Intro to Combinatorics
(“that n choose 2 stuff”)

CMSC 250

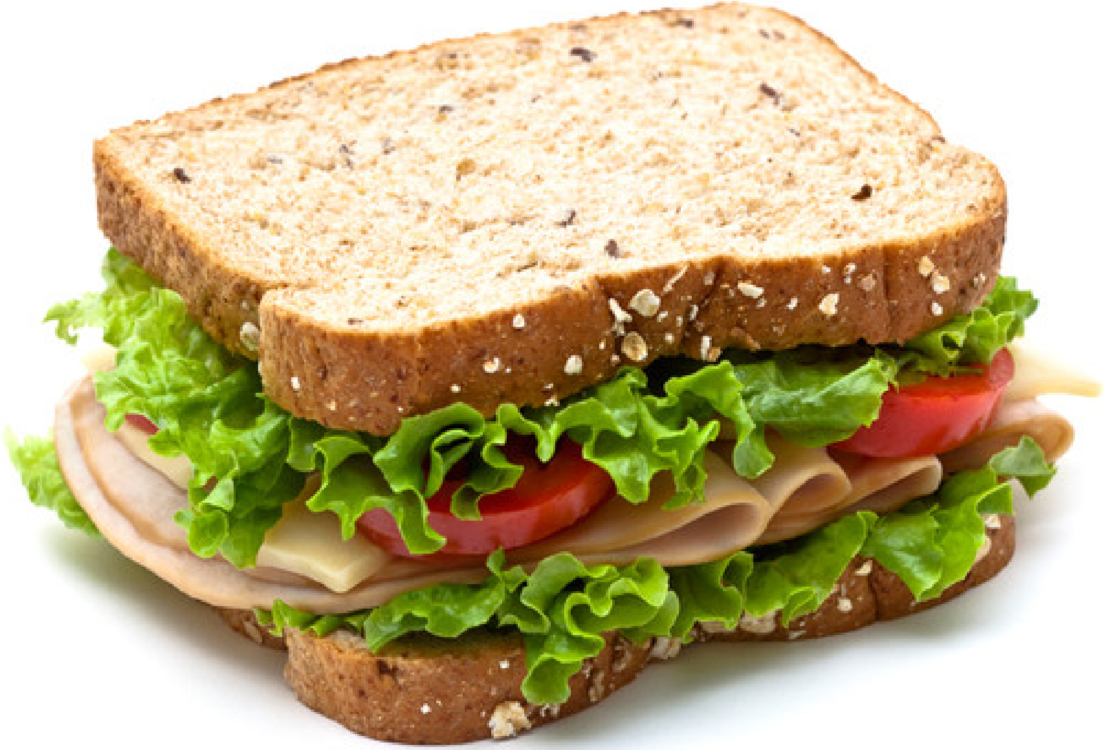
Reminders

- **Exam weight decreased to 70%**
  - Midterms 20%, Final 30%
- **Homework weight increased to 20%**
  - LaTeX EC down to 10% per homework.
  - LaTeX ***not required,*** and choice of submission for a given homework is **not binding** for others.
  - We drop the lowest one.
- **Quiz weight increased to 10%**
  - First one on Friday, autogradable, on Gradescope.
  - Multiple choice, T/F
  - Content: The two lectures. Syllabus, videos, slides, relevant book chapters (they've been posted).
  - We also drop the lowest one.

Next week

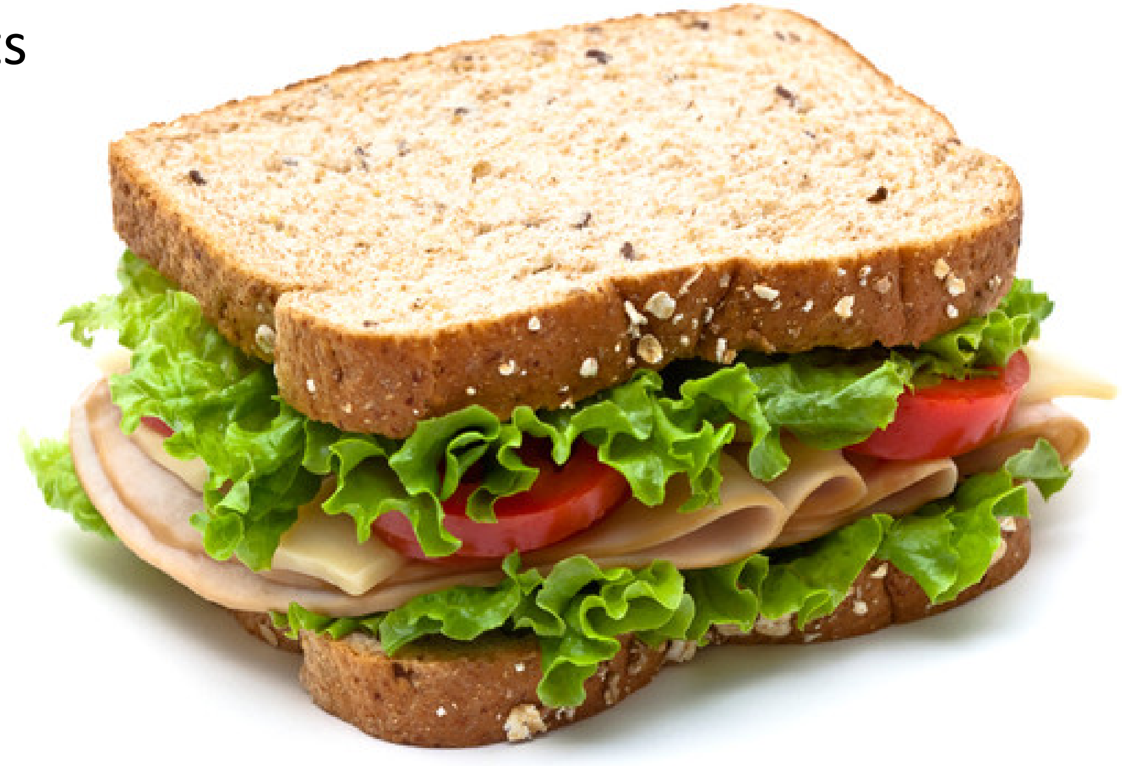
- **Monday**:
  - Discussion session on addition rule, multiplication rule, permutations, r-permutations.
  - You are given your 1st homework.
  - Submit your quiz by 11:59pm!
- **Tuesday**: We finish up perms / combs
- **Wednesday**: Discussion Session with exercises on all things perms / combs.
- **Thursday**: We expand on something that you will see for the first time Monday (in discussion).
- **Friday**: Your 2nd quiz!

Jason’s sandwich



Jason’s sandwich

- Suppose that Jason has the following ingredients to make a sandwich with:
  - White or black bread
  - Butter, Mayo or Honey Mustard
  - Romaine Lettuce, Spinach, Kale
  - Bologna, Ham or Turkey
  - Tomato or egg slices



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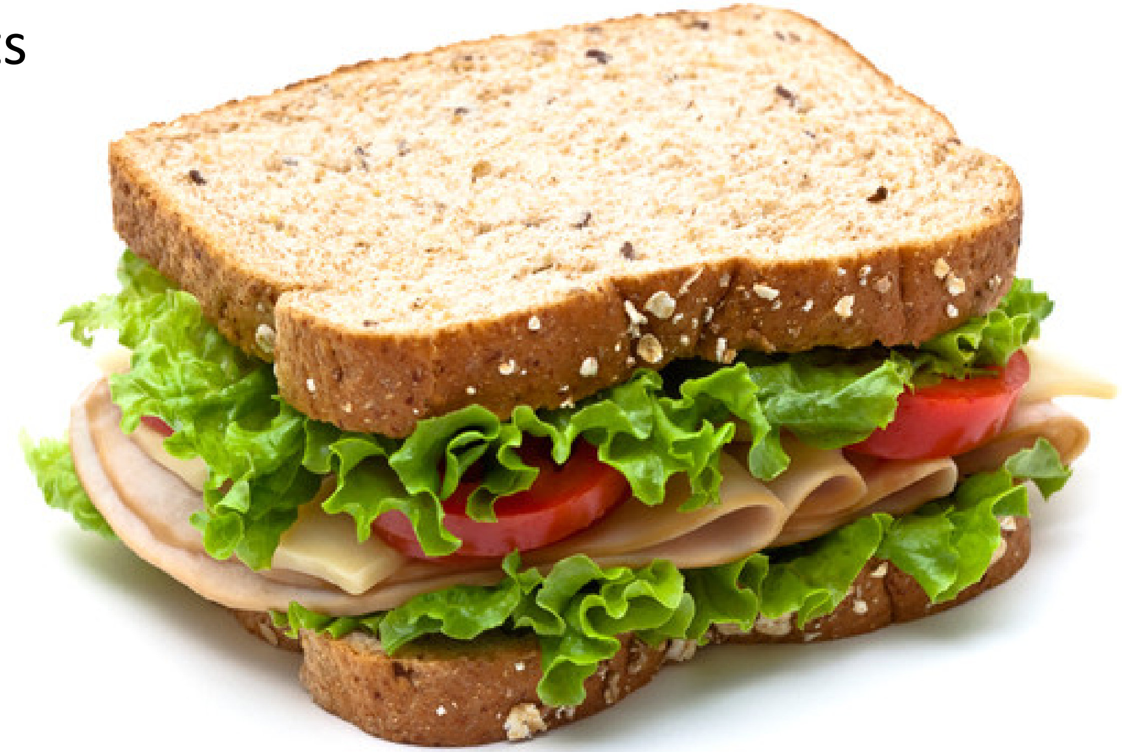
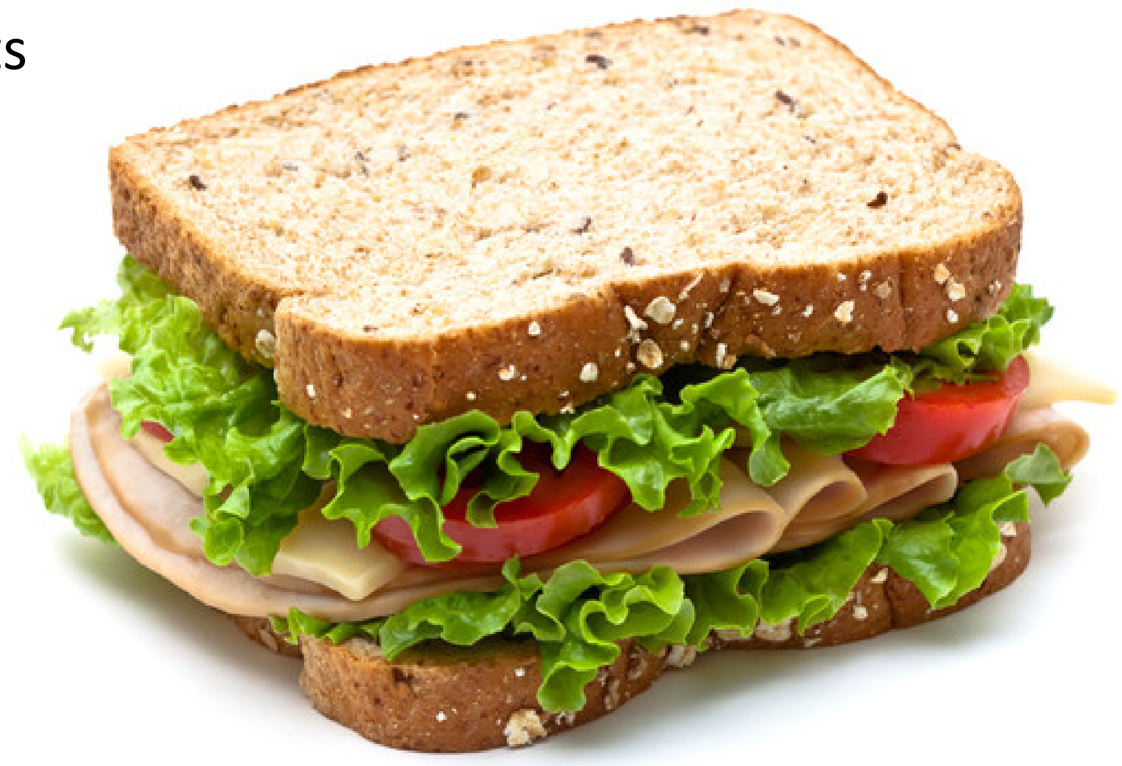


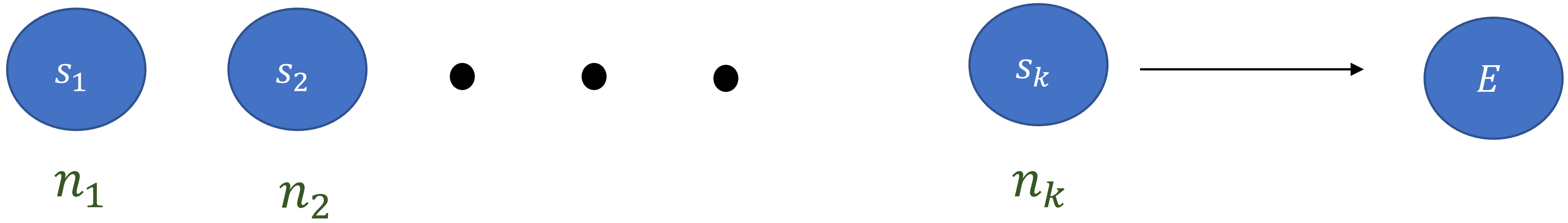
Jason’s sandwich

- Suppose that Jason has the following ingredients to make a sandwich with:
  - White or black bread 2 options
  - Butter, Mayo or Honey Mustard 3 options
  - Romaine Lettuce, Spinach, Kale 3 options
  - Bologna, Ham or Turkey 3 options
  - Tomato or egg slices 2 options
- **How many different sandwiches can Jason make?**
  - $2 \times 3 \times 3 \times 3 \times 2 = 4 \times 27 = 108$



The multiplication rule

- Suppose that E is some experiment that is conducted through k sequential steps $s_1, s_2, \dots, s_k$, where every s_i can be conducted in n_i different ways.



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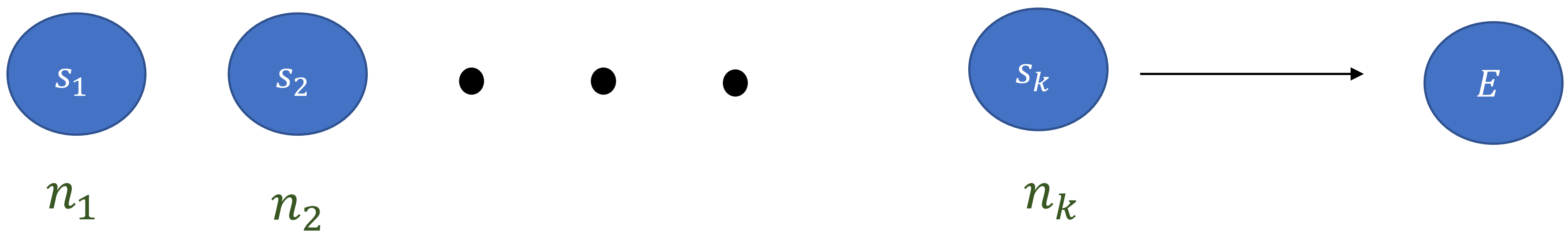

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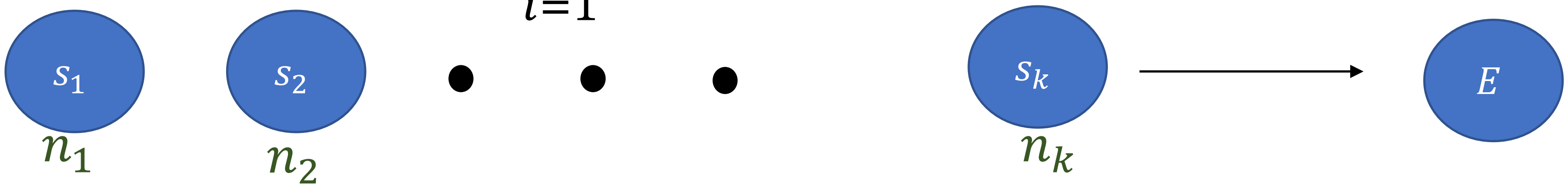
  - Example: $E =$ "*sandwich preparation*", $s_1 =$ "*chop bread*", $s_2 =$ "*choose condiment*", …
- Then, the total number of ways that E can be conducted in is

$$\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k$$



A familiar example

- How many subsets are there of a set of 4 elements?
- Example: $\{a, b, c, d\}$
  - a: in or out. 2 choices.
  - b: in or out. 2 choices.
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$2 \times 2 \times 2 \times 2 = 2^4 = 16$ subsets.

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$2 \times 2 \times 2 \times 2 = 2^4 = 16$ subsets.

- Generalization: there are 2^n subsets of a set of size n.
  - But you already knew this.

Picking projects

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In how many different ways can Murad pick a project?

- By the multiplication rule: $20 \times 15 \times 40 = 12000$

Picking projects

- Suppose now that Murad has to pick one project for CMSC420.
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- There are 20 + 15 + 40 = 75 projects available, so 75 different ways.

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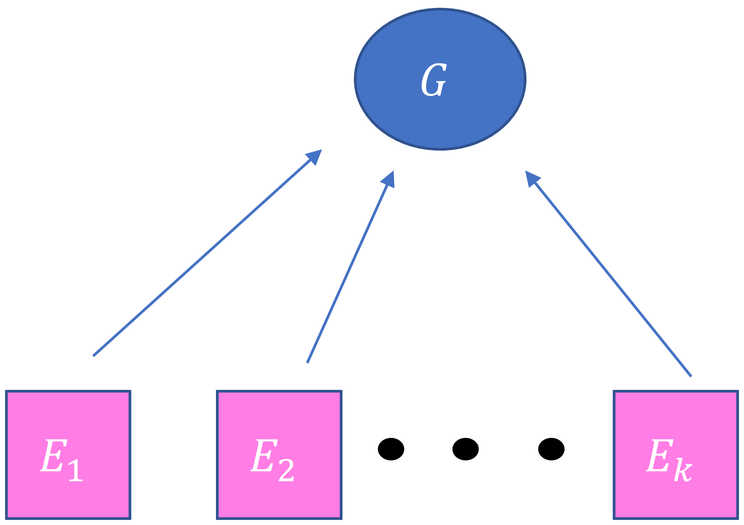
In how many different ways can Murad pick a project now?

- There are 20 + 15 + 40 = 75 projects available, so 75 different ways.
- Note that if a project was shared between two categories, we'd have an overcount! (74 instead of 75)
  - It is ***your responsibility*** to be able to understand when an overcount occurs in a question!

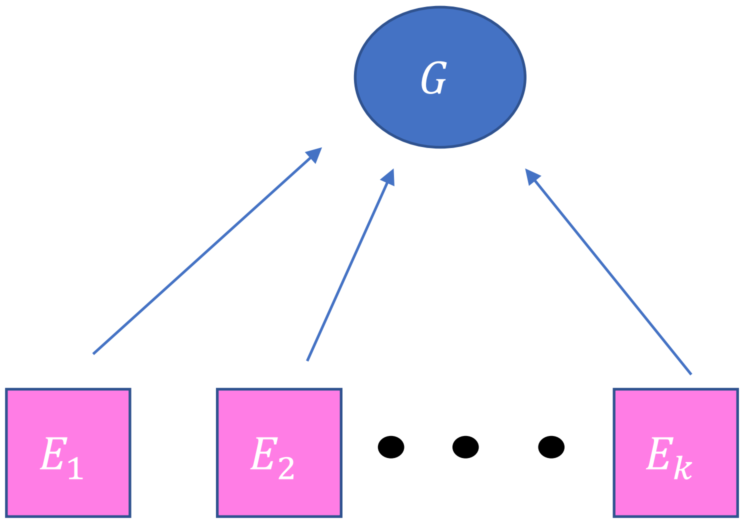
Addition (sum) rule

- Suppose that we have a goal G that can be reached when **any given one** of the experiments E_i succeed:



Addition (sum) rule

- Then, if every E_i can be attained in $|E_i|$ ways, the total number of ways in which G can happen is $|E_1| + |E_2| + \cdots + |E_k| = \sum_{i=1}^{k} |E_i|$

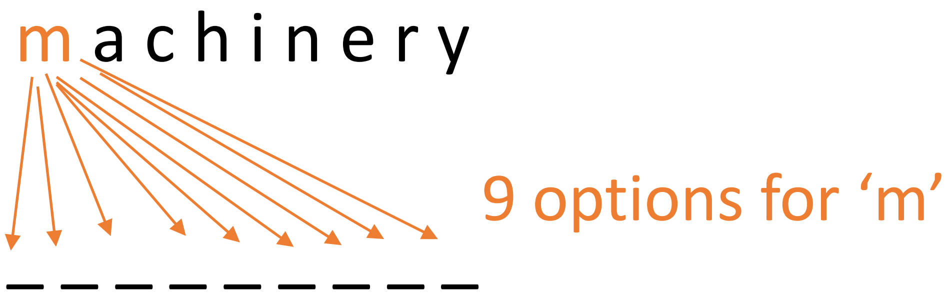


Subtraction rule

Permutations

- The problem of finding a permutation of a string is a *classic* application of the multiplication rule.
- Consider the string "machinery".
- A **permutation** of "machinery" is **a string which results by re-organizing the characters of "machinery" around**.
  - Examples: chyirenma, hcyranemi, machinery (!)
  - Question: How many permutations of "machinery" are there?

Permutations



Permutations

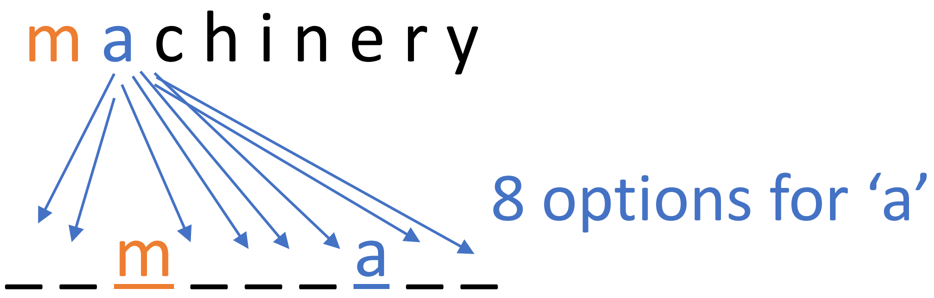
Permutations

m a c h i n e r y

_ _ m _ _ _ _ _ _

8 options for ‘a’

Permutations



Permutations

m a c h i n e r y

7 options for 'c'…

_ _ m _ _ _ _ a _ _

Permutations

m a c h i n e r y

_ _ m _ _ c a _ _

7 options for ‘c’…

Permutations

m a c h i n e r y

6 options for ‘h’…

_ _ <u>m</u> _ _ <u>c</u> <u>a</u> _ _

Permutations

m a c h i n e r y

6 options for ‘h’…

h _ m _ _ c a _ _

Permutations

m a c h i n e r y

5 options for ‘i’

h _ m _ _ c a _ _

Permutations

m a c h i n e r y

5 options for 'i'

h _ m _ _ c a _ i

Permutations

m a c h i n e r y

h _ m _ _ c a _ i

4 options for ‘n’

Permutations

m a c h i n e r y

4 options for ‘n’

h _ m _ n c a _ i

Permutations

m a c h i n e r y

h _ m _ n c a _ i

3 options for ‘e’

Permutations

m a c h i n e r y

3 options for ‘e’

h e m _ n c a _ i

Permutations

m a c h i n e r y

2 options for ‘r’

h e m _ n c a _ i

Permutations

m a c h i n e r y

h e m _ n c a r i

2 options for ‘r’

Permutations

m a c h i n e r y

h e m _ n c a r i

1 option for ‘y’

Permutations

m a c h i n e r y

h e m y n c a r i

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Total #possible permutations $= 9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880$

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That's a lot! (Original string has length 9)

Permutations

m a c h i n e r y

h e m y n c a r i

1 option for ‘y’

Total #possible permutations $= 9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880$

In general, for a string of length n we have ourselves $n!$ different permutations!

That’s a lot! (Original string has length 9)

Break!

Permutations

- Now, consider the string “puzzle”.
- How many permutations are there of this string?
- Note that two letters in puzzle are the same.

Permutations

- Now, consider the string "puzzle".
- How many permutations are there of this string?
- Note that two letters in puzzle are the same.
  - Call the first z z_1 and the second z z_2
- So, one permutation of puz_1z_2le is puz_2z_1le
  - But this is clearly equivalent to $puz_1z_2le,$ so we wouldn't want to count it!
  - So clearly the answer is **not** **6! (6 is the length of "puzzle")**
  - **What is the answer?**

Thought Experiment

- Pretend the two ‘z’s in “puzzle” are different, e.g z_1, z_2
  - Then, 6! permutations, as discussed
  - Now we have the “equivalent” permutations, for instance

$$z_1 pul z_2 e$$
$$z_2 pul z_1 e$$

  - We want to **not doublecount** these!

Thought Experiment

$$z_1 pulz_2 e$$
$$z_2 pulz_1 e$$

We want to **not doublecount** such permutations!

- Then, we need to stop pretending that the ‘z’s are different
  - **Bad news: 6! is overcount ☹**
  - **Good news: 6! is an overcount in a precise way! ☺ Everything is counted <u>exactly twice!</u>**

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  - **Bad news: 6! is overcount ☹**
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  - Answer: $\frac{6!}{2}$

Permutations

- Now, consider the string "scissor".
- **How many permutations of "scissor" are there?**
- **Note that three** letters in "scissor" are the same.
    - As previously discussed, the answer cannot be **7! (7 is the length of "scissor")**

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  - Observe all the possible positions of the various 's's:
    - $s_1cis_2s_3or$
    - $s_1cis_3s_2or$
    - $s_2cis_1s_3or$
    - $s_2cis_3s_1or$
    - $s_3cis_1s_2or$
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    - $s_1 c i s_2 s_3 o r$
    - $s_1 c i s_3 s_2 o r$
    - $s_2 c i s_1 s_3 o r$
    - $s_2 c i s_3 s_1 o r$
    - $s_3 c i s_1 s_2 o r$
    - $s_3 c i s_2 s_1 o r$

3! = 6 different ways to arrange those 3 's's

Final answer

- Think of it like this: *How many times can I fit essentially the same string into the number of permutations of the original string?*
- Therefore, the total #permutations when not assume different 's's is

$$\frac{7!}{3!} = \frac{1 \times \cancel{2 \times 3 \times} 4 \times 5 \times 6 \times 7}{1 \cancel{\times 2 \times 3}} = 20 \times 42 = 840$$

Complex overcounting

- Consider now the string "onomatopoeia".
- 12 letters, with 4 'o's, 2 'a's
- Considering the characters being different, we have:

$$o_1 n o_2 m a t o_3 p o_4 e i a,$$

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How many such positionings of the 'o's are possible?

6	12
16	Something Else

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$$\ldots$$

$4! = 24$ different ways.

Complex overcounting

- However, we also have the two 'a's to consider!
- Fortunately, those equivalent permutations are simpler to count:

$$onoma_1topoeia_2$$
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- Key: **<u>for every one</u>** of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! ***<u>(MULTIPLICATION RULE)</u>***

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- Key: **for every one** of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! ***(MULTIPLICATION RULE)***
- Final answer:

$$\#\text{permutations} = \frac{12!}{4!\cdot 2!} = \frac{5\cdot 6\cdot \ldots \cdot 11\cdot 12}{2} = 5\cdot 6^2\cdot \ldots \cdot 10\cdot 11 = 9{,}979{,}200$$

Important “pedagogical” note

- In the previous problem, we came up with the quantity

$$\frac{12!}{4! \cdot 2!} = 9{,}979{,}200$$

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- **How you should answer in an exam:** $\frac{12!}{4! \cdot 2!}$
- **Don't perform computations, like** $9{,}979{,}200$
  - Helps **you** save time and **us when grading** ☺

For you!

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- How many non-equivalent permutations of “bookkeeper” exist?

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$$\frac{10!}{2! \cdot 2! \cdot 3!}$$

Don’t forget the third ‘e’!

More practice

- What about the #non-equivalent permutations for the word

combinatorics

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combinatorics

$$\frac{13!}{2! \cdot 2! \cdot 2!} = \cdots$$

General template

- Total # permutations of a string σ of letters of length *n* where there are n_a $'a's, n_b$ $'b's, n_c$ $'c's, \ldots n_z$ 'z's

$$\frac{n!}{n_a! \times n_b! \times \cdots \times n_z!}$$

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- Claim: This formula breaks when some letter is **not** in σ.

Yes

No

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$$\frac{n!}{n_a! \times n_b! \times \cdots \times n_z!}$$

- Claim: This formula breaks when some letter is **not** in σ.

Remember:
0! = 1

Yes

No

Sounds legit

Great-gran claims she has drunk nothing but Pepsi for 64 years and wouldn't touch water even if dying

2018-10-14

Jackie Page, 77, downs a can of the fizzy pop every morning and can guzzle up to four a day.

r-permutations

- Warning: permutations (as we've talked about them) are best presented with strings.
- r-permutations: Those are best presented with sets.
  - Note that $r \in \mathbb{N}$
  - So we can have 2-permutations, 3-permutations, etc

r-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where **order of the people matters.**

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- Examples: shortest-to-tallest or tallest-to-shortest or something-in-between

r-permutations: Example

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- Examples: Jenny-Fred-Bob or Fred-Jenny-Bob or Fred-Bob-Jenny

r-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where **order of the people matters.**
- In how many ways can I pick these people?

r-permutations: Example

I need three people for this photo. You guys figure out your order.

r-permutations: Example

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10 ways to pick the first person…

. . .

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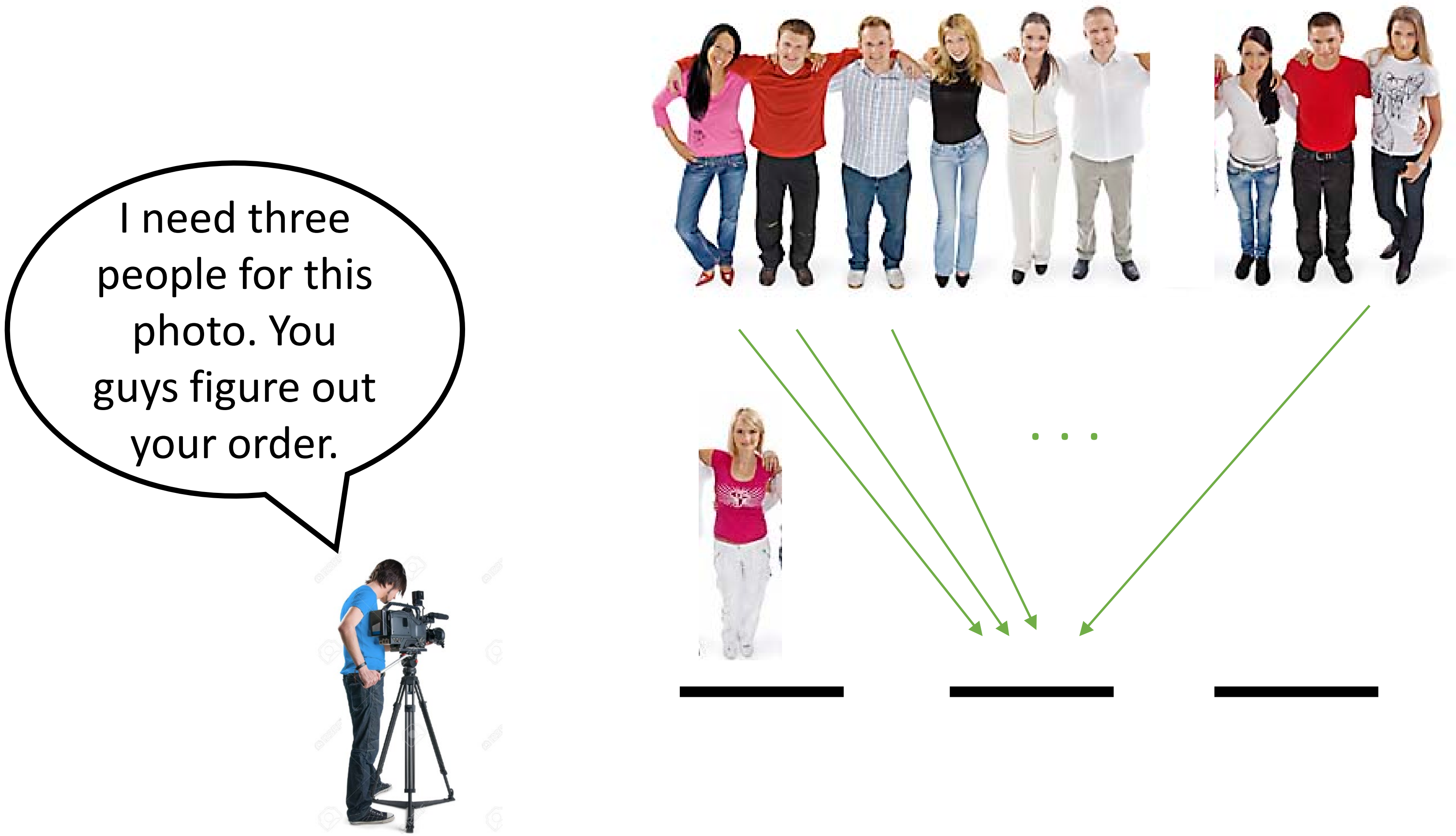

9 ways to pick the **second** person…

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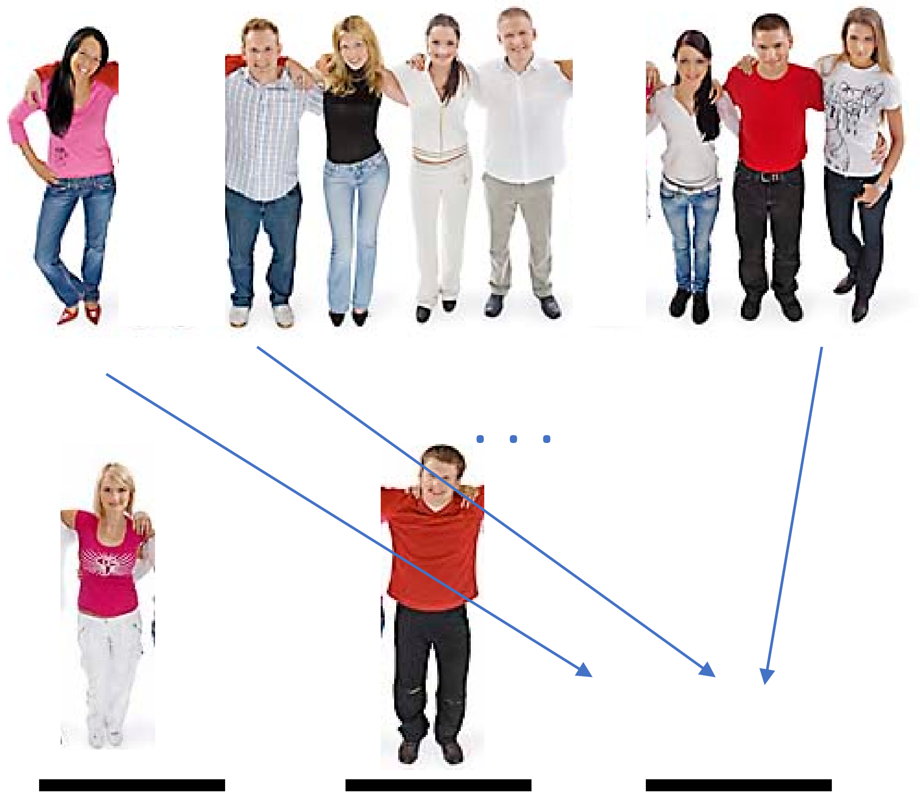

8 ways to pick the **third** person…

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8 ways to pick the **third** person…

r-permutations: Example

For a total of $10 \times 9 \times 8 = 720$ ways.

r-permutations: Example

I need three people for this photo. You guys figure out your order.

For a total of $10 \times 9 \times 8 = 720$ ways.

Note: $10 \times 9 \times 8 = \frac{10!}{(10-3)!}$

Example on Books

- Clyde has the following books on his bookshelf
  - Epp, Rosen, Hughes, Bogart, Davis, Shaffer, Sellers, Scott
- Jason wants to borrow any 5 of them and read them in the order he picks them in.
- In how many ways can Jason get smart by reading those books?

Example on Books

- Clyde has the following books on his bookshelf
  - Epp, Rosen, Hughes, Bogart, Davis, Shaffer, Sellers, Scott
- Jason wants to borrow any 5 of them and read them in the order he picks them in.
- In how many ways can Jason get smart by reading those books?

$$\frac{8!}{(8-5)!} = \frac{8!}{3!}$$

General formula

- Let $n, r \in \mathbb{N}$ such that $0 \leq r \leq n$. The total ways in which we can select r elements from a set of n elements where order matters is equal to:

$$P(n, r) = \frac{n!}{(n-r)!}$$

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"P" for permutation. This quantity is known as the r-permutations of a set with n elements.

Pop quizzes

1) $P(n, 1) = \cdots$

0	1	n	$n!$

Pop quizzes

1) $P(n, 1) = \cdots$ | 0 | 1 | n | $n!$

- Two ways to convince yourselves:
  - Formula: $\frac{n!}{(n-1)!} = n$
  - Semantics of r-permutations: In how many ways can I pick 1 element from a set of n elements? Clearly, I can pick any one of n elements, so n ways.

Pop quizzes

2) $P(n, n) = \cdots$

0	1	n	$n!$

Pop quizzes

2) $P(n,n) = \cdots$ 0   1 n $n!$

- Again, two ways to convince ourselves:
  - Formula: $\frac{n!}{(n-n)!} = \frac{n!}{0!}$
  - Semantics: $n!$ ways to pick all of the elements of a set and put them in order!

Pop quizzes

3) $P(n, 0) = \cdots$

0

1

n

$n!$

Pop quizzes

3) $P(n, 0) = \cdots$ | 0 | 1 | n | $n!$

- Again, two ways to convince ourselves:
  - Formula: $\frac{n!}{(n-0)!} = \frac{n!}{n!} = 1$
  - Semantics: Only one way to pick nothing: just pick nothing and leave!

Practice

1. How many MD license plates are possible to create?

Practice

1. How many MD license plates are possible to create? $26^2 \cdot 10^5$

Practice

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1. How many MD license plates are possible to create? $26^2 \cdot 10^5$
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3. How many words of length 10 can I construct from the English alphabet, where letters can be chosen:

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Remember these phrases!

Caption this

Combinations (that “n choose r” stuff)

- Earlier, we discussed this example:

I need three people for this photo. You guys figure out your order.

- Our goal was to pick three people for a picture, where **order of the people mattered.**

Combinations (that "n choose r" stuff)

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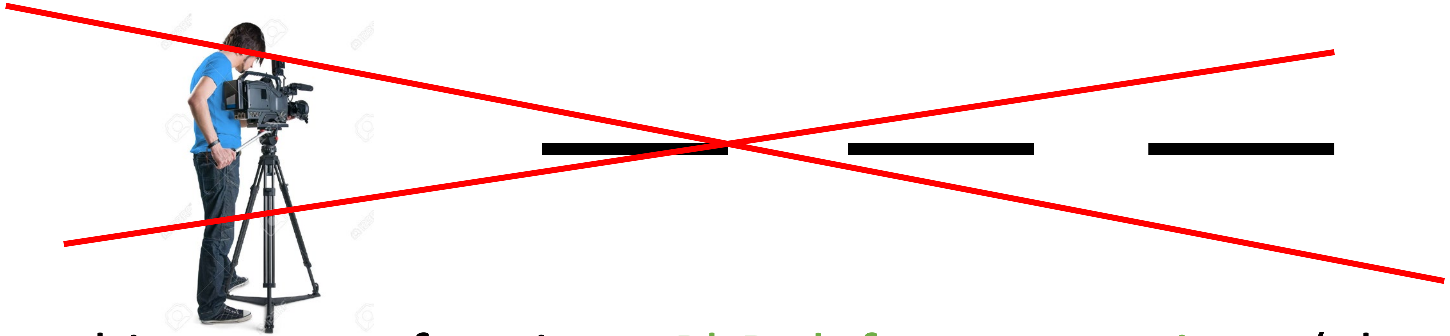

- We now change this setup to forming a PhD defense committee (also 3 people).
- In this setup, does order matter?

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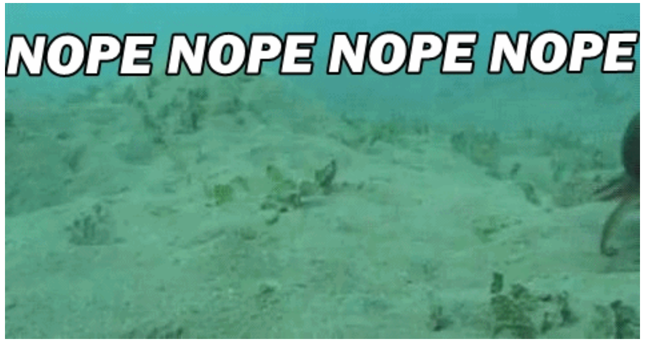

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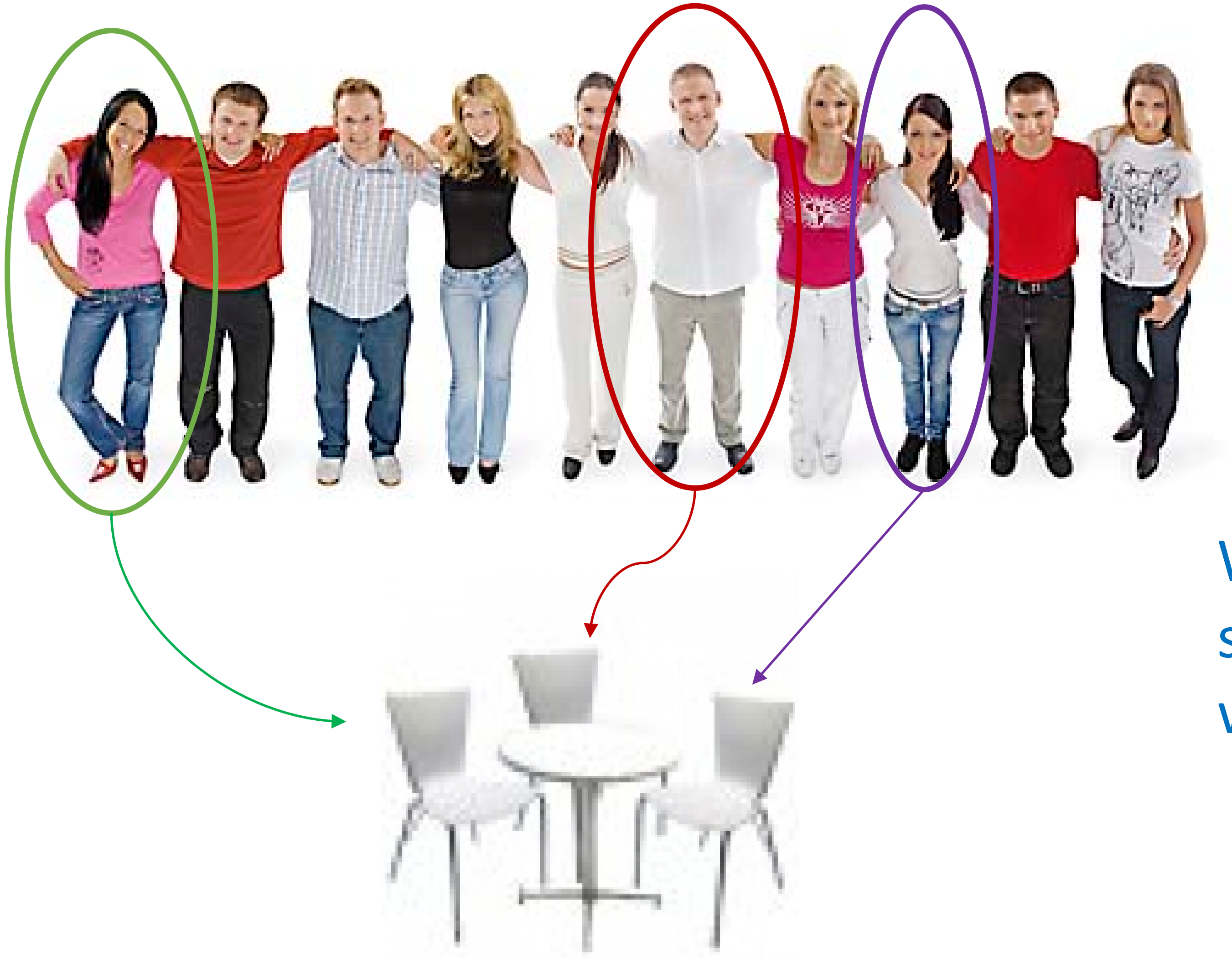
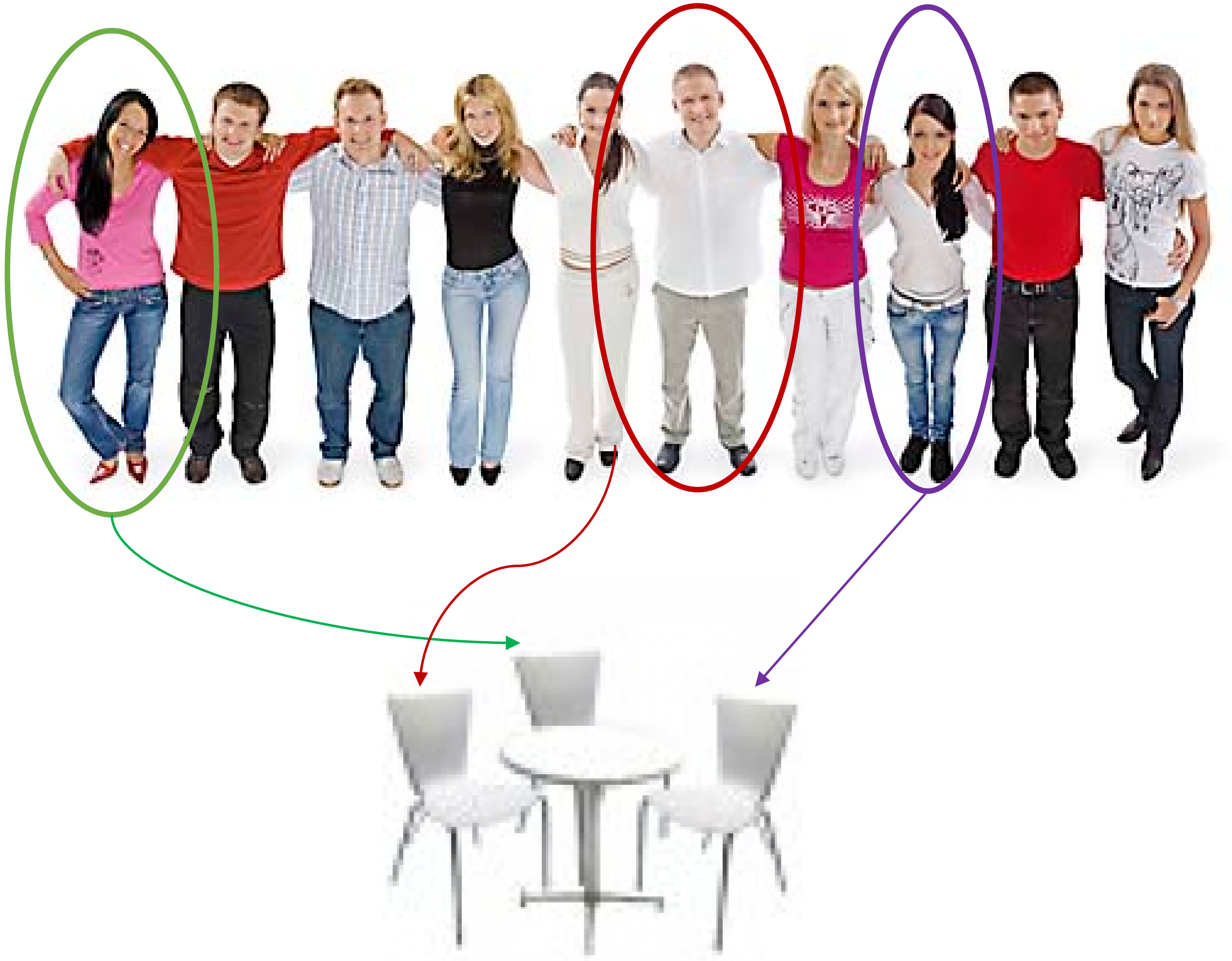


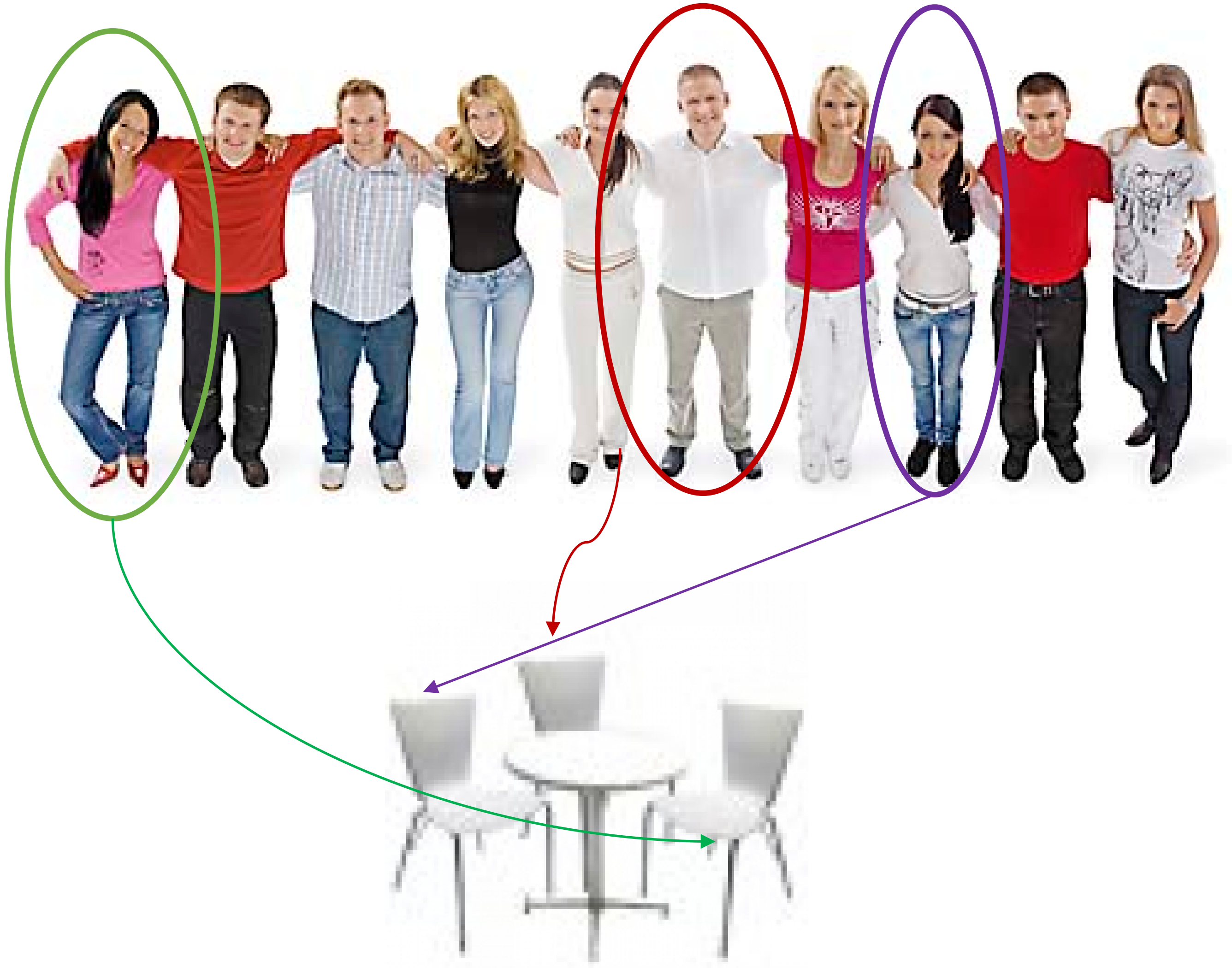
We can make this selection in $P(10, 3)$ ways...

Combinations (that "n choose r" stuff)



We can make this selection in $P(10,3)$ ways… but **since order doesn't matter**, we have 3! permutations of these people that are equivalent.

Combinations (that "n choose r" stuff)



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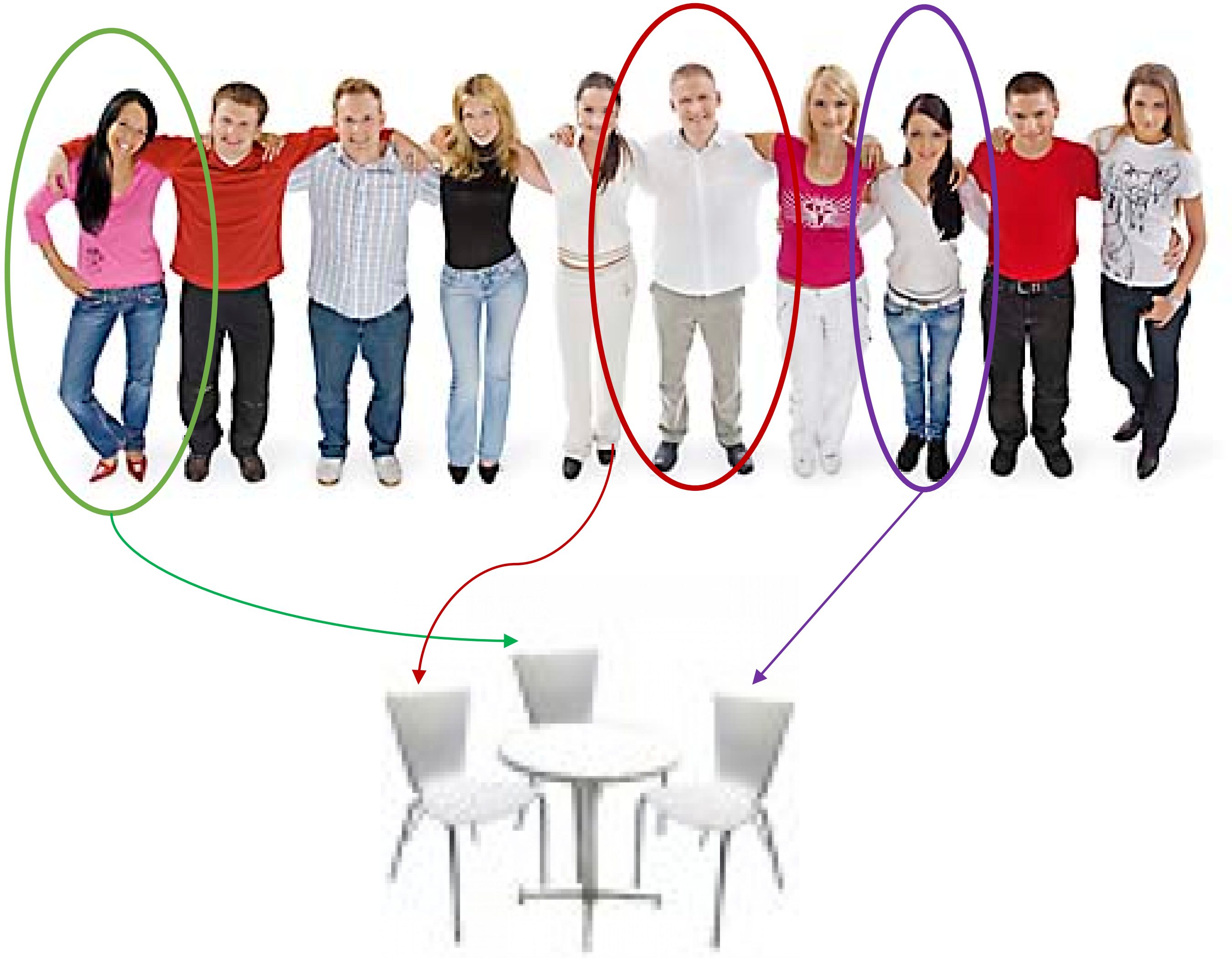

Overcount ☹

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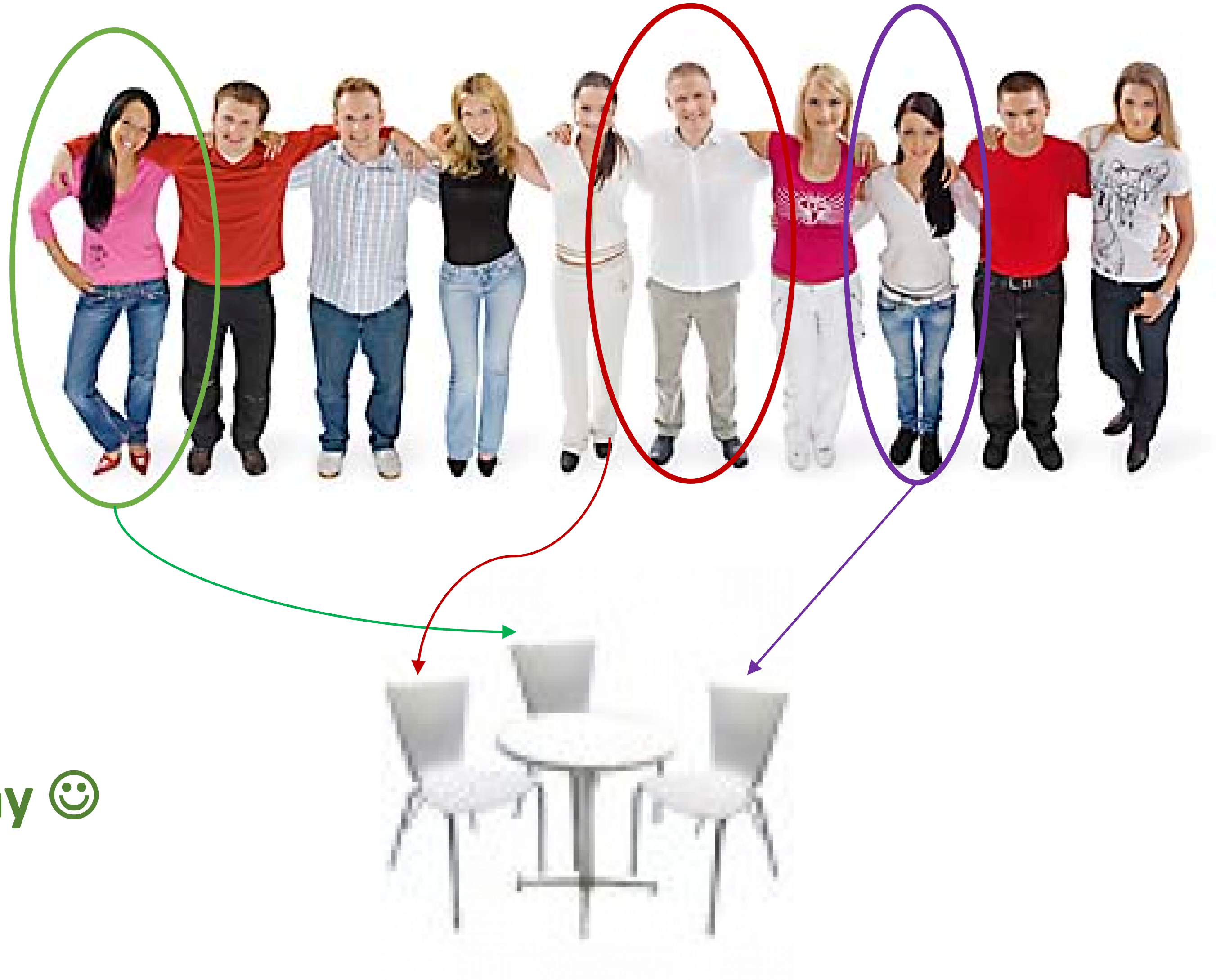

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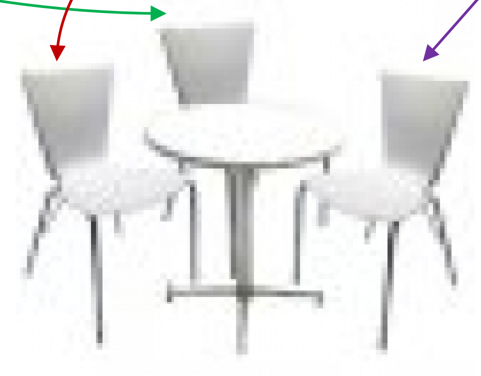

Overcount ☹

In a precise way ☺

$$\frac{P(10,3)}{3!} = \frac{10!}{7! \times 3!}$$

We can make this selection in $P(10,3)$ ways… but **since order doesn't matter**, we have $3!$ permutations of these people that are equivalent.

Closer analysis of example

- Note that essentially we are asking you: Out of a set of 10 people, how many subsets of 3 people can I retrieve?

$\binom{n}{r}$ notation

- The quantity

$$\frac{\boldsymbol{P(10,3)}}{\mathbf{3!}}$$

is the number of *3-combinations* from a set of size 10, denoted thus:

$$\binom{n}{3}$$

and pronounced “n choose 3”.

$\binom{n}{r}$ notation

- Let $n, r \in \mathbb{N}$ with $0 \leq r \leq n$
- Given a set A of size n, the total number of subsets of A of size r is:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

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- Pop quiz: $(\forall n, r \in \mathbb{N})[(0 \leq r \leq n) \Rightarrow (\binom{n}{r} \leq P(n,r))]$

True

False

$\binom{n}{r}$ notation

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- Pop quiz: $(\forall n, r \in \mathbb{N})[(0 \leq r \leq n) \Rightarrow (\binom{n}{r} \leq P(n,r))]$

Recall that

$\binom{n}{r} = \frac{P(n,r)}{r!}$ and $r! \geq 1$

True

False

Quiz

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} =$

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} = n$

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Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$ (Note how this differs from $P(n,n) = n!$)

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$ (Note how this differs from $P(n, n) = n!$)
3. $\binom{n}{0} =$

Quiz

1	n	$n!$	Sth else

1. $\binom{n}{1} = n$
2. $\binom{n}{n} = 1$ (Note how this differs from $P(n,n) = n!$)
3. $\binom{n}{0} = 1$