Quantifiers

CMSC250

Reminders

- Midterm grades: You can expect them tomorrow (Friday 03-12) midnight the latest.
 - Through Friday 03-19 midnight you can submit regrades.
- HW4 grades have been posted; check for any regrade requests you might want to make. Usual Friday 11:59pm deadline for those.

Reminders

- HW5 posted last Monday, due Monday 03-22 11:59pm
 - Normal length, though.
 - Please work on it before Monday 22nd ☺
- Tomorrow: Quiz 6
 - Due also after Springbreak (Monday 03-22 11:59pm).
 - Normal, miniscule, length.
- Slides, textbook chapters and select exercises for material of week of 03-22 available since end of last week.
 - Big-Oh notation
 - Intro to proofs (big cheese in the class)

Existential / universal quantifier

- There exist two quantifiers in formal logic / set theory
 - The universal quantifier: ∀ (read "for all")
 - The existential quantifier: ∃ (read "exists")
- We will see that every quantifier needs a set associated with it, so general syntax of **quantified expressions** will be:

(Quantifier variable ∈ Set)[Some property on variable]

"There exists" (\exists)

- Examples:
 - $(\exists x \in \mathbb{R}) [8x = 1]$

"There exists" (\exists)

- Examples:
 - $(\exists x \in \mathbb{R}) [8x = 1]$ True

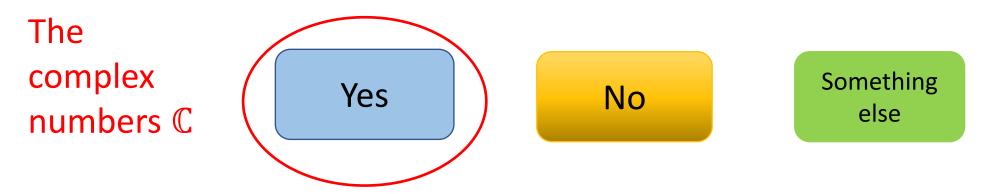
- Examples:
 - $(\exists x \in \mathbb{R}) [8x = 1]$ True
 - $(\exists n \in \mathbb{Z})[n^2 = -1]$

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- Examples:
 - $(\exists x \in \mathbb{R}) [8x = 1]$ True
 - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain *D* where $(\exists n \in D)[n^2 = -1]$ is true?

Yes No Something else

- Examples:
 - $(\exists x \in \mathbb{R}) [8x = 1]$ True
 - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain D where $(\exists n \in D)[n^2 = -1]$ is true?



- The symbol ∀ (LaTeX: \forall) is read "for all".
- Examples:
 - $(\forall x \in \mathbb{N}) [((x > 2) \land (x \text{ is prime})) \Rightarrow (x \text{ is odd})]$

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- Examples:
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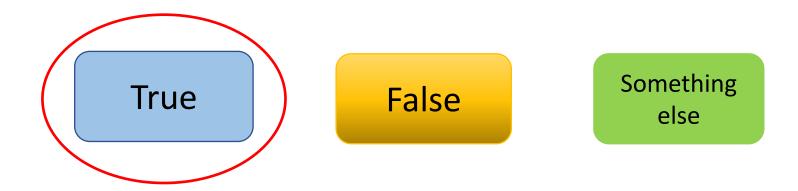
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- Examples:
 - $(\forall x \in \mathbb{N}) [((x > 2) \land (x \text{ is prime})) \Rightarrow (x \text{ is odd})]$ True
 - $(\forall n \in \mathbb{Z}) [n^2 \ge 0]$ True

- Let D be the set of all students in this class who are over 8 feet tall.
- $(\forall x \in D)[x \ has \ perfect \ attendance \ so \ far!]$

True False Something else

- Let D be the set of all students in this class who are over 8 feet tall.
- $(\forall x \in D)[x \ has \ perfect \ attendance \ so \ far!]$



- If disagree, need to find $x \in D$ who missed a class
- Called vacuously true!

 $\bullet (\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$

• $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$ True, $x = \frac{4}{5}, y = \frac{8}{5}$

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$ True, $x = \frac{4}{5}, y = \frac{8}{5}$

- Common abbreviation: $(\exists x, y \in D)[...]$
- Generally: $(\exists x_1, x_2, ..., x_n \in D)[...]$

Alternating nested quantifiers

- Notice the differences between the following:
 - $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$
 - $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$

Alternating nested quantifiers

- Notice the differences between the following:
 - $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$ True (\mathbb{N} unbounded from above)
 - $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$ False (\mathbb{N} bounded from below)

• WHEN QUANTIFIERS ARE DIFFERENT, THEIR ORDER MATTERS!!!!!!

Fill this in!

Statement	True	False
$(\exists n \in \mathbb{N})[n+n=0]$		\bigcirc
$(\exists n \in \mathbb{N})[n+n=1]$		\bigcirc
$(\exists n \in \mathbb{Z})[n+n=1]$		
$(\exists x, y \in \mathbb{Z})[x + y = 1]$	0	\bigcirc
$(\exists x \in \mathbb{R})[x(x+1) = -1]$	0	0
$(\forall x, y, z \in \mathbb{R})[((x < y < z) \Rightarrow (x^2 < y^2 < z^2)]$	0	\circ
$(\forall x, y, z \in \mathbb{R})[((x < y < z) \Rightarrow (x^3 < y^3 < z^3)]$		0
$(\forall x, y \in \mathbb{N})(\exists z \in \mathbb{Q})[(x < y) \Rightarrow (x < z < y)]$	0	\bigcirc

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true $(D = \mathbb{N}, \text{ select } y = x + 1)$

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true $(D = \mathbb{N}, \text{ select } y = x + 1)$
 - 2. Is false

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true $(D = \mathbb{N}, \text{ select } y = x + 1)$
 - 2. Is false $(D = \mathbb{Z}^{\leq 0})$, counter-example is 0)

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true $(D = \mathbb{N}, \text{ select } y = x + 1)$
 - 2. Is false $(D = \mathbb{Z}^{\leq 0})$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \le 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true $(D = \mathbb{N}, \text{ select } y = x + 1)$
 - 2. Is false $(D = \mathbb{Z}^{\leq 0})$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \le 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

1. True for $D = (-\infty, 1)$

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
 - 1. Is true $(D = \mathbb{N}, \text{ select } y = x + 1)$
 - 2. Is false $(D = \mathbb{Z}^{\leq 0})$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \le 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

- 1. True for $D = (-\infty, 1)$
- 2. False for $D = (-\infty, 1]$ (!)

Negated Quantifiers

- It is not the case that Alice comes to all office hours.
- There is an office hour that Alice does not come to.

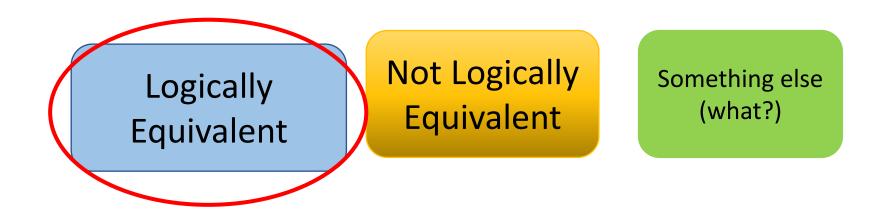
Logically Equivalent

Not Logically Equivalent

Something else (what?)

Negated Quantifiers

- It is not the case that Alice comes to all office hours.
- There is an office hour that Alice does not come to.



Negated Quantifiers

• We can therefore reach the following logical equivalences:

$$\sim (\forall x \in D)[P(x)] \equiv (\exists x \in D)[\sim P(x)]$$
$$\sim (\exists x \in D)[P(x)] \equiv (\forall x \in D)[\sim P(x)]$$

Negating nested quantifiers

Observe how we can negate

$$(\forall x \in D)[(\exists y \in D)[x < y]]$$

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$$\sim (\forall x \in D)[(\exists y \in D)[x < y]] \equiv$$

Negating nested quantifiers

Observe how we can negate

$$(\forall x \in D)[(\exists y \in D)[x < y]]$$

$$(\forall x \in D) [(\exists y \in D)[x < y]] \equiv (\exists x \in D) [\sim (\exists y \in D)[x < y]] \equiv$$

Negating nested quantifiers

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Negating nested quantifiers

Observe how we can negate

$$(\forall x \in D)[(\exists y \in D)[x < y]]$$

$$(\forall x \in D) [(\exists y \in D)[x < y]] \equiv$$

$$(\exists x \in D) [\sim (\exists y \in D)[x < y]] \equiv$$

$$(\exists x \in D) [(\forall y \in D)[\sim (x < y)]] \equiv$$

$$(\exists x \in D) [(\forall y \in D)[x \ge y]]$$

- A set is dense if between any two elements of it there exists another element of it.
- Observe the statement

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

• This statement says: *D* is dense.

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

- 1. Give a D where the statement is true.
- Give a D where the statement is false.
- 3. Negate the statement.

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

- 1. Give a D where the statement is true.
- 2. Give a D where the statement is false
- 3. Negate the statement
- Answers:

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

- 1. Give a D where the statement is true.
- 2. Give a D where the statement is false
- 3. Negate the statement
- Answers:

1.
$$D = \mathbb{R}$$

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

- 1. Give a D where the statement is true.
- 2. Give a D where the statement is false
- 3. Negate the statement
- Answers:
 - 1. $D = \mathbb{R}$
 - 2. $D = \mathbb{N}$

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

- 1. Give a D where the statement is true.
- 2. Give a D where the statement is false
- 3. Negate the statement
- Answers:
 - 1. $D = \mathbb{R}$
 - 2. $D = \mathbb{N}$
 - 3. (See next slide)

$$\sim (\forall x \in D) \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) [x < z < y] \right] \right] \equiv$$

$$\sim (\forall x \in D) \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right] \equiv$$

$$(\exists x \in D) \sim \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right]$$

$$\sim (\forall x \in D) \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right] \equiv$$

$$(\exists x \in D) \sim \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right]$$

$$\equiv (\exists x \in D) \left[(\exists y \in D) \sim \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right]$$

How do we negate an implication?

Negating implications

- Recall that $(p \Rightarrow q) \equiv (\sim p \lor q)$
- Therefore, $\sim (p \Rightarrow q) \equiv \sim (\sim p \lor q) \equiv p \land \sim q$ (by De Morgan's law and double negation)
- So the negation of an implication is a conjunction!
- Intuitive result: If all men are mortal, then we say

$$(\forall x \in D)[x \text{ is } a \text{ } man \Rightarrow x \text{ is } mortal]$$

• If we want to negate this, to say that there exists some man that is immortal, then we say:

$$(\exists x \in D)[(x \text{ is a man}) \land (x \text{ is not mortal})]$$

$$\sim (\forall x \in D) \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right] \equiv$$

$$(\exists x \in D) \sim \left[(\forall y \in D) \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right] \equiv$$

$$(\exists x \in D) \left[(\exists y \in D) \sim \left[(x < y) \Rightarrow (\exists z \in D) \left[x < z < y \right] \right] \right]$$

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$$(\exists x \in D) \left[(\exists y \in D) \left[(x < y) \land (\sim (\exists z \in D) \left[x < z < y \right] \right] \right] \equiv$$