Logical Equivalence

CMSC250

Spring 2020

First short exam tomorrow

- Tomorrow you have your first short exam.
  - You MUST sit for it in your discussion and, generally, attend all discussions in your own discussion!
  - Please read Friday ELMS announcement about how the short exams will be graded, returned to you and how we will adhere to regrade requests.
  - Remember: ***Excused*** absences from minor exams **exist**; makeups **don't.**
  - **NO EXCEPTIONS CAN BE MADE IN SUCH A HUGE CLASS. PLEASE BE AWARE OF THE RULES.**
    - I have asked my TAs to be uploading solutions to those exams Wednesdays at 5:01pm.

Section woes

- Just added the class? Need to switch a section?
- Please contact **your academic advisor.**
  - Remember: because of the merging of sections, some of the sections have few people registered in them yet they still appear closed.
  - Otherwise we would be overflowing the capacity of our CSIC rooms.
  - Refer to our Google workbook for the exact merging of sections.

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  - Otherwise we would be overflowing the capacity of our CSIC rooms.
  - Refer to our Google workbook for the exact merging of sections.
- Exception: If you were registered in the (cancelled) 0308 and need to switch a section, Ms Apitchaya Pimpawathin (AP) can help you (IRB 1132)

Section woes

- Which section / discussion session you are in doesn't matter for lectures.
  - You can attend every lecture you want, as long as you find a seat.
    - 010x and 020x are very crowded, 030x are not.

HW01

- First homework is ready on ELMS.
- Due Monday Feb 10 for full credit, Wednesday Feb 12 for half credit.
- Submission on Gradescope. Re-synced before lecture today. Check if you’re in with your ELMS e-mail.
- You can also pick a hardcopy outside IRB2206
  - Header row for those reads “010x-020x” (typo that stayed from last semester) but homework is good for 030x too.

Equivalences

- Let's observe the following truth table

p	q	$p \wedge q$	$q \wedge p$
F	***F***	***F***	***F***
F	***T***	***F***	***F***
T	***F***	***F***	***F***
T	***T***	***T***	***T***

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- **Conclusion:** $\boldsymbol{p \wedge q \equiv q \wedge p}$

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- **Conclusion: $p \wedge q \equiv q \wedge p$**

This symbol means
"logical equivalence"

An important equivalence

- Please fill – in the following truth table:

p	q	$p \Rightarrow q$	$(\sim p) \vee q$
F	***F***	***?***	***?***
F	***T***	***?***	***?***
T	***F***	***?***	***?***
T	***T***	***?***	***?***

$\Leftrightarrow$ vs $\equiv$

- $\Leftrightarrow$ *("if and only if")* is used to **form statements**, e.g
  - $p \Leftrightarrow (q \wedge (\sim r))$
- $\equiv$ *("logically equivalent to")* **compares two statements**, e.g
  - $(p \wedge q) \equiv (q \wedge p)$

Another important equivalence

- Let's fill in the following truth table :

$\boldsymbol{a}$	$\boldsymbol{b}$	$\sim(\boldsymbol{a} \wedge \boldsymbol{b})$	$(\sim \boldsymbol{a}) \vee (\sim \boldsymbol{b})$
F	***F***	**?**	**?**
F	***T***	**?**	**?**
T	***F***	**?**	**?**
T	***T***	**?**	**?**

Another important equivalence

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F	***T***	*T*	*T*
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- **These columns are the same!**
- **Conclusion:** $\sim(a \wedge b) \equiv (\sim a) \vee (\sim b)$

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This result is known as
De Morgan’s law

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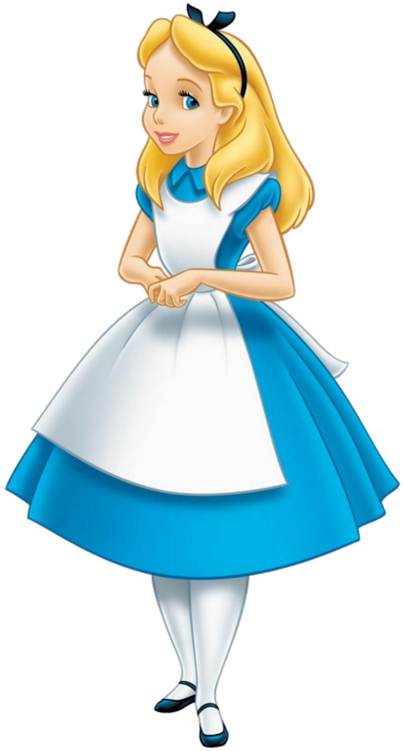
- **Conclusion:** $\sim(a \wedge b) \equiv (\sim a) \vee (\sim b)$

Understanding De Morgan's Law

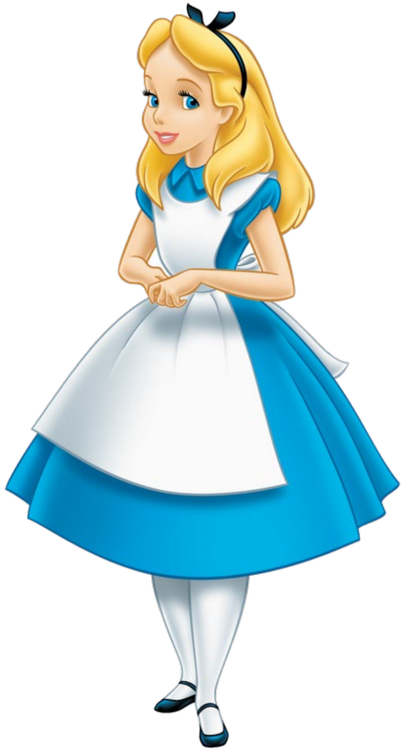
- $\sim$("*Alice is Blonde*" $\wedge$ "*Alice wears Green Dress*"): **Clearly true**



Understanding De Morgan's Law

- $\sim(\text{“}Alice\ is\ Blonde\text{”} \wedge \text{“}Alice\ wears\ Green\ Dress\text{”})$: **Clearly true**

- $(\sim\text{“}Alice\ is\ Blonde\text{”}) \vee (\sim\text{“}Alice\ wears\ Green\ Dress\text{”})$: **Also true!**



De Morgan’s Laws (there’s two of them)

$$\sim (a \vee b) \equiv (\sim a) \wedge (\sim b)$$

$$\sim (a \wedge b) \equiv (\sim a) \vee (\sim b)$$

- Conjunctions flipped to disjunctions, and vice versa
- Negation operator (~) distributed across terms
- These laws give us our first pair of equivalent expressions!

Are these correct equivalences?

$a \Rightarrow b \equiv (\sim b) \Rightarrow (\sim a)$	$a \Rightarrow b \equiv (\sim a) \Rightarrow (\sim b)$	$a \Leftrightarrow b \equiv \big((\sim a) \vee b\big) \wedge ((\sim b) \vee a)$

Left column

Middle column

Right column

Proving equivalences

- How do we prove an equivalence? (e.g $\sim(a \wedge b) \equiv (\sim a) \vee (\sim b))$

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   - Can we do better?

Proving equivalences

- How do we prove an equivalence? (e.g $\sim(a \land b) \equiv (\sim a) \lor (\sim b)$)

1. Truth tables
   - One major problem: for n variables, 2^n rows (input combinations) to enumerate!
   - Can we do better?
2. Laws of logical equivalence in a chain, one after the other!
   - We no longer have to compare 2^n input combinations to ensure that they all map to the same truth value (**T** or **F**). ☺
   - But somebody needs to code the system up!

Table of equivalences

Commutativity of binary operators	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associativity of binary operators	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributivity of binary operators	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity laws	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation laws	$p \vee ({\sim}p) \equiv T$	$p \wedge ({\sim}p) \equiv F$
Double negation	${\sim}({\sim}p) \equiv p$	
Idempotence	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's axioms	${\sim}(p \wedge q) \equiv ({\sim}p) \vee ({\sim}q)$	${\sim}(p \vee q) \equiv ({\sim}p) \wedge ({\sim}q)$
Universal bound laws	$p \vee T \equiv T$	$p \wedge F \equiv F$
Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negations of contradictions / tautologies	${\sim}F \equiv T$	${\sim}T \equiv F$
Contrapositive	$(a \Rightarrow b) \equiv (({\sim}b) \Rightarrow ({\sim}a))$	
Equivalence between biconditional and implication	$a \Leftrightarrow b \equiv (a \Rightarrow b) \wedge (b \Rightarrow a)$	
Equivalence between implication and disjunction	$a \Rightarrow b \equiv {\sim}a \vee b$	

- This exact table will be **given to you** during **all** exams where you might need it, **including the first one tomorrow, Wed Feb 05**. This way you can refer to the various axioms by name without remembering their "exotic" names.

Proving equivalences using laws

- Suppose we want to investigate if

$$(((a \wedge b) \vee q) \wedge (b \wedge a)) \equiv (p \vee \sim p) \wedge ((a \wedge b) \vee ((\sim r) \wedge r))$$

- How many rows would the truth table have?

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  - $2^5 = 32$ ☹ Too much time!

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- How many rows would the truth table have?
  - $2^5 = 32$ ☹ Too much time!
- Let’s see how we could use the laws of logical equivalence to prove this equivalence (doc camera)
  - Important: **document the laws!**

More equivalences

- Let's prove the following equivalences **true** or **false** together.

$$a \Rightarrow b \equiv (\sim b) \Rightarrow (\sim a) \quad \textit{(Contrapositive)}$$

$$a \Rightarrow b \equiv (\sim a) \Rightarrow (\sim b) \quad \textit{(Inverse Error)}$$

$$a \Leftrightarrow b \equiv \big((\sim a) \vee b\big) \wedge ((\sim b) \vee a)$$

Simplifying expressions

- Large expressions can often be **simplified** using the equivalences we discussed earlier.
- Example: Let's simplify $p \land (p \lor q) \land (p \land q)$

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Here's one way
$p \wedge (p \vee q) \wedge (p \wedge q)$ *(Original expression)* $\equiv p \wedge (p \wedge q)$ *(How?)* $\equiv (p \wedge p) \wedge q$ *(How?)* $\equiv p \wedge q$ *(How?)*

Your turn, class!

- Let's simplify the following three expressions.

$(a_1 \vee a_1) \wedge (a_2 \vee a_2) \wedge \cdots \wedge (a_{100} \vee a_{100}) \wedge (\sim a_1 \vee \sim a_1) \wedge (\sim a_2 \vee \sim a_2) \wedge \cdots \wedge (\sim a_{100} \vee \sim a_{100})$	$(p \wedge r) \vee ((p \vee s) \wedge (p \vee a))$	$p \wedge ((p \vee \sim q) \vee (\sim (\sim (z \vee \sim q))))$
Left column	Middle column	Right column

Jason needs to project the table with the equivalences while you solve this exercise. If he doesn't, berate him appropriately.

Solution to 1

$$(a_1 \vee a_1) \wedge (a_2 \vee a_2) \wedge \cdots \wedge (a_{100} \vee a_{100}) \wedge (\sim a_1 \vee \sim a_1) \wedge (\sim a_2 \vee \sim a_2) \wedge \cdots \wedge (\sim a_{100} \vee \sim a_{100})$$

$\equiv a_1 \wedge a_2 \wedge \cdots \wedge (a_{100}) \wedge (\sim a_1) \wedge (\sim a_2) \wedge \cdots \wedge (\sim a_{100})$ *(Idempotence 100 times)*

$\equiv a_1 \wedge (\sim a_1) \wedge \ a_2 \wedge (\sim a_2) \wedge \cdots \wedge (a_{999}) \wedge (\sim a_{999}) \ldots \wedge (a_{100}) \wedge (\sim a_{100})$ *(Commutativity 100 times)*

$\equiv F \wedge F \wedge \cdots \wedge F \ldots \wedge F$ *(Negation 100 times)*

$\equiv F$ *(Idempotence 99 times)*

Solution to 2

$$(p \wedge r) \vee \big((p \vee s) \wedge (p \vee a)\big)$$

$$\equiv (p \wedge r) \vee (p \vee (s \wedge a)) \qquad \textit{(Distributivity)}$$

$$\equiv ((p \wedge r) \vee p) \vee (s \wedge a) \qquad \textit{(Associativity)}$$

$$\equiv (p \vee (p \wedge r)) \vee (s \wedge a) \qquad \textit{(Commutativity)}$$

$$\equiv p \vee (s \wedge a) \qquad \textit{(Absorption)}$$

Solution to 3

$$p \wedge \left((p \vee \sim q) \vee (\sim (\sim (z \vee \sim q))) \right)$$

$$\equiv p \wedge \left((p \vee \sim q) \vee (z \vee \sim q) \right)$$ *(Double Negation)*

$$\equiv p \wedge \left((p \vee z) \vee (\sim q \vee \sim q) \right)$$ *(Associativity)*

$$\equiv p \wedge \left((p \vee z) \vee \sim q \right)$$ *(Idempotence)*

$$\equiv p \wedge (p \vee (z \vee \sim q))$$ *(Associativity)*

$$\equiv p$$ *(Absorption)*