Quantifiers

CMSC250

Reminders

- **Midterm grades**: You can expect them tomorrow (Friday 03-12) midnight the latest.
  - Through Friday 03-19 midnight you can submit regrades.
- HW4 grades have been posted; check for any regrade requests you might want to make. Usual Friday 11:59pm deadline for those.

Reminders

- HW5 posted last Monday, due Monday 03-22 11:59pm
  - Normal length, though.
  - Please work on it before Monday 22nd ☹
- Tomorrow: Quiz 6
  - Due also after Springbreak (Monday 03-22 11:59pm).
  - Normal, miniscule, length.
- Slides, textbook chapters and select exercises for material of week of 03-22 available since end of last week.
  - Big-Oh notation
  - Intro to proofs (big cheese in the class)

Existential / universal quantifier

- There exist two quantifiers in formal logic / set theory
  - The universal quantifier: ∀ (read "for all")
  - The existential quantifier: ∃ (read "exists")
- We will see that every quantifier needs a set associated with it, so general syntax of **quantified expressions** will be:

(Quantifier variable ∈ Set)[Some property on variable]

"There exists" (∃)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$

“There exists” ($\exists$)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True

"There exists" (∃)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$

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  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain *D* where $(\exists n \in D)[n^2 = -1]$ is true?

Yes

No

Something else

"There exists" ($\exists$)

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  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain *D* where $(\exists n \in D)[n^2 = -1]$ is true?

The complex numbers $\mathbb{C}$

Yes

No

Something else

“For all”

- The symbol ∀ *(LaTeX: \forall)* is read “for all”.
- Examples:
  - $(\forall x \in \mathbb{N})\ [((x > 2) \wedge (x\ is\ prime)) \Rightarrow (x\ is\ odd)]$

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“For all”

- Let D be the set of all students in this class who are over 8 feet tall.
- $(\forall x \in D)[x\ has\ perfect\ attendance\ so\ far!]$

True

False

Something else

“For all”

- Let D be the set of all students in this class who are over 8 feet tall.
- $(\forall x \in D)[x\ has\ perfect\ attendance\ so\ far!]$

True

False

Something else

- If disagree, need to find $x \in D$ who missed a class
- Called vacuously true!

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$

Nesting quantifiers

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Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

True, $x = \frac{4}{5}, y = \frac{8}{5}$

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

True, $x = \frac{4}{5}, y = \frac{8}{5}$

- Common abbreviation: $(\exists x, y \in D)[\dots]$
- Generally: $(\exists x_1, x_2, \dots, x_n \in D)[\dots]$

Alternating nested quantifiers

- Notice the differences between the following:
  - $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$
  - $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$

Alternating nested quantifiers

- Notice the differences between the following:
  - $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$ True ($\mathbb{N}$ unbounded from above)
  - $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$ False ($\mathbb{N}$ bounded from below)

- ***WHEN QUANTIFIERS ARE DIFFERENT, THEIR ORDER MATTERS!!!!!!!***

Fill this in!

Statement	True	False
$(\exists n \in \mathbb{N})[n+n=0]$	○	○
$(\exists n \in \mathbb{N})[n+n=1]$	○	○
$(\exists n \in \mathbb{Z})[n+n=1]$	○	○
$(\exists x, y \in \mathbb{Z})[x+y=1]$	○	○
$(\exists x \in \mathbb{R})[x(x+1)=-1]$	○	○
$(\forall x, y, z \in \mathbb{R})[((x<y<z) \Rightarrow (x^2<y^2<z^2)]$	○	○
$(\forall x, y, z \in \mathbb{R})[((x<y<z) \Rightarrow (x^3<y^3<z^3)]$	○	○
$(\forall x, y \in \mathbb{N})(\exists z \in \mathbb{Q})[(x<y) \Rightarrow (x<z<y)]$	○	○

Finding domains

- Give infinite sets *D* such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
    1. Is true ($D = \mathbb{N}$, select $y = x + 1$)

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)

Finding domains

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  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \leq 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

Finding domains

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  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \leq 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

1. True for $D = (-\infty, 1)$

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \leq 1] \wedge (\forall x \in D)(\exists y \in D)[x < y]$$

1. True for $D = (-\infty, 1)$
2. False for $D = (-\infty, 1]$ (!)

Negated Quantifiers

- It is not the case that Alice comes to all office hours.
- There is an office hour that Alice does not come to.

Logically Equivalent

Not Logically Equivalent

Something else (what?)

Negated Quantifiers

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- There is an office hour that Alice does not come to.

Logically Equivalent

Not Logically Equivalent

Something else (what?)

Negated Quantifiers

- We can therefore reach the following logical equivalences:

$$\sim (\forall x \in D)[P(x)] \equiv (\exists x \in D)[\sim P(x)]$$

$$\sim (\exists x \in D)[P(x)] \equiv (\forall x \in D)[\sim P(x)]$$

Negating nested quantifiers

- Observe how we can negate

$$(\forall x \in D)[(\exists y \in D)[x < y]]$$

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$$\sim (\forall x \in D)\big[(\exists y \in D)[x < y]\big] \equiv$$

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$$\sim (\forall x \in D)\big[(\exists y \in D)[x < y]\big] \equiv$$
$$(\exists x \in D)\big[\sim (\exists y \in D)[x < y]\big] \equiv$$

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$$\sim (\forall x \in D)\big[(\exists y \in D)[x < y]\big] \equiv$$
$$(\exists x \in D)\big[\sim (\exists y \in D)[x < y]\big] \equiv$$
$$(\exists x \in D)\big[(\forall y \in D)[\sim (x < y)]\big] \equiv$$

Negating nested quantifiers

- Observe how we can negate

$$(\forall x \in D)[(\exists y \in D)[x < y]]$$

$$\begin{aligned}
&\sim (\forall x \in D)\big[(\exists y \in D)[x < y]\big] \equiv \\
&(\exists x \in D)\big[\sim (\exists y \in D)[x < y]\big] \equiv \\
&(\exists x \in D)\big[(\forall y \in D)[\sim (x < y)]\big] \equiv \\
&(\exists x \in D)\big[(\forall y \in D)[x \geq y]\big]
\end{aligned}$$

Another example

- A set is **dense** if between any two elements of it there exists another element of it.
- Observe the statement

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

- This statement says: D is dense.

Another example

- Observe the statement

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

1. Give a D where the statement is true.
2. Give a D where the statement is false.
3. Negate the statement.

Another example

- Observe the statement

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

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- Answers:

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- Answers:
  1. $D = \mathbb{R}$

Another example

- Observe the statement

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- Answers:
  1. $D = \mathbb{R}$
  2. $D = \mathbb{N}$

Another example

- Observe the statement

$$(\forall x \in D)[(\forall y \in D)[(x < y) \Rightarrow (\exists z \in D)[x < z < y]]]$$

1. Give a D where the statement is true.
2. Give a D where the statement is false
3. Negate the statement

- Answers:
  1. $D = \mathbb{R}$
  2. $D = \mathbb{N}$
  3. (See next slide)

Negating the statement

$$\sim (\forall x \in D)\Big[(\forall y \in D)\big[(x < y) \Rightarrow (\exists z \in D)[x < z < y]\big]\Big] \equiv$$

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$$\equiv (\exists x \in D)\Big[(\exists y \in D) \sim \big[\underbrace{(x < y) \Rightarrow (\exists z \in D)[x < z < y]}\big]\Big]$$

How do we negate an implication?

Negating implications

- Recall that $(\boldsymbol{p} \Rightarrow \boldsymbol{q}) \equiv (\sim \boldsymbol{p} \vee \boldsymbol{q})$
- Therefore, $\sim (p \Rightarrow q) \equiv \sim (\sim p \vee q) \equiv p \wedge \sim q$ *(by De Morgan's law and double negation)*
- So the negation of an implication is a **conjunction**!
- **Intuitive result**: If all men are mortal, then we say

$$(\forall x \in D)[x \text{ } is \text{ } a \text{ } man \Rightarrow x \text{ } is \text{ } mortal]$$

- If we want to negate this, to say that **there exists some man that is immortal**, then we say:

$$(\exists x \in D)[(x \text{ } is \text{ } a \text{ } man) \wedge (x \text{ } is \text{ } not \text{ } mortal)]$$

AND!

Back to our example

$$\sim (\forall x \in D)\Big[(\forall y \in D)\big[(x < y) \Rightarrow (\exists z \in D)[x < z < y]\big]\Big] \equiv$$

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$$\equiv (\exists x \in D)\big[(\exists y \in D)[(x < y) \land (\forall z \in D)(z \leq x) \lor (z \geq y)]\big]$$