Elec4621:
Advanced Digital Signal Processing
Chapter 1: Foundations of Digital Signal Processing

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1 Introduction

For some, the material presented in these notes may seem excessively mathematical: Why do we need to go back to thinking of linear spaces, inner products, etc.? What have matrices got to do with signal processing? Who cares about orthogonality anyway?

For others, however, this presentation may provide insights which link previously disconnected ideas. My purpose is to show you both the simple and the deep. It may be a relief to some of you to see that discrete convolution is such a mind-bogglingly simple concept, that linear operations on sequences of samples are directly analogous to matrix operations on finite length vectors, and that there is nothing mysterious at all about Parseval's theorem. On the other hand, I would like you to try to grapple with the significance of aliasing, the interpretation of frequency in both the discrete and continuous domains and the extent to which discrete Fourier transforms are true frequency transforms. Some of these questions will be answered more fully as the subject progresses, especially when we come to consider random processes, auto-correlation and power spectral densities.

Lecture notes will be handed out regularly in lectures, but the lectures themselves will sometimes follow the material in a different order to that in the notes. This is because the methodical style appropriate to a written presentation is not usually helpful in lectures. For this reason, you are recommended to take your own notes during lectures to remind yourself of the flow of ideas. There is very little point in trying to write everything down. Instead, try to focus on the flow of the lecture, sketching connections, "mind-maps," etc., which emerge as you go. This way, you will have a chance of taking away useful concepts, which will last even after you forget the details.

2 Linear Spaces

2.1 Definitions

A linear space is a set, S, whose elements we identify as vectors, v, having the following properties:

1. Closure under addition: if $v_1, v_2 \in S$, then $v_1 + v_2 \in S$
2. Closure under scalar multiplication: if $v \in S$ and $a \in \mathbb{R}$, then $av \in S$

An inner product space is a vector space endowed with an inner product, $\langle v, w \rangle \in \mathbb{R}$, satisfying

1. Symmetry: $\langle v, w \rangle = \langle w, v \rangle$. Note, however, that if the linear space is defined over the field of complex numbers, $\mathbb{C}$, instead of $\mathbb{R}$, then we have $\langle v, w \rangle = \langle w, v \rangle^*$, where α^* denotes the complex conjugate of α.
2. Linearity: $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ and $\langle v+u, w \rangle = \langle v, w \rangle + \langle u, w \rangle$. By symmetry, we must also have $\langle v, \alpha w \rangle = \alpha \langle v, w \rangle$ and $\langle v, w+u \rangle = \langle v, w \rangle + \langle v, u \rangle$, unless the vector space is defined over $\mathbb{C}$, in which case $\langle v, \alpha w \rangle = \alpha^* \langle v, w \rangle$.
3. Cauchy-Schwarz inequality: $\langle v, w \rangle \leq \sqrt{\langle v, v \rangle \cdot \langle w, w \rangle}$ with equality if and only if $v = \alpha w$ for some $\alpha \in \mathbb{R}$.

Some definitions and consequences:

Norms: We write $\|v\| = \sqrt{\langle v, v \rangle}$ for the norm of vector, v.

- It follows that $\|v\| \geq 0$, with equality if and only if $v = 0$.
- The Cauchy-Schwarz inequality may written $\langle v, w \rangle \leq \|v\| \cdot \|w\|$

Triangle inequality: $\|v + w\| \leq \|v\| + \|w\|$

Orthogonal vectors: We say that v and w are orthogonal if $\langle v, w \rangle = 0$.

Orthonormal vectors: We say that v and w are orthonormal if $\langle v, w \rangle = 0$ and $\|v\| = \|w\| = 1$.

Orthonormal expansions: Let $\{\psi_i\}$ be an orthonormal set of vectors, i.e. ψ_i, ψ_j are orthonormal for any $i \neq j$. Moreover, suppose that vector v can be expressed as a linear combination of these orthonormal vectors, i.e. $v = \sum_i a_i \psi_i$. Then $a_i = \langle v, \psi_i \rangle$ and $\|v\|^2 = \langle v, v \rangle = \sum_i |a_i|^2$.

Linear independence: We say that a collection of vectors, ψ_i, are linearly independent if the only combination of scale factors, $a_i \in \mathbb{R}$, such that $\sum_i a_i \psi_i = 0$, is $a_i = 0, \forall i$.

- It follows that any collection of non-zero, mutually orthogonal vectors must be linearly independent.

Basis: We say that the collection of vectors, $\{\psi_i\}$, forms a basis for S if the vectors are linearly independent and every vector, $v \in S$, may be written as a linear combination of the basis vectors, ψ_i, i.e. $v = \sum_i a_i \psi_i$ for some $a_i \in \mathbb{R}$. The number of elements in the basis, $\{\psi_i\}$, is the dimension of the vector space. If the basis consists of infinitely many elements then we have an infinite dimensional vector space.

- With respect to any particular basis, vectors in S may be equivalently expressed in terms of the scaling factors, a_i.

2.2 Important Examples

2.2.1 Finite Sequences (column/row vectors)

Consider the set of all column vectors (n-tuples),

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

where $v_i \in \mathbb{R}$. In this case, we frequently use boldface, $\mathbf{v}$, to denote a vector, and v_i to denote its elements.

The set of all such n-tuples forms an n-dimensional vector space where vector addition and scalar multiplication are performed element-wise. We define inner products to be the familiar dot-product, i.e.

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i = \mathbf{w}^t \mathbf{v} = \begin{pmatrix} w_1 & w_2 & \cdots & w_n \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

If we define the vector space over $\mathbb{C}$ instead of $\mathbb{R}$, we must define the inner product by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{i=1}^{n} v_i w_i^* = \mathbf{w}^* \mathbf{v} = \begin{pmatrix} w_1^* & w_2^* & \cdots & w_n^* \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Note that $\mathbf{w}^*$ denotes the conjugate transpose of vector $\mathbf{w}$. A trivial basis for the linear space of n-tuples is

$$\boldsymbol{\psi}_i = \underbrace{\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}}_{i}{}^t$$

which is clearly orthonormal. The orthonormal expansion of any vector, $\mathbf{v}$, relative to this basis may be written:

$$\mathbf{v} = \sum_i a_i \boldsymbol{\psi}_i, \quad \text{where} \quad a_i = \langle \mathbf{v}, \boldsymbol{\psi}_i \rangle = v_i$$

That is, the coefficients, a_i, are simply the sample values, v_i.

2.2.2 Infinite Sequences (discrete signals)

This is a natural extension from finite-dimensional vectors discussed before. Consider the set of sequences,

$$\mathbf{v} = (\ldots, v_0, v_1, v_2, v_3, \ldots)$$

where $v_i \in \mathbb{R}$.

The set of all such sequences forms a vector space where again vector addition and scalar multiplication are performed element-wise. If we restrict ourselves to the set of all square-summable sequences, we can define the inner product by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_i v_i w_i$$

The sequence $\mathbf{v}$ is said to be "square-summable" if

$$\sum_i |v_i|^2 < \infty$$

If we define the vector space over $\mathbb{C}$ instead of $\mathbb{R}$, we define the inner product instead by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_i v_i w_i^*$$

Again, an obvious orthonormal basis for the space is the set of singleton sequences,

$$\boldsymbol{\delta}_i = \underbrace{\begin{pmatrix} \cdots & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}}_{i} \tag{1}$$

2.2.3 Functions (continuous signals)

Consider the set of functions, $\mathbf{f} = f(x)$. These form a vector space if we define addition and scalar multiplication of functions point-wise, i.e.

$$\begin{aligned} \mathbf{f} + \mathbf{g} &= \mathbf{h} \Longleftrightarrow h(x) = f(x) + g(x), \, \forall x \\ \alpha \mathbf{f} &= \mathbf{h} \Longleftrightarrow h(x) = \alpha f(x), \, \forall x \end{aligned}$$

If we restrict our attention to the set of all square-integrable functions, we can define the inner product by

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int f(x) g(x) \cdot dx$$

If we define the vector space over $\mathbb{C}$ instead of $\mathbb{R}$, we must use the following more general inner product expression instead:

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int f(x) g^*(x) \cdot dx$$

Thus, the norm of a signal, $\|\mathbf{f}\| = \sqrt{\langle \mathbf{f}, \mathbf{f} \rangle}$, may be interpreted as the square root of its energy.

3 Linear Systems

A linear system is a function, H, which maps vectors $\mathbf{v} \in S$ to vectors, $H(\mathbf{v}) = \mathbf{v}' \in S'$, having the linearity property:

$$H(\mathbf{v} + \alpha \mathbf{w}) = H(\mathbf{v}) + \alpha H(\mathbf{w}), \quad \forall \mathbf{v}, \mathbf{w} \in S_v, \alpha \in \mathbb{R}$$

Often $S = S'$, but there are many useful examples where this is not so. Consider, for example, the unit sampling operator which maps continuous signals $\mathbf{v}$ into sequences $\mathbf{v}'$, such that

$$u_i = v(i)$$

The dual of sampling is interpolation, which maps sequences to continuous signals. In both cases, S and S' are different linear spaces. The Discrete Time Fourier Transform (DTFT) is another example, mapping discrete discrete sequences to functions on the interval $\omega \in (-\pi, \pi)$. We shall consider each of these in detail shortly.

3.1 Finite-Dimensional Linear Systems

If S and S' are finite-dimensional vector spaces with dimension n and n', respectively, then the linear system may be represented using a matrix,

$$\mathbf{v}' = \begin{pmatrix} v'_1 \\ v'_2 \\ \vdots \\ v'_n \end{pmatrix} = H(\mathbf{v}) = \mathbf{H}^* \cdot \mathbf{v} = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,n'} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,n'} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n'} \end{pmatrix}^* \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

It is worth noting that each element of the ouput vector may be written as an inner product:

$$v'_i = \mathbf{h}_i^* \cdot \mathbf{v} = \langle \mathbf{v}, \mathbf{h}_i \rangle$$

where $\mathbf{h}_i$ is the i'th column of $\mathbf{H}$, and the super-script, *, denotes transposition and complex conjugation together. When complex numbers are involved, the transpose operation is almost always accompanied by complex conjugation in virtually every useful mathematical relationship.

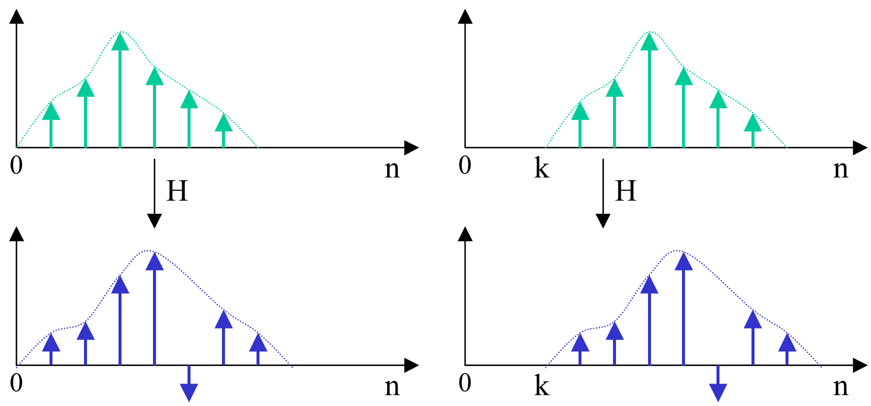


Figure 1: Behaviour of a discrete-time linear time invariant operator.

The output vector may be expressed as a linear combination of n "system responses":

$$\mathbf{v}' = \sum_{i=1}^{n} v_i \cdot \mathbf{h}_i$$

where $\mathbf{h}_i$, the i'th column of $\mathbf{H}$, is the response of the system to an input vector which is zero everywhere except in the i'th element, where it holds a 1.

Thus, the columns of the matrix operator may be interpreted as its system responses, while the rows may be understood as combinatorial weights. The operator itself may be interpreted and implemented either by summing the weighted system responses, or by forming weighted sums of the input samples.

3.2 Discrete-Time LTI Systems

Let $\mathbf{x} \equiv x[n]$ be a discrete-time signal (i.e. a sequence, or a function on $\mathbb{Z}$) and H a linear system with $H(\mathbf{x}) = \mathbf{y}$.

We write $\mathbf{x}_k \equiv x[n-k]$ for the signal obtained by delaying $\mathbf{x}$ by k time steps. H is LTI if for all input signals $\mathbf{x}$, we have

$$H(\mathbf{x}_k) = H(\mathbf{x})_k = \mathbf{y}_k$$

That is, the response of the system to a delayed input signal is a correspondingly delayed version of the response to the original signal. This is illustrated in Figure 1.

Recall that any sequence $\mathbf{x}$ may be written as a linear combination of the orthonormal basis sequences $\boldsymbol{\delta}_i$ defined in equation (1), i.e.

$$\mathbf{x} = \sum_i a_i \boldsymbol{\delta}_i = \sum_i \langle \mathbf{x}, \boldsymbol{\delta}_i \rangle \boldsymbol{\delta}_i = \sum_i x[i] \boldsymbol{\delta}_i$$

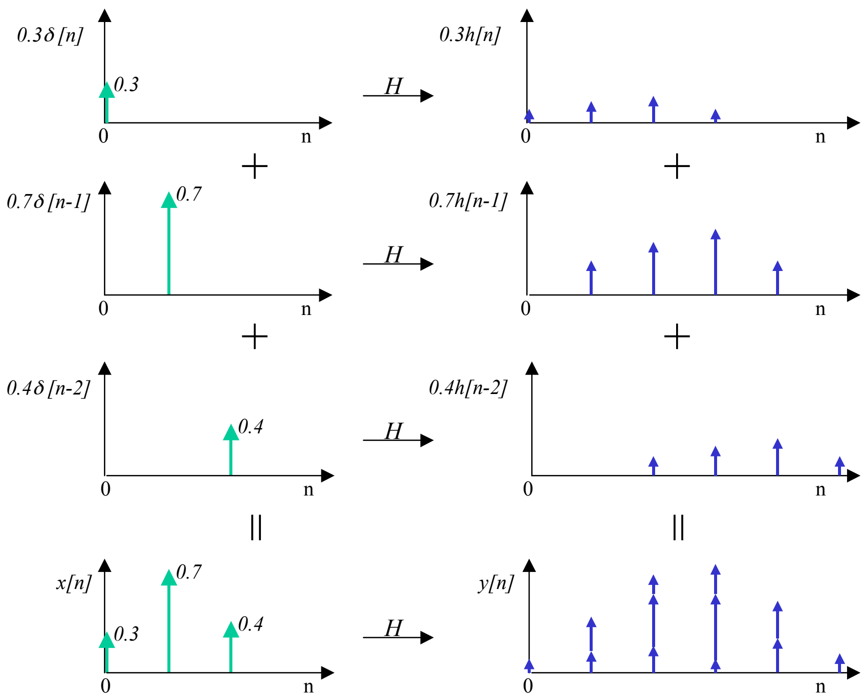


Figure 2: Convolution at work.

Since H is a linear operator we have

$$\mathbf{y} = H(\mathbf{x}) = H\left(\sum_n x[n]\boldsymbol{\delta}_n\right) = \sum_i x[i] \cdot H(\boldsymbol{\delta}_i)$$

Finally, note that $\boldsymbol{\delta}_i \equiv \delta_i[n] = \delta[n-i]$, where $\boldsymbol{\delta} = \boldsymbol{\delta}_0$ is the unit impulse sequence. That is, $\boldsymbol{\delta}_i$ is a delayed version of the unit impulse sequence. Since H is time-invariant, we must have $H(\boldsymbol{\delta}_i) = H(\boldsymbol{\delta})_i$ and so

$$\mathbf{y} = \sum_i x[i] \cdot H(\boldsymbol{\delta})_i = \sum_i x[i]\mathbf{h}_i$$

or

$$y[n] = \sum_i x[i]h[n-i] \underset{k=n-i}{=} \sum_k h[k]x[n-k] = (h \star x)[n] \tag{2}$$

Thus, the LTI system is entirely characterized by its response to a unit impulse, $H(\boldsymbol{\delta}) = \mathbf{h} \equiv h[n]$. The summation in equation (2) is known as convolution. Figure 2 illustrates the convolution principle.

As for finite-dimensional linear systems, this convolution relationship may

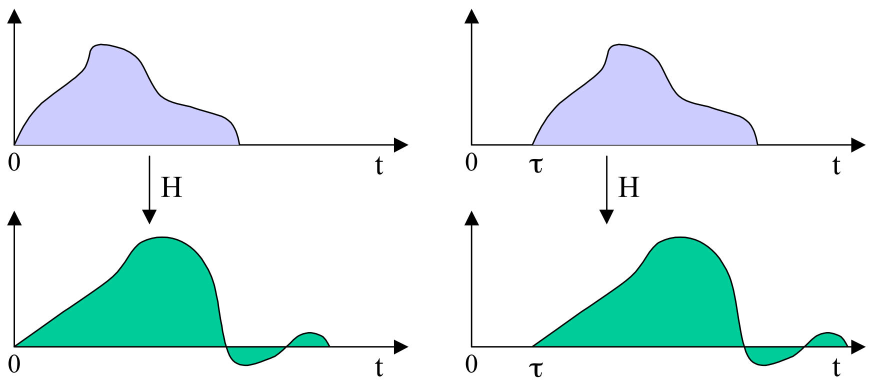


Figure 3: Behaviour of a continuous-time linear time invariant operator.

alternatively be expressed in terms of inner products. In particular, we have

$$y[k] = \sum_i x[i]h[k-i] = \sum_i x[i]\tilde{h}[i-k] = \langle \mathbf{x}, \tilde{\mathbf{h}}_k \rangle$$

where $\tilde{\mathbf{h}} \equiv \tilde{h}[n] = h[-n]$ is a time-reversed version of the impulse response, $\mathbf{h} \equiv h[n]$. Thus, the k^{th} output sample may be obtained by delaying (sliding to the right) $\tilde{\mathbf{h}}$ by k time steps and taking the inner product (pairwise multiplication and addition) of this delayed sequence with the input sequence $\mathbf{x}$.

3.3 Continuous-Time LTI Systems

Let $\mathbf{f} \equiv f(t)$ be a continuous-time signal (i.e. a function on $\mathbb{R}$) and H a linear system with $H(\mathbf{f}) = \mathbf{g}$. We write $\mathbf{f}_\tau \equiv f(t-\tau)$ for the signal obtained by delaying $\mathbf{f}$ by an amount τ. H is LTI if for all input signals, $\mathbf{f}$, we have

$$H(\mathbf{f}_\tau) = H(\mathbf{f})_\tau = \mathbf{g}_\tau$$

That is, the response of the system to a delayed input signal is a correspondingly delayed version of the response to the original signal. This is illustrated in Figure 3.

We can write

$$f(t) = \int f(\tau) \cdot \delta(t-\tau) \cdot d\tau \tag{3}$$

where $\delta(t)$ is the Dirac-delta function. This is not a real function, but a distribution – it “measures” a function at some point. The Dirac-delta function is defined, in fact, by equation (3). We think of $\boldsymbol{\delta} \equiv \delta(t)$ as the unit impulse signal, even though it is not a physically realizable signal. By analogy with the discrete case, we have

$$\mathbf{g} = H(\mathbf{f}) = \int f(\tau) \cdot \mathbf{h}_\tau \cdot d\tau$$

where $\mathbf{h} = H(\boldsymbol{\delta})$ is the response of the LTI system to the unit impulse. This analogy may be made rigorous, but we do not attempt to do so here. Writing the above equation in full we have

$$\begin{aligned} g(t) &= \int f(\tau) \cdot h_\tau(t) \cdot d\tau = \int f(\tau) \cdot h(t-\tau) \cdot d\tau \\ &= \int h(\kappa) \cdot f(t-\kappa) \cdot d\kappa = (h \star f)(t) \end{aligned}$$

which is the well-known convolution integral. The operation of the convolution integral may be pictured in a manner which is analogous to the discrete-time convolution depicted in Figure 2

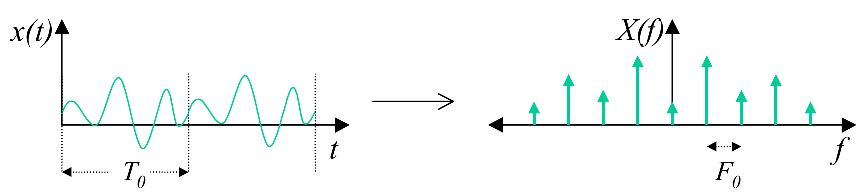
As in the discrete case, the convolution integral may be performed using inner products. Specifically, we have

$$g(p) = \int f(\tau) \cdot h(p-\tau) \cdot d\tau = \int f(\tau) \cdot \tilde{h}_p(\tau) \cdot d\tau = \langle \mathbf{f}, \tilde{\mathbf{h}}_p \rangle$$

where, again, $\tilde{\mathbf{h}} \equiv h(-t)$ is a time-reversed version of the impulse response, $\mathbf{h} \equiv h(t)$. This means that the output value at time p may be obtained by delaying (sliding to the right) $\tilde{\mathbf{h}}$ by time p and taking the inner product of this delayed function with the input function $\mathbf{f}$.

4 Fourier Transforms

4.1 Fourier Series



Consider periodic functions, $x(t)$, with period T_0. Suppose also that the "Dirichlet" conditions are satisfied, i.e.

1. $x(t)$ is absolutely integrable over one period, i.e. $\int_0^{T_0} |x(t)| \cdot dt < \infty$;
2. $x(t)$ has a finite number of extrema (local maxima or minima) within each period; and
3. $x(t)$ has at most a finite number of discontinuities within each period.

Then $x(t)$ may be expanded in a Fourier series with

$$\begin{aligned} x(t) &= \sum_{n=-\infty}^{\infty} X[n] e^{jn2\pi F_0 t} \\ X[n] &= \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jn2\pi F_0 t} \cdot dt \end{aligned}$$

where the fundamental frequency, $F_0 = \frac{1}{T_0}$.

Note that this may also be written as

$$\begin{aligned} \mathbf{x} &= \sqrt{T_0} \cdot \sum_{n=-\infty}^{\infty} X[n] \boldsymbol{\psi}_n \\ X[n] &= \frac{1}{\sqrt{T_0}} \langle \mathbf{x}, \boldsymbol{\psi}_n \rangle \end{aligned}$$

where the functions $\boldsymbol{\psi}_n \equiv \psi_n(t)$ are defined by

$$\psi_n(t) = \frac{1}{\sqrt{T_0}} e^{j2\pi n F_0 t}$$

Here, we are dealing with a linear space of periodic functions and the relevant inner product is given by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \int_0^{T_0} v(t) w(t) \cdot dt$$

It is easily verified that the complex sinusoids $\boldsymbol{\psi}_n$ are mutually orthogonal with unit norm. It follows that the Fourier Series is essentially an orthonormal expansion (up to a scaling of the coefficients by $\sqrt{T_0}$). Moreover, since all periodic signals $\mathbf{x}$ which satisfy the Dirichlet conditions with period T_0 can be written as linear combinations of the $\boldsymbol{\psi}_n$, these must constitute an orthonormal basis.

Recall that for orthonormal expansions, the squared norm of any vector is identical to the sum of the squares of the coefficients in the expansion. That is,

$$\int_0^{T_0} |x|^2 (t) \cdot dt = \langle \mathbf{x}, \mathbf{x} \rangle = \sum_{n=-\infty}^{\infty} |\langle \mathbf{x}, \boldsymbol{\psi}_n \rangle|^2 = T_0 \sum_{n=-\infty}^{\infty} |X[n]|^2$$

This is known as Parseval's relationship for Fourier series. We may identify the left hand side of the above equation with the energy in a single period. Here, we think of $x(t)$ as a voltage waveform across a 1Ω resistor, so that $x^2(t)$ is the instantaneous power dissipated in the resistor and its integral is the dissipated energy. Alternatively, we may write

$$\frac{1}{T_0} \int_0^{T_0} |x|^2 (t) \cdot dt = \sum_{n=-\infty}^{\infty} |X[n]|^2$$

where the left hand side represents the power of the periodic signal.

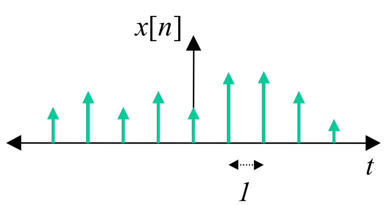
Although we have considered periodic functions, the Fourier Series expansion is equally applicable to the class of finite support signals, defined on the interval $t \in [0, T_0]$, where we have only to view the finite support signal as a single period of a hypothetical infinite support signal; the integrals and inner products defined above remain unchanged in this case. Alternatively, the Fourier Series expansion is applicable to functions defined on any interval of length T_0, e.g. $t \in \left[-\frac{T_0}{2}, \frac{T_0}{2}\right]$.

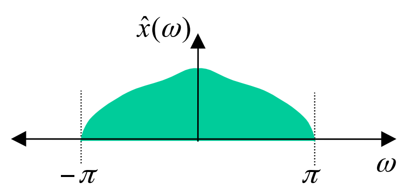
4.1.1 Normalized Fourier Series

A common problem in engineering, is that the connections between related concepts can easily get obscured by the notation. In particular, carrying absolute time scales around in the various formulae only makes them harder to recognize and harder to remember, without offering any particular advantage. In a suitable time scale (why do we have to use seconds?), we can always think of the signal as having a period of $T_0 = 2\pi$. This is probably the most natural period to adopt for a normalized Fourier series, since it is the period of the basic cosine and sine functions. In this normalized framework, we may rewrite the above relationships as follows:

$$
\begin{aligned}
x(t) &= \sum_{n=-\infty}^{\infty} X[n] e^{jnt} \\
X[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-jnt} \cdot dt \\
\sum_{n=-\infty}^{\infty} |X[n]|^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |x|^2 (t) \cdot dt
\end{aligned}
$$

4.2 Discrete-Time Fourier Transform (DTFT)





We have seen that the Fourier Series represents an invertible mapping between the inner product space formed by all finite support functions $x(t)$ defined on $t \in (-\pi, \pi)$ which satisfy the Dirichlet conditions, and infinite length two-sided sequences $X[n]$. By direct substitution, then, it is easy to verify that every sequence $x[n]$ may be represented by a finite support function, $\hat{x}(\omega)$, defined on

$\omega \in (-\pi, \pi)$, as follows:

$$\begin{aligned} \hat{x}(\omega) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \\ x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\omega) e^{j\omega n} \cdot d\omega \end{aligned} \tag{4}$$

This is known as the DTFT. Mathematically, it is no different from the Fourier Series; however, the interpretation which we shall apply is quite different. The interpretation will be that $\hat{x}(\omega)$ is also the true Fourier transform of a continuous signal $x(t)$, bandlimited to the range $\omega = 2\pi f \in (-\pi, \pi)$, having unit-spaced samples $x[n]$. It is not immediately obvious from the above why this is the correct interpretation. To make the interpretation stick, we will need to press on a little further.

Continuing the relationship between the Fourier Series and DTFT, we see that the family of functions $\{\boldsymbol{\psi}_n\}$, with

$$\psi_n(\omega) = \frac{1}{\sqrt{2\pi}} e^{-jn\omega}$$

forms an orthonormal basis for functions satisfying the Dirichlet conditions which are defined on $\omega \in (-\pi, \pi)$, and the DTFT may be written as

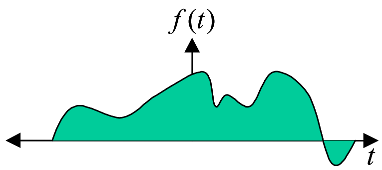
$$\begin{aligned} \hat{x}(\omega) &\equiv \hat{\mathbf{x}} = \sum_{n=-\infty}^{\infty} \langle \hat{\mathbf{x}}, \boldsymbol{\psi}_n \rangle \boldsymbol{\psi}_n = \sqrt{2\pi} \sum_{n=-\infty}^{\infty} x[n] \boldsymbol{\psi}_n \\ x[n] &= \frac{1}{\sqrt{2\pi}} \langle \hat{\mathbf{x}}, \boldsymbol{\psi}_n \rangle \end{aligned}$$

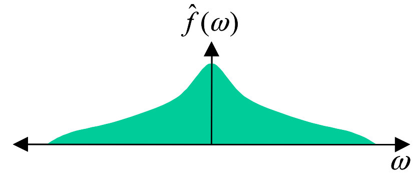
From the properties of orthonormal expansions, then, we have

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \|\hat{\mathbf{x}}\|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{x}(\omega)|^2 \cdot d\omega$$

which is known as Parseval's relationship for discrete-time signals.

4.3 Continuous-Time Fourier Transform (CTFT or just FT)





Suppose the signal $f(t)$ has finite energy. That is, $\left|\int f^2(t)\,dt\right| < \infty$. Then the following Fourier transform (FT) relationship holds:

$$\begin{aligned}\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\cdot dt \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\,e^{j\omega t}\cdot d\omega \end{aligned} \tag{5}$$

4.3.1 Some Properties of the Fourier Transform

Conjugate Symmetry: If $f(t)$ is a real-valued function,

$$\hat{f}(\omega) = \hat{f}^*(-\omega),$$

where a^* denotes the complex conjugate of a.

Time shift: Let $f_\tau(t) = f(t-\tau)$, i.e. $f_\tau(t)$ is obtained by delaying the signal $f(t)$ by time t; then

$$\hat{f}_\tau(\omega) = e^{-j\omega\tau}\hat{f}(\omega).$$

Convolution: Let $g(t) = (h \star f)(t)$ be the convolution of signal $f(t)$ and the impulse response $f(t)$ of an LTI filter; then

$$\hat{g}(\omega) = \hat{h}(\omega)\hat{f}(\omega).$$

We say that $\hat{h}(\omega)$ is the transfer function of the LTI system.

Sinusoidal modulation: Let $f_m(t) = f(t)e^{j\omega_0 t}$; then $\hat{f}_m(\omega) = \hat{f}(\omega-\omega_0)$. Note that we are usually dealing with real-valued signals $f(t)$, but $f_m(t)$ is not generally real-valued. For real signals, let $f_c(t) = f(t)\cos\omega_0 t = \frac{1}{2}f(t)(e^{j\omega_0 t} + e^{-j\omega_0 t})$; then

$$\hat{f}_c(\omega) = \frac{1}{2}\left(\hat{f}(\omega-\omega_0) + \hat{f}(\omega+\omega_0)\right)$$

General modulation: Let $g(t) = f(t)\cdot m(t)$; then

$$\hat{g}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{m}(\phi)\hat{f}(\omega-\phi)\cdot d\phi = \frac{1}{2\pi}\left(\hat{m}\star\hat{f}\right)(\omega)$$

Parseval's relation:

$$\int_{-\infty}^{\infty}|f(t)|^2\cdot dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\hat{f}(\omega)\right|^2\cdot d\omega$$

Generalized Parseval's relation:

$$\int_{-\infty}^{\infty} f(t)g^*(t)\cdot dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega)\hat{g}^*(\omega)\cdot d\omega$$

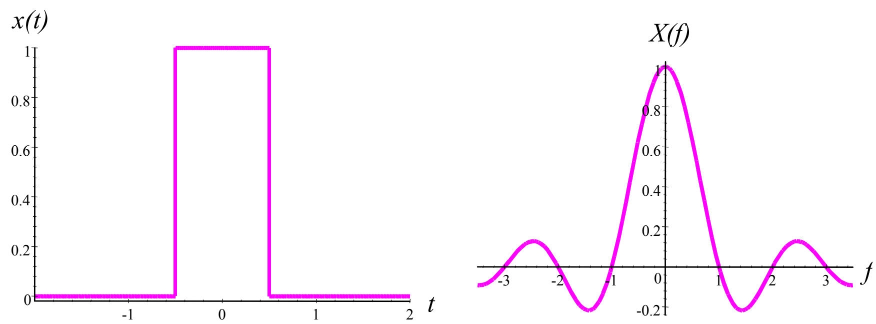


Figure 4: Unit pulse, $x(t) = \Pi(t)$, and its Fourier transform, $X(f) = \hat{x}(\omega)|_{\omega=2\pi f}$.

Differentiation: Let $g(t) = \frac{d}{dt}f(t)$; then

$$\hat{g}(\omega) = j\omega \cdot \hat{f}(\omega)$$

Thus, differentiating $f(t)$ is equivalent to applying an LTI system (filter) whose transfer function is $\hat{h}(\omega) = j\omega$.

4.3.2 Important Examples

Impulse: $f(t) = \delta(t) \implies \hat{f}(\omega) = 1, \forall\omega$. Similarly, $f(t) = 1, \forall t \implies \hat{f}(\omega) = \delta(\omega)$.

Rectangular pulse (time domain): Define the "pulse" function,

$$\Pi(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ 0 & \text{if } |t| \geq \frac{1}{2} \end{cases}$$

Its Fourier transform, $\hat{\Pi}(\omega)$, is given by

$$\hat{\Pi}(\omega) = \text{sinc}(\frac{\omega}{2\pi}) = \frac{\sin\omega/2}{\omega/2}$$

These are illustrated in Figure 4.The way to remember this is that the nulls of the Fourier transform of a unit length pulse lie at multiples of $\omega = 2\pi$. This is because these frequencies correspond to sinusoids which execute a whole number of cycles within the unit duration of the pulse, and hence integrate out.

Rectangular pulse (Fourier domain): Let $\hat{x}(\omega)$ be the unit pulse on the interval, $\omega \in (-\pi, \pi)$,

$$\hat{x}(\omega) = \begin{cases} 1 & \text{if } |\omega| < -\pi \\ 0 & \text{if } |t| \geq \pi \end{cases}$$

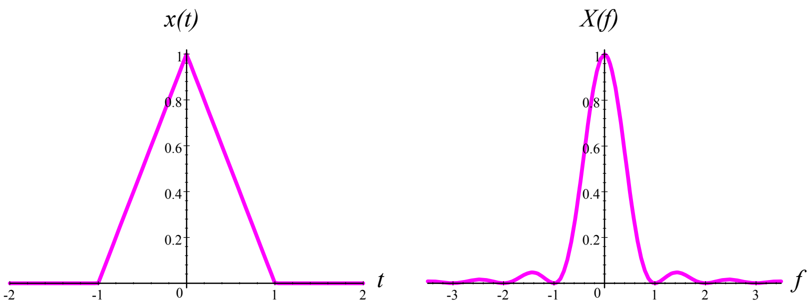


Figure 5: Triangle, $x(t) = \Lambda(t)$, and its Fourier transform, $X(f) = \hat{x}(\omega)|_{\omega=2\pi f}$.

Its inverse Fourier transform $x(t)$, is given by

$$x(t) = \text{sinc}(t) = \frac{\sin \pi t}{\pi t}$$

Triangular waveform: Define the triangular waveform,

$$\Lambda(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

Now observe that $\Lambda(t) = (\Pi \star \Pi)(t)$, so that

$$\hat{\Lambda}(\omega) = \text{sinc}^2(\frac{\omega}{2\pi})$$

These are illustrated in Figure 5.

Raised cosine: Define the raised cosine function,

$$R_c(t) = \begin{cases} \frac{1}{2}(1 + \cos \pi t) & \text{if } |t| < 1 \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

Its Fourier transform is

$$\begin{aligned}
\hat{R}_c(\omega) &= \int_{-1}^{1} \frac{1}{2}(1 + \cos(\pi t)) e^{-j\omega t} dt \\
&= \frac{1}{2} \int_{-1}^{1} \left(e^{-j\omega t} + \frac{1}{2} e^{-j(\omega-\pi)t} + \frac{1}{2} e^{-j(\omega+\pi)t} \right) dt \\
&= \frac{1}{2} \frac{e^{-j\omega} - e^{j\omega}}{-j\omega} + \frac{1}{4} \frac{e^{-j(\omega-\pi)} - e^{j(\omega-\pi)}}{-j(\omega-\pi)} + \frac{1}{4} \frac{e^{-j(\omega+\pi)} - e^{j(\omega+\pi)}}{-j(\omega+\pi)} \\
&= \frac{\sin \omega}{\omega} + \frac{1}{2} \frac{\sin(\omega-\pi)}{\omega-\pi} + \frac{1}{2} \frac{\sin(\omega+\pi)}{\omega+\pi} \\
&= \frac{-\pi^2 \sin \omega}{(\omega-\pi)(\omega+\pi)\omega}
\end{aligned}$$

These are illustrated in Figure6.

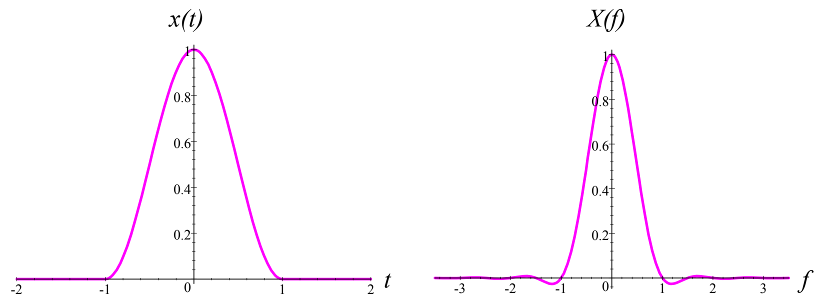


Figure 6: Raised cosine, $x(t) = R_c(t)$, and its Fourier transform, $X(f) = \hat{x}(\omega)|_{\omega=2\pi f}$.

5 Sampling and Interpolation

We have considered three different Fourier transforms:

Fourier Series: Maps periodic or finite support functions to an infinite sequence of discrete frequency components.

Discrete-time Fourier Transform: Maps discrete signals to a continuous frequency representation with finite support.

Fourier Transform: Maps continuous signals to a continuous frequency representation with infinite support.

The relationship between the last two is particularly important for our purposes, since we often need to deal with both discrete-time and continuous signals together. This relationship is embodied by the sampling theorem, which we develop in the following simple steps:

- Let $f(t)$ be a signal with Fourier transform, $\hat{f}(\omega)$.
- Let $x[n]$ be obtained by impulsively sampling $f(t)$ with a unit sampling interval, i.e.
$$x[n] = f(t)|_{t=n} \tag{6}$$
and let $\hat{x}(\omega)$ denote the DTFT of $x[n]$.
- Suppose that $f(t)$ is a bandlimited signal with
$$\hat{f}(\omega) = 0, \ |\omega| \geq \pi \tag{7}$$
- Observe that in this case the inverse DTFT integral and the inverse FT integral, given in equations (4) and (5) respectively, are identical. Specifically, we see that
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{x}(\omega)e^{jn\omega}\cdot d\omega = x[n] = f(t)|_{t=n} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)e^{jn\omega}\cdot d\omega$$

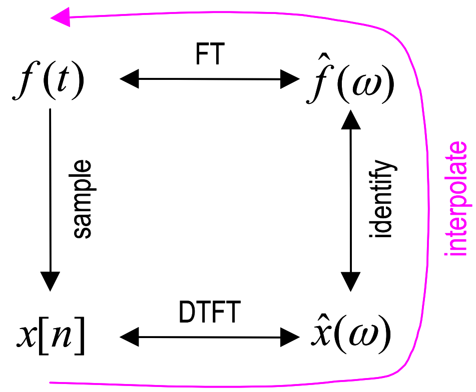


Figure 7: The Nyquist Cycle: Interpolation of a bandlimited continuous signal, $f(t)$, from its impulse samples, $x[n]$, based on the DTFT and FT relationships.

so that $\hat{f}(\omega)$ and $\hat{x}(\omega)$ must be identical. Since knowledge of $\hat{x}(\omega)$ is equivalent to knowledge of $x[n]$ and knowledge of $\hat{f}(\omega)$ is equivalent to knowledge of $f(t)$, we conclude that the sampled sequence $x[n]$ captures all the information in the continuous signal $f(t)$, provided equation (7) is satisfied. More generally, an impulsively sampled sequence of the form $x[n] = f(t)|_{t=nT}$ captures all the information in the continuous sequence $f(t)$, provided the sampling interval satisfies $T \leq \frac{\pi}{\Omega_0}$, where $\hat{f}(\omega)$ is bandlimited to the region $\omega \in (-\Omega_0, \Omega_0)$. This is commonly known as the Nyquist sampling limit.

- To reinforce the above observation, we show how the original continuous signal $f(t)$ may be recovered from $x[n]$. The above reasoning is compactly represented by the commutative diagram in Figure 7.Since we are able to identify $\hat{f}(\omega)$ with $\hat{x}(\omega)$, it must be possible to reconstruct $f(t)$ by first applying the forward DTFT to $x[n]$ and then applying the inverse FT to the result. We obtain

$$\begin{aligned} f(t) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)e^{j\omega t}\cdot d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{x}(\omega)e^{j\omega t}\cdot d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}x[n]e^{-j\omega n}\right)e^{j\omega t}\cdot d\omega \\ &= \sum_{n=-\infty}^{\infty}x[n]\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{j\omega(t-n)}\cdot d\omega \\ &= \sum_{n=-\infty}^{\infty}x[n]\,\mathrm{sinc}\,(t-n) = \sum_{n=-\infty}^{\infty}x[n]\,\mathrm{sinc}_n\,(t) \end{aligned}$$

Thus, $f(t)$ may be obtained directly from $x[n]$, by so-called sinc interpolation. Specifically, we interpolate the samples by translating a sinc function to each sample location, weighting the translated sinc function by the relevant sample value, and summing the weighted, translated sinc functions.

The interpolation formula above provides a basis for the space of signals bandlimited to $\omega \in (-\pi, \pi)$. Specifically, any signal, $\mathbf{f} \equiv f(t)$, in this space, may be represented as a linear combination of the signals, $\boldsymbol{\psi}_n \equiv \mathrm{sinc}\,(t-n)$, with

$$\mathbf{f} = \sum_{n=-\infty}^{\infty} x[n]\boldsymbol{\psi}_n$$

In fact, it is not difficult to show that $\{\boldsymbol{\psi}_n\}$ is an orthonormal basis, so the sampling relationship is an orthonormal expansion of $x(t)$ and we have another Parseval relationship:

$$\sum_{n=-\infty}^{\infty}|x[n]|^2 = \int_{-\infty}^{\infty}|f(t)|^2\cdot dt$$

- The Nyquist cycle in Figure 7 does more than just connect discrete and continuous signals through sampling and interpolation formulae. It also imparts meaning to the frequency representation embodied by the DTFT. What is the meaning of a frequency of 1.0325 rad/s for discrete sequences of samples? The answer is that the DTFT is identical to the Fourier transform of the underlying continuous signal and derives all of its meaning from the continuous signal. When we measure frequency content in the DTFT domain, we are really looking at the frequencies of the underlying continuous signal. Similarly, when we filter a discrete sequence, we are really filtering the underlying continuous signal.

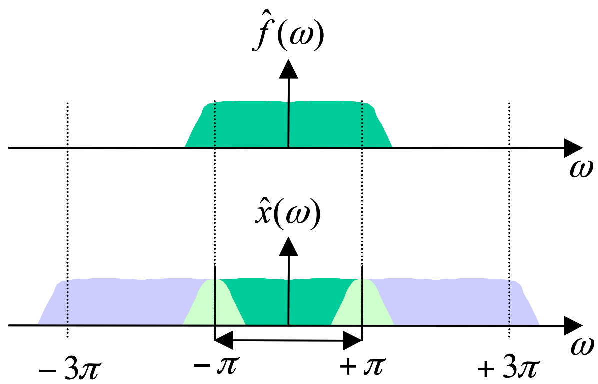


Figure 8: Aliasing contributions to the DTFT, $\hat{y}(\omega)$, of a sequence sampled below the Nyquist rate.

- As a final note, we point out that signals are not generally exactly bandlimited to any frequency range. For non-bandlimited signals, the equality of $\hat{f}(\omega)$ and $\hat{x}(\omega)$ no longer holds. We find more generally that

$$\hat{x}(\omega) = \sum_{k=-\infty}^{\infty} \hat{f}(\omega - 2\pi k)$$

  Thus, the spectrum of the discrete sequence is generally a sum of "aliasing" components, as shown in Figure 8

6 Convolution and Time Invariance

Let H be an LTI filter and $\mathbf{h} \equiv h[n]$ its impulse response. We have seen that the operation $\mathbf{y} = H(\mathbf{x})$ may be expressed in terms of $\mathbf{h}$ as

$$\mathbf{y} = \sum_n x[n]\, \mathbf{h}_n$$

and this is the most natural form of the convolution equation when thinking of $\mathbf{h}$ as the response to a unit impulse at $n = 0$. We have also seen that convolution may be expressed in terms of inner products as

$$y[n] = \left\langle \mathbf{x}, \tilde{\mathbf{h}}_n \right\rangle$$

This expression reveals the fact that convolution may be viewed as a sliding window, weighted averaging operation. The weights are contained in $\tilde{\mathbf{h}}$, the mirror image of $\mathbf{h}$, which slides to the right one position at a time, generating each consecutive output $y[n]$, as it goes.

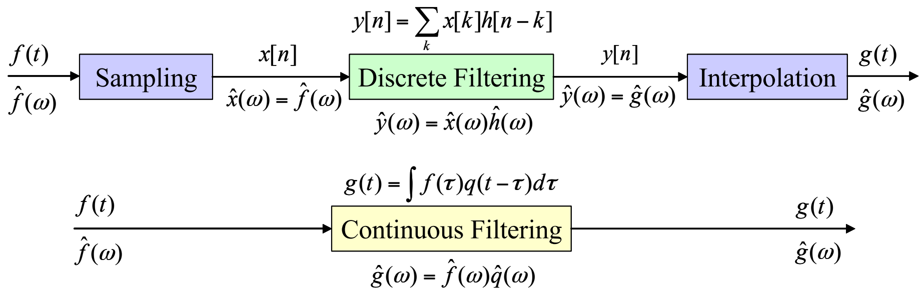


Figure 9: Equivalent discrete and continuous processing systems. The equivalence holds so long as the sampling and interpolation are ideal and $\hat{q}(\omega)$ agrees with $\hat{h}(\omega)$ on $\omega \in (-\pi, \pi)$.

We may express the convolution operation in yet a third way, in terms of translated copies of the input sequence, $\mathbf{x}$. Specifically, $y[n] = \sum_i h[i]\, x[n-i]$ may be expressed as

$$\mathbf{y} = \sum_k h[k]\, \mathbf{x}_k \tag{8}$$

This means that the output sequence is a linear combination of shifted copies of the input sequence.

This perspective provides perhaps the clearest explanation of why the Fourier transform is a useful tool for analysing, designing and even implementing LTI operators. In particular, the Fourier transform is essentially the only invertible linear transform under which shifts become simple scale factors. Specifically,

$$\hat{x}_k(\omega) = \hat{x}(\omega) \cdot \underbrace{e^{-j\omega k}}_{\text{scale factor}}$$

Using this property, together with equation (8) and linearity, we obtain

$$\begin{aligned}\hat{y}(\omega) &= \sum_k h[k]\, \hat{x}(\omega)\, e^{-j\omega k} \\ &= \hat{x}(\omega) \sum_k h[k]\, e^{-j\omega k} \\ &= \hat{x}(\omega)\, \hat{h}(\omega)\end{aligned}$$

The same relationship may be derived for continuous-time signals. In fact, since the discrete and continuous Fourier transforms are identical (assuming Nyquist sampling) all operations which we perform on the discrete signal are equivalent to operations performed on the original continuous signal.

Figure 9 makes this fact concrete. Virtually all signal processing systems start with a continuous signal at the input and produce a continuous signal at the output, but most of the operations in between are performed digitally in the discrete domain. Provided we sample the continuous signal below the Nyquist limit and synthesize the continuous signal using sinc interpolation, or a suitable approximation, the end-to-end system will be linear and time invariant and we have full control over its properties through design of the discrete filters. A digital filter with DTFT $\hat{h}(\omega)$ has exactly the same effect as any continuous filter whose Fourier transform $\hat{q}(\omega)$ agrees with $\hat{h}(\omega)$ over the interval $\omega \in (-\pi, \pi)$. If $\hat{q}(\omega)$ happens to be Nyquist bandlimited also[1], then $\hat{h}(\omega)$ and $\hat{q}(\omega)$ are identical, so that $h[n]$ and $q(t)$ are related through sampling and interpolation, just as the signals themselves are.

[1] Unfortunately, this is never the case when we start with a prototype analog design (e.g., Butterworth, Chebychev, ...) and try to emulate it digitally.