

Sequences, Series and Summation / Product Notation

CMSC 250

Sequences and series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
 - Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$

Sequences and series

- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
 - Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
 - Examples:

Sequences and series

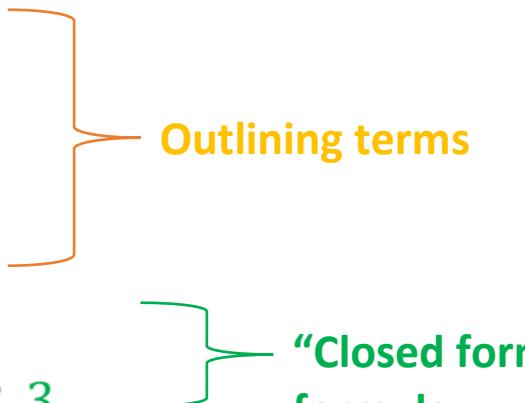
- A ***sequence*** is a **function** from the naturals to the complex numbers (but we often use reals).
 - Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
 - Examples:
 - 1, 2, 3, 4, 5, ...
 - 1.5, 2.5, 3.5, ...
 - 1, 1, 1, 1,
 - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7} \dots$
- 
- Outlining terms

Sequences and series

- A **sequence** is a **function** from the naturals to the complex numbers (but we often use reals).

- Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
- Examples:

- 1, 2, 3, 4, 5, ...
- 1.5, 2.5, 3.5, ...
- 1, 1, 1, 1,
- $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7} \dots$
- $a_n = 2^n, n = 0, 1, 2, \dots$
- $b_k = \log k + 2k, k = 1, 2, 3, \dots$



Sequences and series

- A **sequence** is a **function** from the naturals to the complex numbers (but we often use reals).

- Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
- Examples:

- 1, 2, 3, 4, 5, ...
- 1.5, 2.5, 3.5, ...
- 1, 1, 1, 1, ...
- $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7} \dots$

Outlining terms

- $a_n = 2^n, n = 0, 1, 2, \dots$
- $b_k = \log k + 2k, k = 1, 2, 3, \dots$

“Closed form”
formula

- $F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$

- $T_n = \begin{cases} 1, & \text{if } n = 1, 2 \\ 2, & \text{if } n = 3 \\ T_{n-1} + T_{n-2} + T_{n-3}, & \text{if } n \geq 4 \end{cases}$

Recursive
formula

Sequences and series

- A **sequence** is a **function** from the naturals to the complex numbers (but we often use reals).

- Typical notation: $a: \mathbb{N} \rightarrow \mathbb{C}$
- Examples:

- $1, 2, 3, 4, 5, \dots$
- $1.5, 2.5, 3.5, \dots$
- $1, 1, 1, 1, \dots$
- $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \dots$

- $a_n = 2^n, n = 0, 1, 2, \dots$

- $b_k = \log k + 2k, k = 1, 2, 3, \dots$

- $F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$

- $T_n = \begin{cases} 1, & \text{if } n = 1, 2 \\ 2, & \text{if } n = 3 \\ T_{n-1} + T_{n-2} + T_{n-3}, & \text{if } n \geq 4 \end{cases}$

Outlining terms

“Closed form”
formula

Recursive
formula

All of those are **valid ways** to describe a sequence!

Recursion: good idea?

- Example: Fibonacci

$$F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$$

- We **can** use recursion to compute, say, F_{1000}
- Is it a good idea?

Yes

No

Something
Else

Recursion: good idea?

- Example: Fibonacci

$$F_n = \begin{cases} 1, & \text{if } n = 0, 1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$$

- We **can** use recursion to compute, say, F_{1000}
- Is it a good idea?



- Recomputing terms + hidden memory cost of recursion!

Recursion: Done right

- Is there a better way to compute F_{1000} ?

Yes

No

Something
Else

Recursion: Done right

- Is there a better way to compute F_{1000} ?



1. Store the values of $F_0 = 1$, $F_1 = 1$ in an array A.
2. for $i = 2$ to 1000

$$F_i = A[i - 1] + A[i - 2]$$
$$A[i] = F_i$$

end

- This is a very elementary example of a **very useful technique** called **dynamic programming**.

Closed formula for Fibonacci

- The closed-form formula for F_n is:

$$F_n = \frac{1}{\sqrt{5}} \left(\underbrace{\frac{1 + \sqrt{5}}{2}}_{\phi} \right)^n - \frac{1}{\sqrt{5}} \left(\underbrace{\frac{1 - \sqrt{5}}{2}}_{\psi} \right)^n$$

- Roughly: $F_n \approx \phi^n \approx (1.618)^n$

Recursion vs closed formula

1. Computation:

- Recursion leads to a fast dynamic program.
- Classic recursion is elegant.
- Closed form: faster, but numerical issues arise.

2. Rate of growth:

- Recursion gives no hint as to **how big** F_n is.
- Closed form yields $F_n \approx (1.618)^n$

Summation notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{i=1}^k a_i$$

Summation notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{i=1}^k a_i$$

Index of term with which the sum begins

Index of last term to be included in the sum

The diagram shows the summation symbol \sum with a subscript $i=1$ and a superscript k . A green arrow points from the text "Index of term with which the sum begins" to the $i=1$ part. A blue arrow points from the text "Index of last term to be included in the sum" to the k part.

Examples

$$\sum_{i=1}^2 a_i = a_1 + a_2$$

$$\sum_{i=1}^1 a_i = a_1$$

$$\sum_{i=1}^0 a_i = ?$$

0

1

Something Else

Examples

$$\sum_{i=1}^2 a_i = a_1 + a_2$$

$$\sum_{i=1}^1 a_i = a_1$$

$$\sum_{i=1}^0 a_i = ?$$



$$\sum_{i=1}^0 a_i = 0$$

- The reason for this is that the following should hold **for any ℓ** :

$$\sum_{i=1}^{\ell} a_i + \sum_{i=\ell+1}^n a_i = \sum_{i=1}^n a_i$$

- So, for $\ell = 0$, we have the equality:

$$\sum_{i=1}^0 a_i + \sum_{i=0+1}^n a_i = \sum_{i=1}^n a_i \Rightarrow \sum_{i=1}^0 a_i = 0$$

Product Notation

- The **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

$$\prod_{i=1}^k a_i$$

Index of term with which the product begins

Index of last term to be included in the product

A diagram illustrating the components of a product notation. The symbol \prod is shown with a blue arrow pointing from the variable k above it to the top of the symbol. A red arrow points from the index $i=1$ below the symbol to the bottom of the symbol. To the left of the symbol, the text "Index of term with which the product begins" is written in green. To the right, the text "Index of last term to be included in the product" is written in blue.

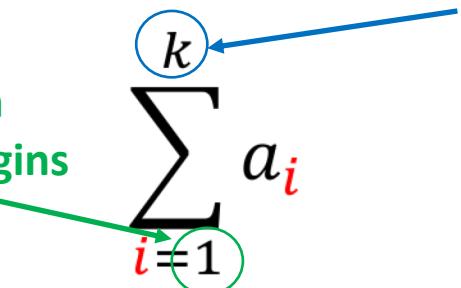
Sum / Product notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{i=1}^k a_i$$

Index of term with which the sum begins

Index of last term to be included in the sum



- Their **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

Sum / Product notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{i=1}^k a_i$$

Index of term with which the sum begins

Index of last term to be included in the sum

- Their **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

$$\prod_{i=1}^k a_i$$

Index of term with which the product begins

Index of last term to be included in the product

Sum / Product notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their **sum**, $a_1 + a_2 + a_3 + \dots + a_k$ is denoted as:

$$\sum_{r=1}^k a_r$$

“Running” (or “looping” indices can be anything we want! (i, j, k, \dots) as long as I use the same variable in the Σ and Π symbols and the variable representing the sequence term!

- Their **product**, $a_1 \cdot a_2 \cdot \dots \cdot a_k$ is denoted as:

$$\prod_{j=1}^k a_j$$

Sum-Product notation

- We can have certain *exclusionary conditions* under the Σ and Π symbols.
- Examples:

$$\sum_{\substack{m=0 \\ m \text{ even}}}^{100} s_m$$

$$\sum_{\substack{m=0 \\ 3 \mid m}}^{100} s_m \neq \sum_{\substack{m=0 \\ 3 \mid s_m}}^{100} s_m$$

Series and partial sums

- A **series** is the sum of all elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \dots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted S_n , is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^n a_i = a_0 + a_1 + a_2 + \dots + a_n$$

Or 1, if we start at 1

Series and partial sums

- A **series** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \dots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted S_n , is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^n a_i = a_0 + a_1 + a_2 + \dots + a_n$$

Or 1, if we start at 1

- Question (1): What is S_0, S_1, S_2, \dots ?

Series and partial sums

- A **series** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \dots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted S_n , is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^n a_i = a_0 + a_1 + a_2 + \dots + a_n$$

Or 1, if we start at 1

- Question (1): What is S_0, S_1, S_2, \dots ? A sequence of partial sums!
- Question (2): What is $S_n - S_{n-1}$?

Series and partial sums

- A **series** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \dots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted S_n , is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

$$S_n = \sum_{i=0}^n a_i = a_0 + a_1 + a_2 + \dots + a_n$$

Or 1, if we start at 1

- Question (1): What is S_0, S_1, S_2, \dots ? A sequence of partial sums!
- Question (2): What is $S_n - S_{n-1}$? a_n

Famous sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, a_0 + d, a_1 + d, a_2 + d \dots \text{where } d \in \mathbb{R}$$

The diagram shows the first four terms of an arithmetic sequence: $a_0, a_0 + d, a_1 + d, a_2 + d$. Below each term, a red bracket groups them together, and a red label $\alpha_1, \alpha_2, \alpha_3$ is placed below the brackets. This visualizes the common difference d as the difference between consecutive terms.

Famous sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, a_0 + d, a_1 + d, a_2 + d \dots \text{where } d \in \mathbb{R}$$

The diagram illustrates an arithmetic sequence starting with a_0 . The first term is a_0 . The second term is $a_0 + d$. The third term is $a_1 + d$. The fourth term is $a_2 + d$. The fifth term is $a_3 + d$. Red brackets are placed under the first three terms: a_0 , $a_0 + d$, and $a_1 + d$. These brackets are labeled α_1 , α_2 , and α_3 respectively, indicating the common difference d between consecutive terms.

- Question: which among the following is the correct characterization for a_n ?

$d \cdot a_{n-1}$

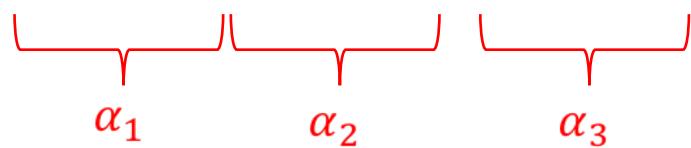
$\alpha_0 + d \cdot a_{n-1}$

$\alpha_0 + n \cdot d$

$\alpha_0 + (n - 1) \cdot d$

Famous sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, a_0 + d, a_1 + d, a_2 + d \dots \text{where } d \in \mathbb{R}$$


- Question: which among the following is the correct characterization for a_n ?

$$d \cdot a_{n-1}$$

$$\alpha_0 + d \cdot a_{n-1}$$

$$\alpha_0 + n \cdot d$$

$$\alpha_0 + (n - 1) \cdot d$$

A question for you

- In the arithmetic progression:

$$a_0, a_0 + d, a_1 + d, a_2 + d \dots \text{where } d \in \mathbb{R}$$

- Should we allow $d = 0$?

YES

NO

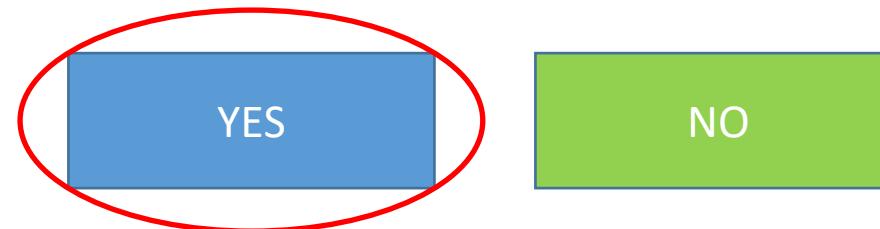
A question for you

- In the arithmetic progression:

$$a_0, a_0 + d, a_1 + d, a_2 + d \dots \text{where } d \in \mathbb{R}$$

- Should we allow $d = 0$?

It will be a pretty
boring sequence, but it
will still be a sequence!



Famous Sequences

- **Geometric sequence (or progression):**

$$a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \dots$$

The diagram illustrates a geometric sequence starting with a_0 . The next term is $m \cdot a_0$, which is grouped with a_0 by a red bracket and labeled α_1 below it. The term $m \cdot a_1$ is grouped with a_1 by a red bracket and labeled α_2 below it. The term $m \cdot a_2$ is grouped with a_2 by a red bracket and labeled α_3 below it. The sequence continues with ellipsis after $m \cdot a_2$.

Famous Sequences

- **Geometric sequence (or progression):**

$$a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \dots , \quad m \in \mathbb{R}$$

The diagram shows a sequence of terms: $a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \dots$. Red brackets are placed under the terms $m \cdot a_0$, $m \cdot a_1$, and $m \cdot a_2$. Below these bracketed terms, the labels α_1 , α_2 , and α_3 are written in red.

- Question: which among the following is the correct characterization for a_n ?

$(m - 1)^n \cdot a_0$

$m^n a_0$

$m^{n-1} a_0$

$m \cdot n \cdot a_0$

Famous Sequences

- **Geometric sequence (or progression):**

$$a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \dots , \quad m \in \mathbb{R}$$

The diagram shows a sequence of terms: $a_0, m \cdot a_0, m \cdot a_1, m \cdot a_2 \dots$. Below the sequence, red brackets group the first three terms: a_0 , $m \cdot a_0$, and $m \cdot a_1$. These brackets are labeled below them as α_1 , α_2 , and α_3 respectively. The term $m \cdot a_2$ is shown without a bracket below it.

- Question: which among the following is the correct characterization for a_n ?

$(m - 1)^n \cdot a_0$

$m^n a_0$

$m^{n-1} a_0$

$m \cdot n \cdot a_0$

The Gauss story



- Gauss was a great mathematician (1777-1855)
- When Gauss was in 1st grade, the class was misbehaving.
- For punishment, the teacher made everyone compute

$$1 + 2 + \cdots + 100$$

- Gauss did it in 2 minutes. Can you?

The Gauss trick

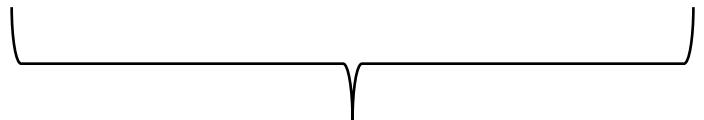


$$S = 1 + 2 + \dots + 100$$

$$S = 100 + 99 + \dots + 1$$

+

$$2S = 101 + 101 + \dots + 101$$

 A brace consisting of two curved lines meeting at a central vertical tick mark, spanning the width of the sequence of 101's.

100 terms

$$\Rightarrow 2S = 101 * 100 = 10100 \Rightarrow S = 5050$$

And now the rest of the story



And now the rest of the story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

And now the rest of the story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
------	-------	--------

And now the rest of the story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$

And now the rest of the story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$
1980	3 rd	$1 + 2 + \dots + 80$

And now the rest of the story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$
1980	3 rd	$1 + 2 + \dots + 80$
2000s	1 st	$1 + 2 + \dots + 100$

And now the rest of the story



- This is a **complete fabrication!**
- This is how this story has progressed over time:

YEAR	GRADE	SERIES
1960	5 th	$1 + 2 + \dots + 60$
1980	3 rd	$1 + 2 + \dots + 80$
2000s	1 st	$1 + 2 + \dots + 100$
2020	Nursery School	$1 + 2 + \dots + 120$

Our conjecture:

Famous Sequences

- **Harmonic:**

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$

- **Fibonacci:** $F_0 = F_1 = 1$ and $\forall n \geq 2, F_n = F_{n-1} + F_{n-2}$

$$1, 1, 2, 3, 5, 8, 13, 21, \dots$$

What we'll do next

- We will have an intro to **induction**.
- The following can be proven via induction:

$$\sum_{i=1}^n i = \frac{n(n + 1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n + 1)(2n + 1)}{6}$$