Relations and Functions

CMSC250

Relations

Definition

- Let A, B be sets. A relation R from A to B is any subset of $A \times B$.

Examples

- $(<, \mathbb{R} \times \mathbb{R})$
  - $\{\ldots, (-1.5, -1.2), (-1.4, -1.2), (\sqrt{2}, \sqrt{3}), (\sqrt{2}, \sqrt{5}), \ldots\}$

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  - $\{\dots, (2, 2), (2, 2.1), (\sqrt{2}, \sqrt{3}), (\sqrt{2}, \sqrt{5}), \dots\}$
- $(R, \mathbb{R} \times \mathbb{N})$
  - $\{(r, n) \mid n$ appears in the decimal expansion of $r\}$
  - E.g: $\{\dots, (\pi, 1),\ (e, 7),\ ({}^{1}/_{3}, 3), \dots\}$
  - We would formally say that all of the above are elements of the relation R

Reflexivity

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- Examples:
  - $(\leq, \mathbb{N} \times \mathbb{N})$ is reflexive, since $(\forall n \in \mathbb{N})[n \leq n]$
  - $(<, \mathbb{N} \times \mathbb{N})$ is **not** reflexive, since $\sim (\forall n \in \mathbb{N})[n < n]$ (in fact, there is no such n)
  - $(R, \mathbb{N} \times \mathbb{N})$ defined as $\{(x, y) \mid x + y \geq 100\}$ is **not** reflexive (e.g $10 \in \mathbb{N}$, but $(10, 10) \notin R$)

Symmetry

- A relation $X \subseteq A \times A$ is **symmetric** if

$$(\forall a_1, a_2 \in A)\Big[\big((a_1, a_2) \in X\big) \Rightarrow ((a_2, a_1) \in X)\Big]$$

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$$(\forall a_1, a_2 \in A)\Big[\big((a_1, a_2) \in X\big) \Rightarrow \big((a_2, a_1) \in X\big)\Big]$$

- Examples:
  - $(\leq, \mathbb{N} \times \mathbb{N})$ is **not** symmetric since $4 \leq 5$ but $\sim (5 \leq 4)$
  - $(<, \mathbb{N} \times \mathbb{N})$ is **not** symmetric (see above)
  - $(R, \mathbb{N} \times \mathbb{N})$ defined as $\{(x, y) \mid x + y \geq 100\}$ **is symmetric** since

$$(x + y \geq 100) \Rightarrow (y + x \geq 100)$$

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$$(\forall a_1, a_2, a_3 \in A)\Big[\Big((a_1, a_2) \in X\Big) \wedge ((a_2, a_3) \in X) \Rightarrow (a_1, a_3) \in X)\Big]$$

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- $(\leq, \mathbb{N} \times \mathbb{N})$ **is transitive** since $\big((x \leq y) \wedge (y \leq z)\big) \Rightarrow (x \leq z)$
- $(<, \mathbb{N} \times \mathbb{N})$ **is transitive** (see above)
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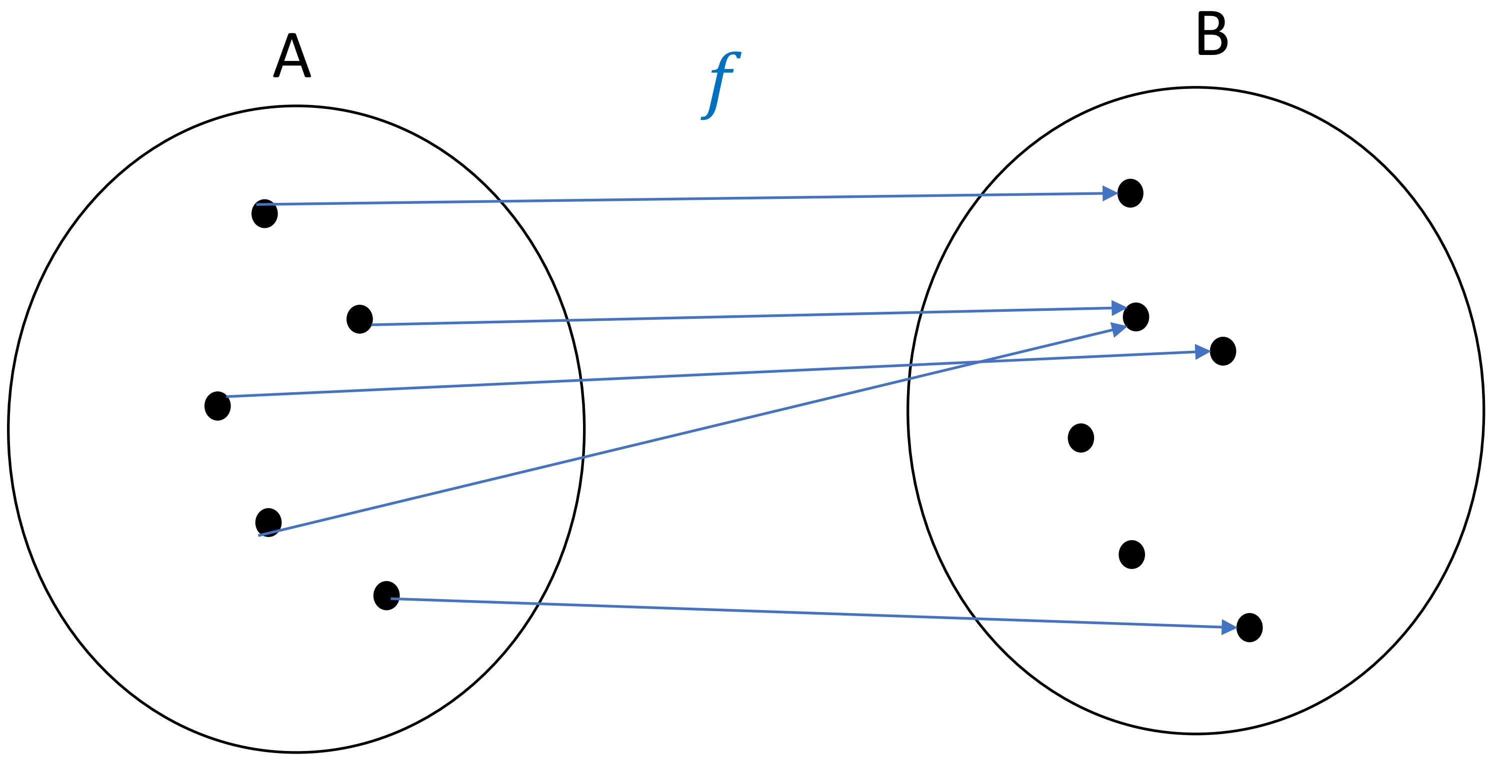
- $(R, \mathbb{N} \times \mathbb{N})$ defined as $\{(x, y) \mid x + y \geq 100\}$ **is not transitive** since (counter-example):

$$((1, 100) \in R) \wedge ((100, 5) \in R), \text{ but } (1, 5) \notin R$$

Functions

Functions

- Most basic representation: **Arrow Diagrams**



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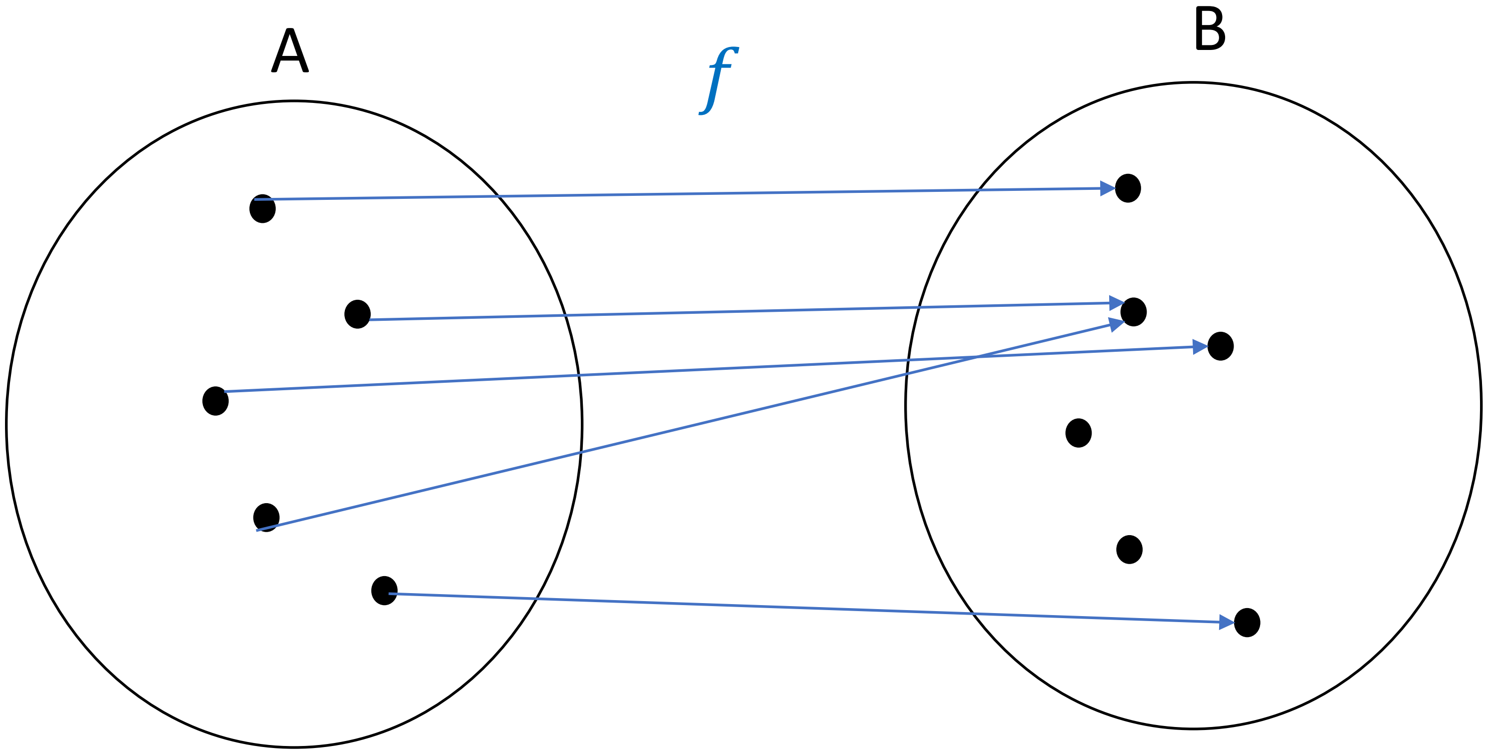

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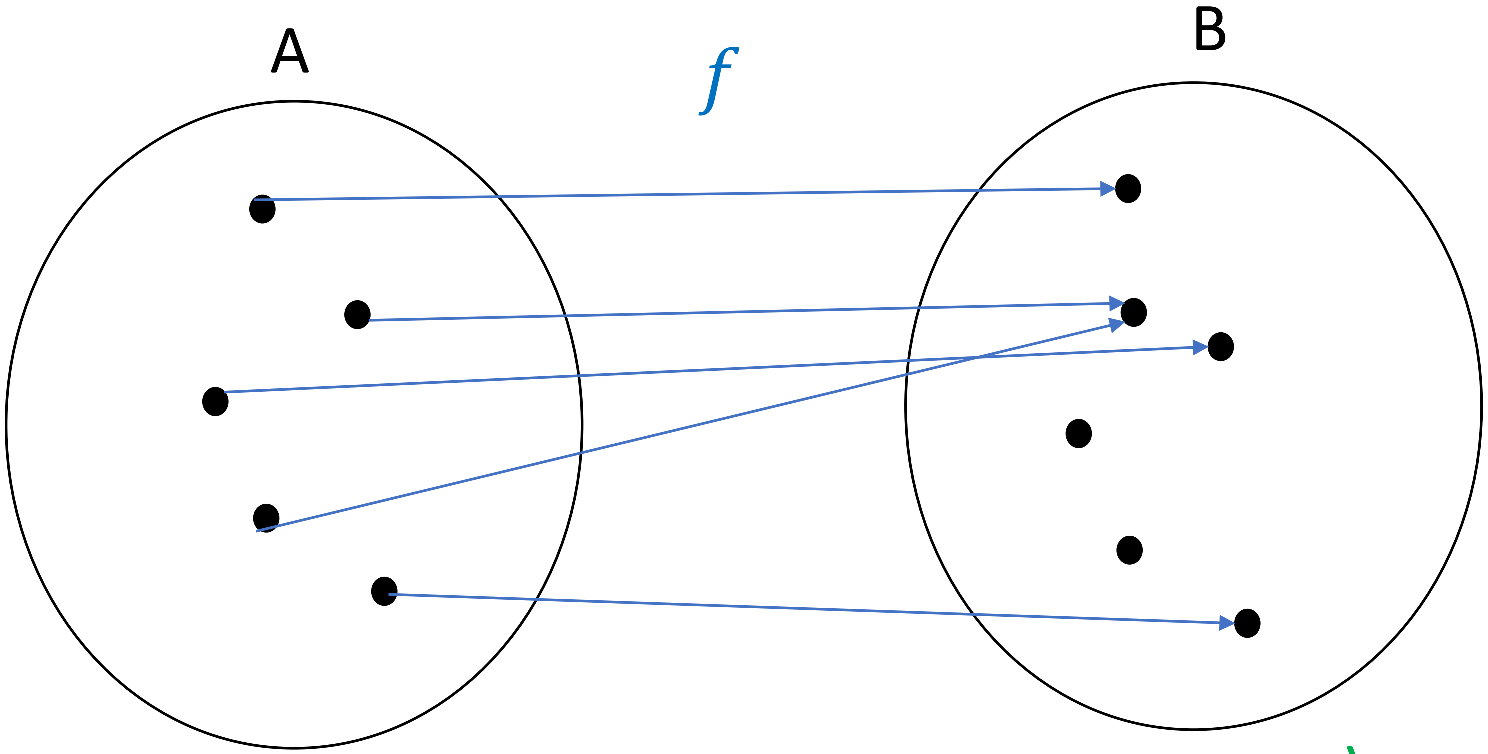

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\mapsto in LaTeX

Example 1

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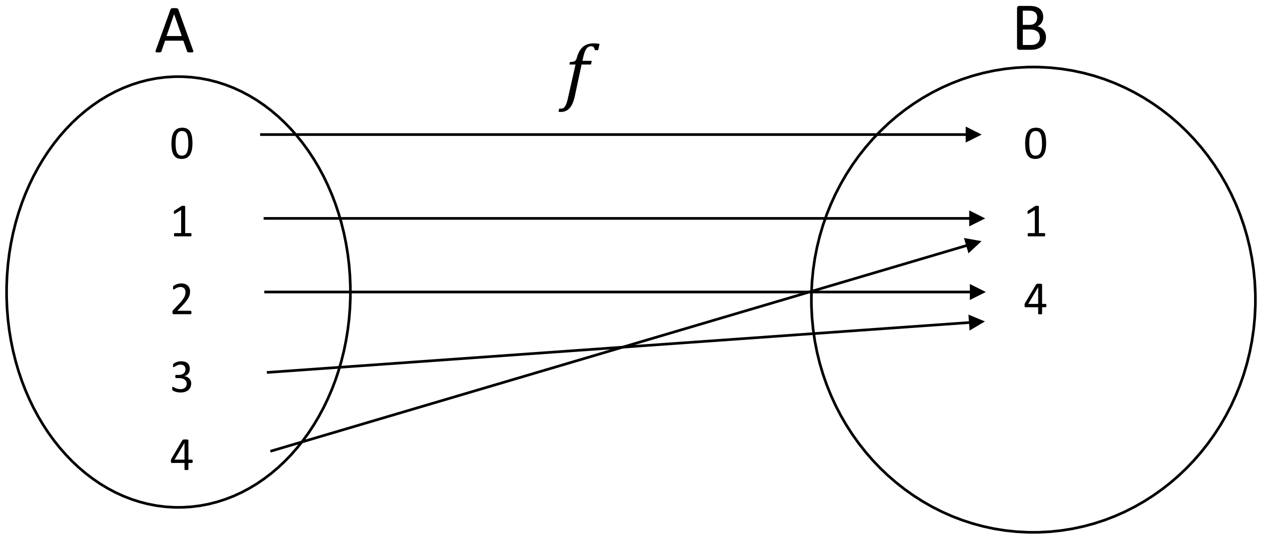

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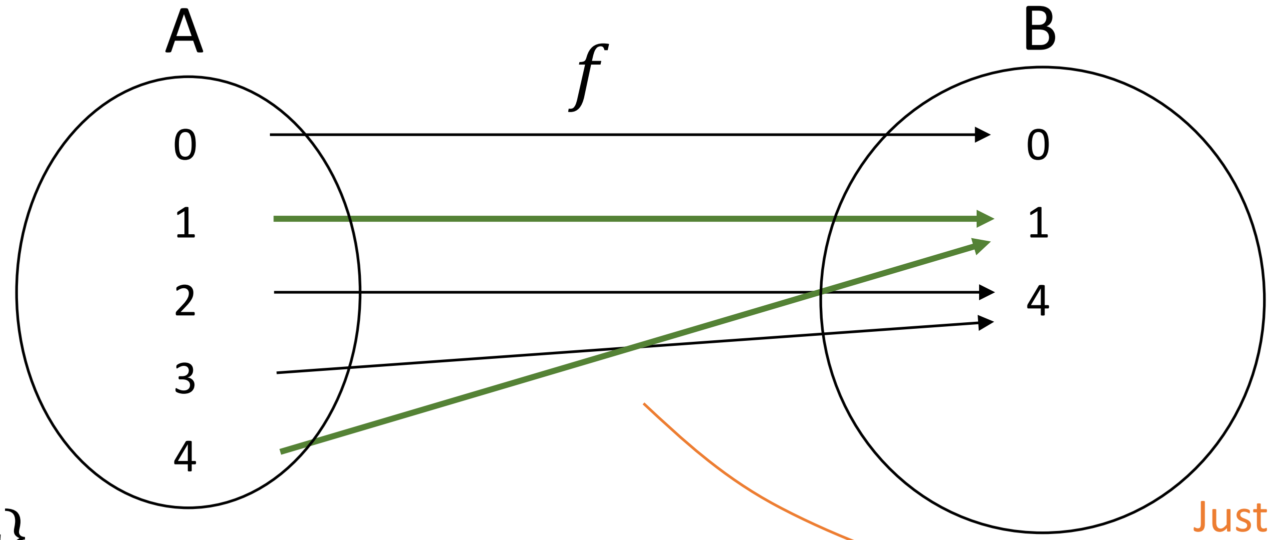

- Domain: $\{0, 1, 2, 3, 4\}$
- Co-domain: $\{0, 1, 4\}$
- Formula (that we came up with): $f(x) = x^2 \textbf{ mod } 5$

Just because two ‘x’s map to the same ‘y’ doesn’t make this a non-function… it just makes it a **non-injective** (not “1-1”) function

Example 2

- Is this a function?

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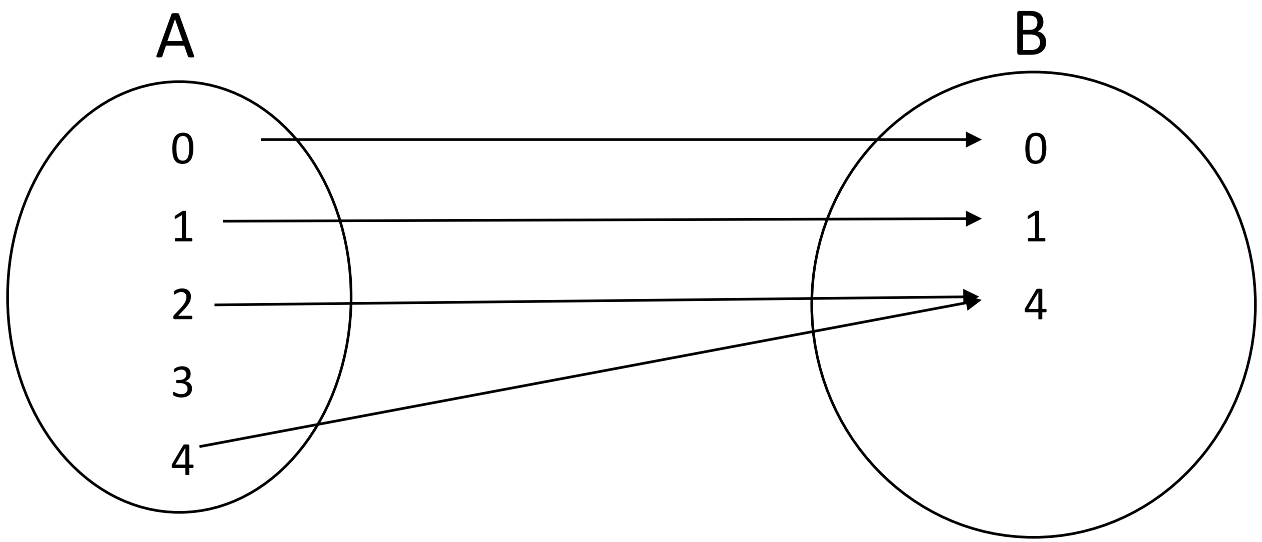

Example 2

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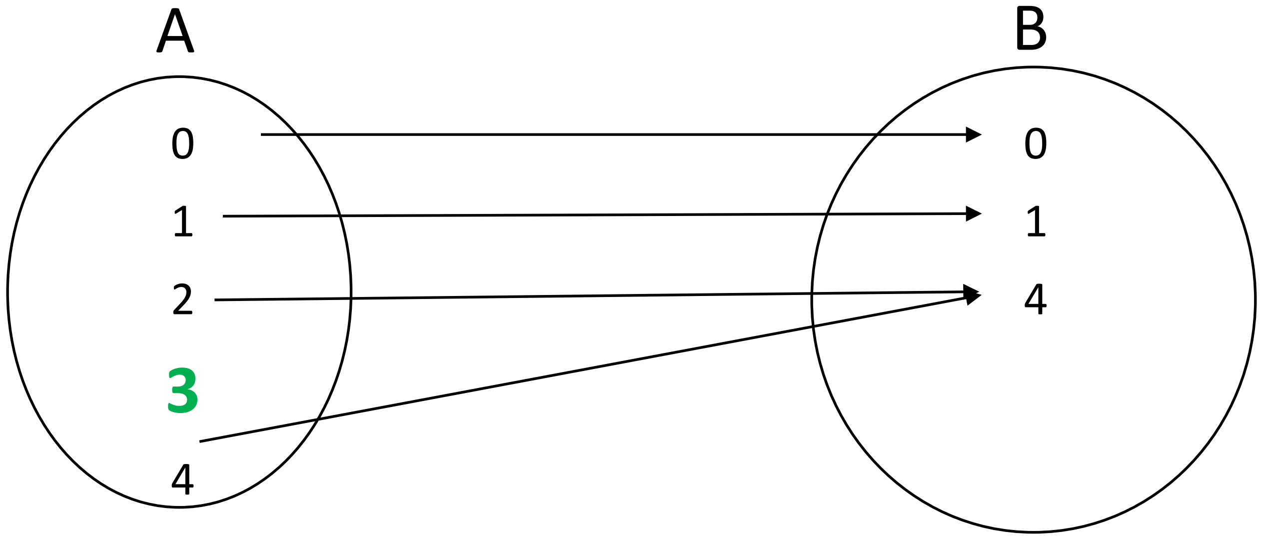

- **Every element of the domain should map to some co-domain element!**

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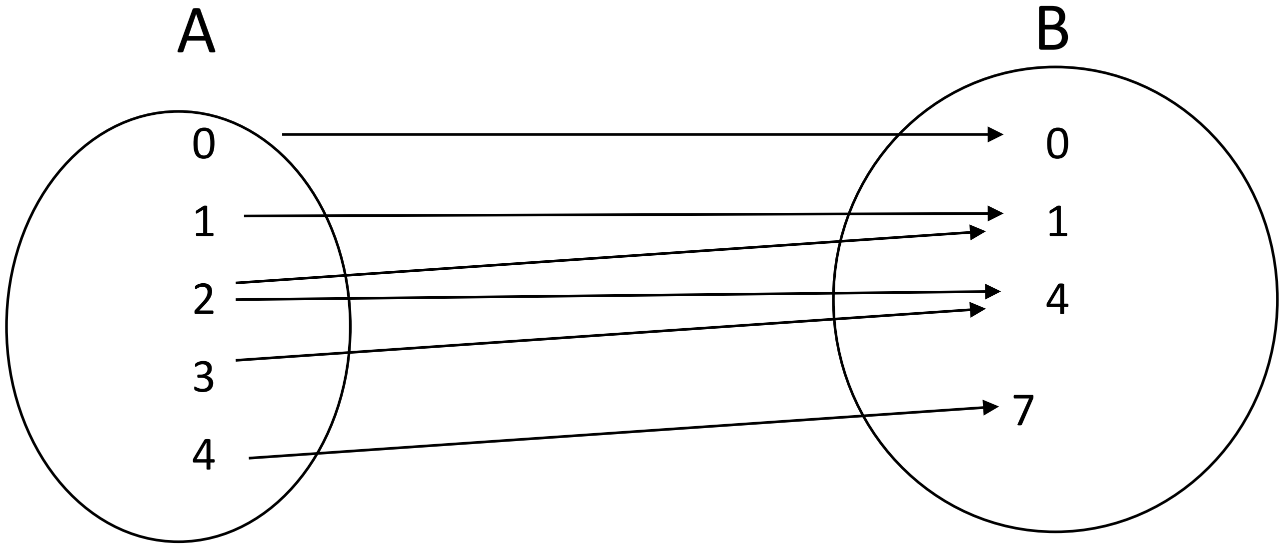

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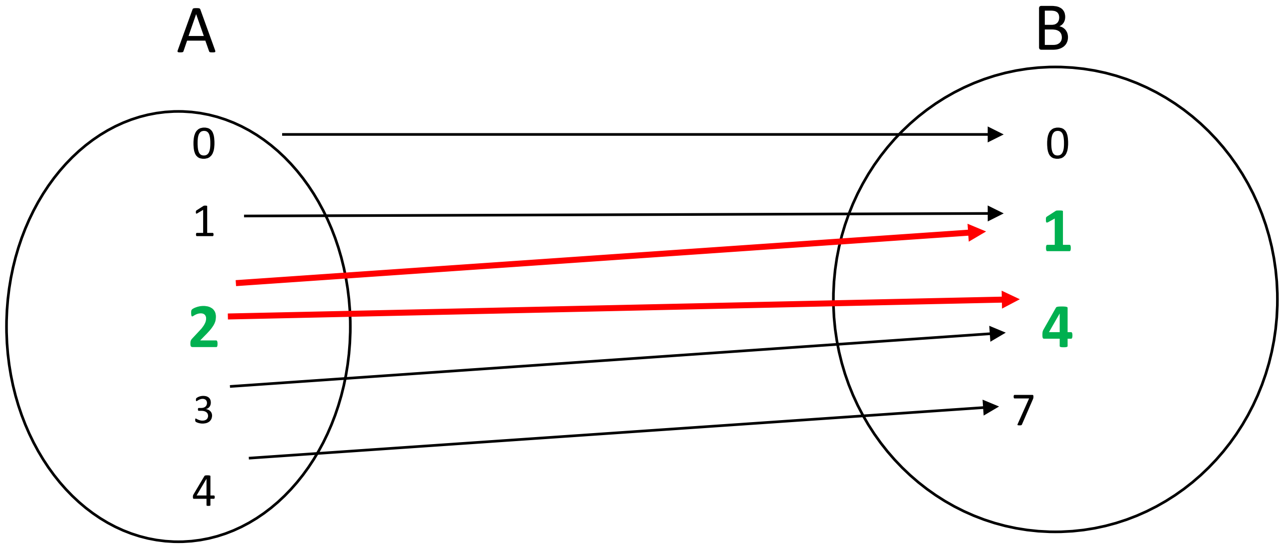

Fails the “vertical line” test (2 different `y’s mapped to by the same `x’)

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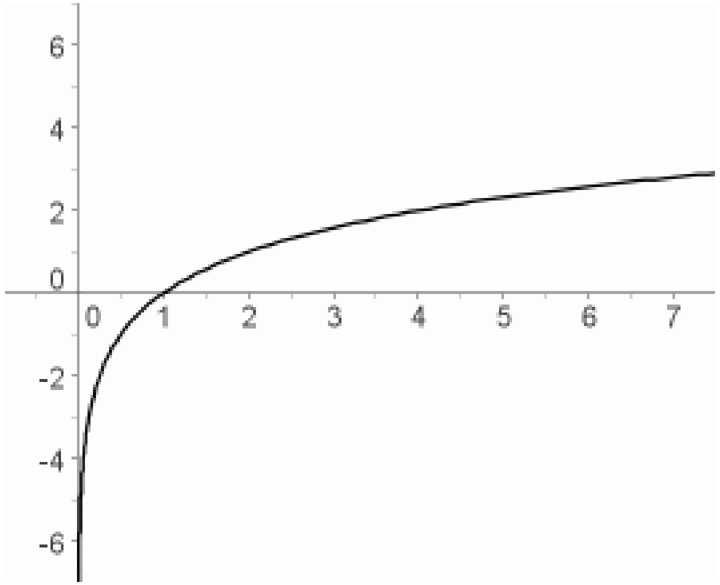

Example 4

Yes

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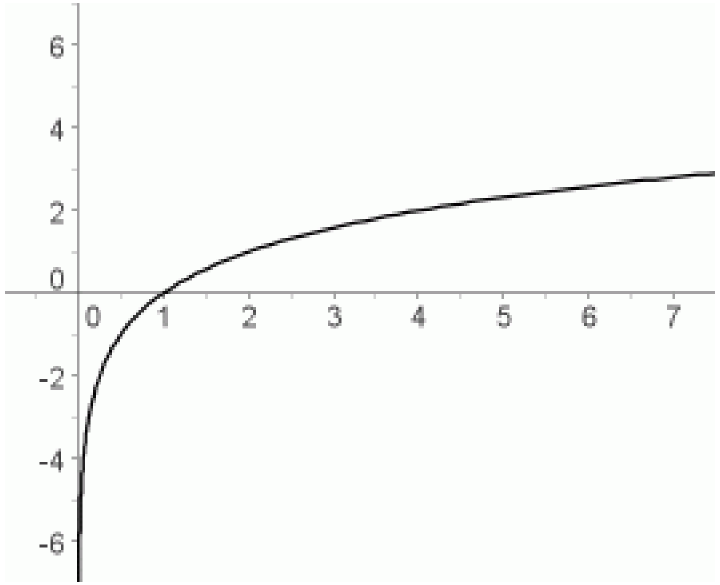
(As long as the domain is $\mathbb{R}^{>0}$!!)



Log function

Example 5

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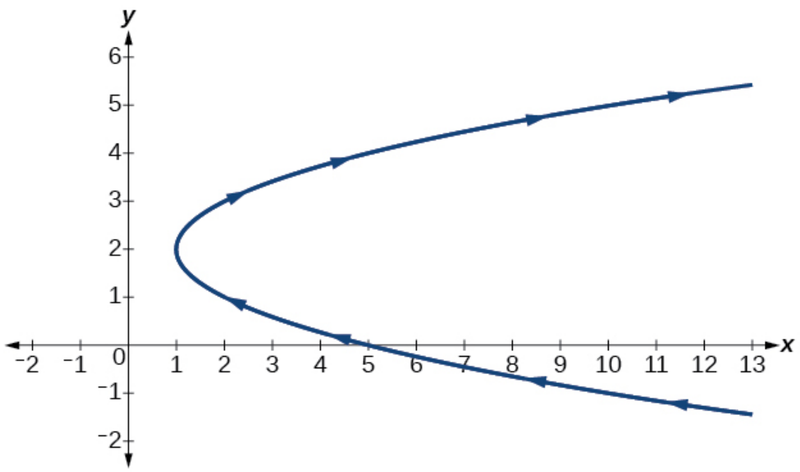

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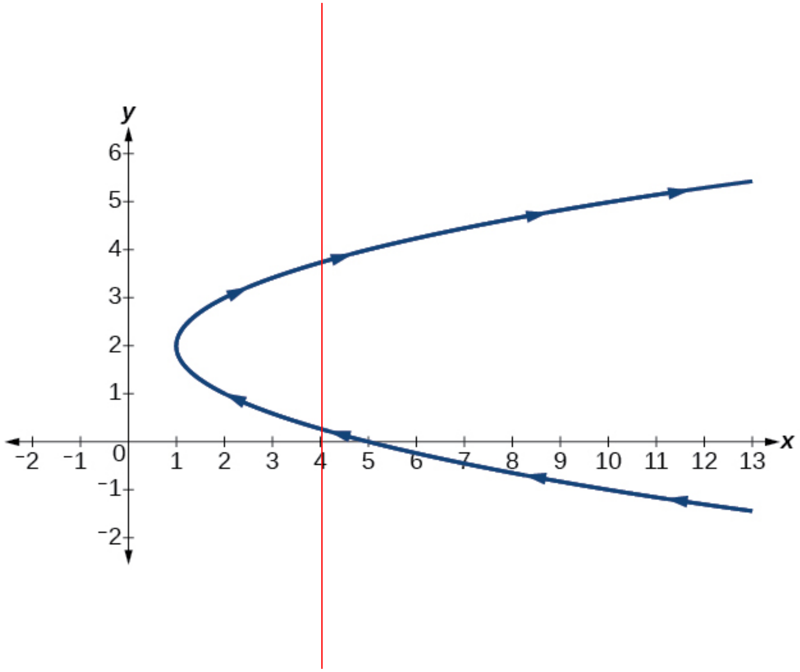

Fails the "vertical line" test

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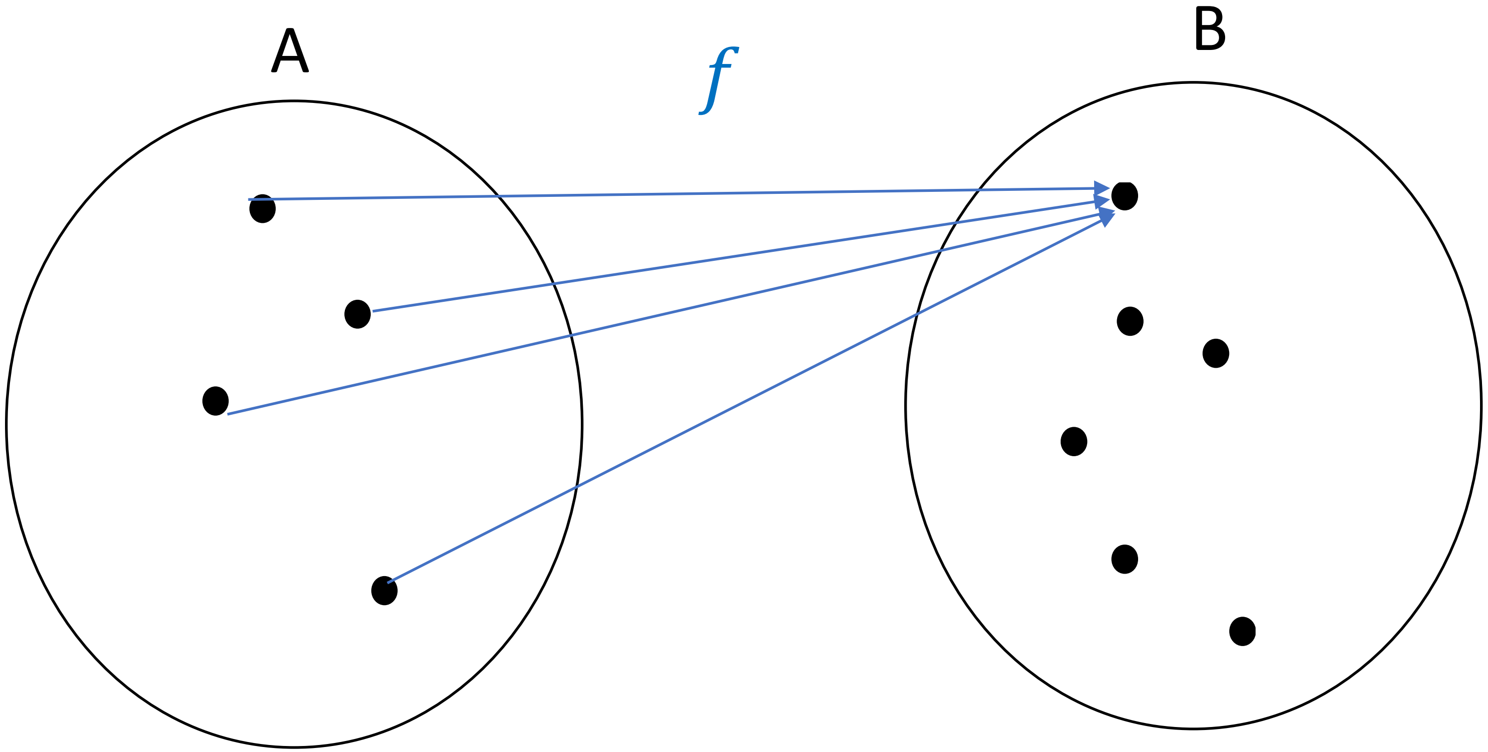

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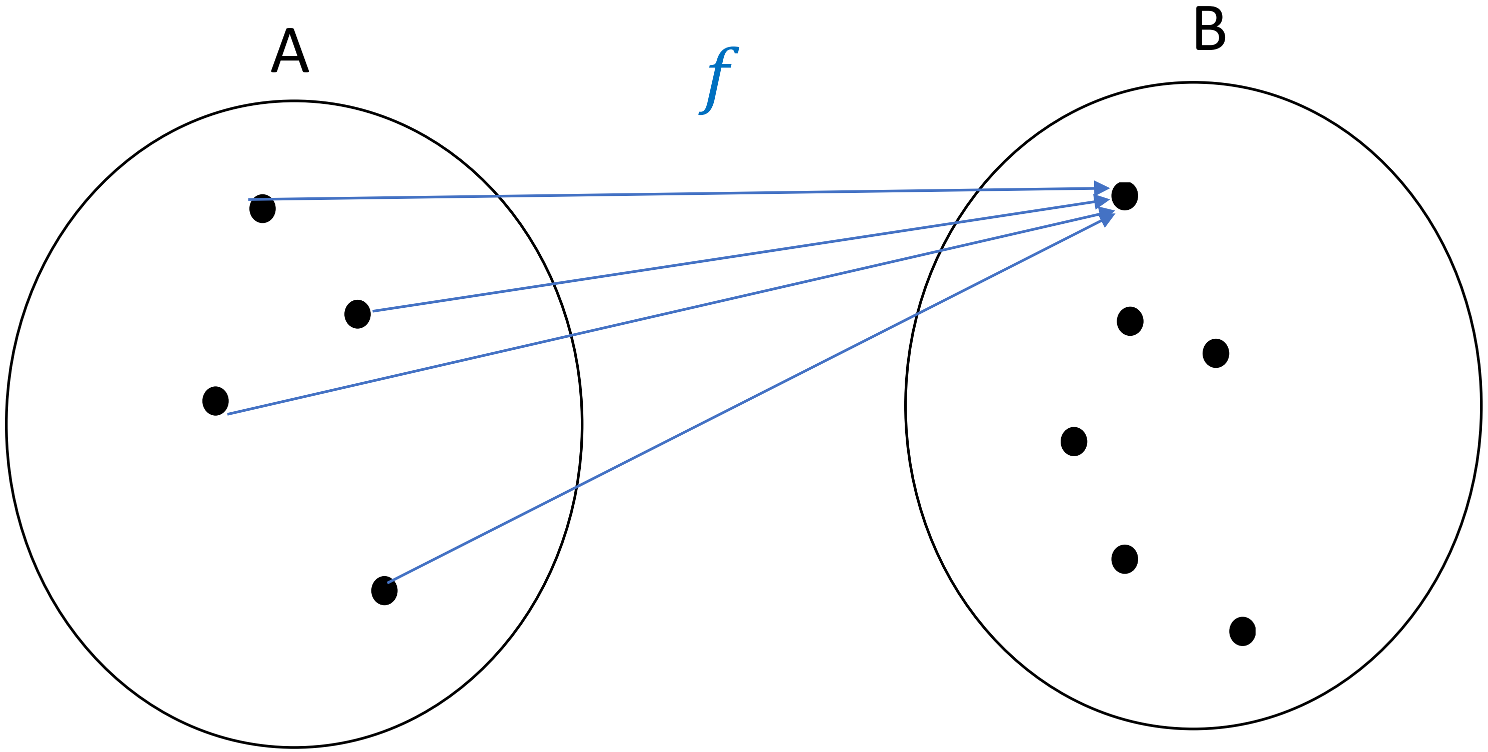

It just happens to not be "1-1"

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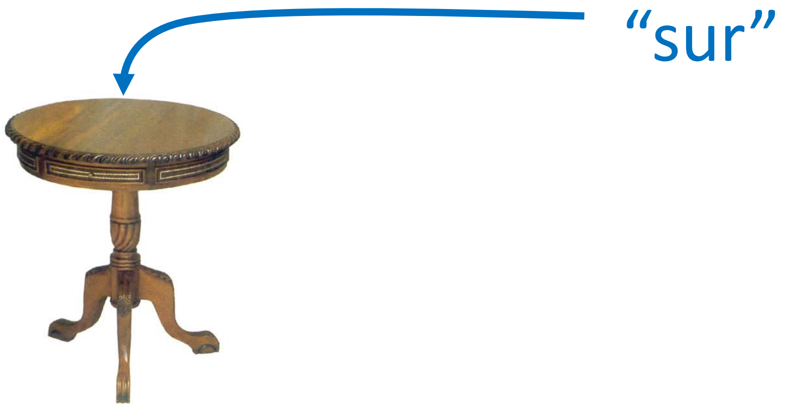

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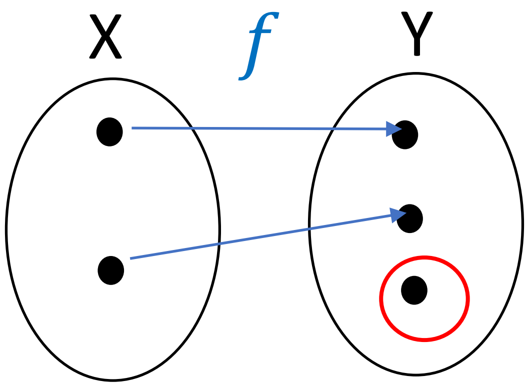
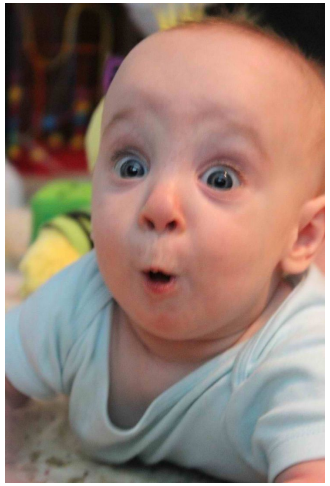
  - ***Why at most one and not exactly one?***
  - ***Because 1-1 but not onto functions are possible!***





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- Can this function **ever** be injective?
  - Yes. Pick $D = \mathbb{N}, C = SQUARES$

Making functions onto or 1-1

- To make a function **onto**, we need to make the **co-domain smaller.**

- To make a function **1-1** , we need to make the **domain smaller.**

Bijective functions

- A function $f: X \mapsto Y$ is called **bijective (or a bijection, or a <u>1-1 correspondence</u>)** iff it is both surjective and injective.
  - We will try to avoid using the term **"1-1 correspondence"** (your book uses it) since it can confuse us with the notion of an injective (or 1-1) **function**.

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  (In all examples, $C = \mathbb{R}$*)*

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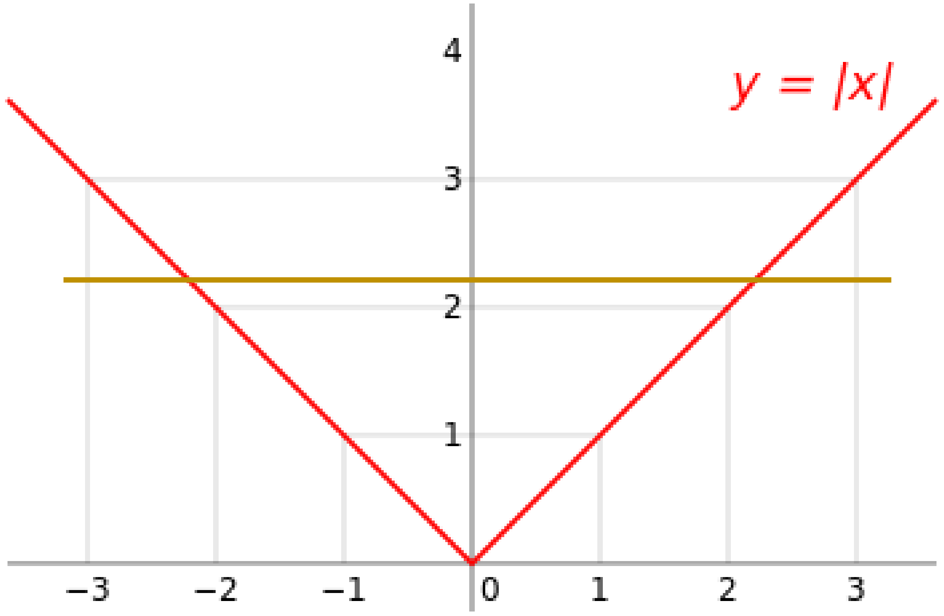

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!!!!

For a = 0, the graph of the function fails the "horizontal line test"!

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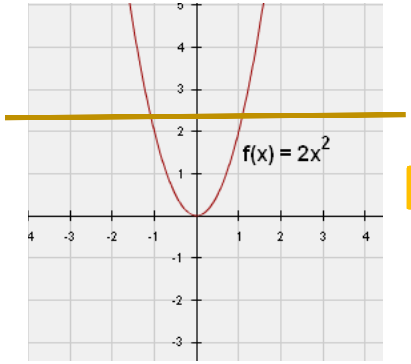

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4. $h(n) = 4n - 1, n \in \mathbb{Z}$

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4. $h(n) = 4n - 1, n \in \mathbb{Z}$ *No*

Non-surjective! Set $h(n) = y$ and solve for n:

$$4n - 1 = y \Rightarrow n = \frac{y+1}{4}$$

There are infinitely many choices of y for which $n \notin \mathbb{Z}$!

Quiz on bijections

- Are the following functions **bijections**?

Yes

No

(In all examples, $C = \mathbb{R}$)

1. $f(x) = |x|, x \in \mathbb{R}$ *No*
2. $f(x) = a \cdot x + b, (\forall a, x, b \in \mathbb{R})$ *No*
3. $g(x) = a \cdot x^2, a, x \in \mathbb{R}, a > 0$ *No*
4. $h(n) = 4n - 1, n \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1, x \in \mathbb{R}$

Quiz on bijections

Yes

No

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  (In all examples, $C = \mathbb{R}$)

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4. $h(n) = 4n - 1, n \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1, x \in \mathbb{R}$ *Yes*

Surjective and injective! Surjective, since, if we set $h(n) = y$ and solve for n:

$$4n - 1 = y \Rightarrow n = \frac{y+1}{4}$$

For every real y, there's always a **real** solution n. Injective, since it's of the form of (2) with $a \neq 0$.

Some special functions

- Are the following Java functions injective, surjective, bijective?
  - `Comparable.compareTo()`
  - `Object.hashCode()`