Sequences, Series and Summation / Product Notation

CMSC 250

Sequences and series

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    - 1.5, 2.5, 3.5, …
    - 1, 1, 1, 1, ….
    - $\sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \sqrt{6}, \sqrt{7}\ldots$

Outlining terms

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    - $a_n = 2^n, n = 0, 1, 2, \ldots$
    - $b_k = logk + 2k, k = 1, 2, 3, \ldots$

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"Closed form" formula

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    - $F_n = \begin{cases} 1, & if\ n = 0, 1 \\ F_{n-1} + F_{n-2}, & if\ n \geq 2 \end{cases}$
    - $T_n = \begin{cases} 1, & if\ n = 1,2 \\ 2, & if\ n = 3 \\ T_{n-1} + T_{n-2} + T_{n-3}, & if\ n \geq 4 \end{cases}$

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Recursive formula

All of those are **valid ways** to describe a sequence!

Recursion: good idea?

- Example: Fibonacci

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- Is it a good idea?

Yes | No | Something Else

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- Recomputing terms + hidden memory cost of recursion!

Recursion: Done right

- Is there a better way to compute F_{1000}?

Yes

No

Something Else

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1. Store the values of $F_0 = 1,\ F_1 = 1$ in an array A.
2. for i = 2 to 1000

   $F_i = A[i-1] + A[i-2]$

   $A[i] = F_i$

   end

- This is a very elementary example of a **very useful technique** called ***dynamic programming.***

Closed formula for Fibonacci

- The closed-form formula for F_n is:

$$F_n = \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1+\sqrt{5}}{2}\right)}_{\phi}^{n} - \frac{1}{\sqrt{5}}\underbrace{\left(\frac{1-\sqrt{5}}{2}\right)}_{\psi}^{n}$$

- Roughly: $F_n \approx \phi^n \approx (1.618)^n$

Recursion vs closed formula

1. Computation:
   - Recursion leads to a fast dynamic program.
   - Classic recursion is elegant.
   - Closed form: faster, but numerical issues arise.
2. Rate of growth:
   - Recursion gives no hint as to **how big** F_n is.
   - Closed form yields $F_n \approx (1.618)^n$

Summation notation

- Suppose I have some terms of a sequence, let's say $a_1, a_2, a_3, \dots, a_k$.
- Their sum, $a_1 + a_2 + a_3 + \cdots + a_k$ is denoted as:

$$\sum_{i=1}^{k} a_i$$

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Index of term **with which the sum begins**

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$$\sum_{i=1}^{2} a_i = a_1 + a_2$$

$$\sum_{i=1}^{1} a_i = a_1$$

$$\sum_{i=1}^{0} a_i = ?$$

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$$\sum_{i=1}^{0} a_i = 0$$

- The reason for this is that the following should hold for any ℓ:

$$\sum_{i=1}^{\ell} a_i + \sum_{\ell+1}^{n} a_i = \sum_{i=1}^{n} a_i$$

- So, for $\ell = 0$, we have the equality:

$$\sum_{i=1}^{0} a_i + \sum_{i=0+1}^{n} a_i = \sum_{i=1}^{n} a_i \Rightarrow \sum_{i=1}^{0} a_i = 0$$

Product Notation

- The **product**, $a_1 \cdot a_2 \cdot \ldots \cdot a_k$ is denoted as:

Index of term **with which the product begins**

$$\prod_{i=1}^{k} a_i$$

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"Running" (or "looping" indices can be anything we want! $(i, j, k, \ldots)$ **as long as I use the same variable in the Σ and Π symbols and the variable representing the sequence term!**

Sum-Product notation

- We can have certain ***exclusionary conditions*** under the Σ and Π symbols.
- Examples:

$$\sum_{\substack{m=0 \\ m\ even}}^{100} s_m$$

$$\sum_{\substack{m=0 \\ 3\ |\ m}}^{100} s_m \neq \sum_{\substack{m=0 \\ 3\ |\ s_m}}^{100} s_m$$

Series and partial sums

- A ***series*** is the **sum** of **all** elements of an **infinite** sequence.

$$\sum_{i=0}^{+\infty} a_i = a_0 + a_1 + a_2 + \cdots$$

Or 1, if we start at 1

- A **partial sum** of a sequence, denoted $\boldsymbol{S_n}$, is the sum ranging from the first up to (and including) the n^{th} term of a (usually infinite) sequence:

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- Question (2): **What is $S_n - S_{n-1}$? a_n**

Famous sequences

- **Arithmetic** (often called the arithmetic **progression**):

$$a_0, \underbrace{a_0 + d}_{\alpha_1}, \underbrace{a_1 + d}_{\alpha_2}, \underbrace{a_2 + d}_{\alpha_3} \ldots \text{ where } \quad d \in \mathbb{R}$$

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$d \cdot a_{n-1}$

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A question for you

- In the arithmetic progression:

$$a_0, a_0 + d, a_1 + d, a_2 + d \ldots \text{where} \quad d \in \mathbb{R}$$

- Should we allow $d = 0$?

YES

NO

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It will be a pretty boring sequence, but it will still be a sequence!

YES

NO

Famous Sequences

- **Geometric** sequence (or **progression**):

$$a_0, \underbrace{m \cdot a_0}_{\alpha_1}, \underbrace{m \cdot a_1}_{\alpha_2}, \underbrace{m \cdot a_2}_{\alpha_3} \ldots$$

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$(m-1)^n \cdot a_0$	$m^n a_0$	$m^{n-1} a_0$	$m \cdot n \cdot a_0$

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The Gauss story

- Gauss was a great mathematician (1777-1855)
- When Gauss was in 1st grade, the class was misbehaving.
- For punishment, the teacher made everyone compute

$$1 + 2 + \cdots + 100$$

- Gauss did it in 2 minutes. Can you?

The Gauss trick

$$S = \quad 1 \; + \quad 2 + \cdots + 100$$
$$S = 100 + 99 + \cdots + 1$$
$$+$$

$$2S = \underbrace{101 + 101 + \cdots + 101}_{\text{100 terms}}$$

$$\Rightarrow 2S = 101 * 100 = 10100 \Rightarrow S = 5050$$

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1980	3^{rd}	1 + 2 + … + 80

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1960	5^{th}	1 + 2 + … + 60
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2000s	1^{st}	1 + 2 + … + 100

And now the rest of the story

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- This is how this story has progressed over time:

	YEAR	GRADE	SERIES
	1960	5th	1 + 2 + … + 60
	1980	3rd	1 + 2 + … + 80
	2000s	1st	1 + 2 + … + 100
Our conjecture:	2020	Nursery School	1 + 2 + … + 120

Famous Sequences

- **Harmonic**:

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

- **Fibonacci**: $F_0 = F_1 = 1$ and $\forall n \geq 2, F_n = F_{n-1} + F_{n-2}$

1, 1, 2, 3, 5, 8, 13, 21, …

What we'll do next

- We will have an intro to **induction.**
- The following can be proven via induction:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$