Elec4621: Advanced Digital Signal Processing
Chapter 2: Z-Transforms, Filters and Oscillators

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March 16, 2015

1 Introduction to the Z-Transform

The Z-transform of a sequence, $x[n]$, is a formal power series, defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{1}$$

For finite sequences, $x[n]$, zero outside the range $0 \leq n \leq N$, the Z-transform is a polynomial of degree N in z^{-1}:

$$X(z) = x[0] + x[1] z^{-1} + \cdots + x[N] z^{-N}$$

Now let $x[n]$ and $h[n]$ be two finite sequences with lengths $N+1$ and $M+1$, respectively. The product of their Z-transform polynomials is a polynomial of degree $N+M$ in z^{-1}:

$$\begin{aligned}
X(z) H(z) &= \left(x[0] + x[1]z^{-1} + \cdots + x[N] z^{-N}\right) \times \left(h[0] + h[1]z^{-1} + \cdots + h[M] z^{-M}\right) \\
&= \sum_k \sum_m x[k] h[m] z^{-(k+m)} \\
&=_{n=m+k} \sum_k x[k] \sum_n h[n-k] z^{-n} \\
&= \sum_n y[n] z^{-n}; \quad \text{where } y[n] = \sum_k x[k] h[n-k]
\end{aligned}$$

That is, $X(z) H(z)$ is the Z-transform of $y[n]$, the convolution of $x[n]$ with $h[n]$. Of course, the above derivation is true for both finite and non-finite sequences, but starting with finite sequences has intuitive appeal, because you are used to working with polynomials. In fact, the convolution principle just derived is a property of polynomial multiplication. Specifically, when you multiply two polynomials in z together and collect terms, the term in z^{-n} is found by summing the products, $x[k]\,h[n-k]$ over all k where the sum is non-zero. You have done this yourself many times in High School.

Example 1 *Consider the polynomials* $X(z) = 1 + \frac{1}{2}z^{-1} + z^{-2}$ *and* $H(z) = 1 + \frac{1}{4}z^{-1}$*. Multiplying these polynomials gives*

$$\begin{aligned} X(z)\,H(z) &= X(z) + \frac{1}{4}z^{-1}X(z) \\ &= 1 + \frac{1}{2}z^{-1} + z^{-2} + \frac{1}{4}z^{-1} + \frac{1}{8}z^{-2} + \frac{1}{4}z^{-3} \\ &= 1 + \left(\frac{1}{2} + \frac{1}{4}\right)z^{-1} + \left(1 + \frac{1}{8}\right)z^{-2} + \frac{1}{4}z^{-3} \end{aligned}$$

We could arrive at the same result by recognizing that $X(z)\,H(z) = Y(z)$ *where* $Y(z)$ *is a polynomial of degree* 3 *in* z^{-1}*,*

$$Y(z) = y_0 + y_1 z^{-1} + y_2 z^{-2} + y_3 z^{-3}$$

where the coefficients are given by

$$y_n = \sum_{k=\max\{0,n-1\}}^{\min\{2,n\}} x_k h_{n-k}$$

This yields

$$\begin{aligned} y_0 &= \sum_{k=0}^{0} x_k h_{0-k} = x_0 h_0 = 1 \\ y_1 &= \sum_{k=0}^{1} x_k h_{1-k} = x_0 h_1 + x_1 h_0 = \frac{1}{4} + \frac{1}{2} \\ y_2 &= \sum_{k=1}^{2} x_k h_{2-k} = x_1 h_1 + x_2 h_0 = \frac{1}{8} + 1 \\ y_3 &= \sum_{k=2}^{2} x_k h_{3-k} = x_2 h_1 = \frac{1}{4} \end{aligned}$$

The key point to observe from the above is that polynomial multiplication is the same thing as convolution, where we identify the coefficients of the polynomial with the elements of the sequences being convolved. Power series are polynomials having an infinite number of terms and the Z-transform is nothing other than an association of the samples in a sequence with the coefficients in a formal power series.

Note that this association does not actually give any meaning to the parameter, z. In a formal power series, z is a formal parameter which is used for manipulative purposes, but has no real meaning. In the example above, for example, we did not need to give z any interpretation. We will find in the next section that there is a useful interpretation which can be associated with z, which derives from the close similarity between the Z-transform and Fourier transform expressions. For now, however, we wish to make a clear distinction between the Z-transform and the Fourier transform:

Remark 1 *The Fourier transform gives an invertible relationship between sequences of finite energy and functions defined on the interval $\omega \in (-\pi, \pi)$. By contrast, the Z-transform is best understood as a formal assocation between the samples in a sequence (not necessarily with finite energy) and the coefficients of a power series (generalized polynomial) for manipulative purposes.*

The manipulative power of the Z-transform representation of sequences becomes particularly apparent when we consider recursive systems, as suggested by the following example.

Example 2 *Consider the discrete linear time invariant system described by*

$$y[n] = x[n] + \alpha y[n-1]$$

where $x[n]$ is the input to the system and $y[n]$ is the output. Such systems are said to be recursive, since the output at time n is dependent upon the output produced at previous time instants. Thinking of sequences as vectors, as in your first set of handouts, the above relationship may be written as

$$\mathbf{y} = \mathbf{x} + \alpha \mathbf{y}_1$$

where $\mathbf{y}_1$ denotes the sequence, $\mathbf{y}$, delayed by 1. Applying the readily verified identity

$$Y(z) = z^{-1} Y_1(z)$$

we find that

$$Y(z) = X(z) + \alpha z^{-1} Y(z)$$

which we manipulate in a trivial way to obtain

$$Y(z) = \frac{X(z)}{1 - \alpha z^{-1}}$$

So the system impulse response, $h[n]$, must have Z-transform,

$$\begin{aligned} H(z) &= \frac{1}{1 - (\alpha z^{-1})} \\ &= 1 + \left(\alpha z^{-1}\right) + \left(\alpha z^{-1}\right)^2 + \left(\alpha z^{-1}\right)^3 + \cdots \end{aligned}$$

The power series expansion above allows us to deduce the impulse response immediately as

$$h[n] = \begin{cases} \alpha^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

2 Z-Transforms and the Fourier Transform

Up to this point we have not given any interpretation to z whatsoever, regarding it as a formal parameter. It is useful, however, to consider what happens if we substitute

$$z = re^{j\omega}$$

into the Z-transform expression of equation (1). We obtain

$$\begin{aligned} X(z) &= \sum_n \underbrace{x[n]\, r^{-n}}_{x_r[n]} e^{-j\omega n} \\ &= \sum_n x_r[n]\, e^{-j\omega n} \\ &= \hat{x}_r(\omega) \end{aligned}$$

That is, $X(z)$ is identical to the DTFT of the modified sequence, $\hat{x}_r(\omega)$, formed by applying the exponential decay term, r^{-n}, to $x[n]$.

Of course, this Fourier transform only exists so long as the sum above converges, for which it is sufficient to know that the sequence, $x_r[n]$, is absolutely summable, i.e.,

$$\sum_n |x_r[n]| = \sum_n \left|r^{-n} x[n]\right| < \infty$$

If this is true for $r = 1$, then the Fourier transform of $x[n]$ exists and we can say that

$$\hat{x}(\omega) = X(z)|_{z=e^{j\omega}}$$

Suppose that $x[n]$ is a causal sequence (or one which extends only a finite distance into the past). This is true of all filter impulse responses in which we will be interested during this subject. It follows that $x_r[n]$ is absolutely summable for all $r > r_0$ where r_0 is the magnitude of the largest pole in $X(z)$. r_0 is called the radius of convergence (ROC).

Remark 2 *So long as the ROC is strictly less than* 1, $x[n]$ *is absolutely summable and its Fourier transform exists, with* $\hat{x}(\omega) = X\left(e^{j\omega}\right)$.

2.1 Z-Transform Inversion

We may use the relationship with the Fourier transform to find an inversion procedure for the Z-transform. As we shall see, however, some caution should be exercised here when using the term "inversion."

Let $X(z)$ be the Z-transform of some sequence, $x[n]$, and let $r > r_0$ where r_0 is the ROC (radius of convergence). In particular, $X(z)$ is finite for all complex z with magnitude $|z| \geq r$. Then $x_r[n]$ is absolutely summable and its Fourier transform (DTFT) exists and is equal to $X\left(re^{j\omega}\right)$. We can use the inverse

DTFT relationship to write

$$\begin{aligned} x[n] &= r^n x_r[n] \\ &= r^n \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(re^{j\omega}\right) e^{j\omega} d\omega \end{aligned} \tag{2}$$

Although this may seem perfectly clean at first sight, there are a number of subtleties. Firstly, this inverse is not actually useful for practical purposes if $r > 1$, since then the term r^n grows rapidly without bound, amplifying the effects of otherwise insignificant calculation errors to the point where the result is useless. But if the ROC includes $r = 1$, we might as well just use the inverse DTFT directly, getting

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X\left(e^{j\omega}\right) e^{j\omega} d\omega$$

which is stable.

Secondly, although one can recover $x[n]$ from $X(z)$, one cannot start with just any function, $X(z)$, and recover a valid sequence, $x[n]$ whose Z-transform is $X(z)$. In fact, the function $X(z)$ must have the property that it is defined by its values on any closed contour inside the ROC. This is clear, since we should recover the same sequence, $x[n]$, using any $r > r_0$ in equation (2). Functions with this property (amongst others) are said to be "Analytic functions".

These observations reinforce the fact that the Z-transform is a tool for analysis, not for numerically representing sequences as functions. By contrast, many of the transforms which we encounter in signal processing are useful for signal representation and for direct numerical computation.

3 Pole-Zero Representation of Filters

Although the word "filter" may be applied to any linear time-invariant (LTI) operator which is stable, not all such operators can be implemented in practice. The family of discrete LTI operators which can be implemented may be described by the following general equation

$$y[n] = \sum_{k=0}^{M} a_k x[n-k] - \sum_{k=1}^{N} b_k y[n-k] \tag{3}$$

where $y[n]$ is the output sequence, $x[n]$ the input sequence and M, N, $\{a_k\}$ and $\{b_k\}$ are parameters, which we can select. This equation describes the discrete LTI operators which may be implemented using a finite number of operations, with a finite amount of memory. For this reason, the term "digital filter" is generally used to refer to operators of this form.

The Z-transform is perfectly suited to representing and analyzing exactly systems of the form given in equation (3). Using the property that a unit delay is equivalent to multiplication by z^{-1} in the Z-transform domain, we may rewrite equation (3) as

$$Y(z) = \sum_{k=0}^{M} a_k z^{-k} X(z) - \sum_{k=1}^{N} b_k z^{-k} Y(z)$$

and hence

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(Z)} = \frac{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_N z^{-N}} \\ &= \frac{a_0 z^M + a_1 z^{M-1} + \cdots + a_M}{z^M} \cdot \frac{z^N}{z^N + b_1 z^{N-1} + \cdots + b_N} \\ &= A \frac{(z - z_1)(z - z_2)\cdots(z - z_M)}{z^M} \cdot \frac{z^N}{(z - p_1)(z - p_2)\cdots(z - p_N)} \end{aligned} \tag{4}$$

Here, $H(z)$ is the Z-transform of the system's impulse response, $h[n]$, and we see that $H(z)$ is a rational polynomial – an expression involving finite polynomials in the numerator and denominator.

According to the Fundamental Theorem of Algebra, every finite polynomial can be factored into simple monomials, as shown in equation (4), where $\{z_k\}$ (zeros of $H(z)$) are the roots of the numerator polynomial and $\{p_k\}$ (poles of $H(z)$) are the roots of the denomenator polynomial. Both the poles and the zeros are complex-valued in general, but they must appear in complex-conjugate pairs, so long as the filter coefficients, $\{a_k\}$ and $\{b_k\}$, are real-valued.

It is instructive to represent $H(z)$ as a cascade of simple segments, each having real-valued coefficients. Specifically, equation (4) may be rearranged as

$$\begin{aligned} H(z) &= A \prod_{k=1}^{M_{\text{real}}} \frac{z - z_k}{z} \prod_{k=M_{\text{real}}+1}^{M_{\text{real}}+M_{\text{conj}}} \frac{(z - z_k)(z - z_k^*)}{z^2} \\ &\quad \times \prod_{k=1}^{N_{\text{real}}} \frac{z}{z - p_k} \prod_{k=N_{\text{real}}+1}^{N_{\text{real}}+N_{\text{conj}}} \frac{z^2}{(z - p_k)(z - p_k^*)} \\ &= A \prod_{k=1}^{M_{\text{real}}} \frac{z - z_k}{z} \prod_{k=M_{\text{real}}+1}^{M_{\text{real}}+M_{\text{conj}}} \frac{z^2 - 2z r_k \cos\theta_k + r_k^2}{z^2} \\ &\quad \times \prod_{k=1}^{N_{\text{real}}} \frac{z}{z - p_k} \prod_{k=N_{\text{real}}+1}^{N_{\text{real}}+N_{\text{conj}}} \frac{z^2}{z^2 - 2z\rho_k \cos\omega_k + \rho_k^2} \end{aligned}$$

where M_{real}, M_{conj}, N_{real} and N_{conj} denote the number of real-valued zeros, the number of complex conjugate pairs of zeros, the number of real-valued poles and the number of complex conjugate pairs of poles, complex-valued zeros are expressed as

$$z_k = r_k e^{j\theta_k}$$

while complex-valued poles are expressed as

$$p_k = \rho_k e^{j\omega_k}$$

The overall filter may thus be understood (and implemented, if desired) as a cascade of first and second order sections and its properties may be understood in terms of the properties of those simple sections.

A filter having no non-trivial poles is composed of first and second order sections of the form

$$\frac{z - z_k}{z} \quad \text{and} \quad \frac{z^2 - 2zr_k \cos\theta_k + r_k^2}{z^2}$$

Such a filter is sometimes called an "all-zero" filter, but more commonly it is identified as an **FIR (Finite Impulse Response)** filter, because its implementation involves no feedback – $N = 0$ in equation (3).

A filter having no non-trivial zeros, has sections of the form

$$\frac{z}{z - p_k} \quad \text{and} \quad \frac{z^2}{z^2 - 2z\rho_k \cos\omega_k + \rho_k^2}$$

Such a filter is sometimes called an "all-pole" filter.

The term "**recursive filter**" is used to describe filters having one or more non-trivial poles, possibly in addition to one or more zeros. The term derives from the presence of feedback terms, b_k, in equation (3). Recursive filters are also called IIR (Infinite Impulse Response) filters since the presence of feedback implies that their impulse responses extend indefinitely, unless poles and zeros cancel perfectly.

Remark 3 *The all-zero and all-pole first and second order sections used to represent any system of the form given in equation (3) each have exactly the same number of poles and zeros. The so-called "all-zero" sections have only trivial poles at $z = 0$, while the "all-pole" sections have only trivial zeros at $z = 0$.*

3.1 First Order Impulse Responses

The first order, all-zero section,

$$\frac{z - z_k}{z} = 1 - z_k z^{-1}$$

represents an FIR filter having two non-zero samples (also called filter taps),

$$h[0] = 1, \quad h[1] = -z_k$$

The first order, all-pole section,

$$\frac{z}{z - p_k} = \frac{1}{1 - p_k z^{-1}} = 1 + p_k z^{-1} + p_k^2 z^{-2} + \cdots + p_k^n z^{-n} + \cdots$$

represents an IIR filter having the exponentially decaying impulse response

$$h[n] = \begin{cases} p_k^n & n \geq 0 \\ 0 & n < 0 \end{cases} = p_k^n u[n]$$

where $u[n]$ is the unit step sequence. As an alternative to the power series manipulation above, we can recognize that this section arises in connection with a recursive system of the form

$$y[n] = x[n] + p_k \cdot y[n-1]$$

To find the impulse response of this system, let $x[n] = \delta[n]$ and $y[n] = 0$ for all $n < 0$. Then, identifying $y[n]$ with the impulse response, $h[n]$, we find that

$$\begin{aligned} h[0] &= x[0] = 1 \\ h[1] &= p_k h[0] = p_k \\ h[2] &= p_k h[1] = p_k^2 \\ &\vdots \\ h[n] &= p_k h[n-1] = p_k^n \\ &\vdots \end{aligned}$$

exactly as before.

3.2 Second Order Impulse Responses

The second order, all-zero section,

$$\frac{z^2 - 2zr_k \cos\theta_k + r_k^2}{z^2} = 1 - 2r_k \cos\theta_k z^{-1} + r_k^2 z^{-2}$$

represents an FIR filter having three filter taps,

$$h[0] = 1, \quad h[1] = -2r_k \cos\theta_k, \quad h[2] = r_k^2$$

The second order, all-pole section,

$$\frac{z^2}{z^2 - 2z\rho_k \cos\omega_k + \rho_k^2} = \frac{z}{z - \rho_k e^{j\omega_k}} \cdot \frac{z}{z - \rho_k e^{-j\omega_k}} \tag{5}$$

represents an IIR filter with impulse response, $h[n]$, equal to the convolution of $h_1[n] = \rho_k^n e^{jn\omega_k} u[n]$ and $h_2[n] = \rho_k^n e^{-jn\omega_k} u[n]$. That is,

$$\begin{aligned} h[n] &= \sum_t h_1[t]\, h_2[n-t] = \sum_{t=0}^{n} \rho_k^t e^{jt\omega_k} \rho_k^{n-t} e^{-j(n-t)\omega_k} \\ &= \rho_k^n e^{-jn\omega_k} \sum_{t=0}^{n} e^{j2t\omega_k} = \rho_k^n e^{-jn\omega_k} \frac{1 - e^{j2(n+1)\omega_k}}{1 - e^{j2\omega_k}} u[n] \\ &= \rho_k^n \frac{e^{-j(n+1)\omega_k} - e^{j(n+1)\omega_k}}{e^{-j\omega_k} - e^{j\omega_k}} u[n] = \rho_k^n \frac{\sin(n+1)\,\omega_k}{\sin\omega_k} u[n] \end{aligned}$$

Exercise 1 *Sketch the second order all-pole impulse response, $h[n]$, given above, assuming that $\rho_k < 1$. What are its main features? Evaluate $h[0]$. What is the peak response if $\omega_k = \pi/6$ and $\rho_k \approx 1$? Can you think why it is that the peak response might grow as ω_k gets close to 0. Hint: sketch the locations of the poles for various values of ω_k with ρ_k close to 1.*

3.3 Frequency Responses

Ignoring the gain factor, A, every realizable digital filter is completely described by its poles and zeros, of which there are a finite number. The frequency response, $\hat{h}(\omega)$, may be readily calculated by substituting $e^{j\omega}$ for z, and we may deduce many useful properties concerning this frequency response by examining the structure of the poles and zeros. Some of the more interesting properties are examined in the next two sections.

Example 3 *Consider the first order all-zero filter,*

$$H(z) = \frac{z - z_0}{z}$$

The frequency response at frequency ω is given by

$$\hat{h}(\omega) = H\left(e^{j\omega}\right) = \frac{e^{j\omega} - z_0}{e^{j\omega} - 0} = \frac{v_1(\omega)}{v_2(\omega)}$$

where $v_1(\omega)$ and $v_2(\omega)$ are the two phasors shown in Figure 1. Thus, the magnitude response, $\left|\hat{h}(\omega)\right| = |v_1(\omega)| / |v_2(\omega)|$, is the ratio of the distance from z_0 to $e^{j\omega}$ and the distance from 0 to $e^{j\omega}$. Similarly, the phase response, $\angle\hat{h}(\omega) = \angle v_1(\omega) - \angle v_2(\omega)$, is the difference between the angle of the phasor from z_0 to $e^{j\omega}$ and the angle of the phasor from 0 to $e^{j\omega}$.

Example 4 *Consider the second order all-pole filter,*

$$H(z) = \frac{z^2}{z^2 - 2\rho_0 z \cos\omega_0 + \rho_0^2} = \frac{z^2}{(z - \rho_0 e^{j\omega_0})(z - \rho_0 e^{-j\omega_0})}$$

The pole-zero plot is shown in Figure 2. The frequency response is given by

$$\hat{h}(\omega) = \frac{\left(e^{j\omega} - 0\right)^2}{(e^{j\omega} - \rho_0 e^{j\omega_0})(e^{j\omega} - \rho_0 e^{-j\omega_0})} = \frac{v_1^2(\omega)}{v_2(\omega)\, v_3(\omega)}$$

where v_1, v_2 and v_3 are the phasors shown in the figure.

4 Stability of Recursive Filters

4.1 BIBO Stability

We say that a system is BIBO (Bounded Input, Bounded Output) stable if it produces a bounded output sequence, in response to every bounded input

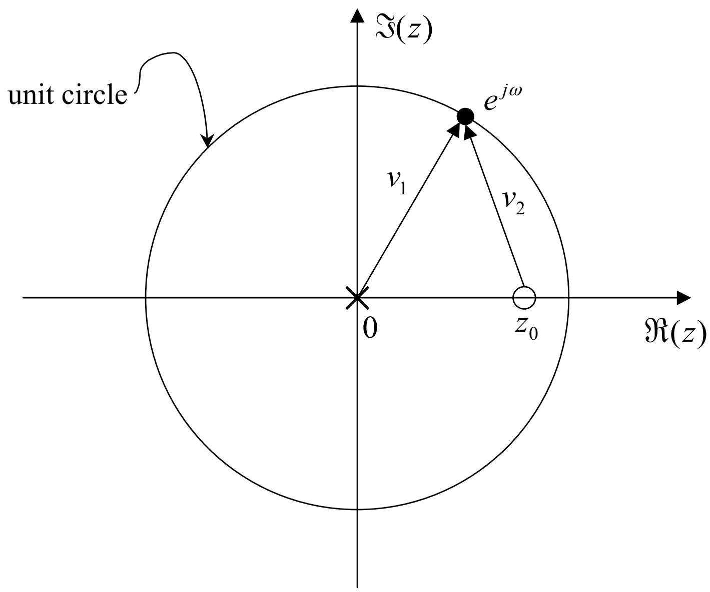

Figure 1: Pole-zero plot for a first order all-zero filter.

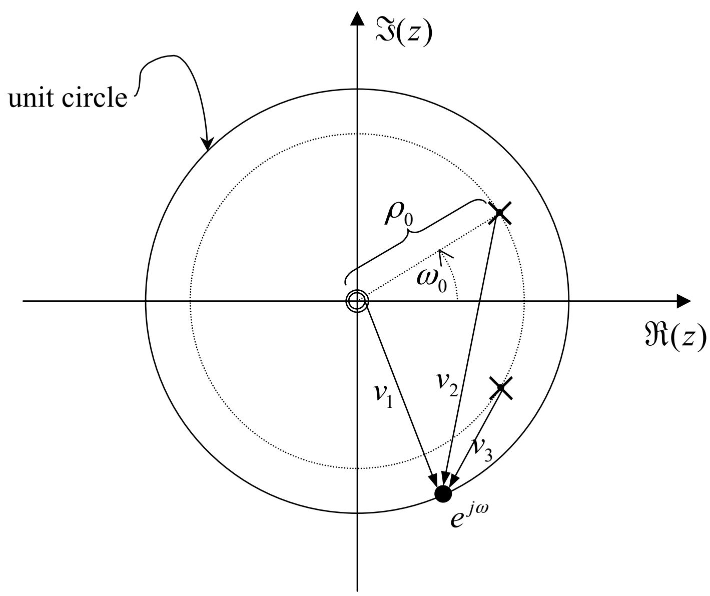

Figure 2: Pole-zero plot of a second order all-pole filter.

sequence. To be more precise, let $x[n]$ denote an input sequence, whose samples are bounded according to

$$|x[n]| \leq B_x, \quad \forall n$$

where B_x is some bound. All practical signals which we will encounter are bounded, because we must acquire them using physical circuits, which cannot accept voltages (or currents) in excess of some implementation-dependent bound. Let $y[n]$ denote the output sequence produced by the system, so that $\mathbf{y} = H(\mathbf{x})$. The operator, $H()$, is BIBO stable so long as there exists a finite bound B_y, for every bound B_x, such that

$$|y[n]| \leq B_y, \quad \forall n$$

When the system under consideration is a filter, the output bound may be found very easily from

$$B_y = B_x \sum_{n=0}^{\infty} |h[n]|$$

To see this, note that this B_y does indeed bound the magnitude of $y[n]$, observe that

$$\begin{aligned} |y[n]| &= \left| \sum_{k=0}^{\infty} h[k]\, x[n-k] \right| \\ &\leq \sum_{k=0}^{\infty} |h[k]\, x[n-k]| \\ &= \sum_{k=0}^{\infty} |h[k]| \cdot |x[n-k]| \\ &\leq B_x \cdot \sum |h[k]| = B_y \end{aligned}$$

To see that the bound, B_y, is tight, choose $x[n]$ to be the causal sequence

$$x[n] = B_x \operatorname{sign}(h[N-n])\, u[n]$$

where N is any positive integer. The output at location N is then given by

$$\begin{aligned} y[N] &= \sum_{k=0}^{N} h[k]\, x[N-k] \\ &= \sum_{k=0}^{N} h[k] \operatorname{sign}(h[k])\, B_x \\ &= B_x \sum_{k=0}^{N} |h[k]| \longrightarrow_{N\to\infty} B_y \end{aligned}$$

Since N is arbitrary, it is always possible to find a causal sequence, $x[n]$, which will produce at least one output, $y[N]$, which approaches B_y arbitrarily closely.

Summary 4 *We conclude that a filter is BIBO stable if and only if*

$$\sum_{n=0}^{\infty} |h[n]| < \infty \tag{6}$$

The value of the left hand side in this expression is known as the filter's BIBO gain.

4.2 Pole Placement

FIR filters are inherently stable because only finitely many terms contribute to the sum in equation (6). Recursive filters, however, may be unstable. Now recall that a filter may be decomposed into a finite cascade of first and second order filters. The entire filter will be stable if and only if each element in this cascade is stable. To see this, simply observe that the BIBO gain of a cascade of stable operators cannot be larger than the product of their individual BIBO gains. It is thus sufficient for us to consider the stability of first and second order all-pole filters, having transfer functions

$$\frac{z}{z-p_0} \quad \text{and} \quad \frac{z^2}{z^2 - 2z\rho_0 \cos\omega_0 + \rho_0^2}$$

In Section 3.1, we found that the impulse response of the first order all-pole filter is $h[n] = p_0^n$, from which we immediately conclude that the filter is BIBO stable if and only if $|p_0| < 1$.

In Section 3.2, we found that the impulse responses of the second order all-pole filter has an envelope of the form ρ_0^n, from which we again conclude that the filter is BIBO stable if and only if $|\rho_0| = |p_0| < 1$.

Summary 5 *The first and second order all-pole filters are each BIBO stable if and only if their poles lie strictly inside the unit circle (magnitude less than* 1*). Since all filters may be composed of first and second order all-pole and all-zero sections, a filter is BIBO stable if and only if all of its poles lie strictly inside the unit circle.*

If an LTI system has one or more poles on the unit circle, it is sometimes called "conditionally stable". As we shall see, this is a desirable condition in the construction of digital oscillators. Note, however, that such a system cannot be called BIBO stable, since it has an infinite BIBO gain.

4.3 The Stability Triangle

Consider a two pole system with transfer function

$$H(z) = \frac{z^2}{z^2 + bz + c} \tag{7}$$

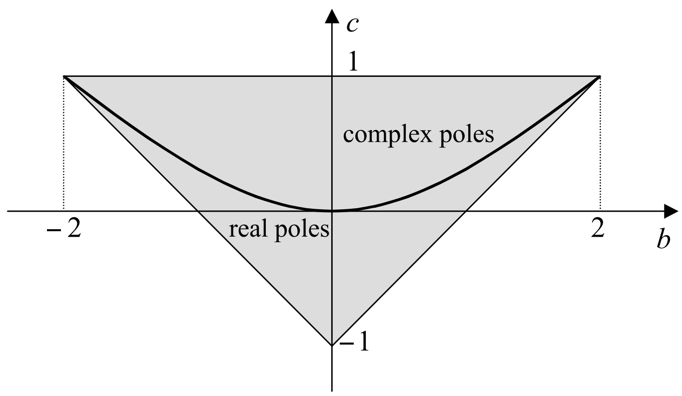

Figure 3: Stability triangle.

where b and c are coefficients. The system has two poles,

$$p_1, p_2 = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

The system is BIBO stable if the poles both lie inside the unit circle. We may consider two cases:

Complex Poles ($|b| < 2\sqrt{c}$): Comparing equation (7) with equation (5), we may immediately identify c with ρ, the magnitude of the poles. So the system is stable only if $c < 1$.

Real Poles ($|b| \geq 2\sqrt{c}$): The system is stable if and only if

$$\left|\frac{b}{2}\right| + \frac{\sqrt{b^2 - 4c}}{2} < 1$$

That is,

$$2 - |b| > \sqrt{|b|^2 - 4c}$$

which clearly requires $|b| \leq 2$ and, squaring both sides,

$$\begin{aligned} 4 - 4|b| + |b|^2 &> |b|^2 - 4c \\ \Longrightarrow |b| - c &< 1 \end{aligned}$$

These conditions may be conveniently represented in the stability diagram shown in Figure 3. Evidently, a stable second order system must have (b, c) lying inside the triangle having vertices at $(-2, 1)$, $(2, 1)$ and $(0, -1)$. Points on the parabola, $|b| = 2\sqrt{c}$, result in identical real-valued poles, while points above the parabola result in complex conjugate poles.

Exercise 2 *Determine by inspection, whether or not each of the following systems is stable.*

$$\frac{z^2}{z^2+2z+3}, \quad \frac{\left(1+4z^2\right)}{\left(3z^2-1\right)}$$

5 Useful Properties of Filters

5.1 Group Delay

Causal filters inevitably involve some kind of delay, since the system response extends from the point of excitation out into the future (not the past). However, not all signals will experience the same delay passing through the filter.

Consider a real-valued signal, $\mathbf{x} \equiv x(t)$. Delaying this signal by an amount, σ, yields $\mathbf{x}_\sigma = x(t-\sigma)$, where the Fourier transforms of $\mathbf{x}$ and $\mathbf{x}_\sigma$ are related by

$$\hat{x}_\sigma(\omega) = e^{-j\sigma\omega}\hat{x}(\omega)$$

So a pure delay by any amount σ is equivalent to filtering $x(t)$ by a filter having the transfer function, $\hat{h}(\omega) = e^{-j\sigma\omega}$. Such a filter is said to have a linear phase response, since the phase, $\angle\hat{h}(\omega) = -\sigma\omega$, is a linear function of ω. The pure delay filter also has unit magnitude.

Now recall that discrete filtering of sampled signals is equivalent to filtering the underlying continuous-time signals which they represent, so long as the continuous-time signal is Nyquist bandlimited. Thus, a digital filter having the transfer function $\hat{h}(\omega) = e^{-j\sigma\omega}$ implements a perfect delay by σ, regardless of whether σ is an integer or not.

More generally, the effect of a digital filter on narrowband signals, with frequencies in the neighbourhood of ω, may be described in terms of two quantities: gain; and delay. The gain is given by $\left|\hat{h}(\omega)\right|$, while the delay (known as the "group delay") is given by

$$t_g(\omega) = -\frac{\partial\theta(\omega)}{\partial\omega}$$

where $\theta(\omega) = \angle\hat{h}(\omega)$ is the phase response of the filter.

5.2 Linear Phase

A linear phase filter is one for which the group delay is constant. If we regard an input signal as a sum of narrowband components, the fact that the group delay is constant means each of the constituent components is delayed by exactly the same amount. Linear phase is important for applications which are sensitive to the relative arrival times of different frequency components. Broad band communications systems (e.g., CDMA systems) are a good example in this regard.

The transfer function of a linear phase filter may be given as

$$\hat{h}(\omega) = \left|\hat{h}(\omega)\right| e^{-j\sigma\omega}$$

where $\sigma = t_g$ is the constant group delay. Noting that $\left|\hat{h}(\omega)\right|$ is an even function of ω, we may deduce that linear phase filters satisfy

$$\hat{h}(-\omega) = \left|\hat{h}(\omega)\right| e^{j\sigma\omega} = \hat{h}(\omega) e^{j2\sigma\omega}$$

Now $\hat{h}(\omega) = H(z)|_{z=e^{j\omega}}$, so $\hat{h}(-\omega) = H\left(z^{-1}\right)\big|_{z=e^{j\omega}}$. Thus, writing the above equation in the Z-transform domain yields

$$H\left(z^{-1}\right) = H(z)\, z^{2\sigma} \tag{8}$$

For a linear phase digital filter to be realizable, $z^{2\sigma}$ must be an integer power of z, meaning that the group delay, σ, must be an integer multiple of $\frac{1}{2}$. In particular, letting $2\sigma = N$, equation (8) becomes

$$H\left(z^{-1}\right) = z^N H(z) \tag{9}$$

We may draw the following conclusions:

1. The non-trivial zeros and poles of a linear phase filter must appear in reciprocal pairs. That is, for each zero at $z_k \neq 0$, there must be another zero at $\frac{1}{z_k}$ and for each pole at $p_k \neq 0$, there must be another pole at $\frac{1}{p_k}$.
2. Realizable linear phase filters may not have any non-trivial poles – i.e., they must be FIR. The reason for this is that any non-trivial pole inside the unit circle must be matched by a reciprocal pole which is necessarily outside the unit circle, rendering the filter unstable.
3. Linear phase filters must be symmetric. To see this, observe that since realizable linear phase filters are FIR, their transfer function can be written as

   $$H(z) = a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}$$

   Combining this with equation (9), we get

   $$H(z) = z^{-N} H\left(z^{-1}\right) = a_0 z^{-N} + a_1 z^{1-N} + \cdots + a_M z^{M-N}$$

   Don't get bogged down in the equations here; all we have done is flip $H(z)$ around and then delay it by N. By assumption, $a_M \neq 0$, so the term $a_M z^{M-N}$ must be non-zero and correspond to one of the terms in the causal $H(z)$. It follows that we must have $M \leq N$. If $a_0 \neq 0$ then the term $a_0 z^{-N}$ must also be one of the terms found in $H(z)$ so $N \leq M$, which means that $N = M$ and $a_n = a_{N-n}$ for all n. More generally, one can have some number of leading terms a_0 throuh a_L from $H(z)$ equal to 0. In this case, we must have $N = M + L = 2\sigma$, where L is the number of leading zeroes in $H(z)$. We will not normally bother considering this special case, since a filter with L leading zeros is just a delayed version (delayed by L) of a filter with no leading zeroes.

4. The transfer function of a linear phase filter can be written as

$$H(z) = A(z)\,A\left(z^{-1}\right)U(z)$$

where $A(z)$ contains all zeroes that lie inside the unit circle, as well as one zero from each pair of complex conjugate zeroes that lie on the unit circle. $A\left(z^{-1}\right)$ accounts for all zeroes that lie outside the unit circle, along with the reciprocal pairs of each zero in $A(z)$ that lies on the unit circle. The term $U(z)$ serves to account for zeroes that 1 and -1, which are their own reciprocals.

5. If $N = 2K$ is even, $U(z)$ has an even order and so can be empty. Such filters can have non-zero gain everywhere on the unit circle – i.e., non-zero $\hat{h}(\omega)$ for all ω. The filter must have $2K + 1$ coefficients, symmetrically arranged as follows:

$$a_0, a_1, \ldots, a_{K-1}, a_K, a_{K-1}, \ldots, a_1, a_0$$

6. If $N = 2K + 1$ is odd, the group delay is an odd multiple of $\frac{1}{2}$, and $U(z)$ is an odd degree polynomial that cannot be empty. That is, $U(z)$ must be zero at least at $z = 1$ or $z = -1$, so that $\hat{h}(\omega)$ must be 0 at DC or at the Nyquist frequency $\omega = \pi$. In fact, $H(z)$ must have a zero at -1 (zero gain at the Nyquist frequency). To see this, note that FIR linear phase filters with odd order have an even number of coefficients:

$$a_0, a_1, \ldots, a_K, a_K, \ldots, a_1, a_0$$

It follows that

$$\begin{aligned} H(-1) &= \sum_{k=0}^{K} a_k (-1)^k + \sum_{k=0}^{K} a_k (-1)^{N-k} \\ &= \underbrace{\left(1 + (-1)^N\right)}_{0,\ \text{since } N \text{ odd}} \cdot \sum_{k=0}^{K} a_k = 0 \end{aligned}$$

Before concluding this section, we note that our treatment of linear phase filters has focussed exclusively on filters for which $H(z) = z^{-N} H\left(z^{-1}\right)$, or in the continuous domain on filters for which $h(t) = h(2\sigma - t)$. These are filters whose impulse response is symmetric about the delay σ, which are then delayed versions of zero phase filters. We have not discussed anti-symmetric filters that are also usually considered under the linear phase category. Anti-symmetric filters have $H(z) = -z^{-n} H\left(z^{-1}\right)$ or $h(t) = -h(2\sigma - t)$; they do not correspond to delayed zero phase filters, but the phase of $\hat{h}(\omega)$ still has constant derivative $-\sigma$ everywhere except at $\omega = 0$ where the phase of $\hat{h}(\omega)$ is discontinuous. Anti-symmetric filters of odd order $N = 2K + 1$, have an even number of anti-symmetric coefficients:

$$a_0, a_1, \ldots, a_K, -a_K, \ldots, -a_1, -a_0$$

and so must have DC gain of 0. Anti-symmetric filters of even order have an odd number of anti-symmetric coefficients, whose central coefficient must be 0:

$$a_0, a_1, \ldots, a_{K-1}, 0, -a_{K-1}, \ldots, -a_1, -a_0$$

It follows that these also have DC gain of 0. Thus $\hat{h}(\omega)$ must be 0 at $\omega = 0$, which is where the phase of $\hat{h}(\omega)$ changes discontinuously. We will not have much cause to consider anti-symmetric filters in this course.

5.3 All Pass

An "all-pass" filter is one whose magnitude response is unity, i.e.,

$$\left|\hat{h}(\omega)\right| = 1, \quad \forall\omega$$

To understand all-pass filters in the Z-transform domain, we begin by observing that

$$1 = \left|\hat{h}(\omega)\right|^2 = \hat{h}(\omega)\,\hat{h}^*(\omega)$$

where * denotes complex conjugation, as usual. But real-valued signals have Fourier transforms which satisfy

$$\hat{h}^*(\omega) = \hat{h}(-\omega)$$

from which we get

$$1 = \hat{h}(\omega)\,\hat{h}(-\omega)$$

and in the Z-transform domain, this becomes

$$H(z)\,H\left(z^{-1}\right) = 1$$

The mapping of $\left|\hat{h}(\omega)\right|^2$ to $H(z)\,H\left(z^{-1}\right)$ is an extremely useful relationship which we will have cause to use many times throughout this subject.

Summary 6 *An all-pass filter satisfies the following relationship*

$$H(z) = \frac{1}{H\left(z^{-1}\right)}$$

This means that every pole, p_k, is matched by a reciprocal zero, $z_k = \frac{1}{p_k}$, and vice-versa. This, in turn, means that non-trivial all-pass filters must have both poles and zeros, so they must have infinite impulse responses. The general form of an all-pass filter is

$$H(z) = \frac{\left(z - \frac{1}{p_1}\right)\left(z - \frac{1}{p_2}\right)\cdots\left(z - \frac{1}{p_N}\right)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

5.4 Minimum Phase

A particularly useful class of filter are the so-called "minimum phase" filters. We will begin by restricting our attention to FIR filters. The general form of a causal linear phase FIR filter's transfer function is

$$H(z) = \frac{z - z_1}{z} \cdot \frac{z - z_2}{z} \cdot \cdots \cdot \frac{z - z_M}{z} \tag{10}$$

and the phase response of such a filter may be found as the sum of the phase responses from each first order section,

$$\angle\hat{h}(\omega) = \angle\hat{h}_1(\omega) + \angle\hat{h}_2(\omega) + \cdots + \angle\hat{h}_M(\omega)$$

where

$$\hat{h}_k(\omega) = \frac{e^{j\omega} - z_k}{e^{j\omega} - 0}$$

Note that z_k may be real or complex-valued.

Exercise 3 *Assuming that z_k is real-valued, in the range $0 < z_k < 1$, convince yourself by drawing pictures or otherwise that as ω travels from 0 to π, the phase of $\hat{h}_k(\omega)$ is always non-negative, increasing from 0 to a maximum value, and then decreasing again back to 0. Sketch the phase as a function of ω.*

Now letting $-1 < z_k < 0$, convince yourself that as ω travels from 0 to π, the phase of $\hat{h}_k(\omega)$ is non-positive, decreasing from 0 to a minimum value, and then returning to 0. Sketch the phase as a function of ω.

Now letting $z_k > 1$, convince yourself that the phase starts at π and decreases monotonically to 0. Sketch the phase as a function of ω.

Finally, letting $z_k < -1$, convince yourself that the phase starts at 0, decreasing monotonically to $-\pi$. Sketch the phase as a function of ω.

The above exercises reveal the fact that filters whose zeros are inside the unit circle have phases which start and end at 0, while filters with zeros outside the unit circle have phases which decrease monotonically. This may be seen as a consequence of the "Argument Principle", according to which the total phase change experienced as z follows a closed trajectory is equal to $2\pi(\#Z - \#P)$, where $\#Z$ and $\#P$ are the number of zeros and poles, respectively, enclosed by the trajectory.

If $|z_k| < 1$, it can be shown that the group delay, $t_g(\omega)$, of the filter

$$\frac{z - z_k}{z}$$

is always smaller than the group delay of the filter

$$\frac{z - \frac{1}{z_k}}{z}$$

Both of these filters, however, have the same magnitude response (with reciprocal DC gains), which may be deduced easily from the fact that

$$\frac{z - z_k}{z - \frac{1}{z_k}}$$

is an all-pass filter.

In general, given any FIR filter of the form in equation (10), one may construct up to 2^M different filters all having exactly the same magnitude response, by selectively replacing zeros by their reciprocals. If the filter has complex zeros, complex-conjugate pairs must be reciprocated together, so there may be only $2^{M/2}$ different filters with the same magnitude response, having real-valued coefficients. Amongst these different filters, the one with minimum delay is that which has all of its zeros inside the unit circle. Such a filter is said to be "minimum phase."

Summary 7 *A filter is said to be minimum phase if all of its zeros lie inside the unit circle. Amongst all filters with the same magnitude response, only one will have minimum phase and this will have the lowest group delay.*

The above property extends from FIR filters to digital filters in general. Minimum phase is a property only of the filter's zeros, not its poles. An obvious benefit of the minimum phase property is that signals are delayed as little as possible as they travel through the filter. Another important benefit, however, is that minimum phase filters can be inverted. Writing

$$H(z) = \frac{A(z)}{B(z)}$$

where $A(z)$ and $B(z)$ are finite polynomials, the inverse filter is

$$\frac{1}{H(z)} = \frac{B(z)}{A(z)}$$

Its poles are the zeros of $H(z)$ and its zeros are the poles of $H(z)$. The inverse filter will be BIBO stable if and only if all of its poles lie inside the unit circle, for which we require $H(z)$ to be minimum phase.

6 Digital Oscillators

We conclude this chapter by considering digital oscillators. In Section 3.2, we saw that an all-pole filter with a pair of complex-valued poles,

$$p_1, p_2 = \rho_0 e^{\pm j\omega_0}$$

has transfer function,

$$H(z) = \frac{z^2}{z^2 - 2z\rho_0 \cos\omega_0 + \rho_0^2}$$

and impulse response

$$h[n] = u[n]\frac{1}{\sin\omega_0}\rho_0^n \sin(n\omega_0 + \omega_0)$$

This impulse response has the form of an exponentially decaying sinusoidal waveform, with frequency $f = \frac{\omega_0}{2\pi}$ cycles per sampling interval.

If we move the poles onto the unit circle so that $\rho_0^n = 1$ and $p_1, p_2 = e^{\pm j\omega_0}$, a single impulse of amplitude $\sin\omega_0$ at time $n = 0$ will excite the system to produce an oscillatory output waveform,

$$\begin{aligned} y[n] &= u[n]\sin(n\omega_0 + \omega_0) \\ &= u[n]\cos\left(n\omega_0 + \omega_0 - \frac{\pi}{2}\right) \end{aligned} \tag{11}$$

which has frequency $\frac{\omega_0}{2\pi}$ and a phase offset of $\theta_0 = \omega_0 - \frac{\pi}{2}$.

The recursive system may be found by writing

$$\begin{gathered} H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2z^{-1}\cos\omega_0 + z^{-2}} \\ \Longrightarrow Y(z) = X(z) + 2\cos\omega_0 \cdot z^{-1}Y(z) - z^{-2}Y(z) \\ \Longrightarrow y[n] = x[n] + 2\cos\omega_0 \cdot y[n-1] - y[n-2] \end{gathered}$$

Since the impulse response in equation (11) is obtained with zero initial conditions, $y[n] = 0$ for $n < 0$, setting $x[n] = \sin\omega_0 \cdot \delta[n]$, the system may be implemented as follows:

- Set $y[0] = \sin\omega_0$
- For each $n > 0$, find $y[n]$ from the previously calculated values according to
$$y[n] = (2\cos\omega_0) \cdot y[n-1] - y[n-2] \tag{12}$$

The complexity of the digital oscillator is one multiplication by $2\cos\omega_0$ (a constant, determined by the frequency), and one subtraction, per output sample. For reference, we should consider the most naive method of constructing a digital oscillator. Namely, we could simply set

$$y[n] = \cos(\omega_0 n + \theta_0)$$

where θ_0 is a fixed phase offset of choice. For each output sample, this would require the following computations:

- computation of $\omega_0 n$, which can be done by adding ω_0 to the previously calculated value, $\omega_0(n-1)$;
- restrict the resulting phase to a practically useful range, say 0 to 2π, which requires a comparison operation and an optional subtraction of 2π;

- either direct evaluation of the trigonometric function, $\cos()$, or (more practically), quantize $\omega_0 n$ and use it to address a lookup table.

For hardware implementations, the cost of implementing a sizeable lookup table can be much higher than that of a multiplication. If high accuracy is required, the cost of numerically evaluating a trigonometric function is vastly higher than that of performing a single multiplication. For these reasons, equation (12) can be a very attractive means of implementing a digital oscillator.

Exercise 4 *Show that it is possible to control the initial phase of the digital oscillator by driving it with an input, $x[n]$, having non-zero values at $n = 0$ and $n = -1$, rather than just $n = 0$. In particular, find values of $x[0]$ and $x[-1]$ such that the phase offset, θ_0, is 0. Show that this is equivalent to driving the filter with an impulse at $n = 0$, but adopting non-zero initial conditions.*

Digital oscillators are key elements in the implementation of communication systems. Digital oscillators are also widely used in the construction of frequency synthesizers, which use a digital circuit operating with a fixed clock frequency to synthesize a new analog or digital clock reference whose frequency can be controlled arbitrarily.