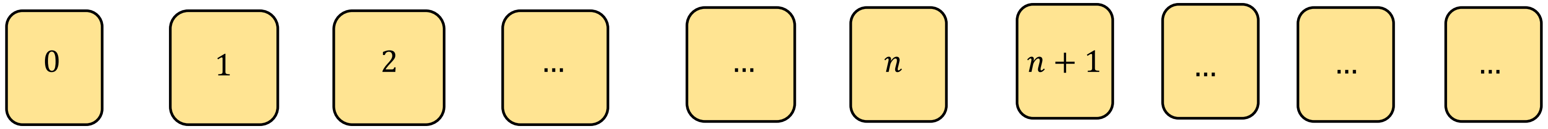
Mathematical Induction: Introduction and basic problems

CMSC 250

The idea behind induction

- Suppose that we want to prove that a proposition $P(n)$ is true for all natural numbers n.



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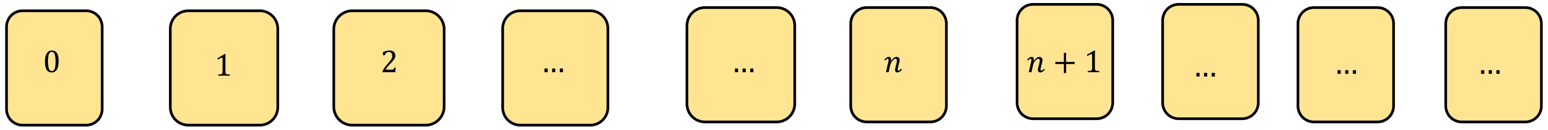

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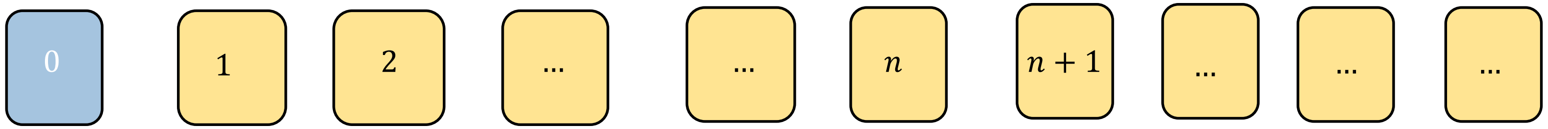

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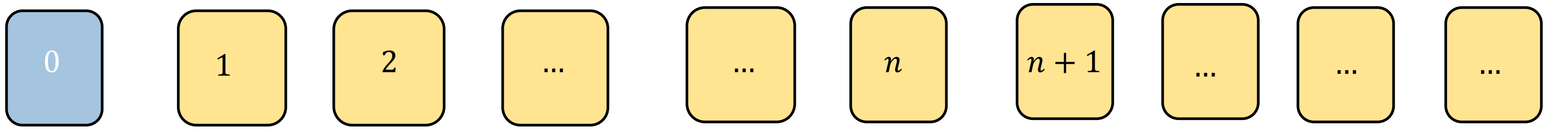

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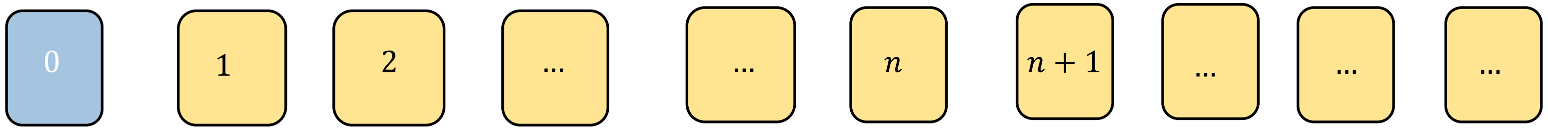

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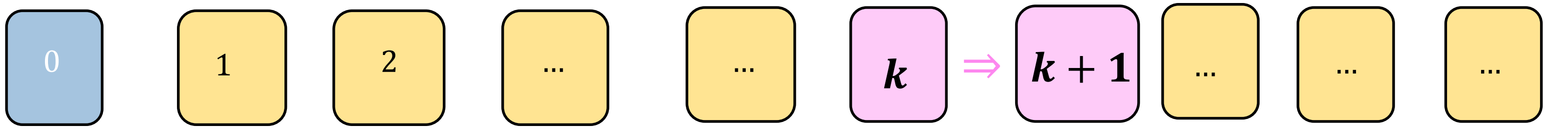
- We will prove two separate things:
  1. $P(0)$ is true.
  2. For $n = k \geq 0$, **if** P is true for k (symb. "$P(k)$", or "$P(k)$ holds"), it **then** must be true for $k + 1$ (symb. "$P(k+1)$", or "$P(k+1)$ holds").



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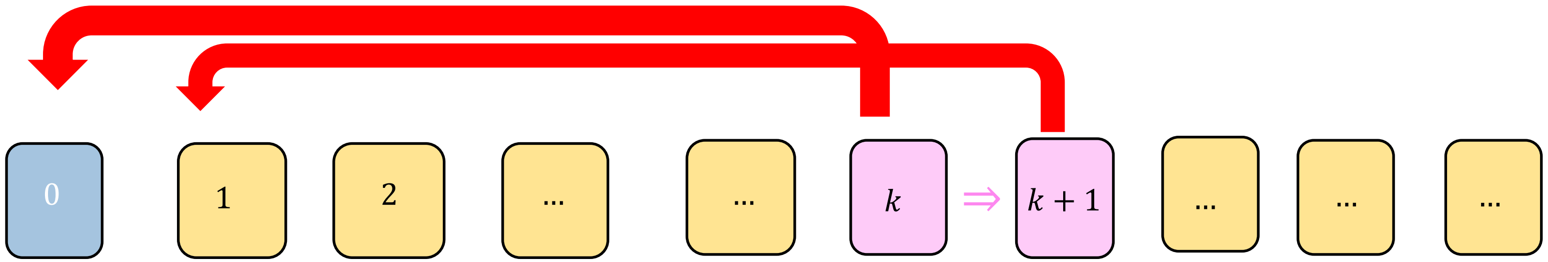

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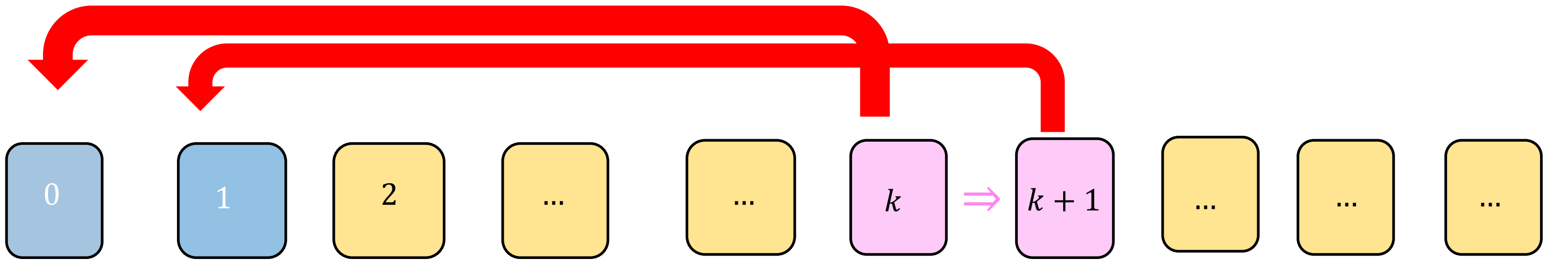

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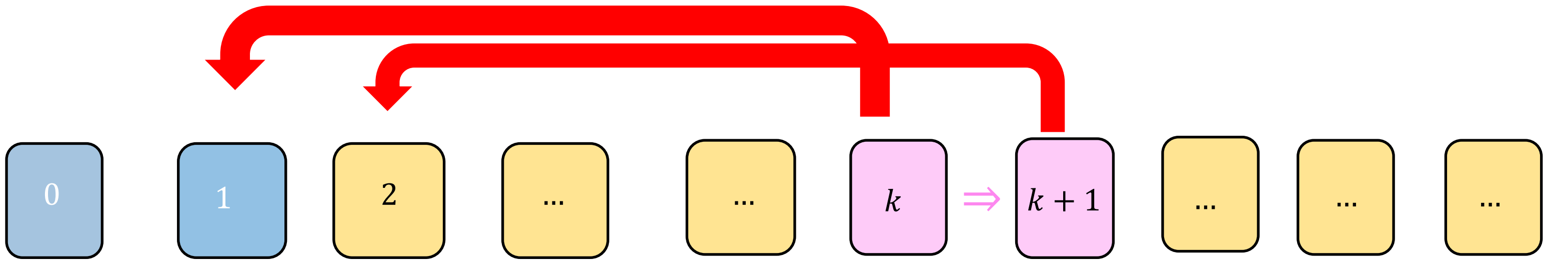

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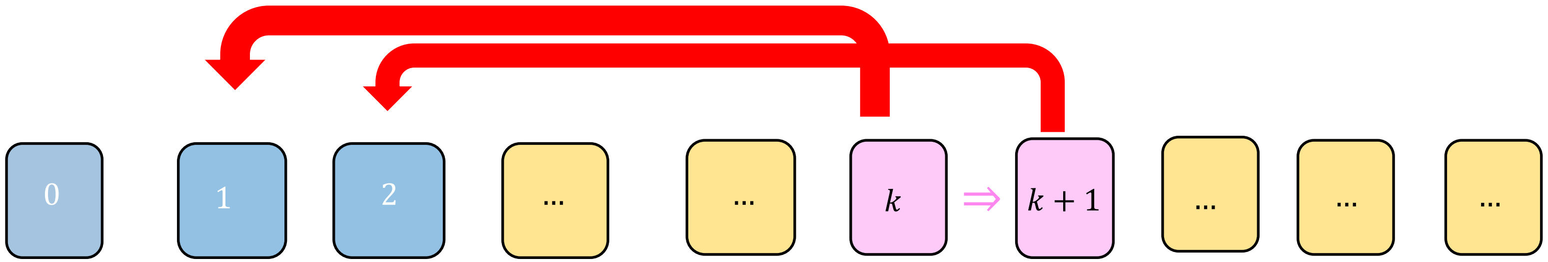

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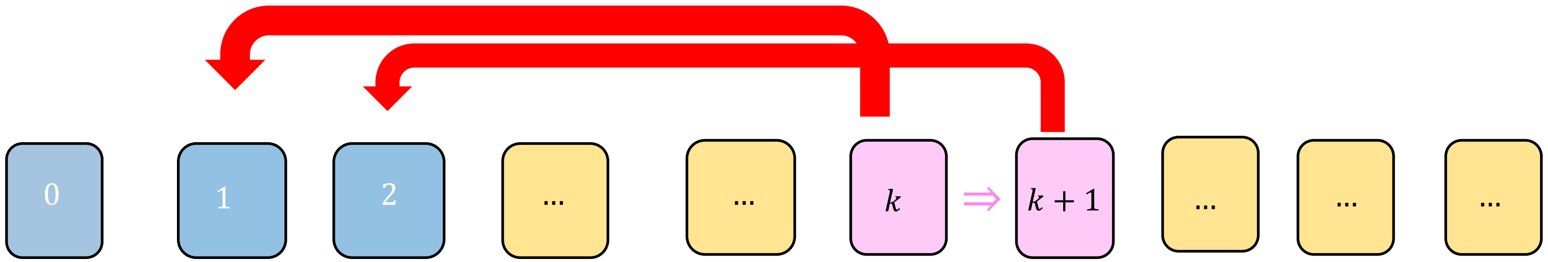

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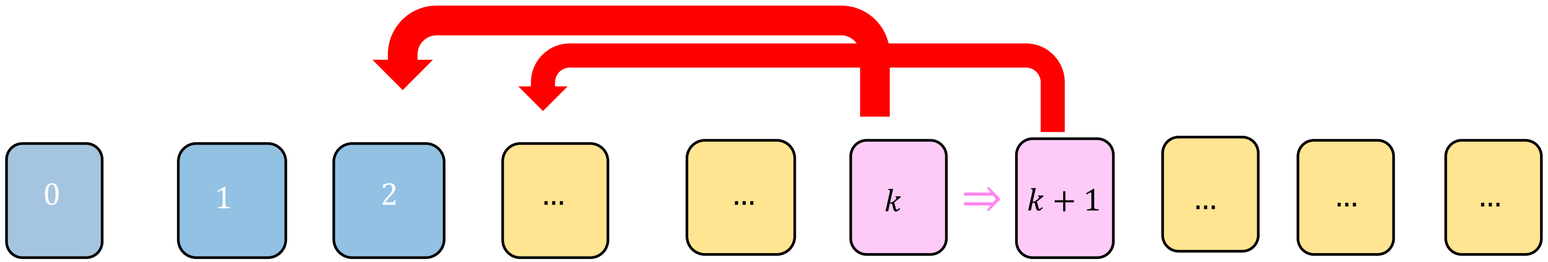

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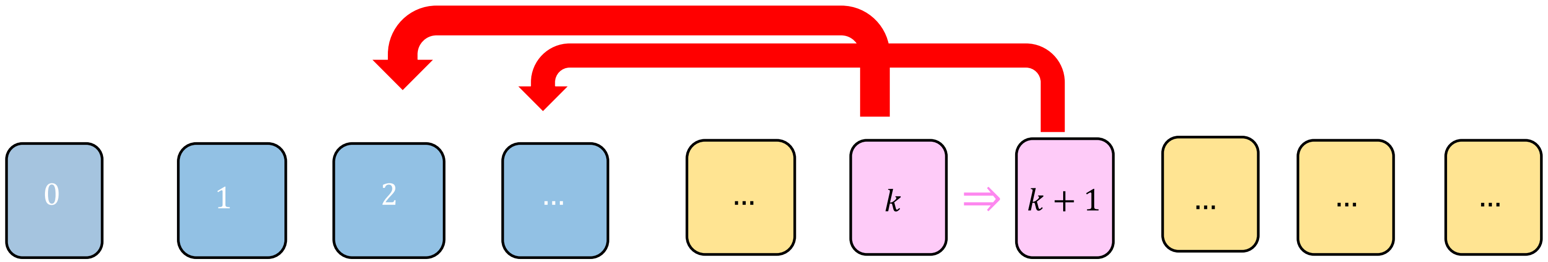

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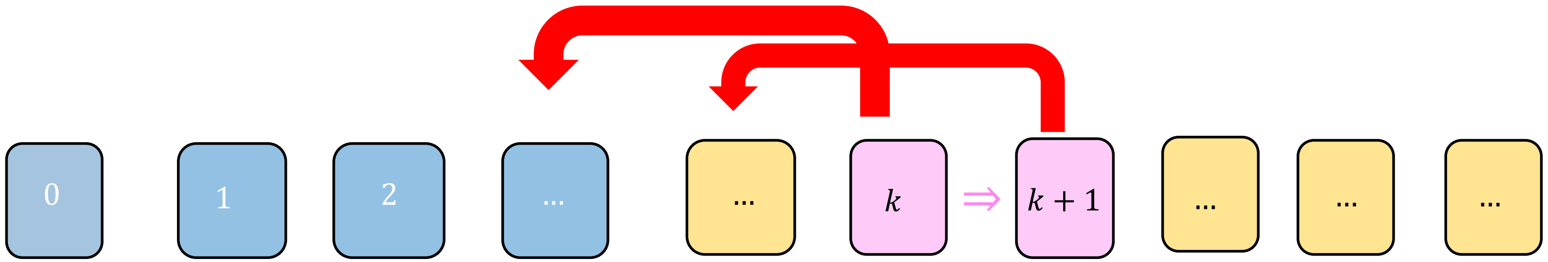

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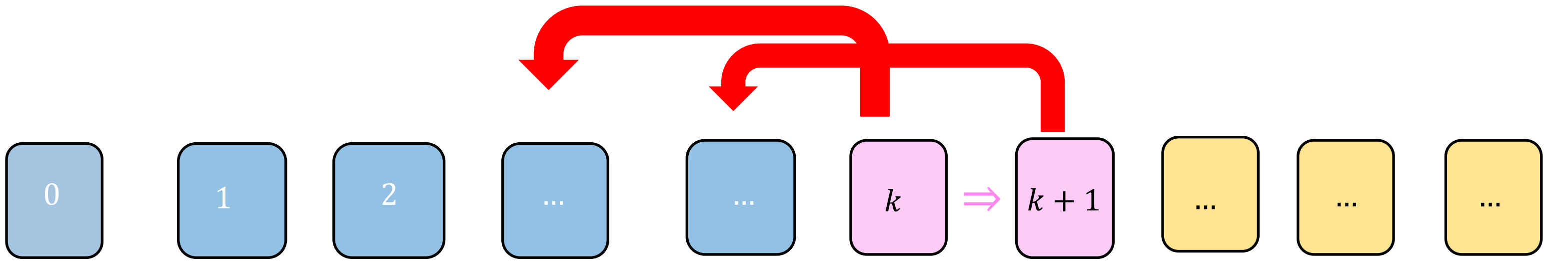

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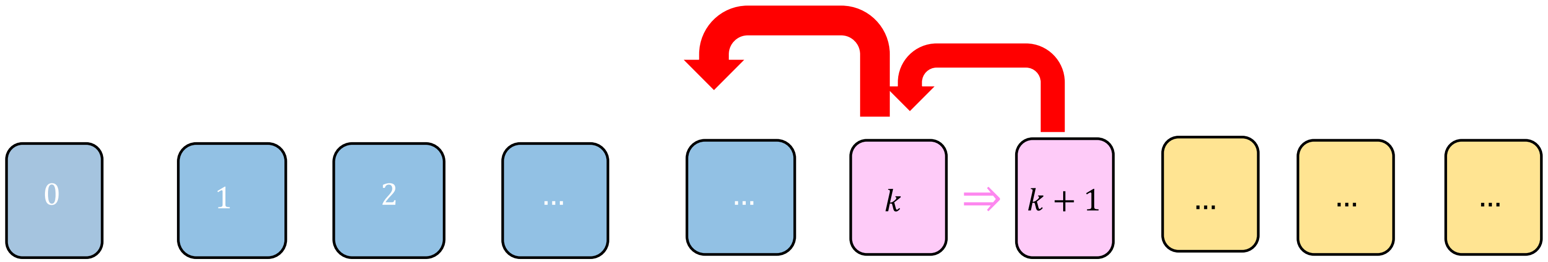

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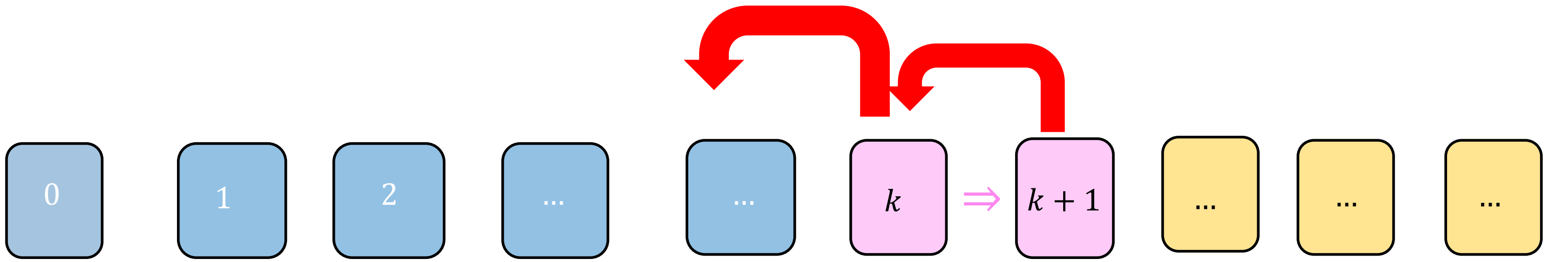

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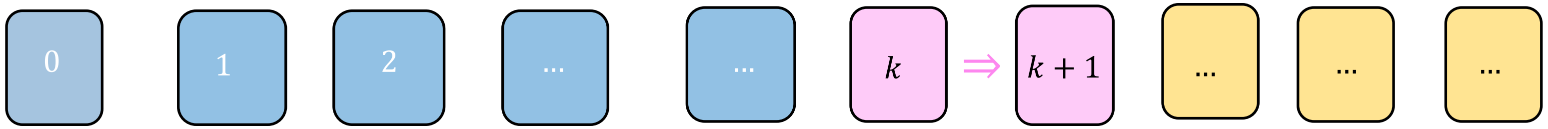

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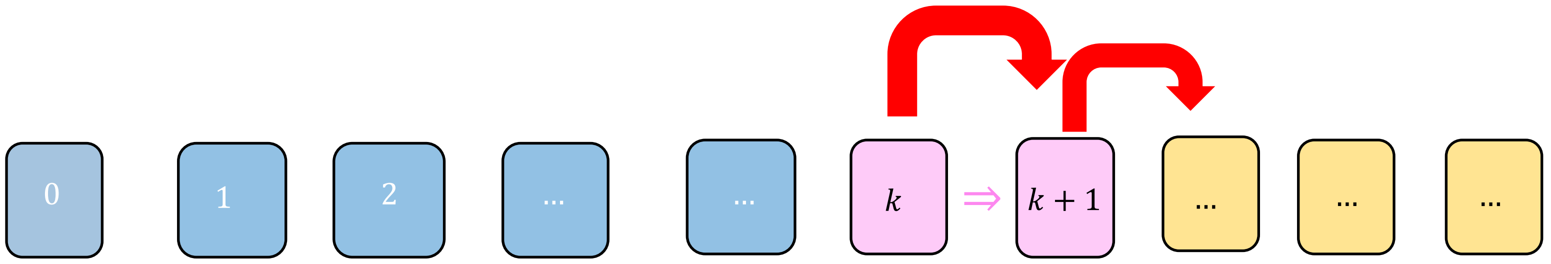

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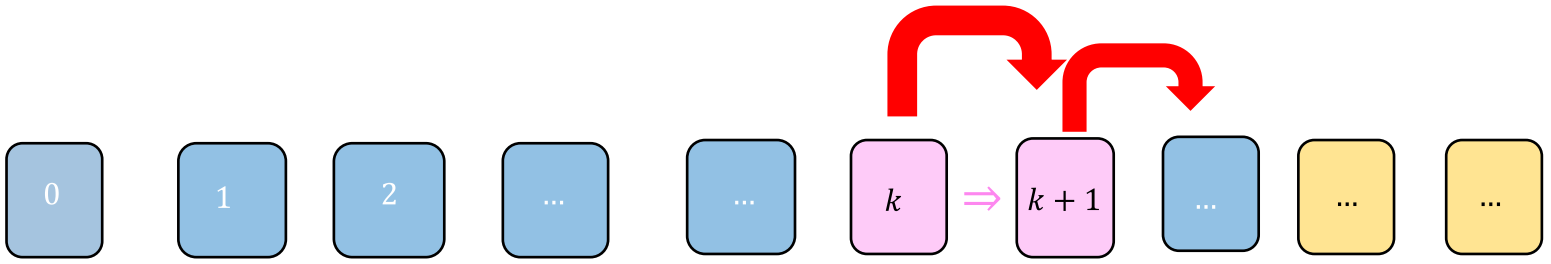

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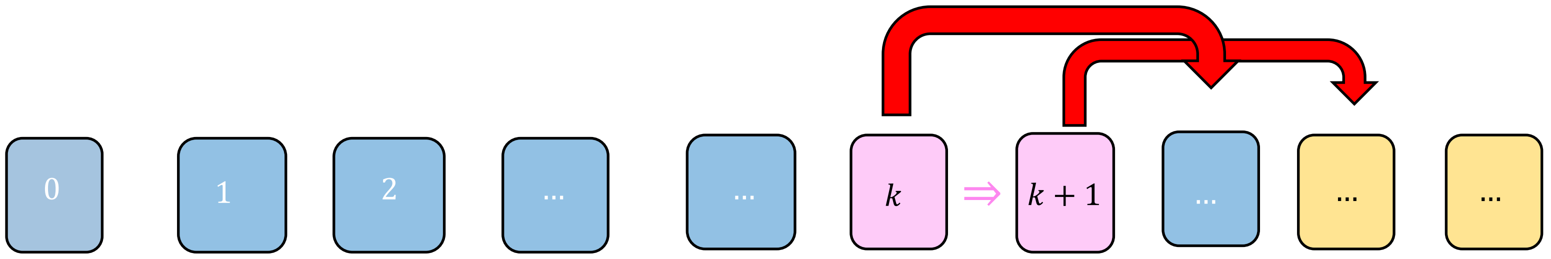

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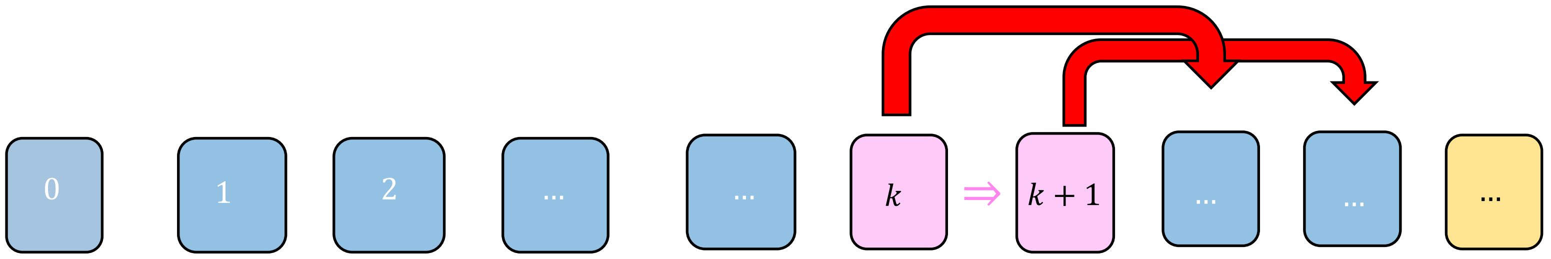

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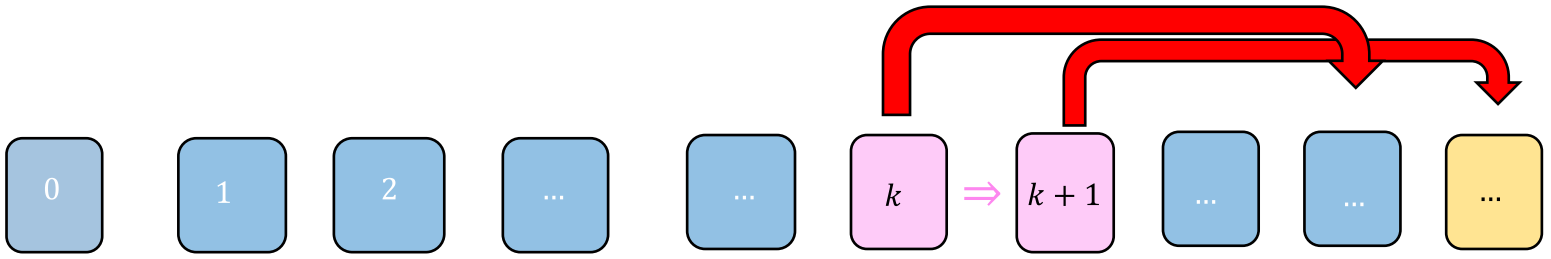

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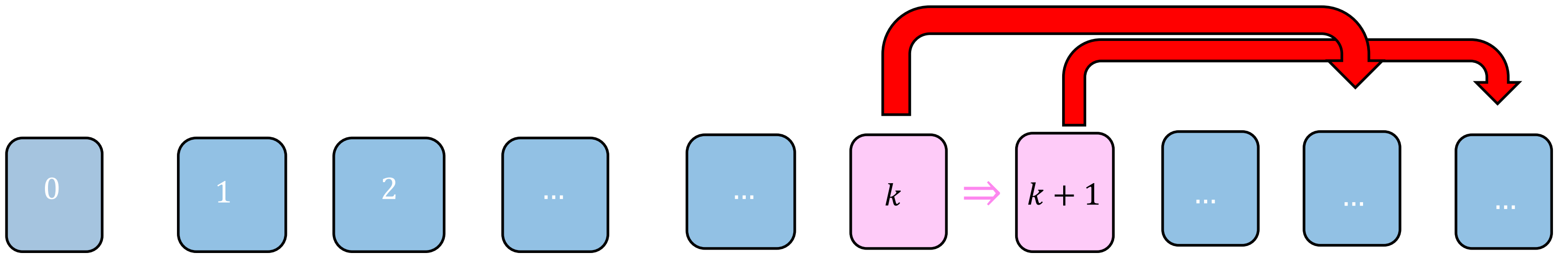

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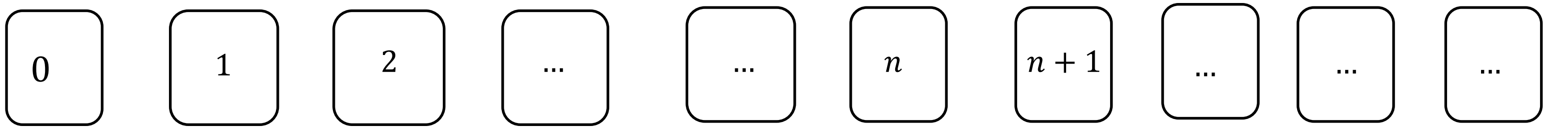

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0	1	2	…	…	n	$n+1$	…	…	…

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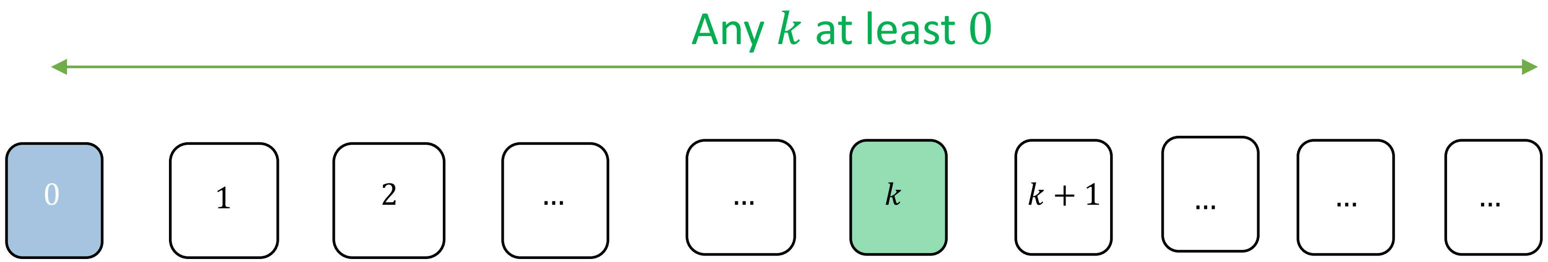

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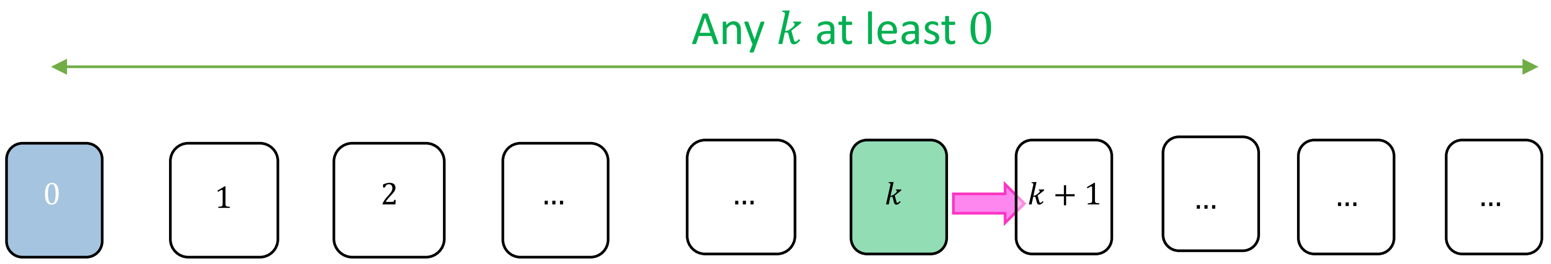

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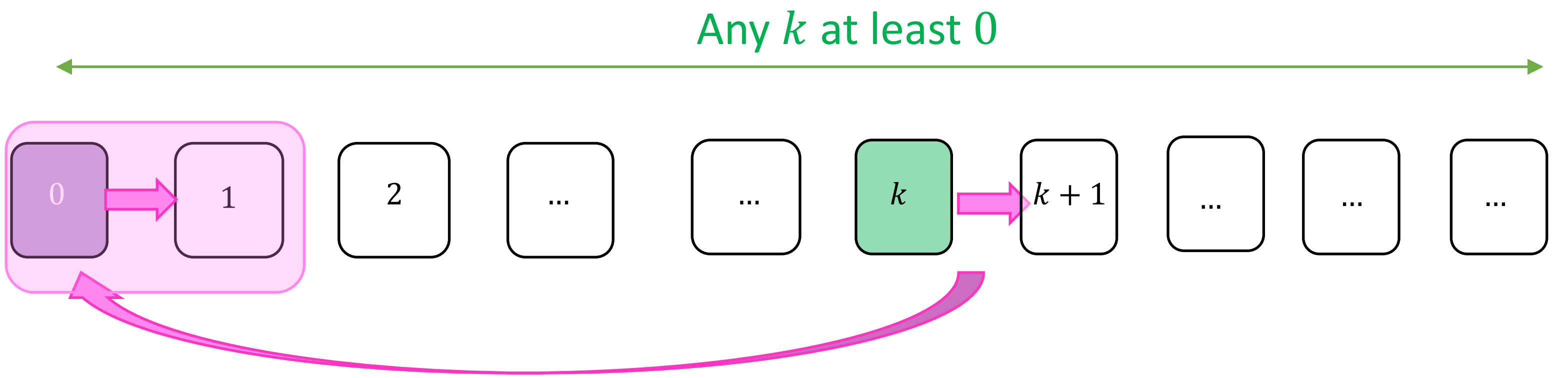

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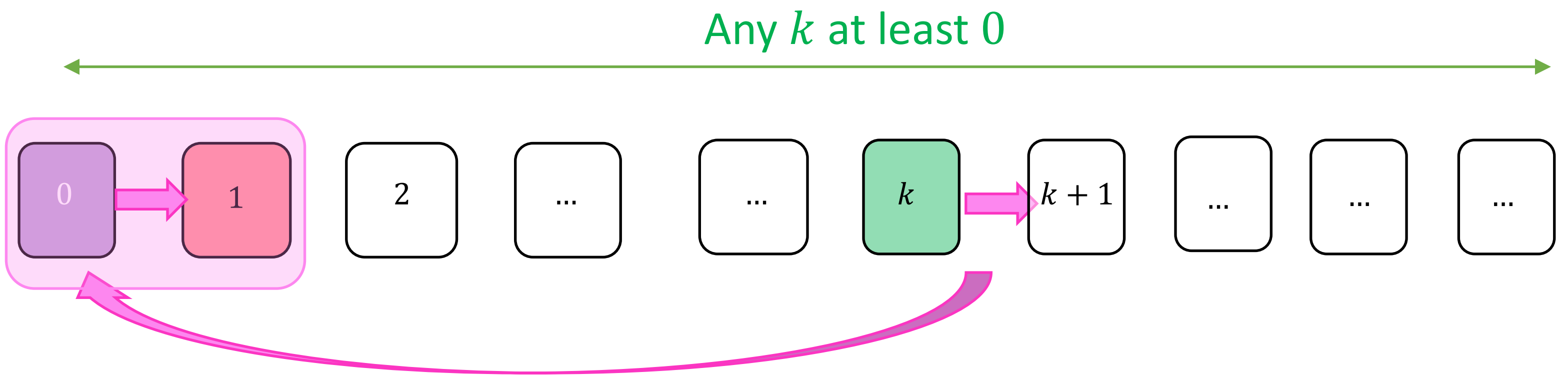

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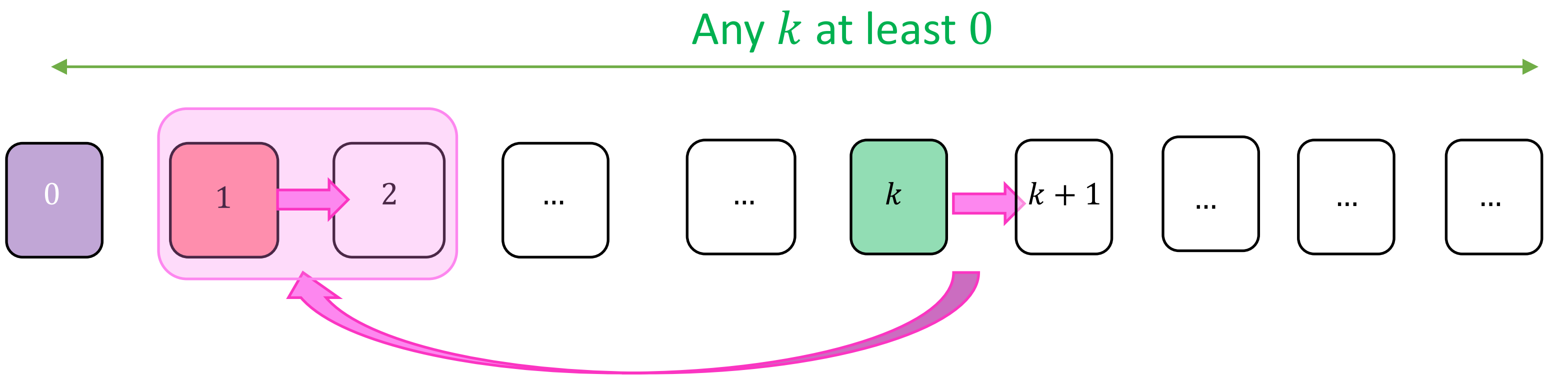

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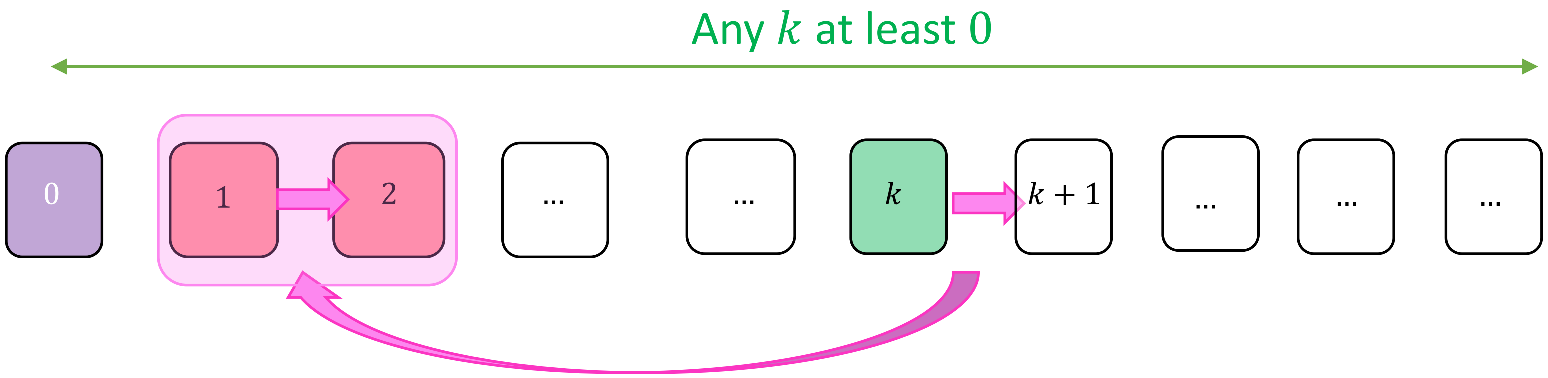

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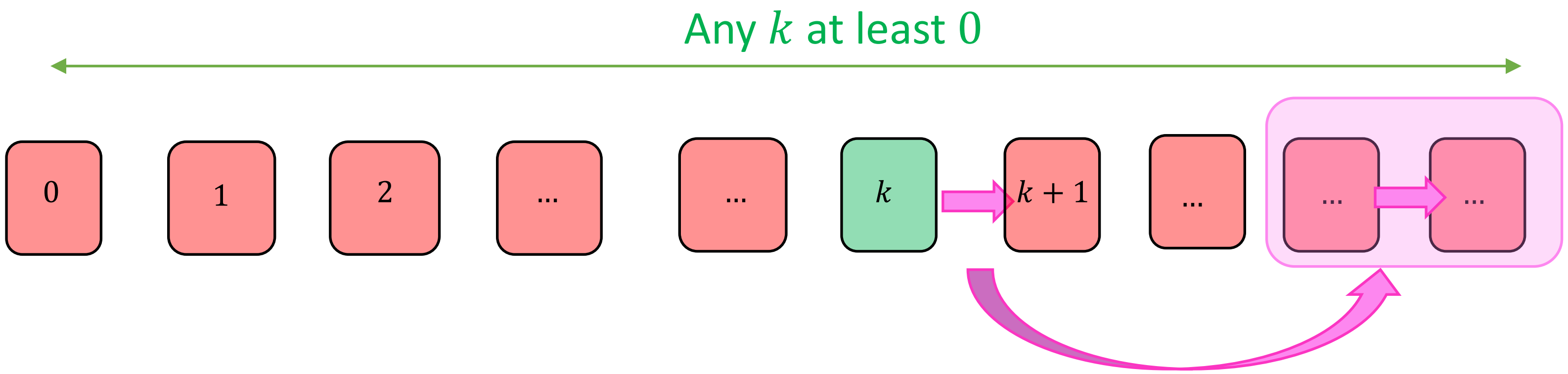

(We fast-forwarded here to save some time.)

An introductory example

- Suppose that we have the sequence a such that:

$$a_n = \begin{cases} 1, & n = 0 \\ 2a_{n-1}, & n \geq 1 \end{cases}$$

- First few terms:

$$1, 2, 4, 8, 16, \ldots$$

- We will prove, via mathematical induction, that for all $n \geq 0$,

$$a_n = 2^n$$

Inductive Base

- For $n = 0$, we will **prove** that $P(0)$ is true, where $P(0)$ is the statement:

$$a_0 = 2^0$$

- This is trivial to prove, since by the base case of the sequence a we have $a_0 = 1 = 2^0$.
- So $P(0)$ is true.

Inductive Hypothesis

- For $n = k \geq 0$, we **assume** that $P(k)$ **is true**:

$$a_k = 2^k$$

Inductive Step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true,where $P(k+1)$ is the statement:

$$a_{k+1} = 2^{k+1}$$

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- Since $k \geq 0$, $k + 1 \geq 1$.

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- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true,where $P(k+1)$ is the statement:

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- Since $k \geq 0$, $k + 1 \geq 1$.
- We can therefore use the **recursive rule** of the sequence’s definition to derive $a_{k+1} = 2 \cdot a_k \; (I)$

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- From our assumption of $P(k)$, we know that $a_k = 2^k$ (II)

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- $(I) \overset{(II)}{\Longrightarrow} a_{k+1} = 2^{k+1}$

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- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true,where $P(k+1)$ is the statement:

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- We can therefore use the **recursive rule** of the sequence's definition to derive $a_{k+1} = 2 \cdot a_k \ (I)$
- From our assumption of $P(k)$, we know that $a_k = 2^k \ (II)$
- $(I) \overset{(II)}{\Longrightarrow} a_{k+1} = 2^{k+1}$
- So $P(k+1)$ is also true and we are done.

Here’s another

- Suppose that we have the sequence s defined as follows:

$$s_n = \begin{cases} 0, & n = 0 \\ s_{n-1} + 10, & n \geq 1 \end{cases}$$

- Using weak induction, prove that $(\forall n \in \mathbb{N})[\, 5 \mid s_n \,]$

Inductive Base

- For $n = 0, s_0 = 0\ (I)$.
- Furthermore, it is the case that $5 \mid 0\ (II)$.
- $(I, II) \Rightarrow 5 \mid s_0 \Rightarrow P(0)\ \ holds$

Inductive Hypothesis

- Suppose that $n = k \geq 0$. We will assume that $P(k)$ holds, i.e:

$$(5 \mid s_k) \Leftrightarrow (\exists r \in \mathbb{Z})[s_k = 5r]$$

Could also use the mod definition!

Inductive Step

- Given $P(k)$, we will now attempt to prove $P(k+1)$, i.e:

$$(5 \mid s_{k+1}) \Leftrightarrow (\exists \ell \in \mathbb{Z})[s_{k+1} = 5\ell]$$

- Since $k \geq 0, k+1 \geq 1$ and we can use the recursive part of the definition of s:

$$s_{k+1} = s_{(k+1)-1} + 10 = s_k + 10 \overset{\text{(By I.H)}}{=} 5 \cdot r + 10 = 5r + 5 * 2 = 5(r+2) = 5\,\ell$$

You do this!

- The sequence b is defined as:

$$b_n = \begin{cases} 1, & n = 0 \\ 4 + b_{n-1}, & n \geq 1 \end{cases}$$

- Prove that for all $n \geq 0$, b_n is odd

The Gaussian Sum

- We will prove that the sum of the first n numbers is equal to $\frac{n(n+1)}{2}$.
- Symbolically:

$$\underbrace{1 + 2 + 3 + \cdots + (n - 1) + n}_{\sum_{i=1}^{n} i} = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Inductive base

- For $n = 0$, we will **prove** that $P(0)$ holds

$$\sum_{i=1}^{0} i \overset{?}{=} \frac{0(0+1)}{2}$$

Remember: $P(n)$ is

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- LHS: $\sum_{i=1}^{0} i = 0$ (recall this fact from our sequences lecture)
- RHS: $\frac{0(0+1)}{2} = 0$
- Since LHS = RHS for $n = 0$, $P(0)$ has been proven true.

Inductive Hypothesis

- For $n = k \geq 0$, we **assume** that $P(k)$ **is true**:

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

So, we **assume** that

$$P(k) \Leftrightarrow \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$$

is true for an arbitrary $k \geq 0$

- Inductive Hypothesis done!

Inductive step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true.

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Inductive step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true.

Just adding 1 to k

Just adding 1 to $k+1$

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

Inductive step

- Given that $P(k)$ is true, we will **prove** that $P(k+1)$ is true.

Just adding 1 to k

Just adding 1 to $k+1$

$$\underbrace{\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \Rightarrow \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}}_{\textbf{This is our goal!}}$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \qquad (1)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \quad (1)$$

- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \qquad (2)$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

- Starting from the LHS of the relation to prove, we have:

$$\sum_{i=1}^{k+1} i = 1 + 2 + \cdots + k + (k+1) = \sum_{i=1}^{k} i + (k+1) \quad (1)$$

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Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$$

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- **From the Inductive Hypothesis**, we have that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} \qquad (2)$$

- Substituting (2) into (1) yields (next slide):

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

$$= RHS$$

Inductive step, contd.

$$\sum_{i=1}^{k+1} i = \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+2)(k+1)}{2}$$

$$= RHS$$

- So, when $P(k)$ is true, $P(k+1)$ was also proven true.
- We conclude that $P(n)$ is true $\forall n \geq 0$. $\square$

And one for you!

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Inductive Base

- For $n = 0$, LHS = $\sum_{i=1}^{0} i^2 = 0$
- RHS=$\frac{0(0+1)(2*0+1)}{2} = 0$
- Since LHS = RHS, $P(0)$ holds and we are done.

Inductive Base

- For $n = 0$, LHS = $\sum_{i=1}^{0} i^2 = 0$
- RHS=$\frac{0(0+1)(2*0+1)}{2} = 0$
- Since LHS = RHS, $P(0)$ holds and we are done.

- You could also start from $n = 1$! LHS = RHS in both cases
  - $n = 0$ sometimes makes the math easier (RHS in this case)

Inductive Hypothesis

- Suppose that $n = k \geq 0$. *(Or 1 in the alternative scenario)*
- We will then assume $P(k)$, i.e:

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive Step

- We will now attempt to prove $P(k+1)$, i.e

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Careful with factoring please!!!

Inductive Step

- We will now attempt to prove $P(k+1)$, i.e

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Careful with factoring please!!!

- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

Inductive Step

- We will now attempt to prove $P(k+1)$, i.e

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Careful with factoring please!!!

- By leveraging associativity of sum, the LHS can be written as follows:

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

We can apply the I.H here!

Inductive Step

- By I.H, we can now write:

$$\sum_{i=1}^{k+1} i^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

- Remember: we **want** this to be equal to

$$\frac{(k+1)(k+2)(2k+3)}{6}$$

- **We will <u>fearlessly</u> manipulate the algebra <u>until it does</u>!**

Inductive Step - Algebra

$$\frac{k(k+1)(2k+1)}{6}+(k+1)^2=\frac{k(k+1)(2k+1)}{6}+\frac{6(k+1)^2}{6}=$$

$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}=\frac{(k+1)[2k^2+7k+6]}{6}$$

- If only we could prove that $2k^2+7k+6=(k+2)(2k+3)$, we'd be done!
- But…. $(k+2)(2k+3)=2k^2+3k+4k+6=2k^2+7k+6$! ☺
- So we're done.

And one with more than 1 variable!

- Prove that the sum of the first n terms of a **geometric sequence** with $m \in (\mathbb{R} - \{1\})$ and $a_0 = 1$ is equal to $\frac{m^n - 1}{m - 1}$.
- Symbolically:

$$\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}$$

And one with more than 1 variable!

- Prove that the sum of the first n terms of a **geometric sequence** with $m \in (\mathbb{R} - \{1\})$ and $a_0 = 1$ is equal to $\frac{m^n - 1}{m - 1}$.
- Symbolically:

$$\sum_{i=0}^{n-1} m^i = \frac{m^n - 1}{m - 1}$$

- In this instance, we have two variables, m and n, and it's spectacularly easy to confuse ourselves about which variable we will be focusing on.
  - So, we will **explicitly** say, at the beginning of our proof, that we will be performing a proof by induction on n.

Proof

- Proof : We attempt to prove $P(n), \ \forall n \in \mathbb{N}$. We proceed via induction on n.
- **Inductive base**: We attempt to prove $P(0)$.

$$P(0): \sum_{i=0}^{0-1} m^i = \frac{m^0 - 1}{m - 1} \Leftrightarrow \sum_{i=0}^{-1} m^i = \frac{m^0 - 1}{m - 1} \Leftrightarrow 0 = 0$$

So $P(0)$ is true.

- **Inductive hypothesis**: Suppose $n = k \geq 0$. We assume $P(k)$, i.e

$$\sum_{i=0}^{k-1} m^i = \frac{m^k - 1}{m - 1}$$

Proof (contd.)

- **Inductive step:** We will attempt to prove $P(k+1)$, i.e

$(k+1)-1$

$$\sum_{i=0}^{k} m^i = \frac{m^{k+1}-1}{m-1}$$

From the LHS to the RHS:

By the I.H

$$LHS = \sum_{i=0}^{k} m^i = \sum_{i=0}^{k-1} m^i + m^k = \frac{m^k-1}{m-1} + m^k = \frac{m-1+m^k(m-1)}{m-1} = \frac{m^{k+1}-1}{m-1} = RHS \ \square$$

A coin problem

- We will prove that every dollar amount ≥ 4 cents can be **exclusively** paid for by 2 and/or 5 cent coins.

Theorem expressed in quantifiers

- All quantifiers implicitly assumed over $\mathbb{N}$.

$$(\forall n \geq 4)(\exists n_1, n_2)[n = 2n_1 + 5n_2]$$

Inductive base

- The least amount of money we are required to prove the statement for is 4¢, so we will attempt to prove $P(4)$.
- For $n = 4$, we have 4¢. Since $4¢ = 2 \times 2¢$, we are done (we have shown that the amount of 4¢ can be exclusively paid for by using only 2 and/or 5 cent coins)

Inductive hypothesis 

- Let $n = k \geq 4$.
- Assume $P(k) \Leftrightarrow (\exists k_1, k_2)[k = 2k_1 + 5k_2]$

Inductive step

- We will prove that $P(k) \Rightarrow P(k+1)$, i.e that we can pay an amount of money equal to $k+1$ cents using **only** 2¢ or 5¢ coins.
- In terms of algebra, what we want to prove is:

$$(\exists k_3, k_4 \in \mathbb{N})\ [k+1 = 2k_3 + 5k_4]$$

Inductive step

- We will prove that $P(k) \Rightarrow P(k+1)$, i.e that we can pay an amount of money equal to $k+1$ cents using **only** 2¢ or 5¢ coins.
- In terms of algebra, what we want to prove is:

$$(\exists k_3, k_4 \in \mathbb{N})\ [k+1 = 2k_3 + 5k_4]$$

Different variables from I.H!

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

1. **Case #1:** $k_1 \geq 2$

- I have at least 2 2¢ coins, so I can take away 2 2¢ coins and add one 5¢ coin

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

$$k = 2k_1 + 5k_2$$

1. **Case #1:** $k_1 \geq 2$

- I have at least 2 2¢ coins, so I can take away two 2¢ coins and add one 5 ¢ coin
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (5 - 2 * 2) =$$
$$= (2k_1 - 4) + (5k_2 + 5) = 2\underbrace{(k_1 - 2)}_{k_3} + 5\underbrace{(k_2 + 1)}_{k_4} = 2k_3 + 5k_4$$

Inductive Step (contd.)

- From the Inductive Hypothesis (I.H), we have that for some specific positive integers k_1 and k_2:

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1. **Case #1:** $k_1 \geq 2$

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- By adding 1 on both sides of the I.H we obtain:

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$$= (2k_1 - 4) + (5k_2 + 5) = 2\underbrace{(k_1 - 2)}_{k_1 - 2 \geq 0 \text{ because } k_1 \geq 2} + 5\underbrace{(k_2 + 1)}_{\text{In } \mathbb{N} \text{ by closure}} = 2k_3 + 5k_4$$

Inductive step

2. Case #2: $k_2 \geq 1$

- I have at least one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 * 2 - 5) =$$

$$= 2\underbrace{(k_1 + 3)}_{k_3} + 5\underbrace{(k_2 - 1)}_{k_4} = 2k_3 + 5k_4$$

Inductive step

2. Case #2: $k_2 \geq 1$

- I have <u>at least</u> one 5¢ coin so I can take away one 5¢ coin and add three 2¢ coins
- By adding 1 on both sides of the I.H we obtain:

$$k + 1 = 2k_1 + 5k_2 + 1 = 2k_1 + 5k_2 + (3 * 2 - 5) =$$

$$= 2\underbrace{(k_1 + 3)}_{\substack{(k_1+3) \in \mathbb{N} \\ \text{by closure}}} + 5\underbrace{(k_2 - 1)}_{\substack{k_2 - 1 \geq 0 \\ \text{because} \\ k_2 \geq 1}} = 2k_3 + 5k_4$$

Inductive step

3. **Case #3:** $(k_1 \leq 1) \wedge (k_2 = 0)$

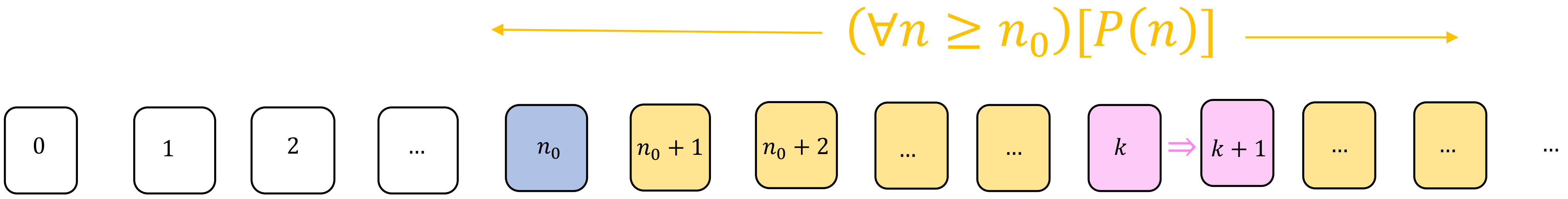
- This case means that we have either 0 or 2¢ at our disposal.
- But this is not possible, since we want to prove the theorem only for values ≥ 4¢
- So we're done. ◻

A note about the penny problem

- Note that we proved the theorem for $n \geq 4$
- Generally speaking, we can use induction to prove statements $P(n)\ \forall n \geq n_0$, where $n_0 \in \mathbb{N}$.
- Most of the time n_0 will be small (0, 1, 2, …)

A note about the penny problem

- Note that we proved the theorem for $n \geq 4$
- Generally speaking, we can use induction to prove statements $P(n)\ \forall n \geq n_0$, where $n_0 \in \mathbb{N}$.
- Most of the time n_0 will be small (0, 1, 2, …)
- If $P(n_0) \wedge (\forall k \geq n_0)[P(k) \Rightarrow P(k+1)]$ is true, then the inductive principle holds and we have the desired statement



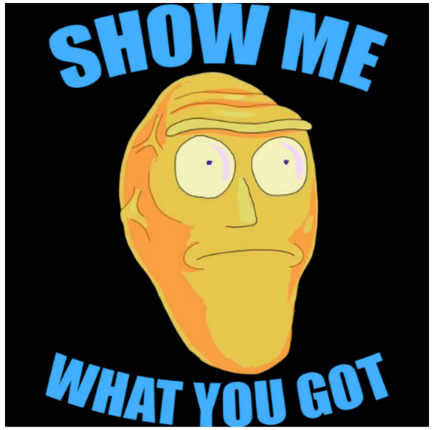
Another!

- Prove that every dollar amount equal to at least 112 cents can be paid for exclusively by 5 and 6 cent coins.

Another!

- Prove that every dollar amount equal to at least 112 cents can be paid for exclusively by 5 and 6 cent coins.
- Let’s do this one together.

A coin problem for **you!**



Prove to me that every dollar amount $\geq$ 20 cents can be **exclusively** paid for through combinations of **5**-cent coins and **6**-cent coins!

Here’s one with an inequality!

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $P(k) \Rightarrow P(k+1)$, i.e

$$(2^k < k!) \Rightarrow (2^{k+1} < (k+1)!)$$

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!\ \text{Done.}$
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Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!\ \mathrm{Done.}$
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   - From the I.H, we have that $2^k < k! \overset{2>0}{\Longleftrightarrow} 2^k \cdot 2 < k! \cdot 2$ (2)

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   - From the I.H, we have that $2^k < k! \overset{2>0}{\Longleftrightarrow} 2^k \cdot 2 < k! \cdot 2$ (2)
   - Since $k \geq 4$, we have that $2 < k + 1 \overset{k!>0}{\Longleftrightarrow} k! \cdot 2 < k!\,(k+1)$ (3)

Inductive Step…

- Prove that for all integers n at least 4, $2^n < n!$

1. **I.B:** We will prove $P(4) \Leftrightarrow 2^4 < 4!$ Done.
2. **I.H:** For $n = k \geq 4$, we assume $P(k)$, i.e $2^k < k!$
3. **I.S:** We will prove $2^{k+1} < (k+1)!$
   - From algebra, we have that $2^{k+1} = 2^k \cdot 2$ (1)
   - From the I.H, we have that $2^k < k! \overset{2>0}{\Longleftrightarrow} 2^k \cdot 2 < k! \cdot 2$ (2)
   - Since $k \geq 4$, we have that $2 < k + 1 \overset{k!>0}{\Longleftrightarrow} k! \cdot 2 < k!\,(k+1)$ (3)
   - $(2) \overset{(3)}{\Rightarrow} 2^k \cdot 2 < (k+1)! \overset{(1)}{\Leftrightarrow} 2^{k+1} < (k+1)!$

An inequality problem for you!

- Using mathematical induction, prove that, for all naturals $n \geq 3$,

$$2n + 1 < 2^n$$