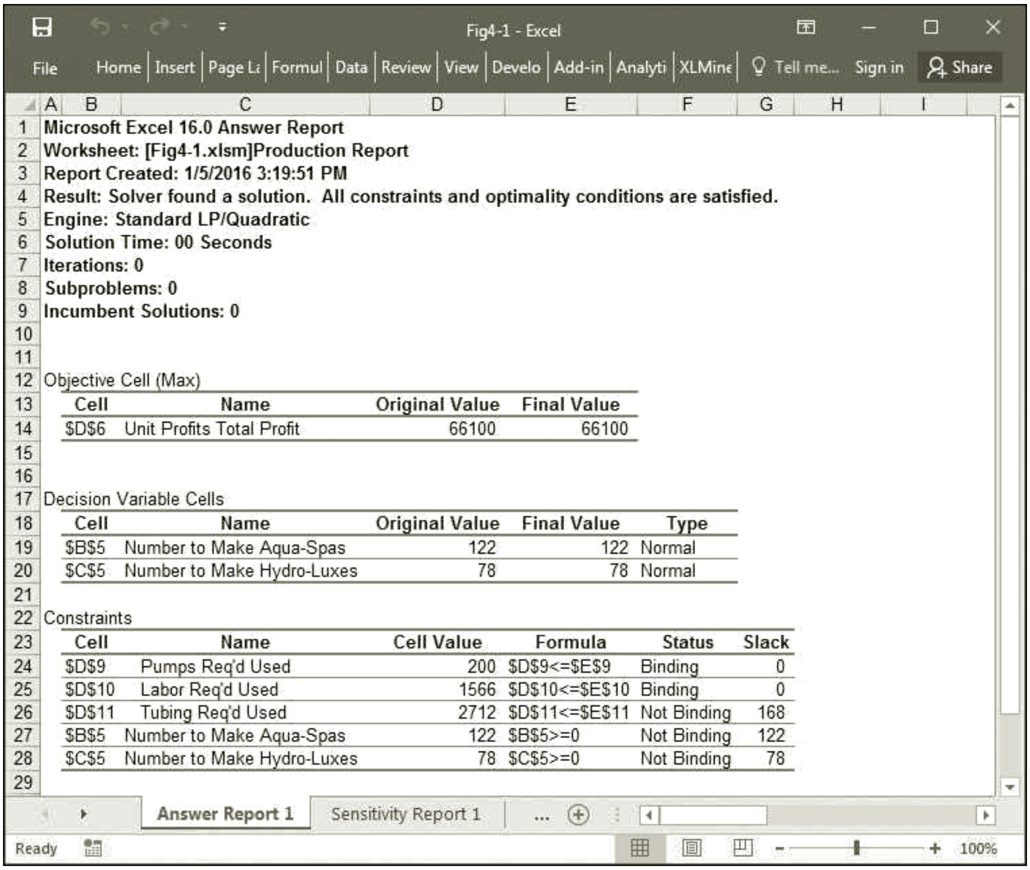
FIGURE 4.2

Answer Report for the hot tub problem



Microsoft Excel 16.0 Answer Report
Worksheet: [Fig4-1.xlsm]Production Report
Report Created: 1/5/2016 3:19:51 PM
Result: Solver found a solution. All constraints and optimality conditions are satisfied.
Engine: Standard LP/Quadratic
Solution Time: 00 Seconds
Iterations: 0
Subproblems: 0
Incumbent Solutions: 0

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$6	Unit Profits Total Profit	66100	66100

Decision Variable Cells

Cell	Name	Original Value	Final Value	Type
$B$5	Number to Make Aqua-Spas	122	122	Normal
$C$5	Number to Make Hydro-Luxes	78	78	Normal

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$9	Pumps Req'd Used	200	$D$9<=$E$9	Binding	0
$D$10	Labor Req'd Used	1566	$D$10<=$E$10	Binding	0
$D$11	Tubing Req'd Used	2712	$D$11<=$E$11	Not Binding	168
$B$5	Number to Make Aqua-Spas	122	$B$5>=0	Not Binding	122
$C$5	Number to Make Hydro-Luxes	78	$C$5>=0	Not Binding	78

The final section of this report provides information about the constraints. In particular, the Cell Value column shows the final (optimal) value assumed by each constraint cell. Note that these values correspond to the final value assumed by the LHS formula of each constraint. The Formula column indicates the upper or lower bounds that apply to each constraint cell. The Status column indicates which constraints are binding and which are nonbinding. A constraint is **binding** if it is satisfied as a strict equality in the optimal solution; otherwise, it is **nonbinding**. Notice that the constraints for the number of pumps and amount of labor used are both binding, meaning that *all* the available pumps and labor hours will be used if this solution is implemented. Therefore, these constraints are preventing Blue Ridge Hot Tubs from achieving a higher level of profit.

Finally, the values in the Slack column indicate the difference between the LHS and RHS of each constraint. By definition, binding constraints have zero slack and nonbinding constraints have some positive level of slack. The values in the Slack column indicate that if this solution is implemented, all the available pumps and labor hours will be used, but 168 feet of tubing will be left over. The slack values for the nonnegativity conditions indicate the amounts by which the decision variables exceed their respective lower bounds of zero.

The Answer Report does not provide any information that could not be derived from the solution shown in the spreadsheet model. However, the format of this report gives a convenient summary of the solution that can be incorporated easily into a word-processing document as part of a written report to management.

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Report Headings

When creating the reports described in this chapter, Solver will try to use various text entries from the original spreadsheet to generate meaningful headings and labels in the reports. Given the various ways in which a model can be implemented, Solver might not always produce meaningful headings. However, you can change any text entry to make the report more meaningful or descriptive.

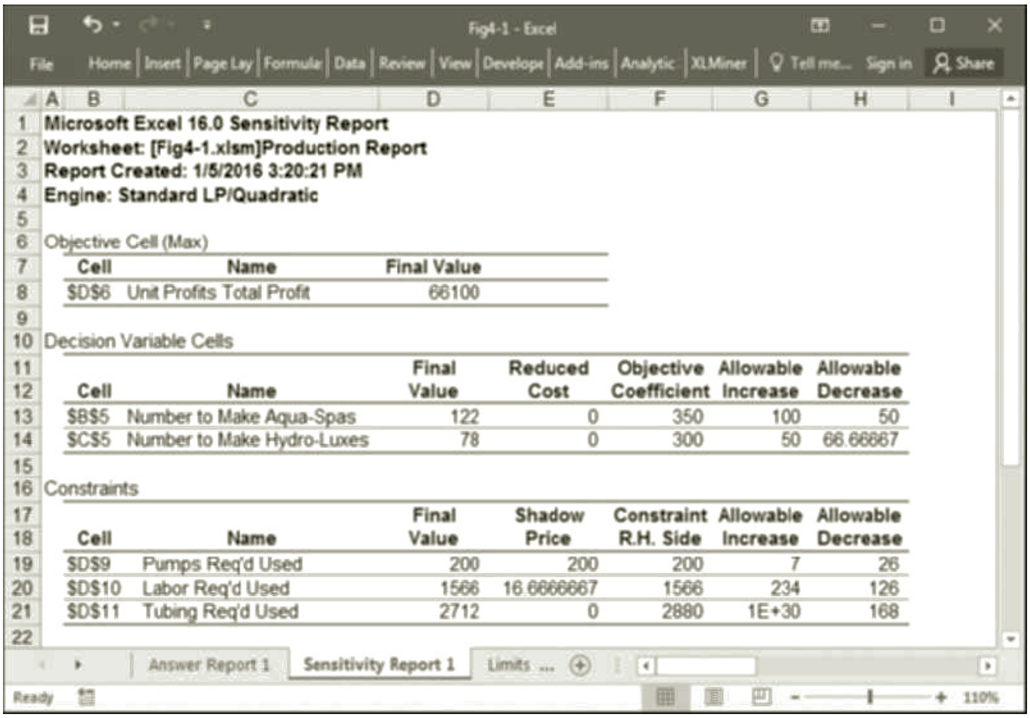
4.5 The Sensitivity Report

Figure 4.3 shows the Sensitivity Report for the Blue Ridge Hot Tubs problem. To create this report, first solve the LP problem in the usual way and then click Reports, Optimization, Sensitivity on the Analytic Solver Platform tab on the ribbon. This report summarizes information about the variable cells and constraints for our model. This information is useful in evaluating how sensitive the optimal solution is to changes in various coefficients in the model.

4.5.1 CHANGES IN THE OBJECTIVE FUNCTION COEFFICIENTS

Chapter 2 introduced the level-curve approach to solving a graphical LP problem and showed how to use this approach to solve the Blue Ridge Hot Tubs problem. This graphical solution is repeated in Figure 4.4 (and file Fig4-4.xlsm that accompanies this book).

The slope of the original level curve in Figure 4.4 is determined by the coefficients in the objective function of the model (the values 350 and 300). In Figure 4.5, we can



Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Fig4-1.xlsm]Production Report
Report Created: 1/5/2016 3:20:21 PM
Engine: Standard LP/Quadratic

Objective Cell (Max)

Cell	Name	Final Value
$D$6	Unit Profits Total Profit	66100

Decision Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to Make Aqua-Spas	122	0	350	100	50
$C$5	Number to Make Hydro-Luxes	78	0	300	50	66.66667

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	Pumps Req'd Used	200	200	200	7	26
$D$10	Labor Req'd Used	1566	16.6666667	1566	234	126
$D$11	Tubing Req'd Used	2712	0	2880	1E+30	168

FIGURE 4.3

Sensitivity Report for the hot tub problem

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FIGURE 4.4 *Graph of original feasible region and optimal solution*

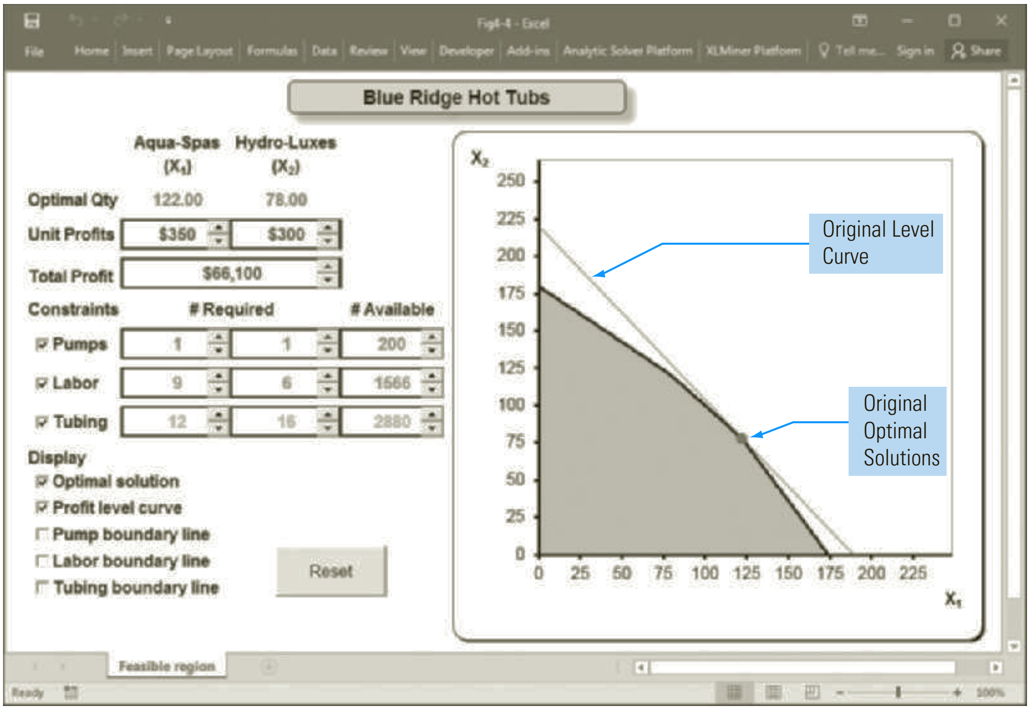
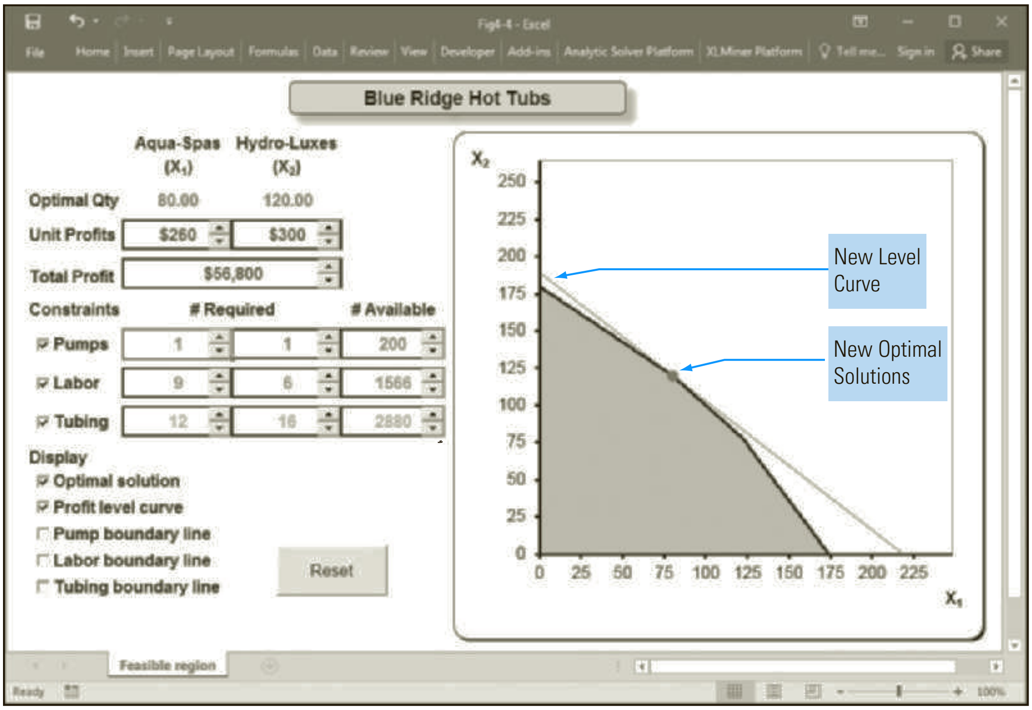


FIGURE 4.5 *How a change in an objective function coefficient can change the slope of the level curve and the optimal solution*



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see that if the slope of the level curve were different, the extreme point represented by $X_1 = 80$, $X_2 = 120$ would be the optimal solution. Of course, the only way to change the level curve for the objective function is to change the coefficients in the objective function. So, if the objective function coefficients are at all uncertain, we might be interested in determining how much these values could change before the optimal solution would change.

For example, if the owner of Blue Ridge Hot Tubs does not have complete control over the costs of producing hot tubs (which is likely because he purchases the fiberglass hot tub shells from another company), the profit figures in the objective function of our LP model might not be the exact profits earned on hot tubs produced in the future. So before the manager decides to produce 122 Aqua-Spas and 78 Hydro-Luxes, he might want to determine how sensitive this solution is to the profit figures in the objective. That is, the manager might want to determine how much the profit figures could change before the optimal solution of $X_1 = 122$, $X_2 = 78$ would change. This information is provided in the Sensitivity Report shown in Figure 4.3.

The original objective function coefficients associated with the variable cells are listed in the Objective Coefficient column in Figure 4.3. The next two columns show the allowable increases and decreases in these values. For example, the objective function value associated with Aqua-Spas (or variable X_1) can increase by as much as \$100 or decrease by as much as \$50 without changing the optimal solution, assuming all other coefficients remain constant. (You can verify this by changing the profit coefficient for Aqua-Spas to any value in the range from \$300 to \$450 and re-solving the model.) Similarly, the objective function value associated with Hydro-Luxes (or variable X_2) can increase by \$50 or decrease by approximately \$66.67 without changing the optimal values of the decision variables, assuming all other coefficients remain constant. (Again, you can verify this by re-solving the model with different profit values for Hydro-Luxes.)

Software Note

When setting up a spreadsheet model for an LP problem for which you intend to generate a Sensitivity Report, it is a good idea to make sure the cells corresponding to RHS values of constraints contain constants or formulas that do not involve the decision variables. Thus, any RHS formula related directly or indirectly to the decision variables should be moved algebraically to the LHS of the constraint before implementing your model. This will help to reduce problems in interpreting the Solver Sensitivity Report.

4.5.2 A COMMENT ABOUT CONSTANCY

The phrase "assuming all other coefficients remain constant" in the previous paragraph underscores the fact that the allowable increases and decreases shown in the Sensitivity Report apply only if *all* the other coefficients in the LP model do not change. The objective coefficient for Aqua-Spas can assume any value from \$300 to \$450 without changing the optimal solution—*but this is guaranteed to be true only if all the other coefficients in the model remain constant* (*including the objective function coefficient for* X_2). Similarly, the objective function coefficient for X_2 can assume any value between \$233.33 and \$350 without changing the optimal solution—*but this is guaranteed to be true only if all the other coefficients in the model remain constant* (*including the objective function coefficient for* X_1). Later in this chapter, you will see how to determine whether

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the current solution remains optimal if changes are made in two or more objective coefficients at the same time.

4.5.3 ALTERNATE OPTIMAL SOLUTIONS

Sometimes, the allowable increase or allowable decrease for the objective function coefficient for one or more variables will equal zero. In the absence of degeneracy (to be described later), this indicates that alternate optimal solutions exist. You can usually get Solver to produce an alternate optimal solution (when they exist) by: (1) adding a constraint to your model that holds the objective function at the current optimal value, and then (2) attempting to maximize or minimize the value of one of the decision variables that had an objective function coefficient with an allowable increase or decrease of zero. This approach sometimes involves some "trial and error" in step 2, but should cause Solver to produce an alternate optimal solution to your problem.

4.5.4 CHANGES IN THE RHS VALUES

As noted earlier, constraints that have zero slack in the optimal solution to an LP problem are called binding constraints. Binding constraints prevent us from further improving (i.e., maximizing or minimizing) the objective function. For example, the Answer Report in Figure 4.2 indicates that the constraints for the number of pumps and hours of labor available are binding, whereas the constraint on the amount of tubing available is nonbinding. This is also evident in Figure 4.3 by comparing the Final Value column with the Constraint R.H. Side column. The values in the Final Value column represent the LHS values of each constraint at the optimal solution. A constraint is binding if its Final Value is equal to its Constraint R.H. Side value.

After solving an LP problem, you might want to determine how much better or worse the solution would be if we had more or less of a given resource. For example, Howie Jones might wonder how much more profit could be earned if additional pumps or labor hours were available. The Shadow Price column in Figure 4.3 provides the answers to such questions.

The **shadow price** for a constraint indicates the amount by which the objective function value changes given a unit *increase* in the RHS value of the constraint, assuming all other coefficients remain constant. If a shadow price is positive, a unit increase in the RHS value of the associated constraint results in an increase in the optimal objective function value. If a shadow price is negative, a unit increase in the RHS value of the associated constraint results in a decrease in the optimal objective function value. To analyze the effects of decreases in the RHS values, you reverse the sign on the shadow price. That is, the negated shadow price for a constraint indicates the amount by which the optimal objective function value changes given a unit *decrease* in the RHS value of the constraint, assuming all other coefficients remain constant. The shadow price values apply provided that the increase or decrease in the RHS value falls within the allowable increase or allowable decrease limits in the Sensitivity Report for each constraint.

For example, Figure 4.3 indicates that the shadow price for the labor constraint is 16.67. Therefore, if the number of available labor hours increased by any amount in the range from 0 to 234 hours, the optimal objective function value changes (increases) by $16.67 for each additional labor hour. If the number of available labor hours decreased by any amount in the range from 0 to 126 hours, the optimal objective function value changes (decreases) by −$16.67 for each lost labor hour. A similar interpretation holds for the shadow price for the constraint on the number of pumps. (It is coincidental that the shadow price for the pump constraint (200) is the same as that constraint's RHS and Final Values.)

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4.5.5 SHADOW PRICES FOR NONBINDING CONSTRAINTS

Now, let's consider the shadow price for the nonbinding tubing constraint. The tubing constraint has a shadow price of zero with an allowable increase of infinity and an allowable decrease of 168. Therefore, if the RHS value for the tubing constraint increases by *any* amount, the objective function value does not change (or changes by zero). This result is not surprising. Because the optimal solution to this problem leaves 168 feet of tubing unused, *additional* tubing will not produce a better solution. Furthermore, because the optimal solution includes 168 feet of unused tubing, we can reduce the RHS value of this constraint by 168 without affecting the optimal solution.

As this example illustrates, the shadow price of a nonbinding constraint is always zero. There is always some amount by which the RHS value of a nonbinding constraint can be changed without affecting the optimal solution.

4.5.6 A NOTE ABOUT SHADOW PRICES

One important point needs to be made concerning shadow prices. To illustrate this point, let's suppose that the RHS value of the labor constraint for our example problem increases by 162 hours (from 1,566 to 1,728) due to the addition of new workers. Because this increase is within the allowable increase listed for the labor constraint, you might expect that the optimal objective function value would increase by $16.67 × 162 = $2,700. That is, the new optimal objective function value would be approximately $68,800 ($66,100 + $16.67 × 162 = $68,800). Figure 4.6 shows the re-solved model after increasing the RHS value for the labor constraint by 162 labor hours to 1,728.

In Figure 4.6, the new optimal objective function value is $68,800, as expected. But this solution involves producing 176 Aqua-Spas and 24 Hydro-Luxes. That is, the

FIGURE 4.6 *Solution to the revised hot tub problem with 162 additional labor hours*

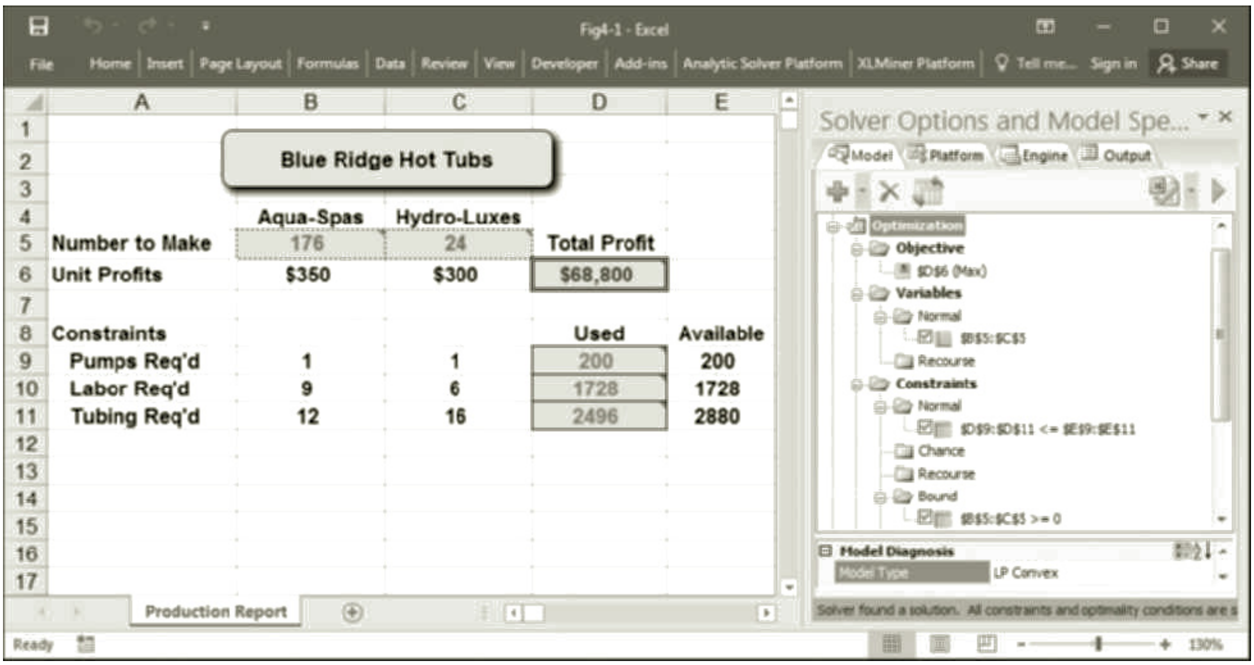

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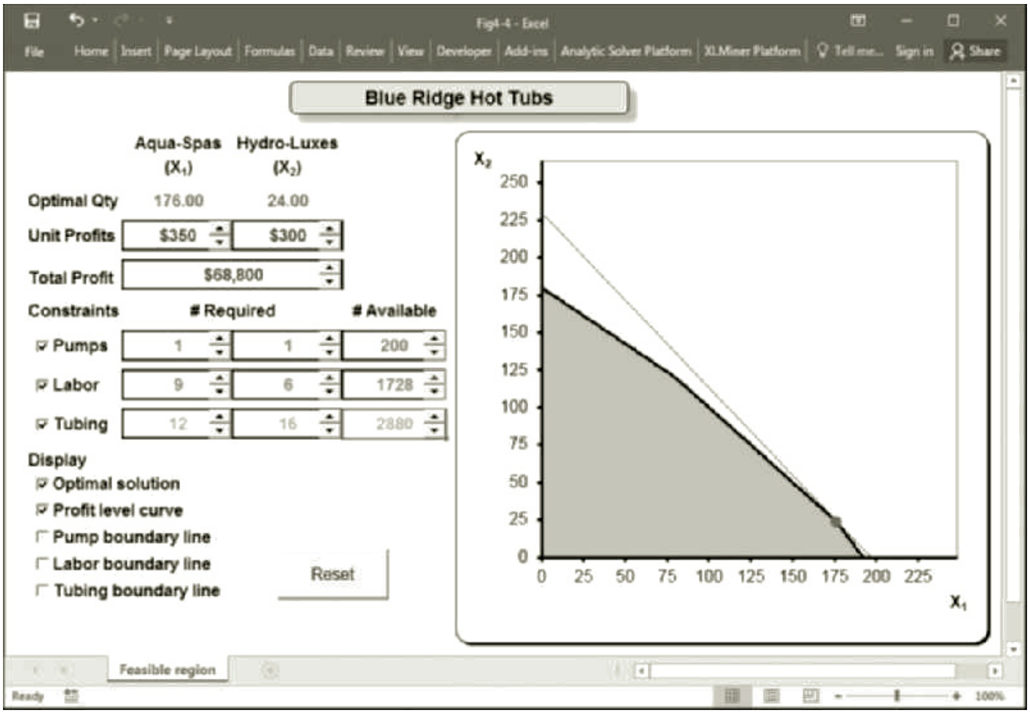
FIGURE 4.7

How a change in the RHS value of the labor constraint changes the feasible region and optimal solution



optimal solution to the revised problem is *different* from the solution to the original problem shown in Figure 4.1. This is not surprising because changing the RHS of a constraint also changes the feasible region for the problem. The effect of increasing the RHS of the labor constraint is shown graphically in Figure 4.7.

So, although shadow prices indicate how the objective function value changes if a given RHS value changes, they *do not* tell you which values the decision variables need to assume in order to achieve this new objective function value. Determining the new optimal values for the decision variables requires that you make the appropriate changes in the RHS value and re-solve the model.

Another Interpretation of Shadow Prices

Unfortunately, there is no one universally accepted way of reporting shadow prices for constraints. In some software packages, the signs of the shadow prices do not conform to the convention used by Solver. Regardless of which software package you use, there is another way to look at shadow prices that should always lead to a proper interpretation. The absolute value of the shadow price always indicates the amount by which the objective function will be *improved* if the corresponding constraint is *loosened*. A less than or equal to constraint is loosened by *increasing* its RHS value, whereas a greater than or equal to constraint is loosened by *decreasing* its RHS value. (The absolute value of the shadow price can also be interpreted as the amount by which the objective will be made *worse* if the corresponding constraint is *tightened*.)

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4.5.7 SHADOW PRICES AND THE VALUE OF ADDITIONAL RESOURCES

In the previous example, an additional 162 hours of labor allowed us to increase profits by $2,700. A question might then arise as to how much we should be willing to pay to acquire these additional 162 hours of labor. The answer to this question is, "It depends. . . ."

If labor is a *variable* cost that was subtracted (along with other variable costs) from the selling price of the hot tubs to determine the marginal profits associated with each type of tub, we should be willing to pay up to $2,700 *above and beyond* what we would ordinarily pay to acquire 162 hours of labor. In this case, notice that both the original and revised profit figures of $66,100 and $68,800, respectively, represent the profit earned *after* the normal labor charge has been paid. Therefore, we could pay a premium of up to $2,700 to acquire the additional 162 hours of labor (or an extra $16.67 per additional labor hour) and still earn at least as much profit as we would have without the additional 162 hours of labor. Thus, if the normal labor rate is $12 per hour, we could pay up to $28.67 per hour to acquire each of the additional 162 hours of labor.

On the other hand, if labor is a sunk cost, which must be paid regardless of how many hot tubs are produced, it would not (or should not) have been subtracted from the selling price of the hot tubs in determining the marginal profit coefficients for each tub produced. In this case, we should be willing to pay a maximum of $16.67 per hour to acquire each of the additional 162 hours of labor.

4.5.8 OTHER USES OF SHADOW PRICES

Because shadow prices represent the marginal values of the resources in an LP problem, they can help us answer a number of other managerial questions that might arise. For example, suppose Blue Ridge Hot Tubs is considering introducing a new model of hot tub called the Typhoon-Lagoon. Suppose that each unit of this new model requires 1 pump, 8 hours of labor, and 13 feet of tubing, and can be sold to generate a marginal profit of $320. Would production of this new model be profitable?

Because Blue Ridge Hot Tubs has limited resources, the production of any Typhoon-Lagoons would consume some of the resources currently devoted to the production of Aqua-Spas and Hydro-Luxes. So, producing Typhoon-Lagoons will reduce the number of pumps, labor hours, and tubing available for producing the other types of hot tubs. The shadow prices in Figure 4.3 indicate that each pump taken away from production of the current products will reduce profits by $200. Similarly, each labor hour taken away from the production of the current products will reduce profits by $16.67. The shadow price for the tubing constraint indicates that the supply of tubing can be reduced without adversely affecting profits.

Because each Typhoon-Lagoon requires 1 pump, 8 hours of labor, and 13 feet of tubing, the diversion of resources required to produce one unit of this new model would cause a reduction in profit of $\$200 \times 1 + \$16.67 \times 8 + \$0 \times 13 = \$333.33$. This reduction would be partially offset by the $320 increase in profit generated by each Typhoon-Lagoon. The net effect of producing each Typhoon-Lagoon would be a $13.33 reduction in profit ($\$320 - \$333.33 = -\$13.33$). Therefore, the production of Typhoon-Lagoons would not be profitable (although the company might choose to produce a small number of Typhoon-Lagoons to enhance its product line for marketing purposes).

Another way to determine whether or not Typhoon-Lagoons should be produced is to add this alternative to our model and solve the resulting LP problem. The LP model for

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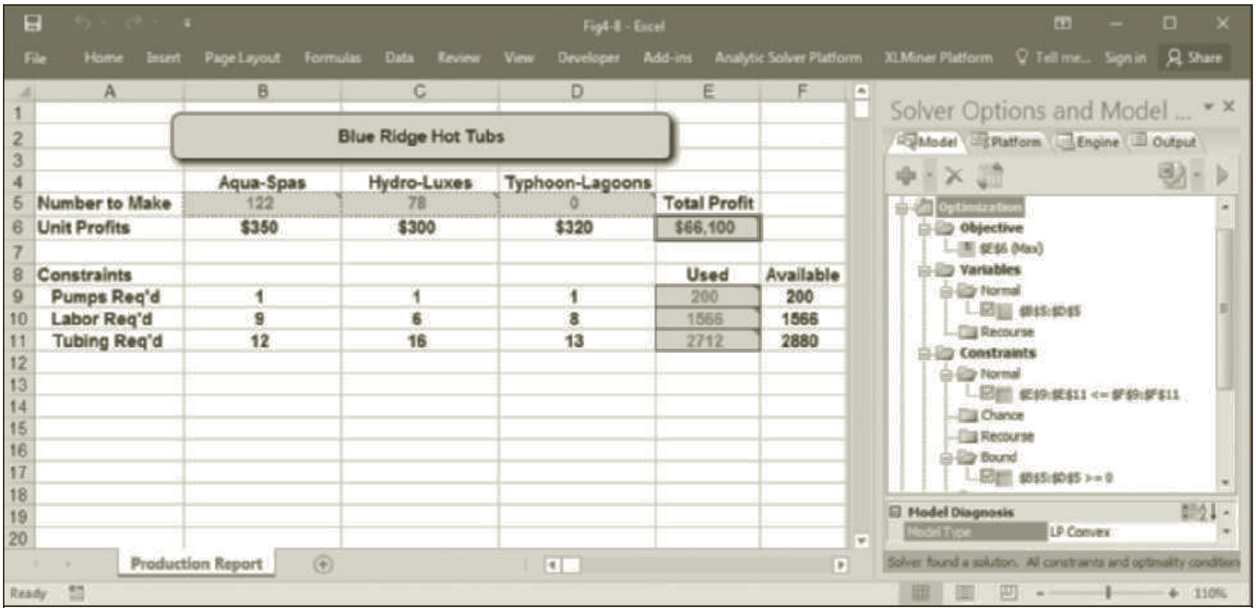
FIGURE 4.8 *Spreadsheet model for the revised product mix problem with three hot tub models*



Key Cell Formulas

Cell	Formula	Copied to
E6	=SUMPRODUCT(B6:D6,$B$5:$D$5)	E9:E11

this revised problem is represented as follows, where X_1, X_2, and X_3 represent the number of Aqua-Spas, Hydro-Luxes, and Typhoon-Lagoons to be produced, respectively:

$$
\begin{array}{llr}
\text{MAX:} & 350X_1 + 300X_2 + 320X_3 & \} \text{ profit} \\
\text{Subject to:} & 1X_1 + 1X_2 + 1X_3 \leq 200 & \} \text{ pump constraint} \\
 & 9X_1 + 6X_2 + 8X_3 \leq 1{,}566 & \} \text{ labor constraint} \\
 & 12X_1 + 16X_2 + 13X_3 \leq 2{,}880 & \} \text{ tubing constraint} \\
 & X_1, X_2, X_3 \geq 0 & \} \text{ nonnegativity conditions}
\end{array}
$$

This model is implemented and solved in the spreadsheet, as shown in Figure 4.8 (and file Fig4-8.xlsm that accompanies this book). Notice that the optimal solution to this problem involves producing 122 Aqua-Spas ($X_1 = 122$), 78 Hydro-Luxes ($X_2 = 78$), and no Typhoon-Lagoons ($X_3 = 0$). So, as expected, the optimal solution does not involve producing Typhoon-Lagoons. Figure 4.9 shows the Sensitivity Report for our revised model.

4.5.9 THE MEANING OF THE REDUCED COSTS

The Sensitivity Report in Figure 4.9 for our revised model is identical to the Sensitivity Report for our original model *except* that it includes an additional row in the decision variable cells section. This row reports sensitivity information on the number of Typhoon-Lagoons to produce. Notice that the Reduced Cost column indicates that

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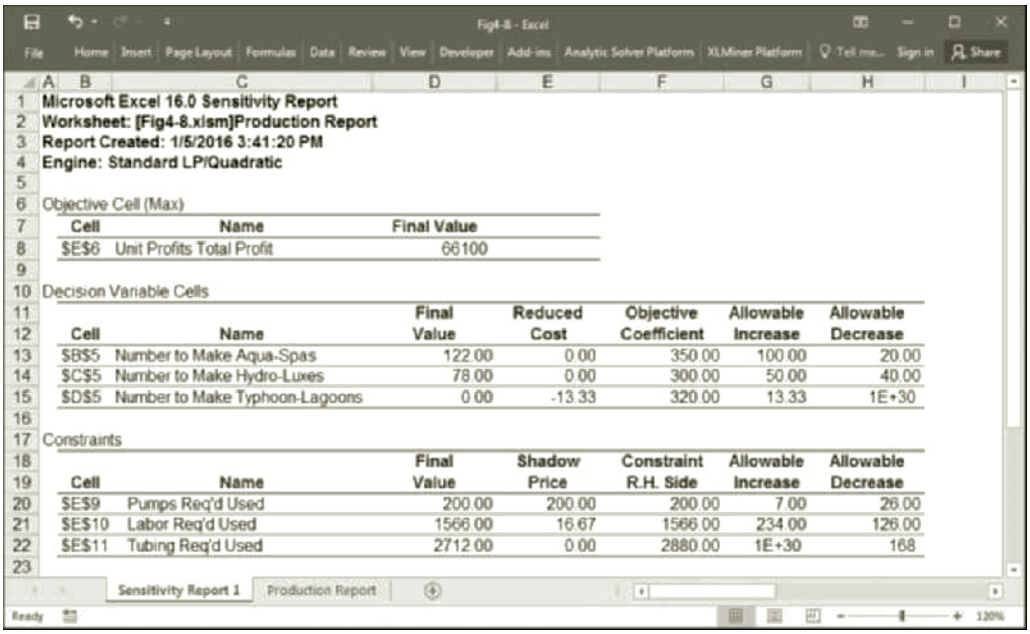
the reduced cost value for Typhoon-Lagoons is −13.33. This is the same number that we calculated in the previous section when determining whether or not it would be profitable to produce Typhoon-Lagoons.

The **reduced cost** for each variable is equal to the per-unit amount the product contributes to profits minus the per-unit value of the resources it consumes (where the consumed resources are priced at their shadow prices). For example, the reduced cost of each variable in this problem is calculated as:

Reduced cost of Aqua-Spas	$= 350 - 200 \times 1 - 16.67 \times 9 - 0 \times 12 = 0$
Reduced cost of Hydro-Luxes	$= 300 - 200 \times 1 - 16.67 \times 6 - 0 \times 16 = 0$
Reduced cost of Typhoon-Lagoons	$= 320 - 200 \times 1 - 16.67 \times 8 - 0 \times 13 = -13.33$

The allowable increase in the objective function coefficient for Typhoon-Lagoons equals 13.33. This means that the current solution will remain optimal provided that the marginal profit on Typhoon-Lagoons is less than or equal to \$320 + \$13.33 = \$333.33 (because this would keep its reduced cost less than or equal to zero). However, if the marginal profit for Typhoon-Lagoons is more than \$333.33, producing this product would be profitable and the optimal solution to the problem would change.

It is interesting to note that the shadow prices (marginal values) of the resources consumed equate exactly with the marginal profits of the products that, at optimality, assume values between their simple lower and upper bounds. This will always be the case. In the optimal solution to an LP problem, the variables that assume values *between* their simple lower and upper bounds always have reduced cost values of zero. (In our example problem, all the variables have implicit simple upper bounds of positive infinity.) The variables with optimal values equal to their simple lower bounds have reduced cost values that are less than or equal to zero for maximization problems, or greater than or equal to zero for minimization problems. Variables with optimal values equal to their simple upper bounds have reduced cost values that are greater than or


Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Fig4-8.xlsm]Production Report
Report Created: 1/5/2016 3:41:20 PM
Engine: Standard LP/Quadratic

Objective Cell (Max)

Cell	Name	Final Value
\$E\$6	Unit Profits Total Profit	66100

Decision Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$5	Number to Make Aqua-Spas	122.00	0.00	350.00	100.00	20.00
\$C\$5	Number to Make Hydro-Luxes	78.00	0.00	300.00	50.00	40.00
\$D\$5	Number to Make Typhoon-Lagoons	0.00	-13.33	320.00	13.33	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$9	Pumps Req'd Used	200.00	200.00	200.00	7.00	26.00
\$E\$10	Labor Req'd Used	1566.00	16.67	1566.00	234.00	126.00
\$E\$11	Tubing Req'd Used	2712.00	0.00	2880.00	1E+30	168

FIGURE 4.9

Sensitivity Report for the revised product mix problem with three hot tub models

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FIGURE 4.10

Summary of optimal reduced cost values

Type of Problem	Optimal Value of Decision Variable	Optimal Value of Reduced Cost
Maximization	at simple lower bound	≤ 0
	between lower and upper bounds	$= 0$
	at simple upper bound	≥ 0
Minimization	at simple lower bound	≥ 0
	between lower and upper bounds	$= 0$
	at simple upper bound	≤ 0

equal to zero for maximization problems, or less than or equal to zero for minimization problems. Figure 4.10 summarizes these relationships.

Generally, at optimality, a variable assumes its largest possible value (or is set equal to its simple upper bound) if this variable helps improve the objective function value. In a maximization problem, the variable's reduced cost must be nonnegative to indicate that if the variable's value increased, the objective value would increase (improve). In a minimization problem, the variable's reduced cost must be nonpositive to indicate that if the variable's value increased, the objective value would decrease (improve).

Similar arguments can be made for the optimal reduced costs of variables at their lower bounds. At optimality, a variable assumes its smallest (lower bound) value if it cannot be used to improve the objective value. In a maximization problem, the variable's reduced cost must be nonpositive to indicate that if the variable's value increased, the objective value would decrease (worsen). In a minimization problem, the variable's reduced cost must be nonnegative to indicate that if the variable's value increased, the objective value would increase (worsen).

Key Points

Our discussion of Solver's Sensitivity Report highlights some key points concerning shadow prices and their relationship to reduced costs. These key points are summarized as:

- The shadow prices of resources equate the marginal value of the resources consumed with the marginal benefit of the goods being produced.
- Resources in excess supply have a shadow price (or marginal value) of zero.
- The reduced cost of a product is the difference between its marginal profit and the marginal value of the resources it consumes.
- Products whose marginal profits are less than the marginal value of the goods required for their production will not be produced in an optimal solution.

4.5.10 ANALYZING CHANGES IN CONSTRAINT COEFFICIENTS

Given what we know about reduced costs and shadow prices, we can now analyze how changes in some constraint coefficients affect the optimal solution to an LP problem. For example, it is unprofitable for Blue Ridge Hot Tubs to manufacture Typhoon-Lagoons assuming that each unit requires 8 hours of labor. However, what would happen if the

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product could be produced in only 7 hours? The reduced cost value for Typhoon-Lagoons is calculated as:

$$\$320 - \$200 \times 1 - \$16.67 \times 7 - \$0 \times 13 = \$3.31$$

Because this new reduced cost value is positive, producing Typhoon-Lagoons would be profitable in this scenario and the solution shown in Figure 4.8 would no longer be optimal. We could also reach this conclusion by changing the labor requirement for Typhoon-Lagoons in our spreadsheet model and re-solving the problem. In fact, we have to do this to determine the new optimal solution if each Typhoon-Lagoon requires only 7 hours of labor.

As another example, suppose that we wanted to know the maximum amount of labor that is required to assemble a Typhoon-Lagoon while keeping its production economically justifiable. The production of Typhoon-Lagoons would be profitable provided that the reduced cost for the product is greater than or equal to zero. If L_3 represents the amount of labor required to produce a Typhoon-Lagoon, we want to find the maximum value of L_3 that keeps the reduced cost for Typhoon-Lagoons greater than or equal to zero. That is, we want to find the maximum value of L_3 that satisfies the inequality:

$$\$320 - \$200 \times 1 - \$16.67 \times L_3 - \$0 \times 13 \geq 0$$

If we solve this inequality for L_3, we obtain:

$$L_3 \leq \frac{120}{16.67} = 7.20$$

Thus, the production of Typhoon-Lagoons would be economically justified provided that the labor required to produce them does not exceed 7.20 hours per unit. Similar types of questions can be answered using knowledge of the basic relationships between reduced costs, shadow prices, and optimality conditions.

4.5.11 SIMULTANEOUS CHANGES IN OBJECTIVE FUNCTION COEFFICIENTS

Earlier, we noted that the values in the Allowable Increase and Allowable Decrease columns in the Sensitivity Report for the objective function coefficients indicate the maximum amounts by which each objective coefficient can change without altering the optimal solution—assuming all other coefficients in the model remain constant. A technique known as The 100% Rule determines whether the current solution remains optimal when more than one objective function coefficient changes. The following two situations could arise when applying this rule:

Case 1. All variables whose objective function coefficients change have non-zero reduced costs.

Case 2. At least one variable whose objective function coefficient changes has a reduced cost of zero.

In case 1, the current solution remains optimal provided that the objective function coefficient of each changed variable remains within the limits indicated in the Allowable Increase and Allowable Decrease columns of the Sensitivity Report.

Case 2 is a bit trickier. In case 2, we must perform the following analysis where:

c_j = the original objective function coefficient for variable X_j

Δc_j = the planned change in c_j

I_j = the allowable increase in c_j given in the Sensitivity Report

D_j = the allowable decrease in c_j given in the Sensitivity Report

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$$r_j = \begin{cases} \dfrac{\Delta c_j}{I_j}, \text{ if } \Delta c_j \geq 0 \\ \dfrac{-\Delta c_j}{D_j}, \text{ if } \Delta c_j 0 \end{cases}$$

Notice that r_j measures the ratio of the planned change in c_j to the maximum allowable change for which the current solution remains optimal. If only one objective function coefficient changed, the current solution remains optimal provided that $r_j \leq 1$ (or, if r_j is expressed as a percentage, it must be less than or equal to 100%). Similarly, if more than one objective function coefficient changes, the current solution will remain optimal provided that $\Sigma r_j \leq 1$. (Note that if $\Sigma r_j > 1$, the current solution might remain optimal, but this is not guaranteed.)

4.5.12 A WARNING ABOUT DEGENERACY

The solution to an LP problem sometimes exhibits a mathematical anomaly known as **degeneracy**. The solution to an LP problem is degenerate if the RHS values of any of the constraints have an allowable increase or allowable decrease of zero. The presence of degeneracy impacts our interpretation of the values on the Sensitivity Report in a number of important ways:

1) When the solution is degenerate, the methods mentioned earlier for detecting alternate optimal solutions cannot be relied upon.
2) When a solution is degenerate, the reduced costs for the variable cells may not be unique. Additionally, in this case, the objective function coefficients for variable cells must change by at least as much as (and possibly more than) their respective reduced costs before the optimal solution would change.
3) When the solution is degenerate, the allowable increases and decreases for the objective function coefficients still hold and, in fact, the coefficients may have to be changed substantially beyond the allowable increase and decrease limits before the optimal solution changes.
4) When the solution is degenerate, the given shadow prices and their ranges may still be interpreted in the usual way but they may not be unique. That is, a different set of shadow prices and ranges may also apply to the problem (even if the optimal solution is unique).

So before interpreting the results on a Sensitivity Report, you should always first check to see if the solution is degenerate because this has important ramifications on how the numbers on the report should be interpreted. A complete description of the degeneracy anomaly goes beyond the intended scope of this book. However, degeneracy is sometimes caused by having redundant constraints in an LP model. *Extreme caution* (and perhaps consultation with an expert in mathematical programming) is in order if important business decisions are being made based on the Sensitivity Report for a degenerate LP problem.

4.6 The Limits Report

The Limits Report for the original Blue Ridge Hot Tubs problem is shown in Figure 4.11. This report lists the optimal value of the objective cell. It then summarizes the optimal values for each variable cell and indicates what values the objective cell assumes

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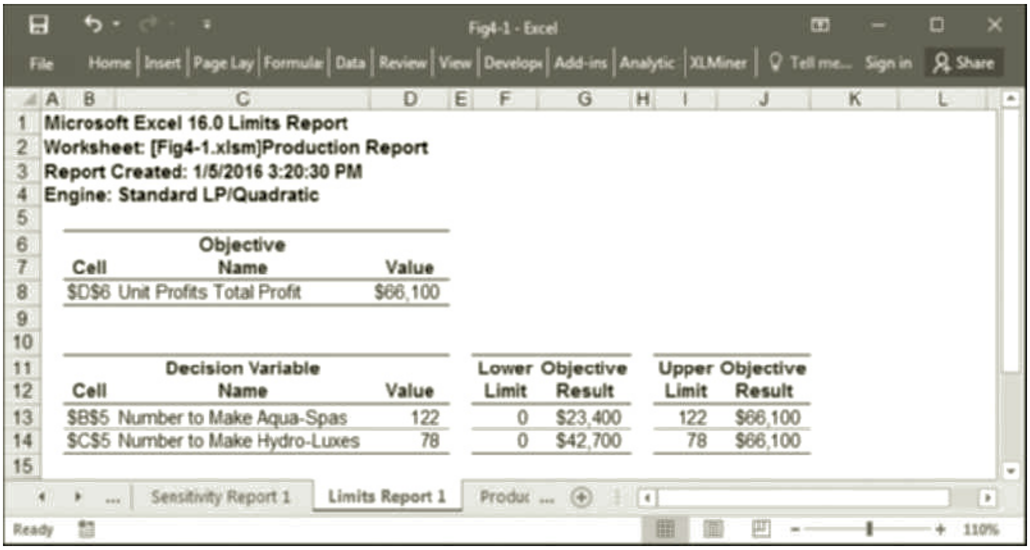

Microsoft Excel 16.0 Limits Report
Worksheet: [Fig4-1.xlsm]Production Report
Report Created: 1/5/2016 3:20:30 PM
Engine: Standard LP/Quadratic

Cell	Objective Name	Value
$D$6	Unit Profits Total Profit	$66,100

Cell	Decision Variable Name	Value	Lower Limit	Objective Result	Upper Limit	Objective Result
$B$5	Number to Make Aqua-Spas	122	0	$23,400	122	$66,100
$C$5	Number to Make Hydro-Luxes	78	0	$42,700	78	$66,100

FIGURE 4.11

Limits Report for the original Blue Ridge Hot Tubs problem

if each variable cell is set to its upper or lower limits. The values in the Lower Limits column indicate the smallest value each variable cell can assume while the values of all other variable cells remain constant and all the constraints are satisfied. The values in the Upper Limits column indicate the largest value each variable cell can assume while the values of all other variable cells remain constant and all the constraints are satisfied.

4.7 Ad Hoc Sensitivity Analysis

Although the standard Sensitivity Reports Solver prepares can be quite useful, they cannot possibly anticipate and provide answers to every question that might arise about the solution to an LP problem and the effects that changes to model parameters might have on the optimal solution. However, Analytic Solver Platform provides a number of powerful features that we can use to address *ad hoc* sensitivity analysis questions when they arise. In this section we will consider two such ad hoc techniques: Spider Plots and Solver Tables. A Spider Plot summarizes the optimal value for one output cell as individual changes are made to various input cells. A Solver Table summarizes the optimal value for multiple output cells as changes are made to a single input cell. As illustrated in the following example, these tools can be helpful in developing an understanding of how changes in various model parameters affect the optimal solution to a problem.

4.7.1 CREATING SPIDER PLOTS AND TABLES

Recall that the optimal solution to the original Blue Ridge Hot Tubs problem involves producing 122 Aqua-Spas and 78 Hydro-Luxes for a total profit of $66,100. However, this solution assumes there will be exactly 200 pumps, 1,566 labor hours, and 2,880 feet of tubing available. In reality, pumps and tubing are sometimes defective, and workers sometimes call in sick. So, the owner of the company may wonder how sensitive the total profit is to changes in these parameters. Although the Solver Sensitivity Report provides some information about this issue, a Spider Plot is sometimes more helpful in communicating this information to management.

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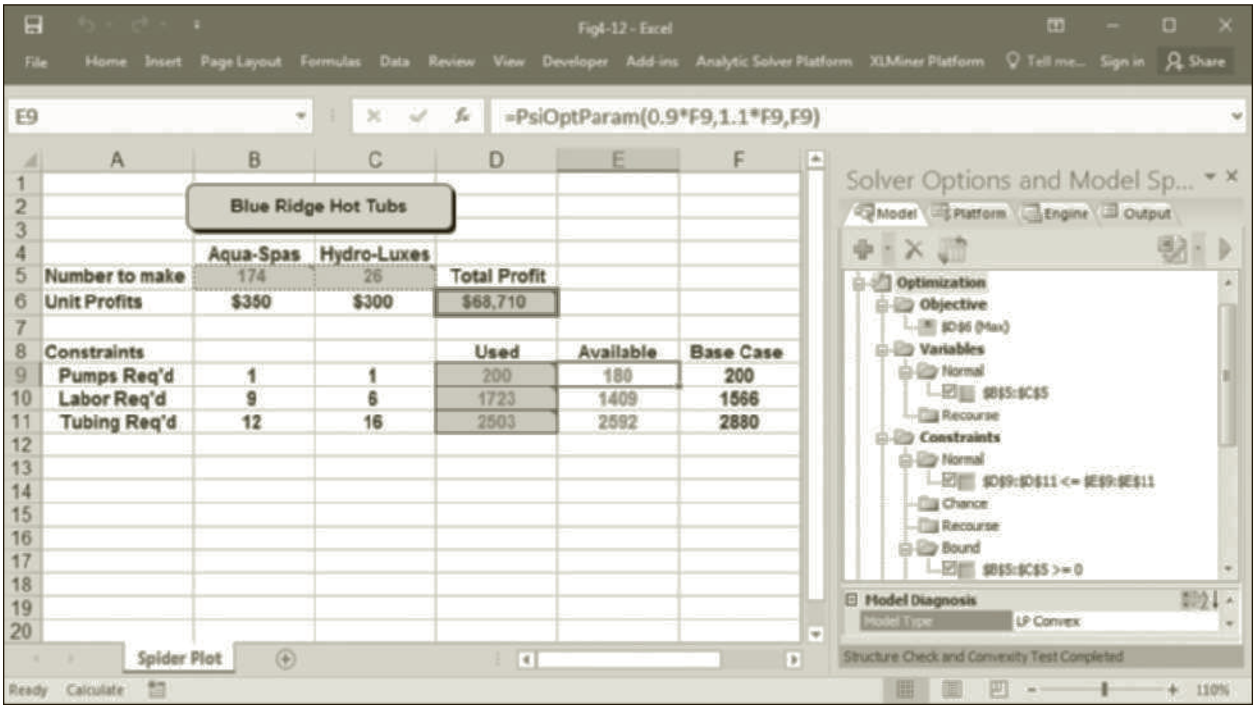
FIGURE 4.12 *Set up for creating a Spider Plot and Table*



Key Cell Formulas

Cell	Formula	Copied to
D6	=SUMPRODUCT(B6:C6,$B$5:$C$5)	D9:D11
E9	=PsiOptParam(0.9*F9,1.1*F9,F9)	E10:E11

Again, a Spider Plot summarizes the optimal value for one output cell as individual changes are made to various model input cells (or parameters) one at a time, while holding the values of the other input cells constant at their original (or "base case") values. In this case, the output cell of interest is cell D6 representing total profit. The parameters of interest are cells E9, E10, and E11 representing, respectively, the availability of pumps, labor, and tubing. Figure 4.12 (and file Fig4-12.xlsm that accompanies this book) shows how to set up a spreadsheet to create a Spider Plot for this problem.

The strategy in Figure 4.12 is to individually (one at a time) vary the availability of pumps, labor, and tubing between 90% and 110% of their original values while holding the remaining resources at their base case levels. The base case values for each of our three parameters are listed in cells F9 through F11. For each of these three parameters we will create and optimize eleven different scenarios (while holding the other two parameters at their base case values) and record the corresponding optimal value for the objective function (cell D6). Therefore, we will use Analytic Solver to solve a total of 33 (i.e., 3 x 11) variations of our Blue Ridge Hot Tubs problem. In the first eleven runs we will change the first parameter (the number of pumps available) between 90% and 110% of its base case value. In the next eleven runs (runs 12 to 22) we will

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change the second parameter (the amount of labor available) between 90% and 110% of its base case value. Finally, in the last eleven runs (runs 23 to 33) we will change the third parameter (the amount of tubing available) between 90% and 110% of its base case value.

Cell E9 contains the following formula that will vary the value in E9 from 90% to 110% of the base case value in cell F9:

Formula for cell E9: =PsiOptParam(0.9*F9, 1.1*F9, F9)
(Copy to E10:E11.)

Note that the first two arguments in the PsiOptParam( ) function specify, respectively, the minimum and maximum values for the cell being parameterized while the third argument defines the base case value for the cell. So the general form of this function is =PsiOptParam(minimum value, maximum value, base case value). Similar formulas in cells E10 and E11 vary the values in those cells between 90% and 110% of their base case values found in cell F10 and F11, respectively.

To solve this problem, we use the same settings for the objective cell, variable cells, and constraint cells as before. However, to make Analytic Solver run the multiple optimizations needed for this problem and chart the results, do the following:

1. On the Analytic Solver Platform tab, click Charts, Multiple Optimizations, Parameter Analysis. (This causes the dialog in Figure 4.13 to appear.)
2. Make the selections indicated in Figure 4.13 and click OK.

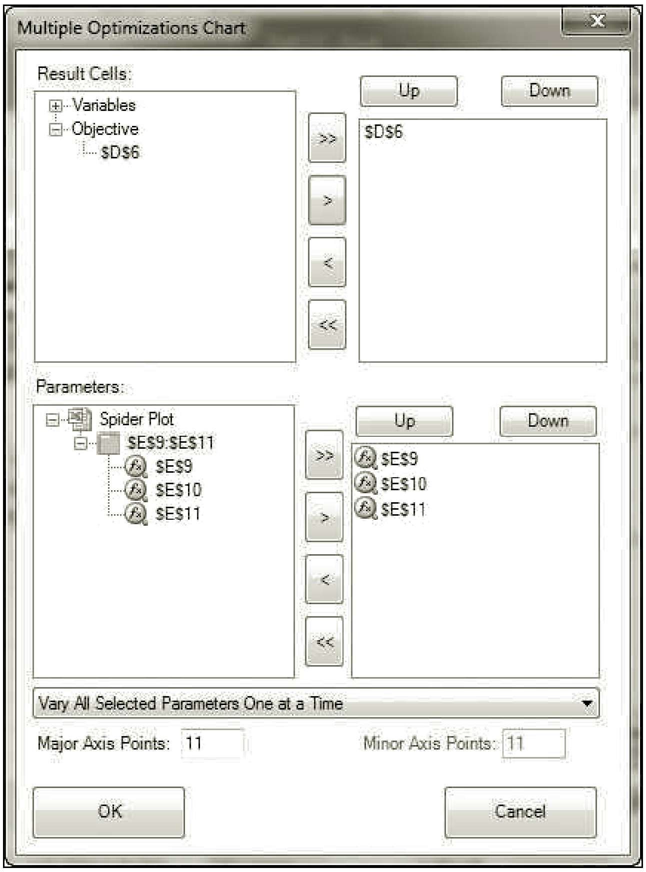

FIGURE 4.13

Dialog box sttings for Spider Plot

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Analytic Solver Platform then runs the 33 optimizations required for this analysis and charts the results named as an aptly Spider Plot as shown in Figure 4.14. (Labels were added manually to the lines in the chart for clarity.) Note that because eleven major axis points were requested in the dialog box in Figure 4.13 and we are varying three parameters, a total of 33 optimizations must be performed. Additionally, because each parameter is being varied over eleven equally spaced values between 90% and 110% of its base case value, the actual values used in this example include 90%, 92%, 94%, 96%, 98%, 100%, 102%, 104%, 106%, 108%, and 110% of the base case value. (More generally, when varying a parameter between a minimum (*min*) and maximum (*max*) percent over n major axis points, the percentage used in optimization number i (P_i) is given by $P_i = min + (i - 1)*(max - min)/(n - 1)$.)

The plot in Figure 4.14 shows the optimal objective function values (from cell D6) for each of our 33 optimization runs. The center point in the graph corresponds to the optimal solution to the original model with 100% of the pumps, labor, and tubing available. Each line in the graph shows the impact on total profit of varying a different resource level from 90% to 110% of its original (base case) value.

It is clear from Figure 4.14 that total profit is relatively insensitive to modest decreases or large increases in the availability of tubing (E11). This is consistent with the sensitivity information regarding tubing shown earlier in Figure 4.9. The optimal solution to the original problem involved using all the pumps and all the labor hours but only 2,712 feet of the 2,880 available feet of tubing. As a result, we could achieve the same level of profit even if the availability of tubing was reduced by 168 feet (or to about 94.2% of its original value). Similarly, because we are not using all of the available tubing, acquiring more tubing would only increase the surplus

FIGURE 4.14

A Spider Plot showing the relationship between profit and the availability of pumps, labor, and tubing

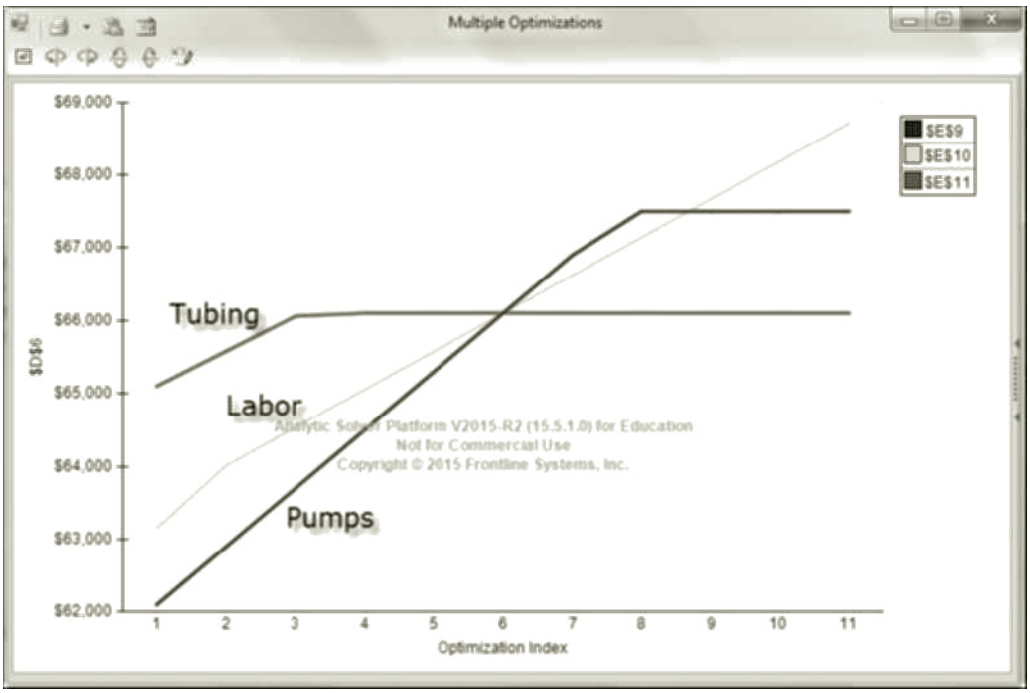

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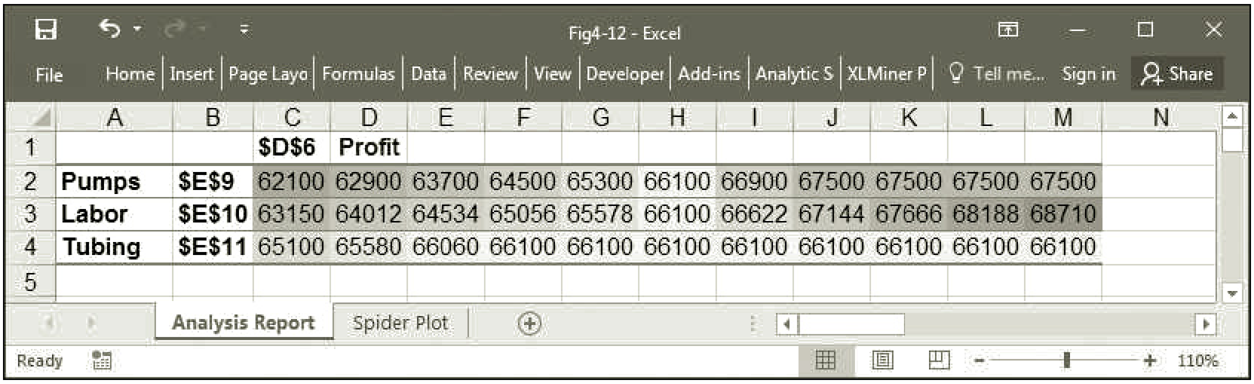
FIGURE 4.15 *A Spider Table summarizing the relationship between profit and the availability of pumps, labor, and tubing*



	A	B	C	D	E	F	G	H	I	J	K	L	M
1			$D$6	Profit									
2	Pumps	$E$9	62100	62900	63700	64500	65300	66100	66900	67500	67500	67500	67500
3	Labor	$E$10	63150	64012	64534	65056	65578	66100	66622	67144	67666	68188	68710
4	Tubing	$E$11	65100	65580	66060	66100	66100	66100	66100	66100	66100	66100	66100
5													

and not allow for any improvement in profit. Thus, our analysis suggests that the availability of tubing probably should not be a top concern in this problem. On the other hand, the Spider Plot suggests that changes in the availability of pumps (E9) and labor (E10) have a more pronounced impact on profit and the optimal solution to the problem.

The data underlying a Spider Plot can be summarized in a Spider Table, which is also easy to create using Analytic Solver Platform. To create the Spider Table for our example problem do the following:

1. On the Analytic Solver Platform tab, click Reports, Optimization, Parameter Analysis. (This also causes a dialog box like the one shown in Figure 4.13 to appear.)
2. Make the selections indicated in Figure 4.13 and click OK.

The resulting Spider Table is shown in Figure 4.15. Note that labels were added to the table manually for clarity. Additionally, conditional formatting was used to apply a "heat map" type of format to this table making it easier to distinguish the larger and smaller values in the table. (This was accomplished using the Conditional Formatting, Color Scales command on the Home tab on the ribbon.) This table provides the numeric detail for each of the lines drawn in the Spider Plot.

4.7.2 CREATING A SOLVER TABLE

The Spider Plot in Figure 4.14 suggests that the total profit earned is most sensitive to changes in the available supply of pumps. We can create a Solver Table to study in greater detail the impact of changes in the available number of pumps. Recall that a Solver Table summarizes the optimal value of multiple output cells as changes are made to a single input cell. In this case, the single input we want to change is cell E9 representing the number of pumps available. We might want to track what happens to several output cells, including the optimal number of Aqua-Spas and Hydro-Luxes (cells B5 and C5), the total profit (cell D6), and the total amount pumps, labor, and tubing used (cells D9, D10, and D11). Figure 4.16 (and file Fig4-16.xlsm that accompanies this book) shows how to set up the Solver Table for this problem.

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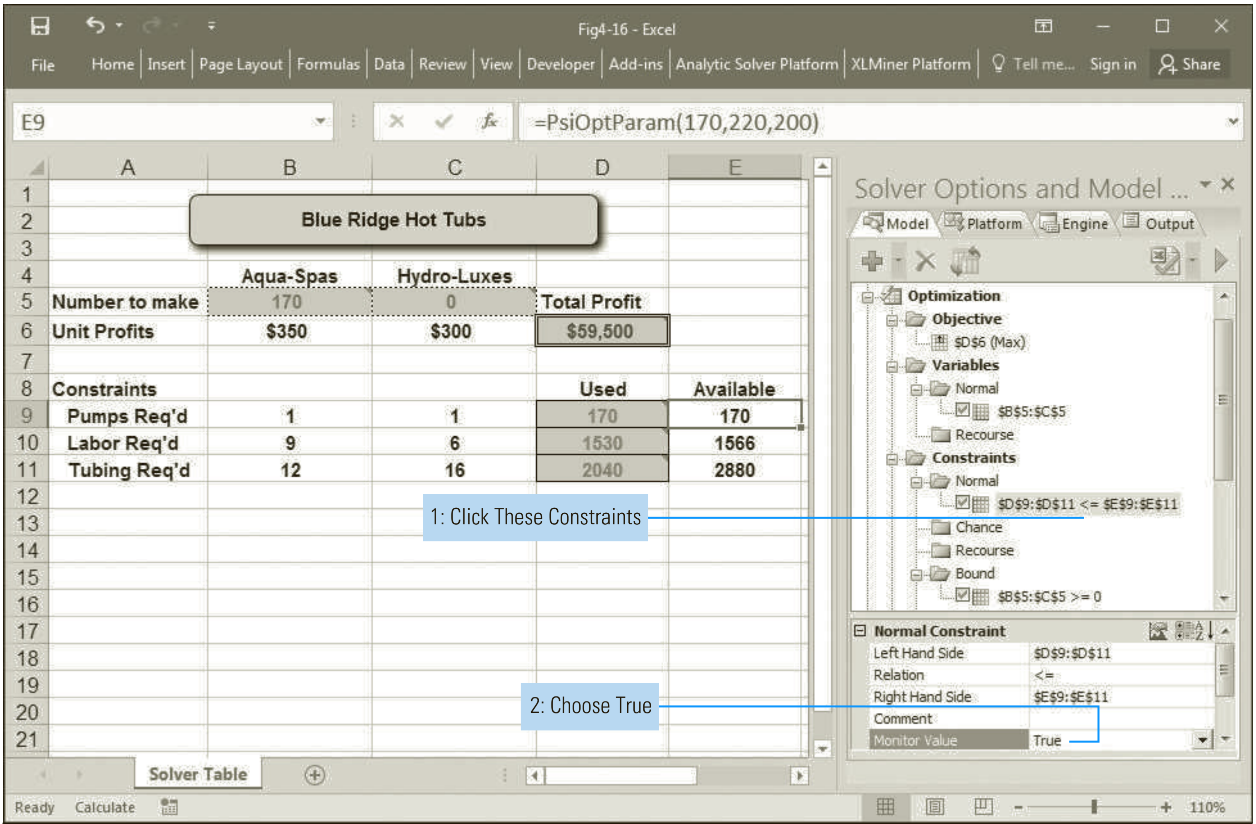
FIGURE 4.16 *Setup for creating a Solver Table*



Key Cell Formulas

Cell	Formula	Copied to
D6	=SUMPRODUCT(B6:C6,$B$5:$C$5)	D9:D11
E9	=PsiOptParam(170, 220, 200)	E10:E11

In this problem we want to perform eleven optimizations, varying the number of pumps available from 170 to 220 in each successive run. The following formula in cell E9 will "parameterize" the number of pumps so that its value changes as each successive optimization is run:

Formula for cell E9: =PsiOptParam(170,220,200)

In Figure 4.16, also notice that the "Monitor Value" property has been set to True for the constraints $D$9:$D$11<= $E$9:$E$11. This instructs Analytic Solver Platform to keep track of the final values of the left hand side of this constraint (i.e., the final values of cells D9, D10, and D11, corresponding to the quantity of pumps, labor, and tubing used in each optimization). The "Monitor Value" property for the objective cell and variables cells is set to True by default.

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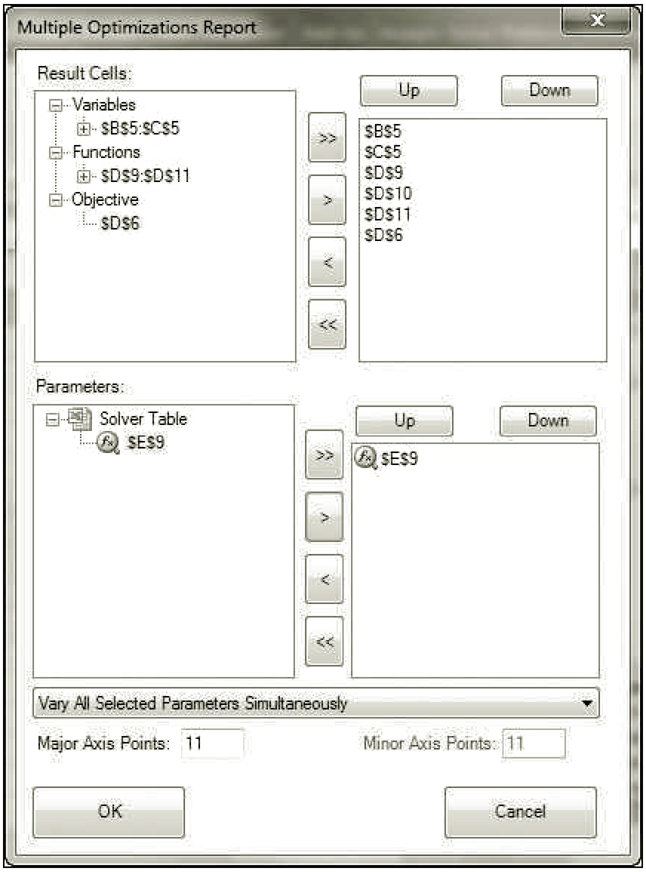

FIGURE 4.17

Dialog box settings for creating a Solver Table

To solve this problem, we use the same settings for the objective cell, variable cells, and constraint cells as before. However, to make Analytic Solver run the multiple optimizations needed for this problem and summarize the results, do the following:

1. On the Analytic Solver Platform tab, click Reports, Optimization, Parameter Analysis. (This causes the dialog box in Figure 4.17 to appear.)
2. Make the selections indicated in Figure 4.17 and click OK.

Analytic Solver Platform then runs the 11 optimizations required for this analysis and creates the results table shown in Figure 4.18. (The column titles in row 1 of this table were added manually for clarity.) Additionally, because each parameter is being varied over 11 equally spaced values between 170 and 220 the actual values used in this example include 170, 175, 180, 185, 190, 195, 200, 205, 210, 215 and 220. (More generally, when varying a parameter between a minimum (*min*) and maximum (*max*) over n major axis points, the parameter value used in optimization number i (V_i) is given by $V_i = min + (i - 1)*(max - min)/(n - 1)$.)

A number of interesting insights emerge from Figure 4.18. First, comparing columns A and D, as the number of available pumps increases from 170 up to 205, they are always all used. With about 175 pumps, we also begin to use all the available labor. However, when the number of available pumps increases to 210 or more, only 207 pumps can be used because we run out of both tubing and labor at that point. This suggests that the company should not be interested in getting more than 7 additional pumps unless it can also increase the amount of tubing and/or labor available.

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FIGURE 4.18 *Solver Table showing changes in the optimal solution, profit, and resource usage and the number of pumps changed*

Pumps Available	Aqua-Spas	Hydro-Luxes	Pumps Used	Labor Used	Tubing Used	Total Profit
$E$9	$B$5	$C$5	$D$9	$D$10	$D$11	$D$6
170	170	0	170	1530	2040	$59,500
175	172	3	175	1566	2112	$61,100
180	162	18	180	1566	2232	$62,100
185	152	33	185	1566	2352	$63,100
190	142	48	190	1566	2472	$64,100
195	132	63	195	1566	2592	$65,100
200	122	78	200	1566	2712	$66,100
205	112	93	205	1566	2832	$67,100
210	108	99	207	1566	2880	$67,500
215	108	99	207	1566	2880	$67,500
220	108	99	207	1566	2880	$67,500

Also note that the addition or subtraction of 5 pumps from the initial supply of 200 causes the optimal objective function value (column E) to change by $1,000. This suggests that if the company has 200 pumps, the marginal value of each pump is about $200 (i.e., $1000/5 = $200). Of course, this is equivalent to the ***shadow price*** of pumps shown earlier in Figure 4.9.

Finally, it is interesting to note that when the availability of pumps is between 175 and 205, each increase of 5 pumps causes the optimal number of Aqua-Spas to decrease by 10 and the optimal number of Hydro-Luxes to increase by 15. Thus, one advantage of the Solver Table over the Sensitivity Report is that it tells you not only how much the optimal value of the objective function changes as the number of pumps change, but it can also tell you how the optimal solution changes.

4.7.3 COMMENTS

Additional Solver Tables and Spider Plots/Tables could be constructed to analyze every element of the model, including objective function and constraint coefficients. However, these techniques are considered ‘computationally expensive’ because they require the LP model to be solved repeatedly. For small problems like Blue Ridge Hot Tubs, this is not really a problem. But as problem size and complexity increases, this approach to sensitivity analysis can become burdensome.

4.8 Robust Optimization

As we have seen, an optimal solution to an LP problem will occur on the boundary of its feasible region. While these boundaries can be determined very precisely for a given set of data, any uncertainties or changes in the data will result in uncertainties or changes

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in the boundaries of the feasible region. Thus, the optimal solution to an LP problem can be somewhat fragile and could actually become infeasible (and a costly mistake) if any of the coefficients in an LP model are incorrect or differ from the real-world phenomena being modeled. In recent years, this reality has led a number of researchers and practitioners to consider (and often prefer) robust solutions to optimization problems. A robust solution to an LP problem is a solution in the interior of the feasible region (rather than on the boundary of the feasible region) that has a reasonably good objective function value. Clearly, such a solution will not maximize (or minimize) the objective function value (except in trivial cases), so it is not an optimal solution in the traditional sense of the word. However, a robust solution will generally remain feasible if modest perturbations or changes occur to the coefficients in the model.

Analytic Solver Platform offers a number of powerful tools for identifying robust solutions to optimization problems. In the case of LP problems, we can easily accommodate uncertainties in constraint coefficients using uncertainty set (USet) chance constraints. To illustrate this, recall that the original Blue Ridge Hot Tubs problem assumed each Aqua-Spa required 1 pump, 9 hours of labor, and 12 feet of tubing while each Hydro-Lux required 1 pump, 6 hours of labor, and 16 feet of tubing. The optimal solution to this problem was to make 122 Aqua-Spas and 78 Hydro-Luxes (see Figure 4.1). Clearly, each hot tub will require 1 pump, so there is really no uncertainty about that constraint. However, the actual amount of labor and tubing might vary a bit from the values assumed earlier. So, suppose the amount of labor required per hot tub might vary from their originally assumed values by 15 minutes (or .25 hours) and the amount of tubing required might vary by 6 inches (or 0.5 feet). That is, the amount of labor required per Aqua-Spa is uncertain but can reasonably be expected to vary uniformly between 8.75 and 9.25 hours and the labor required per Hydro-Lux is expected to vary uniformly from 5.75 to 6.25 hours. Similarly, the amount of tubing required per Aqua-Spa is uncertain but can reasonably be expected to vary uniformly between 11.5 and 12.5 feet and the tubing required per Hydro-Lux is expected to vary uniformly from 15.5 to 16.5 feet. Figure 4.19 (and the file Fig4-19.xlsm that accompanies this book) shows a revised version of the Blue Ridge Hot Tubs problem that accounts for these uncertainties in the labor and tubing constraints.

In Figure 4.19, note that the previous numeric constants in cells B10 through C11 have been replaced by the following random number generators:

Formula for cell B10:	=PsiUniform(8.75,9.25)
Formula for cell C10:	=PsiUniform(5.75,6.25)
Formula for cell B11:	=PsiUniform(11.5,12.5)
Formula for cell C11:	=PsiUniform(15.5,16.5)

The PsiUniform(Lower,Upper) function is an Analytic Solver Platform function that returns a random value from a uniform distribution between a specified lower and upper limit each time the spreadsheet is recalculated. Because the numbers are randomly generated (or randomly sampled) the actual numbers on your computer's screen will likely differ from those in Figure 4.19 (and will change if you recalculate your spreadsheet by pressing the F9 function key). Also notice that the optimal solution of 122 Aqua-Spas and 78 Hydro-Luxes actually violates the labor constraint in the scenario shown in Figure 4.19. So, as stated earlier, the optimal solution to an LP problem can actually end up being infeasible if uncertainty exists about the values of one or more model coefficients that we simply ignore or assume away. The spreadsheet in Figure 4.19 does not ignore the uncertainties in the labor and tubing coefficients but, instead, models them explicitly.

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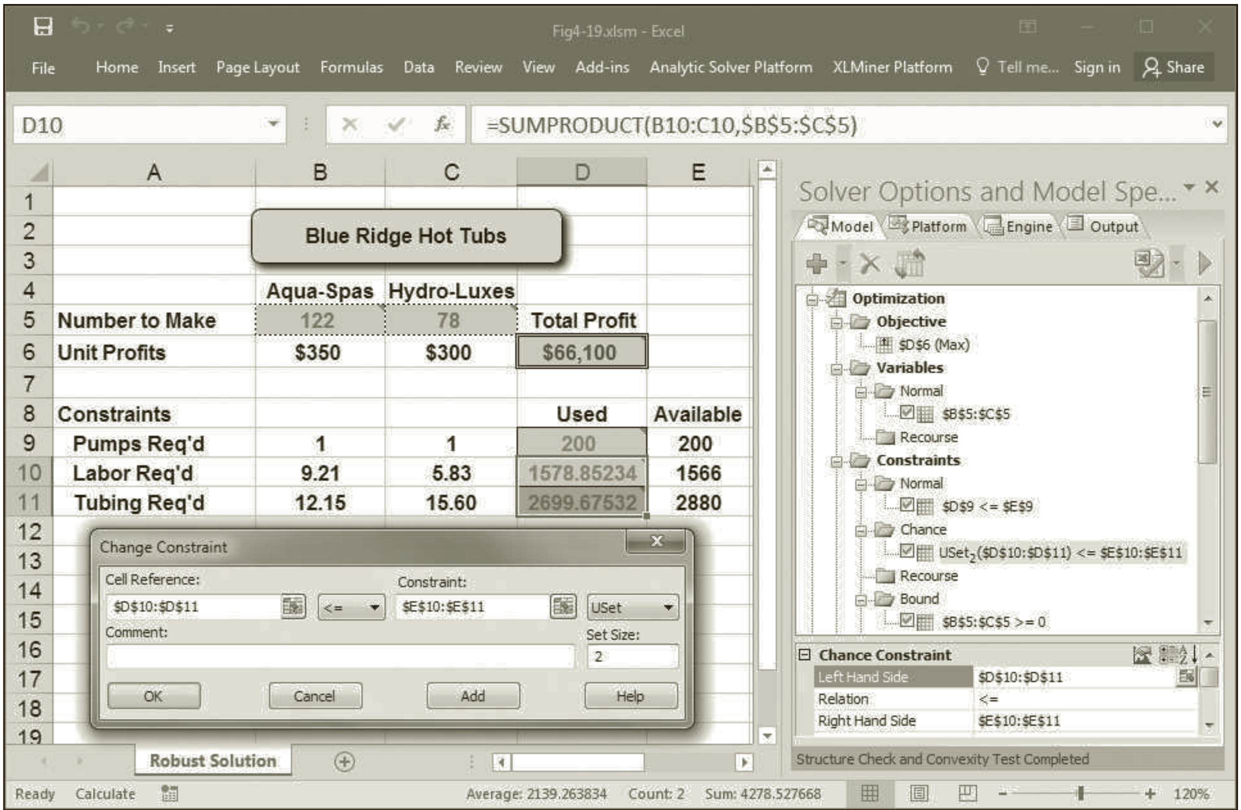
FIGURE 4.19 *A robust optimization spreadsheet for the Blue Ridge Hot Tubs problem*



Key Cell Formulas

Cell	Formula	Copied to
D6	=SUMPRODUCT(B6:C6,$B$5:$C$5)	D9:D11
B10	=PsiUniform(8.75, 9.25)	--
C10	=PsiUniform(5.75, 6.25)	--
B11	=PsiUniform(11.5, 12.5)	--
C11	=PsiUniform(15.5, 16.5)	--

In Figure 4.19, notice that the constraint on the number of pumps used was defined in the usual way and appears as a “normal” constraint in the Analytic Solver task pane Model tab. When defining the labor and tubing constraints we must select the USet constraint type and specify a value for the Set Size. (Notice that these constraints appear as Chance constraints in the Analytic Solver task pane Model tab.) The Set Size is sometimes referred to as the budget of uncertainty for the constraint. There is no prescribed way of determining the set size value. Generally speaking, as the set size value increases the solution obtained becomes more conservative (or more robust).

When we solve this problem, Analytic Solver actually formulates a larger LP problem with several different coefficient values for the USet constraints. It then solves this problem to obtain a solution that satisfies all of these possible constraint configurations. Figure 4.20 shows the first solution Analytic Solver found for this

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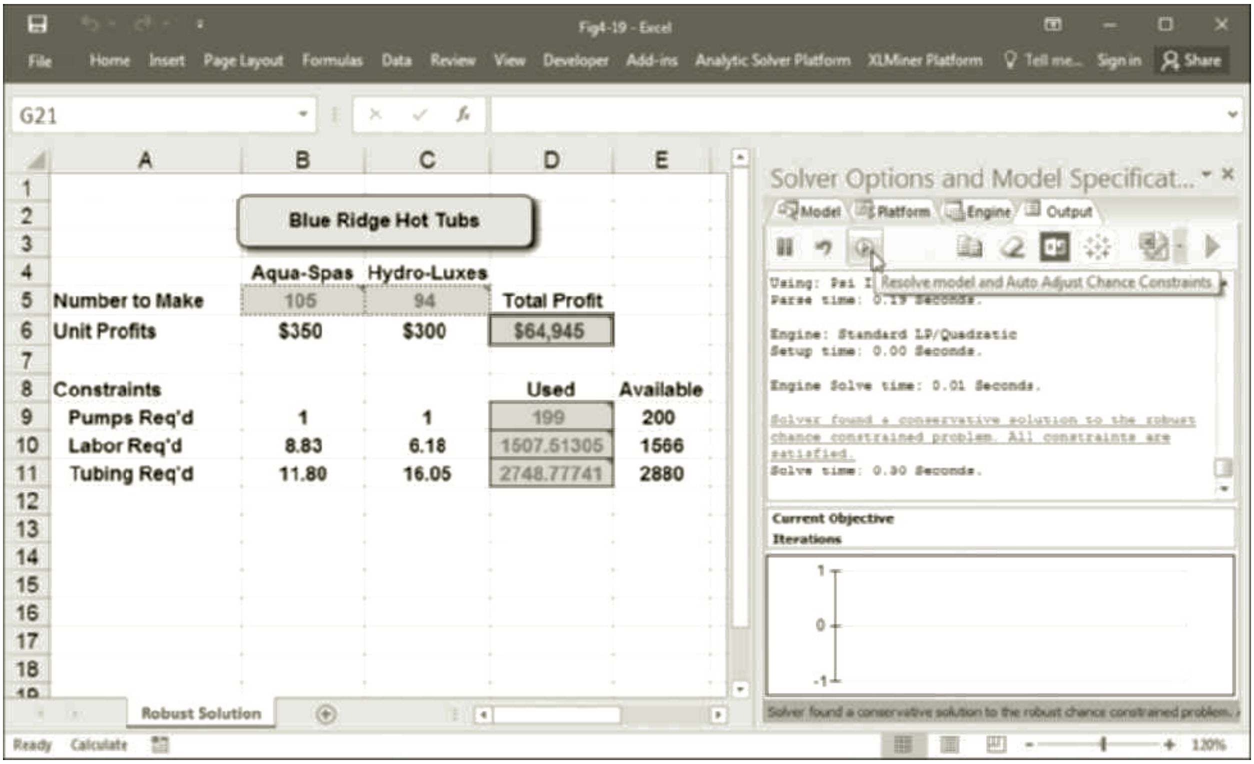
FIGURE 4.20 *A conservative robust solution to the Blue Ridge Hot Tubs problem*



problem. This solution involves producing about 105 Aqua-Spas and 94 Hydro-Luxes for a profit of $64,945. If you evaluate this solution under our original deterministic assumptions where Aqua-Spas require 9 hours of labor and 12 feet of tubing and Hydro-Luxes require 6 hours of labor and 16 feet of tubing you will see that this solution uses about 1,509 of the available 1,566 hours of labor and about 2,764 feet of the available 2,880 feet of tubing. So as indicated on the Output tab in the Analytic Solver task pane, this is a "conservative solution" that does not make use of all the available resources. If you click the smaller green arrow on the Output tab, Analytic Solver will display a less conservative solution to the problem as shown in Figure 4.21.

The solution in Figure 4.21 involves producing about 115 Aqua-Spas and 85 Hydro-Luxes for a profit of $65,769. If you evaluate this solution under our original deterministic assumptions (with 9 hours of labor and 12 feet of tubing for Aqua-Spas and 6 hours of labor and 16 feet of tubing for Hydro-Luxes) you will see that this solution uses about 1,545 of the available 1,566 hours of labor and about 2,740 feet of the available 2,880 feet of tubing. This solution is better than the previous one shown in Figure 4.20 from a profit perspective but takes us much closer to the boundary of the labor constraint.

As you can see, robust optimization is a very powerful technique but requires a bit of trial and error on the part of the decision maker in terms of specifying the set size and evaluating the trade-offs between satisfying the constraints by a comfortable margin (to allow for uncertainties) and sacrificing objective function value. A complete discussion

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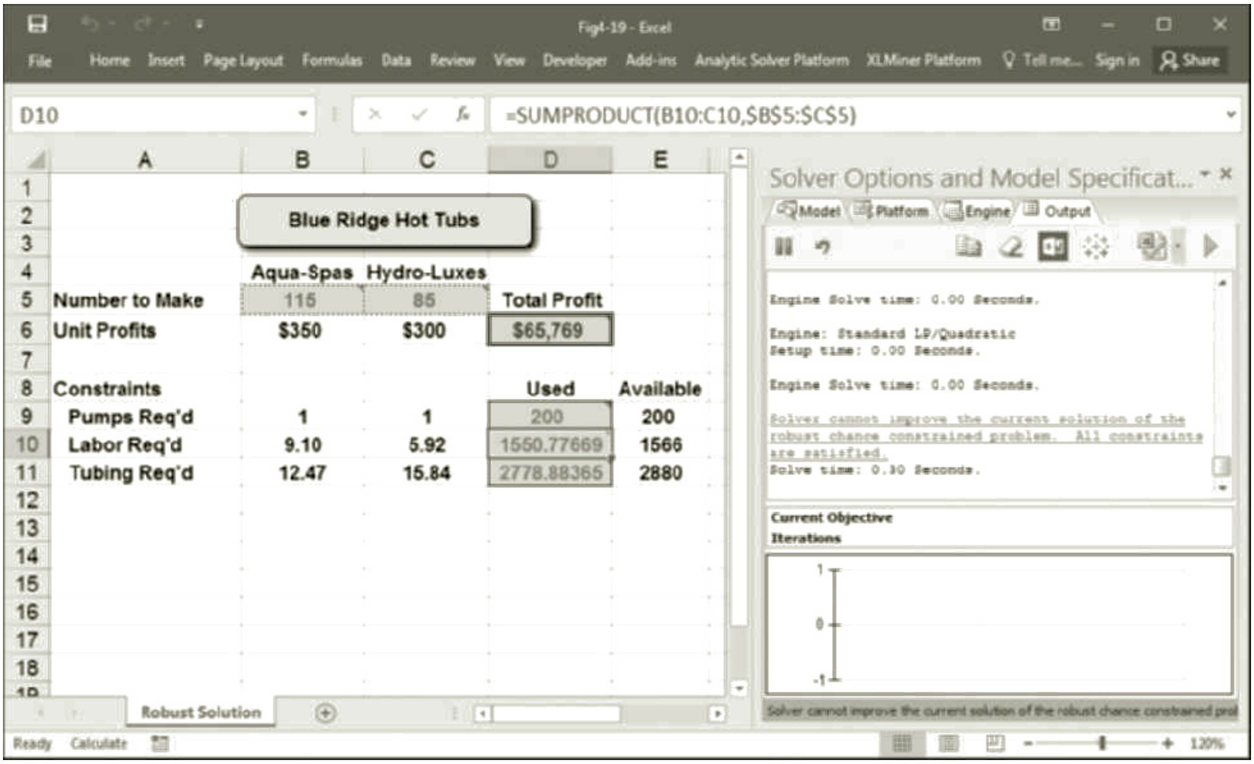
FIGURE 4.21 *A less conservative robust solution to the Blue Ridge Hot Tubs problem*



of robust optimization is beyond the scope of this book, but you can read the Analytic Solver Platform User Manual for more information and additional reference resources. We will also explore the idea of optimization under uncertainty in chapter 12.

4.9 The Simplex Method

We have repeatedly mentioned that the simplex method is the preferred method for solving LP problems. This section provides an overview of the simplex method and shows how it relates to some of the items that appear on the Answer Report and the Sensitivity Report.

4.9.1 CREATING EQUALITY CONSTRAINTS USING SLACK VARIABLES

Because our original formulation of the LP model for the Blue Ridge Hot Tubs problem has only two decision variables (X_1 and X_2), you might be surprised to learn that Solver actually used five variables to solve this problem. As you saw in chapter 2 when we plotted the boundary lines for the constraints in an LP problem, it is easier to work with equal to conditions rather than less than or equal to, or greater than or equal to conditions. Similarly, the simplex method requires that all constraints in an LP model be expressed as equalities.

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To solve an LP problem using the simplex method, Solver temporarily turns all inequality constraints into equality constraints by adding one new variable to each less than or equal to constraint and subtracting one new variable from each greater than or equal to constraint. The new variables used to create equality constraints are called slack variables.

For example, consider the less than or equal to constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \ \ldots \ + a_{kn}X_n \leq b_k$$

Solver can turn this constraint into an equal to constraint by adding the nonnegative slack variable S_k to the LHS of the constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \ \ldots \ + a_{kn}X_n + S_k = b_k$$

The variable S_k represents the amount by which $a_{k1}X_1 + a_{k2}X_2 + \ \ldots \ + a_{kn}X_n$ is less than b_k. Now consider the greater than or equal to constraint:

$$a_{k1}X1 + a_{k2}X_2 + \ \ldots \ + a_{kn}X_n \geq b_k$$

Solver can turn this constraint into an equal to constraint by subtracting the nonnegative slack variable S_k from the LHS of the constraint:

$$a_{k1}X_1 + a_{k2}X_2 + \ \ldots \ + a_{kn}X_n - S_k = b_k$$

In this case, the variable S_k represents the amount by which $a_{k1}X_1 + a_{k2}X_2 + \ \ldots \ + a_{kn}X_n$ exceeds b_k.

To solve the original Blue Ridge Hot Tubs problem using the simplex method, Solver actually solved the following modified problem involving *five* variables:

$$\begin{array}{llrcl}
\text{MAX:} & 350X_1 + 300X_2 & & & \} \text{ profit} \\
\text{Subject to:} & 1X_1 + 1X_2 + S_1 & = & 200 & \} \text{ pump constraint} \\
 & 9X_1 + 6X_2 + S_2 & = & 1{,}566 & \} \text{ labor constraint} \\
 & 12X_1 + 16X_2 + S_3 & = & 2{,}880 & \} \text{ tubing constraint} \\
 & X_1, X_2, S_1, S_2, S_3 & \geq & 0 & \} \text{ nonnegativity conditions}
\end{array}$$

We will refer to X_1 and X_2 as the **structural variables** in the model to distinguish them from the slack variables.

Recall that we did not set up slack variables in the spreadsheet or include them in the formulas in the constraint cells. Solver automatically sets up the slack variables it needs to solve a particular problem. The only time Solver even mentions these variables is when it creates an Answer Report like the one shown in Figure 4.2. The values in the Slack column in the Answer Report correspond to the optimal values of the slack variables.

4.9.2 BASIC FEASIBLE SOLUTIONS

After all the inequality constraints in an LP problem have been converted into equalities (by adding or subtracting appropriate slack variables), the constraints in the LP model represent a system (or collection) of linear equations. If there are a total of n variables in a system of m equations, one strategy for finding a solution to the system of equations is to select any m variables and try to find values for them that solve the system, assuming all other variables are set equal to their lower bounds (which are usually zero). This strategy requires more variables than constraints in the system of equations—or that $n \geq m$.

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The *m* variables selected to solve the system of equations in an LP model are sometimes called **basic variables**, while the remaining variables are called **nonbasic variables**. If a solution to the system of equations can be obtained using a given set of basic variables (while the nonbasic variables are all set equal to zero), that solution is called a **basic feasible solution**. Every basic feasible solution corresponds to one of the extreme points of the feasible region for the LP problem, and we know that the optimal solution to the LP problem also occurs at an extreme point. So, the challenge in LP is to find the set of basic variables (and their optimal values) that produce the basic feasible solution corresponding to the optimal extreme point of the feasible region.

Because our modified problem involves three constraints and five variables, we could select three basic variables in ten different ways to form possible basic feasible solutions for the problem. Figure 4.22 summarizes the results for these ten options.

The first five solutions in Figure 4.22 are feasible and, therefore, represent basic feasible solutions to this problem. The remaining solutions are infeasible because they violate the nonnegativity conditions. The best feasible alternative shown in Figure 4.22 corresponds to the optimal solution to the problem. In particular, if X_1, X_2, and S_3 are selected as basic variables and S_1 and S_2 are nonbasic and assigned their lower

FIGURE 4.22

Possible basic feasible solutions for the original Blue Ridge Hot Tubs problem

	Basic Variables	Nonbasic Variables	Solution	Objective Value
1	S_1, S_2, S_3	X_1, X_2	$X_1=0, X_2=0,$ $S_1=200, S_2=1566, S_3=2{,}880$	0
2	X_1, S_1, S_3	X_2, S_2	$X_1=174, X_2=0,$ $S_1=26, S_2=0, S_3=792$	60,900
3	X_1, X_2, S_3	S_1, S_2	$X_1=122, X_2=78,$ $S_1=0, S_2=0, S_3=168$	66,100
4	X_1, X_2, S_2	S_1, S_3	$X_1=80, X_2=120,$ $S_1=0, S_2=126, S_3=0$	64,000
5	X_2, S_1, S_2	X_1, S_3	$X_1=0, X_2=180,$ $S_1=20, S_2=486, S_3=0$	54,000
6*	X_1, X_2, S_1	S_2, S_3	$X_1=108, X_2=99,$ $S_1=-7, S_2=0, S_3=0$	67,500
7*	X_1, S_1, S_2	X_2, S_3	$X_1=240, X_2=0,$ $S_1=-40, S_2=-594, S_3=0$	84,000
8*	X_1, S_2, S_3	X_2, S_1	$X_1=200, X_2=0,$ $S_1=0, S_2=-234, S_3=480$	70,000
9*	X_2, S_2, S_3	X_1, S_1	$X_1=0, X_2=200,$ $S_1=0, S_2=366, S_3=-320$	60,000
10*	X_2, S_1, S_3	X_1, S_2	$X_1=0, X_2=261,$ $S_1=-61, S_2=0, S_3=-1{,}296$	78,300

Note: * denotes infeasible solutions

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bound values (zero), we try to find values for X_1, X_2, and S_3 that satisfy the following constraints:

$$\begin{aligned} 1X_1 + 1X_2 \quad &= 200 \quad &\text{\} pump constraint} \\ 9X_1 + 6X_2 \quad &= 1{,}566 \quad &\text{\} labor constraint} \\ 12X_1 + 16X_2 + S_3 &= 2{,}880 \quad &\text{\} tubing constraint} \end{aligned}$$

Notice that S_1 and S_2 in the modified equal to constraints are not included in the above constraint equations because we are assuming that the values of these nonbasic variables are equal to zero (their lower bounds). Using linear algebra, the simplex method determines that the values $X_1 = 122$, $X_2 = 78$, and $S_3 = 168$ satisfy the equations given above. So, a basic feasible solution to this problem is $X_1 = 122$, $X_2 = 78$, $S_1 = 0$, $S_2 = 0$, $S_3 = 168$. As indicated in Figure 4.22, this solution produces an objective function value of \$66,100. (Notice that the optimal values for the slack variables S_1, S_2, and S_3 also correspond to the values shown in the Answer Report in Figure 4.2 in the Slack column for constraint cells D9, D10, and D11.) Figure 4.23 shows the relationships between the basic feasible solutions listed in Figure 4.22 and the extreme points of the feasible region for this problem.

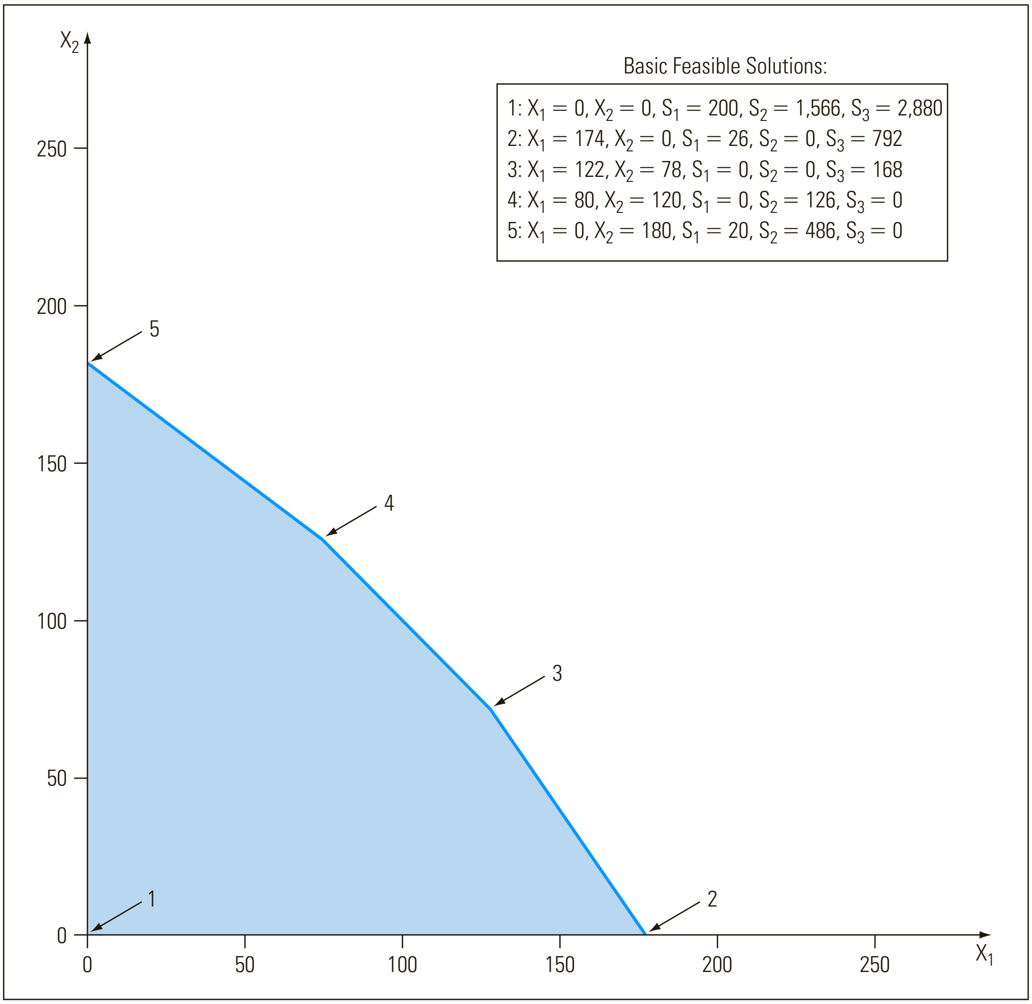

FIGURE 4.23

Illustration of the relationship between basic feasible solutions and extreme points

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4.9.3 FINDING THE BEST SOLUTION

The simplex method operates by first identifying any basic feasible solution (or extreme point) for an LP problem, and then moving to an adjacent extreme point, if such a move improves the value of the objective function. When no adjacent extreme point has a better objective function value, the current extreme point is optimal and the simplex method terminates.

The process of moving from one extreme point to an adjacent one is accomplished by switching one of the basic variables with one of the nonbasic variables to create a new basic feasible solution that corresponds to the adjacent extreme point. For example, in Figure 4.23, moving from the first basic feasible solution (point 1) to the second basic feasible solution (point 2) involves making X_1 a basic variable and S_2 a nonbasic variable. Similarly, we can move from point 2 to point 3 by switching basic variables with non-basic variables. So, starting at point 1 in Figure 4.23, the simplex method could move to point 2, then to the optimal solution at point 3. Alternatively, the simplex method could move from point 1 through points 5 and 4 to reach the optimal solution at point 3. Thus, although there is no guarantee that the simplex method will take the shortest route to the optimal solution of an LP problem, it will find the optimal solution eventually.

To determine whether switching a basic and nonbasic variable will result in a better solution, the simplex method calculates the reduced cost for each nonbasic variable to determine if the objective function could be improved if any of these variables are substituted for one of the basic variables. (Note that unbounded solutions are detected easily in the simplex method by the existence of a nonbasic variable that could improve the objective value by an infinite amount if it were made basic.) This process continues until no further improvement in the objective function value is possible.

4.10 Summary

This chapter described the methods for assessing how sensitive an LP model is to various changes that might occur in the model or its optimal solution. The impact of changes in an LP model can be analyzed easily by re-solving the model. Solver also provides a significant amount of sensitivity information automatically. For LP problems, the maximum amount of sensitivity information is obtained by solving the problem using the simplex method. Before using the information on the Sensitivity Report, you should always first check for the presence of degeneracy because this can have a significant impact on how one should interpret the numbers on this report. While the information available in the Sensitivity Report is useful, it does not provide answers to all the questions an analyst might have about the optimal solution to an LP problem. As a result, various ad hoc techniques like Spider Plots/Tables and Solver Tables are available to help address specific questions that might arise.

The simplex method considers only the extreme points of the feasible region, and is an efficient way of solving LP problems. In this method, slack variables are first introduced to convert all constraints to equal to constraints. The simplex method systematically moves to better and better corner point solutions until no adjacent extreme point provides an improved objective function value.

4.11 References

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