Pigeonhole Principle

CMSC250

Look at these pigeons.

Figure: Look.

Examples first

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3. Is there a pair of New Yorkers with the same number of hairs on their heads?

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1. Is there a pair of you with the same birthday month? Yes, since there are more than 12 of you!
2. Is there a pair of you with the same birthday week?Yes, since there are more than 52 of you!
3. Is there a pair of New Yorkers with the same number of hairs on their heads? Yes! Number of hairs on your head $\leq 300,000$, New Yorkers $\geq 8,000,000$.

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4. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$. If I pick 5 integers, is it the case that at least one pair of integers has a sum of 9? Yes. Pigeonholes = pairs of ints that sum to 9:

$$(1, 8)$$
$$(2, 7)$$
$$(3, 6)$$
$$(4, 5)$$

   and pigeons = ints to pick.

Examples first

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5. Let $A \subseteq \{1, 2, \ldots, 10\}$, and $|A| = 6$. Is there a pair of subsets of A that have the same sum? Yes.
   There are $2^6 = 64$ subsets of A. Max sum: $10 + 9 + \cdots + 5 = 45$
   Min sum: 0
   46 different sums (pigeonholes)
   64 different subsets (pigeons).

Examples first

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6. Is it true that within a group of 700 people, there must be 2 who have the same **first** and **last** initials? Yes.
   There are $26^2 = 676$ different sets of first and last initials (pigeonholes)
   There are 700 people (pigeons).

Formal Statement of the principle

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Let $m, n \in \mathbb{N}^{\geq 1}$. If n pigeons fly into m pigeonholes and $n > m$, then **at least one** pigeonhole will contain more than one pigeon.

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  Absolutely. Only thing we need is one pigeonhole with at least 2 pigeons.
- Example: There might not be somebody with initials (X, Y).

Pigeonhole Principle (in functions)

Let A and B be finite sets such that $|A| > |B|$. Then, there does not exist a one-to-one function $f : A \mapsto B$.

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2. If there are 105 of you, are there at least **3** of you with the same birthday week? Yes. If there are at most 2, then $2 \times 52 = 104 < 105$
3. Is it true that within a group of 86 people, there exist **at least 4** with the same **last initial** (e.g **B** for Justin **B**ieber).

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2. If there are 105 of you, are there at least **3** of you with the same birthday week? Yes. If there are at most 2, then $2 \times 52 = 104 < 105$
3. Is it true that within a group of 86 people, there exist **at least 4** with the same **last initial** (e.g **B** for Justin **B**ieber). Yes. Pigeonholes = #initials=26. For $k = 3$, $86 > 3 \times 26 = 78$

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   $1000 + 999 + \cdots + 981 = \sum_{i=1}^{1000} i - \sum_{i=1}^{980} i \overset{\text{Gauss}}{=\!=} \frac{1000 \cdot 1001}{2} - \frac{980 \cdot 981}{2} =$
   19810. The min sum is 0, corresponding to $\emptyset \subseteq A$. So 19811 sums.
   Since $\lceil {}^{2^{20}}/_{19811} \rceil = 53$ (yes, you may totally use a calculator here), there are 53 subsets of A that sum to the same number.

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Non-constructive! It proves that it's a **logical necessity** that 53 subsets map to the same sum, but doesn't tell you **anything** (e.g cardinality) of the subsets.

Generalization

Generalized Pigeonhole Principle

Let n and m be positive integers. Then, if there exists a positive integer k such that $n > km$ and n pigeons fly into m pigeonholes, there will be **at least one** pigeonhole with **at least** $k + 1$ pigeons.

- Our second example set consisted of examples of the **generalized** form of the principle.