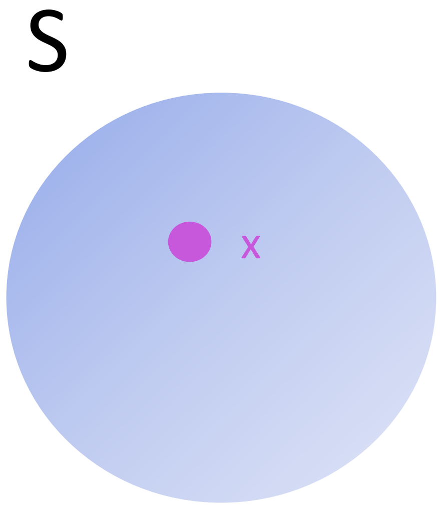
Sets & Quantifiers

CMSC250

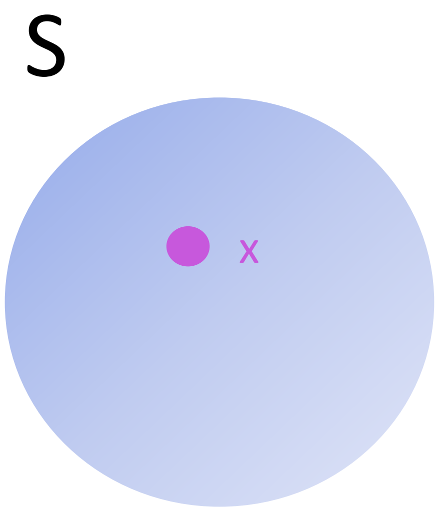
What is a set?

- A set is a collection of **distinct** objects.
- We use the notation $x \in S$ to say that S contains x.
- We'd like to know if $x \in S$ fast!
- Unless explicitly specified otherwise, **sets are unordered**.



What is a set?

- A set is a collection of **distinct** objects.
- We use the notation $x \in S$ to say that S contains x.
- We'd like to know if $x \in S$ fast!
- Unless explicitly specified otherwise, **sets are unordered**.
- Given the last two requirements, what's the best possible data structure to implement a set in memory?



Doubly Linked List

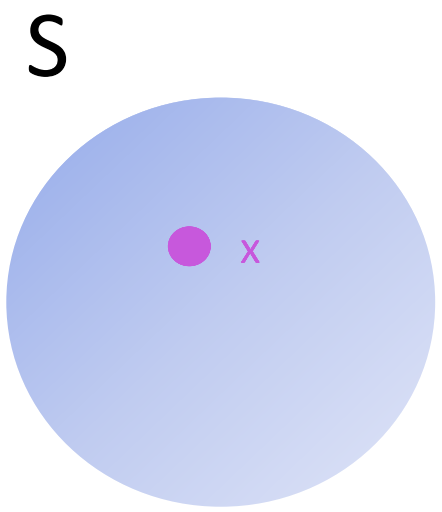
Binary Tree

Stack

Something else (what?)

What is a set?

- A set is a collection of **distinct** objects.
- We use the notation $x \in S$ to say that S contains x.
- We'd like to know if $x \in S$ fast!
- Unless explicitly specified otherwise, **sets are unordered**.
- Given the last two requirements, what's the best possible data structure to implement a set in memory?



Doubly Linked List

Binary Tree

Stack

Something else (what?)

Hash table!

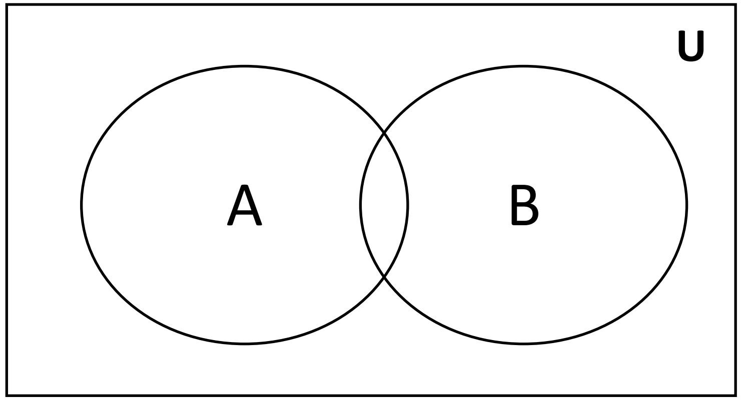
Elementary number sets

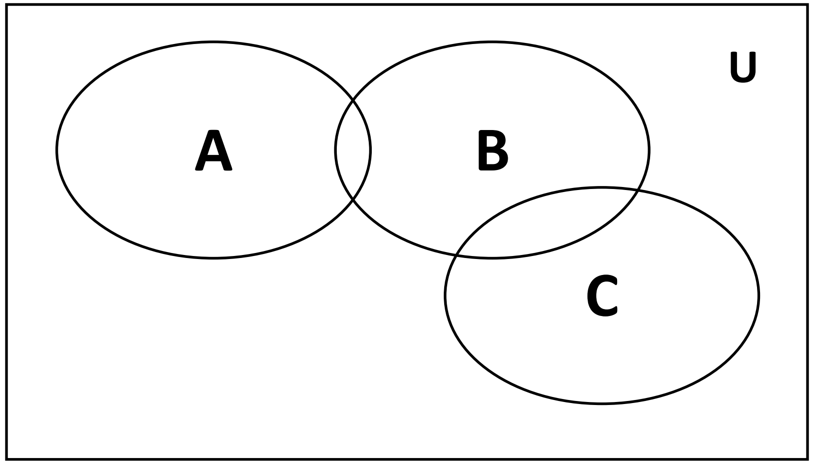
- $\mathbb{N}$: the **natural** numbers
  - $\mathbb{N} = \{\mathbf{0}, 1,2,3, \dots .\}$. In our class, $0 \in \mathbb{N}$!
- $\mathbb{Z}$: the **integers**
  - $\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$
- $\mathbb{Q}$: the **rationals**
  - $\mathbb{Q} = \{\frac{a}{b}, (a \in \mathbb{Z}) \land (b \in \mathbb{Z}) \land (b \neq 0)$
  - Any number that can be written as a ratio of integers!
- $\mathbb{R}$: the reals
  - This will typically be our "upper limit" in 250.
  - That is, we don't usually care about $\mathbb{C}$, the set of **complex** numbers

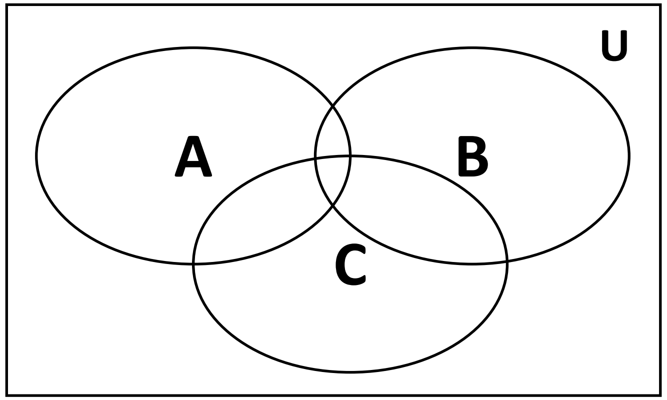
Fill those in!

	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
0	☐	☐	☐	☐
-1	☐	☐	☐	☐
$1/2$	☐	☐	☐	☐
$-1/2$	☐	☐	☐	☐
$0.333333\ldots$	☐	☐	☐	☐
$0.333333\ldots/0.11111111\ldots$	☐	☐	☐	☐
π	☐	☐	☐	☐
i, such that $i^2 = -1$	☐	☐	☐	☐

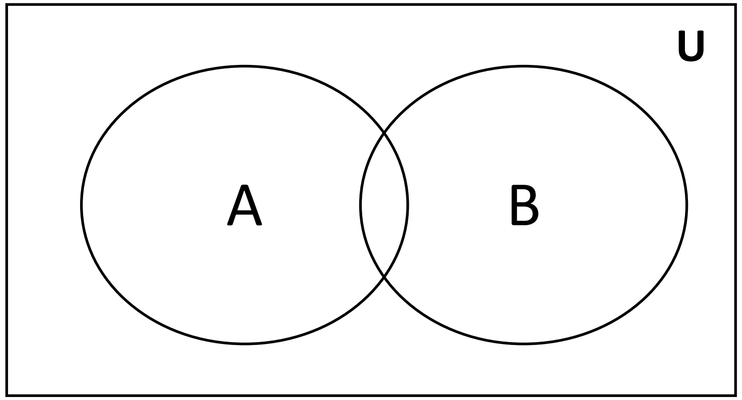
Venn Diagrams

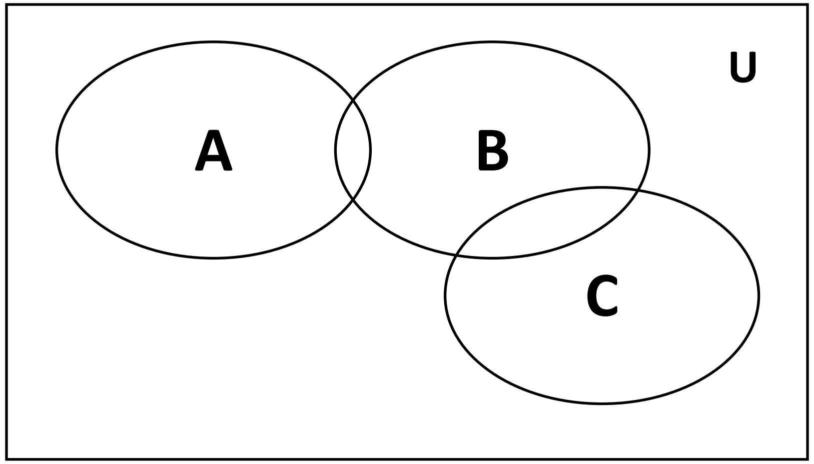


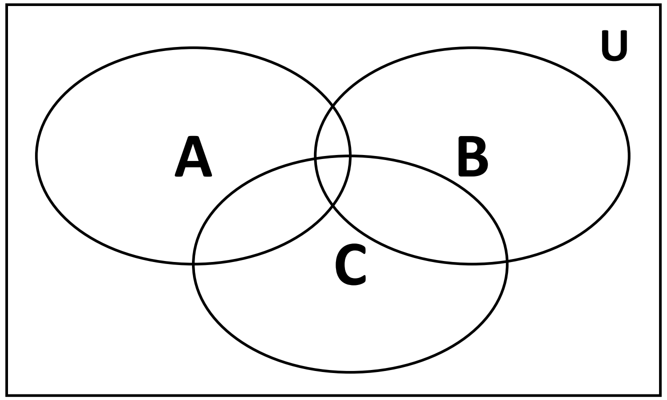




Venn Diagrams







- U is the *Universal Domain:* a set that we imagine holds every *conceivable* element.

- When talking about sets of numbers, U is usually $\mathbb{R}$, the reals.

“There exists” (∃)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$

“There exists” (∃)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True

“There exists” ($\exists$)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$

“There exists” (∃)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False

"There exists" ($\exists$)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain *D* where $(\exists n \in D)[n^2 = -1]$ is true?

Yes

No

Something else

"There exists" ($\exists$)

- Examples:
  - $(\exists x \in \mathbb{R})\ [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain *D* where $(\exists n \in D)[n^2 = -1]$ is true?

The complex numbers $\mathbb{C}$

Yes

No

Something else

“For all”

- The symbol ∀ *(LaTeX: \forall)* is read “for all”.
- Examples:
  - $(\forall x \in \mathbb{N})\ [((x > 2) \wedge (x\ is\ prime)) \Rightarrow (x\ is\ odd)]$

“For all”

- The symbol ∀ *(LaTeX: \forall)* is read “for all”.
- Examples:
  - $(\forall x \in \mathbb{N})\ [((x > 2) \wedge (x\ is\ prime)) \Rightarrow (x\ is\ odd)]$
    True

“For all”

- The symbol $\forall$ is read “for all”.
- Examples:
  - $(\forall x \in \mathbb{N})\ [((x > 2) \land (x\ is\ prime)) \Rightarrow (x\ is\ odd)]$
    True
  - $(\forall n \in \mathbb{Z})\ [n^2 \geq 0]$

“For all”

- The symbol $\forall$ is read “for all”.
- Examples:
  - $(\forall x \in \mathbb{N})\ [((x > 2) \wedge (x\ is\ prime)) \Rightarrow (x\ is\ odd)]$ True
  - $(\forall n \in \mathbb{Z})\ [n^2 \geq 0]$ True

“For all”

- Let D be the set of all students in this class who are over 8 feet tall.
- $(\forall x \in D)[x\ has\ perfect\ attendance\ so\ far!]$

True

False

Something else

“For all”

- Let D be the set of all students in this class who are over 8 feet tall.
- $(\forall x \in D)[x\ has\ perfect\ attendance\ so\ far!]$

True

False

Something else

- If disagree, need to find $x \in D$ who missed a class
- Called vacuously true!

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

True, $x = \frac{4}{5}, y = \frac{8}{5}$

Nesting quantifiers

- $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
- $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$

  True, $x = \frac{4}{5}, y = \frac{8}{5}$

- Common abbreviation: $(\exists x, y \in D)[\ldots]$
- Generally: $(\exists x_1, x_2, \ldots, x_n \in D)[\ldots]$

Alternating nested quantifiers

- Notice the differences between the following:
  - $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$
  - $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$

Alternating nested quantifiers

- Notice the differences between the following:
  - $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$ True ($\mathbb{N}$ unbounded from above)
  - $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$ False ($\mathbb{N}$ bounded from below)

- ***WHEN QUANTIFIERS ARE DIFFERENT, THEIR ORDER MATTERS!!!!!!!***

Fill this in!

Statement	**True**	**False**
$(\exists n \in \mathbb{N})[n + n = 0]$	○	○
$(\exists n \in \mathbb{N})[n + n = 1]$	○	○
$(\exists n \in \mathbb{Z})[n + n = 1]$	○	○
$(\exists x, y \in \mathbb{Z})[x + y = 1]$	○	○
$(\exists x \in \mathbb{R})[x(x + 1) = -1]$	○	○
$(\forall x, y, z \in \mathbb{R})[((x < y < z) \Rightarrow (x^2 < y^2 < z^2)]$	○	○
$(\forall x, y, z \in \mathbb{R})[((x < y < z) \Rightarrow (x^3 < y^3 < z^3)]$	○	○
$(\forall x, y \in \mathbb{N})(\exists z \in \mathbb{Q})[(x < y) \Rightarrow (x < z < y)]$	○	○

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
    1. Is true

Finding domains

- Give infinite sets *D* such that $(\forall x \in D)(\exists y \in D)[x < y]$
    1. Is true $(D = \mathbb{N}$, select $y = x + 1)$

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false

Finding domains

- Give infinite sets *D* such that $(\forall x \in D)(\exists y \in D)[x < y]$
    1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
    2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)

Finding domains

- Give infinite sets *D* such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \leq 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \leq 1] \wedge (\forall x \in D)(\exists y \in D)[x < y]$$

1. True for $D = (-\infty, 1)$

Finding domains

- Give infinite sets D such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}^{\leq 0}$, counter-example is 0)
- Do the same thing for

$$(\forall x \in D)[x \leq 1] \wedge (\forall x \in D)(\exists y \in D)[x < y]$$

1. True for $D = (-\infty, 1)$
2. False for $D = (-\infty, 1]$ (!)

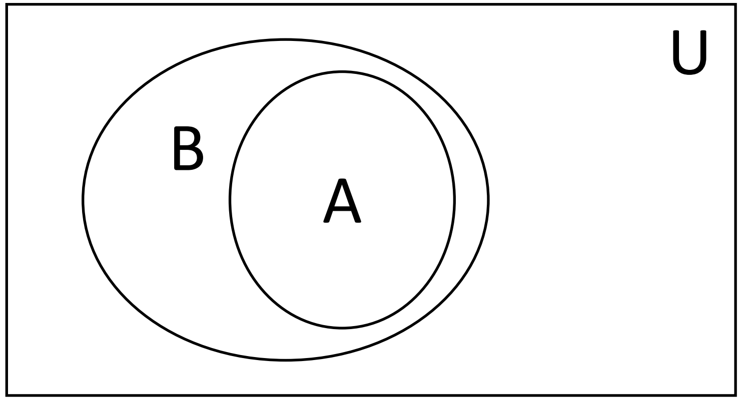
Subset

- We say that A is a subset of B $(A \subseteq B)$ iff

$$(\forall x \in A)[x \in B]$$

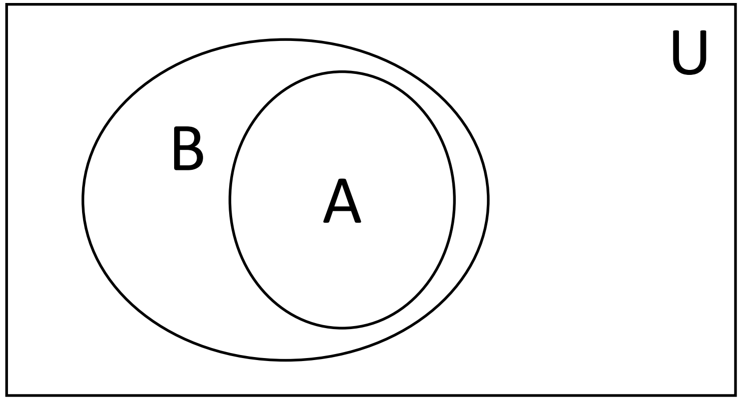
$$\Leftrightarrow$$

$$(\forall x \in U)[(x \in A) \Rightarrow (x \in B)]$$



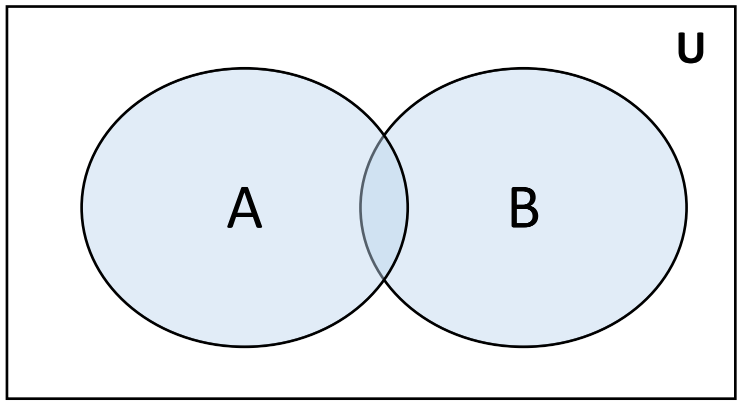
Superset and proper subset/superset

- We say that B is a **superset** of A ($B \supseteq A$) iff $A \subseteq B$.
- We say that A is a **proper subset** of B ($A \subset B$) iff $(A \subseteq B) \wedge (A \neq B)$.
- We say that B is a **proper superset** of A ($B \supset A$) iff $A \subset B$



Union

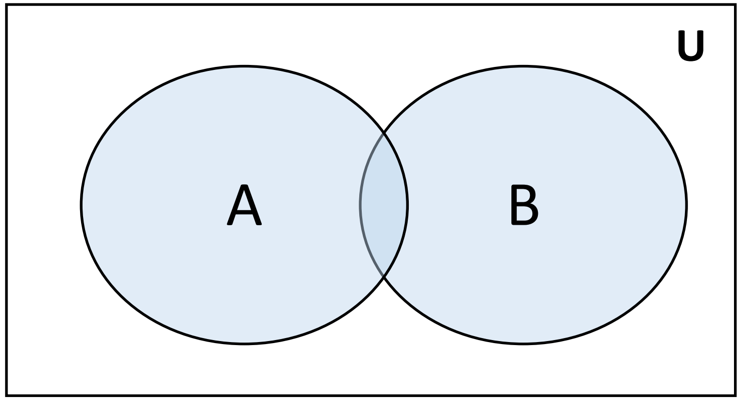
$$A \cup B = \{(x \in A) \vee (x \in B)\}$$



Union

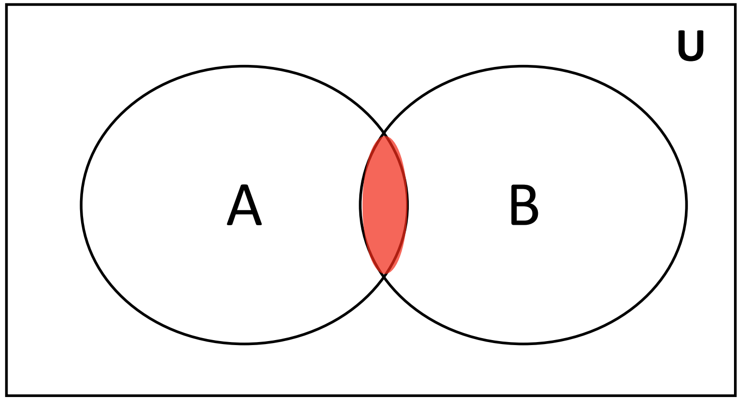
$$A \cup B = \{(x \in A) \vee (x \in B)\}$$

Connection between union and logical disjunction!



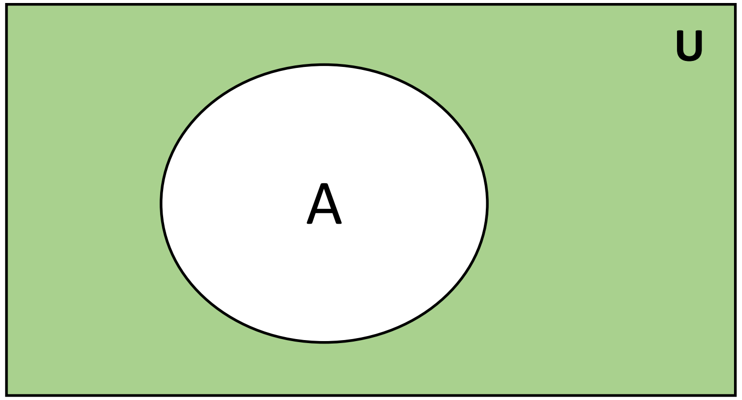
Intersection

$$A \cap B = \{(x \in A) \land (x \in B)\}$$



Absolute complement

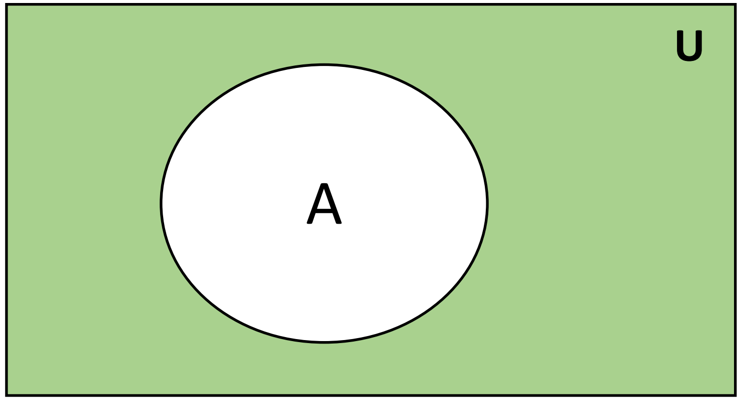
$$A^c = \{(x \notin A)\} = \{(x \in U) \wedge (\sim(x \in A))\}$$



Absolute complement

$$A^c = \{(x \notin A)\} = \{(x \in U) \land (\sim(x \in A))\}$$

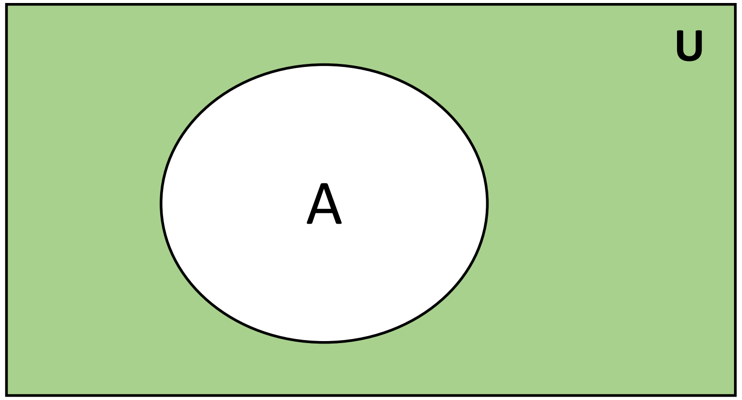
Connection between absolute complement and logical negation!



Absolute complement

$$A^c = \{(x \notin A)\} = \{(x \in U) \wedge (\sim(x \in A))\}$$

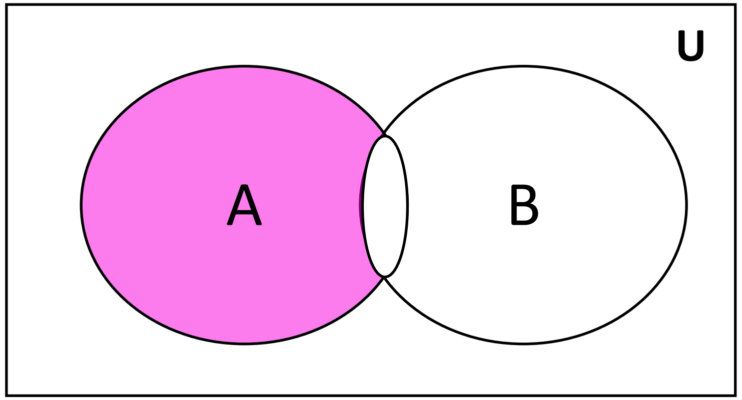
Please use this notation instead of A' or $\overline{A}$ since it's Epp-friendly



Connection between absolute complement and logical negation!

Relative Complement

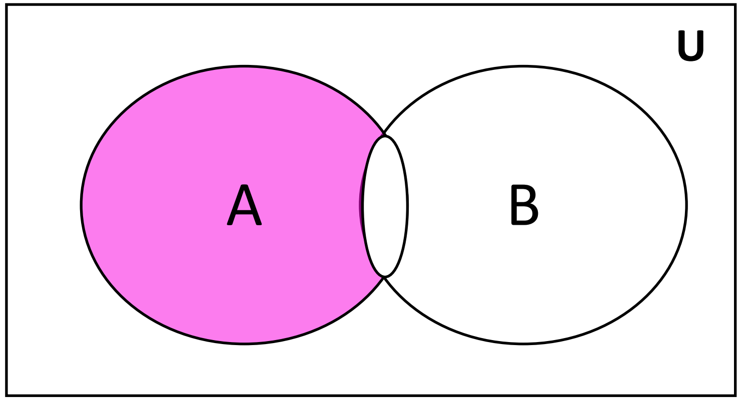
$$A - B = \{(x \in A) \wedge (x \notin B)\}$$



Relative Complement

$$A - B = \{(x \; \in A) \wedge (x \notin B)\}$$

Please use this notation instead of $A \setminus B$ since it's Epp-friendly!



Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$ T
2. $1 \in \{-2, 0, \{1\}, 3\}$

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$ T
2. $1 \in \{-2, 0, \{1\}, 3\}$ F
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$ T
2. $1 \in \{-2, 0, \{1\}, 3\}$ F
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$ F, in fact, not even mathematically correct syntax
4. $\{1\} \subseteq \{-2, 0, \{1\}, 3\}$

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$ T
2. $1 \in \{-2, 0, \{1\}, 3\}$ F
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$ F, in fact, not even mathematically correct syntax
4. $\{1\} \subseteq \{-2, 0, \{1\}, 3\}$ F
5. $\{1\} \in \{-2, 0, \{1\}, 3\}$

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$ T
2. $1 \in \{-2, 0, \{1\}, 3\}$ F
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$ F, in fact, not even mathematically correct syntax
4. $\{1\} \subseteq \{-2, 0, \{1\}, 3\}$ F
5. $\{1\} \in \{-2, 0, \{1\}, 3\}$ T
6. $\{1\} \subseteq \{-2, 0, 1, 3\}$

Careful about membership and subset!

- Be careful to distinguish between **members** of a set and **subsets** of a set…

True

False

1. $1 \in \{-2, 0, 1, 3\}$ T
2. $1 \in \{-2, 0, \{1\}, 3\}$ F
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$ F, in fact, not even mathematically correct syntax
4. $\{1\} \subseteq \{-2, 0, \{1\}, 3\}$ F
5. $\{1\} \in \{-2, 0, \{1\}, 3\}$ T
6. $\{1\} \subseteq \{-2, 0, 1, 3\}$ T

The empty set ($\emptyset$, { })

- The empty set, denoted either $\emptyset$ or { }, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

The empty set ($\emptyset$, { })

- The empty set, denoted either $\emptyset$ or { }, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

The empty set ($\emptyset$, { })

- The empty set, denoted either $\emptyset$ or { }, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

1. $\emptyset \subseteq \mathbb{N}$

The empty set ($\emptyset, \{\ \}$)

- The empty set, denoted either $\emptyset$ or $\{\ \}$, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

1. $\emptyset \subseteq \mathbb{N}$ T
2. $\emptyset \subseteq A$ for **any set** A

The empty set ($\emptyset, \{ \}$)

- The empty set, denoted either $\emptyset$ or $\{ \}$, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

1. $\emptyset \subseteq \mathbb{N}$ T
2. $\emptyset \subseteq A$ for **any set** A T
3. $\emptyset \subset A$ for **any set** A

The empty set ($\emptyset$, { })

- The empty set, denoted either $\emptyset$ or { }, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

1. $\emptyset \subseteq \mathbb{N}$
2. $\emptyset \subseteq A$ for **any set** A T
3. $\emptyset \subset A$ for **any set** A *F*
4. $\emptyset \subseteq \emptyset$

The empty set ($\emptyset, \{\ \}$)

- The empty set, denoted either $\emptyset$ or $\{\ \}$, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

1. $\emptyset \subseteq \mathbb{N}$ T
2. $\emptyset \subseteq A$ for **any set** A T
3. $\emptyset \subset A$ for **any set** A *F*
4. $\emptyset \subseteq \emptyset$ T

The empty set ($\emptyset$, { })

- The empty set, denoted either $\emptyset$ or { }, is the **unique** set with **no elements.**
  - Uniqueness can be proven, through a proof by contradiction!

True

False

1. $\emptyset \subseteq \mathbb{N}$ T
2. $\emptyset \subseteq A$ for **any set** A T
3. $\emptyset \subset A$ for **any set** A *F*
4. $\emptyset \subseteq \emptyset$ T

The powerset

- Given a set A, the powerset $\mathcal{P}(A)$ is **the set of all subsets of A.**
  - $\mathcal{P}(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
  - $\mathcal{P}(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}$
  - $\mathbb{N}^{2k}, \mathbb{N}^{2k+1}, \mathbf{P}, \ \mathbf{SQUARES} \in \mathcal{P}(\mathbb{N})$
    - And lots more…

Facts about the powerset

- The following are **facts** about the powerset:
  - Since $\emptyset \subseteq A$ for all sets A, $\emptyset \in \mathcal{P}(A)$ for all sets A
  - Since $A \subseteq A$ for all sets A, $A \in \mathcal{P}(A)$ for all sets A

Powerset quizzing

- Let $A = \{1, 2, \ldots, n\}$
- Then, $|P(A)|$

$\approx n \cdot logn$	$= n^2$	$= 2^n$	$= n!$

Powerset quizzing

- Let $A = \{1, 2, \ldots, n\}$
- Then, $|P(A)|$

$\approx n \cdot logn$

$= n^2$

$= 2^n$

$= n!$

Powerset quizzing

- $P(\{1\}) =$

Powerset quizzing

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $P(P(\{1\})) =$

Powerset quizzing

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $P(P(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
- $P(\emptyset) =$

Powerset quizzing

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $P(P(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
- $P(\emptyset) = \{\emptyset\}$
- $P(\{\emptyset\}) =$

Powerset quizzing

- $P(\{1\}) = \{\emptyset, \{1\}\}$
- $P(P(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
- $P(\emptyset) = \{\emptyset\}$
- $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$