# Inclusion / Exclusion principle

CMSC 250

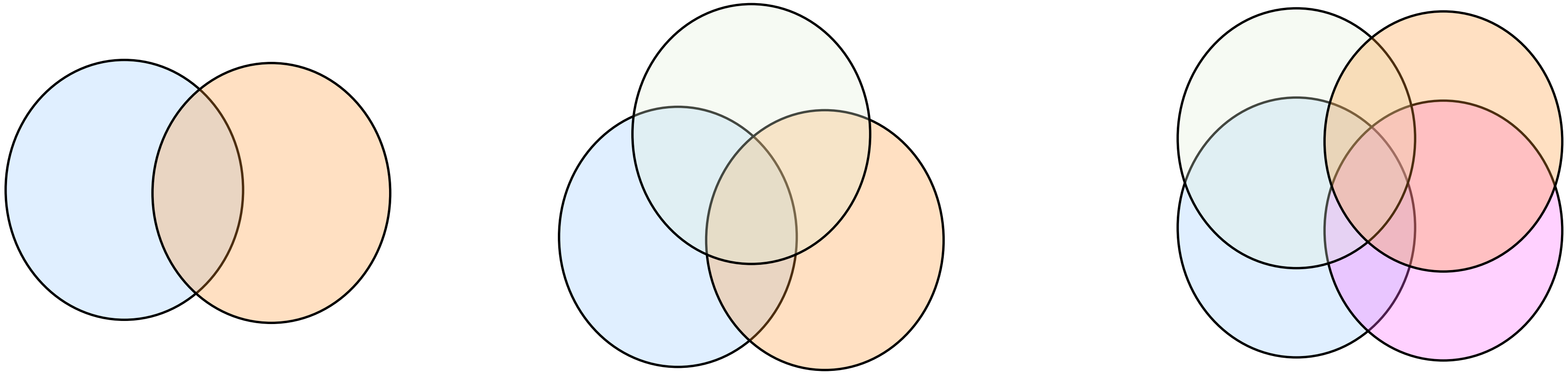
# Schedule

- Today: We practice the addition rule
- Generalizing into **inclusion – exclusion principle**.
- We follow up with more combinatorial practice, introducing *probability* in the mix!

# Inclusion / Exclusion principle

- The inclusion / exclusion principle is effectively a generalization of the "law of addition / subtraction".
- It allows us to calculate the cardinalities (sizes) of unions:

$$A_1 \cup A_2 \cup \cdots A_n$$



2 sets  3 sets  4 sets  ....

# Picking passwords

- Suppose that we want to register for some website, and we have to pick a password.
- The website's pretty old-tech, so it tells us that the password should be between 4 and 6 symbols long, with English lowercase or uppercase characters, digits, as well as any one of the "special" characters #, *, _, -, @, &, !

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  - ***How many different passwords can the website store in its database?***
  - If we call the sets of different passwords $N_4, N_5, N_6$, we have:

$$|N_4| + |N_5| + |N_6|$$

- Letters, lowercase and uppercase
- Digits
- #, *, _, -, @, &, !

# Calculating...

$|N_4|$ =

$P(69, 4)$

$69^4$

$\binom{69}{4}$

$4^{69}$

$|N_5|$ =

$|N_6|$ =

- Letters, lowercase and uppercase
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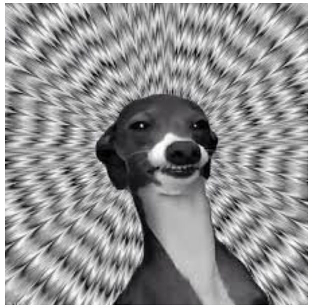
***That's about 109.5 billion different passwords!***



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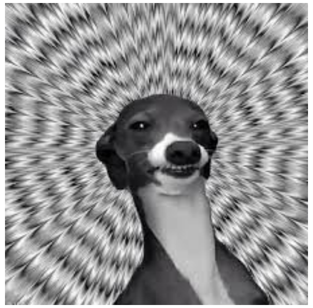

Finally, notice that $N_4, N_5$ and $N_6$ are pairwise disjoint sets (why?)

# Picking different passwords

- Suppose now that the website tells us that our passwords should not have repeated characters.
- Call our new sets $M_4, M_5, M_6$.
- The total #passwords is still yielded as:

$$|M_4| + |M_5| + |M_6|$$

- Letters, lowercase and uppercase
- Digits
- #, *, _, -, @, &, !
- NO REPEATED CHARS

# Calculating...

$|M_4| =$

$P(69,4)$ | $69^4$ | $\binom{69}{4}$ | $4^{69}$

$|M_5| =$

$|M_6| =$

- Letters, lowercase and uppercase
- Digits
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# Calculating...

$|M_4| = P(69, 4)$

$P(69, 4)$ | $69^4$ | $\binom{69}{4}$ | $4^{69}$

$|M_5| =$

$|M_6| =$

# Calculating...

- Letters, lowercase and uppercase
- Digits
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- **NO REPEATED CHARS**

$$|M_4| = P(69, 4)$$

$P(69,4)$ | $69^4$ | $\binom{69}{4}$ | $4^{69}$

$$|M_5| = P(69, 5)$$

$$|M_6| = P(69, 6)$$

- Letters, lowercase and uppercase
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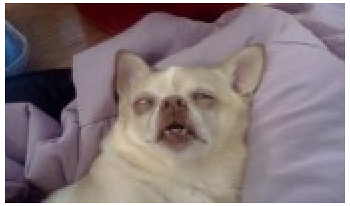

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# Calculating…

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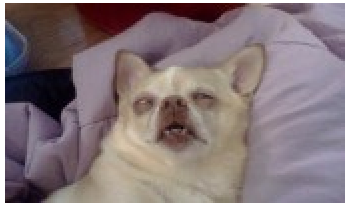

Finally, notice that $M_4, M_5$ and $M_6$ are still disjoint sets.

# The addition rule:a special case

- The previous examples were instances of the so-called **addition rule**.
- It's just a special case of inclusion – exclusion, where all sets are ***pairwise disjoint!***
- Formally, the rule is stated as follows:

Let $n \in \mathbb{N}^{>0}$. If $A_1, A_2, \dots, A_n$ are finite, pairwise disjoint sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{i=1}^{n} |A_i|$$

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- In our examples,

$$|N_4 \cup N_5 \cup N_6| = \sum_{i=4}^{6} |N_i| \quad (= 69^4 + 69^5 + 69^6)$$

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$$|M_4 \cup M_5 \cup M_6| = \sum_{i=4}^{6} |M_i| \quad (= \mathrm{P}(69,4) + \mathrm{P}(69,5) + \mathrm{P}(69,6))$$

# A not – so – special case

- Once again, our passwords can use: English lowercase or uppercase characters, digits, as well as any one of the "special" characters #, *, _, -, @, &, !
  - 69 characters total.
- Alice likes passwords of length 6 that start with an 'A'.
- Bob likes passwords of length 6 that end with a 'B'.
- Both are security-conscious, so they never use the same character.

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- Alice likes passwords of length 6 that start with an 'A'.
- Bob likes passwords of length 6 that end with a 'B'.
- Both are security-conscious, so they never use the same character.
- What is the total number of passwords that either Alice or Bob use?

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  - What is $|P_A|$?

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| $P(69, 6)$ | $P(69, 5)$ | $69^5$ | Something Else |

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- What am I looking for?

| | | | |
|---|---|---|---|
| $\lvert P_A \cap P_B \rvert$ | $\lvert P_A \cup P_B \rvert$ | $\lvert P_A - P_B \rvert$ | $\lvert P_B - P_A \rvert$ |

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Remember: I'm looking for the #passwords that **either Alice OR Bob use.**

# Practice

- **You** told us that we're looking for $|P_A \cup P_B|$
- By the addition rule, $|P_A \cup P_B| = |P_A| + |P_B| = 2 * P(68, 5)$

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You've been fooled!

- $A1234B$ was counted twice!

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- $A1234B$ was counted twice!
- Many passwords were counted twice
  - How many?

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$P(69, 4)$

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Something Else

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- How many passwords do both Alice and Bob like?

$P(69, 4)$ | $P(68, 4)$ | $P(67, 4)$ | Something Else

A _ _ _ _ B

- 4 positions
- Cannot choose 'A' and 'B' because they've been used already!
  - So 67 characters available
- Order matters.

# $|P_A \cup P_B|$

- From the rule we supplied earlier:

$$|P_A \cup P_B| = |P_A| + |P_B| - |P_A \cap P_B| = 2 * P(68,5) - P(67,4) =$$

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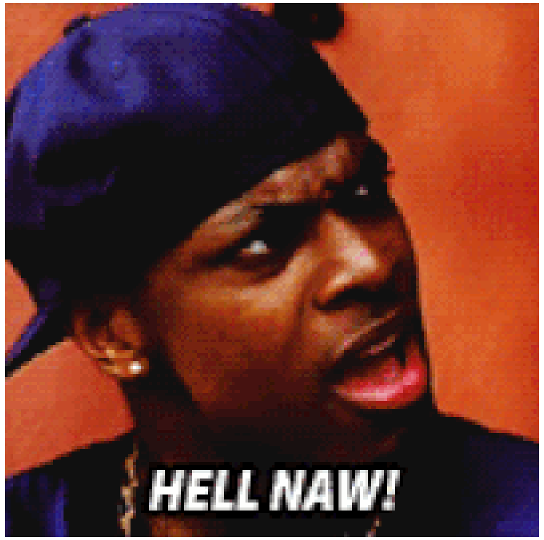
NOPE, WE'RE BUSY PEOPLE



# General Rule

- For any finite sets $A, B$:

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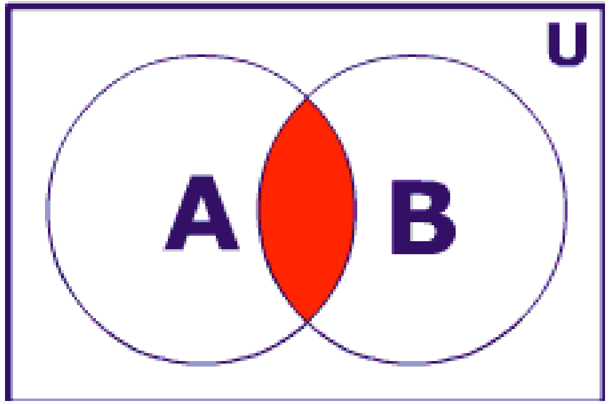
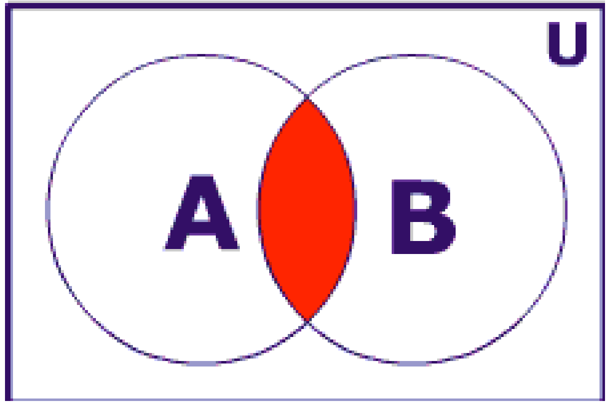


# General Rule

- For any finite sets $A, B$:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- This is the **inclusion-exclusion principle for two variables,** also known as the difference (subtraction) Rule.
- So this "rule" boils down to being another special case of inclusion / exclusion principle!



# A number-theoretic problem

- How many numbers between 1 and 1000 are divisible by either 2 or 3?

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- How many numbers between 1 and 1000 are divisible by either 2 or 3?
- $A_2 = \{x \in \mathbb{N} | (1 \leq x \leq 1000) \wedge (x \equiv 0 \ (mod\ 2))\}$
- $A_3 = \{x \in \mathbb{N} | (1 \leq x \leq 1000) \wedge (x \equiv 0 \ (mod\ 3))\}$
- Generally, $A_i = \{x \in \mathbb{N} | (1 \leq x \leq 1000) \wedge (x \equiv 0 \ (mod\ i))\}$
- $|A_2| = \lfloor {}^{1000}/_{2} \rfloor = 500$
- $|A_3| = \lfloor {}^{1000}/_{3} \rfloor = 333$
- $|A_i| = \lfloor {}^{1000}/_{i} \rfloor$

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- $|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = 833 \ - |A_2 \cap A_3|$

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# A number–theoretic problem

- $|A_2 \cup A_3| = |A_2| + |A_3| - |A_2 \cap A_3| = 833 \; - |A_2 \cap A_3|$
  - **What is the set $A_2 \cap A_3$?**
  - It's just $A_6$.
- $|A_6| = \lfloor {}^{1000}/_{6} \rfloor = 166$
- So $|A_2 \cup A_3| = 833 \; - 166 = 667$

# What about 3 variables?

- Some Discrete Mathematics students were polled about their past Computer Science & Mathematics course experience.
  - 30 had taken precalculus
  - 18 had taken calculus
  - 26 had taken Java
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VENN DIAGRAM TIME!

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  - We count students in $(P \cap J), (P \cap C), (J \cap C)$ **twice.**
- Is $|P \cup J \cup C| = |P| + |J| + |C| - (|P \cap J| + |P \cap C| + |J \cap C|)$?

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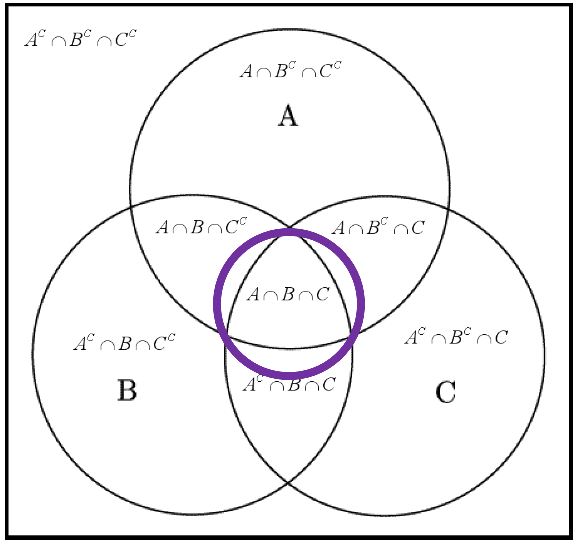
**NO. We are losing the students in $(P \cap C \cap J)$!**



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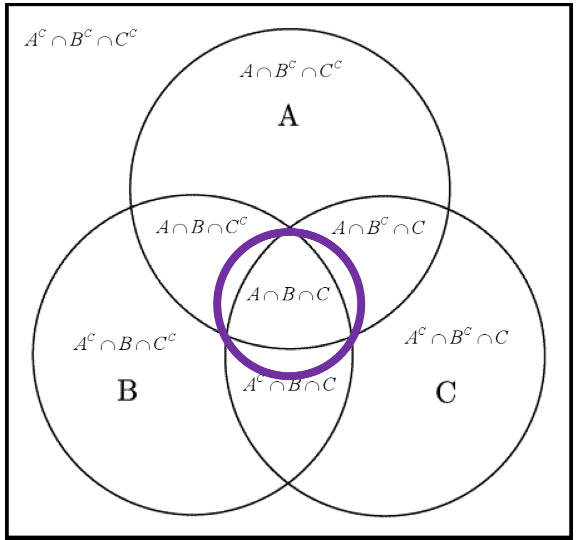
**NO. We are losing the students in $(\boldsymbol{P \cap C \cap J})$!**

So we need to add them back:

$$|P \cup J \cup C| =$$
$$|P| + |J| + |C| - (|P \cap J| + |P \cap C| + |J \cap C|) + (\boldsymbol{P \cap C \cap J})$$



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| **Problem givens** | **Translation into sets** |
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| 18 had taken calculus | $\lvert C \rvert = 18$ |
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| 9 had taken **both** precalculus and calculus | $\lvert P \cap C \rvert = 9$ |
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- We can then answer:

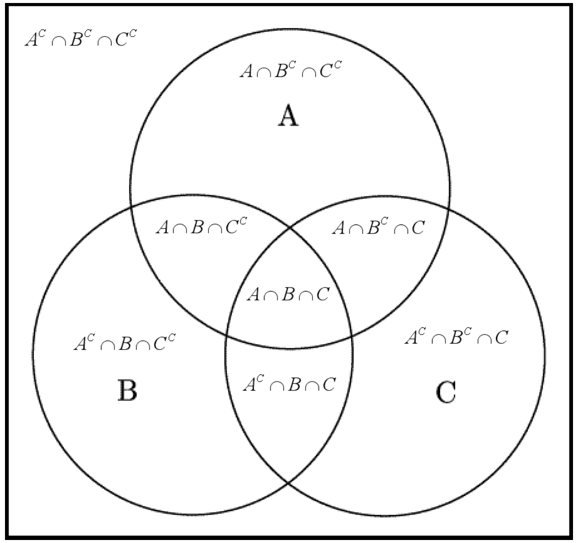
$$|\boldsymbol{P} \cup \boldsymbol{J} \cup \boldsymbol{C}| = |P| + |J| + |C| - (|P \cap J| + |P \cap C| + |J \cap C|) + (\boldsymbol{P} \cap \boldsymbol{C} \cap \boldsymbol{J})$$
$$= 30 + 26 + 18 - ( \quad 16 \quad + \quad 9 \quad + \quad 8) \quad + \quad 5 \quad = \mathbf{46}$$

# A generalizable framework

- For three finite sets $A, B, C$, we have:

$$|A \cup B \cup C| =$$
$$|A| + |B| + |C| - (|A \cap B| + |B \cap C| + |C \cap A|) + |A \cap B \cap C|$$

- This is the inclusion-exclusion principle for 3 sets.



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- Therefore:

$$|A_2 \cup A_3 \cup A_5| =$$

$$= |A_2| + |A_3| + |A_5| - (|A_2 \cap A_3| + |A_2 \cap A_5| + |A_3 \cap A_5|) + |A_2 \cap A_3 \cap A_5|$$

$$= |A_2| + |A_3| + |A_5| - (|A_6| + |A_{10}| + |A_{15}|) + |A_{30}|$$

$$= \lfloor {}^{1000}/_{2} \rfloor + \lfloor {}^{1000}/_{3} \rfloor + \left\lfloor {}^{1000}/_{5} \right\rfloor - \left(\lfloor {}^{1000}/_{6} \rfloor + \lfloor {}^{1000}/_{10} \rfloor + \left\lfloor {}^{1000}/_{15} \right\rfloor\right) + \lfloor {}^{1000}/_{6} \rfloor$$

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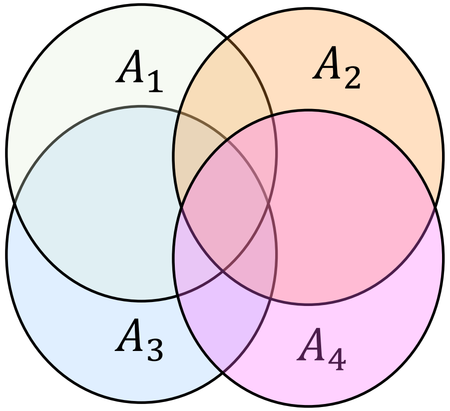
$$= 500 + 333 + 200 - 166 \ - 100 \ - 66 + 33 = 734$$

# Here's one for you

- Inclusion-Exclusion rule for 4 (four) sets $A_1, A_2, A_3, A_4$



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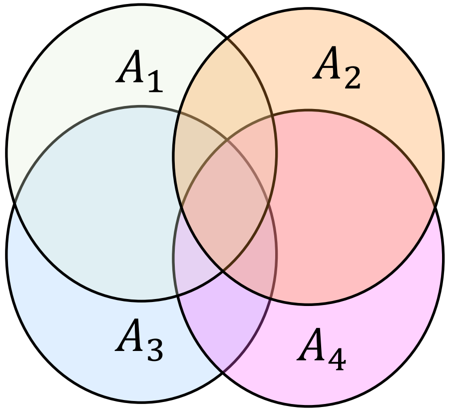


$|A_1 \cup A_2 \cup A_3 \cup A_4| =$

$$|A_1| + |A_2| + |A_3| + |A_4|$$

$$-(|A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| + |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|)$$

$$+(|A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_2 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_4|)$$

$$-|\mathrm{A}_1 \cap A_2 \cap A_3 \cap A_4|$$