List arrays and Skiplists

CMSC 420

Central question

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  - When you have n nodes, you scan $\frac{n}{2}$ nodes to find the middle element.
  - When you have $\frac{n}{2}$, you scan $\frac{n}{4}$….
  - When you have $\frac{n}{4}$, you scan $\frac{n}{8}$….
  - …
  - When you have $1 = \frac{n}{n} = \frac{n}{2^{\log_2 n}}$, you scan $1 = \frac{n}{\log_2 n}$ nodes….

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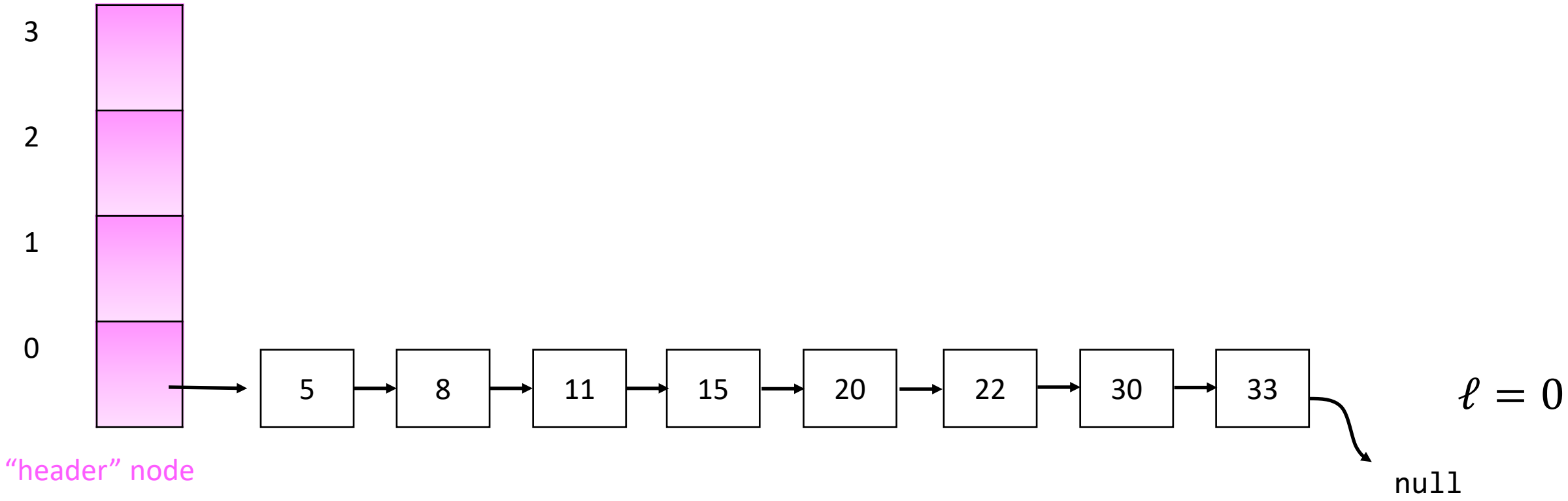
  - When you have $\frac{n}{2}$, you scan $\frac{n}{4}$….
  - When you have $\frac{n}{4}$, you scan $\frac{n}{8}$….
  - …
  - When you have $\frac{n}{2^{\log_2 n}}$, you scan $\frac{n}{n} = 1$ nodes….

Total #nodes scanned = $\frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^{\log_2 n}} = n\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{\log_2 n}}\right) = \mathcal{O}(n)$

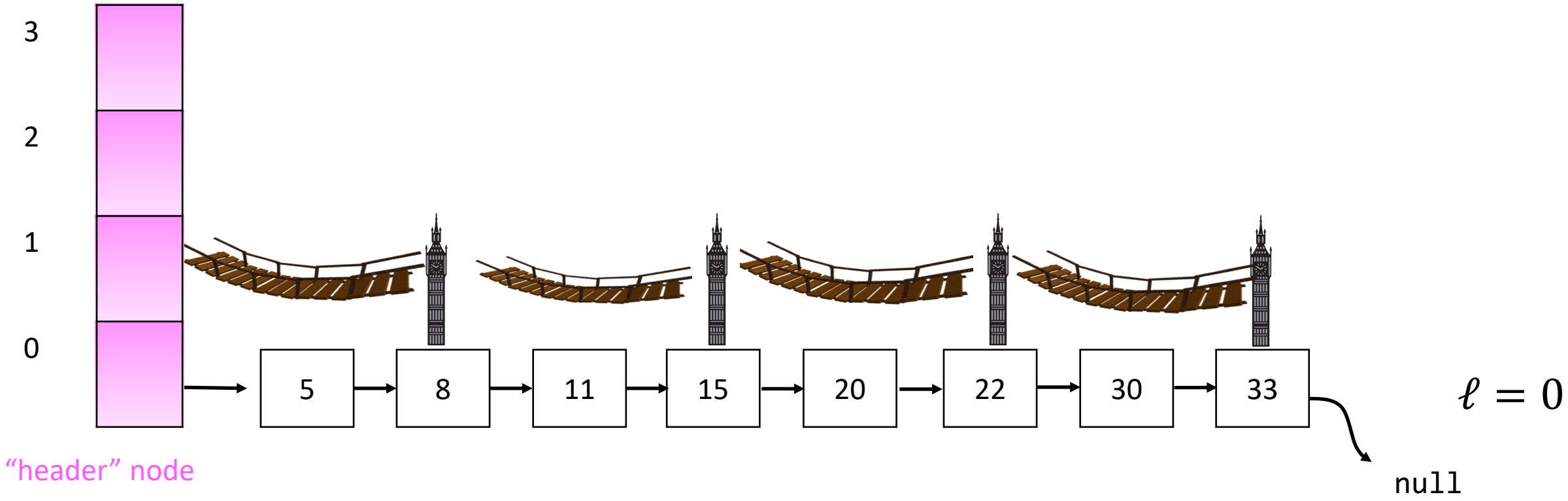
Central question

- Idea to improve complexity: Consider every node to be a "tower" consisting of a certain number of levels, different for each node.
  - The levels themselves will be connected by linked lists
    - Think of them as bridges!
  - The "tallest" tower will have height $\log_2 n$!
  - First level will be the original list: connects every $2^0 = 1$ nodes.
  - Second level will connect every $2^1 = 2$ nodes.
  - ….
  - i^{th} level will connect every 2^i nodes.

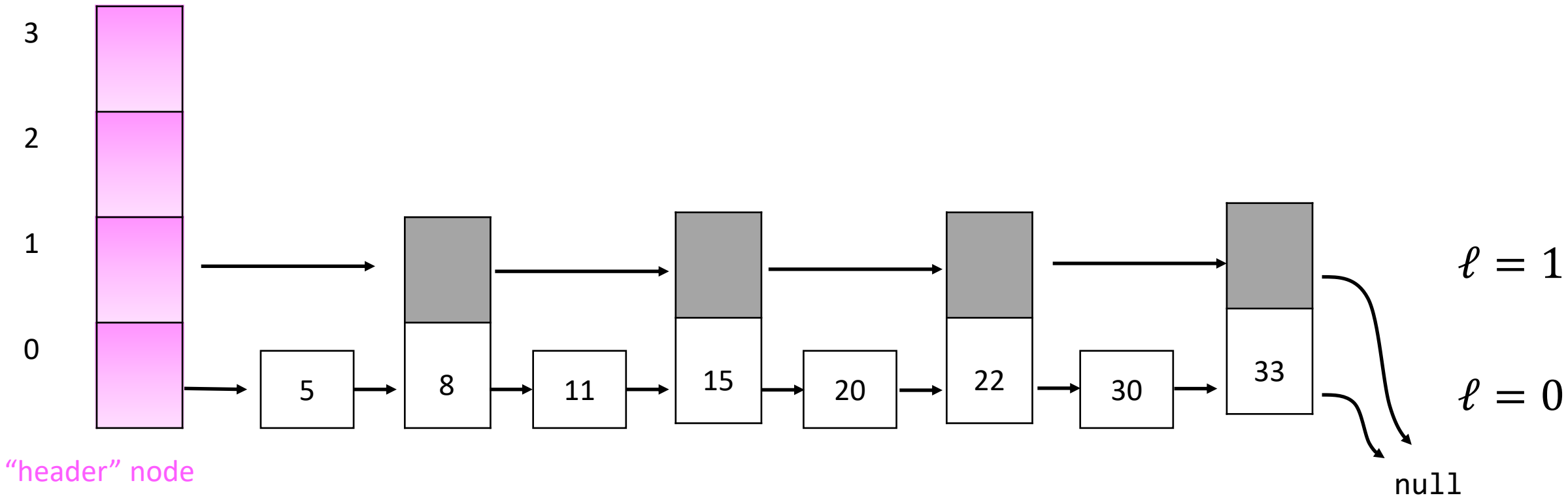
List Arrays



List Arrays

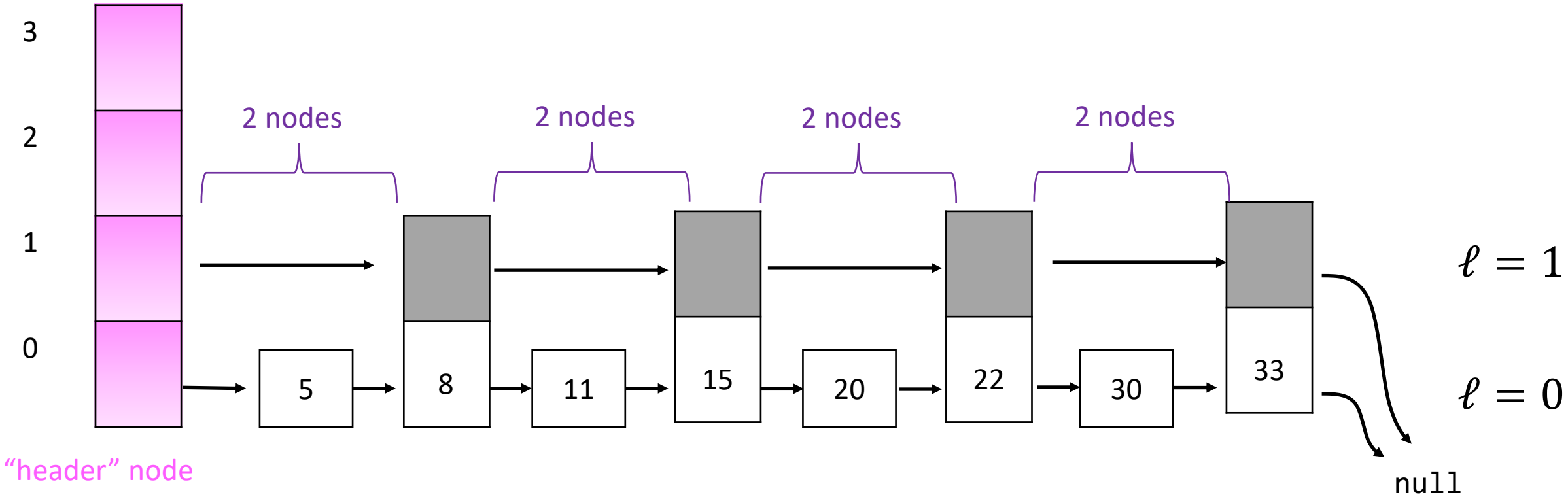


List Arrays

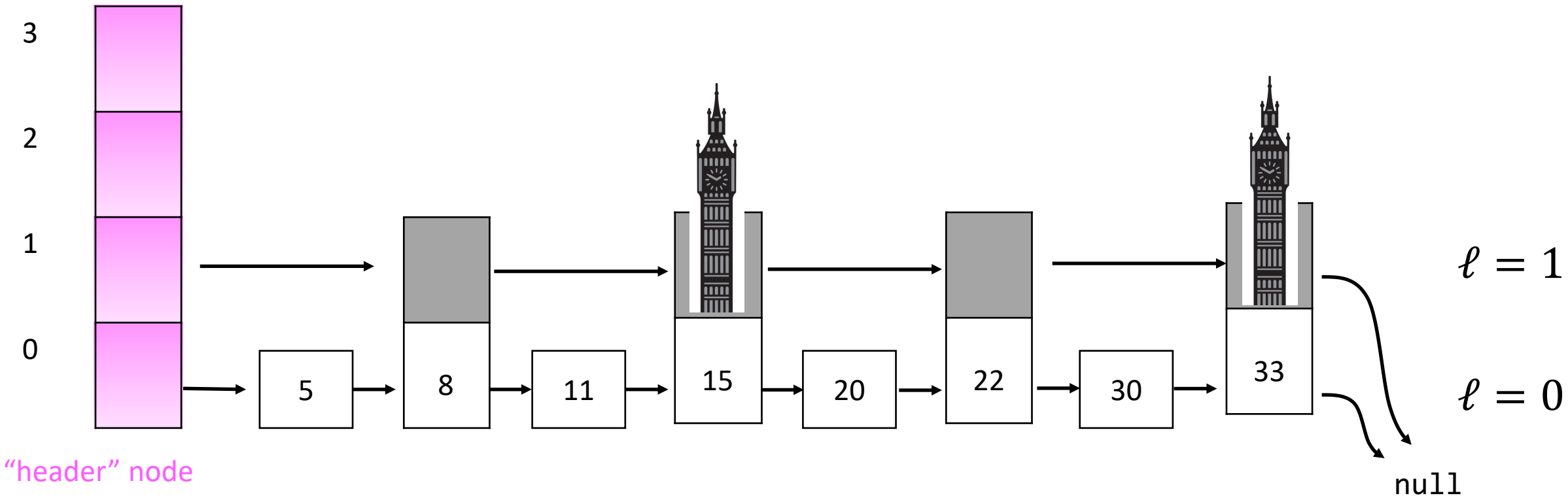


List Arrays

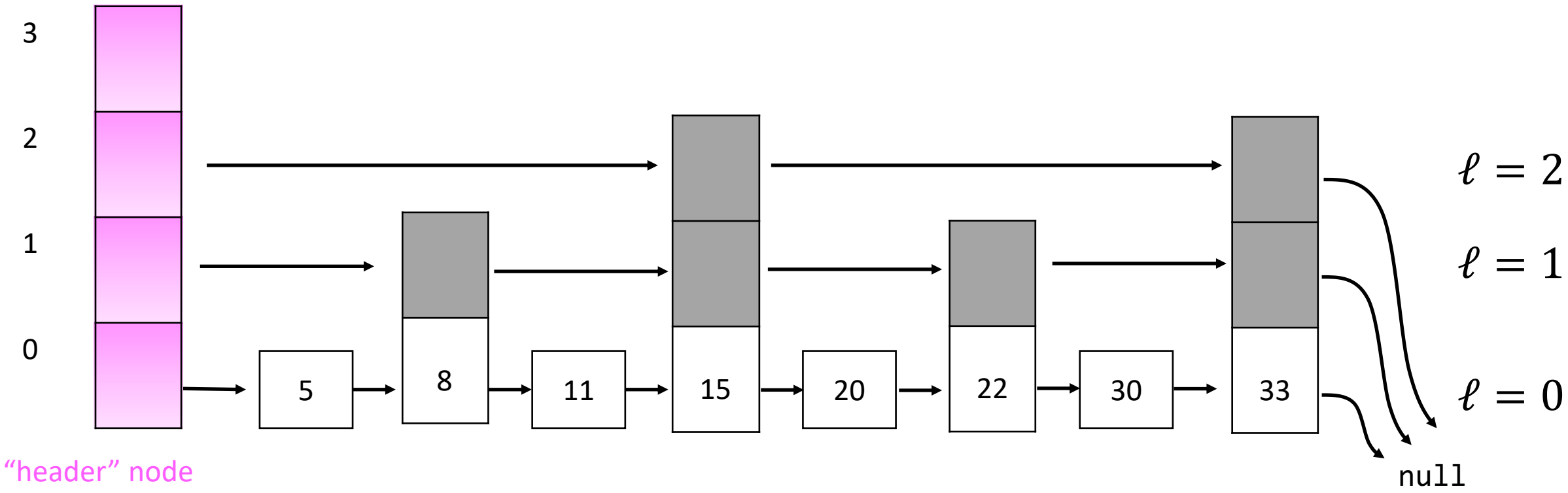
Second level ($\ell = 1$) connects every 2^1 = 2 nodes!



List Arrays

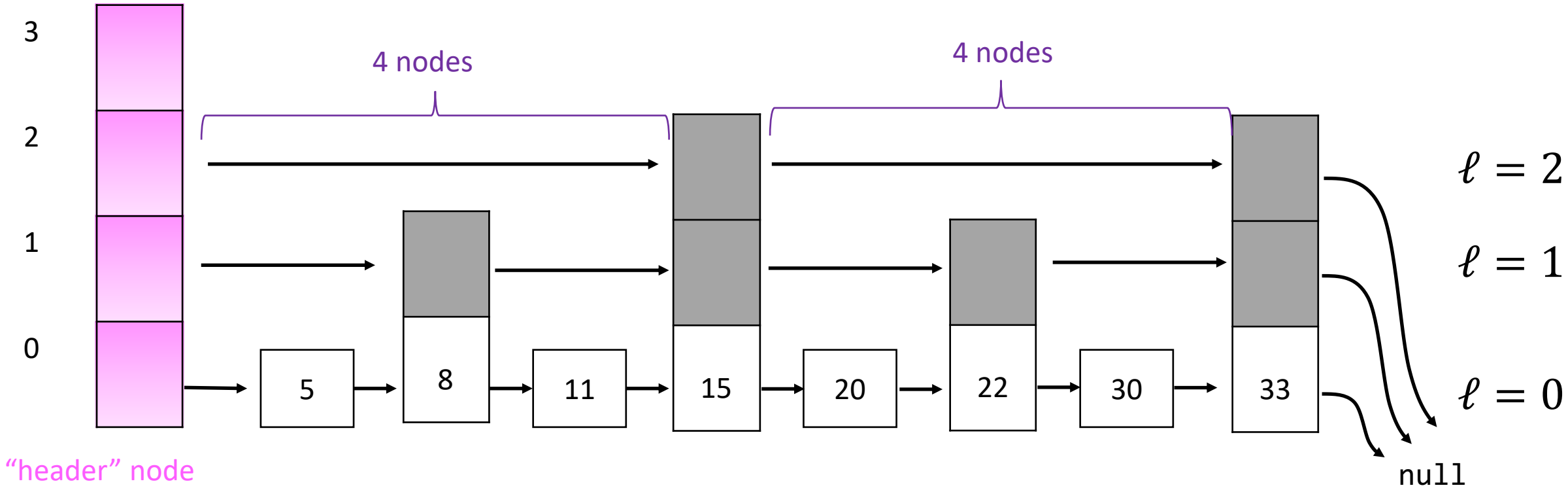


List Arrays

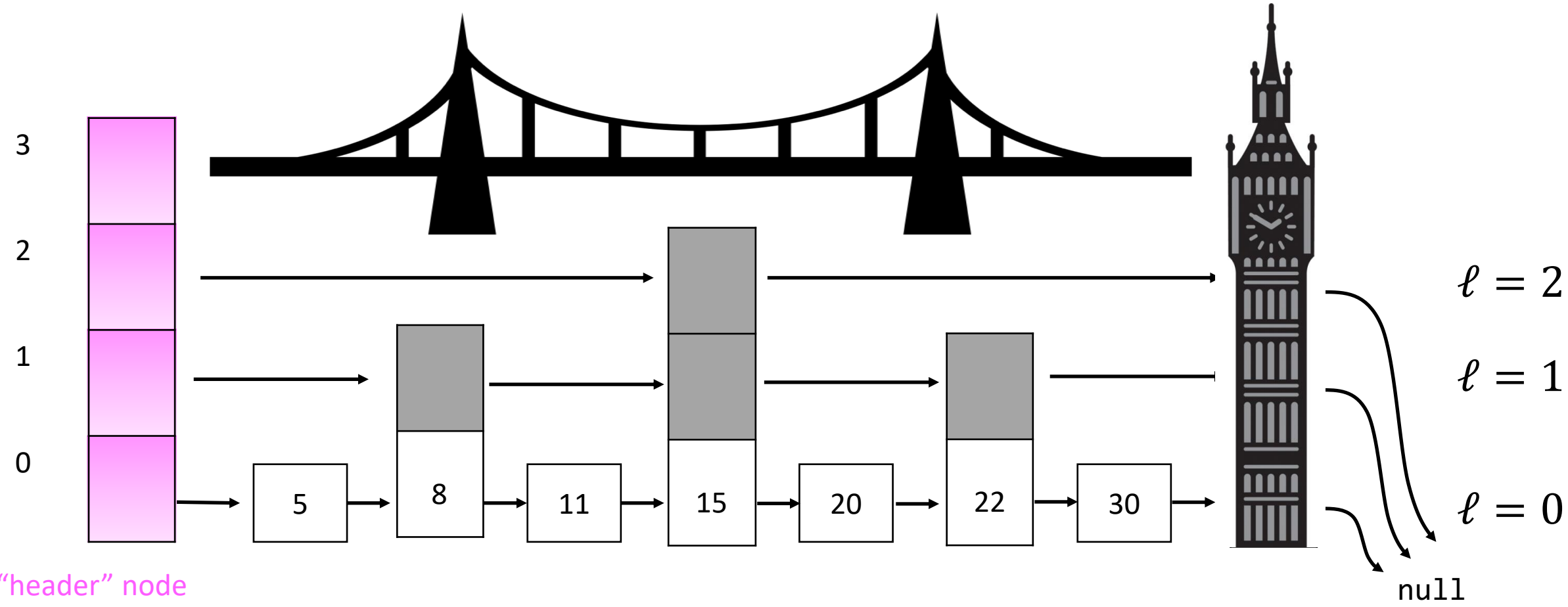


List Arrays

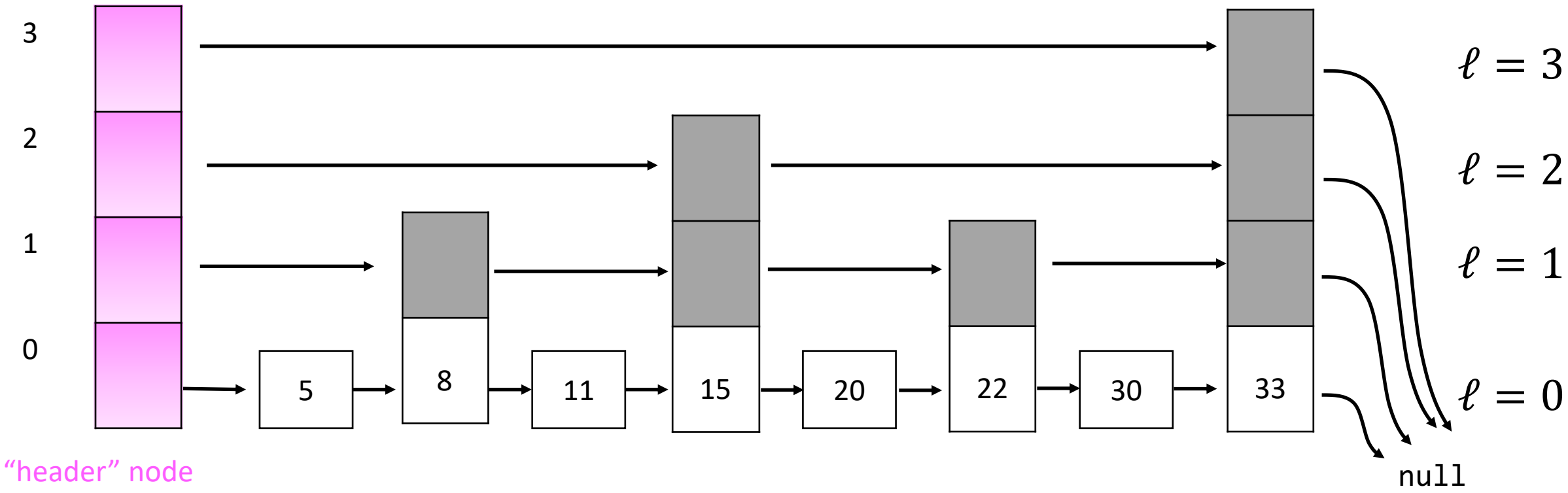
Third level ($\ell = 2$) connects every 2^2 = 4 nodes!



List Arrays

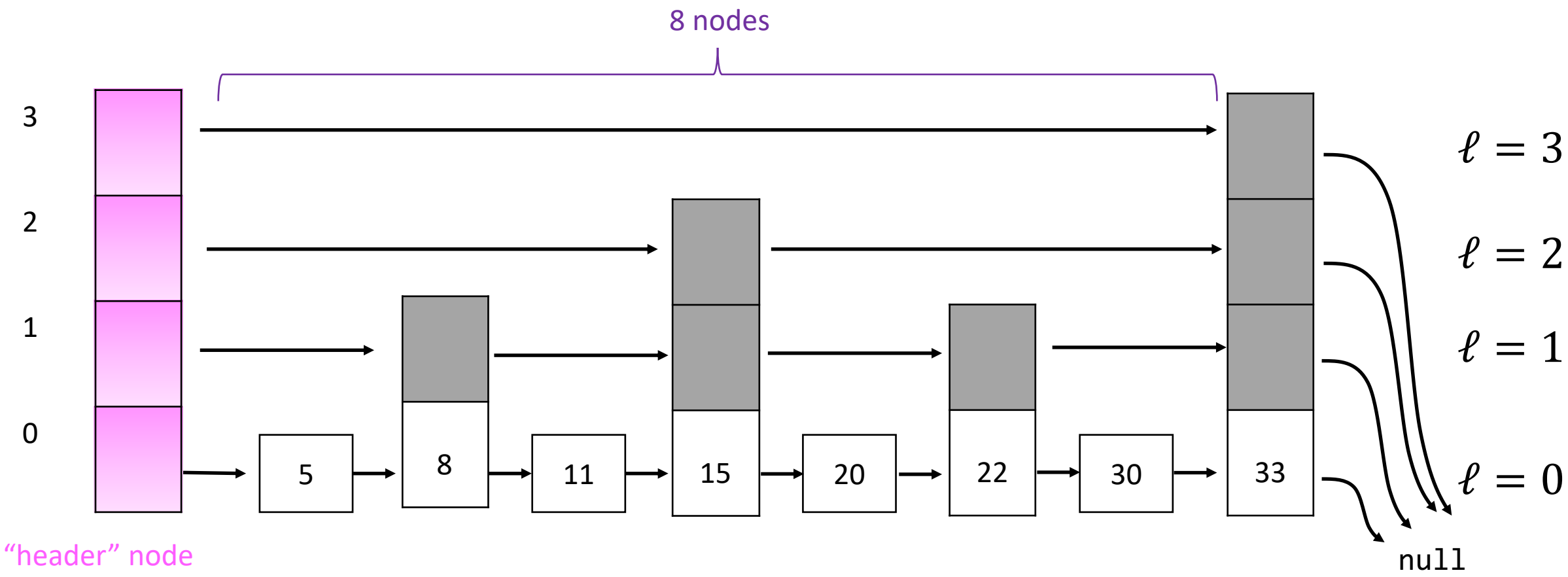


List Arrays



List Arrays

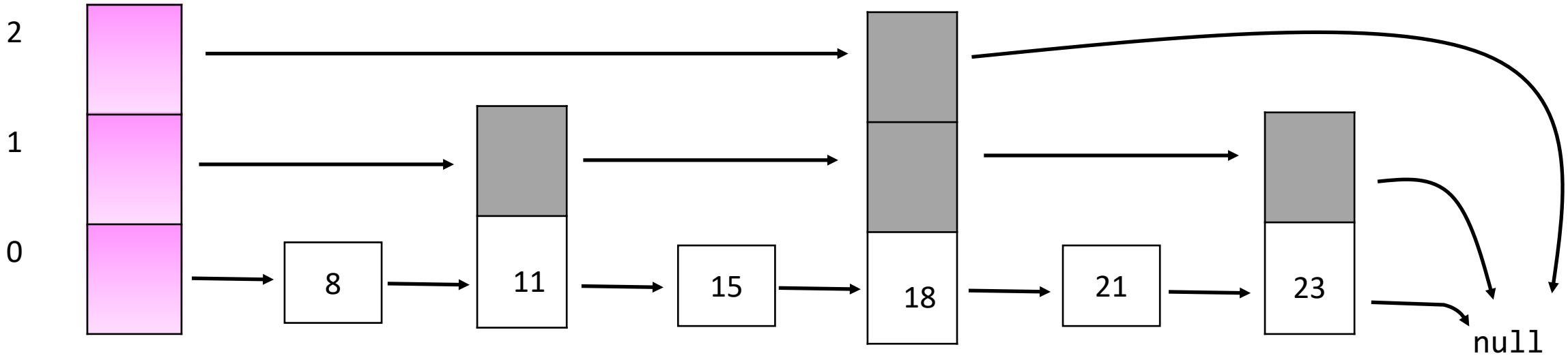
Fourth level ($\ell = 3$) connects every 2^3 = 8 nodes!



Extension to other lengths

- What if my list has a length that is not a power of 2?
- No problem: we then have $\lceil \log_2 n \rceil$ levels. Example:

$$(n = 6) \Rightarrow (\lceil \log_2 n \rceil = 3)$$

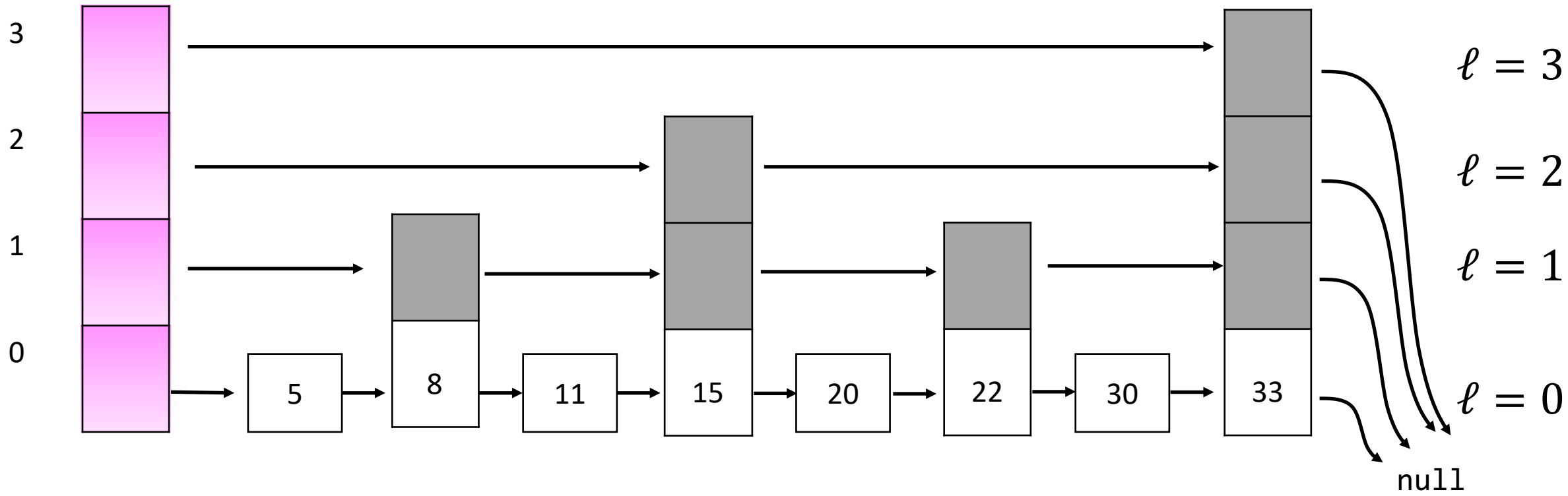


Searching a list array

- Every one of the individual arrays has the data element stored at position 0.
- Searching works as follows:
  1. Begin at `header[levels - 1]`. Traverse the current level until:
     a) You find the element (search **hit**, congrats)
     b) You find a tower whose element is greater, or:
     c) You hit `null`.
  2. If either (b) or (c) happens, descend to the level below and goto (1).
     - If there is no level below, you have a search **miss**. At this point you should consider how this makes you feel.

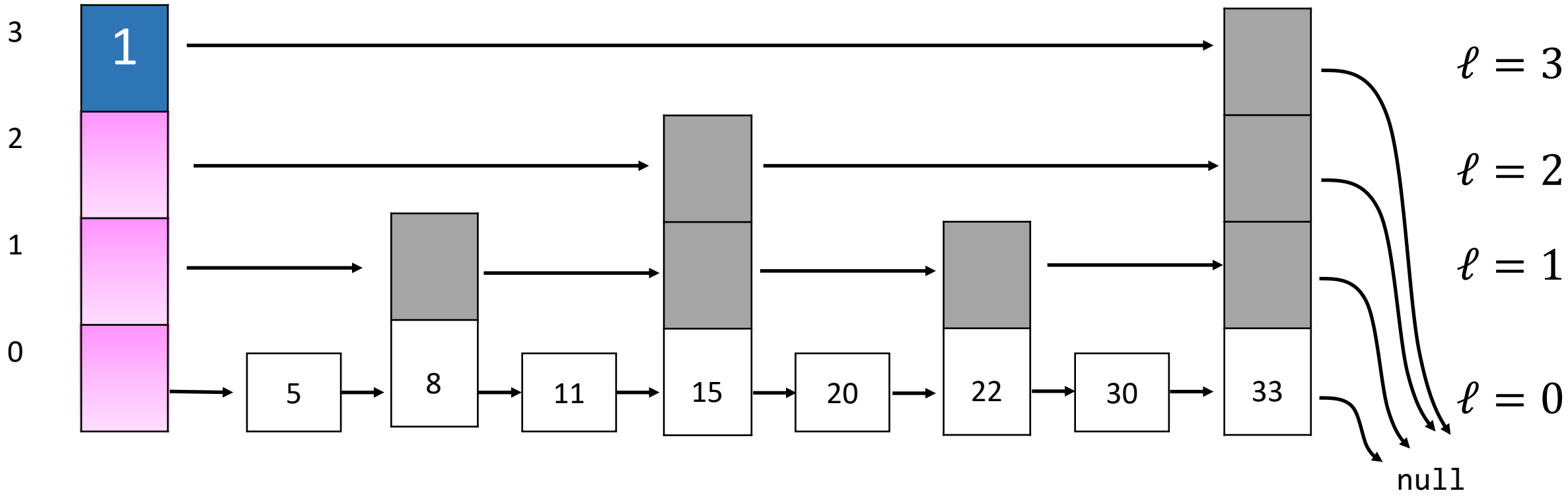
Search example

Search for “20” – should be a successful search



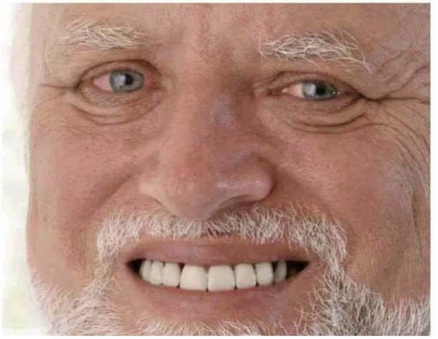
Search example

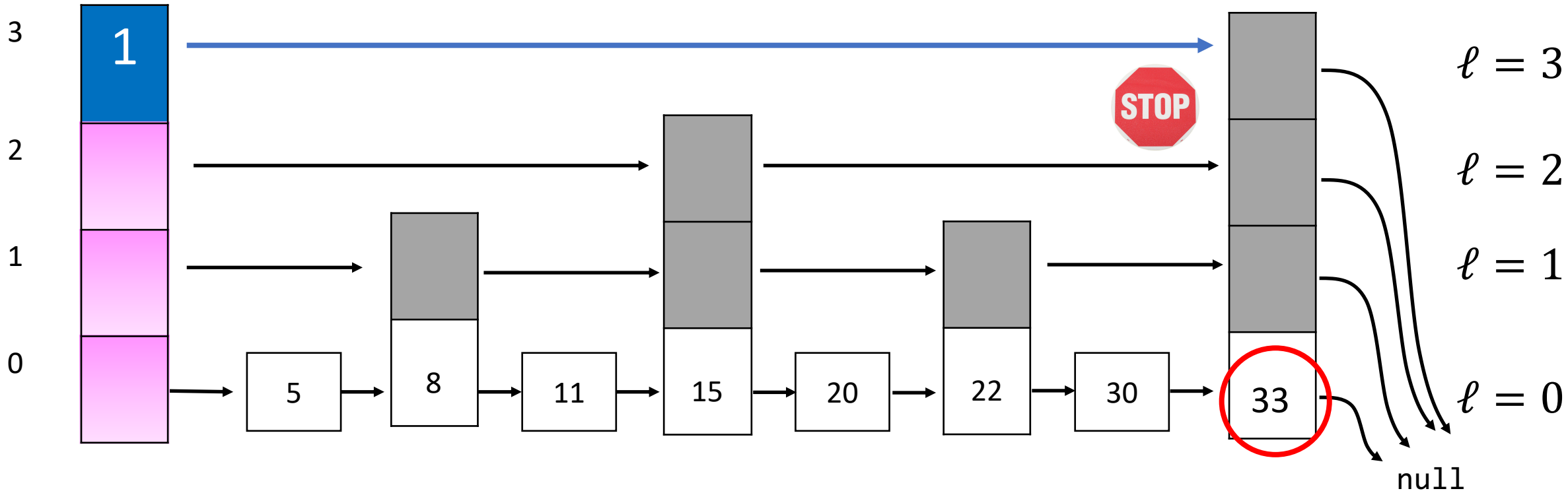
Start at header[levels-1]…



Search example

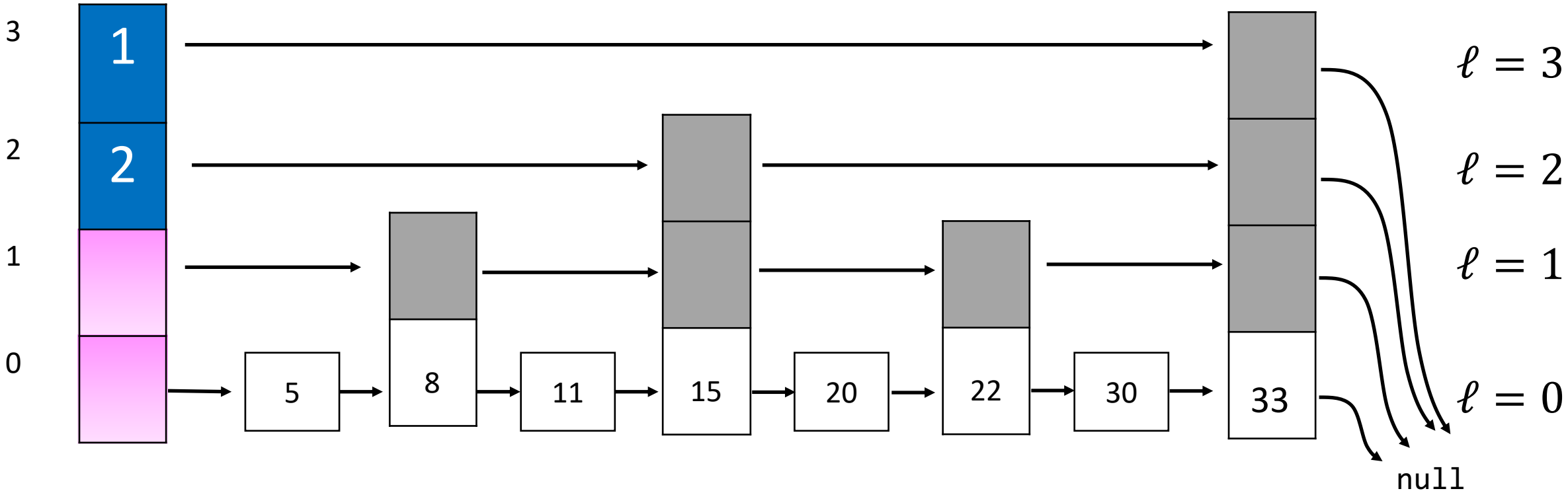
The pointer points to a tower with element greater than 20





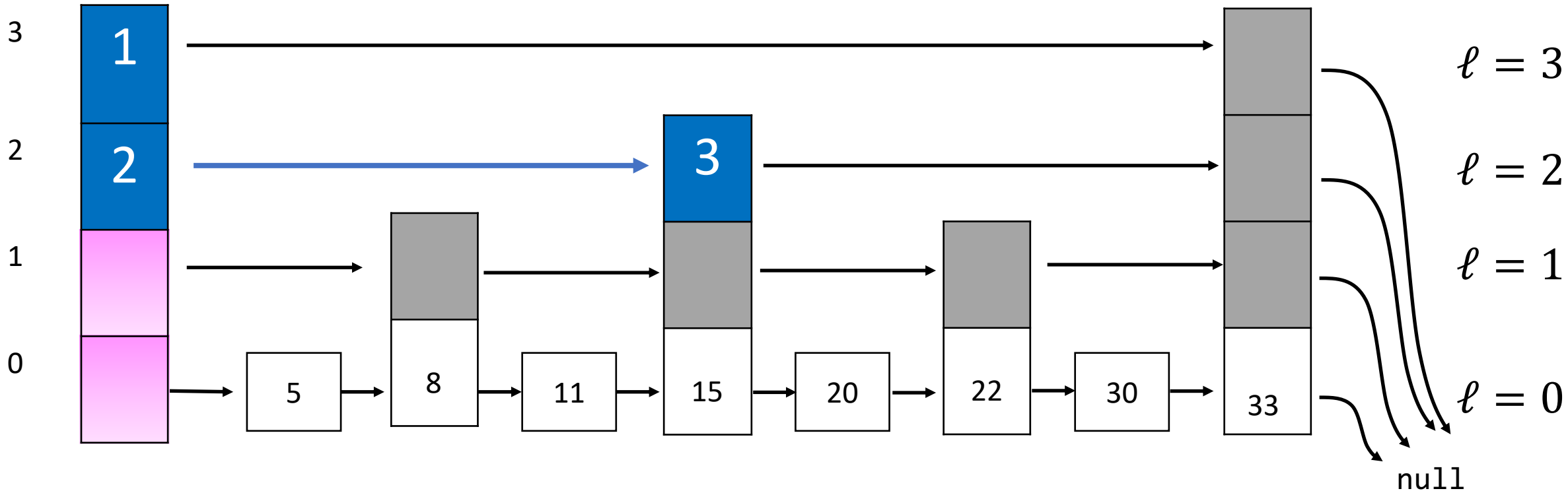
Search example

Solution: decrease your level!



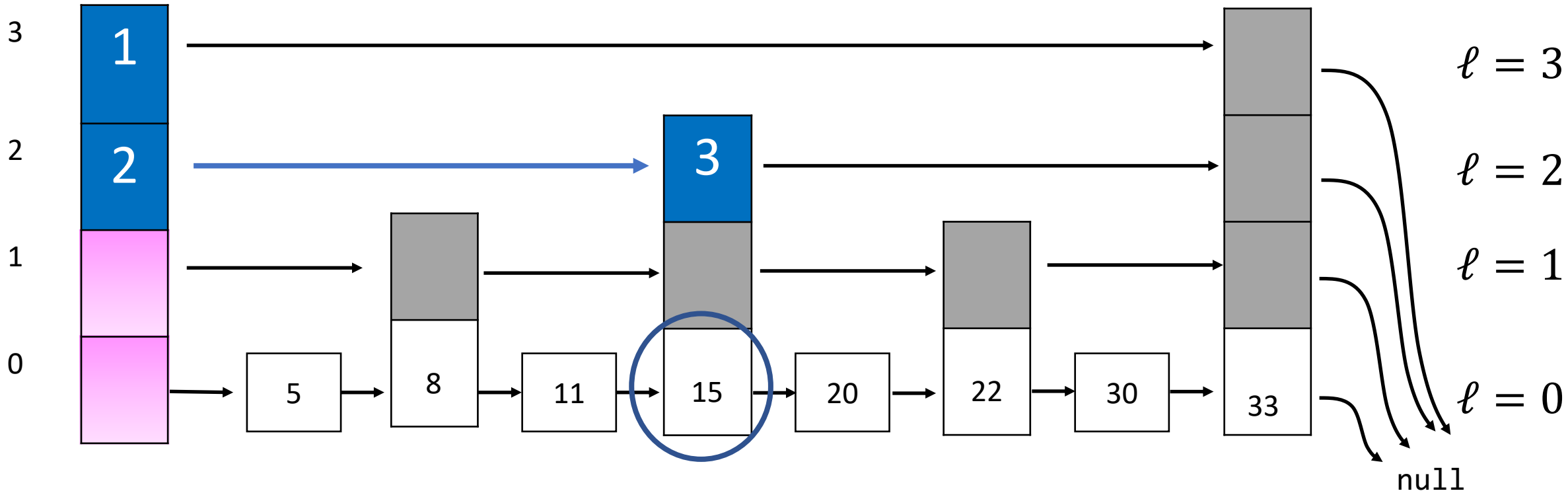
Search example

Dereference pointer.



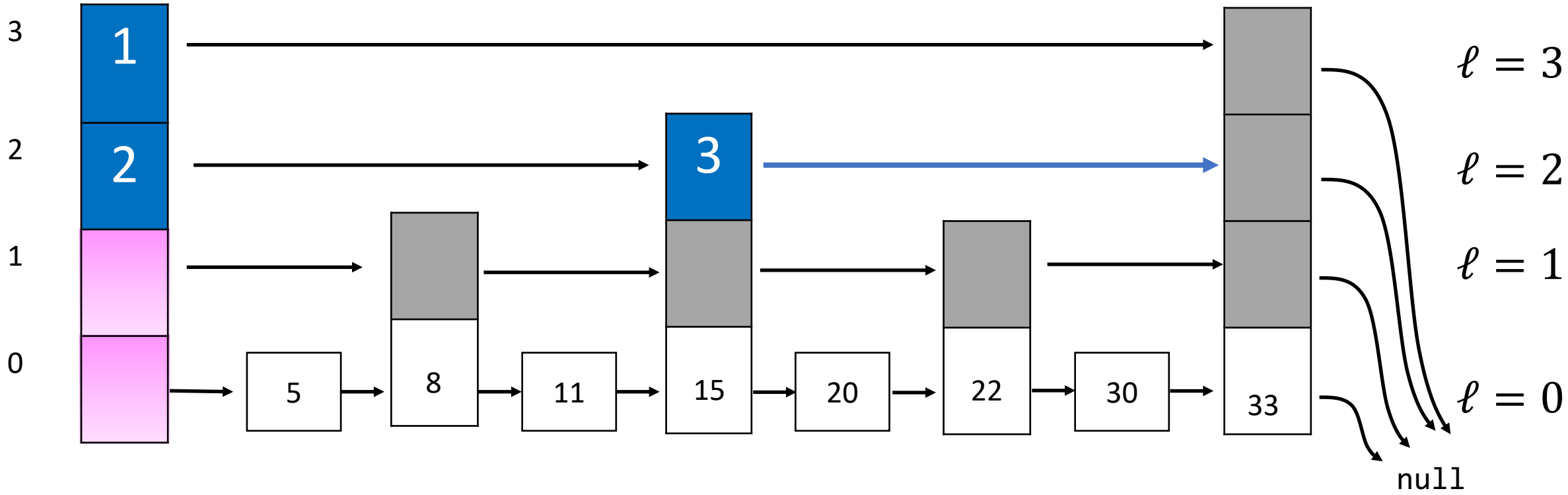
Search example

*Key (15) **smaller** than desired key (20)*



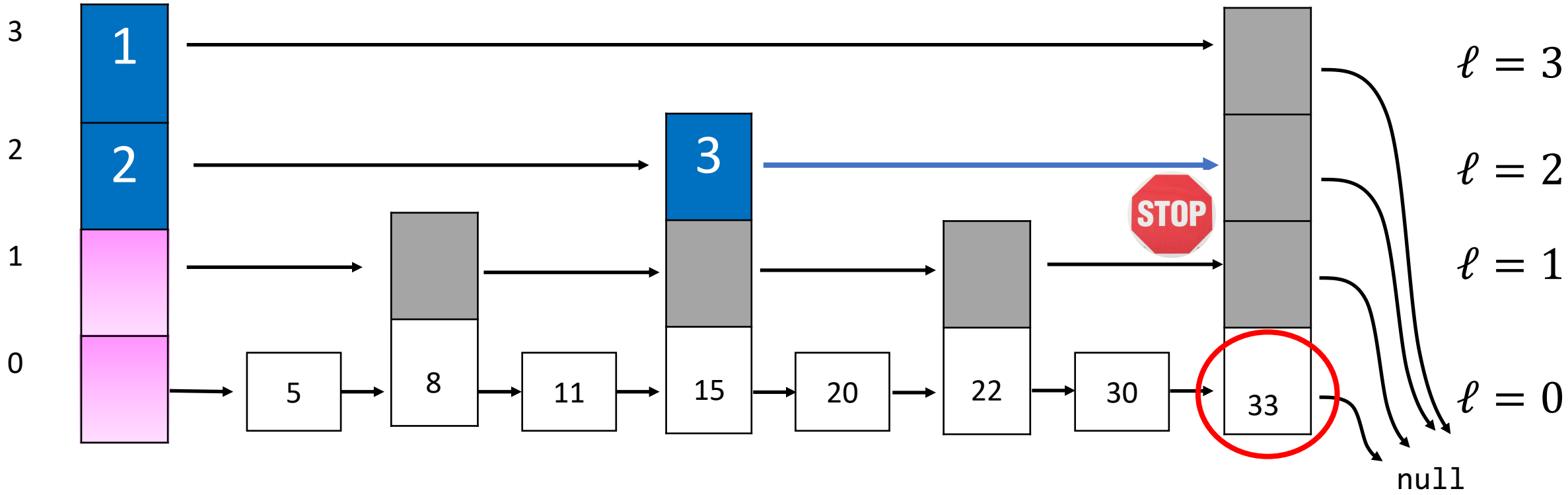
Search example

So keep going on the same level!



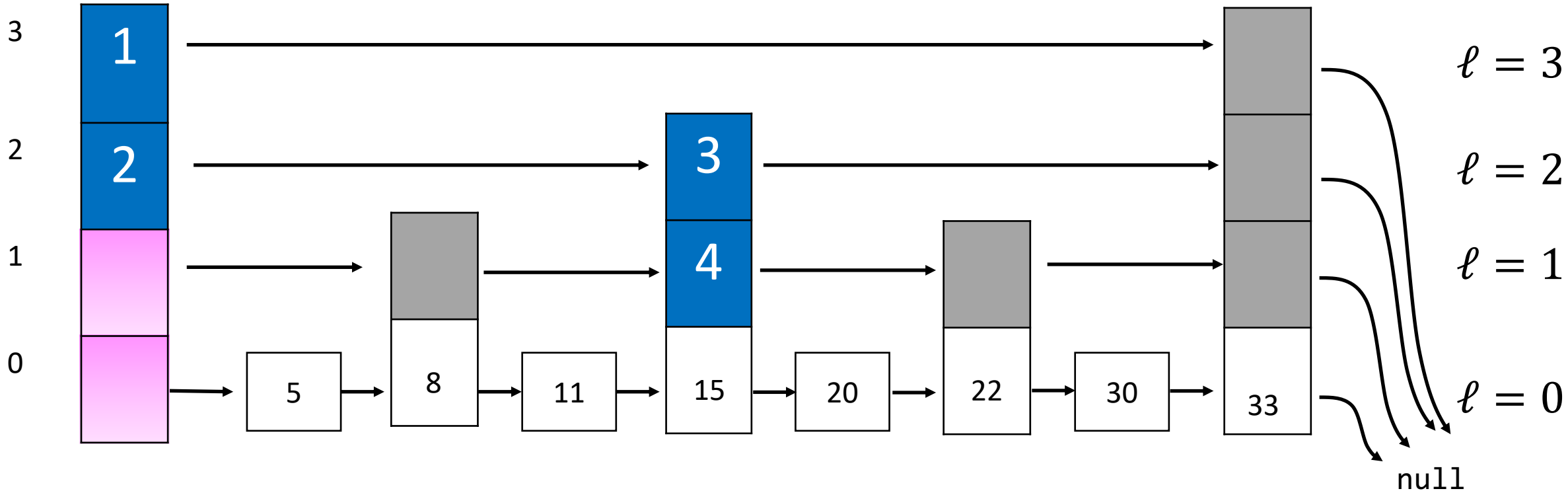
Search example

Contained key (33) **greater** than sought key (20) again!



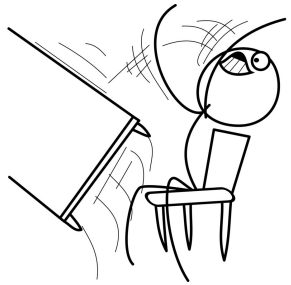
Search example

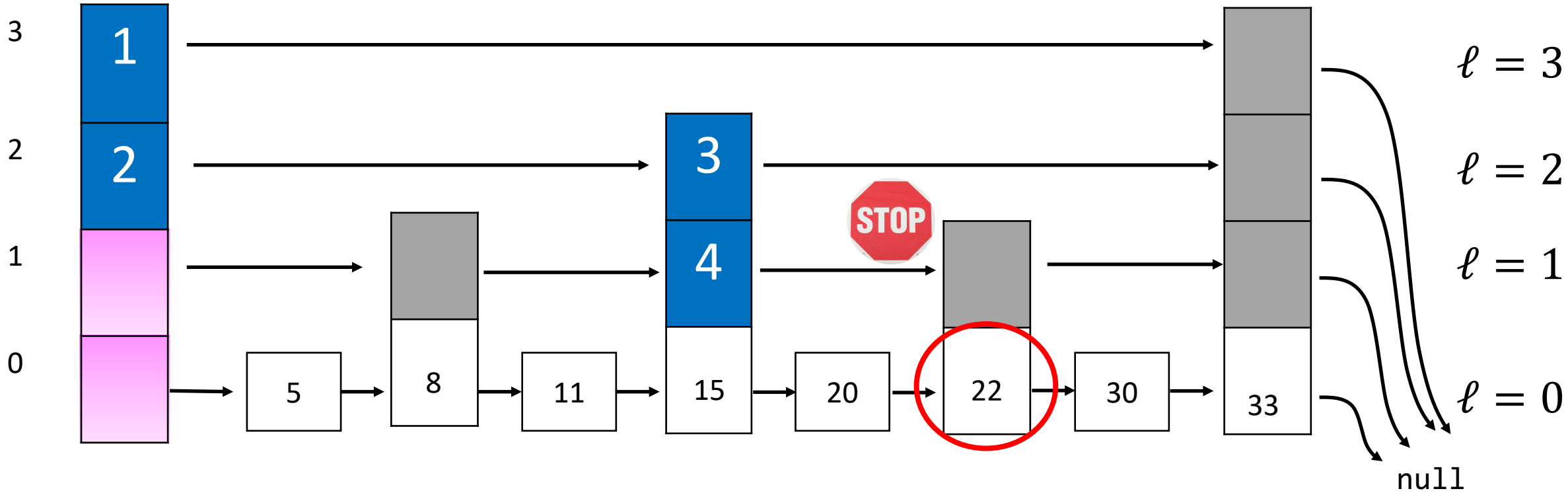
So descend a level again!



Search example

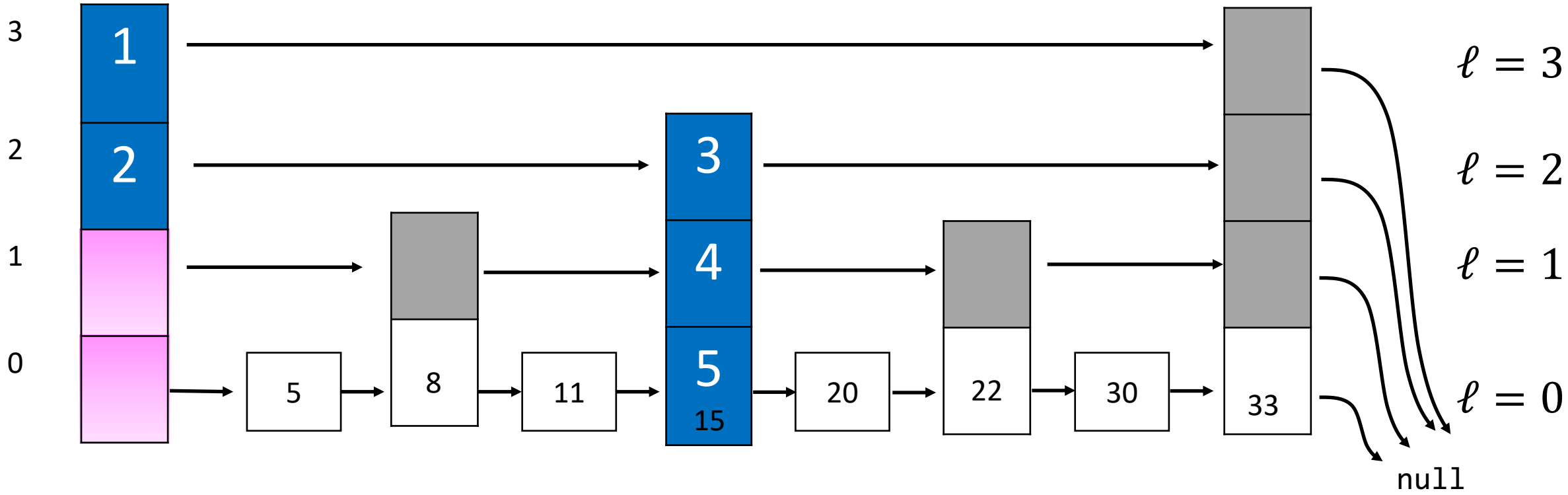
Dereferencing pointer leads us to a tower with a key too big…





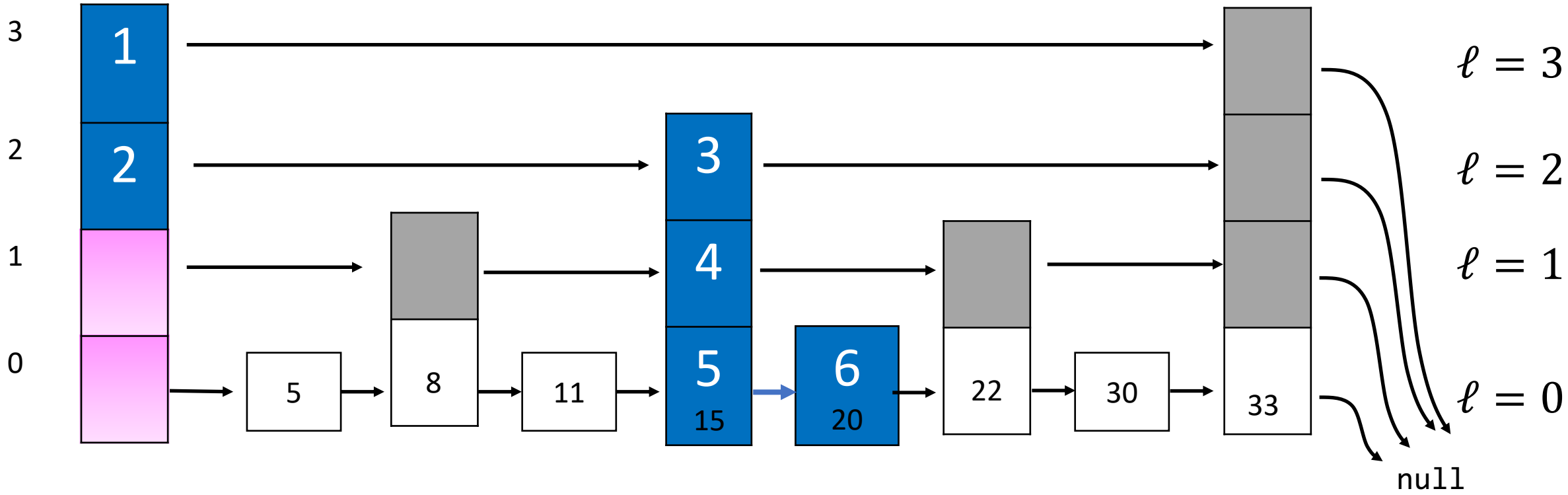
Search example

So descend another level…



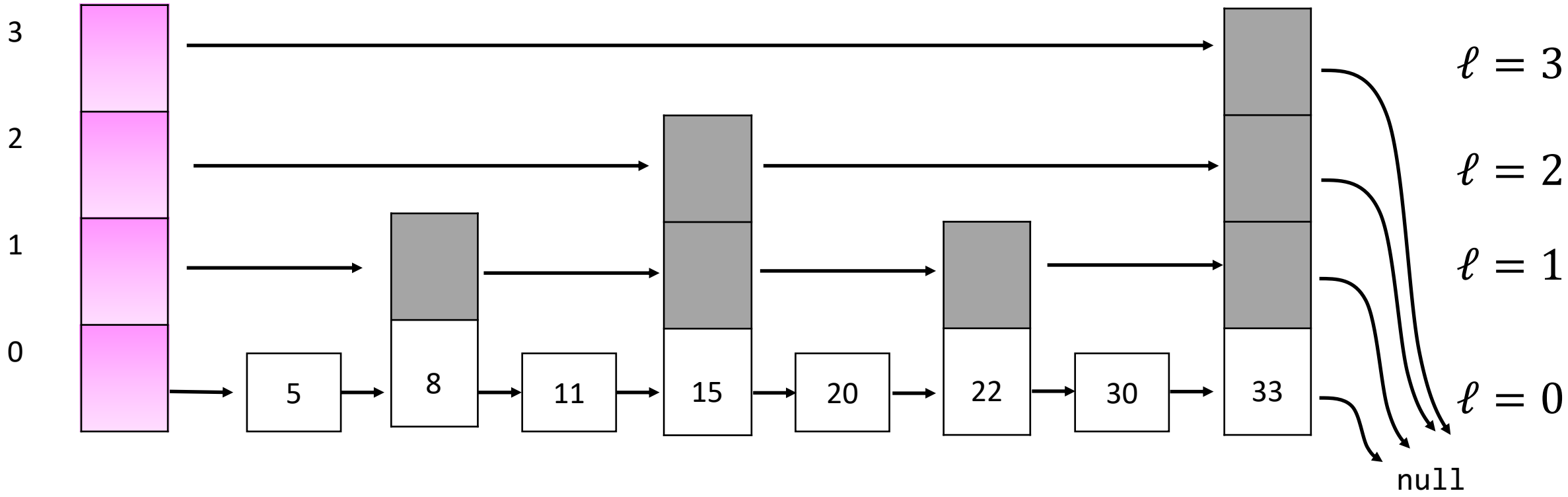
Search example

Final pointer dereference gives us our key. Search **hit.**

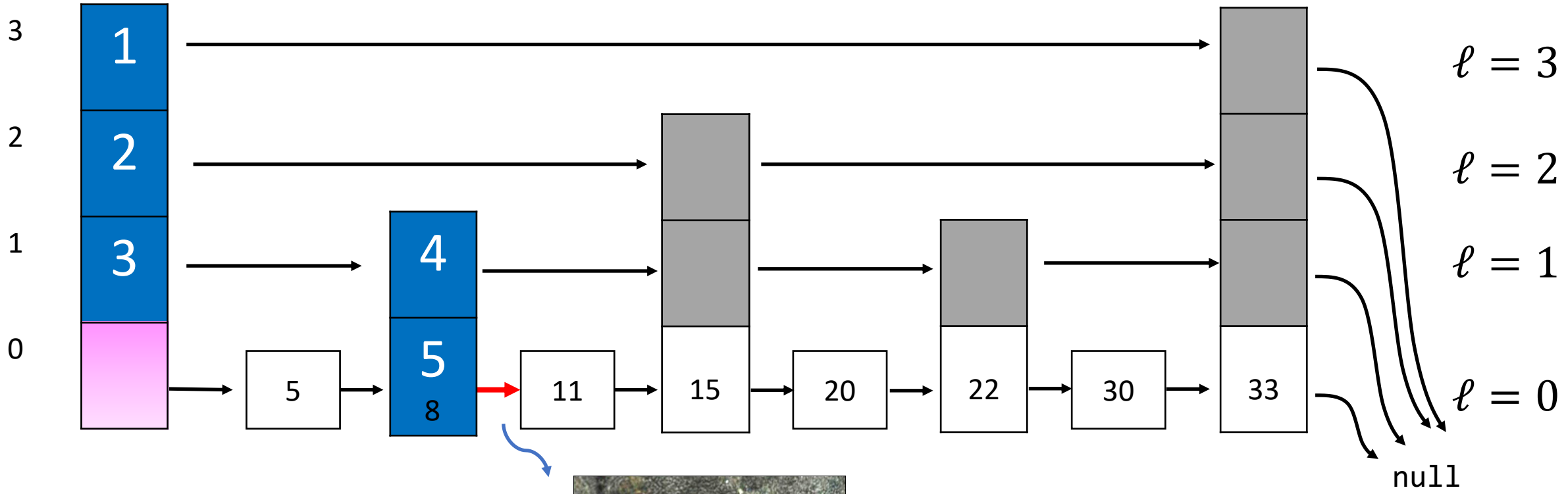


Another search example – for you!

- For the same list array, sequentially mark the nodes traversed by searching for 9 (nine), which should fail!



Solution



Space analysis

- To make matters simpler, assume that n is some power of 2.
  - So, $n/2$ will be an integer, as will be $n/4 = n/2^2$, $n/8 = n/2^3$, etc
- Alongside the list level, we have $\log_2 n + 1$ levels.
- The list level (0^{th} level) has n nodes.
- The 1st level has $n/2$ nodes.
- The 2nd level has $n/4$ nodes.
- …
- …
- Level # $\log_2 n$ has 1 nodes

Total # nodes:

$$n + \frac{n}{2} + \frac{n}{4} + \cdots + 1 = n\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots \frac{1}{n}\right) =$$

$$= 2^{-(\log_2 n - 1)}$$

$$= n(2^0 + 2^{-1} + 2^{-2} + \cdots + 2^{-\log_2 n})$$

Sum of geometric progression with $a_0 = 2^0 = 1$, $r = 1/2$ and $\log_2 n + 1$ terms!

Unpacking the sum…

- $2^0 + 2^{-1} + 2^{-2} + \cdots + 2^{-(\log_2 n)} = \frac{2^0\left[1-(1/2)^{\log_2 n+1}\right]}{1-1/2} = 2\left[1 - (1/2)^{\log_2 n+1}\right]$
- $(1/2)^{\log_2 n+1} = \frac{1}{2} \cdot \frac{1}{2^{\log_2 n}} = \frac{1}{2n}$
- Substituting the green stuff onto the red stuff yields:
- $2^0 + 2^{-1} + 2^{-2} + \cdots + 2^{-(\log_2 n)} = 2\left(1 - \frac{1}{2n}\right) = \frac{2n-1}{n}$
- Note that this tends to 2 as n goes to infinity, which is no surprise since the geometric series $\sum_{i=0}^{n} 2^{-i}$ converges to $\frac{1}{1-r} = \frac{1}{1-1/2} = 2$

Final answer

- Num nodes = $n\left(2^0 + 2^{-1} + 2^{-2} + \cdots + 2^{-\log_2 n}\right) =$

$$\cancel{n}\frac{2n-1}{\cancel{n}} = 2n - 1$$

- We also have the "header" node which contributes $\log_2 n + 1$ nodes…

- Final answer: $2n - 1 + \log_2 n + 1 = 2n + \log_2 n = \mathcal{O}(n)$

Search efficiency analysis

- We have a sorted linked list ℓ with n elements and we build a list array A on top.
- We want to search for an element e.
- Unit cost (what n counts): dereferencing a pointer.
- What is the computational complexity of searching for e in ℓ using A?

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$O(n)$

$O(\log_2 n)$

$O(\log_2 (\log_2 n))$

Something else (what?)

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Now let's determine the constant…

Search efficiency analysis

- We have a sorted linked list ℓ with n elements and we build a list array A on top.
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- What is the maximum number of links we will scan to succeed or fail the search for e in ℓ using A?

$\log_2 n$

$1.5 \log_2 n$

$2 \log_2 n$

Something else (what?)

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$2\log_2 n + 1$

Also acceptable

$\log_2 n$

$1.5\log_2 n$

$2\log_2 n$

Something else (what?)

At every level, we visit at most 2 nodes (why?) and at level 0 we might have one additional pointer dereference.

Huge issue with list arrays

- Inserting a new element or deleting from it requires re-building the entire list array ☹
- Pay $O(n)$ cost to do so.
- For the insertion and/or deletion of n elements, the cost ends up being quadratic, which is unacceptable.
- Conclusion: a list array is **not** a good dictionary because of sub-par insertion and deletion.
  - But for ***immutable*** sorted lists, they work pretty well!

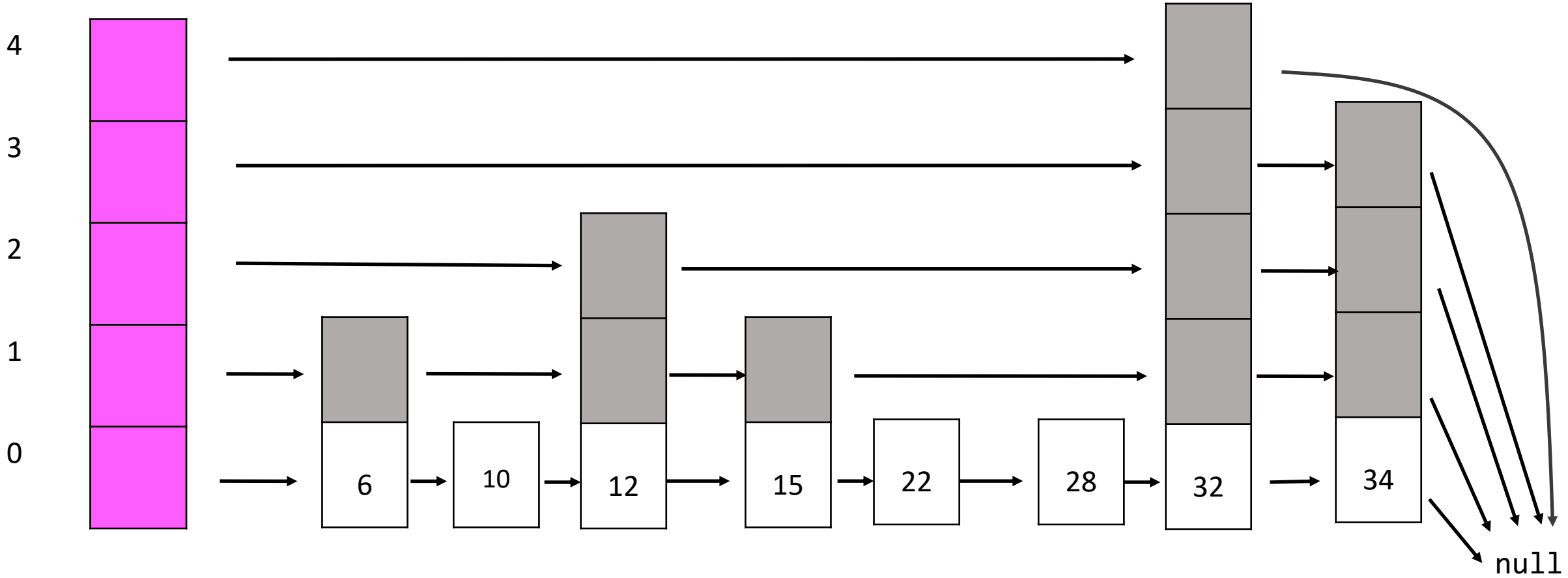
Binary Search “vs” list arrays

Binary Search	List array
Used in contiguous storage (arrays)	Used in discontiguous storage (linked lists)
Requires up to $\log_2 n$ comparisons	Requires up to $2 \log_2 n + 1$ comparisons
Doesn’t need extra space	Needs $O(n)$ extra space
Doesn’t require adjustments after mutations	Needs to be re-built after every mutation

Skiplist: relaxing the constraint

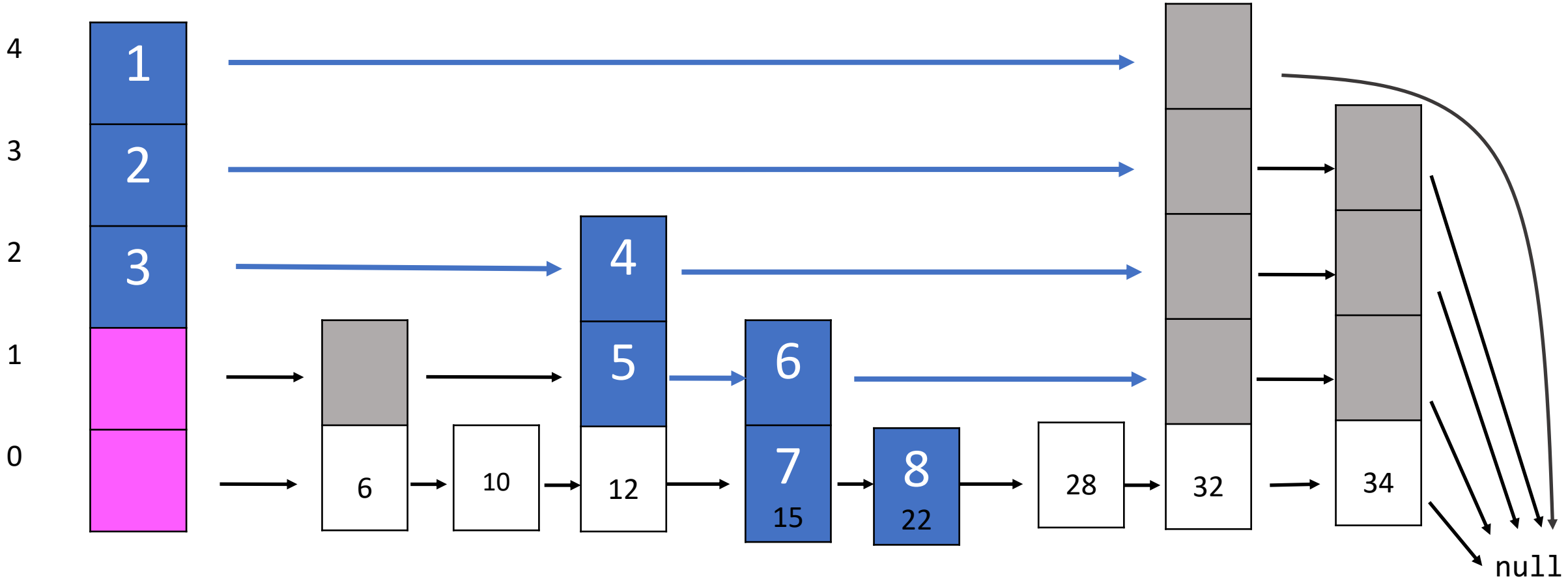
- Observation: in a list array, half ($^1/_2$) of the "towers" that extend to level i also extend to level $i + 1$. Therefore:
  - $^1/_2$ of the list's keys are represented in level 1
  - $^1/_4$ are represented in level 2
  - ….
  - $(^1/_2)^i$ are represented in level i.
- Solution: probabilistically build the towers by randomly sampling the new height of an inserted node before actually inserting it!
- Use a geometric distribution: $P(level = i) = \left(\frac{1}{2}\right)^{i+1}$

Skiplist example



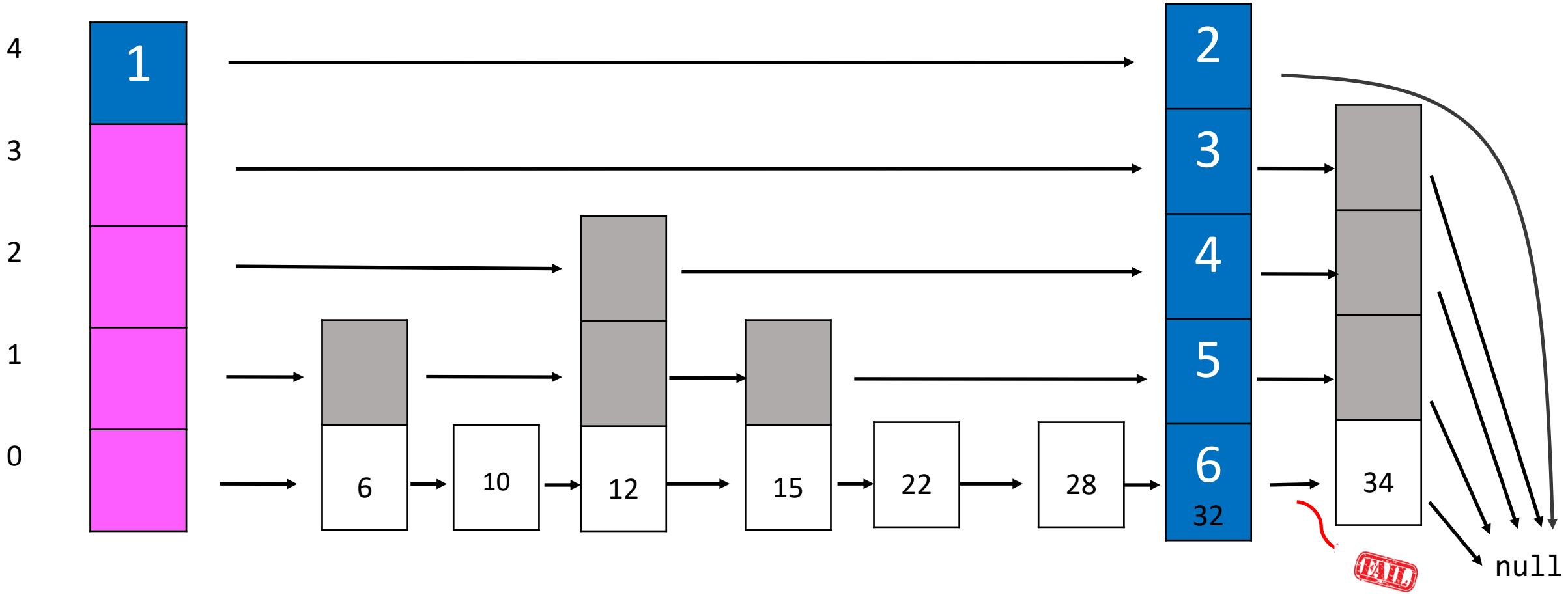
Skiplist example

Searching for 22 (should be a search ***hit***)



Skiplist example

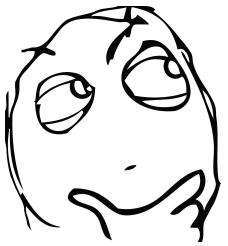
Searching for 33 (should be a search ***miss***)

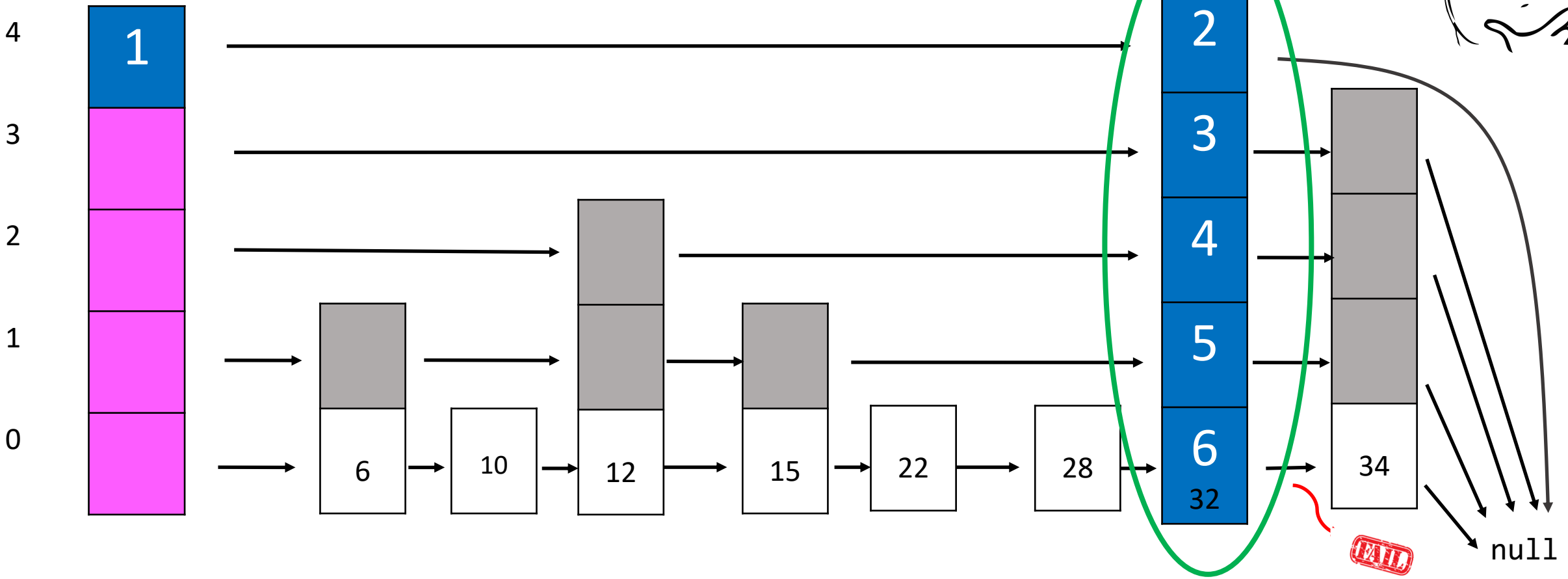


Skiplist example

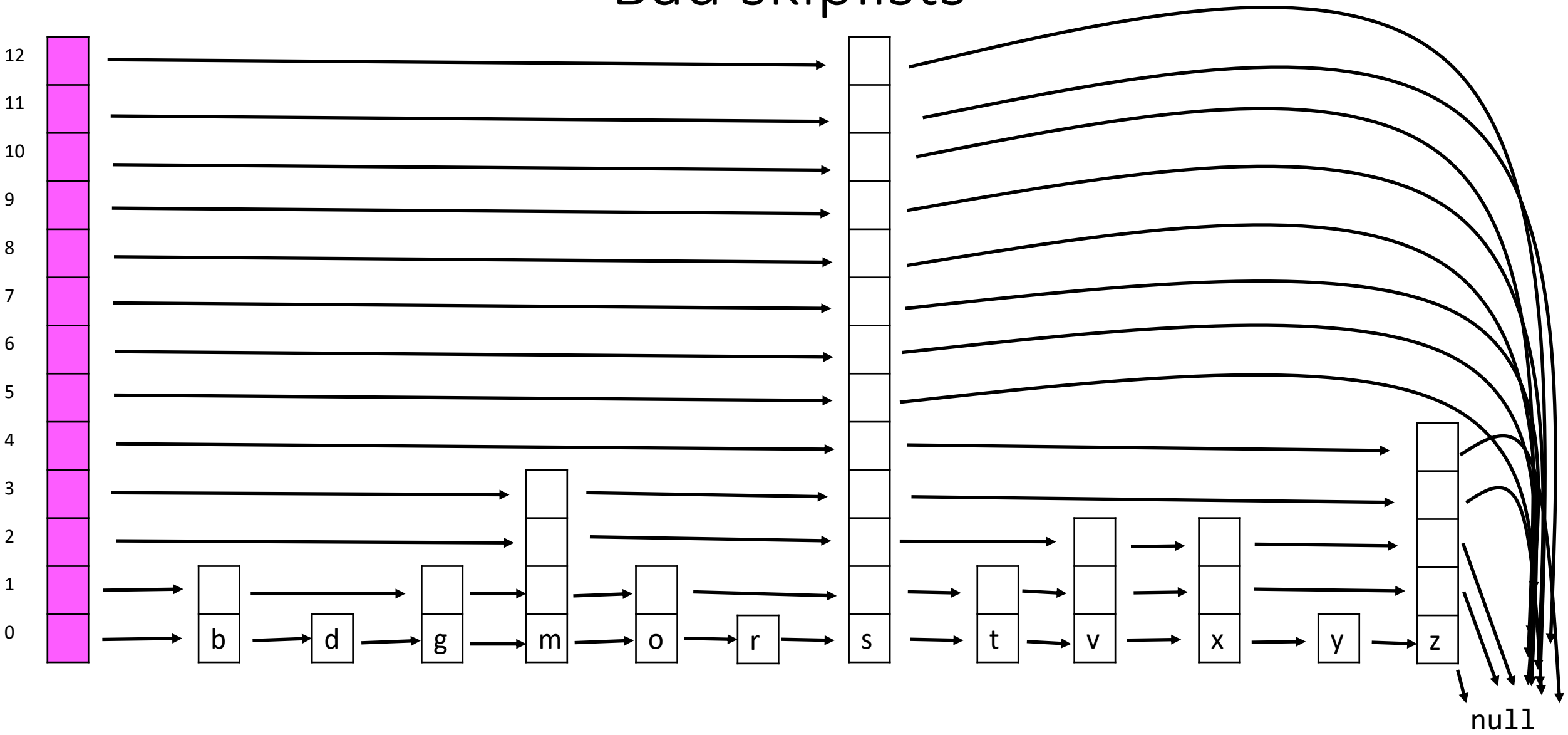
Searching for 33 (should be a search ***miss***)

It certainly looks like the height of this tower is slowing us down…





Bad skiplists



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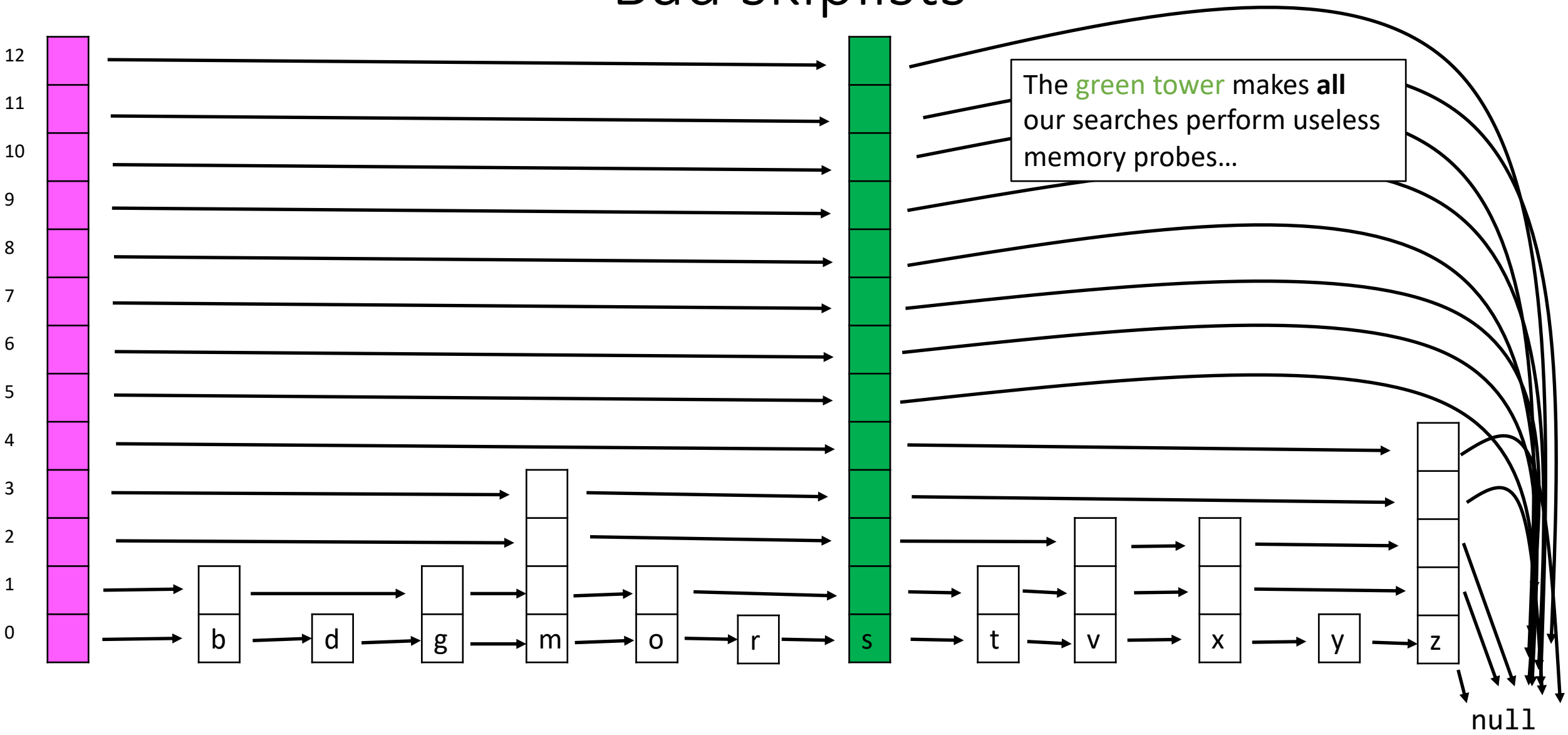
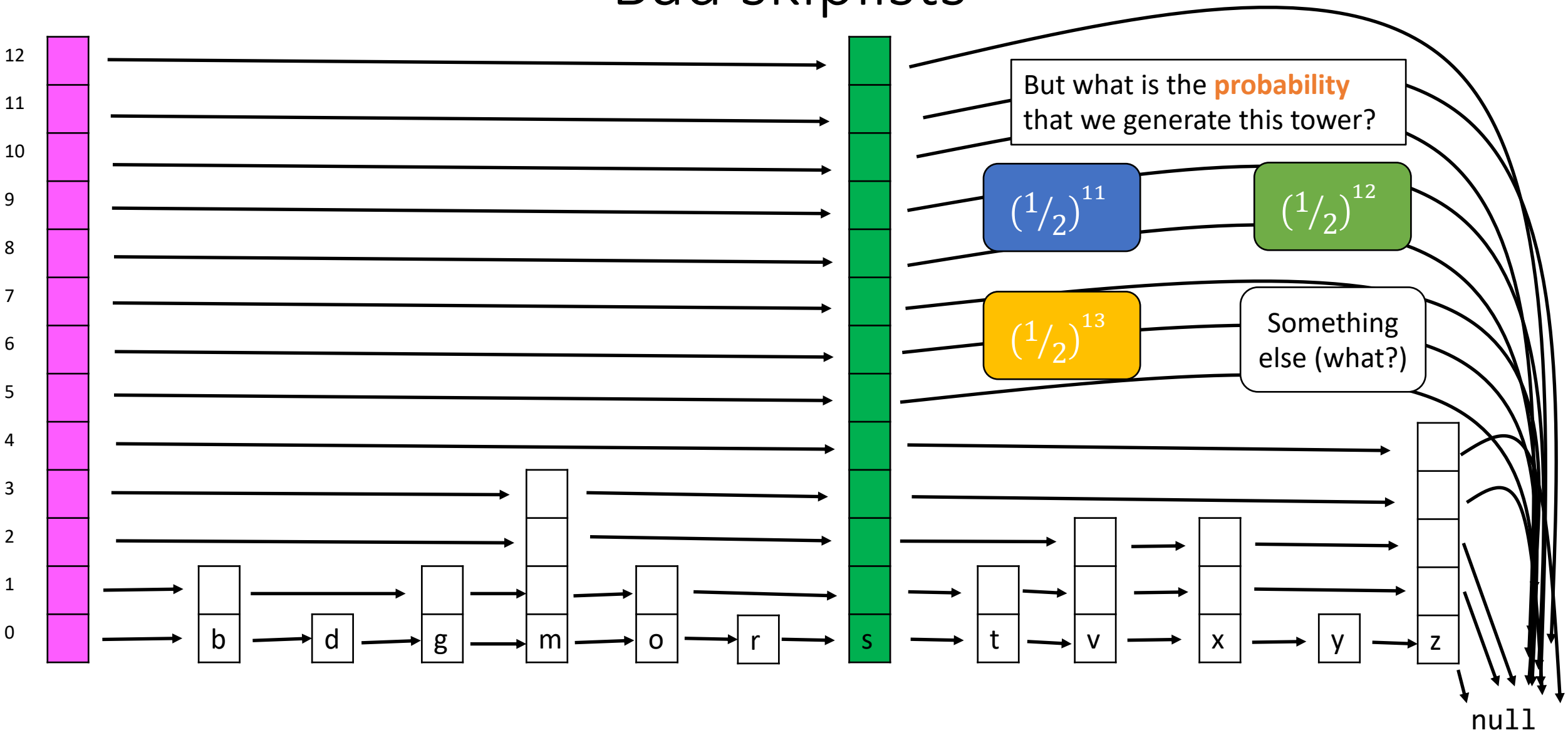


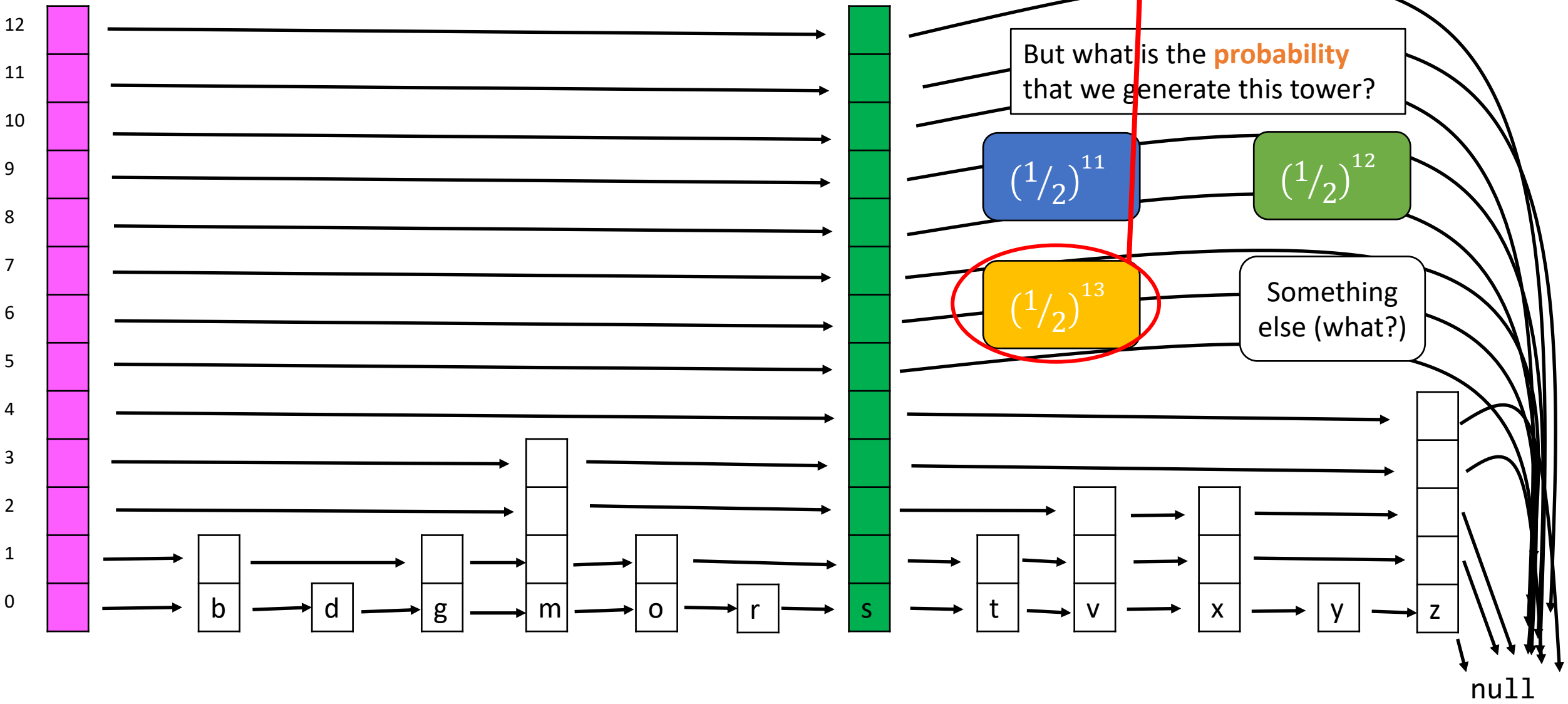
Bad skiplists



We have 12 flips to make the tower have 13 levels and another to keep it there!

Bad skiplists

= 0.012207031%



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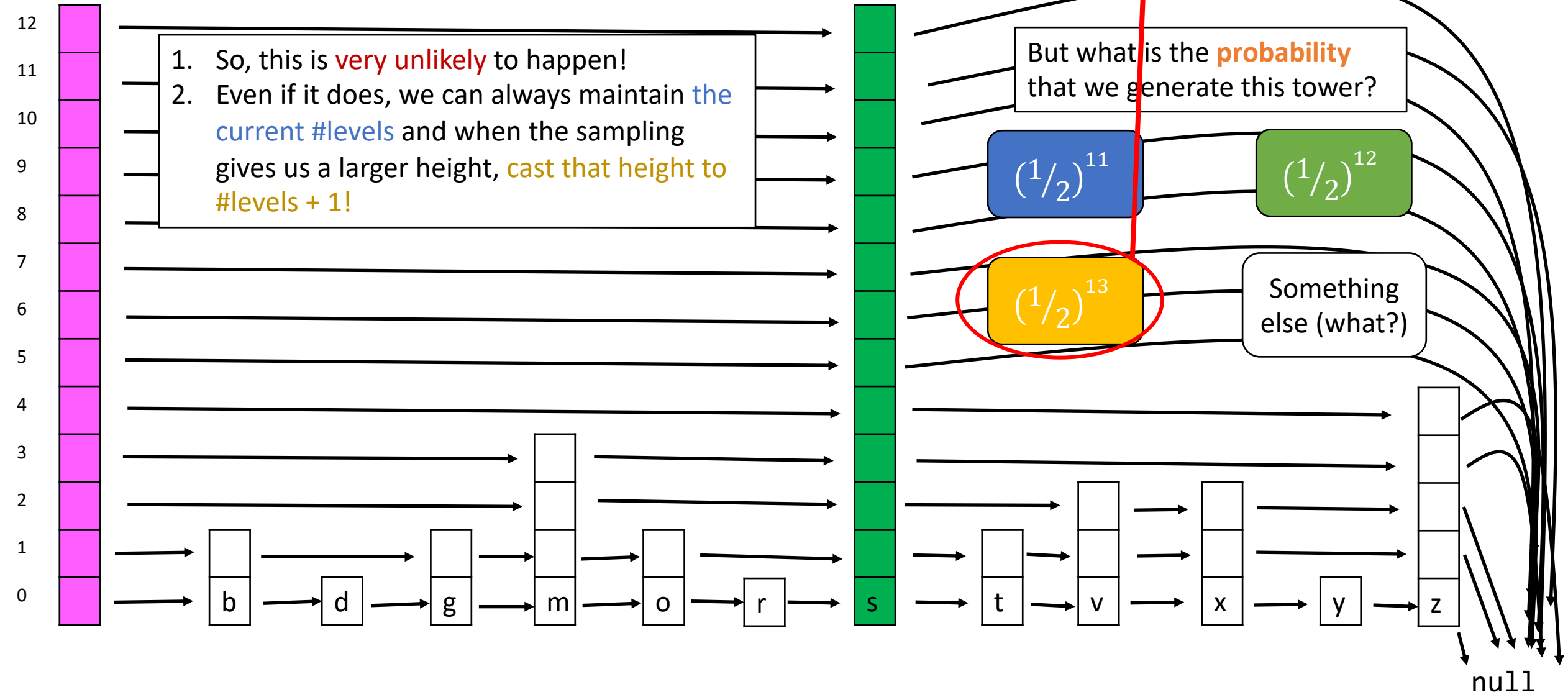
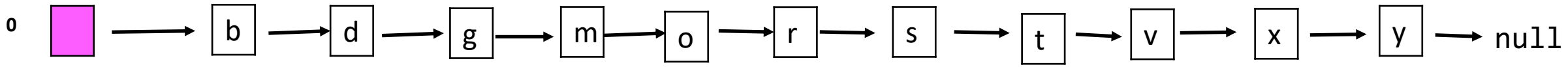
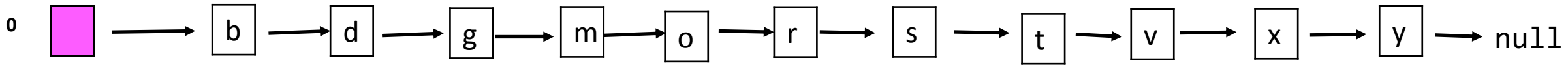


Bad skiplists



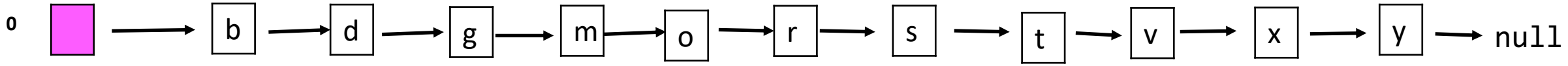
Bad skiplists

- Degenerate skiplist: is equivalent to the linked list of its elements.



Bad skiplists

- Degenerate skiplist: is equivalent to the linked list of its elements.
- But how likely is such a skiplist?



Bad skiplists

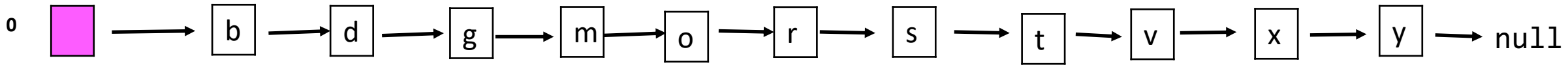
- Degenerate skiplist: is equivalent to the linked list of its elements.
- But how likely is such a skiplist?

$(1/2)^{11}$

$(1/2)^{12}$

$(1/2)^{13}$

Something else (what?)



Bad skiplists

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- But how likely is such a skiplist?

Very unlikely!

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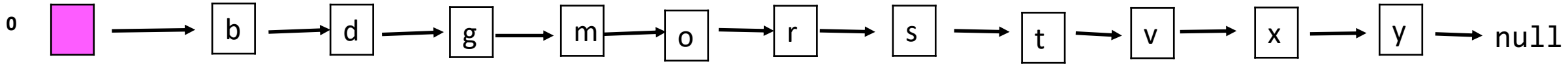
$(1/2)^{13}$

Something else (what?)



Insertion

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Insertion

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  3. When you find the position, **splice**: insert the tower in place by using `update` to update the relevant pointers.

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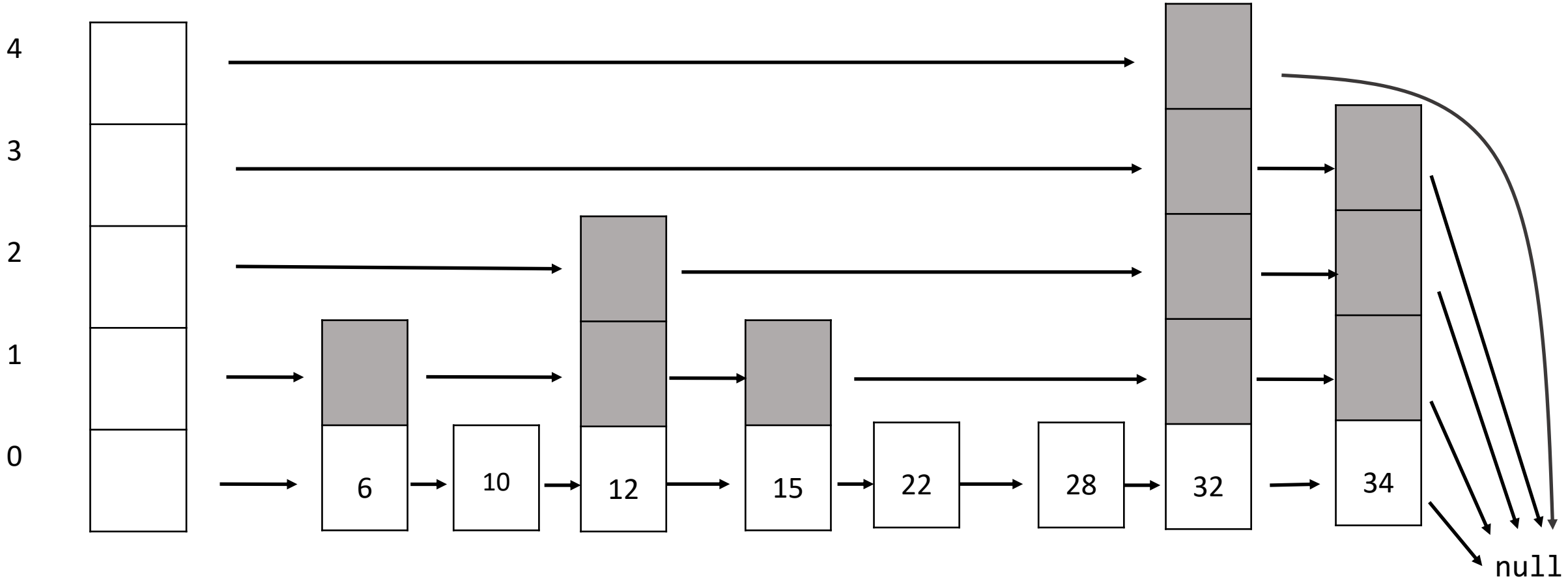
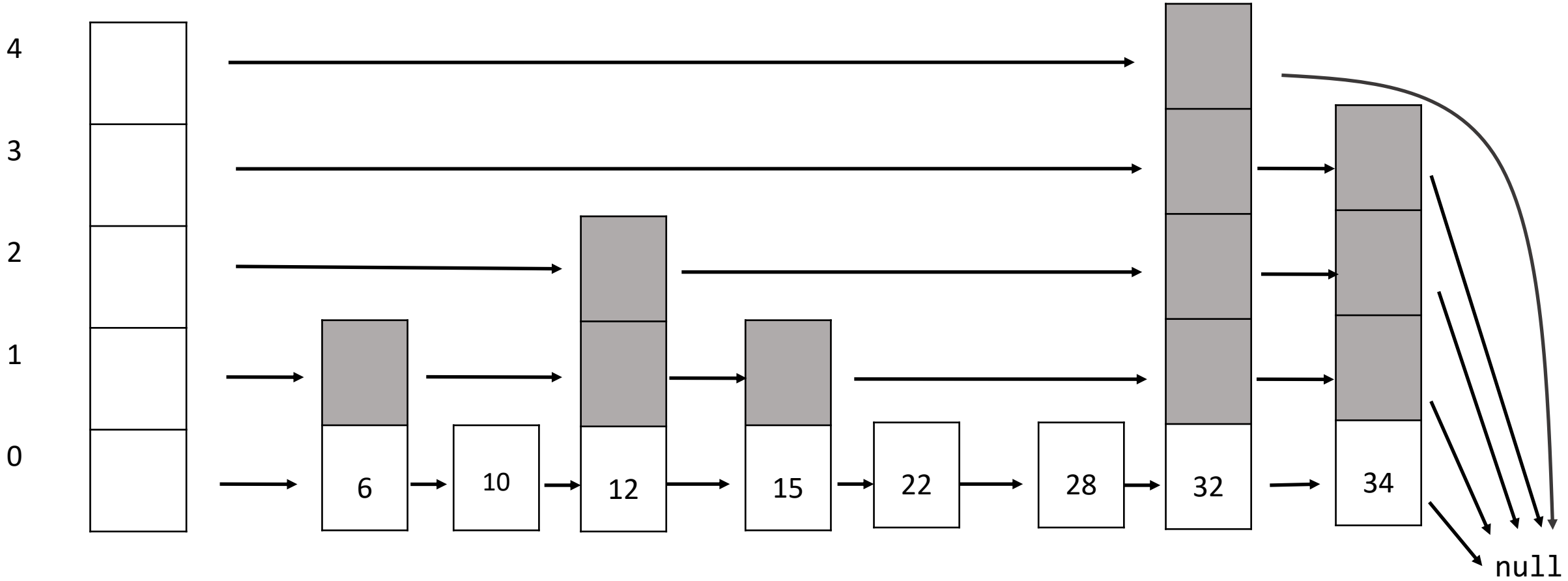
  3. When you find the position, **splice**: insert the tower in place by using `update` to update the relevant pointers.
- We employ zero-indexing: the list level is at level 0, one level above is level 1, etc.

Insertion Example



Insertion Example

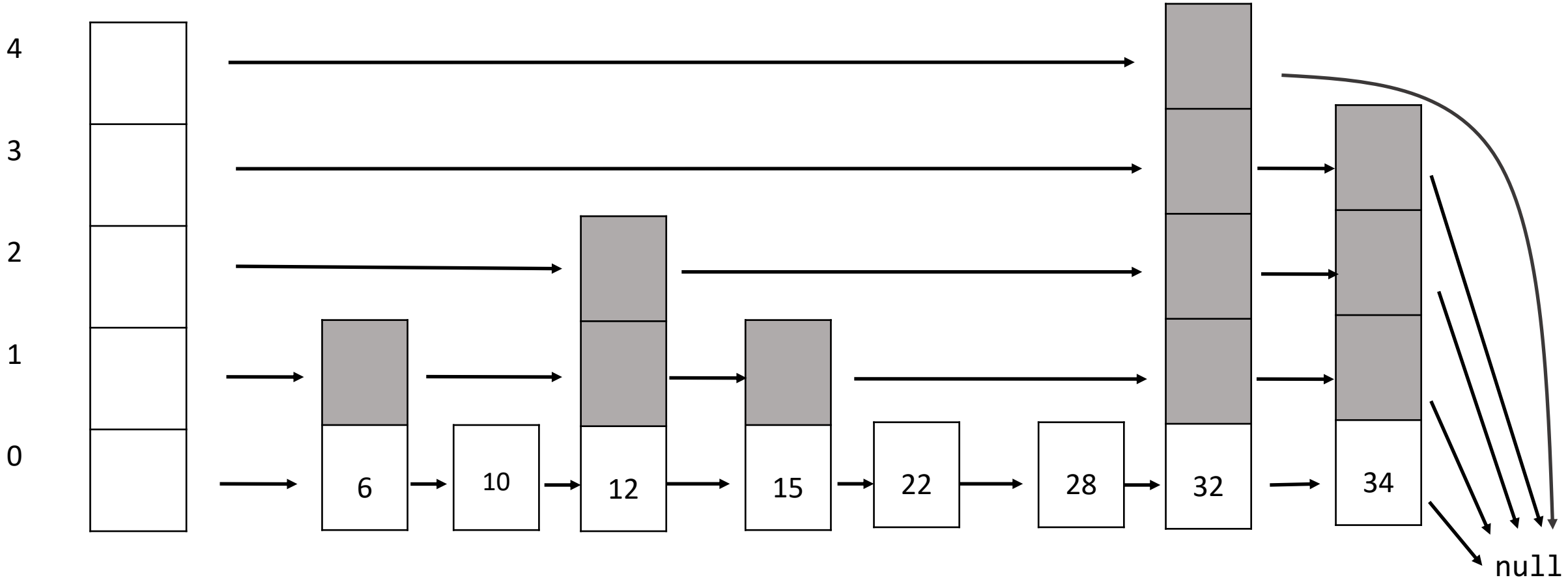
Suppose we want to insert 25.



Task: Insert 25

Insertion Example

Step 1: Sample a #levels “on top” of the key. Suppose we sample 3 (probability?)



Task: Insert 25

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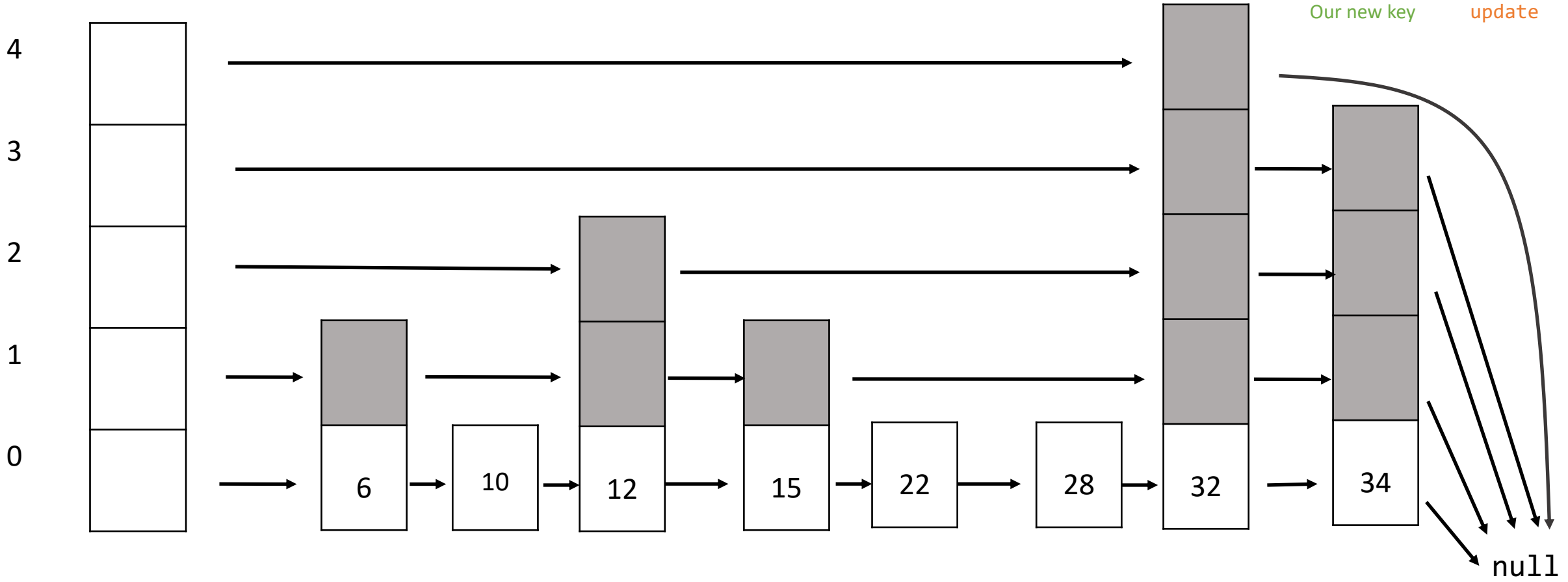
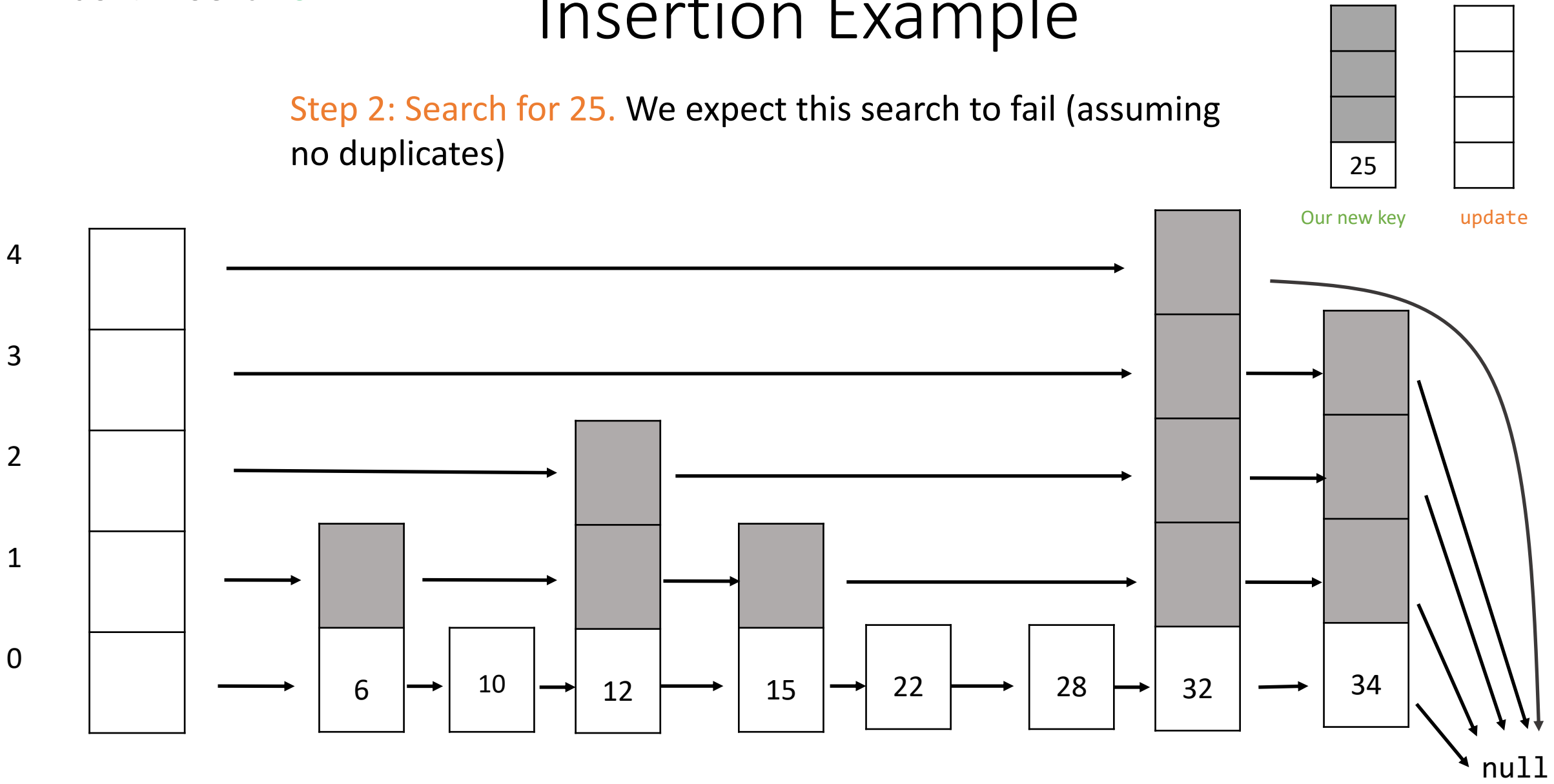
Step 1: Sample a #levels “on top” of the key. Suppose we sample 3 (probability?)



Task: Insert 25

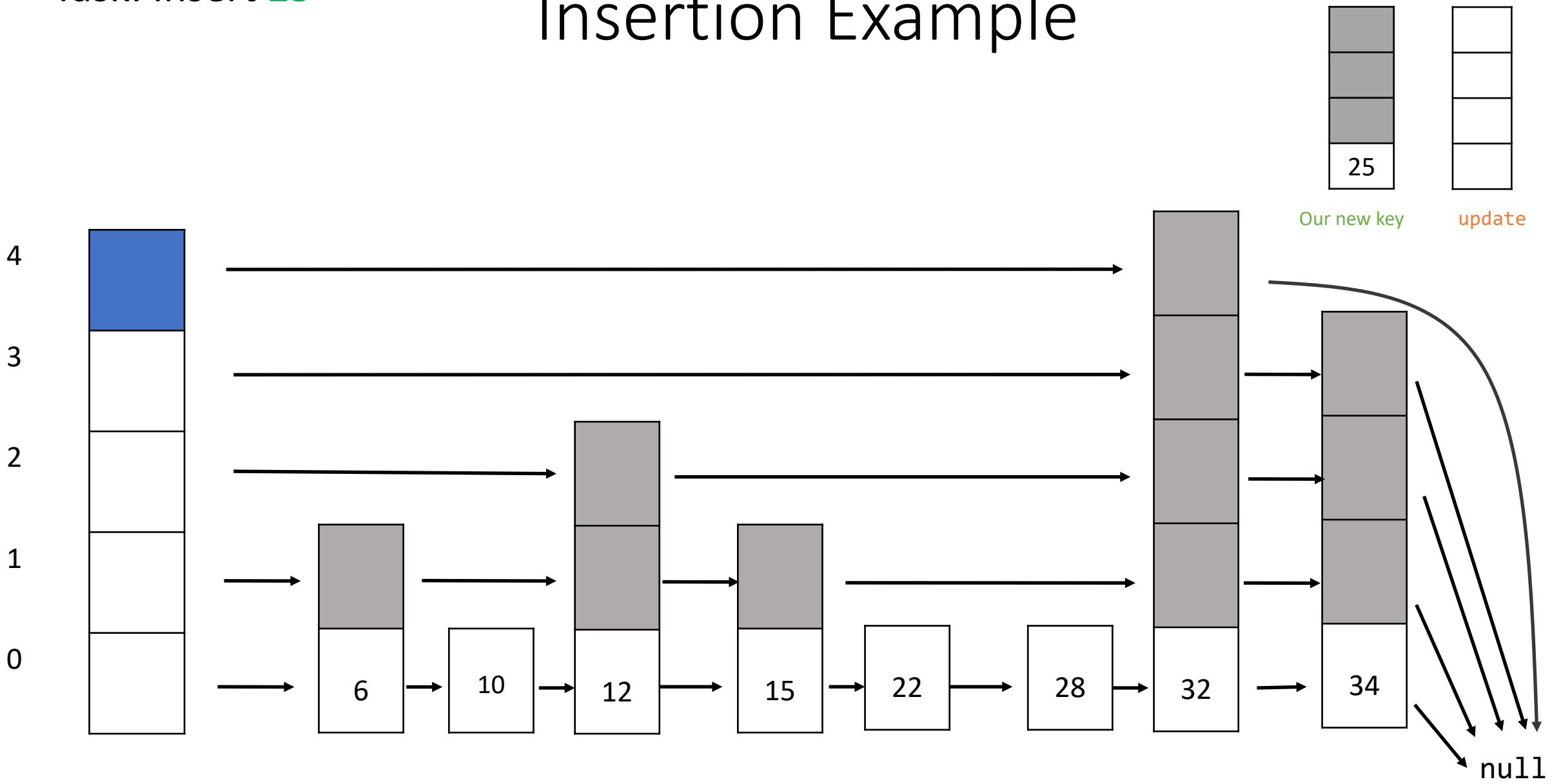
Insertion Example

Step 2: Search for 25. We expect this search to fail (assuming no duplicates)



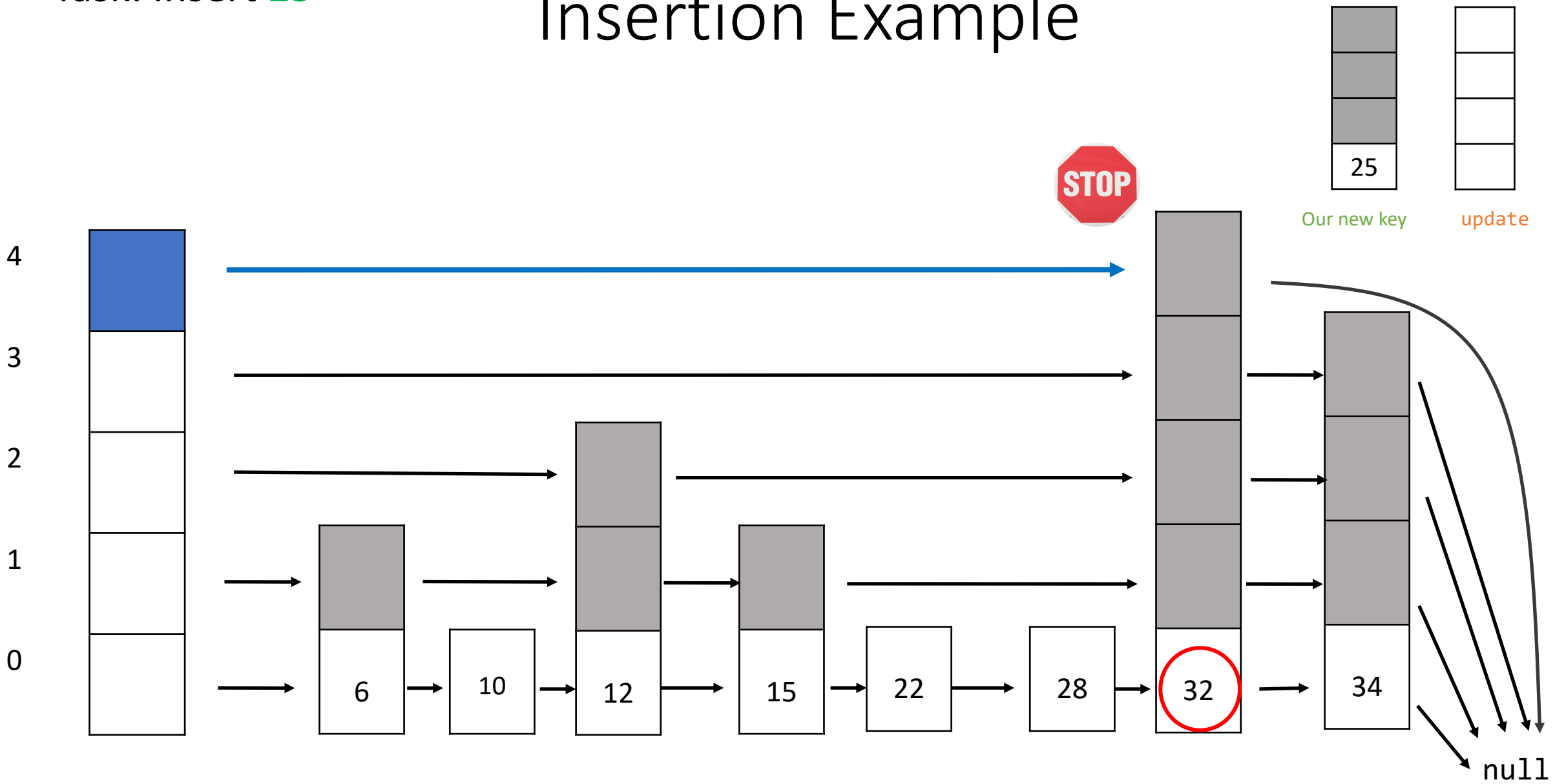
Task: Insert 25

Insertion Example



Task: Insert 25

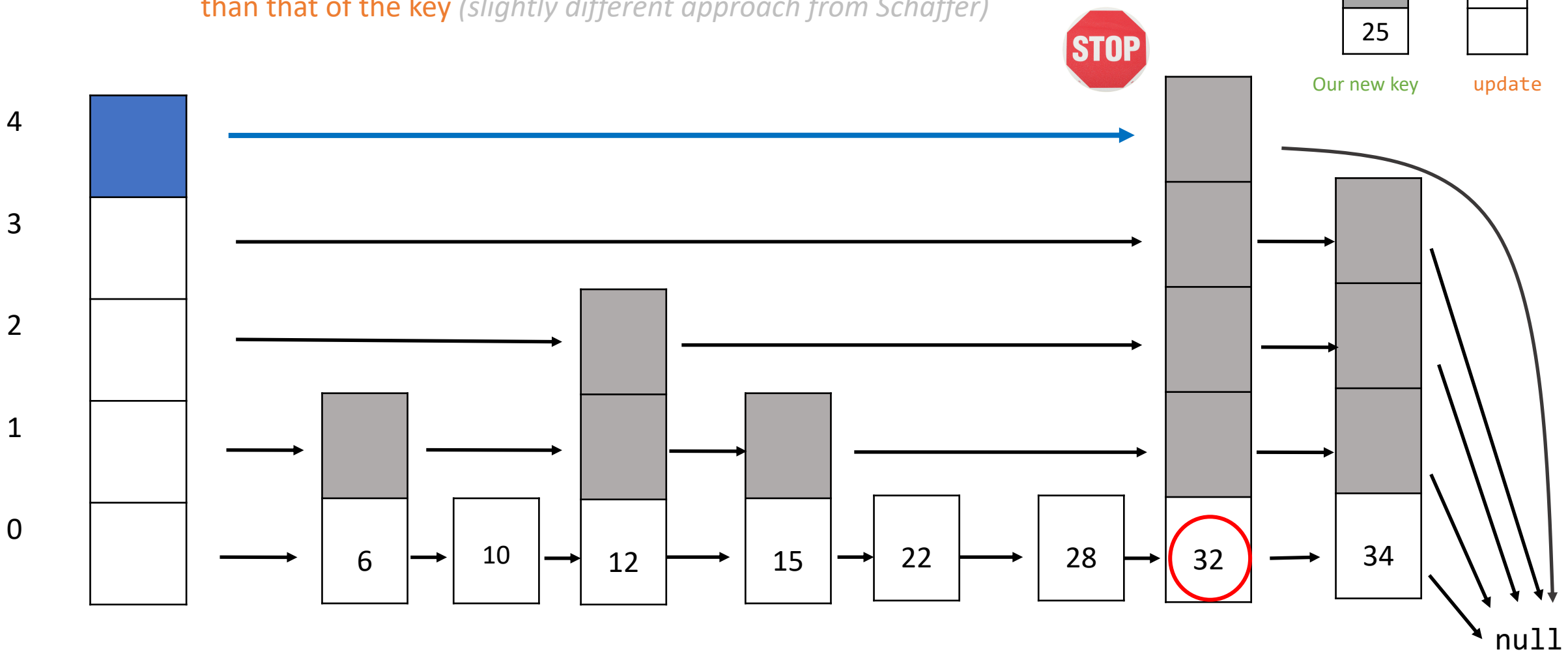
Insertion Example



Task: Insert 25

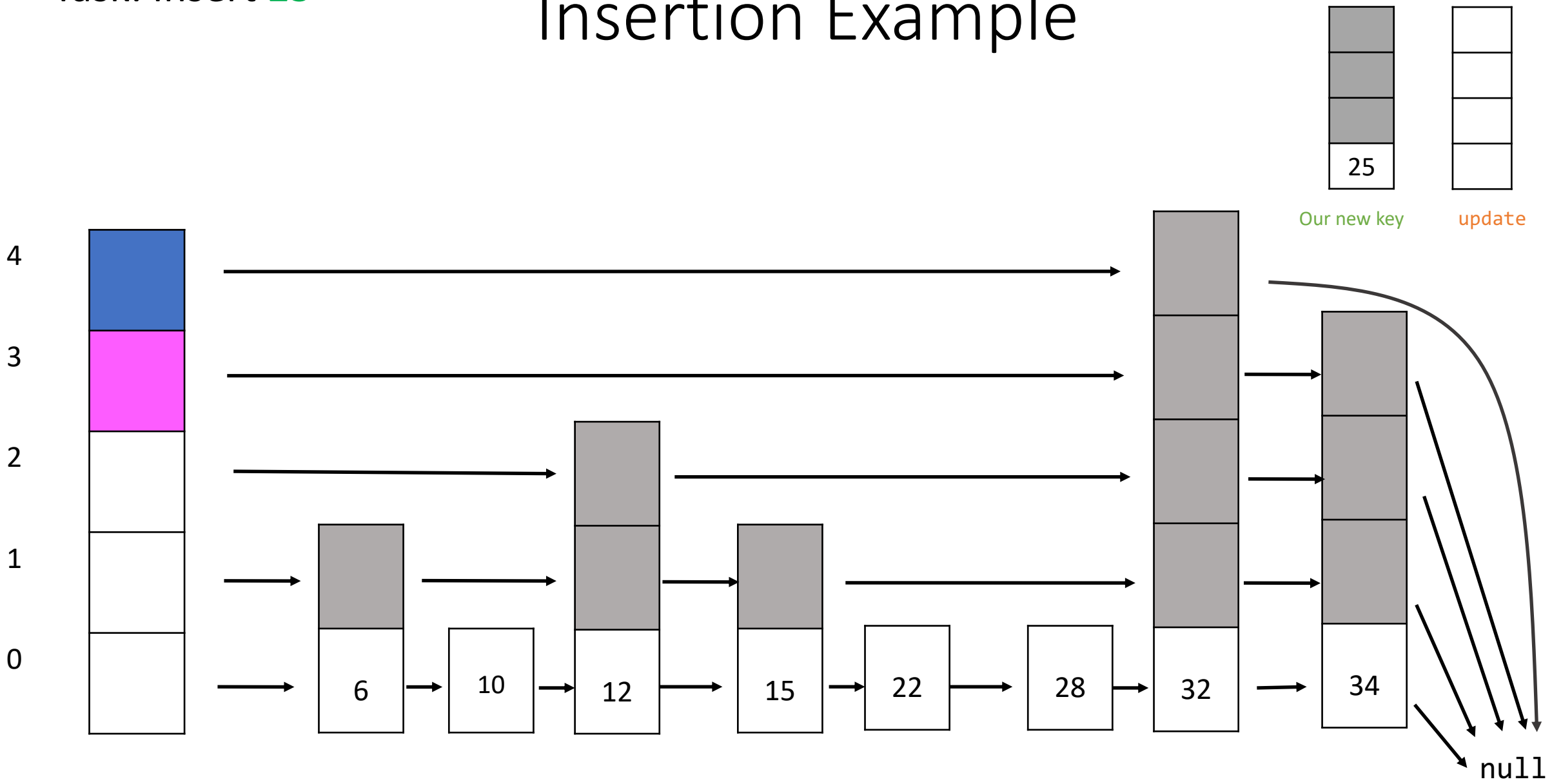
Insertion Example

We don't have to touch `update` yet, since the current #levels is 4, greater than that of the key *(slightly different approach from Schaffer)*



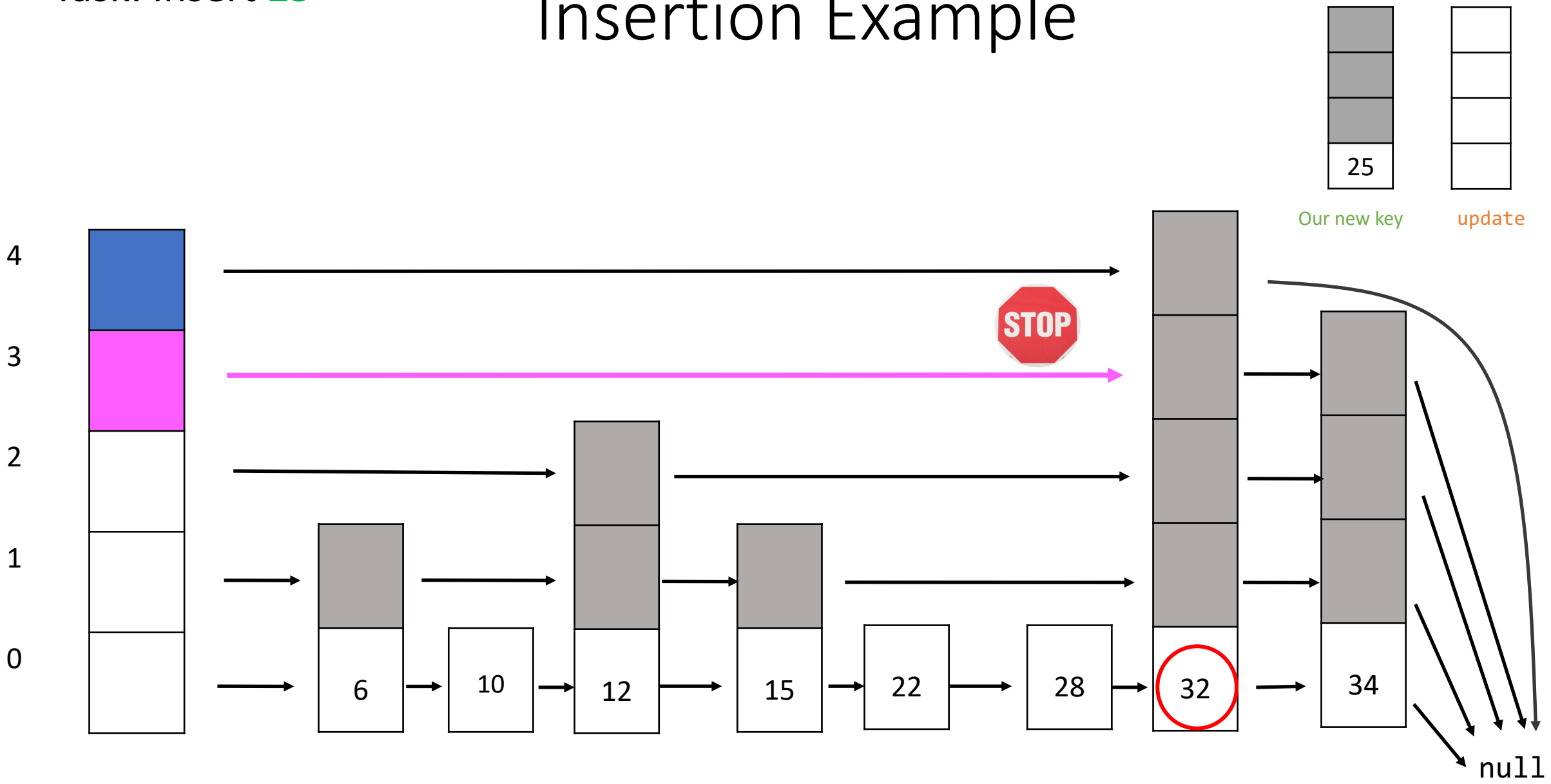
Task: Insert 25

Insertion Example



Task: Insert 25

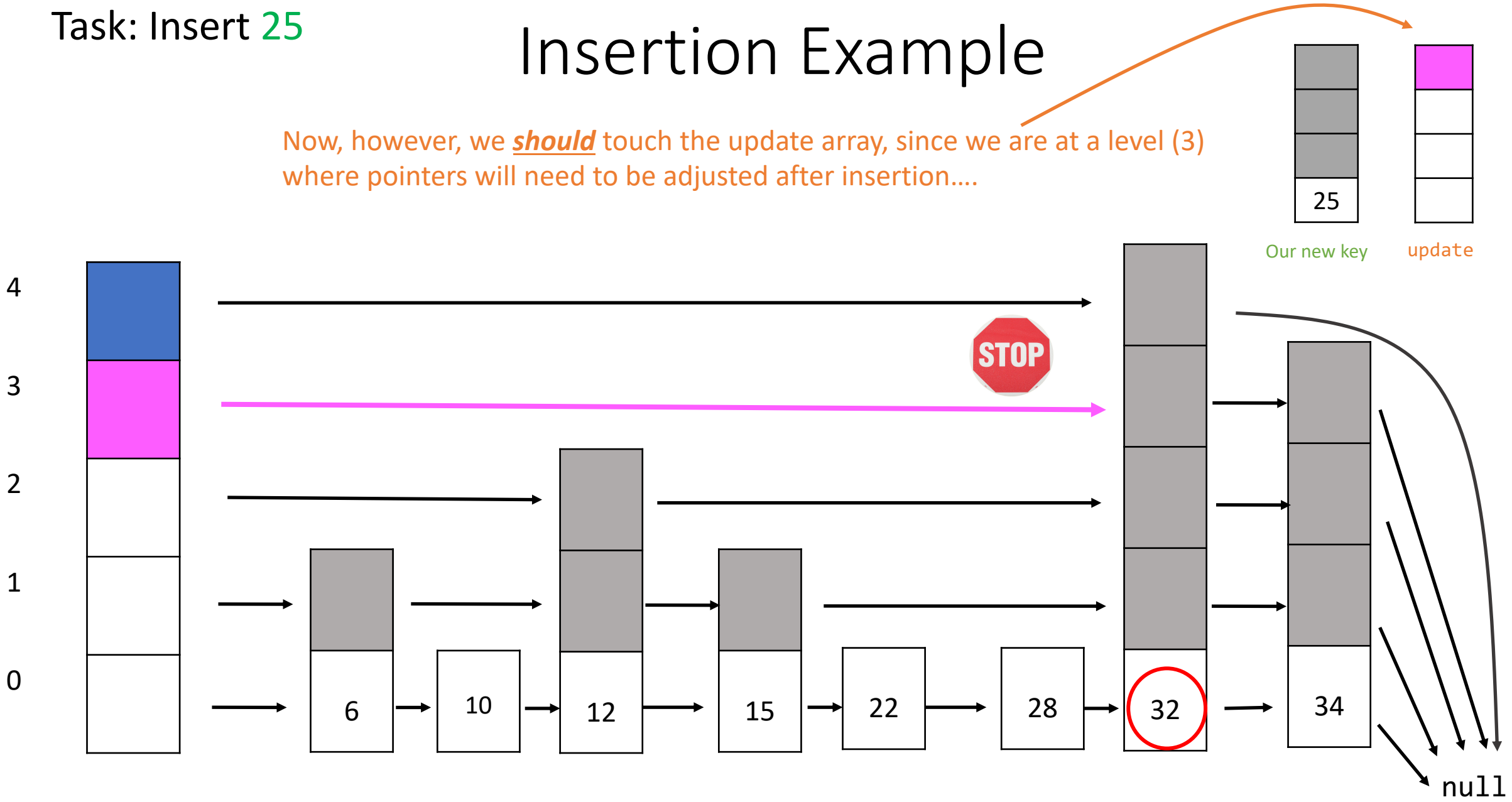
Insertion Example



Task: Insert 25

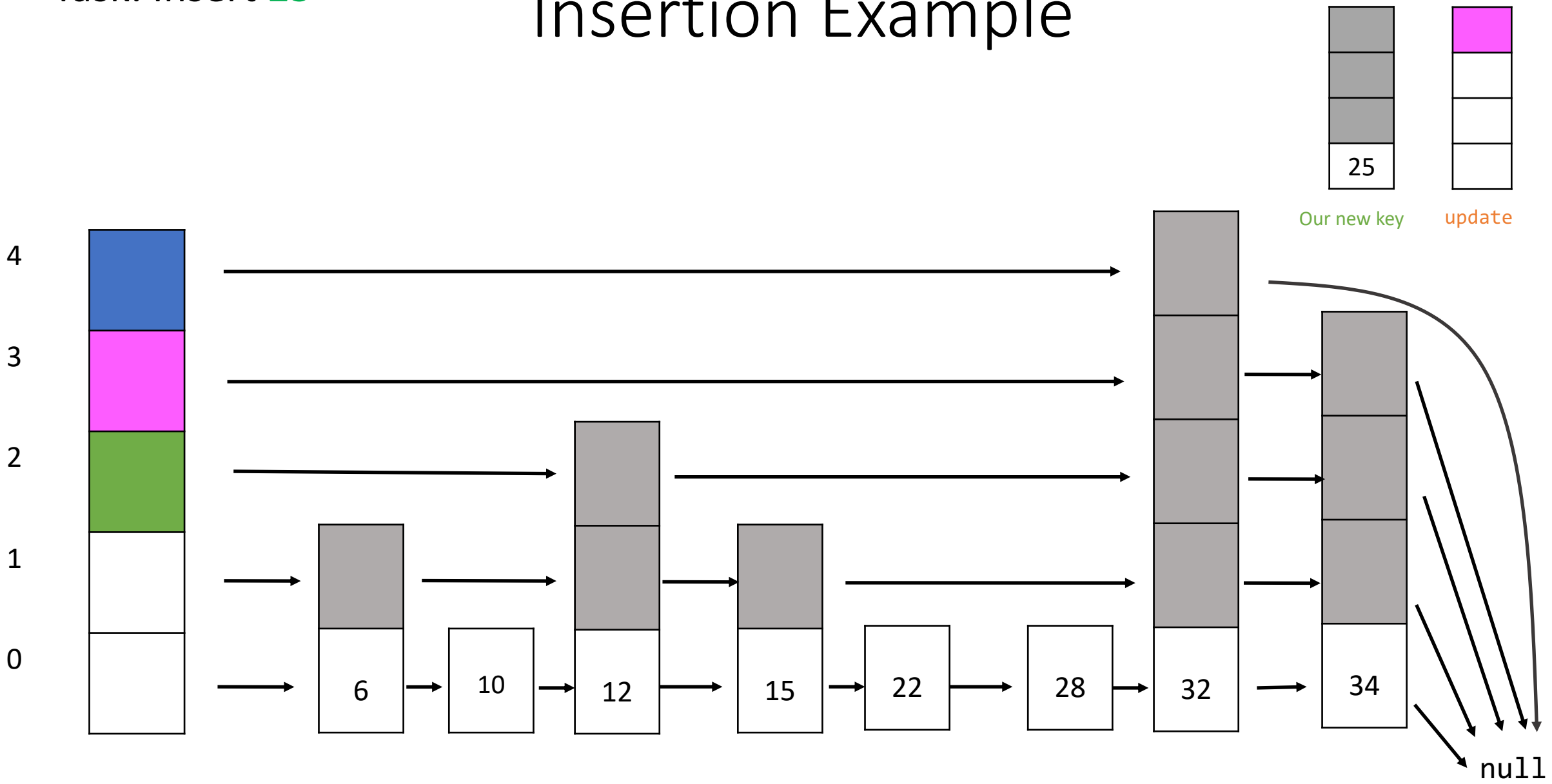
Insertion Example

Now, however, we ***should*** touch the update array, since we are at a level (3) where pointers will need to be adjusted after insertion….



Task: Insert 25

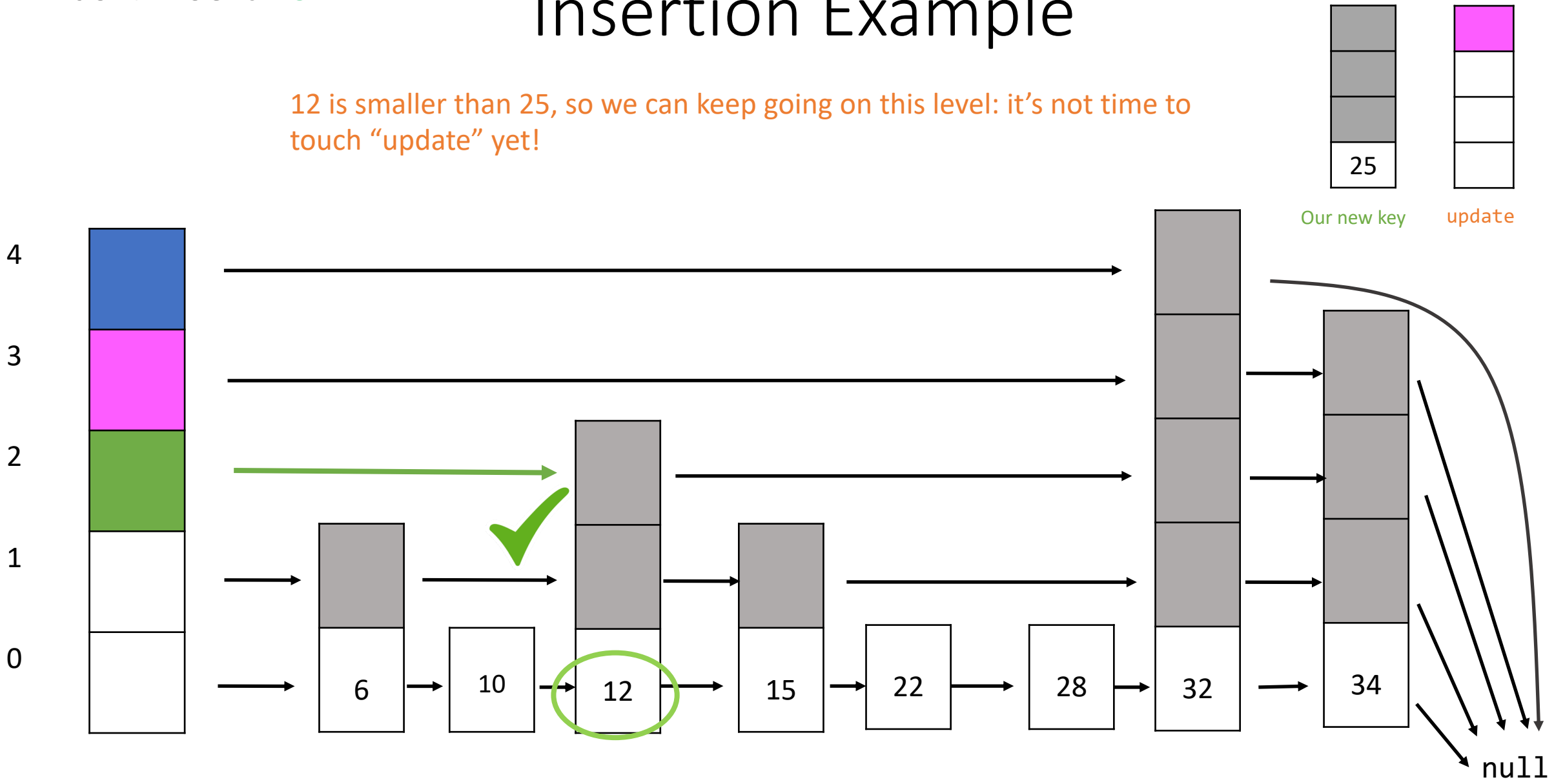
Insertion Example



Task: Insert 25

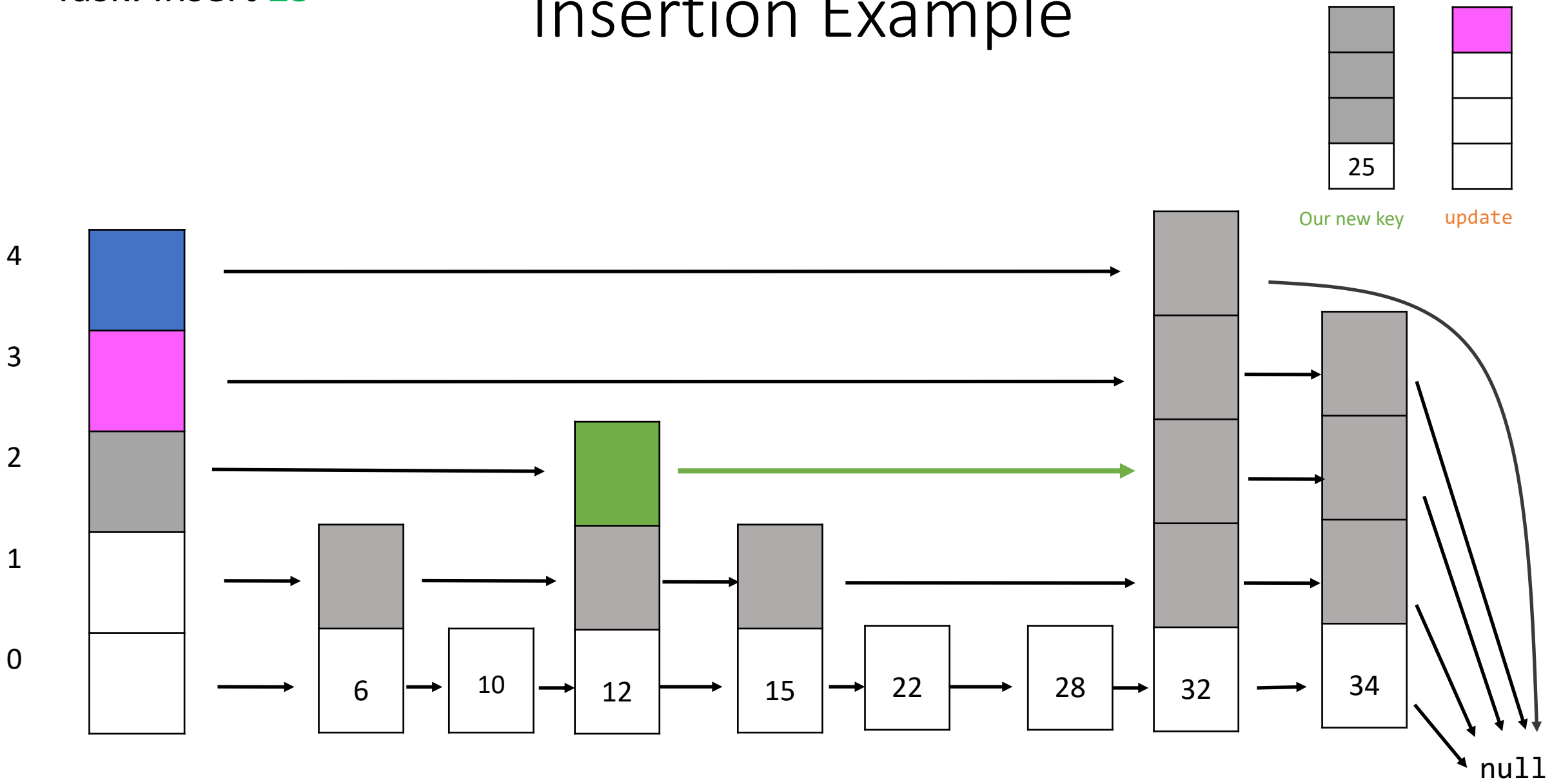
Insertion Example

12 is smaller than 25, so we can keep going on this level: it's not time to touch "update" yet!



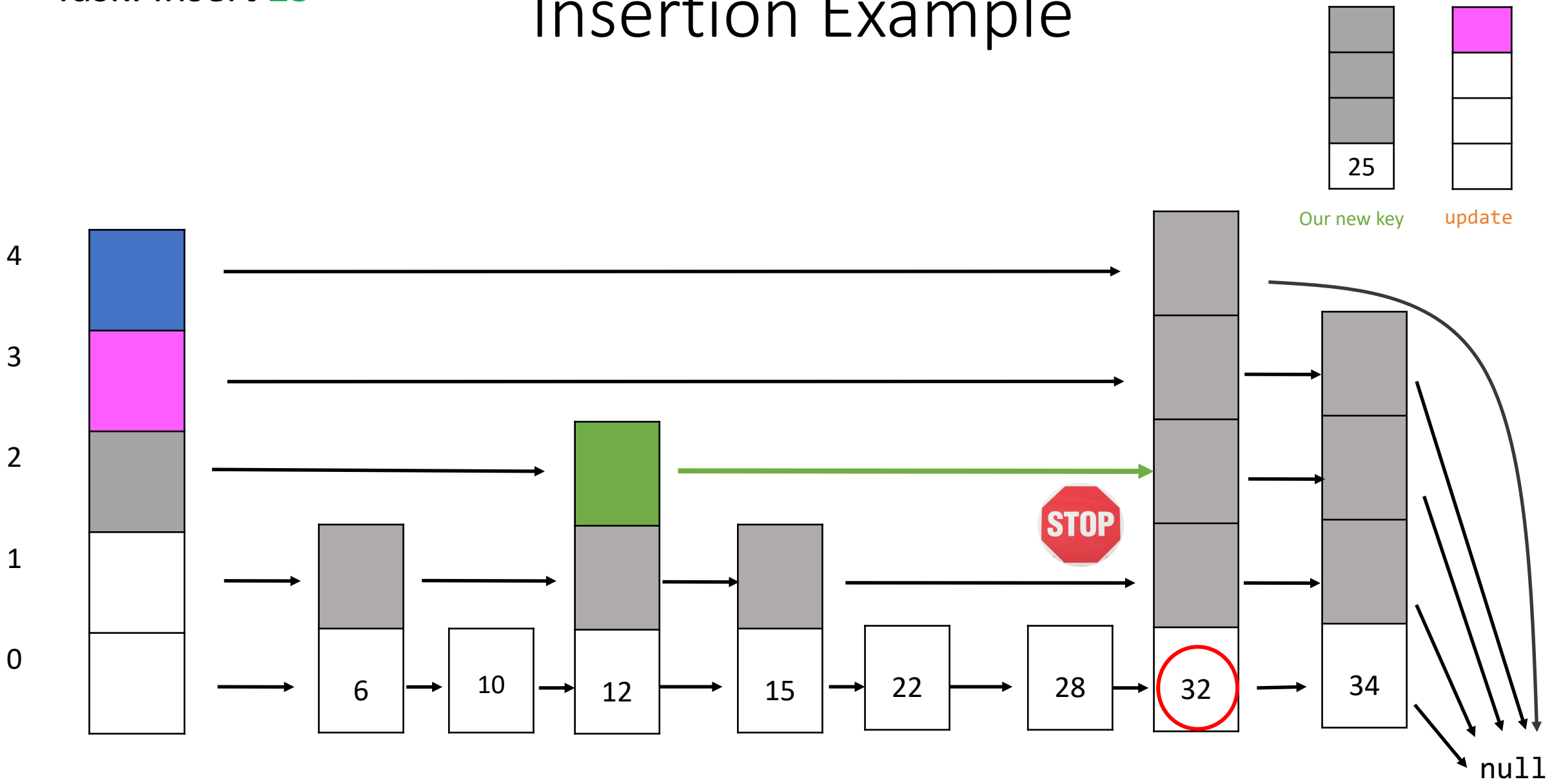
Task: Insert 25

Insertion Example



Task: Insert 25

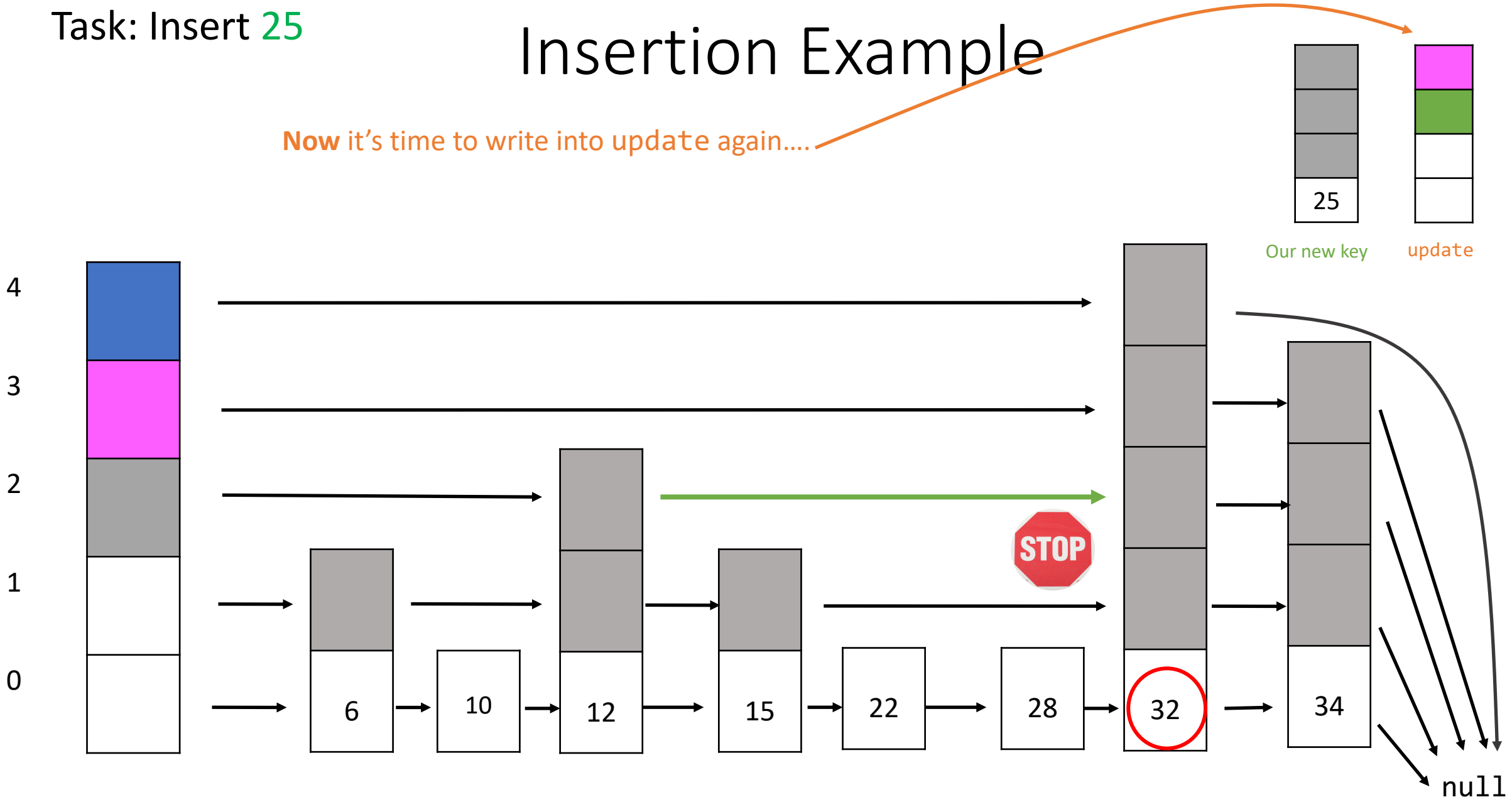
Insertion Example



Task: Insert 25

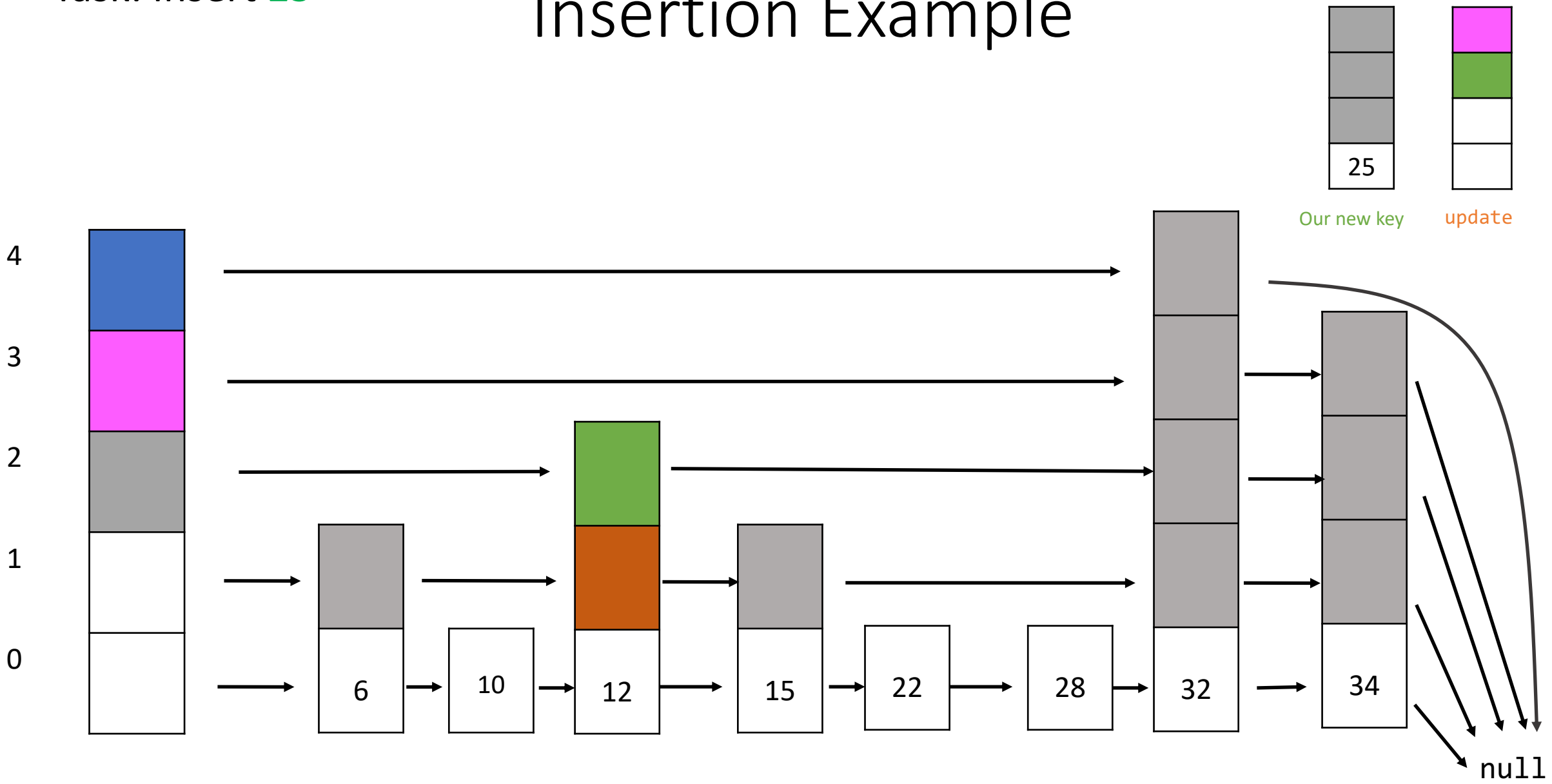
Insertion Example

Now it's time to write into update again….



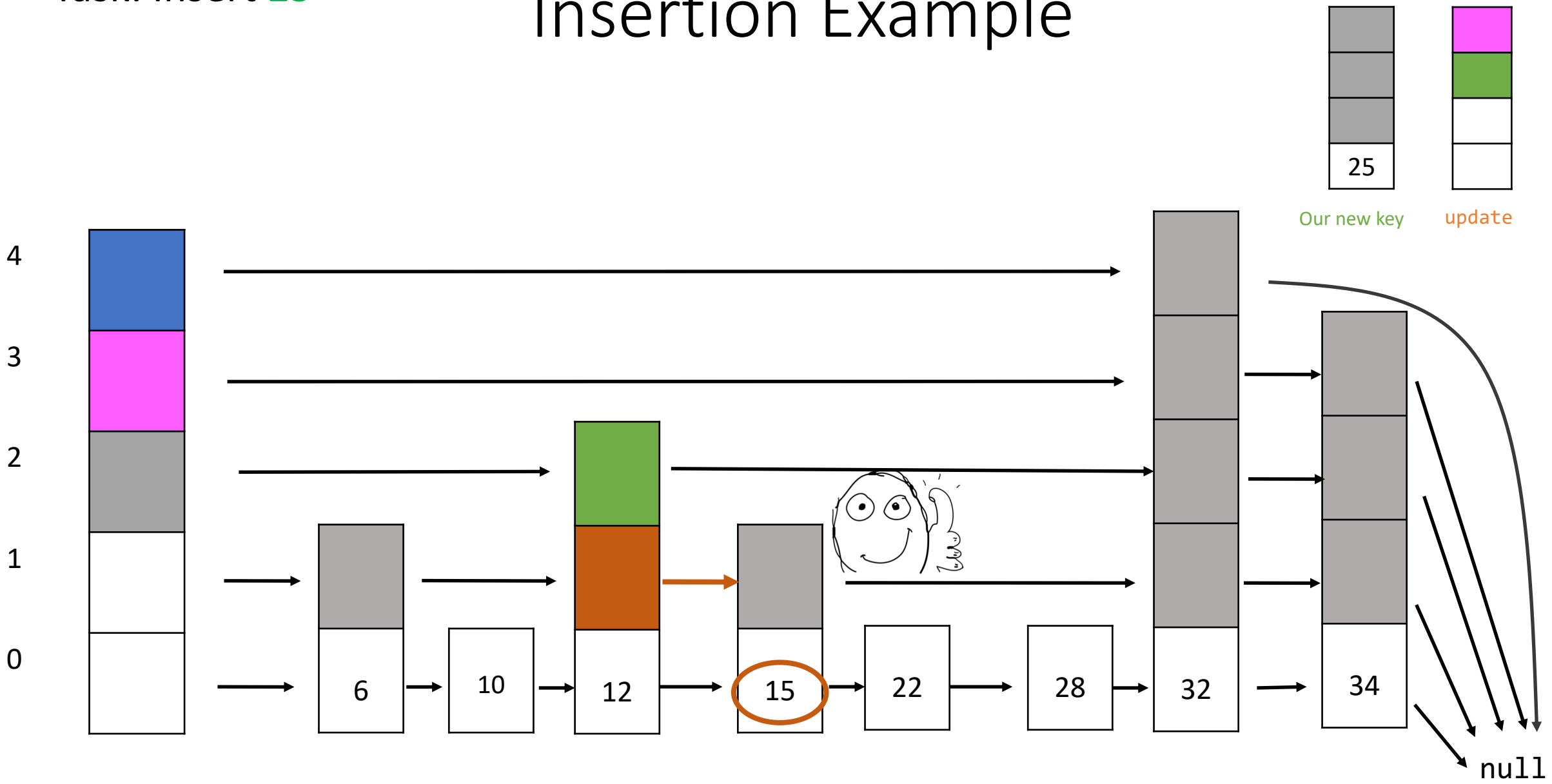
Task: Insert 25

Insertion Example



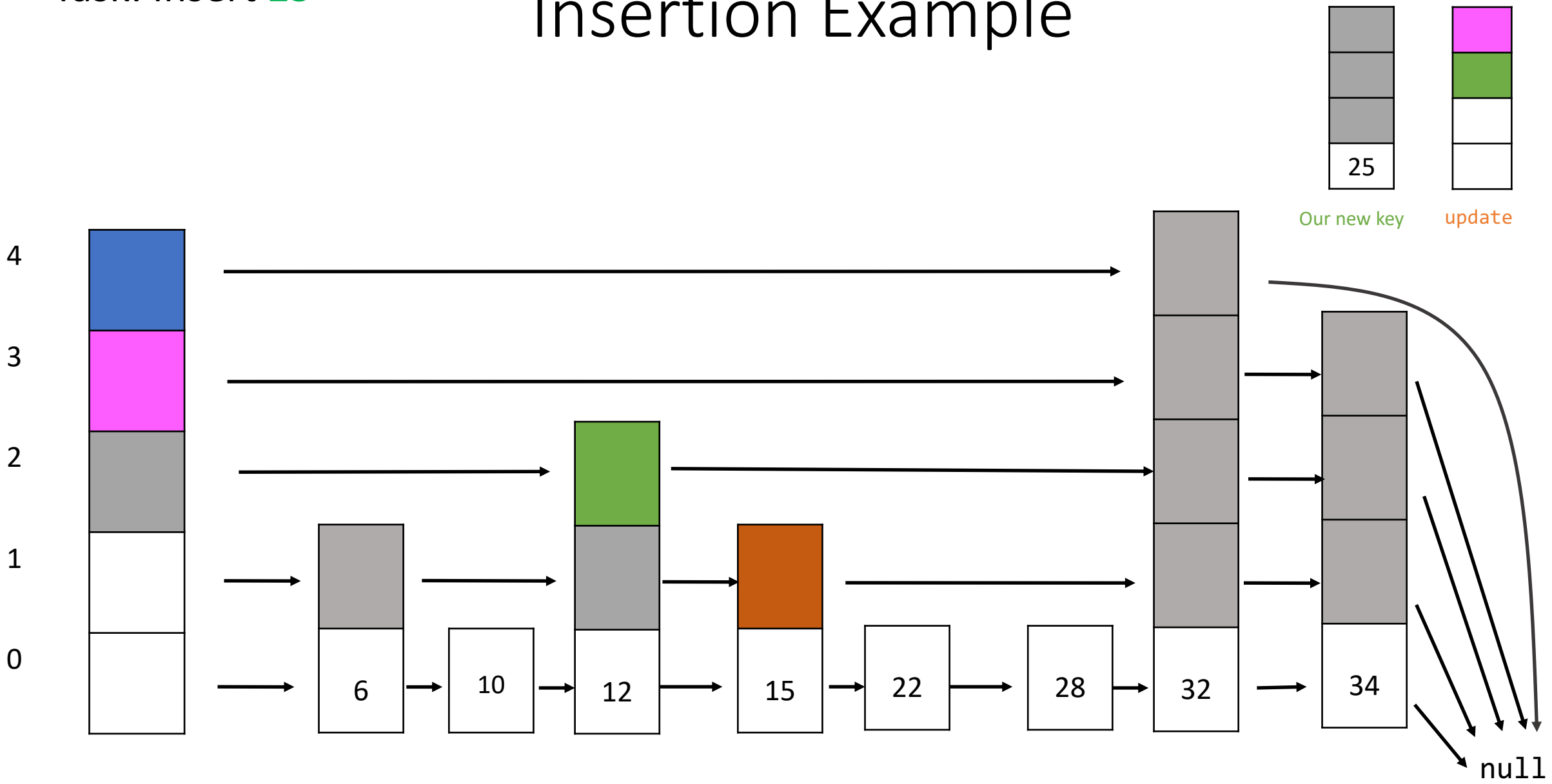
Task: Insert 25

Insertion Example



Task: Insert 25

Insertion Example



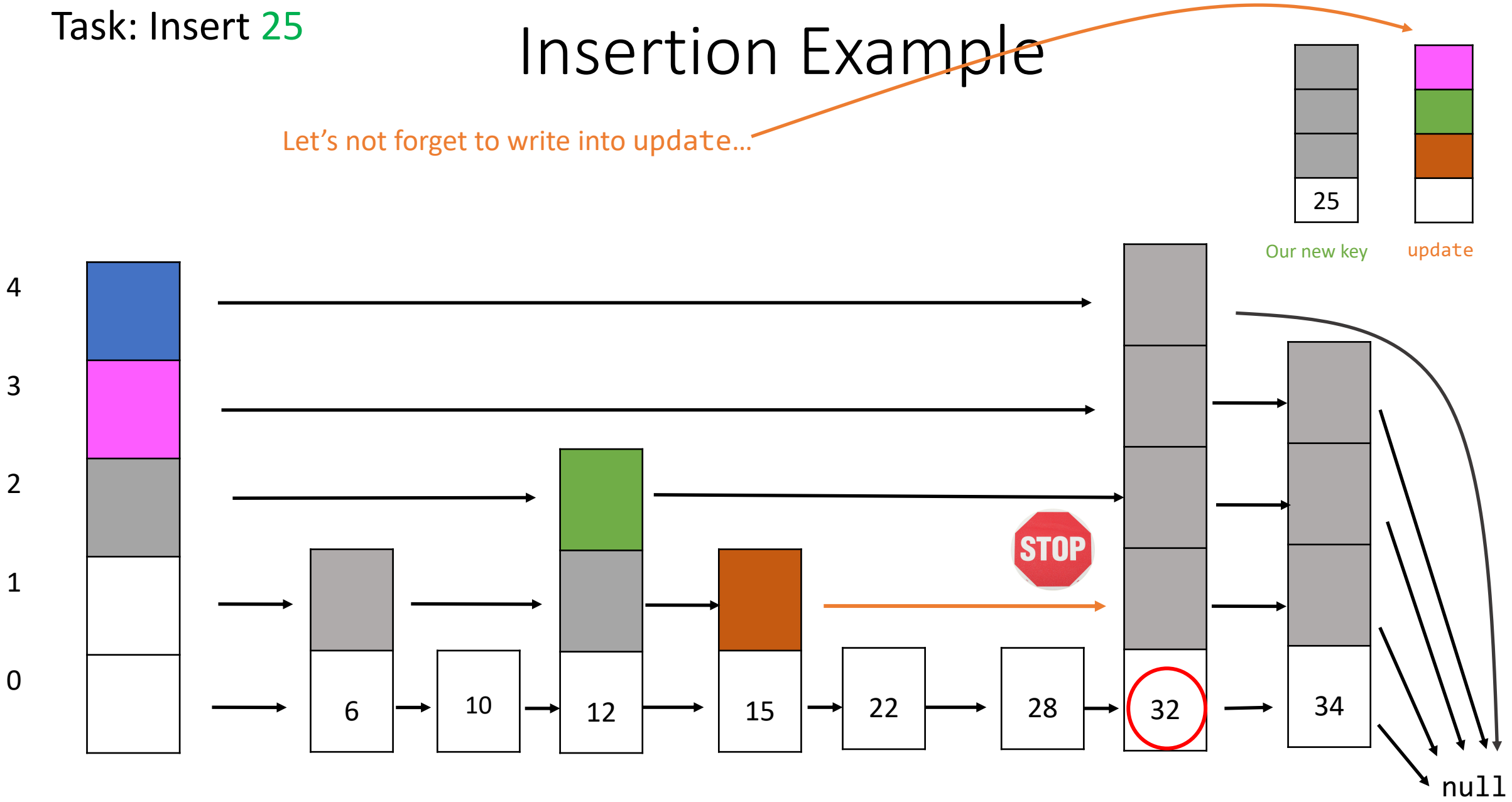
Task: Insert 25

Insertion Example

Task: Insert 25

Insertion Example

Let's not forget to write into update…



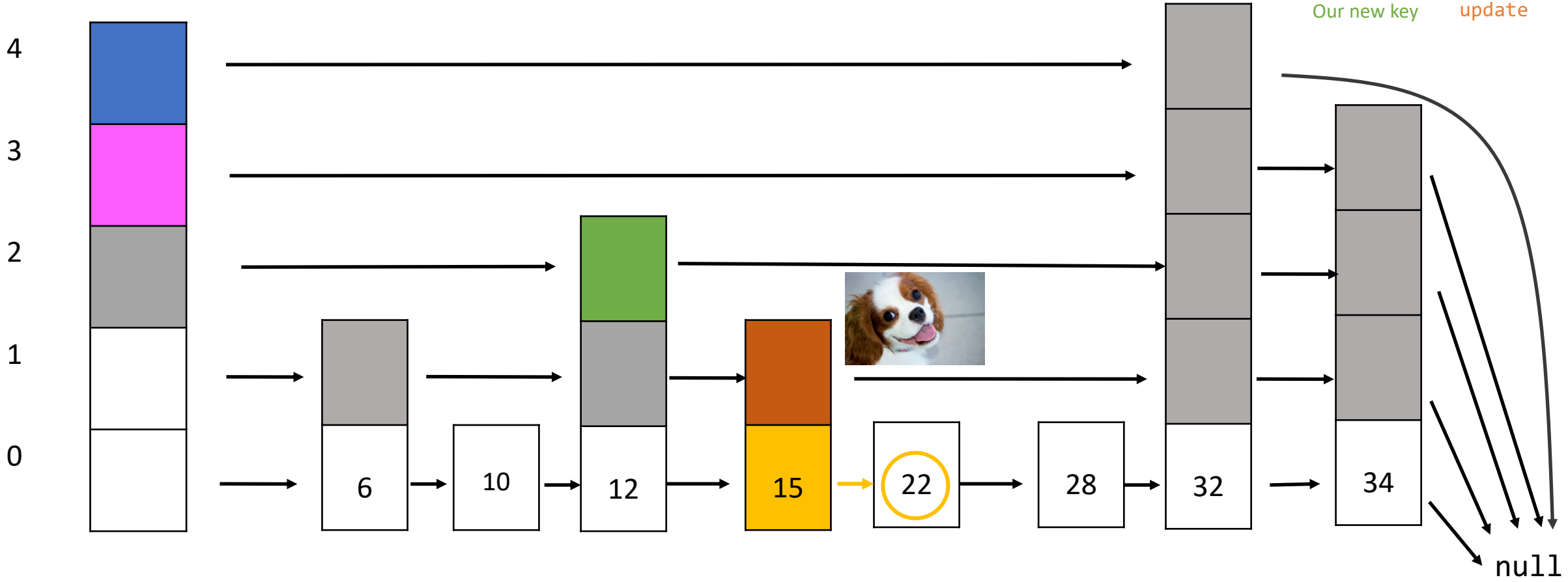
Task: Insert 25

Insertion Example

Task: Insert 25

Insertion Example

22 < 25, keep going….



Task: Insert 25

Insertion Example

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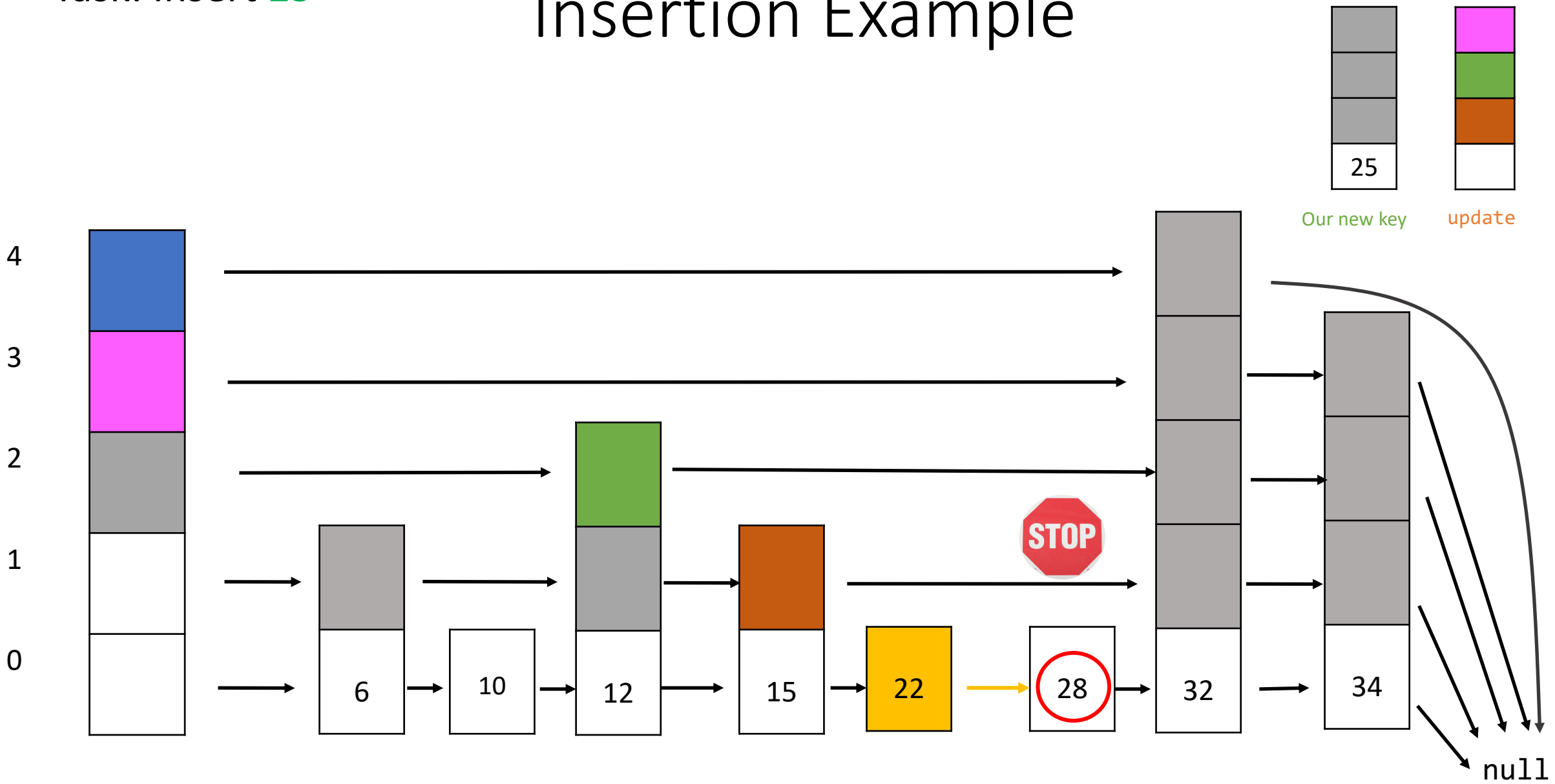
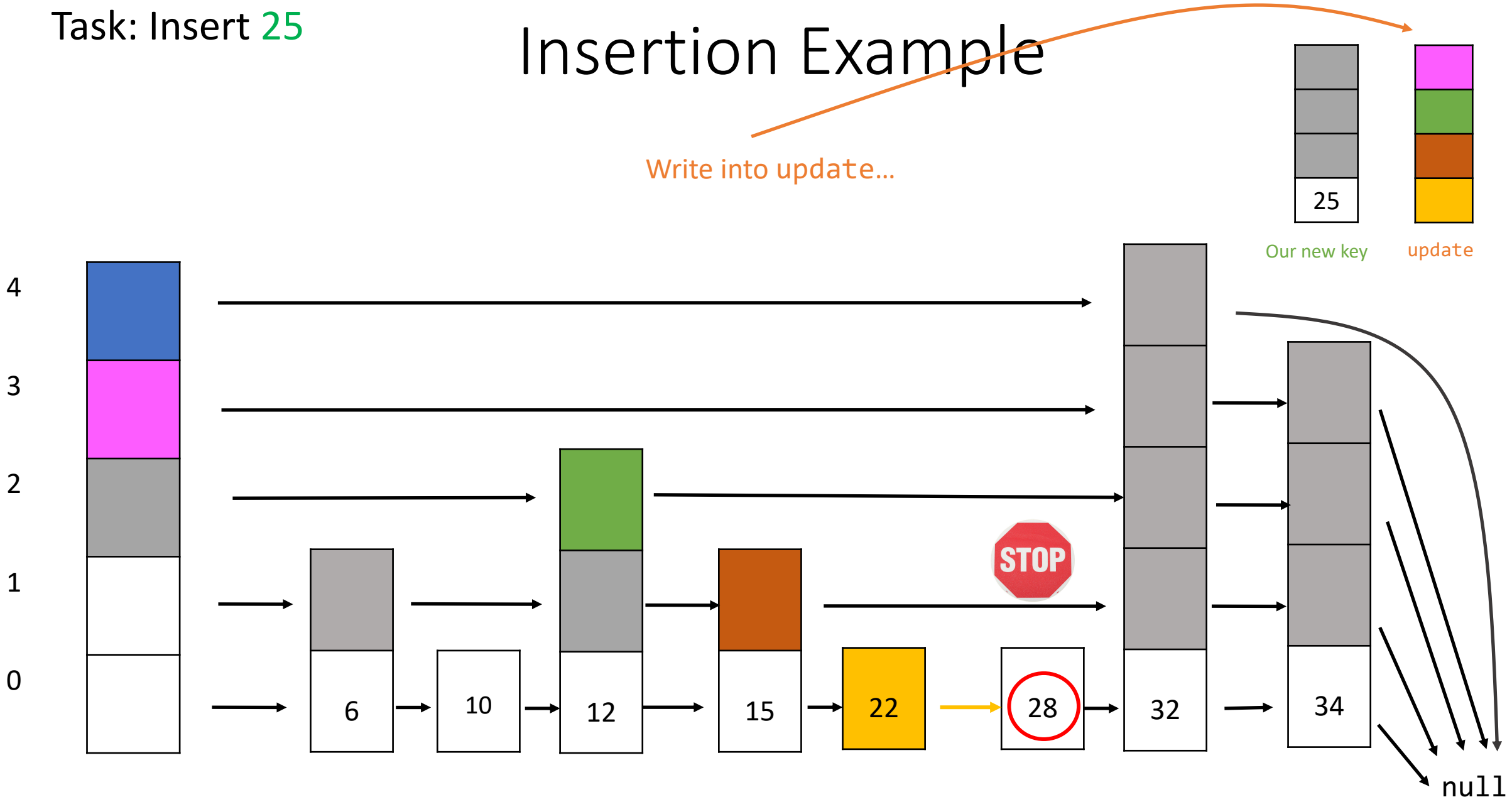
Insertion Example



Task: Insert 25

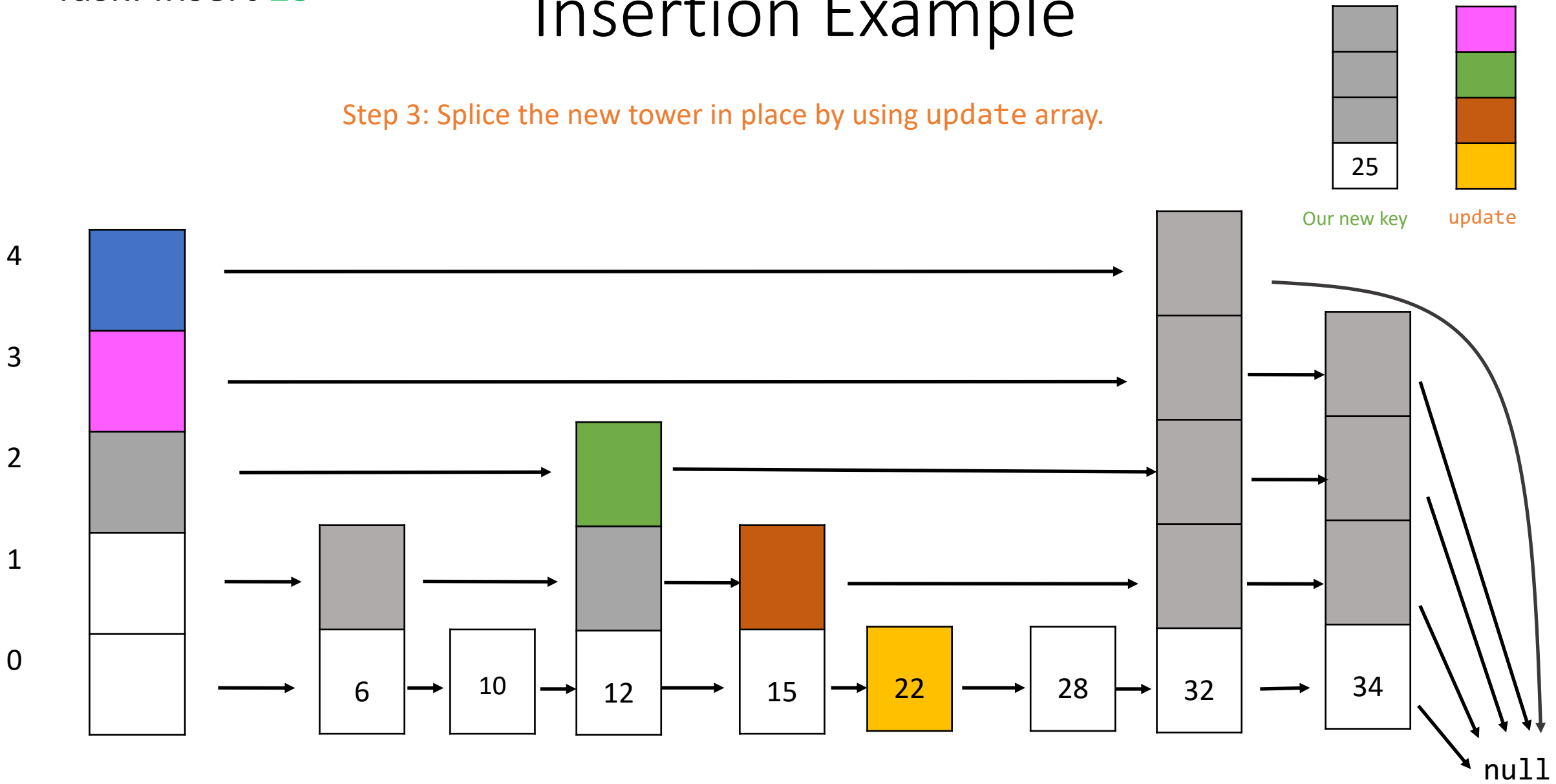
Insertion Example



Task: Insert 25

Insertion Example

Step 3: Splice the new tower in place by using `update` array.

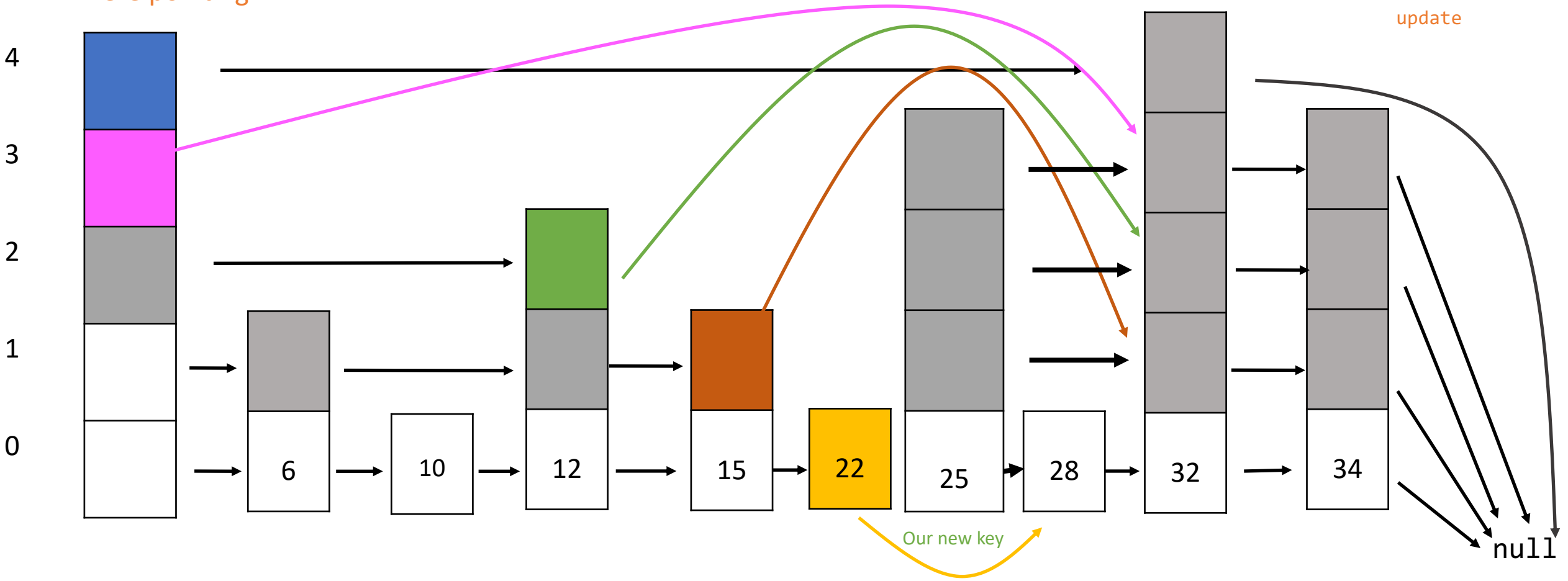


Task: Insert 25

Insertion Example

Step 3: Splice the new tower in place by using `update` array.

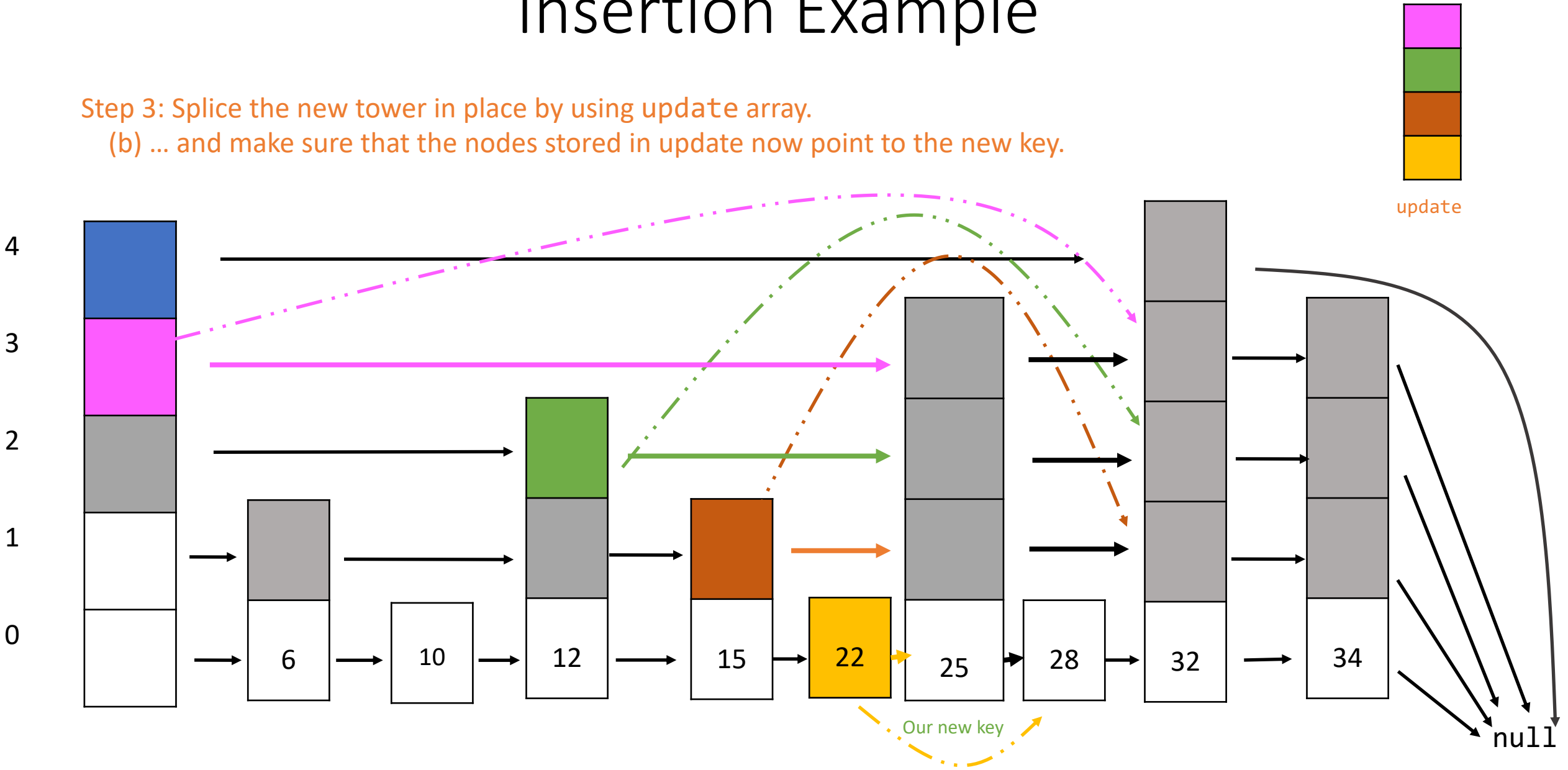
(a) Make all the forward pointers of the new key point wherever the nodes stored in `update` were pointing….



Task: Insert 25

Insertion Example

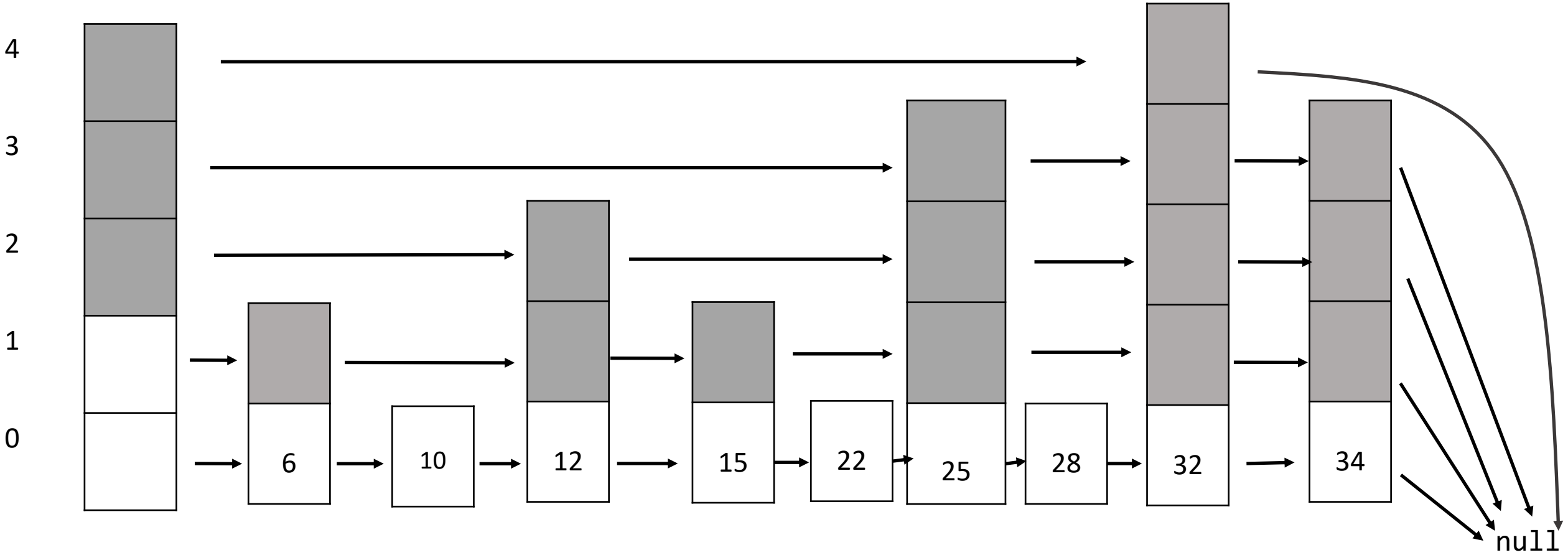
Step 3: Splice the new tower in place by using `update` array.
(b) … and make sure that the nodes stored in update now point to the new key.



Task: Insert 25

Insertion Example

Done! ☺



Live insertion demo!

- Using only the pseudocode from Shaffer's book (p. 519 in the Java edition) and a coin, let's see if we can insert some ints into an initially empty Skiplist!

```
/** Insert a record into the skiplist */
public void insert(Key k, E newValue) {
  int newLevel = randomLevel();  // New node's level
  if (newLevel > level)          // If new node is deeper
    AdjustHead(newLevel);        //   adjust the header
  // Track end of level
  SkipNode<Key,E>[] update =
        (SkipNode<Key,E>[])new SkipNode[level+1];
  SkipNode<Key,E> x = head;        // Start at header node
  for (int i=level; i>=0; i--) { // Find insert position
    while((x.forward[i] != null) &&
          (k.compareTo(x.forward[i].key()) > 0))
      x = x.forward[i];
    update[i] = x;               // Track end at level i
  }
  x = new SkipNode<Key,E>(k, newValue, newLevel);
  for (int i=0; i<=newLevel; i++) {      // Splice into list
    x.forward[i] = update[i].forward[i]; // Who x points to
    update[i].forward[i] = x;            // Who y points to
  }
  size++;                        // Increment dictionary size
}
```

Figure 16.4 Implementation for the Skip List `Insert` function.

```
/** Pick a level using a geometric distribution */
int randomLevel() {
  int lev;
  for (lev=0; DSutil.random(2) == 0; lev++); // Do nothing
  return lev;
}
```

Skiplist insertion demo!

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}

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int randomLevel() {
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  return lev;
}
```

Worksheet exercise!

Deletion

- Deletion works very much like insertion.
- Two differences:
  1. We don't have a need to sample a height.
  2. We use the information in the ***update*** array differently: instead of connecting nodes "before" the new tower to it and connecting the new tower to subsequent towers, we remove the existing pointers.

Analysis

- Searching for an appropriate position dominates the cost for both insertions and deletions.
- Unit cost: visiting a node on our way to the appropriate position.
- Therefore, the length of the path that connects the sought key with `header[levels-1]` dominates our complexity.
- It can be proven (e.g check out "Analysis of expected search cost" here) that, with very high probability, this length is bounded above by a scalar multiple of $\log_2 n$.

Analysis

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- Unit cost: visiting a node on our way to the appropriate position.
- Therefore, the length of the path that connects the sought key with `header[levels-1]` dominates our complexity.
- It can be proven (e.g check out "Analysis of expected search cost" here) that, with very high probability, this length is bounded above by a scalar multiple of $\log_2 n$.
- So, with very high probability, searching a key k in a skiplist ℓ is $\mathcal{O}(\log_2 n)$!

Why skiplists?

- Excellent distributed performance.
  - Locked **and** lock-free implementations exist
  - E.g check out Java's `ConcurrentSkipList`
- Compare with AVL trees and Red-Black trees where locking entire paths or even the entire tree might be necessary because of successive rotations.
- Local updates ensure that you only need to lock at most two towers per insertion, and only at the very end.
- Header node immutable; never needs to be locked.

Why skiplists?

- Insensitivity in insertion order.
  - AVL + Red-Black trees might make balance guarantees, but a bad insertion order can affect performance of insertions!
- Ease of implementation
  - No recursion overhead: core of search algorithm is a double loop (while inside a for)
- Linearity of data and local updates allow for distribution of data across nodes (very good horizontal scalability)
  - Compare with AVL and Red-Black trees where storing different subtrees in different nodes might make it hard to propagate rotations from one subtree to the other.

Disadvantages of skiplists

- $O(n)$ storage overhead.
- Probabilistic instead of deterministic guarantees
  - Compare with AVL and Red-Black trees which offer guarantees of $\log_2 n$ and $2 \cdot \log_2 n$ respectively.
- Worst-case search constant bigger than AVL *(AVL-1)* Trees and B-Trees, on par with Red-Black trees.