Countability

CMSC 250

Motivation

- Two toddlers want to compare their marbles to see who has more.
- They cannot count yet.

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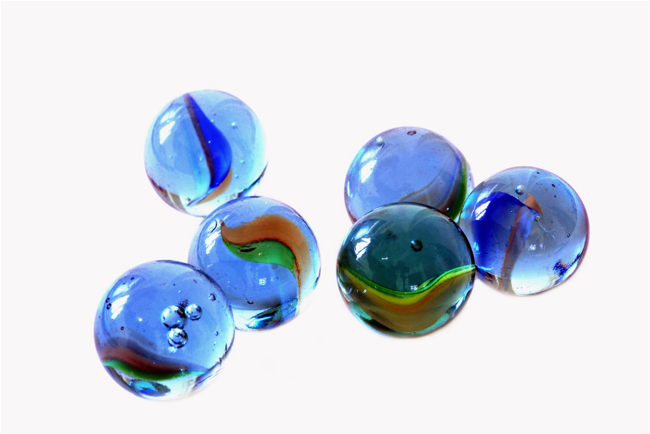
- Two toddlers want to compare their marbles to see who has more.
- They cannot count yet.

How do they find out who has more?

Motivation

- They line them up and compare!



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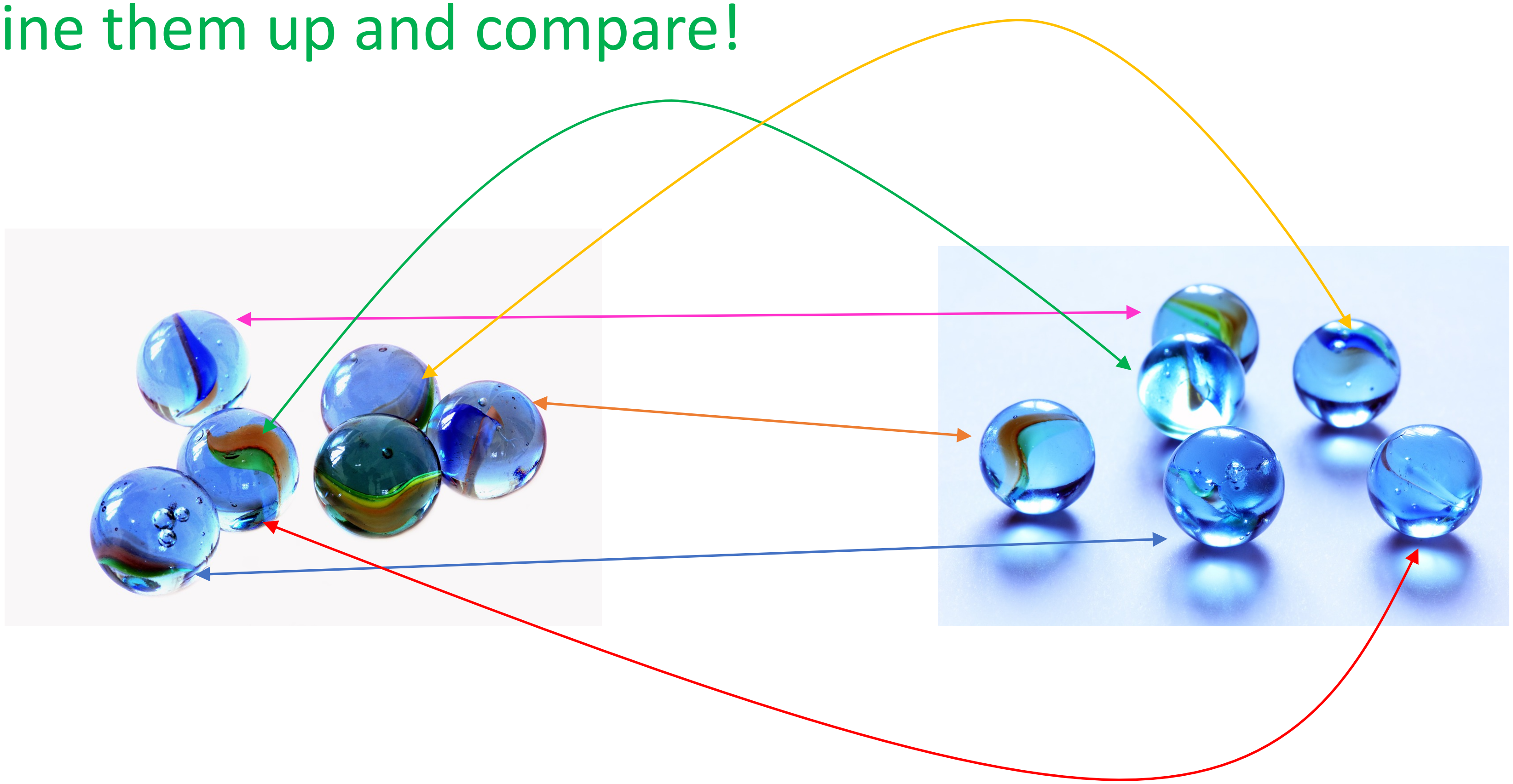

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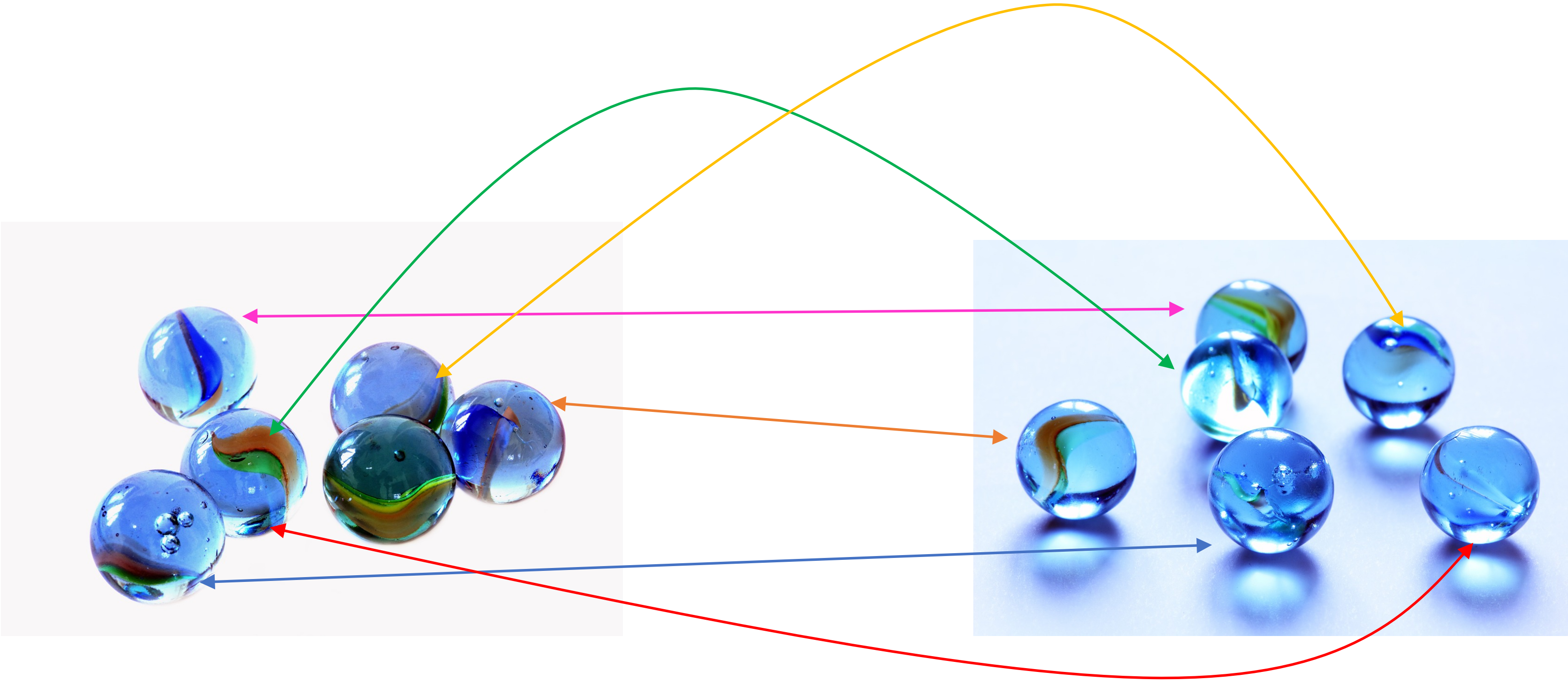
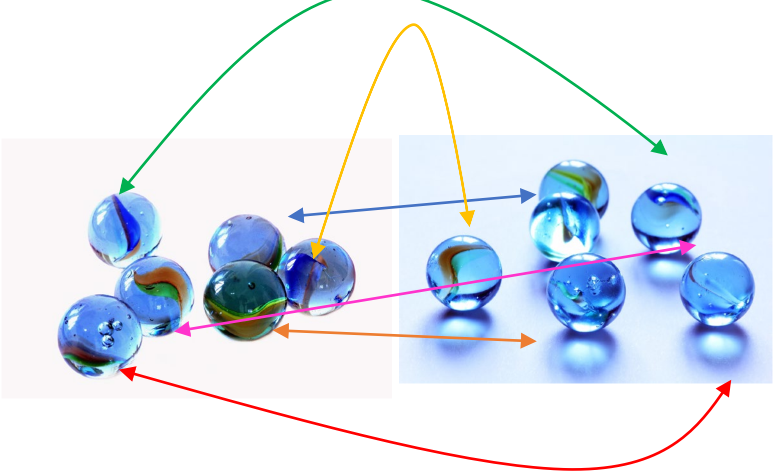


- Intuition for us: If we can find such a mapping between two (infinite) sets, we will say that they have the same *(infinite)* cardinality *(or size)*.

Motivation



- This matching of marbles
  - Every two different marbles on left go to two different marbles on right
  - Every marble on right is matched by some marble on the left

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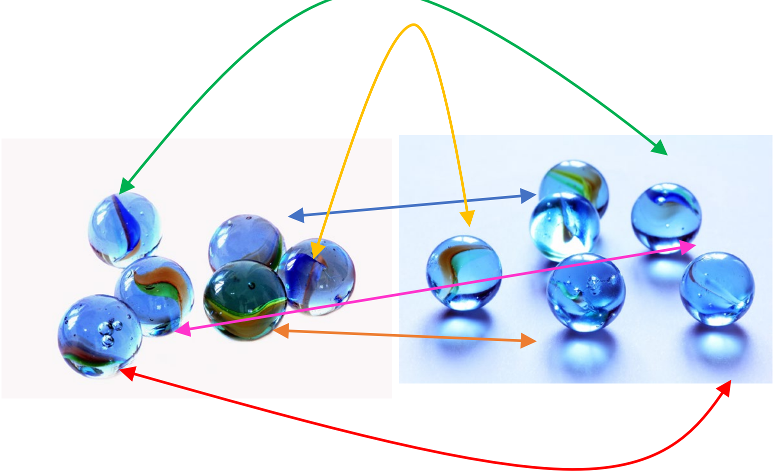

- This matching of marbles
  - Every two different marbles on left go to two different marbles on right
  - Every marble on right is matched by some marble on the left
- This is a bijection!
- **WE DEFINE TWO SETS TO BE THE SAME SIZE IF THERE IS A BIJECTION BETWEEN THEM.**

Refresher on bijections

All domains and codomains $\mathbb{R}$, unless otherwise stated

Refresher on bijections

- Are the following functions **bijections**?

Yes

No

1. $f(x) = |x|$

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Quiz on bijections

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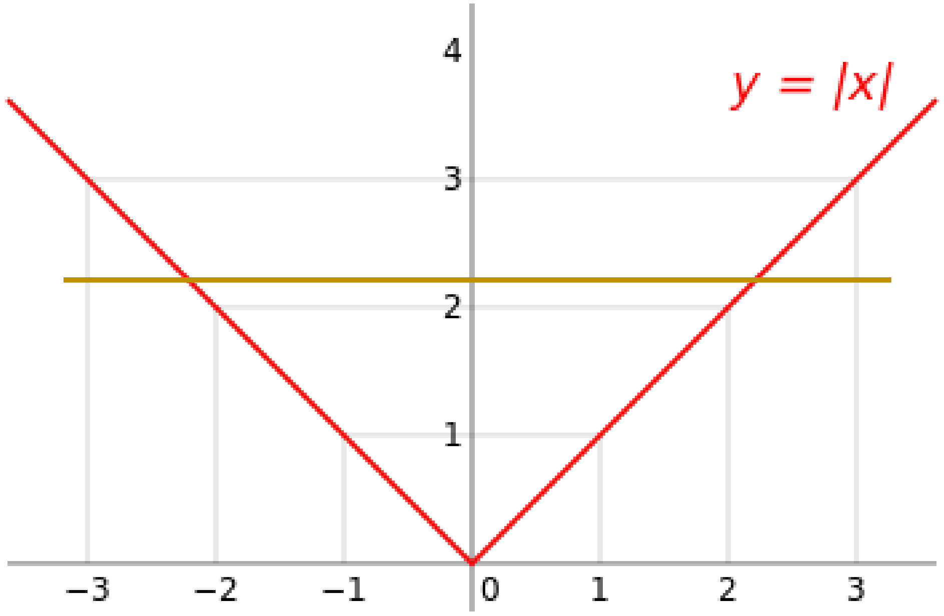

Non-injective!

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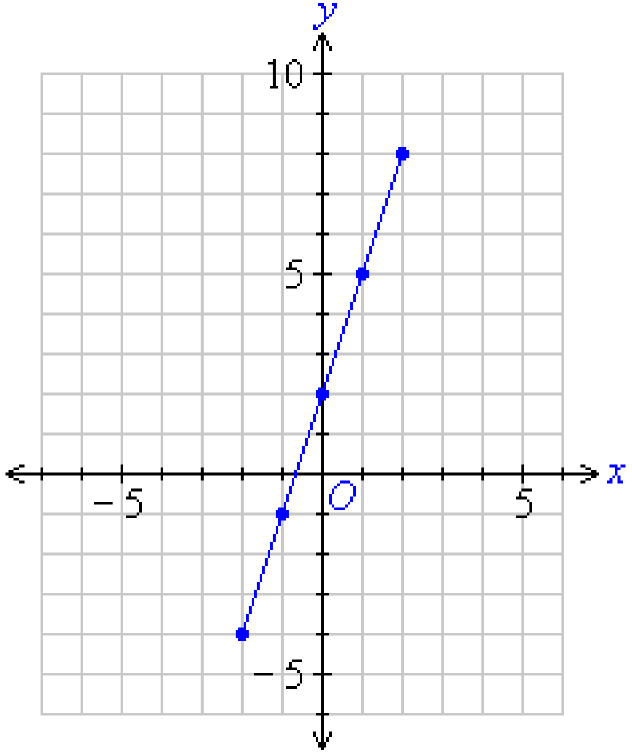

Straight line in coordinate plane

All domains and codomains $\mathbb{R}$, unless otherwise stated

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2. $f(x) = a \cdot x + b, b \in \mathbb{R}, a \in \mathbb{R}^{\neq 0}$ *Yes*
3. $g(x) = a \cdot x^2, a > 0$

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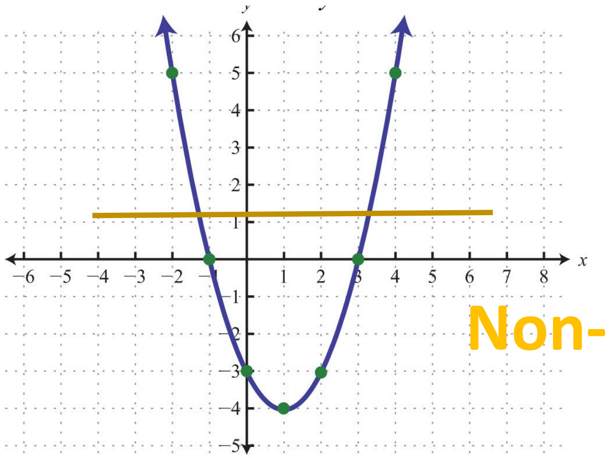

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4. $h(n) = 4n - 1, \boldsymbol{n} \in \mathbb{Z}$

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3. $g(x) = a \cdot x^2, a > 0$ No
4. $h(n) = 4n - 1, \boldsymbol{n} \in \mathbb{Z}$ No

Non-surjective! Set $h(n) = y$ and solve for n:

$$(4n - 1 = y) \Rightarrow n = \frac{y+1}{4}$$

There are infinitely many choices of y for which $n \notin \mathbb{Z}$!

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4. $h(n) = 4n - 1, \boldsymbol{n} \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1,\ x \in \mathbb{R}$

All domains and codomains $\mathbb{R}$, unless otherwise stated

Refresher on bijections

- Are the following functions **bijections**?

Yes

No

1. $f(x) = |x|$ *No*
2. $f(x) = a \cdot x + b, b \in \mathbb{R}, a \in \mathbb{R}^{\neq 0}$ *Yes*
3. $g(x) = a \cdot x^2, a > 0$ *No*
4. $h(n) = 4n - 1, n \in \mathbb{Z}$ *No*
5. $h(x) = 4x - 1, x \in \mathbb{R}$
6. *Yes*

Surjective and injective! Surjective, since, if we set $h(n) = y$ and solve for n:

$$(4x - 1 = y) \Rightarrow x = \frac{y+1}{4}$$

For every real y, there's always a **real** solution n. Injective, since it's of the form of (2) with $a \neq 0$.

Countable sets

- **Definition:** A set S is said to be **countable** if **there exists a bijection from a subset of $\mathbb{N}^{\geq 1}$ to S.**
  - Sometimes, this bijection is called an **enumeration.**
  - Alternatively, yet still rigorously: **If we can form some sequence out of its elements** (or, if we can **enumerate** its elements)
  - Equivalently, blending in Physics: If every one of its elements can be reached in **finite time.**

Finite sets and countability

- **Every finite set is countable.**

Finite sets and countability

- **Every finite set is countable.**
  - ***Why?***

Finite sets and countability

- Every finite set is countable.
  - ***Why?***
  - Suppose that S is a finite set. Since it's finite, it contains n elements, for $n \in \mathbb{N}$. This means that S can be **enumerated,** like so:

$$S = \{s_1, s_2, s_3, \dots, s_n\}$$

But this means that there exists a bijection from $\{1, 2, \dots, n\}$ to S, where $\{1, 2, \dots, n\} \subseteq \mathbb{N}$!

Infinite sets and countability

- Since all finite sets are countable, might as well limit ourselves to the exploration of **infinite sets** that might also be **countable**.
  - We call those "countably infinite" sets.
- Let such a set be called S. Then, to prove that it's countable, we need to find some bijection b from $\mathbb{N}^{\geq 1}$ to S.

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- Is $b: \mathbb{N}^{\geq 1} \mapsto \mathbb{N}^{\geq 1}$ such that

$$b(n) = n$$

a bijection?

Yes

No

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a bijection?

Conclusion: $\mathbb{N}^{\geq 1}$ is countably infinite

Yes

No

Countability of $\mathbb{N}$

- Is $\mathbb{N}$ countable? (recall, $0 \in \mathbb{N}$)

Yes

No

Countability of $\mathbb{N}$

- Is $\mathbb{N}$ countable? (recall, $0 \in \mathbb{N}$)

Yes

No

- Through the bijection $f(n) = n - 1$, like so:

$$\mathbb{N}^{\geq 1}: 1, 2, 3, 4, , \dots, 56, \dots$$

$$\downarrow f$$

$$\mathbb{N}: 0, 1, 2, 3, \dots, 55, \dots$$

Countability of other $\mathrm{A} \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \wedge (x \geq 17)\}$ countable?

Yes

No

Countability of other $A \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \wedge (x \geq 17)\}$ countable?

Yes **No**

- Through the bijection $f(n) = n + 16$, like so:

$$\mathbb{N}^{\geq 1}: 1, 2, 3, 4, \ldots, 56, \ldots$$

$$f$$

$$\mathbb{N}^{\geq 17}: 17, 18, 19, 20, \ldots, 72 \ \ldots$$

Countability of other $\mathrm{A} \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \wedge (x \equiv 0 \ (mod\ 2)\}$ countable?

Yes

No

Countability of other $A \subseteq \mathbb{N}$

- Is the set $\{x \mid (x \in \mathbb{N}) \wedge (x \equiv 0 \ (mod\ 2)\}$ countable?

Yes

No

$\mathbb{N}^{\geq 1}$: $1, 2, 3, 4, \ldots$

$\mathbb{N}^{even}$: $0, 2, 4, 6, \ldots$

Countability of other $\mathrm{A} \subseteq \mathbb{N}$

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Yes

No

$$\mathbb{N}^{\geq 1}: \quad 1, 2, 3, 4, \ldots$$

$$\downarrow f$$

$$\mathbb{N}^{even}: \quad 0, 2, 4, 6, \ldots$$

$$f(n) = 2(n-1)$$

Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

Yes

No

Countability of $\mathbb{Z}$

- Is $\mathbb{Z}$ countable?

Yes

No

$$0, 1, -1, 2, -2, 3, -3, \dots$$

1 2 3 4 5 6 7

Countability of $\mathbb{Z}$

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No

$0, 1, -1, 2, -2, 3, -3, \ldots$

1 2 3 4 5 6 7 $f = ?$

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$$f: \mathbb{N}^{\geq 1} \mapsto \mathbb{Z}, f(n) = \begin{cases} 0, & n = 1 \\ \frac{n}{2}, & n\ even \\ \frac{-(n-1)}{2}, & n\ odd \end{cases}$$

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- f is…
  - onto, since every integer is mapped to

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- f is…
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  - 1-1, since no two naturals map to the same integer
  - So it's a bijection, and $\mathbb{Z}$ is countable!

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

Yes

No

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

Yes

No

$$0, 2, -2, 4, -4, 6, -6 \ldots$$

1 2 3 4 5 6 7

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

Yes

No

$$0, 2, -2, 4, -4, 6, -6 \ldots$$

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$$f(n) = \begin{cases} 0, & n = 1 \\ n, & n = 2, 4, 6, \ldots \\ -n+1, & n = 3, 5, 7, \ldots \end{cases}$$

Countability of $\mathbb{Z}^{even}$

- Is $\mathbb{Z}^{even}$ countable?

Yes

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$$f(n) = \begin{cases} 0, & n = 1 \\ n, & n = 2, 4, 6, \ldots \\ -n+1, & n = 3, 5, 7, \ldots \end{cases}$$

Both onto and 1-1

Countability of $\mathbb{Z}^{even}$

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Countability of $\mathbb{Z}^{even}$

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- So, since we know that $\mathbb{Z}$ is countable…

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  - i.e that there's a bijection from $\mathbb{N}^{\geq 1}$ to $\mathbb{Z}$…

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    - ***If we find a bijection*** from $\mathbb{Z}$ to $\mathbb{Z}^{even}$ …

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      - We will have a bijection from $\mathbb{N}^{\geq 1}$ to $\mathbb{Z}^{even}$, and $\mathbb{Z}^{even}$ is, therefore, countable!

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$$\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots$$

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

$$f(n) = 2 * n$$

Countability of $\mathbb{Z}^{even}$

- If f and g are bijections, then $f(g(x)) = (f \circ g)(x)$ is also a bijection
  - Prove this at home!
- So, since we know that $\mathbb{Z}$ is countable…
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$$f(n) = 2 * n$$

clearly bijective

Countability of $\mathbb{Q}^{>0}$

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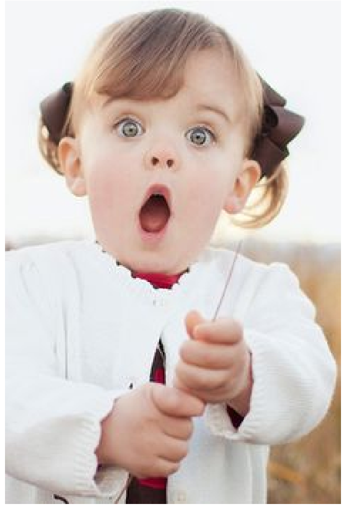

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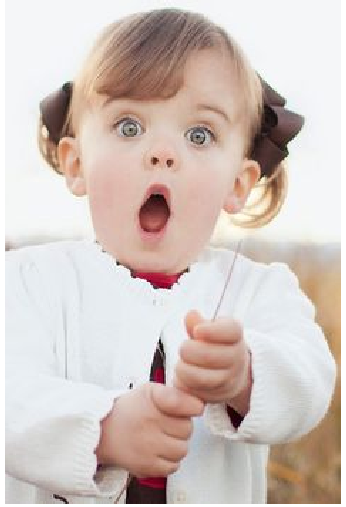

- Is $\mathbb{Q}^{>0}$ countable?

Yes

No

	1	2	3	4	…
1	$^1/_1$	$^1/_2$	$^1/_3$	$^1/_4$	…
2	$^2/_1$	$^2/_2$	$^2/_3$	$^2/_4$	…
3	$^3/_1$	$^3/_2$	$^3/_3$	$^3/_4$	….
4	$^4/_1$	$^4/_2$	$^4/_3$	$^4/_4$	….
…	…	…	….	….	….

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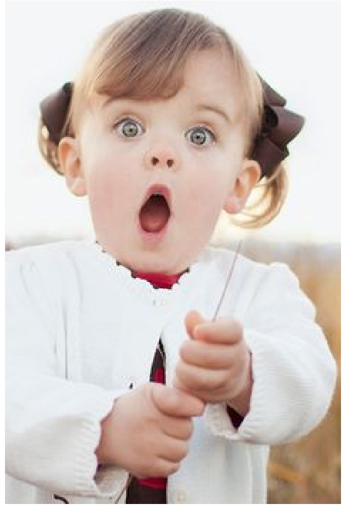

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3	$^3/_1$	$^3/_2$	$^3/_3$	$^3/_4$	….
4	$^4/_1$	$^4/_2$	$^4/_3$	$^4/_4$	….
…	…	…	….	….	….

All strictly positive rationals are counted **exactly once** *(skipping repetitions like* $\frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \cdots$*)*, so this "snaking" is a bijection

Countability of $\mathbb{Q}^{>0}$

- If you don't like the proof involving this "snaking" pattern, ProofWiki has 4 (!) different proofs here: http://www.homeschoolmath.net/teaching/rational-numbers-countable.php
  - (1) tries to prove the "snaking" pattern in a way that I don't find very rigorous
  - 2, 3, 4 assume other facts that we won't prove today, but are easy to prove
    - E.g the cartesian product of countable sets is also countable, or the union of countable sets is also a countable set!

Countability of $\mathbb{Q}^{>0}$

- Another way to prove this fact formally is focusing on the diagonals, and realize that the i^{th} diagonal has an interesting property!

$$d_i = \{\frac{a}{b} \mid (a, i \in \mathbb{N}) \wedge (b \in \mathbb{N}^{\neq 0}) \wedge (a + b) = i + 2\}$$

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- If we can prove that:

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- If we can prove that:
  1. $(\forall i \in \mathbb{N})[d_i \text{ is countable}]$
  2. The **union** of countable sets is **also countable**

- We then have:

$$d_0 \cup d_1 \cup d_2 \cup \cdots = \bigcup_{i=0}^{+\infty} d_i \ = \mathbb{Q}^{>0} \text{ is countable!}$$

Some theorems on countability

- Suppose A is a countable set and $e \notin A$. Is $A \cup \{e\}$ countable?

Yes

No

Unknown to science

Some theorems on countability

- Suppose A is a countable set and $e \notin A$. Is $A \cup \{e\}$ countable?

Yes

No

Unknown to science

- Suppose a_1, a_2, a_3,… is an enumeration of A.
- We then define a new enumeration b of $A \cup \{e\}$, like so:

$$b_n = \begin{cases} e, & n = 1 \\ a_{n-1}, & n \geq 2 \end{cases}$$

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Pretty much like in the case of $\mathbb{N}$, we just "move one index over"!

Some theorems on countability

- Suppose A and B are countable sets. Is $A \cup B$ countable?

Yes

No

Unknown to science

Some theorems on countability

- Suppose A and B are countable sets. Is $A \cup B$ countable?

Yes | No | Unknown to science

- For simplicity, assume A and B are countably infinite.

$$\begin{array}{ccccc} 1, & 2, & 3, & 4, & \dots. \\ \downarrow & \downarrow & \downarrow & \downarrow & f \\ a_1, & b_1, & a_2, & b_2 & \dots \end{array}$$

Some theorems on countability

- Suppose A and B are countable sets. Is $A \cup B$ countable?

Yes | No | Unknown to science

- For simplicity, assume A and B are countably infinite.

$$\begin{array}{cccc} 1, & 2, & 3, & 4, \ldots \\ \downarrow & \downarrow & \downarrow & \downarrow \; f \\ a_1, & b_1, & a_2, & b_2 \ldots \end{array}$$

$$f(n) = \begin{cases} a_{(n+1)/2}, & n \text{ odd} \\ b_{n/2}, & n \text{ even} \end{cases}$$

What if A or B (or both) finite?

- Caveat: the previous will **not** work if A or B end before the other ends.
  - Because some a_i, b_i might not exist.
- We leave it to you to iron out the details of what happens then.

Countability of $\mathbb{R}$

- Is $\mathbb{R}$ countable?

Yes

No

Unknown to science

Countability of $\mathbb{R}$

- Is $\mathbb{R}$ countable?

Yes

No

Unknown to science

Countability of $\mathbb{R}$

- Is $\mathbb{R}$ countable?

Yes | No | Unknown to science

- Cantor's famous **diagonal argument!**

Countability of ℝ

- Is ℝ countable?

Yes | No | Unknown to science

- Cantor's famous **diagonal argument!**
- The argument actually proves that the interval [0,1] is uncountable, but the result generalizes to the entirety of ℝ

Cantor’s diagonal argument

- **Proof by contradiction:** Suppose that $[0, 1]$ is countable. Then, there exists some bijection from $\mathbb{N}^{\geq 1}$ to $[0, 1]$, i.e the reals can be enumerated in a sequence:

1. 0.28422856231…..
2. 0.28422856232…..
3. 0.28922856233…..

……………………………

……………………………

n. 0.28422856001……

Cantor’s diagonal argument

$$
\begin{aligned}
r_1 &= 0.28422856231 \ldots \\
r_2 &= 0.28422856232 \ldots \\
r_3 &= 0.28922856233 \ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
r_n &= 0.28422856001 \ldots
\end{aligned}
$$

- Let’s create the real number $r = 0.a_1 a_2 a_3 \ldots a_n \ldots$ where

$$
a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \le r_{i_i} < 9 \end{cases}
$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

Cantor's diagonal argument

$$r_1 = 0.28422856231\ldots$$
$$r_2 = 0.28422856232\ldots$$
$$r_3 = 0.28922856233\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001\ldots$$

- Let's create the real number $r = 0.a_1a_2a_3\ldots a_n\ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \le r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.$____________

Cantor's diagonal argument

$$r_1 = 0.\mathbf{2}8422856231\ldots$$
$$r_2 = 0.28422856232\ldots$$
$$r_3 = 0.28922856233\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001\ldots$$

- Let's create the real number $r = 0.a_1a_2a_3\ldots a_n\ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \leq r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.$**3**__________

Cantor's diagonal argument

$$
\begin{aligned}
r_1 &= 0.28422856231\ldots\\
r_2 &= 0.28422856232\ldots\\
r_3 &= 0.28922856233\ldots\\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
r_n &= 0.28422856001\ldots
\end{aligned}
$$

- Let's create the real number $r = 0.a_1a_2a_3\ldots a_n\ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \le r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.3$__________

First digit yielded by the application of the second rule…

Cantor's diagonal argument

$$r_1 = 0.\mathbf{2}8422856231 \ldots$$
$$r_2 = 0.2\mathbf{8}422856232 \ldots$$
$$r_3 = 0.28922856233 \ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001 \ldots$$

- Let's create the real number $r = 0.a_1 a_2 a_3 \ldots a_n \ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \le r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.39$ __________

Cantor’s diagonal argument

$$r_1 = 0.\mathbf{2}8422856231\ldots$$
$$r_2 = 0.2\mathbf{8}422856232\ldots$$
$$r_3 = 0.28922856233\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001\ldots$$

- Let’s create the real number $r = 0.a_1a_2a_3\ldots a_n\ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \le r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

Second digit yielded by second rule again!

- In this example, $r = 0.39$______

Cantor's diagonal argument

$$r_1 = 0.\mathbf{2}8422856231 \ldots$$
$$r_2 = 0.2\mathbf{8}422856232 \ldots$$
$$r_3 = 0.28\mathbf{9}22856233 \ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001 \ldots$$

- Let's create the real number $r = 0.a_1a_2a_3 \ldots a_n \ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \leq r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.390$__________

Cantor's diagonal argument

$$r_1 = 0.\mathbf{2}8422856231\ldots$$
$$r_2 = 0.2\mathbf{8}422856232\ldots$$
$$r_3 = 0.28\mathbf{9}22856233\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001\ldots$$

- Let's create the real number $r = 0.a_1a_2a_3\ldots a_n\ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \leq r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.390$________

This time we have to use the first rule, since $r_{3_3} = 9$.

Cantor’s diagonal argument

$$
\begin{aligned}
r_1 &= 0.\mathbf{2}8422856231\ldots \\
r_2 &= 0.2\mathbf{8}422856232\ldots \\
r_3 &= 0.28\mathbf{9}22856233\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots \\
r_n &= 0.28422856001\ldots
\end{aligned}
$$

The proof moves ***diagonally***, hence the term “diagonal” argument!

- Let’s create the real number $r = 0.a_1a_2a_3 \ldots a_n \ldots$ where

$$a_i = \begin{cases} 0, & r_{i_i} = 9 \\ r_{i_i} + 1, & 0 \leq r_{i_i} < 9 \end{cases}$$

Note: r_{i_i} is the i^{th} digit of the i^{th} real.

- In this example, $r = 0.390$__________

Cantor’s diagonal argument

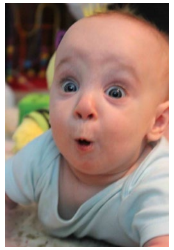
$$r_1 = 0.\mathbf{2}8422856231\ldots$$
$$r_2 = 0.2\mathbf{8}422856232\ldots$$
$$r_3 = 0.28\mathbf{9}22856233\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_n = 0.28422856001\ldots$$

- Somebody claims that $r = 0.395\ldots$ is the 17th real in the list.

Cantor's diagonal argument

$$
\begin{aligned}
r_1 &= 0.\mathbf{2}8422856231\ldots \\
r_2 &= 0.2\mathbf{8}422856232\ldots \\
r_3 &= 0.28\mathbf{9}22856233\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots \\
r_n &= 0.28422856001\ldots
\end{aligned}
$$

- Somebody claims that $r = 0.395\ldots$ is the 17th real in the list.
- But this cannot be true, since our real number was constructed such that it differs from the 17th real in the 17th decimal digit!



Cantor's diagonal argument

$$
\begin{aligned}
r_1 &= 0.\mathbf{2}8422856231\ldots \\
r_2 &= 0.2\mathbf{8}422856232\ldots \\
r_3 &= 0.28\mathbf{9}22856233\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots \\
\vdots &= \ldots\ldots\ldots\ldots\ldots\ldots \\
r_n &= 0.28422856001\ldots
\end{aligned}
$$

- Somebody claims that $r = 0.395\ldots$ is the 17$^{\text{th}}$ real in the list.
- But this cannot be true, since our real number was constructed such that it differs from the 17$^{\text{th}}$ real in the 17$^{\text{th}}$ decimal digit!
- Generally speaking, r will differ from the i^{th} real in the i^{th} digit!
  - So we can't find an $k \in \mathbb{N}$ such that $0.395\ldots = r_k$.
  - **Contradiction, since we assumed we can enumerate all reals in [0,1].**

Is $|\mathbb{N}| < |\mathbb{R}|$?

- It has to be, since any given natural maps to a real, but we just proved that ***not all*** reals **can** be mapped to a natural!
- But how can we say this rigorously?
- **Defn:** $|A| \leq |B|$ if there is an **injection** from A into B
- **Defn:** $|A| < |B|$ if there is an **injection** from A into B but there is **no surjection** from A into B!

More theorems on countability

- Is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ countable?

Yes

No

Unknown to science

More theorems on countability

- Is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ countable?

Yes

No

Unknown to science

- Proof: $f: \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}$ such that $f(x) = (x, 1)$ is an **injection (1-1)**
  - Hence, $\mathbb{R} \times \mathbb{R}$ is **at least as big** as $\mathbb{R}$, and $\mathbb{R}$ is uncountable.
  - So , $\mathbb{R} \times \mathbb{R}$ is uncountable.

More theorems on countability

- Is $\mathbb{C}$ (set of complex numbers) countable?

Yes

No

Unknown to science

More theorems on countability

- Is $\mathbb{C}$ (set of complex numbers) countable?

Yes | No | Unknown to science

- Remember: complex numbers defined as $a + b \cdot i$ for $a, b \in \mathbb{R}$.
  - $f: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ such that $f\big((a, b)\big) = a + b \cdot i$ is a bijection from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$
  - But we know that $\mathbb{R} \times \mathbb{R}$ is uncountable. Therefore, $\mathbb{C}$ is uncountable.

More theorems on countability

- Let A be any uncountable set. Is there any (infinite) $B \subseteq A$ that is countable?

Yes

No

Unknown to science

More theorems on countability

- Let A be any uncountable set. Is there any (infinite) $B \subseteq A$ that is countable?

Yes

No

Unknown to science

- Consider: $[0,1]$ and $\left\{\frac{1}{x} \mid x \in \mathbb{N}^{\geq 1}\right\} \subseteq [0,1]$

More theorems on countability

- Let A be any uncountable set. Is there any (infinite) $B \subseteq A$ that is countable?

Yes

No

Unknown to science

- Consider: $[0,1]$ and $\left\{\frac{1}{x} \mid x \in \mathbb{N}^{\geq 1}\right\} \subseteq [0,1]$

All these are positive rationals!

More theorems on countability

- Let $\{0, 1\}^\infty$ be the set of infinite bitstrings
  - Is it countable?

Yes

No

Unknown to science

More theorems on countability

- Let $\{0,1\}^\infty$ be the set of infinite bitstrings
  - Is it countable?

Yes

No

Unknown to science

- Cantor-like proof in next slide!

The set of infinite bit-strings is uncountable

- Assume that the set is countable, then the strings can be enumerated:

1: 000111010101010…

2: 0101011110001101…

…

n: 010101000011100…

- Construct bit-string s which *differs from the i^{th} string in the list in the i^{th} digit.*
- Since this string is not in the list, we can't enumerate them all. Contradiction.

More theorems on countability

- Let A_1, A_2, A_3 … be an **infinite** sequence of **count**able sets.
- Is $A_1 \times A_2 \times A_3 \times \cdots$ countable?

Yes

No

Unknown to science

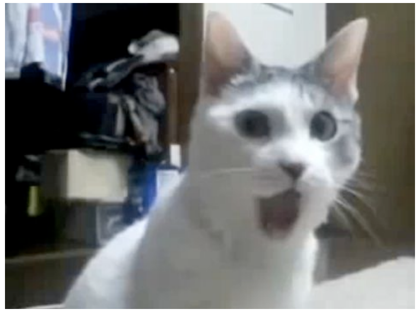
More theorems on countability

- Let A_1, A_2, A_3 ... be an **infinite** sequence of **count**able sets.
- Is $A_1 \times A_2 \times A_3 \times \cdots$ countable?

Yes

No

Unknown to science



- Cantor-like proof in next slide!

Set of **infinite** cartesian product of **countable sets** is **uncountable**

- Notation: $a_i = \{a_{i_1}, a_{i_2}, a_{i_3}, \dots\}$
- Suppose that the set is countable. Then, enumeration:

$$
\begin{gathered}
(a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \dots), \\
(a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \dots), \\
(a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \dots), \\
\dots
\end{gathered}
$$

Set of **infinite** cartesian product of **countable sets** is **uncountable**

- Notation: $a_i = \{a_{i_1}, a_{i_2}, a_{i_3}, \dots\}$
- Suppose that the set is countable. Then, enumeration:

$$
\begin{gathered}
(a_{1_1}, a_{1_2}, a_{1_3}, a_{1_4}, \dots),\\
(a_{2_1}, a_{2_2}, a_{2_3}, a_{2_4}, \dots),\\
(a_{3_1}, a_{3_2}, a_{3_3}, a_{3_4}, \dots),\\
\dots
\end{gathered}
$$

Construct infinite tuple $(a_{1_{x_1}}, a_{2_{x_2}}, a_{3_{x_2}}, \dots)$ such that x_i is an element of A_i different from the element used in the i^{th} position of the i^{th} tuple!

- This tuple cannot be in the list, etc etc etc

More reading

- Epp, 7.4
- Rosen, section 2.5 (Rosen presents Set Theory / Countability **much** earlier than Epp)
- If you're interested in finding out more about the **different kinds of infinities**, check out these VSauce videos:
  - https://www.youtube.com/watch?v=SrU9YDoXE88
  - https://www.youtube.com/watch?v=s86-Z-CbaHA