Discrete Probability

CMSC 250

Informal definition of probability

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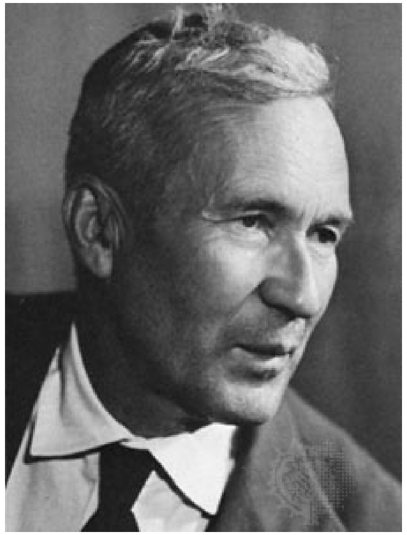
- This definition is owed to Andrey Kolmogorov, and assumes *that all possibilities are equally likely!*



First examples

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$\frac{1}{3}$

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Something else

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  - Why?
    - Set of different *events?*
      - $\{HHH, HHT, HTH, HTT, THH, THT, TTH, \boldsymbol{TTT}\}$ (8 of them)
    - Set of events with **no heads:**
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Implicit assumption: all individual outcomes (HHH, HHT, HTH, ….) ***are considered equally likely (probability 1/8)***

Practice

- Experiment #2: I roll two dice.
  - Probability that I hit seven = ?

$\frac{1}{12}$	$\frac{1}{6}$	$\frac{7}{12}$	Something else

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  - Why?
    - Set of different *events?*
      - $\{(1,1),(1,2),\ldots,(6,1)\}$ (36 of them)
    - Set of events where we hit 7.
      - $\{(2,5),(5,2),(3,4),\ (4,3),(1,6),\ (6,1)\}$ (6 of them)
    - Hence the answer: $\frac{6}{36} = \frac{1}{6}$

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  - Probability that I hit two= ?
    - Same procedure

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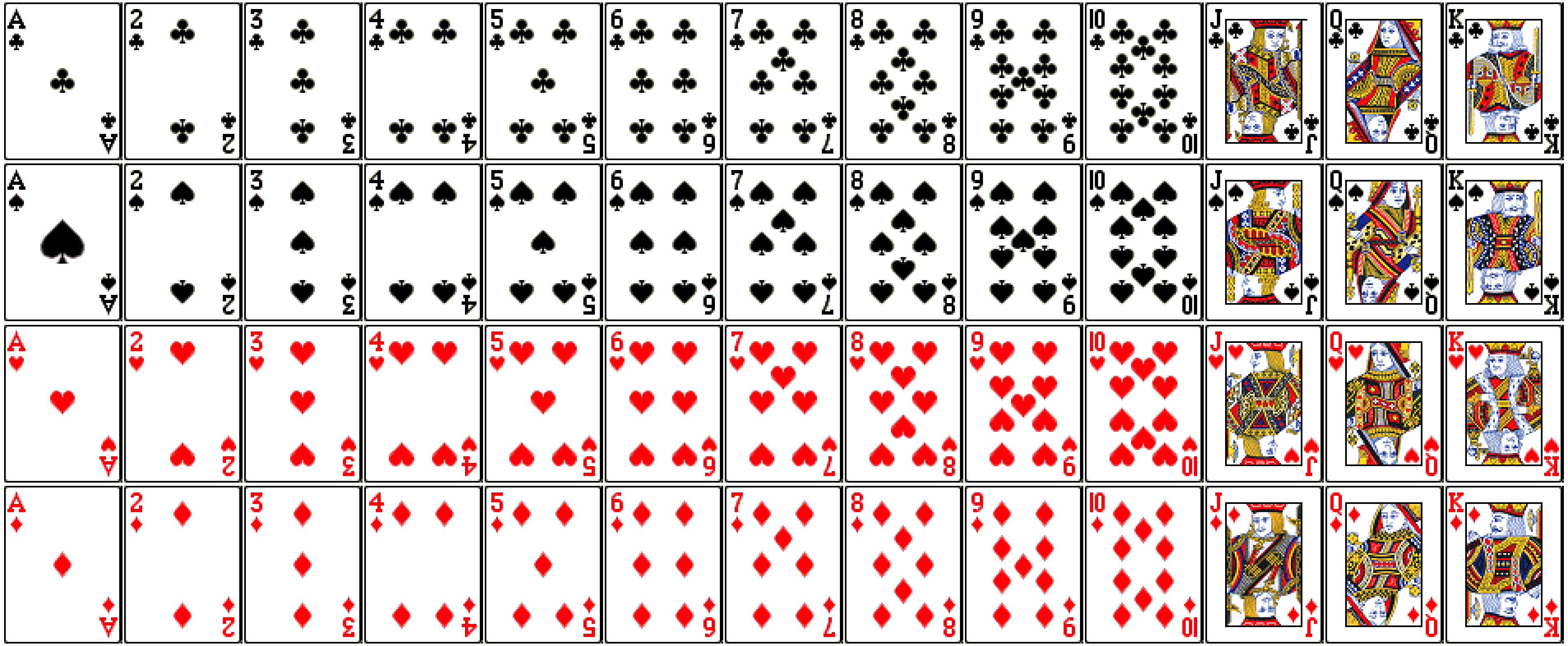
$\frac{1}{36}$

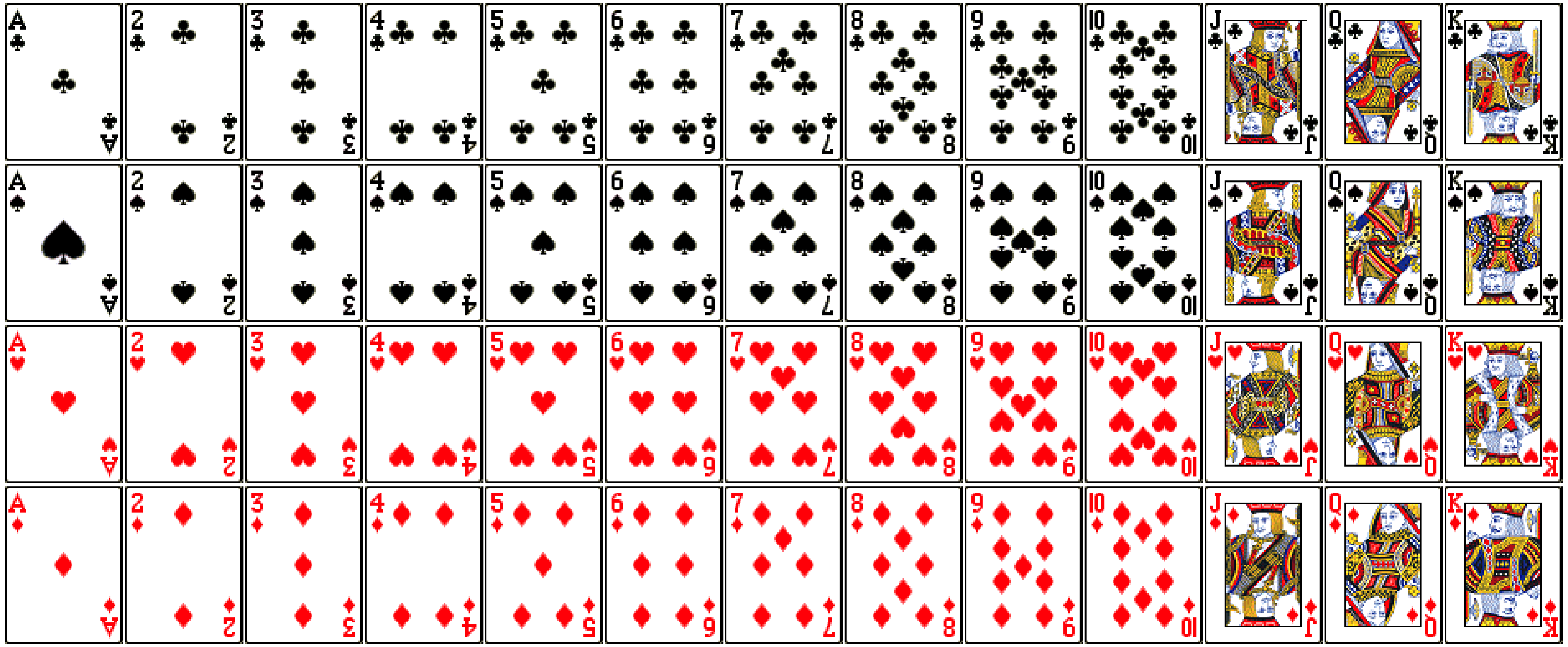
Poker Practice

- Full deck = 52 cards, 13 of each suit:



Poker Practice

- Full deck = 52 cards, 13 of each suit:
- **Flush**: 5 cards of the same suit
- What is the probability of getting a flush?



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  - So $\mathbf{4} * \binom{\mathbf{13}}{\mathbf{5}}$

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- So, probability of being dealt a flush is

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- How likely is this?
  - Not at all likely: $\approx 0.002 = 0.2\%$ ☹

Likelihood of a straight

- Straights are 5 cards of ***consecutive rank***
  - Ace can be ***either end*** (high or low)
  - ***No wrap-arounds*** (e.g Q K A 2 3, suits don't matter)
- What is the probability that we are dealt a straight?

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  - Pick lower rank in 10 ways (A-10)
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That's $10 * 4^5$ ways.
So, probability of a straight = $\frac{10*4^5}{\binom{52}{5}}$

Caveat on flushes

- Wikipedia says we're wrong about flushes!
- Formally, our flushes included (for example) 3h 4h 5h 6h 7h
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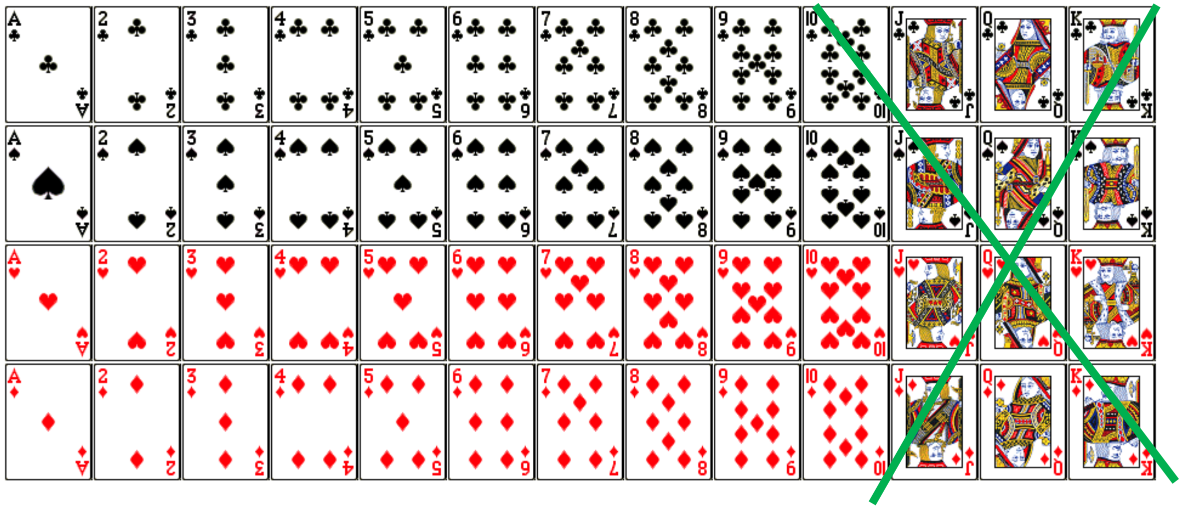
  - Hands like these are called straight flushes and Wikipedia does not include them.
  - ***How many straight flushes are there?***
  - 40. Here's why:
    - Pick rank: A through 10 (higher ranks don't allow straights) in 10 ways
    - Pick suit in 4 ways



Probability of non-straight flush…

$$\frac{4 * \binom{13}{5} - 40}{\binom{52}{5}} = 0.001965$$

- This is how Wikipedia defines the probability of a flush. ☺

Probability of a straight flush…

$$\frac{40}{\binom{52}{5}} = 0.0000138517$$

Probability of a straight flush…

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The expected # hands you need to play to get a straight flush is then

$$\left\lceil \frac{1}{0.0000138517} \right\rceil = 72{,}194$$

Same caveat for straights

- From the #straights we computed we will have to subtract the 40 possible straight flushes to get…

$$\frac{10 * 4^5 - 40}{\binom{52}{5}} = 0.003925$$

Same caveat

- From the #straights we computed we will have to subtract the 40 possible straight flushes to get…

$$\frac{10*4^5-40}{\binom{52}{5}} = 0.003925 > 0.001965 = \text{probability of flush}$$

- *Flushes, being more rare, beat straights in poker.*

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  - The denominator will be $\binom{52}{5}$ (easy), so let's focus on the numerator:
    1. First choose rank in 13 ways.
    2. Then, choose two of four suits in $\binom{4}{2} = 6$ ways.
    3. Then, choose 3 cards out of 50 in $\binom{50}{3}$ ways.

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Yes

No

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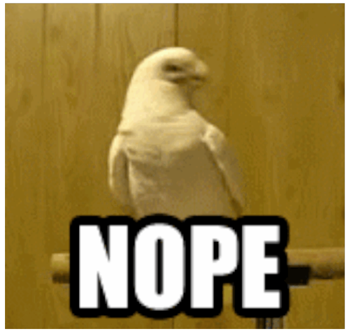

Is this accurate?

Severe overcount!

Yes

No

Don’t count better hands!

- In the computation before, we included:
  - 3-of-a-kind
  - 4-of-a-kind
  - etc
- To properly compute, we would have to subtract **all** better hands possible with at least one pair.

Joint probability (“AND” of two events)

- The probability that two events A and B occur **simultaneously** is known as the **joint** probability of A and B and is denoted in a number of ways:
  - $P(A \cap B)$ (Most useful from a set-theoretic perspective; we’ll be using this)
  - $P(A, B)$ (One sees this a lot in Physics books)
  - $P(AB)$ (Perhaps most convenient, therefore most common)

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$

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- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6

Calculating joints

- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is $\frac{1}{3}$

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- Probability that the first coin toss is heads and the second coin toss is tails $\frac{1}{2} \times \frac{1}{2}$
- Probability that the first die is at most a 2 and the second one is 5 or 6
  - # outcomes of die roll is 6
  - # outcomes where first die is at most 2 is 2
    - Hence, probability of first die roll being at most 2 is $\frac{1}{3}$
  - Similarly, probability of second die roll being 5 or 6 is $\frac{1}{3}$.
  - Hence, probability that **both** events happen (joint probability) is $\frac{1}{9}$.

Calculating joints

- Jason's going to flip a coin and then pick a card from a 52-card deck.
  - Probability that the coin is heads and the card has rank 8?

$\frac{1}{2}$	$\frac{1}{26}$	$\frac{1}{32}$	Something else

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- This is because $P(coin = H) = \frac{1}{2}$ and $P(card\_rank = 8) = \frac{4}{52} = \frac{1}{13}$
  - So their joint probability is $\frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$

The law of joint probability

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{i=1}^{n} P(A_i)$$

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- Unfortunately, this "law" is not always applicable!
- It is applicable only when all the different events A_i are *independent* (sometimes called *marginally independent*) of each other.
- Let's look at an example.

What if the events influence each other?

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  - ***NO!***
    - What is the probability that the die is even and the die is 2?

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$

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  - ***NO!***
    - What is the probability that the die is even and the die is 2?

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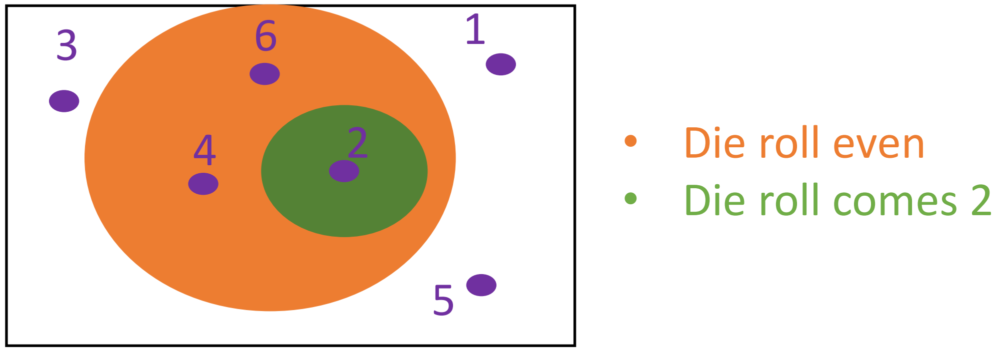
Set-theoretic interpretation

- Notice that the event A: “Die roll is even” is a superset of the event B: “Die roll comes 2”



- Since $A \cap B = A$, $P(A \cap B) = P(A) = \frac{1}{6}$

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- Events such as these are called *disjoint* or *mutually disjoint*.

Set-theoretic interpretation

- A = “Jason gets an A in USND’s 250”
- G=“Jason gets a G in USND’s 250”

- Note that $A \cap G = \emptyset$, so there are no common outcomes.
  - So $P(A \cap G) = 0$

Calculating joints

- I have my original die again.
  - Probability that it comes up 1, 2 or 3 = $\frac{1}{2}$
  - Probability that it comes up 3, 4 or 5 = $\frac{1}{2}$
  - What is the probability that it comes up 1, 2 or 3 **and** 3, 4 or 5?

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$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$

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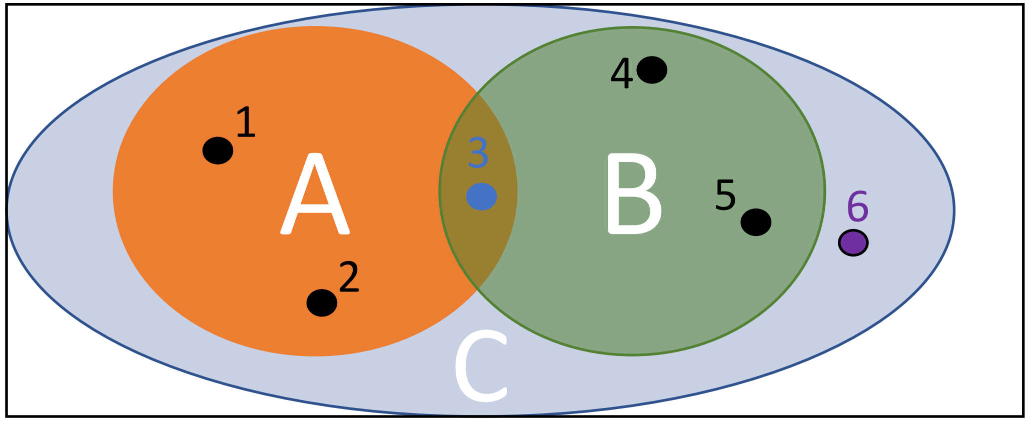
$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$

  - Note that the only common outcome between the two events is **3**, which can come up only **once** out of **six** possibilities.

Set-theoretic interpretation

- Let A = dice comes up 1, 2, or 3
- Let B = dice comes up 3, 4, or 5
- Let C = dice comes up 1, 2, 3, 4, 5 OR 6



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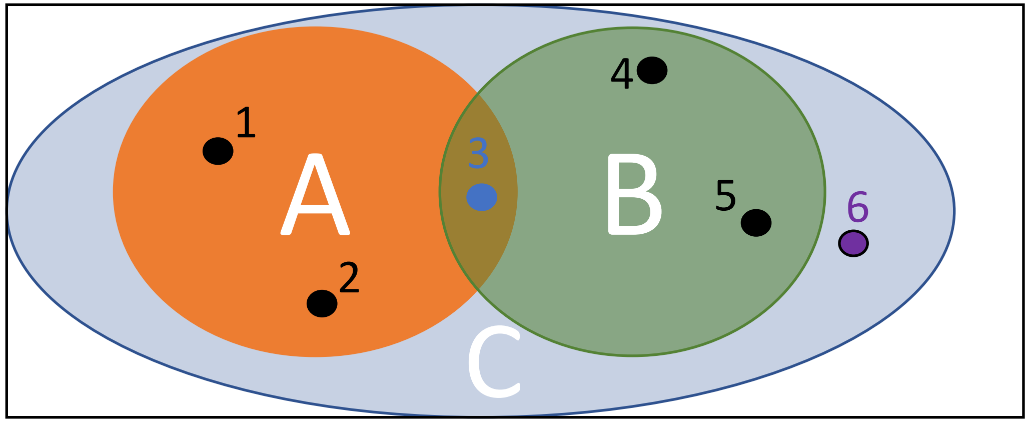

- Then, probability that the dice comes up 3 = $\frac{1}{6}$

Independent events *(informally)*

- Two events are independent if **one does not influence the other.**
- Examples:
  - The event E1 = “first coin toss” and E2 = “second coin toss”
  - With the same die, the events E1 = “roll 1”, E2 = “roll 2”, E3 = “roll 3”
  - Jason flips a coin and then picks a card.
- Counter-examples:
  - E1 = “Die is even”, E2=“Die is 6”
  - E1= “Grade in 250” and “Passing 250”

Law of joint probability *(informally)*

- Two events are independent if **one does not influence the other.**
  - This definition is a but too informal, so mathematicians tend to avoid it.
- Formally, we define that A and B are independent if

$$P(A \cap B) = P(A) \cdot P(B)$$

Disjoint or independent?

1. E_1 = "It rains in College Park, MD today"
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3. E_1 = Die **#1** comes at most 4
   E_2= Die **#2** comes at least 5

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Disjoint or independent?

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   E_2 = "It rains in Athens, Greece today"

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   Weather is weird!

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Recap: “Disjoint” vs “independent”

- Friends don’t let friends get confused between “disjoint” and “independent”!

Disjoint	**Independent**
Has a set-theoretic interpretation!	Has a causality interpretation!
Means that $P(A \cap B) = 0$	Means that $P(A \cap B) = P(A) \cdot P(B)$
Means that $P(A \cup B) = P(A) + P(B)$	Means that $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$

Disjoint Probability ("OR" of two events)

- Jason rolls two dice.
  - **What is the probability that he rolls a 7 or a 9?**

Disjoint Probability ("OR" of two events)

- Jason rolls two dice.
  - **What is the probability that he rolls a 7 or a 9?**
  - #Ways to roll a 7 is 6.
  - #Ways to roll a 9 is 4: (6, 3), (5, 4), (4, 5), (3, 6)
  - #Ways to roll a 7 OR a 9 is then 10.
  - Therefore, the probability is $\frac{10}{36} = \frac{5}{18}$
  - Key: Rolling a 7 and a 9 are **disjoint events**.

Disjoint probability (“OR”)

- 52-card deck
- Probability of drawing a face card (J, Q, K) or a heart

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  - Use law of inclusion / exclusion!

$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 \ - \mathbf{3} = \mathbf{22}$$

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$$|F \cup H| = |F| + |H| - |F \cap H| = 12 + 13 \; - \mathbf{3} = \mathbf{22}$$

- So probability = $\frac{22}{52} = \frac{11}{26}$.

Alternative viewpoint

- $P(F) = \frac{12}{52}$
- $P(H) = \frac{13}{52}$
- $P(F \cap H) = \frac{3}{52}$
- $P(F \cup H) = P(F) + P(H) - P(F \cap H)$

Probability of unions

$$P(A \cup B) = P(A) + P(B) - P\,(A \cap B)$$

- If A and B are independent, we have

$$P(A \cup B) = P(A) + P(B) - P\,(A) \cdot P(B)$$

- If A and B are disjoint, we have

$$P(A \cup B) = P(A) + P(B)$$

Probability of unions of 3 sets

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

- If A, B and C are pairwise independent , we have :

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A) \cdot P(B) - P(B) \cdot P(C) - P(A) \cdot P(C) + P(A \cdot B \cdot C)$$

- If A, B and C are pairwise disjoint (so $A \cap B = A \cap C = B \cap C = \emptyset$, so clearly $A \cap B \cap C = \emptyset$), we have

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

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- If A occurs, then is B
    a) More likely?
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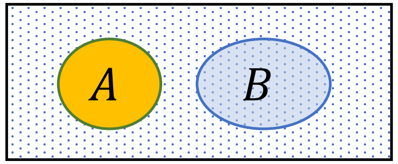
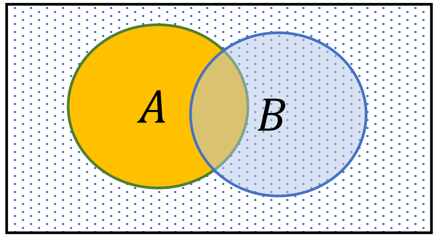


OR



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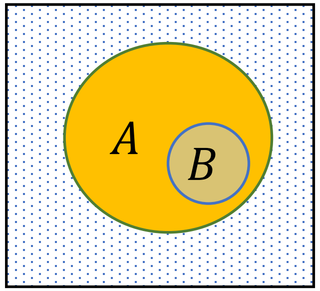

OR …

Examples

- We roll two dice
  - Event A = "Sum of the dice $S \equiv 0 \ (mod\ 4)$"
    - Note that $P(A) = \frac{9}{36} = \frac{1}{4}$, since we have nine rolls of the dice that sum to a multiple of 4:

      (1, 3), (2, 2), (3, 1), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)
  - Event B = "The first die comes up 3"
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  - Therefore, the probability of A given B is $\frac{2}{6} = \frac{1}{3}$

Examples

- We once again two roll dice
  - Event A = “Sum of the dice is $\geq$ 8”
  - Event B = “ First die is 4”

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Let’s see if your intuition was correct!

Examples

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  - Event A = “Sum of the dice is $\geq$ 8” $P(A) = ?$ (work on it)
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  - Event A = "Sum of the dice is $\geq$ 8" $P(A) = \frac{15}{36} = \frac{5}{12}$
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Examples

- We once again two roll dice
  - Event A = “Sum of the dice is ≥ 8” $P(A) = \frac{15}{36} = \frac{5}{12}$
  - Event B = “First die is a 4” $P(B) = \frac{1}{6}$
- Prob of A given B = Prob second dice is 4, 5, or 6 = $\frac{3}{6} = \frac{1}{2} > \frac{5}{12}$

By just $\frac{1}{12}$...

Go up | Go down | Stay the same | Unknown to science

Conditional probability

- Let A, B be two events. The conditional probability of A ***given*** B, denoted $P(A \mid B)$ is defined as follows:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Re-thinking independent events

- **Alternative definition of independent events**: Two events A and B will be called marginally independent, or just independent for short, if and only if

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- Applying the definition of $P(A|B)$ we have:
  - $\frac{P(A \cap B)}{P(B)} = P(A) \Rightarrow P(A \cap B) = P(A) \cdot P(B)$, which is a relationship we had reached **earlier** when discussing the joint probability.

Complex probabilities

- Suppose that I have two dice: a six-sided one and a ten-sided one.
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Complex probabilities

- Suppose that I have two dice: a six-sided one and a ten-sided one.
- I pick either one of them with probability $\frac{1}{2}$ and roll it.
  - What's the probability that the die comes up 6? (work on this yourselves **NOW**)

Complex probabilities

- Suppose that I have two dice: a six-sided one and a ten-sided one.
- I pick either one of them with probability $\frac{1}{2}$
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$$P(Roll = 6) = P(Roll = 6, Die = 6) + P(Roll = 6, Die = 10) =$$

$$= P(Roll = 6|Die = 6) \times P(Die = 6) + P(Roll = 6|Die = 10) \times P(Die = 10) =$$

$$= \frac{1}{6} \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2} = \frac{1}{12} + \frac{1}{20} = \frac{2}{15} \approx 0.1333\ldots$$

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- Suppose that I have two dice: a six-sided one and a ten-sided one.
- Now **we change the problem** so that we pick the ten-sided die with prob $\frac{5}{9}$ and the six-sided die with prob $\frac{4}{9}$.
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$$P(Roll = 6) = P(Roll = 6, Die = 6) \;+ P(Roll = 6, Die = 10) =$$

$$= P(Roll = 6 | Die = 6) \times P(Die = 6) + P(Roll = 6, Die = 10) \times P(Die = 10) =$$

$$= \frac{1}{6} \times \frac{4}{9} + \frac{1}{10} \times \frac{5}{9} = \frac{2}{27} + \frac{1}{18} = \frac{7}{54} \approx 0.130 < \mathbf{0.133}$$

Bayes' Law

- Suppose A and B are events in a sample space Ω. Then, the following is an identity:

$$P(A|B) = P(B|A)\frac{P(A)}{P(B)}$$

known as **Bayes' Law**

Questions

- If $P(A|B) = P(A)$, is it the case that $P(B|A) = P(B)$?

Yes

No

Questions

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Yes

No

- Substituting $P(A|B)$ with $P(A)$ in the formulation of Bayes' Law, we have:

$$\cancel{P(A)} = P(B \mid A) \cdot \frac{\cancel{P(A)}}{P(B)} \Rightarrow 1 = \frac{P(B|A)}{P(B)} \Rightarrow P(B|A) = P(B)$$

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(A *ind* B) iff (B *ind* A)

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- If $P(B) = 0$, then is $P(A|B)$ also 0?

Yes

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No

- It is **undefined**, since $P(A \mid B) = P(B \mid A) \cdot \frac{P(A)}{P(B)}$