k-nomial theorem and Pascal's Triangle

CMSC 250

Video #1

The binomial theorem and some computational challenges.

The binomial theorem

- Recall the following identities from highschool:
- $(x+y)^2 = x^2 + 2xy + y^2$
- $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- $(x+y)^4 = x^4 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + y^4$

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- Is there a pattern here? Can we easily generate the coefficients?
  - (Some of you might already know **how**, but we doubt that you know **why**)

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

- What is the coefficient of x^2y^3?

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- What is the coefficient of x^2y^3?
- There are $2^5 = 32$ terms total (many combine, eg $xxyyy$, $xyxyy$ are both of form x^2y^3).
- How many of those terms have 2 ‘x’s and 3 ‘y’s?

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot\ (x+y)\cdot(x+y)\cdot(x+y)$

$xxyyy,\quad xyxyy,\quad xyyxy,\quad xyyyx,$

$yxxyy,\quad yxyxy,\quad yxyyx,$

$yyxxy,\quad yyxyx,$

$yyyxx$

$(x+y)^5$

- $(x+y)^5 = (x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)\cdot(x+y)$

$xxyyy$, $xyxyy$, $xyyxy$, $xyyyx$,
$yxxyy$, $yxyxy$, $yxyyx$,
$yyxxy$, $yyxyx$,
$yyyxx$

All terms of form x^2y^3

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$$\left.\begin{array}{llll} xxyyy, & xyxyy, & xyyxy, & xyyyx, \\ yxxyy, & yxyxy, & yxyyx, & \\ yyxxy, & yyxyx, & & \\ yyyxx & & & \end{array}\right\}\ \text{All terms of form } x^2y^3$$

- This is just choosing 2 slots out of 5 to put the 'x's in.
- There are $\binom{5}{2} = 10$ ways of doing this.

You do this now

- What is the coefficient of x^3y^4 in $(x+y)^7$?

You do this **now**

- What is the coefficient of x^3y^4 in $(x+y)^7$?

$$\frac{7!}{3!\cdot 4!} = \binom{7}{3}$$

$(x + y)^{n}$

- We now generalize the previous results:
- $(x + y)^n = (x + y) \cdot (x + y) \cdot \ldots \cdot \ (x + y)$

- Co-efficient of $x^r y^{n-r}$ $=$ # of ways to select r ‘x’s from n slots = $\binom{n}{r}$

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- $(x+y)^n = (x+y)\cdot(x+y)\cdot \ldots \cdot\ (x+y)$

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- Binomial Theorem:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r}\ x^r y^{n-r}$$

How to find the coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

- Approach #1: Compute **directly** via formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
- Problem: Large intermediary numbers, **even if $\boldsymbol{n}$, $\boldsymbol{r}$ and $\boldsymbol{\binom{n}{r}}$ are relatively small!**
  - Example: $\binom{20}{10} = \frac{20!}{10! \cdot 10!} = \frac{1 \times 2 \times \cdots \times 10 \times 11 \times 12 \times \cdots \times 20}{(1 \times 2 \times \cdots \times 10) \cdot (1 \times 2 \times \cdots \times 10)}$

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    Not too large!
  - Is our computer **smart enough** to cancel out the stuff in green?
    - Not every computer is!

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    - But assuming that ours is, we still have to compute $11 \times 12 \times \cdots \times 20$, which is **quite large, even though the final result is small!**

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- Can we do better?
  - Yes, through Pascal's triangle!

END OF
VIDEO #1

Video #2

- Using Pascal's identity and triangle to calculate any $\binom{n}{r}$ ***fast***.
- Expanding binomial theorem to trinomial, quadrinomial, …., k-nomial

An easy combinatorial identity

We will prove that

$$(\forall n, r \in \mathbb{N})\left[(r \leq n) \Rightarrow \binom{n}{r} = \binom{n}{n-r}\right]$$

in two different ways!

Another combinatorial identity

$$(\forall n, r \in \mathbb{N}^{\geq 1})\left[(r \leq n) \Rightarrow \binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}\right]$$

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1. Algebraic proof
2. Combinatorial proof!

A combinatorial proof of $\binom{n}{r} = \boxed{\binom{n-1}{r-1} + \binom{n-1}{r}}$

- LHS: #ways to pick r people from a set of n people.
- RHS: Focus on one person, call him $Jason$.
  - If we pick $Jason$, then we are left with $n-1$ people to decide if we want to pick or not, from which we now have to pick $r-1$ people **(first term of RHS)**
  - OR, if we don't pick $Jason$, we are left with $n-1$ people to decide if we want to pick or not, yet still r people that we need to pick (second term of RHS).

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  - OR, if we don't pick $Jason$, we are left with $n-1$ people to decide if we want to pick or not, yet still r people that we need to pick (second term of RHS).
- This is a **combinatorial proof**!
- A combinatorial proof is a type of proof where we show two quantities are equal because they solve the same problem.

Pascal’s Triangle

$$\binom{0}{0}$$

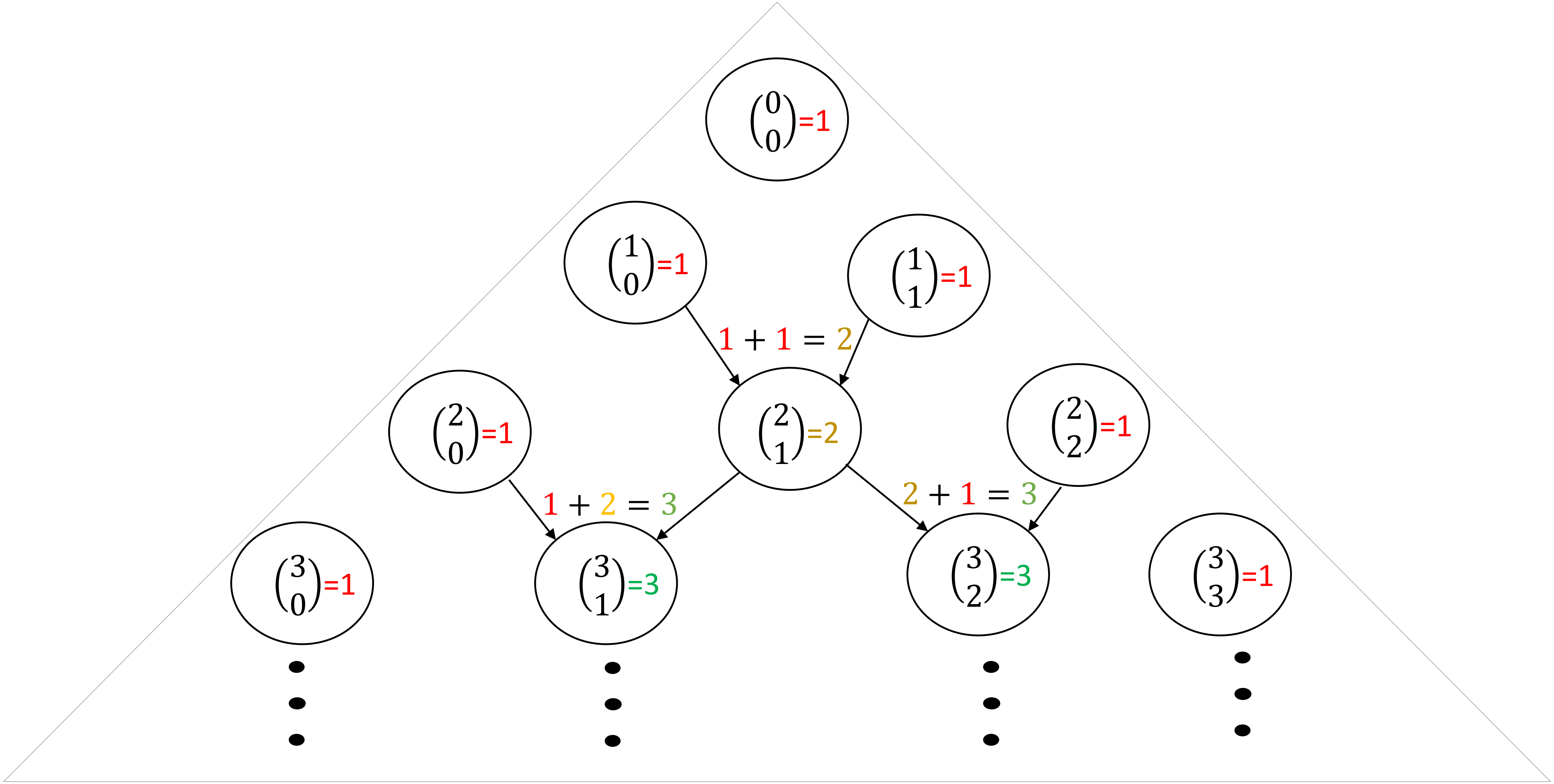
$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Pascal’s Triangle



Upshot

- Use combinatorial identity

generate Pascal's triangle

generate binomial coefficients $\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$

use in the expansion of $(x + y)^n$

Efficiency of Pascal's triangle

- We avoid the intermediary large numbers problem
- i^{th} level of triangle gives us all coefficients $\binom{i}{0}, \binom{i}{1}, \ldots, \binom{i}{i}$
- Compute the value of every node as the sum of its two parents
  - Note that the diagonal "edges" of the triangle always 1.

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- How many permutations of $x^a y^b$ are there?

$$\frac{(a+b)!}{a! \cdot b!}$$

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$$\frac{(a+b)!}{a! \cdot b!} = \frac{n!}{a! \cdot (n-a)!} = \binom{n}{a}$$

An exercise for you to do **now**

- Expand $(x + y + z)^2$

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$$x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$$

Trinomial theorem

- $(x+y+z)^5 = (x+y+z)\cdot(x+y+z)\cdot(x+y+z)\cdot(x+y+z)\cdot(x+y+z)$

- The expansion will have terms of form

$$x^a y^b z^c, \text{ where } a+b+c=5$$

- What should the coefficients be?

Trinomial theorem

$x^a y^b z^c$, where $a + b + c = 5$

- Once again, let's view $x^a y^b z^c$ as a string.
- #permutations of this string =

$$\frac{(a+b+c)!}{a! \cdot b! \cdot c!}$$

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- #permutations of this string =

$$\frac{(a+b+c)!}{a! \cdot b! \cdot c!} = \frac{5!}{a! \cdot b! \cdot c!}$$

Trinomial theorem

$$(x+y+z)^n = \sum_{\substack{a+b+c=n \\ 0\le a,b,c\le n}} \frac{n!}{a!\,b!\,c!} x^a y^b z^c$$

k-nomial theorem

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 0\le a_1,a_2,\ldots,a_k\le n}} \frac{n!}{a_1!\,a_2!\ldots a_k!} x_1^{a_1} x_2^{a_2} \ldots x_k{}^{a_k}$$

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$$\Leftrightarrow \left(\sum_{i=1}^{k} x_i\right)^n = \sum_{\substack{a_1+a_2+\cdots+a_k=n \\ 0\le a_1,a_2,\ldots,a_k\le n}} \frac{n!}{\prod_{i=1}^{k} a_i!} \prod_{i=1}^{k} x_i^{a_i}$$

END OF
VIDEO #2