Mechanism Design and Approximation

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Contents

Author's Note

This text is suitable for advanced undergraduate or graduate courses; it has been developed at Northwestern U. as the primary text for such a course since 2008.

This text provides a look at select topics in economic mechanism design through the lens of approximation. It reviews the classical economic theory of mechanism design wherein a Bayesian designer looks to find the mechanism with optimal performance in expectation over the distribution from which the preferences of the participants are drawn. It then adds to this theory practical constraints such as simplicity, tractability, and robustness. The central question addressed is whether these practical mechanisms are good approximations of the optimal ones. The resulting theory of approximation in mechanism design is based on results that come mostly from the theoretical computer science literature. The results presented are the ones that are most directly compatible with the classical (Bayesian) economic theory and are not representative of the entirety of the literature.

– Jason D. Hartline

1
Mechanism Design and Approximation

Our world is an interconnected collection of economic and computational systems. Within such a system, individuals optimize their actions to achieve their own, perhaps selfish, goals; and the system combines these actions with its basic laws to produce an outcome. Some of these systems perform well, e.g., the national residency matching program which assigns medical students to residency programs in hospitals, e.g., auctions for online advertising on Internet search engines; and some of these systems perform poorly, e.g., financial markets during the 2008 meltdown, e.g., gridlocked transportation networks. The success and failure of these systems depends on the basic laws governing the system. Financial regulation can prevent disastrous market meltdowns, congestion protocols can prevent gridlock in transportation networks, and market and auction design can lead to mechanisms for allocating and exchanging goods or services that yield higher profits or increased value to society.

The two sources for economic considerations are the preferences of individuals and the performance of the system. For instance, bidders in an auction would like to maximize their gains from buying; whereas, the performance of the system could (i.e., from the perspective of the seller) be measured in terms of the revenue it generates. Likewise, the two sources for computational considerations are the individuals who must optimize their strategies, and the system which must enforce its governing rules. For instance, bidders in the auction must figure out how to bid, and the auctioneer must calculate the winner and payments from the bids received. While these calculations may seem easy when auctioning a painting, they both become quite challenging when, e.g., the Federal Communications Commission (FCC) auctions cell phone spectrum for which individual lots have a high degree of complementarities.

These economic and computational systems are complex. The space of individual strategies is complex and the space of possible rules for the system is complex. Optimizing among strategies or system rules in complex environments should lead to complex strategies and system rules, yet the individuals' strategies or system rules that are successful in practice are often remarkably simple. This simplicity may be a consequence of individuals and designers preference for ease of understanding and optimization (i.e., tractability) or robustness to variations in the scenario, especially when these desiderata do not significantly sacrifice performance.

This text focuses on a combined computational and economic theory for the study and design of mechanisms. A central theme will be the tradeoff between optimality and other desirable properties such as simplicity, robustness, computational tractability, and practicality. This tradeoff will be quantified by a theory of approximation which measures the loss of performance of a simple, robust, and practical approximation mechanism in comparison to the complicated and delicate optimal mechanism. The theory provided does not necessarily suggest mechanisms that should be deployed in practice, instead, it pinpoints salient features of good mechanisms that should be a starting point for the practitioner.

In this chapter we will explore mechanism design for routing and congestion control in computer networks as an example. Our study of this example will motivate a number of questions that will be addressed in subsequent chapters of the text. We will conclude the chapter with a formal discussion of approximation and the philosophy that underpins its relevance to the theory of mechanism design.

1.1 An Example: Congestion Control and Routing in Computer Networks

We will discuss novel mechanisms for congestion control and routing in computer networks to give a preliminary illustration of the interplay between strategic incentives, approximation, and computation in mechanism design. In this discussion, we will introduce basic questions that will be answered in the subsequent chapters of this text.

Consider a hypothetical computer network where network users reside at computers and these computers are connected together through a network of routers. Any pair of routers in this network may be connected by a network link and if such a network link exists then each router can route a message directly through the other router. We will assume that the network is completely connected, i.e., there is a path of network links between all pairs of users. The network links have limited capacity; meaning, at most a fixed number of messages can be sent across the link in any given interval of time. Given this limited capacity the network links are a resource that may be over demanded. To enable the sending of messages between users in the network we will need mechanisms for *congestion control*, i.e., determining which messages to route when a network link is over-demanded, and *routing*, i.e., determining which path in the network each message should take.

There are many complex aspects of this congestion control problem: dynamic demands, complex networks, and strategic user behavior. Let us ignore the first two issues at first and focus on the latter: strategic user behavior. Consider a static version of this routing problem over a single network link with unit capacity: each user wishes to send a message across the link, but the link only has capacity for one message. How shall the routing protocol select which message to route?

That the resource that the users (henceforth: agents) are vying for is a network link is not important; we will therefore cast the problem as a more general single-item resource allocation problem. An implicit assumption in this problem is that it is better to allocate the item to some agents over others. For instance, we can model the agents as having value that each gains for receiving the item and it would be better if the item went to an agent that valued it highly.

Definition 1.1 The *single-item allocation* problem is given by

- a single indivisible *item* available,
- n strategic *agents* competing for the item, and
- each agent i has a *value* v_i for receiving the item.

The objective is to maximize the *social surplus*, i.e., the value of the agent who receives the item.

The social surplus is maximized if the item is allocated to the agent with the highest value, denoted $v_{(1)}$. If the values of the agent are publicly known, this would be a simple allocation protocol to implement. Of course, e.g., in our routing application, it is rather unlikely that values are publicly known. A more likely situation is that each agent's value is known privately to that agent and unknown to all other parties. A mechanism that wants to make use of this private information must then solicit it. Consider the following mechanism as a first attempt at a single-item allocation mechanism:

(i) Ask the agents to report their values ($\Rightarrow$ agent i reports b_i),
(ii) select the agent $i^\star$ with highest report ($\Rightarrow i^\star = \operatorname{argmax}_i b_i$), and
(iii) Allocate the item to agent $i^\star$.

Suppose you were one of the agents and your value was $10 for the item; how would you bid? What should we expect to happen if we ran this mechanism? It should be pretty clear that there is no reason your bid should be at all related to your value; in fact, you should always bid the highest number you can think of. The winner is the agent who thinks of and reports the highest number. The unpredictability of the outcome of the mechanism will make it hard to reason about its performance. There are two natural ways to try to address this unpredictability. First, we can accept that the bids are meaningless, ignore them (or not even solicit them), and pick the winner randomly. Second, we could attempt to penalize the agents for bidding a high amount, for instance, with a monetary payment.

Definition 1.2 The *lottery mechanism* is:

(i) select a uniformly random agent, and
(ii) allocate the item to this agent.

The *social surplus* of a mechanism is total value it generates. In this routing example the social surplus is the value of the message routed. It is easy to calculate the expected surplus of the lottery. It is $1/n \sum_i v_i$. This surplus is a bit disappointing in contrast to the surplus available in the case where the values of the messages were publicly known, i.e., $v_{(1)} = \max_i v_i$. In fact, by setting $v_1 = 1$; $v_i = \epsilon$ (for $i \neq 1$); and letting ϵ go to zero we can observe that the surplus of the lottery approaches $v_{(1)}/n$; therefore, its worst-case is at best an n approximation to the optimal surplus $v_{(1)}$. Of course, the lottery always obtains at least an nth of $v_{(1)}$; therefore, its worst-case approximation factor is exactly n. It is fairly easy to observe, though we will not discuss the details here, that this approximation factor is the best possible by any mechanism without payments.

Theorem 1.1 *The surplus of the lottery mechanism is an n approximation to the highest agent value.*

If instead it is possible to charge payments, such payments, if made proportionally to the agents' bids, could discourage low-valued agents from making high bids. This sort of dynamic allocation and pricing mechanism is referred to as an *auction*.

Definition 1.3 A *Single-item auction* is a solution to the single-item allocation problem that solicits bids, picks a winner, and determines payments.

A natural allocation and pricing rule that is used, e.g., in government procurement auctions, is the *first-price auction*.

Definition 1.4 The *first-price auction* is:

(i) ask agents to report their values ($\Rightarrow$ agent i reports b_i),
(ii) select the agent $i^\star$ with highest report ($\Rightarrow i^\star = \operatorname{argmax}_i b_i$),
(iii) allocate the item to agent $i^\star$, and
(iv) charge this winning agent her bid, $b_{i^\star}$.

To get some appreciation for the strategic elements of the first price auction, note that an agent who wins wants to pay as little as possible, thus bidding a low amount is desirable. Of course, if the agent bids too low, then she probably will not win. Strategically, this agent must figure out how to balance this tradeoff. Of course, since agents may not report their true values, the agent with the highest bid may not be the agent with the highest-valued message. See Figure 1.1.

We will be able to analyze the first-price auction and we will do so in Chapter 2. However, for two reasons, there is little hope of generalizing it beyond the single-network-link special case (i.e., to large asymmetric computer networks) which is our eventual goal. First, calculating equilibrium strategies in general asymmetric environments is not easy; consequently, there would be little reason to believe that agents would play by the equilibrium. Second, it would be a challenge to show that the equilibrium is any good. Therefore, we turn to auctions that are strategically simpler.

The ascending-price auction is a stylized version of the auction popularized by Hollywood movies; art, antiques, and estate-sale auction houses such as Sotheby's and Christie's; and Internet auction houses such as eBay.

Definition 1.5 The *ascending-price auction* is:[1]

[1] The ascending-price auction is also referred to as the *English auction* and it contrasts to the Dutch (descending-price) auction.

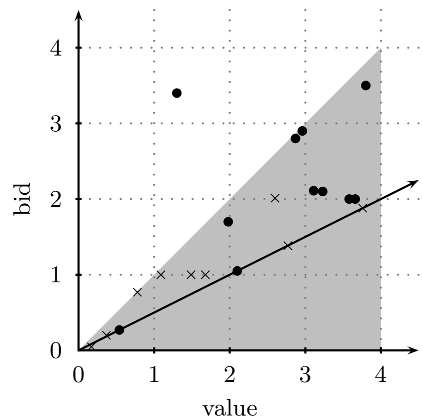


Figure 1.1 An in-class experiment: 21 student were endowed with values uniformly drawn from the interval $[0, 4]$ (denoted as $v_i \sim U[0, 4]$), they were told their own values and that the distribution of values was $U[0, 4]$, they were asked to submit bids for a two-agent one-item first-price auction. The bids of the students were collected and randomly paired for each auction; the winner was paid the difference between his value and his bid in dollars (real money). Winning bids are shown as "•" and losing bids are shown as "×". The grey area denotes strategies that are not dominated. The black line $b = v/2$ denotes the equilibrium strategy in theory. In economic experiments, just like our in class experiment, bidders tend to overbid the equilibrium strategy. A few students knew the equilibrium strategy in advance of the in-class experiment.

(i) gradually raise an offered price up from zero,
(ii) allow agents to drop out when they no longer wish to win at the offered price,
(iii) stop at the price where the second-to-last agent drops out, and
(iv) allocate the item to the remaining agent and charges her the stopping price.

Strategically this auction is much simpler than the first-price auction. What should an agent with value v do? A good strategy would be "drop when the price exceeds v." Indeed, regardless of the actions of the other agents, this is a good strategy for the agent to follow, i.e., it is a *dominant strategy*. It is reasonable to assume that an agent with an obvious dominant strategy will follow it.

Since we know how agents are behaving, we can now make conclusions as to what happens in the auction. The second-highest-valued agent will drop out when the ascending prices reaches her value, $v_{(2)}$. The highest-valued agent will win the item at this price. We can conclude that this auction maximizes the *social surplus*, i.e., the sum of the utilities of all parties. Notice that the utility of losers are zero, the utility of the winner is $v_{(1)} - v_{(2)}$, and the utility of the seller (e.g., the router in the congestion control application) is $v_{(2)}$, the payment received from the winner. The total is simply $v_{(1)}$, as the payment occurs once positively (for the seller) and once negatively (for the winner) and these terms cancel. Of course $v_{(1)}$ is the optimal surplus possible; we could not give the item to anyone else and get more value out of it.

Theorem 1.2 *The ascending-price auction maximizes the social surplus in dominant strategy equilibrium.*

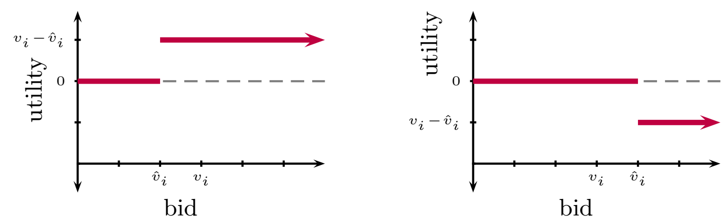
What is striking about this result is that it shows that there is essentially no loss in surplus imposed by the assumption that the agents' values are privately known only to themselves. Of course, we also saw that the same was not true of routing mechanisms that cannot require the winner to make a payment in the form of a monetary *transfer* from the winner to the seller. Recall, the lottery mechanism could be as bad as an n approximation. A conclusion we should take from this exercise is that transfers are very important for surplus maximization when agents have private values.

Unfortunately, despite the good properties of the ascending-price auction there are two drawbacks that will prevent our using it for routing and congestion control in computer networks. First, mechanisms for sending messages in computer networks must be very fast. Ascending auctions are slow and, thus, impractical. Second, the ascending-price auction does not generalize to give a routing mechanisms in networks beyond the single-network-link special case. Challenges arise because ascending prices would not generally find the social surplus maximizing set of messages to route. A solution to these problems comes from Nobel laureate William Vickrey who observed that if we simulate the ascending-price auction with sealed bids we arrive at the same outcome in equilibrium without the need to bother with an ascending price.

Definition 1.6 The *second-price auction* is:[2]

(i) accept sealed bids,
(ii) allocate the item to the agent with the highest bid, and
(iii) charge this winning agent the second-highest bid.

[2] The second-price auction is also referred to as the Vickrey auction.



(a) Case 1: $v_i > \hat{v}_i$ (b) Case 2: $v_i < \hat{v}_i$

Figure 1.2 Utility as a function of bid in the second-price auction.

In order to predict agent behavior in the second-price auction, notice that its outcome can be viewed as a simulation of the ascending-price auction. Via this viewpoint, there is a one-to-one correspondence between bidding b in the second-price auction and dropping out at price b is the ascending-price auction. Since the dominant strategy in the ascending-price auction is for an agent to drop out at when the price exceeds her value; it is similarly a dominant strategy for the agent to bid her true value in the second-price auction. While this intuitive argument can be made formal, instead we will argue directly that *truthful* bidding is a dominant strategy in the second-price auction.

Theorem 1.3 *Truthful bidding is a dominant strategy in the second-price auction.*

Proof We show that truthful bidding is a dominant strategy for agent i. Fix the bids of all other agents and let $\hat{v}_i = \max_{j \neq i} v_j$. Notice that given this $\hat{v}_i$ there are only two possible outcomes for agent i. If she bids $b_i > \hat{v}_i$ then she wins, pays $\hat{v}_i$ (which is the second-highest bid), and has utility $u_i = v_i - \hat{v}_i$. On the other hand, if she bids $b_i < \hat{v}_i$ then she loses, pays nothing, and has utility $u_i = 0$. This analysis allows us to plot the utility of agent i as a function of her bid in two relevant cases, the case that $v_i < \hat{v}_i$ and the case that $v_i > \hat{v}_i$. See Figure 1.2.

Agent i would like to maximize her utility. In Case 1, this is achieved by any bid greater than $\hat{v}_i$. In Case 2, it is achieved by any bid less than $\hat{v}_i$. Notice that in either case bidding $b_i = v_i$ is a good choice. Since the same bid is a good choice regardless of which case we are in, the same bid is good for any $\hat{v}_i$. Thus, bidding truthfully, i.e., $b_i = v_i$, is a dominant strategy. $\square$

Notice that, in the proof of the theorem, $\hat{v}_i$ is the infimum of bids that the bidder can make and still win, and the price charge to such a winning bidder is exactly $\hat{v}_i$. We henceforth refer to $\hat{v}_i$ as agent i's *critical value*. It should be clear that the proof above can be easily generalized, in particular, to any auction where each agent faces such a critical value that is a function only of the other agents' reports. This observation will allow the second-price auction to be generalized beyond single-item environments.

Corollary 1.4 *The second-price auction maximizes the social surplus in dominant strategy equilibrium.*

Proof By the definition of the second-price auction, the agent with the highest bid wins. By Theorem 1.3 is a dominant strategy equilibrium for agents to bid their true values. Thus, in equilibrium the agent with the highest bid is identically the agent with the highest value. The social surplus is maximized. □

In the remainder of this section we explore a number of orthogonal issues related to practical implementations of routing and congestion control. Each of these vignettes will conclude with motivating questions that will be addressed in the subsequent chapters. First, we address the issue of payments. The routing protocol in today's Internet, for instance, does not allow the possibility of monetary payments. How does the routing problem change if we also disallow monetary payments? The second issue we address is speed. While the second-price auction is faster than the ascending-price auction, still the process of soliciting bids, tallying results, and assigning payments may be too cumbersome for a routing mechanism. A simpler posted-pricing mechanism would be faster, but how can we guarantee good performance with a posted pricing? Finally, the single-link case is far from providing a solution to the question of routing and congestion control in general networks. How can we extend the second-price auction to more general environments?

1.1.1 Non-monetary payments

Most Internet mechanisms, including its congestion control mechanisms, do not currently permit monetary transfers. There are historical, social, and infrastructural reasons for this. The Internet was initially developed as a research platform and its users were largely altruistic. Since its development, the social norm is for Internet resources and services to be free and unbiased. Indeed, the "net neutrality" debates of the early

2000's were largely on whether to allow differentiated service in routers based on the identity of the source or destination of messages (and based on contracts that presumably would involve payments). Finally, micropayments in the Internet would require financial infrastructure which is currently unavailable at reasonable monetary and computational overhead.

One solution that has been considered, and implemented (but not widely adopted) for similar resource allocation tasks (e.g., filtering unsolicited electronic mail, a.k.a., spam) is *computational payments* such as "proofs of work." With such a system an agent could "prove" that her message was high-valued by having her computer perform a large, verifiable, but otherwise, worthless computational task. Importantly, unlike monetary payments, computational payments would not represent utility transferred from the winner to the router. Instead, computational payments are utility lost to society.

The *residual surplus* of a mechanism with computational payments is the total value generated less any payments made. The residual surplus for a single-item auction is thus the value of the winner less her payment. For the second-price auction, the residual surplus is $v_{(1)} - v_{(2)}$. For the lottery, the residual surplus is $\frac{1}{n}\sum_i v_i$, which is the same as the surplus as there are no payments.

While the second-price auction maximizes surplus (among all mechanisms) regardless of the values of the agents, for the objective of residual surplus it is clear that neither the second-price auction nor the lottery mechanism is best regardless of agent values. Consider the bad input for the lottery, where $v_1 = 1$ and $v_i = \epsilon$ (for $i \neq 1$). If we let ϵ go to zero, the second-price auction has residual surplus $v_{(1)} = 1$ (which is certainly optimal) and the lottery has expected surplus $1/n$ (which is far from optimal). On the other hand, if we consider the all-ones input, i.e., $v_i = 1$ for all i, then the residual surplus of the second-price auction is $v_{(1)} - v_{(2)} = 0$ (which is far from optimal), whereas the lottery surplus is $v_{(1)} = 1$ (which is clearly optimal). Of course, on the input with $v_1 = v_2 = 1$ and $v_i = \epsilon$ (for $i \geq 3$) both the lottery and the second-price auction have residual surplus far from what we could achieve if the values were publicly known or monetary transfers were allowed.

The underlying fact in the above discussion that separates the objectives of surplus and residual surplus is that for surplus maximization there is a single mechanism that is optimal for any profile of agent values, namely the second-price auction; whereas there is no such mechanism for residual surplus. Since there is no absolute optimal mechanism we must trade-off performance across possible profiles of agent values. There are two ways to do this. The first approach is to assume a distribution over value profiles and then optimize residual surplus in expectation over this distribution. Thus, we might trade off low residual surplus on a rare input for high residual surplus on a likely input. This approach results in a different "optimal mechanism" for different distributions. The second approach begins with the solution to the first approach and asks for a single mechanism that bests approximates the optimal mechanism in worst-case over distributions. This second approach may be especially useful for applications of mechanism design to computer networks because it is not possible to change the routing protocol to accommodate changing traffic workloads.

Question 1.1 In what settings does the second-price auction maximize residual surplus? In what settings does the lottery maximize residual surplus?

Question 1.2 For any given distribution over agent values, what mechanism optimizes residual surplus for the distribution?

Question 1.3 If the optimal mechanism for a distribution is complicated or unnatural, is there a simple or natural mechanism that approximates it?

Question 1.4 In worst-case over distributions of agent values, what single mechanism best approximates the optimal mechanism for the distribution?

1.1.2 Posted Pricing

Consider again the original single-item allocation problem to maximize surplus (with monetary payments). Unfortunately, even a single-round, sealed-bid auction such as the second-price auction may be too complicated and slow for congestion control and routing applications. An even simpler approach would be to just post a take-it-or-leave-it price. Consider the following mechanism.

Definition 1.7 For a given price $\hat{v}$, the *uniform-pricing* mechanism serves the first agent willing to pay $\hat{v}$ (breaking ties in arrival order randomly).

For instance, if we assumed all agents arrive at once and $\hat{v} = 0$ this uniform pricing mechanism is identical to the aforementioned lottery.

Recall that the lottery mechanism is very bad when there are many low-valued agents and a few high-valued agents. The bad example had one agent with value one, and the remaining $n-1$ agents with value ϵ. This uniform-pricing mechanism, however, is more flexible. For instance, for this example we could set $\hat{v} = 2\epsilon$, only the high-valued agent will want to buy, and the surplus would be one. Such a posted-pricing mechanism is very practical and, therefore, especially appropriate for our application to Internet routing.

Of course, the price $\hat{v}$ needs to be chosen well. Fortunately in the routing example where billions of messages are sent every day, it is reasonable to assume that there is some distributional knowledge of the demand. Imagine that the value of each agent i is drawn independently and identically from distribution F. The *cumulative distribution function* for random variable v drawn from distribution $\boldsymbol{F}$ specifies the probability that it is at most z, denoted $F(z) = \mathbf{Pr}_{v\sim F}[v < z]$. For example the uniform distribution on interval $[0, 1]$ is denoted $U[0, 1]$ and its cumulative distribution function is $F(z) = z$.

There is a very natural way to choose $\hat{v}$: mimic the outcome of the second-price auction as much as possible. Notice that with n identically distributed agents, the *ex ante* (meaning: before the values are drawn) probability that any particular agent wins is $1/n$. To mimic the outcome of the second-price auction on any particular agent we could set a price $\hat{v}$ so that the probability that the agent's value is above $\hat{v}$ is exactly $1/n$, this price can be found by inverting the cumulative distribution function as $\hat{v} = F^{-1}(1 - 1/n)$. For the uniform distribution, the solution to this inverse is $\hat{v} = 1 - 1/n$. Unlike the second-price auction, posting a uniform price of $\hat{v}$ may result in no winners (if all agent values are below $\hat{v}$) or an agent other than that with the highest value may win (if there are more than one agents with value above $\hat{v}$).

Theorem 1.5 *For values drawn independently and identically from any distribution F, the uniform pricing of $\hat{v} = F^{-1}(1-1/n)$ is an $e/e-1 \approx$ 1.58 approximation to the optimal social surplus.*

Proof The main idea of this proof is to compare three mechanisms. Let REF denote the second-price auction and its surplus (our reference mechanism). Let APX denote the uniform pricing and its surplus (our approximation mechanism). The second-price auction, REF, optimizes surplus, subject to the *ex post* (meaning: after the mechanism is run) *supply constraint* that at most one agent wins, and chooses to sell to each agent with ex ante probability $1/n$. Consider for comparison a third mechanism UB that maximizes surplus subject to the constraint that each agent is served with ex ante probability at most $1/n$, but has no supply constraint, i.e., UB can serve multiple agents if it so chooses.

The first step in the proof is the simple observation that UB upper bounds REF, i.e., UB $\geq$ REF. This is clear as both mechanisms serve each agent with ex ante probability $1/n$, but REF has an ex post supply constraint whereas UB does not. UB could simulate REF and get the exact same surplus, or it could do something even better. Conclude,

$$\text{UB} \geq \text{REF}\,. \tag{1.1}$$

In fact, UB will do something better than REF. First, observe that UB's optimization is independent between agents. Second, observe that the socially optimal way to serve an agent with ex ante probability $1/n$ is to offer her price $\hat{v} = F^{-1}(1 - 1/n)$. We now wish to calculate UB's *expected surplus*. Let $\mathbf{E}[v \mid v \geq \hat{v}]$ denote the expected value of an agent given that her value v is above the price $\hat{v}$. If we sell to an agent and all we know is that her value is above the price, this quantity is the expected surplus generated. By the choice of price $\hat{v}$, the probability than an agent has a value v that exceeds the price $\hat{v}$ is $\mathbf{Pr}[v \geq \hat{v}] = 1/n$, and when an agent's value is below the price her surplus is zero. Thus, her (total) expected surplus in UB is exactly $\mathbf{E}[v \mid v \geq \hat{v}] \cdot \mathbf{Pr}[v \geq \hat{v}]$. By linearity of expectation, UB's (total) expected surplus is just the sum over the n agents of the surplus of each agent's surplus. Therefore,

$$\begin{aligned} \text{UB} &= n \cdot \mathbf{E}[v \mid v \geq \hat{v}] \cdot \mathbf{Pr}[v \geq \hat{v}] \\ &= \mathbf{E}[v \mid v \geq \hat{v}]\,. \end{aligned} \tag{1.2}$$

Finally, we get a lower bound on APX's surplus that we can relate to REF via its upper bound UB. Recall that the price in the uniform-pricing mechanism is selected so that the probability that any given agent has value exceeding the price is exactly $1/n$. The probability that there are no agents who are above the price is equal to the probability that all agents are below the price, which is equal to the product of the probabilities that each agent is below the threshold, i.e., $(1-1/n)^n \leq 1/e$.[3] Therefore, the probability that the item is sold by uniform pricing is at least $1 - 1/e$. If the item is sold, it is sold to an arbitrary agent with value conditioned to be at least $\hat{v}$, and the expected value of any such agent

[3] The *natural number* is $e \approx 2.178$. That $\lim_{n\to\infty}(1 - 1/n)^n = 1/e$ can be verified by taking the natural logarithm and applying L'Hopital's rule; the non-negativity of the derivative of $(1 - 1/n)^n$ implies it is is monotone non-decreasing; therefore, $1/e$ is an upper bound on $(1 - 1/n)^n$ for any finite n.

is $\mathbf{E}[v \mid v \geq \hat{v}]$. Therefore, the expected surplus of uniform pricing is,

$$\mathrm{APX} \geq (1 - 1/e)\mathbf{E}[v \mid v \geq \hat{v}]\,. \tag{1.3}$$

Combining equations (1.1), (1.2), and (1.3) it is apparent that $\mathrm{APX} \geq (1 - 1/e)\,\mathrm{REF}$. □

Question 1.5 When are simple, practical mechanisms like posted pricing a good approximation to the optimal mechanism?

1.1.3 General Routing Mechanisms

Finally we are ready to propose a mechanism for congestion control and routing in general networks. The main idea in the construction is the notion of critical values that was central to showing that the second-price auction has truthtelling as a dominant strategy (Theorem 1.3). In fact, that proof generalizes to any auction wherein each agent faces a critical value (that is not a function of her bid), the agent wins and pays the critical value if her bid exceeds it, and otherwise she loses.

Definition 1.8 The *second-price routing mechanism* is:

(i) solicit sealed bids,
(ii) find the set of messages that can be routed simultaneously with the largest total value, and
(iii) charge the agents of each routed message their critical values.

Theorem 1.6 *The second-price routing mechanism has truthful bidding as a dominant strategy.*

Corollary 1.7 *The second-price routing mechanism maximizes the social surplus.*

The proof of the theorem is similar to the analogous result for the second-price single-item auction, but we will defer its proof to Chapter 3. The corollary follows because the bids are equal to the agents' values, the mechanism is defined to be optimal for the reported bids, and the payments cancel.

Unfortunately, this is far from the end of the story. Step ((ii)) of the mechanism is known as *winner determination*. To understand exactly what is happening in this step we must be more clear about our model for routing in general networks. For instance, in the Internet, the route that messages take in the network is predetermined by the Border Gateway Protocol (BGP), which enforces that all messages routed to the same destination through any given router follow the same path. There are no opportunities for load-balancing, i.e., for sending messages to the same destination across different paths so as to keep the loads on any given path at a minimum. Alternatively, we could be in a novel network where the routing can determine which messages to route and which path to route them on.

Once we fix a model, we need to figure out how to solve the optimization problem implied by winner determination. Namely, how do we find the subset of messages with the highest total value that can be simultaneously routed? In principle, we are searching over subsets that meet some complicated feasibility condition. Purely from the point of optimization, this is a challenging task. The problem is related to the infamous *disjoint paths* problems: given a set of pairs of vertices in a graph, find a subset of pairs that can be connected via disjoint paths. This problem is *NP hard* to solve. Meaning: it is at least as hard as any problem in the equivalence class of *NP-complete* problems for which it is widely believed that finding optimal solutions is computationally intractable.

Theorem 1.8 *The disjoint-paths problem is NP hard.*

If we believe it is impossible for a designer to implement a mechanism for which *winner determination* is computationally intractable, we cannot accept the second-price routing mechanism as a solution to the general network routing problem.

Algorithmic theory has an answer to intractability: if computing the optimal solution is intractable, try instead to compute an approximately optimal solution.

Question 1.6 Can we replace Step ((ii)) in the mechanism with an approximation algorithm and still retain the dominant-strategy incentive property?

Question 1.7 If not, can we (by some other method) design a computationally tractable approximation mechanism for routing?

Question 1.8 Is there a general theory for designing approximation mechanisms from approximation algorithms?

1.2 Mechanism Design

Mechanism design gives a theory for the design of protocols, services, laws, or other "rules of interaction" in which selfish behavior leads to good outcomes. "Selfish behavior" means that each participant, hereafter *agent*, individually tries to maximize her own utility. Such behavior we define as rational. "Leads" means *in equilibrium.* A set of agent strategies is in equilibrium if no agent prefers to unilaterally change her strategy. Finally, the "good"-ness of an outcome is assessed with respect to the criteria or goals of the designer. Natural economic criteria are *social surplus*, the sum of the utilities of all parties; and *profit*, the total payments made to the mechanism less any cost for providing the outcome.

A theory for mechanism design should satisfy the following four desiderata:

Informative: It pinpoints salient features of the environment and characteristics of good mechanisms therein.

Prescriptive: It gives concrete suggestions for how a good mechanism should be designed.

Predictive: The mechanisms that the theory predicts should be the same as the ones observed in practice.

Tractable: The theory should not assume super-natural ability for the agents or designer to optimize.

Notice that optimality is not one of the desiderata, nor is suggesting a specific mechanism to a practitioner. Instead, intuition from the theory of mechanism design should help guide the design of good mechanisms in practice. Such guidance is possible through informative observations about what good mechanisms do. Observations that are robust to variations in modeling details are especially important.

Sometimes the theory of *optimal mechanism design* meets the above desiderata. The question of designing an optimal mechanism can be viewed as a standard optimization problem: given incentive constraints, imposed by game theoretic strategizing; feasibility constraints, imposed by the environment; and the distribution of agent preferences, optimize the designer's given objective. In ideal environments the given constraints may simplify and, for instance, allow the mechanism design problem to be reduced to a natural optimization problem without incentive constraints or distribution. We saw an example of this for routing in general networks: in order to invoke the second-price mechanism we only needed to find the optimal set of messages to route. Unfortunately, there are many environments and objectives where the optimal mechanism design problem not simplify as nicely.

1.3 Approximation

In environments where optimal mechanisms do not meet the desiderata above, approximation can provide a remedy. In the formal definition of an approximation, below, a good mechanism is one with a small approximation factor.

Definition 1.9 For an environment given implicitly, denote an *approximation mechanism* and its performance by APX, and a *reference mechanism* and its performance by REF.

(i) For any environment, APX is a β *approximation* to REF if APX $\geq \frac{1}{\beta}$ REF.

(ii) For any class of environments, a class of mechanisms is a β approximation to REF if for any environment in the class there is a mechanism APX in the class that is a β approximation to REF.

(iii) For any class of environments, a mechanism APX is a β approximation to REF if for any environment in the class APX is a β approximation to REF.

In the preceding section we saw each of these types of approximation. For i.i.d. $U[0, 1]$, n-agent, single-item environments, posting a uniform price of $\hat{v} = 1 - 1/n$ is a $e/e-1$ approximation to the second-price auction. More generally, for any i.i.d. single-item environment, uniform pricing is a $e/e-1$ approximation to the second-price auction. Finally, for any single-item environment the lottery gives an n approximation to the social surplus of the second-price auction.

Usually we will employ the approximation framework with REF representing the optimal mechanism. For instance, in the preceding section we compared a posted-pricing mechanism to the surplus-optimal second-price auction for i.i.d., single-item environments. For such a comparison, clearly REF $\geq$ APX, and therefore the approximation factor is at least one. It is often instructive to compare the approximation ability of one class of mechanisms to another. For instance, in the preceding section we compared the surplus of a lottery, as the optimal mechanism without payments, to the surplus of the second-price auction, the optimal mechanism (in general). This kind of apples-to-oranges comparison is useful for understanding the relative importance of various features of a mechanism or environment.

1.3.1 Philosophy of Approximation

While it is, no doubt, a compelling success of the theory of mechanism design that its mechanisms are so prevalent in practice, optimal mechanism design cannot claim the entirety of the credit. These mechanisms are employed by practitioners well beyond the environments for which they are optimal. Approximation can explain why: the mechanisms that are optimal in ideal environments may continue to be approximately optimal much more broadly. It is important for the theory to describe how broadly these mechanisms are approximately optimal and how close to optimal they are. Thus, the theory of approximation can complement the theory of optimality and justify the wide prevalence of certain mechanisms. For instance, in Chapter 4 and Chapter 7 we describe how the widely prevalent reserve-price-based mechanisms and posted pricings are corroborated by their approximate optimality.

There are natural environments for mechanism design wherein every "undominated" mechanism is optimal. If we consider only optimal mechanisms we are stuck with the full class from which we can make no observations about what makes a mechanism good; on the other hand, if we relax optimality, we may be able to identify a small subclass of mechanisms that are approximately optimal, i.e., for any environment there is a mechanism in the subclass that approximates the optimal mechanism. This subclass is important in theory as we can potentially observe salient characteristics of it. It is important in practice because, while it is unlikely for a real mechanism designer to be able to optimize over all mechanisms, optimizing over a small class of, hopefully, natural mechanisms may be possible. For instance, a conclusion that we will make precise in Chapter 4 and Chapter 7 is that reserve-price-based mechanisms and posted pricings are approximately optimal in a wide range of environments including those with multi-dimensional agent preferences.

Approximation provides a lens with which to explore the salient features of an environment or mechanism. Suppose we wish to determine whether a particular feature of a mechanism is important. If there exists a subclass of mechanisms without that feature that gives a good approximation to the optimal mechanism, then the feature is perhaps not that important. If, on the other hand, there is no such subclass then the feature is quite important. For instance, previously in this chapter we saw that mechanisms without transfers cannot obtain better than a linear approximation to the optimal social surplus in single-item environments. This result suggests that transfers are very important for mechanism

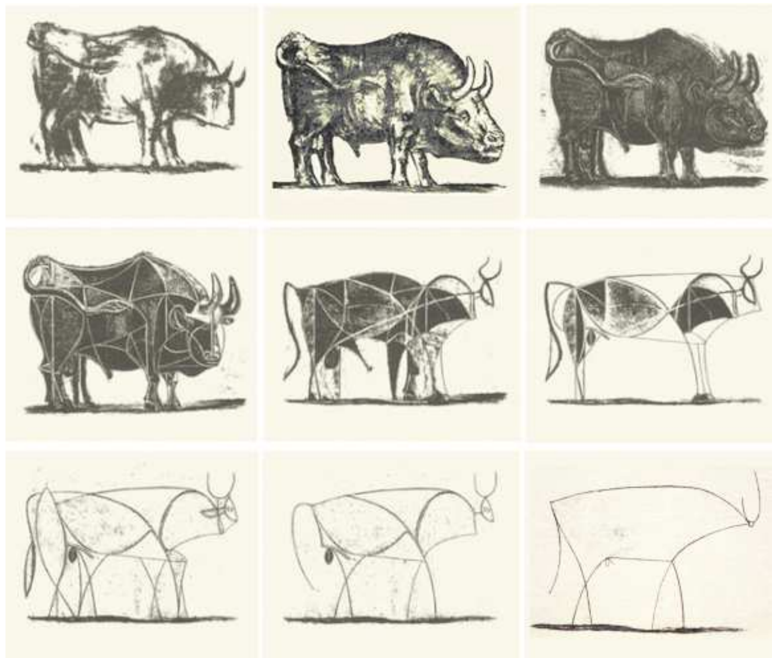


Figure 1.3 Picasso's December, 1945 to January, 1946 abstractionist study of a bull highlights one of the main points of approximation: identifying the salient features of the object of study. Picasso drew these in order from left to right, top to bottom.

design. On the other hand, we also saw that posted-pricing mechanism could obtain an $e/e-1$ approximation to the surplus-optimal mechanism. Posted pricings do not make use of competition between agents, therefore, we can conclude that competition between agents is not that important. Essentially, approximation provides a means to determine which aspect of an environment are details and which are not details. The approximation factor quantifies the relative importance on the spectrum between unimportant details to salient characteristics. Approximation, then allows for design of mechanisms that are not so dependent on details of the setting and therefore more robust. See Figure 1.3 for an illustration of this principle. In particular, in Chapter 4 we will formally observe that revenue-optimal auctions when agent values are drawn from a distribution can be approximated by a mechanism in which the only distributional dependence is a single number; moreover, in Chapter 5 we will observe that some environments permit a single (prior-independent) mechanism to approximate the revenue-optimal mechanism under any distributional assumption.

Suppose the seller of an item is worried about collusion, risk attitudes, after-market effects, or other economic phenomena that are usually not included in standard ideal models for mechanism design. One option would be to explicitly model these effects and study optimal mechanisms in the augmented model. These complicated models are difficult to analyze and optimal mechanisms may be overly influenced by insignificant-seeming modeling choices. Optimal mechanisms are precisely tuned to details in the model and these details may drive the form of the optimal mechanism. On the other hand, we can consider approximations that are robust to various out-of-model phenomena. In such an environment the comparison between the approximation and the optimal mechanism is unfair because the optimal mechanism may suffer from out-of-model phenomena that the approximation is robust to. In fact, this "optimal mechanism" may perform much worse than our approximation when the phenomena are explicitly modeled. For example, Chapter 4 and Chapter 7 describe posted pricing mechanisms that are approximately optimal and robust to timing effects; for this reason an online auction house, such as eBay, may prefer its sellers to use "buy it now" posted pricings instead of auctions.

Finally, there is an issue of non-robustness that is inherent in any optimization over a complex set of objects, such as mechanisms. Suppose the designer does not know the distribution of agent preferences exactly but can learn about it through, e.g., market analysis. Such a market analysis is certainly going to be noisy; exactly optimizing a mechanism to the market analysis may "over fit" to this noise. Both statistics and machine learning theory have techniques for addressing this sort of overfitting. Approximation mechanisms also provide such a robustness. Since the class of approximation mechanisms is restricted from the full set, for these mechanisms to be good, they must pay less attention to details and therefore are robust to sampling noise. Importantly, approximation allows for design and analysis mechanisms for small (a.k.a., *thin*) markets where statistical and machine learning methods are less applicable.

1.3.2 Approximation Factors

Depending on the problem and the approximation mechanism, approximation factors can range from $(1+\epsilon)$, i.e., arbitrarily close approximations, to linear factor approximations (or sometimes even worse). Notice a linear factor approximation is one where, as some parameter in the environment grows, i.e., more agents or more resources, the approximation factor gets worse. As examples, we saw earlier an environment in which uniform pricing is a *constant approximation* and the lottery is a *linear approximation*.[4]

In this text we take constant versus super-constant approximation as the separation between good and bad. We will view a proof that a mechanism is a constant approximation as a positive result and a proof that no mechanism (in a certain class) is a constant approximation as a negative result. Constant approximations tend to represent a tradeoff between simplicity and optimality. Properties of constant approximation mechanisms can, thus, be quite informative. Of course, there are many non-mechanism-design environments where super-constant approximations are both useful and informative; however, for mechanism design super-constant approximations tend to be indicative of (a) a bad mechanism, (b) failure to appropriately characterize optimal mechanisms, or (c) an imposition of incompatible modeling assumptions or constraints.

If you were approached by a seller (henceforth: principal) to design a mechanism and you returned to triumphantly reveal an elegant mechanism that gives her a two approximation to the optimal profit, you would probably find her a bit discouraged. After all, your mechanism leaves half of her profit on the table. In the context of this critique we outline the main points of constant, e.g., two, approximations for the practitioner. First, a two approximation provides informative conclusions that can guide the design of even better mechanisms for specific environments. Second, the approximation factor of two is a theoretical result that holds in a large range of environments, in specific environments the mechanism may perform better. It is easy, via simulation, to evaluate the mechanism performance on specific settings to see how close to optimal it actually is. Third, in many environments the optimal mechanism is not understood at all, meaning the principal's alternative to your two approximation is an ad hoc mechanism with no performance guarantee. This principal is of course free to simulate your mechanism and her mechanism in her given environment and decide to use the bet-

[4] Recall that the approximation factor for uniform pricing bounded by $e/e-1$, an absolute constant that does not increase with various parameters of the auction such as the number of agents. In contrast the approximation factor of the lottery could be as bad as n, the number of agents. As the number of agents increases, so does the approximation bound guaranteed by the lottery.

ter of the two. In this fashion the principal's ad hoc mechanism, if used, is provably a two approximation as well. Fourth, mechanisms that are two approximations in theory arise in practice. In fact, that it is a two approximation explains why the mechanism arises. Even though it is not optimal, it is close enough. If it was far from being optimal the principal (hopefully) would have figured this out and adopted a different approach.

Sometimes it is possible do obtain schemas for approximating the optimal mechanism to within a $(1+\epsilon)$ factor for any ϵ. These schemas tend to be computational approaches that are useful for addressing potential computational intractability of the optimal mechanism design problem. While they do not tend to yield simple mechanisms, they are relevant in complex environments. Often these approximation schemes are based on (a) identifying a restricted class of mechanisms wherein a near-optimal mechanism can be found and (b) conducting a brute-force search over this restricted class. While very little is learned from such a brute-force search, properties of the restricted class of mechanisms can be informative. Many of the optimal mechanisms we describe can in practice only be implemented as approximation schemes.

Chapter Notes

Routing and congestion control are a central problems in computer systems such as the Internet; see Leiner et al. (1997) for a discussion of design criteria. Demers et al. (1989) analyze "fair queuing" which is a lottery-based mechanism for congestion control. Griffin et al. (2002) discuss the Border Gateway Protocol (BGP) which determines the routes messages take in the Internet. The NP-completeness of the disjoint paths problem (and the related problem of integral multi-commodity flow) was established by Even et al. (1976).

William Vickrey's 1961 analysis of the second-price auction is one of the pillars of mechanism design theory. The second-price routing mechanism is a special case of the more general Vickrey-Clarke-Groves (VCG) mechanism which is attributed additionally to Edward Clarke (1971) and Theodore Groves (1973).

Computational payments were proposed as means for fighting unsolicited electronic mail by Dwork and Naor (1992). Hartline and Roughgarden (2008) consider mechanism design with the objective of residual surplus and describe distributional assumptions under which the lottery is optimal, the second-price auction is optimal, and when neither are optimal. They also give a single mechanism that approximates the optimal mechanism for any distribution of agent values.

Vincent and Manelli (2007) showed that there are environments for mechanism design wherein every "undominated" mechanism is optimal for some distribution of agent preferences. This result implies that optimality cannot be used to identify properties of good mechanisms. Robert Wilson (1987) suggested that mechanisms that are less dependent on the details of the environment are likely to be more relevant. This suggestion is known as the "Wilson doctrine."

The $e/e-1$ approximation via a uniform pricing (Theorem 1.5) is a consequence of Chawla et al. (2010). Wang et al. (2008) and Reynolds and Wooders (2009) discuss why the "buy it now" (i.e., posted-pricing) mechanism is replacing the second-price auction format in eBay.

2
Equilibrium

The theory of *equilibrium* attempts to predict what happens in a game when players behave strategically. This is a central concept to this text as, in mechanism design, we are optimizing over games to find games with good equilibria. Here, we review the most fundamental notions of equilibrium. They will all be static notions in that players are assumed to understand the game and will play once in the game. While such foreknowledge is certainly questionable, some justification can be derived from imagining the game in a dynamic setting where players can learn from past play.

This chapter reviews equilibrium in both complete and incomplete information games. As games of incomplete information are the most central to mechanism design, special attention will be paid to them. In particular, we will characterize equilibrium when the private information of each agent is single-dimensional and corresponds, for instance, to a value for receiving a good or service. We will show that auctions with the same equilibrium outcome have the same expected revenue. Using this so-called *revenue equivalence* we will describe how to solve for the equilibrium strategies of standard auctions in symmetric environments.

Our emphasis will be on demonstrating the central theories of equilibrium and not on providing the most comprehensive or general results. For that readers are recommended to consult a game theory textbook.

2.1 Complete Information Games

In games of compete information all players are assumed to know precisely the payoff structure of all other players for all possible outcomes of the game. A classic example of such a game is the *prisoner's dilemma*, the story for which is as follows.

Two prisoners, Bonnie and Clyde, have jointly committed a crime and are being interrogated in separate quarters. Unfortunately, the interrogators are unable to prosecute either prisoner without a confession. Bonnie is offered the following deal: If she confesses and Clyde does not, she will be released and Clyde will serve the full sentence of ten years in prison. If they both confess, she will share the sentence and serve five years. If neither confesses, she will be prosecuted for a minimal offense and receive a year of prison. Clyde is offered the same deal.

This story can be expressed as the following *bimatrix game* where entry (a, b) represents row player's payoff a and column player's payoff b.

	silent	confess
silent	(-1,-1)	(-10,0)
confess	(0,-10)	(-5,-5)

A simple thought experiment enables prediction of what will happen in the prisoners' dilemma. Suppose the Clyde is silent. What should Bonnie do? Remaining silent as well results in one year of prison while confessing results in immediate release. Clearly confessing is better. Now suppose that Clyde confesses. Now what should Bonnie do? Remaining silent results in ten years of prison while confessing as well results in only five. Clearly confessing is better. In other words, no matter what Clyde does, Bonnie is better of by confessing. The prisoners dilemma is hardly a dilemma at all: the *strategy profile* (confess, confess) is a *dominant strategy equilibrium.*

Definition 2.1 A *dominant strategy equilibrium* (DSE) in a complete information game is a strategy profile in which each player's strategy is as least as good as all other strategies regardless of the strategies of all other players.

Dominant strategy equilibrium is a strong notion of equilibrium and is therefore unsurprisingly rare. For an equilibrium notion to be complete it should identify equilibrium in every game. Another well studied game is *chicken.*

James Dean and Buzz (in the movie *Rebel without a Cause*) face off at opposite ends of the street. On the signal they race their cars on a collision course towards each other. The options each have are to swerve or to stay their course. Clearly if they both stay their course they crash. If they both swerve (opposite directions) they escape with their lives but the match is a draw.

Finally, if one swerves and the other stays, the one that stays is the victor and the other the loses.[1]

A reasonable bimatrix game depicting this story is the following.

	stay	swerve
stay	(-10,-10)	(1,-1)
swerve	(-1,1)	(0,0)

Again, a simple thought experiment enables us to predict how the players might play. Suppose James Dean is going to stay, what should Buzz do? If Buzz stays they crash and Buzz's payoff is -10, but if Buzz swerves his payoff is only -1. Clearly, of these two options Buzz prefers to swerve. Suppose now that Buzz is going to swerve, what should James Dean do? If James Dean stays he wins and his payoff is one, but if he swerves it is a draw and his payoff is zero. Clearly, of these two options James Dean prefers to stay. What we have shown is that the strategy profile (stay, swerve) is a mutual best response, a.k.a., a *Nash equilibrium.* Of course, the game is symmetric so the opposite strategy profile (swerve, stay) is also an equilibrium.

Definition 2.2 A *Nash equilibrium* in a game of complete information is a strategy profile where each players strategy is a best response to the strategies of the other players as given by the strategy profile.

In the examples above, the strategies of the players correspond directly to actions in the game, a.k.a., *pure strategies.* In general, Nash equilibrium strategies can be randomizations over actions in the game, a.k.a., *mixed strategies* (see Exercise 2.1).

2.2 Incomplete Information Games

Now we turn to the case where the payoff structure of the game is not completely known. We will assume that each agent has some private information and this information affects the payoff of this agent in the game. We will refer to this information as the agent's type and denote it by t_i for agent i. The profile of types for the n agents in the game is $\boldsymbol{t} = (t_1, \ldots, t_n)$.

A *strategy* in a game of incomplete information is a function that maps

[1] The actual chicken game depicted in *Rebel without a Cause* is slightly different from the one described here.

an agent's type to any of the agent's possible actions in the game (or a distribution over actions for mixed strategies). We will denote by $s_i(\cdot)$ the strategy of agent i and $\boldsymbol{s} = (s_1, \dots, s_n)$ a *strategy profile*.

The auctions described in Chapter 1 were games of incomplete information where an agent's private type was her value for receiving the item, i.e., $t_i = v_i$. As we described, strategies in the ascending-price auction were $s_i(v_i) =$ "drop out when the price exceeds v_i" and strategies in the second-price auction were $s_i(v_i) =$ "bid $b_i = v_i$." We refer to this latter strategy as *truthtelling*. Both of these strategy profiles are in *dominant strategy equilibrium* for their respective games.

Definition 2.3 A *dominant strategy equilibrium* (DSE) is a strategy profile $\boldsymbol{s}$ such that for all i, t_i, and $\boldsymbol{b}_{-i}$ (where $\boldsymbol{b}_{-i}$ generically refers to the actions of all players but i), agent i's utility is maximized by following strategy $s_i(t_i)$.

Notice that aside from strategies being defined as a map from types to actions, this definition of DSE is identical to the definition of DSE for games of complete information.

2.3 Bayes-Nash Equilibrium

Naturally, many games of incomplete information do not have dominant strategy equilibria. Therefore, we will also need to generalize Nash equilibrium to this setting. Recall that equilibrium is a property of a strategy profile. It is in equilibrium if each agent does not want to change her strategy given the other agents' strategies. For an agent i, we want to the fix other agent strategies and let i optimize her strategy (meaning: calculate her best response for all possible types t_i she may have). This is an ill specified optimization as just knowing the other agents' strategies is not enough to calculate a best response. Additionally, i's best response depends on i's beliefs on the types of the other agents. The standard economic treatment addresses this by assuming a common prior.

Definition 2.4 Under the *common prior assumption*, the agent types $\boldsymbol{t}$ are drawn at random from a *prior distribution* $\boldsymbol{F}$ (a joint probability distribution over type profiles) and this prior distribution is *common knowledge*.

The distribution $\boldsymbol{F}$ over $\boldsymbol{t}$ may generally be correlated. Which means that an agent with knowledge of her own type must do *Bayesian updating* to determine the distribution over the types of the remaining bidders. We denote this conditional distribution as $\boldsymbol{F}_{-i}\big|_{t_i}$. Of course, when the distribution of types is independent, i.e., $\boldsymbol{F}$ is the *product distribution* $F_1 \times \cdots \times F_n$, then $\boldsymbol{F}_{-i}\big|_{t_i} = \boldsymbol{F}_{-i}$.

Notice that a prior $\boldsymbol{F}$ and strategies $\boldsymbol{s}$ induces a distribution over the actions of each of the agents. With such a distribution over actions, the problem each agent faces of optimizing her own action is fully specified.

Definition 2.5 A *Bayes-Nash equilibrium (BNE)* for a game G and common prior $\boldsymbol{F}$ is a strategy profile $\boldsymbol{s}$ such that for all i and t_i, $s_i(t_i)$ is a best response when other agents play $\boldsymbol{s}_{-i}(\boldsymbol{t}_{-i})$ when $\boldsymbol{t}_{-i} \sim \boldsymbol{F}_{-i}\big|_{t_i}$.

To illustrate Bayes-Nash equilibrium, consider using the first-price auction to sell a single item to one of two agents, each with valuation drawn independently and identically from the uniform distribution on $[0, 1]$, i.e., the common prior distribution is $\boldsymbol{F} = F \times F$ with $F(z) = \mathbf{Pr}_{v\sim F}[v < z] = z$. Here each agent's type is her valuation. We will calculate the BNE of this game by the "guess and verify" technique. First, we guess that there is a symmetric BNE with $s_i(z) = z/2$ for $i \in \{1, 2\}$. Second, we calculate agent 1's expected utility with value v_1 and bid b_1 under the standard assumption that the agent's utility u_i is her value less her payment (when she wins). In this calculation v_1 and b_1 are fixed and $b_2 = v_2/2$ is random. By the definition of the first-price auction:

$$\mathbf{E}[u_1] = (v_1 - b_1) \times \mathbf{Pr}[1 \text{ wins with bid } b_1]\,.$$

Calculate $\mathbf{Pr}[1 \text{ wins with } b_1]$ as

$$\begin{aligned}\mathbf{Pr}[b_2 \le b_1] &= \mathbf{Pr}[v_2/2 \le b_1] = \mathbf{Pr}[v_2 \le 2b_1] = F(2b_1)\\ &= 2b_1.\end{aligned}$$

Thus,

$$\begin{aligned}\mathbf{E}[u_1] &= (v_1 - b_1) \times 2b_1\\ &= 2v_1b_1 - 2b_1^2.\end{aligned}$$

Third, we optimize agent 1's bid. Agent 1 with value v_1 should maximize $2v_1b_1 - 2b_1^2$ as a function of b_1, and to do so, can differentiate the function and set its derivative equal to zero. The result is $\frac{\mathrm{d}}{\mathrm{d}b_1}(2v_1b_1 - 2b_1^2) = 2v_1 - 4b_1 = 0$ and we can conclude that the optimal bid is $b_1 = v_1/2$.

This proves that agent 1 should bid as prescribed if agent 2 does; and vice versa. Thus, we conclude that the guessed strategy profile is in BNE.

In Bayesian games it is useful to distinguish between stages of the game in terms of the knowledge sets of the agents. The three stages of a Bayesian game are *ex ante*, *interim*, and *ex post*. The ex ante stage is before values are drawn from the distribution. Ex ante, the agents know this distribution but not their own types. The interim stage is immediately after each agent learns her own type, but before playing in the game. In the interim, an agent assumes the other agent types are drawn from the prior distribution conditioned on her own type, i.e., via *Bayesian updating*. In the ex post stage, the game is played and the actions of all agents are known.

2.4 Single-dimensional Games

We will focus on a conceptually simple class of single-dimensional games that is relevant to the auction problems we have already discussed. In a single-dimensional game, each agent's private type is her value for receiving an abstract service, i.e., $t_i = v_i$. The distribution over types is independent (i.e., a product distribution). A game has an outcome $\boldsymbol{x} = (x_1, \ldots, x_n)$ and payments $\boldsymbol{p} = (p_1, \ldots, p_n)$ where x_i is an indicator for whether agent i indeed received their desired service, i.e., $x_i = 1$ if i is served and 0 otherwise. Price p_i will denote the payment i makes to the mechanism. An agent's value can be positive or negative and an agent's payment can be positive or negative. An agent's utility is linear in her value and payment and specified by $u_i = v_i x_i - p_i$. Agents are risk-neutral expected utility maximizers.

Definition 2.6 A *single-dimensional linear utility* is defined as having utility $u = vx - p$ for service-payment outcomes (x, p) and private value v; a *single-dimensional linear agent* possesses such a utility function.

A game G maps actions $\boldsymbol{b}$ of agents to an outcome and payment. Formally we will specify these outcomes and payments as:

- $x_i^G(\boldsymbol{b})$ = outcome to i when actions are $\boldsymbol{b}$, and
- $p_i^G(\boldsymbol{b})$ = payment from i when actions are $\boldsymbol{b}$.

Given a game G and a strategy profile $\boldsymbol{s}$ we can express the outcome and payments of the game as a function of the valuation profile. From the point of view of analysis this description of the the game outcome is much more relevant. Define

- $x_i(\boldsymbol{v}) = x_i^G(\boldsymbol{s}(\boldsymbol{v}))$, and
- $p_i(\boldsymbol{v}) = p_i^G(\boldsymbol{s}(\boldsymbol{v}))$.

We refer to the former as the *allocation rule* and the latter as the *payment rule* for G and $\boldsymbol{s}$ (implicit). Consider an agent i's interim perspective. She knows her own value v_i and believes the other agents values to be drawn from the distribution $\boldsymbol{F}$ (conditioned on her value). For G, $\boldsymbol{s}$, and $\boldsymbol{F}$ taken implicitly we can specify agent i's interim allocation and payment rules as functions of v_i.

- $x_i(v_i) = \mathbf{Pr}[x_i(v_i) = 1 \mid v_i] = \mathbf{E}[x_i(\boldsymbol{v}) \mid v_i]$, and
- $p_i(v_i) = \mathbf{E}[p_i(\boldsymbol{v}) \mid v_i]$.

With linearity of expectation we can combine these with the agent's utility function to write

- $u_i(v_i) = v_i x_i(v_i) - p_i(v_i)$.

Finally, we say that a strategy $s_i(\cdot)$ is *onto* if every action b_i agent i could play in the game is prescribed by s_i for some value v_i, i.e., $\forall b_i \exists v_i \, s_i(v_i) = b_i$. We say that a strategy profile is *onto* if the strategy of every agent is onto. For instance, the truthtelling strategy in the second-price auction is onto. When the strategies of the agents are onto, the interim allocation and payment rules defined above completely specify whether the strategies are in equilibrium or not. In particular, BNE requires that each agent (weakly) prefers playing the action corresponding (via their strategy) to her value than the action corresponding to any other value.

Proposition 2.1 *When values are drawn from a product distribution* $\boldsymbol{F}$*; single-dimensional game* G *and strategy profile* $\boldsymbol{s}$ *is in BNE only if for all* i, v_i, *and* z,

$$v_i x_i(v_i) - p_i(v_i) \geq v_i x_i(z) - p_i(z).$$

If the strategy profile is onto then the converse also holds.

Notice that in Proposition 2.1 the distribution $\boldsymbol{F}$ is required to be a product distribution. If $\boldsymbol{F}$ is not a product distribution, then when agent i's value is v_i then $x_i(z)$ is not generally the probability that she will win when she follows her designated strategy for value z. This distinction arises because the conditional distribution of the other agents values need not be the same when i's value is v_i or z.

2.5 Characterization of Bayes-Nash Equilibrium

We now discuss what Bayes-Nash equilibria look like. For instance, when given G, $\boldsymbol{s}$, and $\boldsymbol{F}$ we can calculate the interim allocation and payment rules $x_i(v_i)$ and $p_i(v_i)$ of each agent. We want to succinctly describe properties of these allocation and payment rules that can arise as BNE.

Theorem 2.2 *When values are drawn from a continuous product distribution $\boldsymbol{F}$; single dimensional G and strategy profile $\boldsymbol{s}$ are in BNE only if for all i,*

(i) *(monotonicity) $x_i(v_i)$ is monotone non-decreasing, and*
(ii) *(payment identity) $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z)\,\mathrm{d}z + p_i(0)$,*

where often $p_i(0) = 0$. If the strategy profile is onto then the converse also holds.

Proof We will prove the theorem in the special case where the support of each agent i's distribution is $[0, \infty]$. Focusing on a single agent i, who we will refer to as Alice, we drop subscripts i from all notations.

We break this proof into three pieces. First, we show, by picture, that the game is in BNE if the characterization holds and the strategy profile is onto. Next, we will prove that a game is in BNE only if the monotonicity condition holds. Finally, we will prove that a game is in BNE only if the payment identity holds.

Note that if Alice with value v deviates from the equilibrium and takes action $s(v^\dagger)$ instead of $s(v)$ then she will receive outcome and payment $x(v^\dagger)$ and $p(v^\dagger)$. This motivates the definition,

$$u(v, v^\dagger) = v x(v^\dagger) - p(v^\dagger),$$

which corresponds to Alice utility when she makes this deviation. For Alice's strategy to be in equilibrium it must be that for all v, and $v^\dagger$, $u(v, v) \geq u(v, v^\dagger)$, i.e., Alice derives no increased utility by deviating. The strategy profile $\boldsymbol{s}$ is in equilibrium if and only if the same condition holds for all agents. (The "if" direction here follows from the assumption that strategies map values onto actions. Meaning: for any action in the game there exists a value $v^\dagger$ such that $s(v^\dagger)$ is that action.)

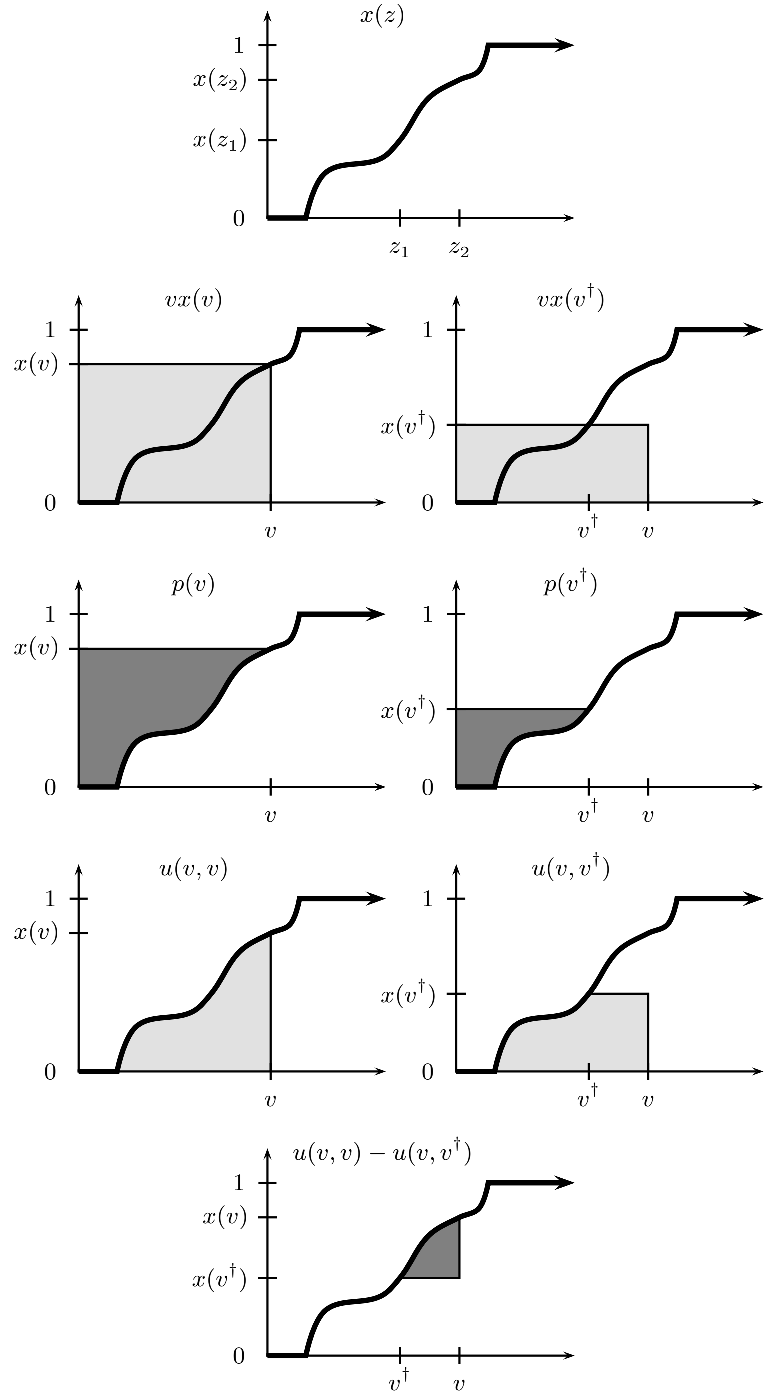


Figure 2.1 The left column shows (shaded) the surplus, payment, and utility of Alice playing action $s(v = z_2)$. The right column shows (shaded) the same for Alice playing action $s(v^\dagger = z_1)$. The final diagram shows (shaded) the difference between Alice's utility for these strategies. Monotonicity implies this difference is non-negative.

(i) G, $\boldsymbol{s}$, and $\boldsymbol{F}$ are in BNE if $\boldsymbol{s}$ is onto and monotonicity and the payment identity hold.

We prove this by picture. Though the formulaic proof is simple, the pictures provide useful intuition. We consider two possible values z_1 and z_2 with $z_1 < z_2$. Supposing Alice has the high value, $v = z_2$, we argue that Alice does not benefit by simulating her strategy for the lower value, $v^\dagger = z_1$, i.e., by playing $s(v^\dagger)$ to obtain outcome $x(v^\dagger)$ and payment $p(v^\dagger)$. We leave the proof of the opposite, that when $v = z_1$ and Alice is considering simulating the higher strategy $v^\dagger = z_2$, as an exercise for the reader.

To start with this proof, we assume that $x(v)$ is monotone and that $p(v) = vx(v) - \int_0^v x(z)\,\mathrm{d}z$.

Consider the diagrams in Figure 2.1. The first diagram (top, center) shows $x(\cdot)$ which is indeed monotone as per our assumption. The column on the left shows Alice's surplus, $vx(v)$; payment, $p(v)$, and utility, $u(v) = vx(v) - p(v)$, assuming that she follow the BNE strategy $s(v = z_2)$. The column on the right shows the analogous quantities when Alice follows strategy $s(v^\dagger = z_1)$ but has value $v = z_2$. The final diagram (bottom, center) shows the difference in the Alice's utility for the outcome and payments of these two strategies. Note that as the picture shows, the monotonicity of the allocation function implies that this difference is always non-negative. Therefore, there is no incentive for Alice to simulate the strategy of a lower value.

As mentioned, a similar proof shows that Alice has no incentive to simulate her strategy for a higher value. We conclude that she (weakly) prefers to play the action given by the BNE $s(\cdot)$ over any other action in the range of her strategy function; since $s(\cdot)$ is onto this range includes all actions.

(ii) G, $\boldsymbol{s}$, and $\boldsymbol{F}$ are in BNE only if the allocation rule is monotone.

If we are in BNE then for all valuations, v and $v^\dagger$, $u(v, v) \geq u(v, v^\dagger)$. Expanding we require

$$vx(v) - p(v) \geq vx(v^\dagger) - p(v^\dagger).$$

We now consider z_1 and z_2 with $z_1 < z_2$ and take turns setting $v = z_1$, $v^\dagger = z_2$, and $v^\dagger = z_1$, $v = z_2$. This yields the following two inequalities:

$$v = z_2, v^\dagger = z_1 \Longrightarrow z_2 x(z_2) - p(z_2) \geq z_2 x(z_1) - p(z_1), \text{ and} \quad (2.1)$$

$$v = z_1, v^\dagger = z_2 \Longrightarrow z_1 x(z_1) - p(z_1) \geq z_1 x(z_2) - p(z_2). \quad (2.2)$$

Adding these inequalities and canceling the payment terms we have,

$$z_2 x(z_2) + z_1 x(z_1) \geq z_2 x(z_1) + z_1 x(z_2).$$

Rearranging,

$$(z_2 - z_1)(x(z_2) - x(z_1)) \geq 0.$$

For $z_2 - z_1 > 0$ it must be that $x(z_2) - x(z_1) \geq 0$, i.e., $x(\cdot)$ is monotone non-decreasing.

(iii) G, $\boldsymbol{s}$, and $\boldsymbol{F}$ are in BNE only if the payment rule satisfies the payment identity.

We will give two proofs that payment rule must satisfy $p(v) = vx(v) - \int_0^v x(z)\,\mathrm{d}z + p(0)$; the first is a calculus-based proof under the assumption that and each of $x(\cdot)$ and $p(\cdot)$ are differentiable and the second is a picture-based proof that requires no assumption.

Calculus-based proof: Fix v and recall that $u(v, z) = vx(z) - p(z)$. Let $u'(v, z)$ be the partial derivative of $u(v, z)$ with respect to z. Thus, $u'(v, z) = vx'(z) - p'(z)$, where $x'(\cdot)$ and $p'(\cdot)$ are the derivatives of $p(\cdot)$ and $x(\cdot)$, respectively. Since BNE implies that $u(v, z)$ is maximized at $z = v$. It must be that

$$u'(v, v) = vx'(v) - p'(v) = 0.$$

This formula must hold true for all values of v. For remainder of the proof, we treat this identity formulaically. To emphasize this, substitute $z = v$:

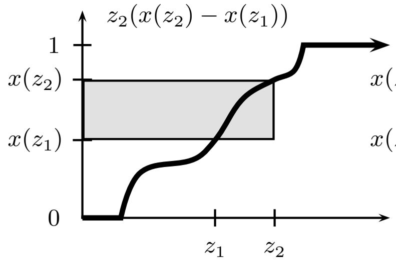
$$zx'(z) - p'(z) = 0.$$

Solving for $p'(z)$ and then integrating both sides of the equality from 0 to v we have,

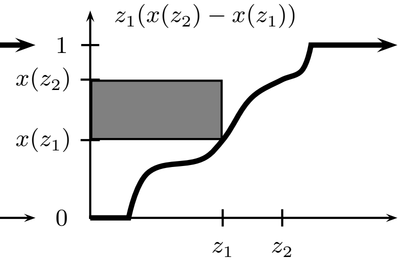
$$p'(z) = zx'(z), \text{ so}$$
$$\int_0^v p'(z)\mathrm{d}z = \int_0^v zx'(z)\,\mathrm{d}z.$$

Simplifying the left-hand side and adding $p(0)$ to both sides,

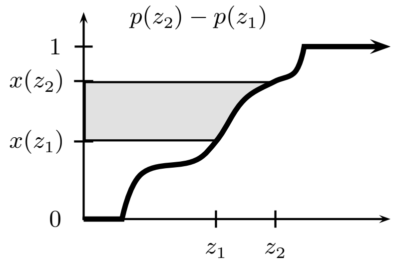
$$p(v) = \int_0^v zx'(z)\,\mathrm{d}z + p(0).$$



(a) Payment upper bound



(b) Payment lower bound



(c) Payment identity

Figure 2.2 Upper (top, left) and lower bounds (top, right) for the difference in payments for two strategies z_1 and z_2 imply that the difference in payments (bottom) must satisfy the payment identity.

Finally, we obtained the desired formula by integrating the right-hand side by parts,

$$\begin{aligned} p(v) &= \Big[zx(z)\Big]_0^v - \int_0^v x(z)\,\mathrm{d}z + p(0) \\ &= vx(v) - \int_0^v x(z)\,\mathrm{d}z + p(0). \end{aligned}$$

Picture-based proof: Consider equations (2.1) and (2.2) and solve for $p(z_2) - p(z_1)$ in each:

$$z_2(x(z_2) - x(z_1)) \geq p(z_2) - p(z_1) \geq z_1(x(z_2) - x(z_1)).$$

The first inequality gives an upper bound on the difference in payments for two types z_2 and z_1 and the second inequality gives a lower bound. It is easy to see that the only payment rule that satisfies these upper and lower bounds for all pairs of types z_2 and z_1 has payment difference exactly equal to the area to the left of the allocation rule between $x(z_1)$ and $x(z_2)$. See Figure 2.2. The payment identity follows by taking $z_1 = 0$ and $z_2 = v$. □

As we conclude the proof of the BNE characterization theorem, it is important to note how little we have assumed of the underlying game. We did not assume it was a single-round, sealed-bid auction. We did not assume that only a winner will make payments. Therefore, we conclude for any potentially wacky, multi-round game the outcomes of all Bayes-Nash equilibria have a nice form.

2.6 Characterization of Dominant Strategy Equilibrium

Dominant strategy equilibrium is a stronger equilibrium concept than Bayes-Nash equilibrium. All dominant strategy equilibria are Bayes-Nash equilibria, but as we have seen, the opposite is not true; for instance, there is no DSE in the first-price auction. Recall that a strategy profile is in DSE if each agent's strategy is optimal for her regardless of what other agents are doing. The DSE characterization theorem below follows from the BNE characterization theorem.

Theorem 2.3 *G and $\boldsymbol{s}$ are in DSE only if for all i and $\boldsymbol{v}$,*

(i) *(monotonicity) $x_i(v_i, \boldsymbol{v}_{-i})$ is monotone non-decreasing in v_i, and*
(ii) *(payment identity) $p_i(v_i, \boldsymbol{v}_{-i}) = v_i x_i(v_i, \boldsymbol{v}_{-i}) - \int_0^{v_i} x_i(z, \boldsymbol{v}_{-i})\,\mathrm{d}z + p_i(0, \boldsymbol{v}_{-i})$,*

where $(z, \boldsymbol{v}_{-i})$ denotes the valuation profile with the ith coordinate replaced with z. If the strategy profile is onto then the converse also holds.

It was important when discussing BNE to explicitly refer to $x_i(v_i)$ and $p_i(v_i)$ as the probability of allocation and the expected payments because a game played by agents with values drawn from a distribution will inherently, from agent i's perspective, have a randomized outcome and payment. In contrast, for games with DSE we can consider outcomes and payments in a non-probabilistic sense. A deterministic game, i.e., one with no internal randomization, will result in deterministic outcomes and payments. For our single-dimensional game where an agent is either served or not served we will have $x_i(\boldsymbol{v}) \in \{0, 1\}$. This specification along with the monotonicity condition implied by DSE implies that the function $x_i(v_i, \boldsymbol{v}_{-i})$ is a step function in v_i. The reader can easily verify that the payment required for such a step function is exactly the critical value, i.e., $\hat{v}_i$ at which $x_i(\cdot, \boldsymbol{v}_{-i})$ changes from 0 to 1. This gives the following corollary.

Corollary 2.4 *A deterministic game G and deterministic strategies $\boldsymbol{s}$ are in DSE only if for all i and $\boldsymbol{v}$,*

(i) *(step-function) $x_i(v_i, \boldsymbol{v}_{-i})$ steps from 0 to 1 at some $\hat{v}_i(\boldsymbol{v}_{-i})$, and*

(ii) *(critical value)* $p_i(v_i, \boldsymbol{v}_{-i}) = \begin{cases} \hat{v}_i(\boldsymbol{v}_{-i}) & \text{if } x_i(v_i, \boldsymbol{v}_{-i}) = 1 \\ 0 & \text{otherwise} \end{cases} + p_i(0, \boldsymbol{v}_{-i}).$

If the strategy profile is onto then the converse also holds.

Notice that the above theorem deliberately skirts around a subtle tie-breaking issue. Consider the truthtelling DSE of the second-price auction on two agents. What happens when $v_1 = v_2$? One agent should win and pay the other's value. As this results in a utility of zero, from the perspective of utility maximization, both agents are indifferent as to which of them it is. One natural tie-breaking rule is the lexicographical one, i.e., in favor of agent 1 winning. For this rule, agent 1 wins when $v_1 \in [v_2, \infty)$ and agent 2 wins when $v_2 \in (v_1, \infty)$. The critical values are $t_1 = v_2$ and $t_2 = v_1$. We will usually prefer the randomized tie-breaking rule because of its symmetry.

2.7 Revenue Equivalence

We are now ready to make one of the most significant observations in auction theory. Namely, mechanisms with the same outcome in BNE have the same expected revenue. In fact, not only do they have the same expected revenue, but each agent has the same expected payment in each mechanism. This result is in fact a direct corollary of Theorem 2.2. The payment identity means that the payment rule is precisely determined by the allocation rule and the payment of the lowest type, i.e., $p_i(0)$.

Corollary 2.5 *For any two mechanisms where 0-valued agents pay nothing, if the mechanisms have the same BNE outcome then they have same expected revenue.*

We can now quantitatively compare the second-price and first-price auctions from a revenue standpoint. Consider the case where the agent's values are distributed independently and identically. What is the equilibrium outcome of the second-price auction? The agent with the highest valuation wins. What is the equilibrium outcome of the first-price auction? This question requires a little more thought. Since the distributions are identical, it is reasonable to expect that there is a symmetric equilibrium, i.e., one where $s_i = s_{i'}$ for all i and i'. Furthermore, it is reasonable to expect that the strategies are monotone, i.e., an agent with a higher value will out bid an agent with a lower value. Under these assumptions, the agent with the highest value wins. Of course, in both auctions a 0-valued agent will pay nothing. Therefore, we can conclude that the two auctions obtain the same expected revenue.

As an example of revenue equivalence consider first-price and second-price auctions for selling a single item to two agents with values drawn from $U[0, 1]$. The expected revenue of the second-price auction is $\mathbf{E}\left[v_{(2)}\right]$. In Section 2.3 we saw that the symmetric strategy of the first-price auction in this environment is for each agent to bid half her value. The expected revenue of first-price auction is therefore $\mathbf{E}\left[v_{(1)}/2\right]$. An important fact about uniform random variables is that in expectation they evenly divide the interval they are over, i.e., $\mathbf{E}\left[v_{(1)}\right] = 2/3$ and $\mathbf{E}\left[v_{(2)}\right] = 1/3$. How do the revenues of these two auctions compare? Their revenues are identically 1/3.

Corollary 2.6 *When agents' values are independent and identically distributed according to a continuous distribution, the second-price and first-price auction have the same expected revenue.*

Of course, much more bizarre auctions are governed by revenue equivalence. As an exercise the reader is encourage to verify that the *all-pay auction*; where agents submit bids, the highest bidder wins, and all agents pay their bids; is revenue equivalent to the first- and second-price auctions.

2.8 Solving for Bayes-Nash Equilibrium

While it is quite important to know what outcomes are possible in BNE, it is also often important to be able to solve for the BNE strategies. For instance, suppose you were a bidder bidding in an auction. How would you bid? In this section we describe an elegant technique for calculating BNE strategies in symmetric environments using revenue equivalence. Actually, we use something a little stronger than revenue equivalence: *interim payment equivalence*. This is the fact that if two mechanisms have the same allocation rule, they must have the same payment rule (because the payment rules satisfy the payment identity). As described previously, the interim payment of agent i with value v_i is $p_i(v_i)$.

Suppose we are to solve for the BNE strategies of mechanism M. The approach is to express an agent's payment in M as a function of the agent's action, then to calculate the agent's expected payment in a strategically-simple mechanism M' that is revenue equivalent to M (usually a "second-price implementation" of M). Setting these terms equal and solving for the agents action gives the equilibrium strategy.

We give the high level the procedure below. As a running example we will calculate the equilibrium strategies in the first-price auction with two $U[0,1]$ agents, in doing so we will use a calculation of expected payments in the strategically-simple second-price auction in the same environment.

(i) *Guess* what the outcome might be in Bayes-Nash equilibrium.

E.g., in the BNE of the first-price auction with two agents with values $U[0,1]$, we expect the agent with the highest value to win. Thus, guess that the highest-valued agent always wins.

(ii) *Calculate* the interim payment of an agent in the auction in terms of the strategy function.

E.g., we calculate below the payment of agent 1 in the first-price auction when her bid is $s_1(v_1)$ in expectation when agent 2's value v_2 is drawn from the uniform distribution.

$$\begin{aligned} p_1^{\mathrm{FP}}(v_1) = \mathbf{E}[p_1^{\mathrm{FP}}(v_1,v_2) \mid 1 \text{ wins}]\, \mathbf{Pr}[1 \text{ wins}] \\ + \mathbf{E}[p_1^{\mathrm{FP}}(v_1,v_2) \mid 1 \text{ loses}]\, \mathbf{Pr}[1 \text{ loses}]\,. \end{aligned}$$

Calculate each of these components for the first-price auction where agent 1 follows strategy $s_1(v_1)$:

$$\mathbf{E}\Big[p_1^{\mathrm{FP}}(v_1,v_2) \mid 1 \text{ wins}\Big] = s_1(v_1).$$

This by the definition of the first-price auction: if you win you pay your bid.

$$\mathbf{Pr}[1 \text{ wins}] = \mathbf{Pr}[v_2 < v_1] = v_1.$$

The first equality follows from the guess that the highest-valued agent wins. The second equality is because v_2 is uniform on $[0,1]$.

$$\mathbf{E}\Big[p_1^{\mathrm{FP}}(v_1) \mid 1 \text{ loses}\Big] = 0.$$

This is because a loser pays nothing in the first-price auction. This means that we do not need to calculate $\mathbf{Pr}[1 \text{ loses}]$. Plug these into the equation above to obtain:

$$p_1^{\mathrm{FP}}(v_1) = s_1(v_1) \cdot v_1.$$

(iii) *Calculate* the interim payment of an agent in a strategically-simple auction with the same equilibrium outcome.

E.g., recall that it is a dominant strategy equilibrium (a special case of Bayes-Nash equilibrium) in the second-price auction for each agent to bid her value. I.e., $b_1 = v_1$ and $b_2 = v_2$. Thus, in the second-price auction the agent with the highest value to wins. We calculate below the payment of agent 1 in the second-price auction when her value is v_1 in expectation when agent 2's value v_2 is drawn from the uniform distribution.

$$\begin{aligned} p_1^{\mathrm{SP}}(v_1) = \mathbf{E}[p_1^{\mathrm{SP}}(v_1, v_2) \mid 1 \text{ wins}]\, \mathbf{Pr}[1 \text{ wins}] \\ + \mathbf{E}[p_1^{\mathrm{SP}}(v_1, v_2) \mid 1 \text{ loses}]\, \mathbf{Pr}[1 \text{ loses}]\,. \end{aligned}$$

Calculate each of these components for the second-price auction:

$$\begin{aligned} \mathbf{E}\Big[p_1^{\mathrm{SP}}(v_1, v_2) \mid 1 \text{ wins}\Big] &= \mathbf{E}[v_2 \mid v_2 < v_1] \\ &= v_1/2. \end{aligned}$$

The first equality follows by the definition of the second-price auction and its dominant strategy equilibrium (i.e., $b_2 = v_2$). The second equality follows because in expectation a uniform random variable evenly divides the interval it is over, and once we condition on $v_2 < v_1$, v_2 is $U[0, v_1]$.

$$\mathbf{Pr}[1 \text{ wins}] = \mathbf{Pr}[v_2 < v_1] = v_1.$$

The first equality follows from the definition of the second-price auction and its dominant strategy equilibrium. The second equality is because v_2 is uniform on $[0, 1]$.

$$\mathbf{E}[p_1(v_1) \mid 1 \text{ loses}] = 0.$$

This is because a loser pays nothing in the second-price auction. This means that we do not need to calculate $\mathbf{Pr}[1 \text{ loses}]$. Plug these into the equation above to obtain:

$$\mathbf{E}\Big[p_1^{\mathrm{SP}}(v_1)\Big] = v_1^2/2.$$

(iv) *Solve* for bidding strategies from expected payments.

E.g., the interim payments calculated in the previous steps must be equal, implying:

$$p_1^{\mathrm{FP}}(v_1) = s_1(v_1) \cdot v_1 = v_1^2/2 = p_1^{\mathrm{SP}}(v_1).$$

We can solve for $s_1(v_1)$ and get

$$s_1(v_1) = v_1/2.$$

(v) Verify initial guess was correct. If the strategy function derived is not onto, verify that actions out of the range of the strategy function are dominated.

E.g., if agents follow symmetric strategies $s_1(z) = s_2(z) = z/2$ then the agent with the highest value wins. With this strategy function, bids are in $[0, 1/2]$ and any bid above $s_1(1) = 1/2$ is dominated by bidding $s_1(1)$. All such bids win with certainty, but of these the bid $s_1(1) = 1/2$ gives the lowest payment.

In the above first-price auction example it should be readily apparent that we did slightly more work than we had to. In this case it would have been enough to note that in both the first- and second-price auction a loser pays nothing. We could therefore simply equate the expected payments conditioned on winning:

$$\mathbf{E}[p_1(v_1) \mid 1 \text{ wins}] = \underbrace{v_1/2}_{\text{second-price}} = \underbrace{s_1(v_i)}_{\text{first-price}}.$$

We can also work through the above framework for the *all-pay* auction where the agents submit bids, the highest bid wins, but all agents pay their bid. The all-pay auction is also is revenue equivalent to the second-price auction. However, now we compare the total expected payment (regardless of winning) to conclude:

$$\mathbf{E}[p_1(v_1)] = \underbrace{v_1^2/2}_{\text{second-price}} = \underbrace{s_1(v_i)}_{\text{all-pay}}.$$

I.e., the BNE strategies for the all-pay auction are $s_i(z) = z^2/2$. Remember, of course, that the equilibrium strategies solved for above are for single-item auctions and two agents with values uniform on $[0, 1]$. For different distributions or numbers of agents the equilibrium strategies will generally be different.

We conclude by observing that if we fail to exhibit a Bayes-Nash equilibrium via this approach then our original guess is contracted and there is no equilibrium of the given mechanism that corresponds to the guess. Conversely, if the approach succeeds then the equilibrium found is the only equilibrium consistent with the guess. As an example, we can conclude the following for first-price auctions.[2]

Proposition 2.7 *When agents' values are independent and identically distributed from a continuous distribution, the first-price auction has a unique Bayes-Nash equilibrium for which the highest-valued agent always wins.*

2.9 Uniqueness of Equilibria

As equilibrium attempts to make a prediction of what will happen in a game or mechanism, the uniqueness of equilibrium is important. If there are multiple equilibria then the prediction is to a set of outcomes not a single outcome. In terms of mechanism design, some of these outcomes could be good and some could be bad. There are also questions of how the players coordinate on an equilibrium.

As an example, in the second-price auction for two agents with values uniformly distributed on $[0, 1]$ there is the dominant strategy equilibrium where agents truthfully report their values. This outcome is good from the perspective of social surplus in that the item is awarded to the highest-valued agent. There are, however, other Bayes-Nash equilibria. For instance, it is also a BNE for agent 1 (Alice) to bid one and agent 2 (Bob) to bid zero (regardless of their values). Alice is happy to win and pay zero (Bob's bid); Bob with any value $v_2 \leq 1$ is at least as happy to lose and pay zero versus winning and paying one (Alice's bid). Via examples like this the social surplus of the worst BNE in the second-price auction can be arbitrarily worse than the social surplus of the best BNE (Exercise 2.8). This latter equilibrium is not dominant strategy as if Bob were to bid his value (a dominant strategy), then Alice would no longer prefer to bid one. Because of this non-robustness of non-DSE in games that possess DSE, we can assume that agents follow DSE if there exists one.

[2] In the next section we will strengthen Proposition 2.7 and show that for the first-price auction (with independent, identical, and continuous distributions) there are no equilibria where the highest-valued agent does not win. Thus, the equilibrium solved for is the unique Bayes-Nash equilibrium.

In contrast, the first-price auction for independent and identical prior distributions does not suffer from multiplicity of Bayes-Nash equilibria. Specifically, the method described in the previous section for solving for the symmetric equilibrium in symmetric auction-like games gives the unique BNE. We describe this result as two parts. First, we exclude the possibility of multiple symmetric equilibria. Second, we exclude the existence of asymmetric equilibria.

Lemma 2.8 *For agents with values drawn independently and identically from a continuous distribution, the first-price auction admits exactly one symmetric Bayes-Nash equilibrium.*

Proof Consider a symmetric strategy profile $\boldsymbol{s} = (s, \ldots, s)$. First, the common strategy $s(\cdot)$ must be non-decreasing (otherwise BNE is contradicted by Theorem 2.2).

Second, if the strategy is non-strictly increasing then there is a point mass some bid b in the bid distribution. Symmetry with respect to this strategy implies that all agents will make a bid equal to this point mass with some measurable (i.e., strictly positive) probability. All but one of these bidders must lose (perhaps via random tie-breaking). Winning, however, must be strictly preferred to losing for some of the values in the interval (as an agent with value v is only indifferent to winning or losing when $v = b$). Such a losing agent has a deviation of bidding $b + \epsilon$, and for ϵ approaching zero this deviation is strictly better than bidding b. This is a contradiction to the existence of such a non-strictly increasing equilibrium.

Finally, for a strictly increasing strategy s the highest-valued agent must always win; therefore, Proposition 2.7 implies that there is only one such equilibrium. $\square$

We now make much the same argument as we did in solving for equilibrium (Section 2.8) to exclude the possibility of asymmetric equilibria in the first-price auction. The main idea in this argument is that there are two formulas for the interim utility of an agent in the first-price auction in terms of the allocation rule $x(\cdot)$. The first formula is from the payment identity of Theorem 2.2, the second formula is from the definition of the first-price auction (i.e., in terms of the agent's strategy).

They are,

$$u(v) = \int_0^v x(z)\,\mathrm{d}z, \text{ and} \tag{2.3}$$

$$u(v) = (v - s(v)) \cdot x(v). \tag{2.4}$$

The uniqueness of the symmetric Bayes-Nash equilibrium in the first-price auction follows from the following lemma.

Lemma 2.9 *For $n = 2$ agents with values drawn independently and identically from a continuous distribution F, the first-price auction with an unknown random reserve from known distribution G admits no asymmetric Bayes-Nash equilibrium.*

Theorem 2.10 *For $n \geq 2$ agents with values drawn independently and identically from a continuous distribution F, the first-price auction there is a unique Bayes-Nash equilibrium that is symmetric.*

Proof By Lemma 2.8 there is exactly one symmetric Bayes-Nash equilibrium of an n-agent first-price auction. If there is an asymmetric equilibrium there must be two agents whose strategies are distinct. We can view the n-agent first-price auction in BNE, from the perspective of this pair of agents, as a two-agent first-price auction with a random reserve drawn from the distribution of BNE bids of the other $n - 2$ agents. Lemma 2.9 then contradicts the distinctness of these two strategies. □

Proof of Lemma 2.9 We will prove this lemma for the special case of strictly-increasing and continuous strategies (for the general argument, see Exercise 2.12). Agent 1 is Alice and agent 2 is Bob.

If the BNE utilities of the agents are the same at all values, i.e., $u_1(v) = u_2(v)$ for all v in the distribution's range, then the payment identity of Theorem 2.2 implies that the strategies are the same at all values. For a contradiction then, fix a strictly-increasing continuous strategy profile $\boldsymbol{s} = (s_1, s_2)$ for which $u_1(v) > u_2(v)$ at some v. By equation (2.3) there must be a measurable interval of values $I = (a, b)$, i.e., with $\mathbf{Pr}[v \in I] > 0$, containing this value v and for which $x_1(v) \geq x_2(v)$ (assume I is the maximal such interval).

A first claim for strictly-increasing continuous strategies is that $s_1(v) > s_2(v)$ if any only if $x_1(v) > x_2(v)$. See Figure 2.3 for a graphical representation of the following argument. Since the strategies are continuous and strictly increasing, the inverses of the strategies are well defined.

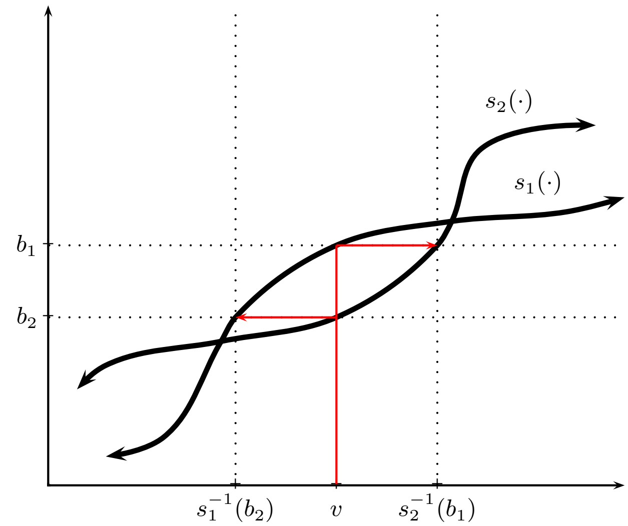


Figure 2.3 Graphical depiction of the first claim in the proof of Lemma 2.9 with $b_i = s_i(v)$. Clearly, $s_2^{-1}(b_1) > s_1^{-1}(b_2)$. Strict monotonicity of the distribution function $F(\cdot)$ then implies that $F(s_2^{-1}(b_1)) > F(s_1^{-1}(b_2))$.

Calculate Alice's interim allocation probability x_1 at value v, for Bob's value $v_2 \sim F$ and reserve bid $\hat{b} \sim G$, as:

$$\begin{aligned} x_1(v) &= \mathbf{Pr}\Big[s_1(v) > s_2(v_2) \wedge s_1(v) > \hat{b}\Big] \\ &= \mathbf{Pr}\Big[s_2^{-1}(s_1(v)) > v_2 \wedge s_1(v) > \hat{b}\Big] \\ &= F(s_2^{-1}(s_1(v))) \cdot G(s_1(v)). \end{aligned}$$

Likewise, Bob's interim allocation probability is

$$x_2(v) = F(s_1^{-1}(s_2(v))) \cdot G(s_2(v)).$$

For $s_1(v) \geq s_2(v)$ then the last term in the allocation probabilities satisfies $G(s_1(v)) \geq G(s_2(v))$ (as the distribution function $G(\cdot)$ is non-decreasing). Similarly, strict monotonicity of the strategy functions and distribution function imply that for $s_1(v) \geq s_2(v)$ the first term in the allocation probabilities satisfies $F(s_2^{-1}(s_1(v))) \geq F(s_1^{-1}(s_2(v)))$; moreover, either both inequalities are strict or both are equality.

A second claim is that the low-bidding Bob on the interval $I = (a, b)$

obtains (weakly) at most the utility of high-bidding Alice at the endpoint a and (weakly) at least the utility of the high-bidding Alice at the endpoint b. We argue the claim for b, the case of a is similar. The key to this claim is that there are not higher values $v > b$ where $s_2(v) < s_1(b)$. This is either because $s_1(b) = s_2(b)$ (and the strategies are monotonically increasing) or because b is the maximum value in the support of the value distribution F. In the first case, by the above claim $x_1(b) = x_2(b)$ so by (2.4) the agents' utilities are equal. In the second case, Bob with value b could deviate and bid $s_1(b)$ and obtain the same allocation probability as Alice with the same value. By equation (2.4) such a deviation would give Bob (with value b) the same utility as Alice (with value b). Existence of such a deviation gives a lower bound on Bob's utility.

Finally, we complete the lemma by writing the difference in utilities of each of Alice and Bob with values a and a. By the second claim, above, this difference is (weakly) greater for Bob than Alice (relative to Alice's utility, Bob's utility is no higher at a and no lower at b).

$$u_1(b) - u_1(a) \leq u_2(b) - u_2(a)$$

However, by the first claim and equation (2.3), Alice has a strictly higher allocation rule on I and therefore strictly higher change in utility.

$$\int_a^b x_1(z)\,\mathrm{d}z > \int_a^b x_2(z)\,\mathrm{d}z$$

These observations give a contradiction. □

2.10 The Revelation Principle

We are interested in designing mechanisms and, while the characterization of Bayes-Nash equilibrium is elegant, solving for equilibrium is still generally quite challenging. The final piece of the puzzle, and the one that has enabled much of modern mechanism design is the *revelation principle*. The revelation principle states, informally, that if we are searching among mechanisms for one with a desirable equilibrium we may restrict our search to single-round, sealed-bid mechanisms in which truthtelling is an equilibrium.

Definition 2.7 A *direct revelation* mechanism is single-round, sealed bid, and has action space equal to the type space, (i.e., an agent can bid any type she might have)

Definition 2.8 A direct revelation mechanism is *Bayesian incentive compatible* (BIC) if truthtelling is a Bayes-Nash equilibrium.

Definition 2.9 A direct revelation mechanism is *dominant strategy incentive compatible* (DSIC) if truthtelling is a dominant strategy equilibrium.

Theorem 2.11 *Any mechanism $\mathcal{M}$ with good BNE (resp. DSE) can be converted into a BIC (resp. DSIC) mechanism $\mathcal{M}'$ with the same BNE (resp. DSE) outcome.*

Proof We will prove the BNE variant of the theorem. Let $\boldsymbol{s}$, $\boldsymbol{F}$, and $\mathcal{M}$ be in BNE. Define single-round, sealed-bid mechanism $\mathcal{M}'$ as follows:

(i) Accept sealed bids $\boldsymbol{b}$.
(ii) Simulate $\boldsymbol{s}(\boldsymbol{b})$ in $\mathcal{M}$.
(iii) Output the outcome of the simulation.

We now claim that $\boldsymbol{s}$ being a BNE of $\mathcal{M}$ implies truthtelling is a BNE of $\mathcal{M}'$ (for distribution $\boldsymbol{F}$). Let $\boldsymbol{s}'$ denote the truthtelling strategy. In $\mathcal{M}'$, consider agent i and suppose all other agents are truthtelling. This means that the actions of the other players in $\mathcal{M}$ are distributed as $\boldsymbol{s}_{-i}(\boldsymbol{s}'_{-i}(\boldsymbol{v}_{-i})) = \boldsymbol{s}_{-i}(\boldsymbol{v}_{-i})$ for $\boldsymbol{v}_{-i} \sim \boldsymbol{F}_{-i}\big|_{v_i}$. Of course, in $\mathcal{M}$ if other players are playing $\boldsymbol{s}_{-i}(\boldsymbol{v}_{-i})$ then since $\boldsymbol{s}$ is a BNE, i's best response is to play $s_i(v_i)$ as well. Agent i can play this action in the simulation of $\mathcal{M}$ is by playing the truthtelling strategy $s'_i(v_i) = v_i$ in $\mathcal{M}'$. □

Notice that we already, in Chapter 1, saw the revelation principle in action. The second-price auction is the revelation principle applied to the ascending-price auction.

Because of the revelation principle, for many of the mechanism design problems we consider, we will look first for Bayesian or dominant-strategy incentive compatible mechanisms. The revelation principle guarantees that, in our search for optimal BNE mechanisms, it suffices to search only those that are BIC (and likewise for DSE and DSIC). The following are corollaries of our BNE and DSE characterization theorems.

We defined the allocation and payment rules $\boldsymbol{x}(\cdot)$ and $\boldsymbol{p}(\cdot)$ as functions of the valuation profile for an implicit game G and strategy profile $\boldsymbol{s}$. When the strategy profile is truthtelling, the allocation and payment rules are identical the original mappings of the game from actions to allocations and prices, denoted $\boldsymbol{x}^G(\cdot)$ and $\boldsymbol{p}^G(\cdot)$. Additionally, let

$x_i^G(v_i) = \mathbf{E}\Big[x_i^G(\boldsymbol{v}) \mid v_i\Big]$ and $p_i^G(v_i) = \mathbf{E}\Big[p_i^G(\boldsymbol{v}) \mid v_i\Big]$ for $\boldsymbol{v} \sim \boldsymbol{F}$. Furthermore, the truthtelling strategy profile in a direct-revelation game is onto.

Corollary 2.12 *A direct mechanism $\mathcal{M}$ is BIC for distribution $\boldsymbol{F}$ if and only if for all i,*

(i) *(monotonicity) $x_i^{\mathcal{M}}(v_i)$ is monotone non-decreasing, and*
(ii) *(payment identity) $p_i^{\mathcal{M}}(v_i) = v_i x_i^{\mathcal{M}}(v_i) - \int_0^{v_i} x_i^{\mathcal{M}}(z)\,\mathrm{d}z + p_i^{\mathcal{M}}(0)$.*

Corollary 2.13 *A direct mechanism $\mathcal{M}$ is DSIC if and only if for all i and $\boldsymbol{v}$,*

(i) *(monotonicity) $x_i^{\mathcal{M}}(v_i, \boldsymbol{v}_{-i})$ is monotone non-decreasing in v_i, and*
(ii) *(payment identity) $p_i^{\mathcal{M}}(v_i, \boldsymbol{v}_{-i}) = v_i x_i^{\mathcal{M}}(v_i, \boldsymbol{v}_{-i}) - \int_0^{v_i} x_i^{\mathcal{M}}(z, \boldsymbol{v}_{-i})\,\mathrm{d}z + p_i^{\mathcal{M}}(0, \boldsymbol{v}_{-i})$.*

Corollary 2.14 *A direct, deterministic mechanism $\mathcal{M}$ is DSIC if and only if for all i and $\boldsymbol{v}$,*

(i) *(step-function) $x_i^{\mathcal{M}}(v_i, \boldsymbol{v}_{-i})$ steps from 0 to 1 at some $\hat{v}_i(\boldsymbol{v}_{-i})$, and*
(ii) *(critical value)* $p_i^{\mathcal{M}}(v_i, \boldsymbol{v}_{-i}) = \begin{cases} \hat{v}_i(\boldsymbol{v}_{-i}) & \textit{if } x_i^{\mathcal{M}}(v_i, \boldsymbol{v}_{-i}) = 1 \\ 0 & \textit{otherwise} \end{cases} + p_i^{\mathcal{M}}(0, \boldsymbol{v}_{-i})$.

When we construct mechanisms we will use the "if" directions of these theorems. When discussing incentive compatible mechanisms we will assume that agents follow their equilibrium strategies and, therefore, each agent's bid is equal to her valuation.

Between DSIC and BIC clearly DSIC is a stronger incentive constraint and we should prefer it over BIC if possible. Importantly, DSIC requires fewer assumptions on the agents. For a DSIC mechanisms, each agent must only know her own value; while for a BIC mechanism, each agent must also know the distribution over other agent values. Unfortunately, there will be some environments where we derive BIC mechanisms where no analogous DSIC mechanism is known.

The revelation principle fails to hold in some environments of interest. We will take special care to point these out. Two such environments, for instance, are where agents only learn their values over time, or where the designer does not know the prior distribution (and hence cannot simulate the agent strategies).

Exercises

2.1 Find a symmetric mixed strategy equilibrium in the chicken game described in Section 2.1. I.e., find a probability ρ such that if James Dean stays with probability ρ and swerves with probability $1-\rho$ then Buzz is happy to do the same.

2.2 Give a characterization of Bayes-Nash equilibrium for discrete single-dimensional type spaces for agents with linear utility. Assume that $T = \{v^0, \ldots, v^N\}$ with the probability that an agent's value is $v \in T$ given by probability mass function $f(v)$. Assume $v^0 = 0$. You will not get a payment identity; instead characterize for any BNE allocation rule, the maximum payments.

(a) Give a characterization for the special case where the values are uniform, i.e., $v^j = j$ for all j.

(b) Give a characterization for the special case where the probabilities are uniform, i.e., $f(v^j) = 1/N$ for all j.

(c) Give a characterization for the general case.

(Hint: You should end up with a very similar characterization to that for continuous type spaces.)

2.3 In Section 2.3 we characterized outcomes and payments for BNE in single-dimensional games. This characterization explains what happens when agents behave strategically.

Suppose instead of strategic interaction, we care about fairness. Consider a valuation profile, $\boldsymbol{v} = (v_1, \ldots, v_n)$, an allocation vector, $\boldsymbol{x} = (x_1, \ldots, x_n)$, and payments, $\boldsymbol{p} = (p_1, \ldots, p_n)$. Here x_i is the probability that i is served and p_i is the expected payment of i regardless of whether i is served or not.

Allocation $\boldsymbol{x}$ and payments $\boldsymbol{p}$ are *envy-free* for valuation profile $\boldsymbol{v}$ if no agent wants to unilaterally swap allocation and payment with another agent. I.e., for all i and j,

$$v_i x_i - p_i \geq v_i x_j - p_j.$$

Characterize envy-free allocations and payments (and prove your characterization correct). Unlike the BNE characterization, your characterization of payments will not be unique. Instead, characterize the minimum payments that are envy-free. Draw a diagram illustrating your payment characterization. (Hint: You should end up with a very similar characterization to that of BNE.)

2.4 AdWords is a Google Inc. product in which the company sells the placement of advertisements along side the search results on its search results page. Consider the following *position auction* environment which provides a simplified model of AdWords. There are m advertisement slots that appear along side search results and n advertisers. Advertiser i has value v_i for a click. Slot j has *click-through rate* w_j, meaning, if an advertiser is assigned slot j the advertiser will receive a click with probability w_j. Each advertiser can be assigned at most one slot and each slot can be assigned at most one advertiser. If a slot is left empty, all subsequent slots must be left empty, i.e., slots cannot be skipped. Assume that the slots are ordered from highest click-through rate to lowest, i.e., $w_j \geq w_{j+1}$ for all j.

(a) Find the envy-free (see Exercise 2.3) outcome and payments with the maximum social surplus. Give a description and formula for the envy-free outcome and payments for each advertiser. (Feel free to specify your payment formula with a comprehensive picture.)

(b) In the real AdWords problem, advertisers only pay if they receive a click, whereas the payments calculated, i.e., $\boldsymbol{p}$, are in expected over all outcomes, click or no click. If we are going to charge advertisers only if they are clicked on, give a formula for calculating these payments $\boldsymbol{p}'$ from $\boldsymbol{p}$.

(c) The real AdWords problem is solved by auction. Design an auction that maximizes the social surplus in dominant strategy equilibrium. Give a formula for the payment rule of your auction (again, a comprehensive picture is fine). Compare your DSE payment rule to the envy-free payment rule. Draw some informal conclusions.

2.5 Consider the first-price auction for selling a single item to two agents whose values are independent but not identical. In each of the settings below prove or disprove the claim that there is a Bayes-Nash equilibrium wherein the item is always allocated to the agent with the highest value.

(a) Agent 1 has value $U[0, 1]$ and agent 2 has value $U[0, 1/2]$.

(b) Agent 1 has value $U[0, 1]$ and agent 2 has value $U[1/2, 1]$.

2.6 Consider the first-price auction for selling k units of an item to n unit-demand agents. This auction solicits bids and allocates one unit to each of the k highest-bidding agents. These winners are charged their bids. This auction is revenue equivalent to the k-unit "second-price" auction where the winners are charged the $(k+1)$st highest bid, $b_{(k+1)}$. Solve for the symmetric Bayes-Nash equilibrium strategies in the first-price auction when the agent values are i.i.d. $U[0,1]$.

2.7 Consider the position auction environment with $n = m = 2$ (see Exercise 2.4). Consider running the following first-price auction: The advertisers submit bids $\boldsymbol{b} = (b_1, b_2)$. The advertisers are assigned to slots in order of their bids. Advertisers pay their bid when clicked. Use revenue equivalence to solve for BNE strategies $\boldsymbol{s}$ when the values of the advertisers are drawn independent and identically from $U[0,1]$.

2.8 Prove that in a two-agent second-price auction for a single-item, that the best Bayes-Nash equilibrium can have a social surplus (i.e., the expected value of the winner) that is arbitrarily larger than the worst Bayes-Nash equilibrium. (Hint: Show that for any fixed β that there is a value distribution $\boldsymbol{F}$ and two BNE where the social surplus in one BNE is strictly larger than a β fraction of the social surplus of the other BNE.)

2.9 Show that with independent, identical, and continuously distributed values, the two-agent all-pay auction (where agents bid, the highest-bidder wins, and all agents pay their bids) admits exactly one strictly continuous Bayes-Nash equilibrium.

2.10 Show that with independent, identical, and continuously distributed values, the two-agent first-price position auction (cf. Exercise 2.4; where agents bid, the highest bidder is served with given probability w_1, the second-highest bidder is served with given probability $w_2 \leq w_1$, and all agents pay their bids when they are served) admits exactly one strictly continuous Bayes-Nash equilibrium.

2.11 Consider the following auction with first-price payment semantics. Agents bid, any agent whose bid is (weakly) higher than all other bids wins, all winners are charged their bids. Notice that in the case of a tie in the highest bid, all of the tied agents win. Prove that there are multiple Bayes-Nash equilibria when agents have values that are independently, identically, and continuously distributed.

2.12 Prove Lemma 2.9: For two agents with values drawn independently and identically from a continuous distribution F with support $[0,1]$, the first-price auction with an unknown random reserve from known distribution G admits no asymmetric Bayes-Nash equilibrium. I.e., remove the assumption of strictly-increasing and continuous strategies from the proof given in the text.

Chapter Notes

The formulation of Bayesian games is due to Harsanyi (1967). The characterization of Bayes-Nash equilibrium, revenue equivalence, and the revelation principle come from Myerson (1981). Parts of the BNE characterization proof presented here come from Archer and Tardos (2001). Amann and Leininger (1996), Bajari (2001), Maskin and Riley (2003), and Lebrun (2006) studied the uniqueness of equilibrium in the first-price and all-pay auctions. The revenue-equivalence-based uniqueness proof presented here is from Chawla and Hartline (2013).

The position auction was formulated by Edelman et al. (2007) and Varian (2007); see Jansen and Mullen (2008) for the history of auctions for advertisements on search engines. Envy freedom has been considered in algorithmic (e.g., Guruswami et al., 2005) and economic (e.g., Jackson and Kremer, 2007) contexts. Hartline and Yan (2011) characterized envy-free outcomes for single-dimensional agents.

3
Optimal Mechanisms

In this chapter we discuss the objectives of social surplus and profit. As we will see, the economics of designing mechanisms to maximize social surplus is relatively simple. The optimal mechanism is a simple generalization of the second-price auction we have already discussed. Furthermore, it is dominant strategy incentive compatible and prior-free, i.e., it is not dependent on distributional assumptions. Social surplus maximization is unique among economic objectives in this regard.

The objective of profit maximization, on the other hand, adds significant new challenge: for profit there is no single optimal mechanism. For any mechanism, there is a distribution over agent preferences and another mechanism where this new mechanism has strictly larger profit than the first one.

This non-existence of an absolutely optimal mechanism requires a relaxation of what we consider a good mechanism. To address this challenge, this chapter follows the traditional economics approach of Bayesian optimization. We will assume that the distribution of the agents' preferences is common knowledge, even to the mechanism designer. This designer should then search for the mechanism that maximizes her expected profit when preferences are indeed drawn from the distribution.

As an example, consider two agents with values drawn independently and identically from $U[0, 1]$. The second-price auction obtains revenue equal to the expected second-highest value, $\mathbf{E}\left[v_{(2)}\right] = {}^{1}/_{3}$. A natural question is whether more revenue can be had. As a first step, it is similarly easy to calculate that the second-price auction with reserve ${}^{1}/_{2}$ obtains an expected revenue of ${}^{5}/_{12}$ (which is higher than ${}^{1}/_{3}$).[1] Above,

[1] There are three cases: (i) ${}^{1}/_{2} > v_{(1)} > v_{(2)}$, (ii) $v_{(1)} > {}^{1}/_{2} > v_{(2)}$, and (iii), $v_{(1)} > v_{(2)} > {}^{1}/_{2}$. Case (i) happens with probability ${}^{1}/_{4}$ and has no revenue; case

perhaps surprisingly, a seller makes more money by sometimes not selling the item even when there is a buyer willing to pay. In this chapter we show that the second-price auction with reserve $1/2$ is indeed optimal for this two agent example and furthermore we give a concise characterization of the revenue-optimal auction for any single-dimensional agent environment.

3.1 Single-dimensional Environments

In our previous discussion of Bayes-Nash equilibrium we focused on the agents' incentives. Single-dimensional linear agents each have a single private value for receiving some abstract service and linear utility, i.e., the agent's utility is her value for the service less her payment (Definition 2.6). Recall that the outcome of a single-dimensional game is an allocation $\boldsymbol{x} = (x_1, \ldots, x_n)$, where x_i is an indicator for whether agent i is served, and payments $\boldsymbol{p} = (p_1, \ldots, p_n)$, where p_i is the payment made by agent i. Here we formalize the designer's constraints and objectives.

Definition 3.1 A *general cost environment* is one where the designer must pay a service cost $c(\boldsymbol{x})$ for the allocation $\boldsymbol{x}$ produced. A *general feasibility environment* is one where there is a feasibility constraint over the set of agents that can be simultaneously served. A *downward-closed* feasibility constraint is one where subsets of feasible sets are feasible.

Of course, downward-closed environments are a special case of general feasibility environments which are a special case of general cost environments. We can express general feasibility environments as general costs environments were $c(\cdot) \in \{0, \infty\}$. We can similarly express downward-closed feasibility environments as the further restriction where $\boldsymbol{x}^\dagger \leq \boldsymbol{x}$ (i.e., for all i, $x_i^\dagger \leq x_i$) and $c(\boldsymbol{x}) = 0$ and implies that $c(\boldsymbol{x}^\dagger) = 0$. We will be aiming for general mechanism design results and the most general results will be the ones that hold in the most general environments. We will pay special attention to restrictions on the environment that enable illuminating observations about optimal mechanisms.

(ii) happens with probability $1/2$ and has revenue $1/2$; and case (iii) happens with probability $1/4$ and has expected revenue $\mathbf{E}\big[v_{(2)} \mid \text{case (iii) occurs}\big] = 2/3$. The calculation of the expected revenue in case (iii) follows from the conditional values being $U[1/2, 1]$ and the fact that, in expectation, uniform random variables evenly divide the interval they are over. The total expected revenue can then be calculated as $5/12$.

The two most fundamental designer objectives are social surplus, a.k.a., social welfare,[2] and profit.

Definition 3.2 The *social surplus* of an allocation is the cumulative value of the agents served less the service cost:

$$\text{Surplus}(\boldsymbol{v}, \boldsymbol{x}) = \sum\nolimits_i v_i \cdot x_i - c(\boldsymbol{x}).$$

The *profit* of allocation and payments is the cumulative payment of the agents less the service cost:

$$\text{Profit}(\boldsymbol{p}, \boldsymbol{x}) = \sum\nolimits_i p_i - c(\boldsymbol{x}).$$

Implicit in the definition of social surplus is the fact that the payments from the agents are transferred to the service provider and therefore do not affect the objective.[3]

The single-item and routing environments that were discussed in Chapter 1 are special cases of downward-closed environments. Single-item environments have

$$c(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \sum_i x_i \leq 1, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

In routing environments, recall, each agent has a message to send between a source and destination in the network.

$$c(\boldsymbol{x}) = \begin{cases} 0 & \text{if messages with } x_i = 1 \text{ can be simultaneously routed, and} \\ \infty & \text{otherwise.} \end{cases}$$

We have yet to see any examples of general cost environments. One natural one is that of a *multicast auction*. The story for this problem comes from live video steaming. Suppose we wish to stream live video to viewers (agents) in a computer network. Because of the high-bandwidth nature of video streaming the content provider must lease the network links. Each link has a publicly known cost. To serve a set of agents, the designer must pay the cost of network links that connect each agent, located at different nodes in the network, to the “root”, i.e., the origin

[2] A mechanism that optimizes social surplus is said to be *economically efficient*; though, we will not use this terminology because of possible confusion with *computational efficiency*. A mechanism is computationally efficient if it computes its outcome quickly (see Chapter 8).

[3] An alternative notion would be to consider only the total value derived by the agents, i.e., the surplus less the total payments. This *residual surplus* was discussed in detail in Chapter 1; mechanisms for optimizing residual surplus are the subject of Exercise 3.1.

of the multicast. The nature of multicast is that the messages need only be transmitted once on each edge to reach the agents. Therefore, the total cost to serve these agents is the minimum cost of the *multicast tree* that connects them.[4]

3.2 Social Surplus

We now derive the optimal mechanism for social surplus. To do this we walk through a standard approach in mechanism design. We completely relax the Bayes-Nash equilibrium incentive constraints and ask and solve the remaining non-game-theoretic optimization question. We then verify that this solution does not violate the incentive constraints. We conclude that the resulting mechanism is optimal.

The non-game-theoretic optimization problem of maximizing surplus for input $\boldsymbol{v} = (v_1, \ldots, v_n)$ is that of finding $\boldsymbol{x}$ to maximize $\mathrm{Surplus}(\boldsymbol{v}, \boldsymbol{x}) = \sum_i v_i x_i - c(\boldsymbol{x})$. Let OPT be an optimal algorithm for solving this problem. We will care about both the allocation that OPT selects, i.e., $\mathrm{argmax}_{\boldsymbol{x}} \mathrm{Surplus}(\boldsymbol{v}, \boldsymbol{x})$ and its surplus $\max_{\boldsymbol{x}} \mathrm{Surplus}(\boldsymbol{v}, \boldsymbol{x})$. Where it is unambiguous we will use notation $\mathrm{OPT}(\boldsymbol{v})$ to denote either of these quantities. Notice that the formulation of OPT has no mention of Bayes-Nash equilibrium incentive constraints.

We know from our characterization that the allocation rule of any BNE is monotone, and that any monotone allocation rule can be implemented in BNE with the appropriate payment rule. Thus, relative to the non-game-theoretic optimization, the mechanism design problem of finding a BIC mechanism to maximize surplus has an added monotonicity constraint. As it turns out, even though we did not impose a monotonicity constraint on OPT, it is satisfied anyway.

Lemma 3.1 *For each agent i and all values of other agents $\boldsymbol{v}_{-i}$, the allocation rule of* OPT *for agent i is a step function.*

Proof Consider any agent i. There are two situations of interest. Either i is served by $\mathrm{OPT}(\boldsymbol{v})$ or i is not served by $\mathrm{OPT}(\boldsymbol{v})$. We write out the surplus of OPT in both of these cases. Below, notation $(z, \boldsymbol{v}_{-i})$ denotes the vector $\boldsymbol{v}$ with the ith coordinate replaced with z.

[4] In combinatorial optimization this problem is known as the *weighted Steiner tree* problem. It is a computationally challenging variant of the *minimum spanning tree* problem.

Case 1 ($i \in \mathrm{OPT}$):

$$\begin{aligned}\mathrm{OPT}(\boldsymbol{v}) &= \max_{\boldsymbol{x}} \mathrm{Surplus}(\boldsymbol{v}, \boldsymbol{x}) \\ &= v_i + \max_{\boldsymbol{x}_{-i}} \mathrm{Surplus}((0, \boldsymbol{v}_{-i}), (1, \boldsymbol{x}_{-i})).\end{aligned}$$

Define $\mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i})$, the optimal surplus from agents other than i assuming that i is served, as the second term on the right hand side. Thus,

$$\mathrm{OPT}(\boldsymbol{v}) = v_i + \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i}).$$

Notice that $\mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i})$ is not a function of v_i.

Case 2 ($i \notin \mathrm{OPT}$):

$$\begin{aligned}\mathrm{OPT}(\boldsymbol{v}) &= \max_{\boldsymbol{x}} \mathrm{Surplus}(\boldsymbol{v}, \boldsymbol{x}) \\ &= \max_{\boldsymbol{x}_{-i}} \mathrm{Surplus}((0, \boldsymbol{v}_{-i}), (0, \boldsymbol{x}_{-i})).\end{aligned}$$

Define $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$, the optimal surplus from agents other than i assuming that i is not served, as the term on the right hand side. Thus,

$$\mathrm{OPT}(\boldsymbol{v}) = \mathrm{OPT}(0, \boldsymbol{v}_{-i}).$$

Notice that $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$ is not a function of v_i.

OPT chooses whether or not to allocate to agent i, and thus which of these cases we are in, so as to optimize the surplus. Therefore, OPT allocates to i whenever the surplus from Case 1 is greater than the surplus from Case 2. I.e., when

$$v_i + \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i}) \geq \mathrm{OPT}(0, \boldsymbol{v}_{-i}).$$

Solving for v_i we conclude that OPT allocates to i whenever

$$v_i \geq \mathrm{OPT}(0, \boldsymbol{v}_{-i}) - \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i}).$$

Notice that neither of the terms on the right hand side contain v_i. Therefore, the allocation rule for i is a step function with critical value $\hat{v}_i = \mathrm{OPT}(0, \boldsymbol{v}_{-i}) - \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i})$. □

Since the allocation rule induced by OPT is a step function, it satisfies our strongest incentive constraint: with the appropriate payments (i.e., the "critical values") truthtelling is a dominant strategy equilibrium (Corollary 2.14). The resulting surplus maximization mechanism is often referred to as the *Vickrey-Clarke-Groves* (VCG) mechanism, named after William Vickrey, Edward Clarke, and Theodore Groves.

Definition 3.3 The *surplus maximization* (SM) mechanism is:

(i) Solicit and accept sealed bids $\boldsymbol{b}$.
(ii) find the optimal outcome $\boldsymbol{x} \leftarrow \mathrm{OPT}(\boldsymbol{b})$, and
(iii) set prices $\boldsymbol{p}$ as

$$p_i \leftarrow \begin{cases} \mathrm{OPT}(0, \boldsymbol{b}_{-i}) - \mathrm{OPT}_{-i}(\infty, \boldsymbol{b}) & \text{if } i \text{ is served} \\ 0 & \text{otherwise.} \end{cases}$$

An intuitive description of the critical value $\hat{v}_i = \mathrm{OPT}(0, \boldsymbol{v}_{-i}) - \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i})$ is the *externality* that agent i imposes on the other agents by being served. In other words, because i is served the other agents obtain total surplus $\mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i})$ instead of the surplus $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$ that they would have received if i was not served. We can similarly write $p_i = \mathrm{OPT}(0, \boldsymbol{v}_{-i}) - \mathrm{OPT}_{-i}(\boldsymbol{v})$ as the externality agent i imposes by being present in the mechanism (regardless of whether she is served or not). Note that if she is not served then the second term is equal to the first and the externality she imposes is zero. Hence, we can interpret the surplus maximization mechanism as serving agents to maximize the social surplus and charging each agent the externality she imposes on the others.

By Corollary 2.14 and Lemma 3.1 we have the following theorem, and by the optimality of OPT and the assumption that agents follow the dominant truthtelling strategy we have the following corollary.

Theorem 3.2 *The surplus maximization mechanism is dominant strategy incentive compatible.*

Corollary 3.3 *The surplus maximization mechanism optimizes social surplus in dominant strategy equilibrium.*

Example 3.4 The second-price routing auction from Chapter 1 is an instantiation of the surplus maximization mechanism where feasible outcomes are subsets of agents whose messages can be simultaneously routed.

It is useful to view the surplus maximization mechanism as a reduction from the mechanism design problem to the non-game-theoretic optimization problem. Given an algorithm that solves the non-game-theoretic optimization problem, i.e., OPT, we can construct the surplus maximization mechanism from it.

Surplus maximization is singular among objectives in that there is a single mechanism that is optimal regardless of distributional assumptions. Essentially: the agents' incentives are already aligned with the designer's objective and one only needs to derive the appropriate payments, i.e., the critical values. For general objectives, e.g., in the next section we will discuss profit maximization, the optimal mechanism is distribution dependent.

There are other ways to implement surplus maximization besides that of Definition 3.3. By revenue equivalence, the payment rule of the surplus maximization mechanism is unique up to the payments each agent would make if her value was zero, i.e., $p_i(0, \boldsymbol{v}_{-i})$ for agent i. For instance $p_i = \mathrm{OPT}_{-i}(\boldsymbol{v})$ is an DSIC payment rule as well with $p_i(0, \boldsymbol{v}_{-i}) = \mathrm{OPT}(0, \boldsymbol{v}_{-i})$. This payment rule does not satisfy the natural *no-positive-transfers* condition which requires that agents not be paid to participate. It is also possible to design BNE mechanisms, e.g., with first-price semantics, that implement the same outcome in equilibrium as the surplus maximization mechanism (see Exercise 3.2), though unlike the surplus maximization mechanism given above, design of such a BNE mechanism requires distributional knowledge.

3.3 Profit

A non-game-theoretic optimization problem looks to maximize some objective subject to feasibility. Given the input, we can search over feasible outcomes for the one with the highest objective value for this input. The outcome produced on one input need not bear any relation to the outcome produced on an (even slightly) different input. Mechanisms, on the other hand, additionally must address agent incentives which impose constraints over the outcomes that the mechanism produces across all possible misreports of the agents. In other words, the mechanism's outcome on one input is constrained by its outcome on similar inputs. Therefore, a mechanism may need to tradeoff its objective performance across inputs.

When the distribution of agent values is specified, e.g., by a common prior (Definition 2.5) and the designer has knowledge of this prior, such a tradeoff can be optimized. In particular, the prior assigns a probability to each input and the designer can then optimize expected objective value over this probability distribution. The mechanism that results from such an optimization is said to be *Bayesian optimal*. In this section we derive the Bayesian optimal mechanism for the objective of profit. Other objectives that are linear in social surplus and payments can be similarly considered (e.g., residual surplus, see Exercise 3.1).

We will use agents with values drawn from the following distributions as examples.

Example 3.5 A *uniform agent* has single-dimensional linear utility with value v drawn uniformly from $[0, 1]$, i.e., $F(z) = z$ and $f(z) = 1$.

Example 3.6 A *bimodal agent* has single-dimensional linear utility with value v drawn uniformly from $[0, 3]$ with probability $3/4$ and uniformly from $(3, 8]$ with probability $1/4$, i.e., the distribution defined by density function $f(v) = 1/4$ for $v \in [0, 3]$ and $f(v) = 1/20$ for $v \in (3, 8]$ (see Figure 3.4, page 71).

Mathematical Note. At various points in the remainder of this chapter it will be convenient to write the expectations of discontinuous distributions via the integral of their density function which is, at their discontinuity, not well defined. We will then reinterpret the expectation via integration by parts. This notational convenience can be made precise via the Dirac delta function which integrates to a step function; however, we will not describe these details formally.

Consider, as an example, the following which is taken from the construction of Proposition 3.15 on page 74. Draw a random variable $\hat{q} \in [0, 1]$ from a distribution G with distribution function $G(q)$. If G is continuous then its density $g(q) = \frac{\mathrm{d}}{\mathrm{d}q} G(q)$ is well defined and we can write the expectation of some function $P(\cdot)$ of $\hat{q}$ as $\mathbf{E}_{\hat{q} \sim G}[P(\hat{q})] = \int_0^1 P(q) g(q) \, \mathrm{d}q$. If G is discontinuous (i.e., it possesses point masses) the same formula is correct when the density g contains the appropriate Dirac delta function.

A change of variables allows any integral over $[0, 1]$ to be reinterpreted as the expectation of a function of a uniform random variable. From the above example,

$$\mathbf{E}_{\hat{q} \sim G}[P(\hat{q})] = \mathbf{E}_{q \sim U[0,1]}[P(q) g(q)] \, .$$

Finally, integration by parts gives, for example, the following formula for rearranging an integral, with $\frac{\mathrm{d}}{\mathrm{d}q} P(q)$ denoted by $p(q)$,

$$\int_0^1 P(q) g(q) \, \mathrm{d}q = \Big[P(q) G(q) \Big]_0^1 - \int_0^1 p(q) G(q) \, \mathrm{d}q.$$

When $P(0) = P(1) = 0$ the first term on the right-hand side is identically zero. If not, we can set $P(0) = P(1) = 0$ which will introduce a discontinuity in to $P(\cdot)$ which we can express in $p(\cdot)$ via the Dirac delta function as described above. Formulaically, this modification allows the first term of the right-hand side to be accounted for by the integral. We can, as above, write these integrals as expectations of functions of a uniform random variable. Integration by parts can be thus expressed for $q \sim U[0,1]$ as:

$$\mathbf{E}[P(q)g(q)] = \mathbf{E}[-p(q)G(q)]\,.$$

3.3.1 Highlevel Approach: Amortized Analysis

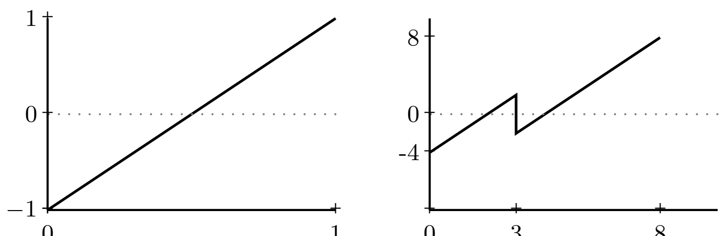
The profit of a mechanism is given by the sum of the agents' payments (minus the cost of serving them) which, via the payment identity of Theorem 2.2, namely

$$p(v) = v \cdot x(v) - \int_0^v x(v^\dagger)\,\mathrm{d}v^\dagger, \tag{3.1}$$

depends on the allocation rule of each agent (in particular, on $x(v^\dagger)$ for $v^\dagger \leq v$ for an agent with value v). In other words, what the mechanism chooses to do when the agent's value is $v^\dagger < v$ affects the revenue the mechanism obtains when her value is v.

This dependence of the payment on the allocation that the agent would receive if she had a lower value implies that there is no pointwise optimal mechanism (as there was for social surplus maximization, cf. Section 3.2). Consider selling an item to a single agent with value v drawn uniformly from $[0,1]$ (Example 3.5). If her value is 0.2, then it is pointwise optimal to offer her the item at price 0.2. This corresponds to the allocation rule which steps from zero to one at 0.2. Similarly if her value is 0.7, then it is pointwise optimal to offer her the item at price 0.7. Of course, offering a 0.7-valued agent a price of 0.2 or a 0.2-valued agent a price of 0.7 is not optimal. There is no single mechanism that is pointwise optimal on both of these inputs. On the other hand, given a distribution over the agent's value, we can easily optimize for the price with maximum expected revenue: post the price $\hat{v}$ that maximizes $\hat{v} \cdot (1 - F(\hat{v}))$. For the uniform agent where $F(z) = z$, this optimal price is $\hat{v}^\star = 1/2$.[5]

[5] Set $\frac{\mathrm{d}}{\mathrm{d}\hat{v}}[\hat{v} \cdot (1 - \hat{v})] = 1 - 2\hat{v} = 0$ and solve for $\hat{v}$ to get the optimal price to post of $\hat{v}^\star = 1/2$.



(a) Uniform agent virtual value. (b) Bimodal agent virtual value.

Figure 3.1 Depicted are virtual value functions $\phi(v) = v - \frac{1-F(v)}{f(v)}$ for the uniform and bimodal agent examples (Example 3.5 and Example 3.6). Notice that the virtual value function in the uniform example is monotone non-decreasing in value while in the bimodal example it is not. For reference, the line $v_2 = v_1$ is depicted (grey dotted line).

The payment identity (3.1) gives a formula for the expected payment that a v-valued agent makes in terms of her allocation rule. As is evident from the integral form of the payment identity, an agent's payment at a given value depends on the allocation probability she would have obtained with a lower value. In fact, her payment is highest when the allocation to lower values is the lowest. Our approach to optimizing profit will be via an *amortized analysis* where we charge the loss in revenue from high values due to high allocation probability at low values to the low values themselves. Via such an approach, the amortized benefit from serving an agent with a given value is her value less a deduction that accounts for the lowered the payment for higher values. We will refer to this amortized benefit as *virtual value* and we will show that the problem of optimizing profit in expectation over the distribution of values reduces to the problem of maximizing *virtual surplus* pointwise.

A straightforward approach to such an amortized analysis (given subsequently in Section 3.3.4) will give virtual value function

$$\phi(v) = v - \frac{1 - F(v)}{f(v)}. \tag{3.2}$$

In equation (3.2), v is the revenue from serving the agent with value v (at a price of v) and $\frac{1-F(v)}{f(v)}$ represents the loss of revenue from serving higher values. We will see that such a formulation satisfies

$$\mathbf{E}_{v\sim F}[p(v)] = \mathbf{E}_{v\sim F}[\phi(v) \cdot x(v)] \tag{3.3}$$

for any allocation and payment rules (x, p) that satisfy the Bayes-Nash equilibrium characterization (Theorem 2.2; i.e., monotonicity of x and the payment identity (3.1)). Equation (3.3) can be derived simply by applying the definition of expectation (as an integral) to the payment identity and simplifying (see Exercise 3.3); we will give a less direct but more economically intuitive construction subsequently in Section 3.3.4.

From equation (3.2) the virtual value function for the uniform agent example is $\phi(v) = 2v - 1$; for the bimodal agent example it is depicted in Figure 3.1. Notice that $\phi(0) < 0$ as there is no value from serving an agent with value zero but serving such an agent lowers the price that she could be charged if her value were higher. Notice that the highest virtual value is always equal to the highest value as there is no amortized deduction necessary to account for lower prices obtained by higher values as no higher values exist, e.g., the uniform agent with values on interval $[0, 1]$ has $\phi(1) = 1$ and the bimodal agent with values on interval $[0, 8]$ has $\phi(8) = 8$.

The importance of equation (3.3) is that it enables the non-pointwise optimization of expected payments to be recast as a pointwise optimization of virtual surplus. The non-game-theoretic optimization problem of maximizing virtual surplus is that of finding $\boldsymbol{x}$ to maximize $\text{Surplus}(\boldsymbol{\phi}(\boldsymbol{v}), \boldsymbol{x}) = \sum_i \phi_i(v_i) \cdot x_i - c(\boldsymbol{x})$.[6] Let OPT again be the surplus maximizing algorithm. We will care about both the allocation that $\text{OPT}(\boldsymbol{\phi}(\boldsymbol{v}))$ selects, i.e., $\text{argmax}_{\boldsymbol{x}} \text{Surplus}(\boldsymbol{\phi}(\boldsymbol{v}), \boldsymbol{x})$ and its virtual surplus $\max_{\boldsymbol{x}} \text{Surplus}(\boldsymbol{\phi}(\boldsymbol{v}), \boldsymbol{x})$. Where it is unambiguous we will use notation $\text{OPT}(\boldsymbol{\phi}(\boldsymbol{v}))$ to denote either of these quantities. Note that this formulation of OPT has no mention of the incentive constraints.

We now give the first part of the derivation of the optimal mechanism for virtual surplus (and, hence, for profit). To do this we again walk through a standard approach in mechanism design. We completely relax the incentive constraints and solve the remaining non-game-theoretic optimization problem. Since expected profit equals expected virtual surplus, this non-game-theoretic optimization problem is to optimize virtual surplus. We then verify that this solution does not violate the incentive constraints (under some conditions). We conclude that (under the same conditions) the resulting mechanism is optimal.

We know from the BIC characterization (Corollary 2.12) that incentive constraints require that the allocation rule be monotone. Thus, the mechanism design problem of finding a BIC mechanism to maximize virtual surplus has an added monotonicity constraint. Notice that, even

[6] Here, $\boldsymbol{\phi}(\boldsymbol{v})$ denotes the profile of virtual values $(\phi_1(v_1), \ldots, \phi_n(v_n))$.

though we did not impose a monotonicity constraint on $\mathrm{OPT}(\boldsymbol{\phi}(\cdot))$, if the virtual valuation functions $\phi_i(\cdot)$ for each agent i are monotone then $\mathrm{OPT}(\boldsymbol{\phi}(\cdot))$ is monotone.

Lemma 3.7 *For any profile of virtual value functions $\boldsymbol{\phi}$, monotonicity of $\phi_i(\cdot)$ implies the monotonicity of the allocation to agent i of $\mathrm{OPT}(\boldsymbol{\phi}(z, \boldsymbol{v}_{-i}))$ with respect to z.*

Proof Let $\boldsymbol{x}(\cdot)$ be the allocation rules of OPT, i.e., $\boldsymbol{x}(\boldsymbol{v}) = \mathrm{argmax}_{\boldsymbol{x}^\dagger} \mathrm{Surplus}(\boldsymbol{v}, \boldsymbol{x}^\dagger)$. Recall from Lemma 3.1 that maximizing surplus is monotone in that $x_i(z, \boldsymbol{v}_{-i})$ is monotone in z. Therefore $x_i(\phi_i(z), \boldsymbol{\phi}_{-i}(\boldsymbol{v}_{-i}))$ is monotone in $\phi_i(z)$, i.e., increasing $\phi_i(z)$ does decrease x_i. By assumption $\phi_i(z)$ is monotone in z; therefore, increasing z cannot decrease $\phi_i(z)$ which cannot decrease $x_i(\phi_i(z), \boldsymbol{\phi}_{-i}(\boldsymbol{v}_{-i}))$. □

For many distributions the virtual value function $v - \frac{1-F(v)}{f(v)}$ of equation (3.2) is monotone, e.g., uniform (Example 3.5), normal, and exponential distributions. We refer to these as regular distributions. For regular distributions the approach suggested above is sufficient for describing the optimal mechanism.

Definition 3.4 A distribution F is *regular* if $v - \frac{1-F(v)}{f(v)}$ is monotone non-decreasing.

On the other hand, many relevant distributions are irregular, e.g., bimodal (Example 3.6; Figure 3.1(b)). For irregular distributions a more sophisticated amortized analysis is needed to derive the appropriate virtual values. To obtain a mechanism that optimizes non-monotone virtual value functions we cannot initially relax the monotonicity constraint; instead we must optimize virtual surplus subject to monotonicity. In Section 3.3.5 we will describe a generic procedure for *ironing* a non-monotone virtual value function to obtain a monotone (ironed) virtual value function. For ironed virtual values from this procedure, pointwise optimization of the ironed virtual surplus is equivalent to optimization of the original virtual surplus subject to monotonicity. We conclude that, even for irregular distributions, the design of optimal mechanisms in expectation for a known distribution on values is equivalent to the pointwise optimization of a virtual surplus that is given by monotone virtual value functions.

3.3.2 The Virtual Surplus Maximization Mechanism

As revenue-optimal mechanism are virtual surplus maximizers, we now give a generic and formal description of this sort of mechanism. For monotone virtual value functions, Lemma 3.7 implies that virtual surplus maximization gives a monotone allocation rule for each agent and any fixed values of the other agents; therefore, it satisfies our strongest incentive constraint. With the appropriate payments (i.e., the "critical values") truthtelling is a dominant strategy equilibrium (recall Corollary 2.14). One way to view the suggested virtual surplus maximization mechanism is as a reduction to surplus maximization, which is solved by the SM mechanism (Definition 3.3; also known as VCG).

Definition 3.5 The *virtual surplus maximization* (VSM) mechanism for single-dimensional linear agents and monotone virtual value functions $\boldsymbol{\phi}$ is:

(i) Solicit and accept sealed bids $\boldsymbol{b}$,
(ii) simulate the surplus maximization mechanism on virtual bids

$$(\boldsymbol{x}, \boldsymbol{p}^{\dagger}) \leftarrow \mathrm{SM}(\boldsymbol{\phi}(\boldsymbol{b})),$$

(iii) set prices $\boldsymbol{p}$ from critical values as

$$p_i \leftarrow \begin{cases} \phi_i^{-1}(p_i^{\dagger}) & \text{if } i \text{ is served,} \\ 0 & \text{otherwise, and} \end{cases}$$

(iv) output outcome $(\boldsymbol{x}, \boldsymbol{p})$.

Notice that the payments $\boldsymbol{p}$ calculated by VSM can be viewed as follows. SM on virtual values outputs virtual prices $\boldsymbol{p}^{\dagger}$. For winners these correspond to the minimum virtual value that the agent must have to win. The price an agent pays is the minimum value that she must have to win, this can be calculated from these virtual prices via the inverse virtual valuation function. (For virtual value functions $\phi(\cdot)$ that are discontinuous or not strictly increasing this inverse virtual value function is defined as $\phi^{-1}(z) = \inf\{v^{\dagger} : \phi(v^{\dagger}) \geq z\}$.)

Theorem 3.8 *For monotone virtual value functions $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_n)$, the virtual surplus maximization mechanism* VSM *is dominant strategy incentive compatible.*

Proof The theorem follows from Lemma 3.7 applied to each agent, the definition of VSM, and Corollary 2.14. □

Corollary 3.9 *For monotone virtual value functions $\boldsymbol{\phi}$, the virtual surplus maximization mechanism optimizes virtual surplus in dominant strategy equilibrium.*

Notice that the approach above was for optimization of an objective in expectation in Bayes-Nash equilibrium. The mechanism we obtained, in fact, satisfies the stronger dominant strategy incentive compatibility condition. Moreover, even though possibly randomized mechanisms were optimized over, the optimal mechanism is deterministic. When there are ties in virtual surplus, i.e., by multiple distinct outcomes each of which gives the same virtual surplus, these ties can be broken arbitrarily; we may, however, prefer the symmetry of random tie breaking.

To employ Corollary 3.9 for optimizing a given objective, it remains to find a virtual value function for which pointwise optimization of virtual surplus corresponds to optimization of the expected objective value.

Definition 3.6 A *virtual value function* $\phi(\cdot)$ for a given objective is a weakly monotone function that maps a value to a virtual value for which expected optimal virtual surplus is equal to the optimal expected objective value.

3.3.3 Single-item Environments

The above description of the virtual surplus maximization mechanisms does not offer much in the way of intuition. To get a clearer picture, we consider optimal mechanisms the special case of single-item environments, i.e., where the feasible outcomes serve at most one agent. We will consider here four special cases: a single agent, multiple (generally asymmetric) agents, multiple agents with a symmetric strictly-increasing virtual value function, and multiple agents with a symmetric (not strictly) increasing virtual value function.

For a single agent with a monotone virtual value function $\phi(\cdot)$, there is some value $\hat{v}^\star = \phi^{-1}(0)$ where the function crosses zero. For example, for the uniform agent this value is $\hat{v}^\star = 1/2$, see Figure 3.1(a). Maximizing virtual surplus is simple: if $v \geq \hat{v}^\star$ then serve the agent; otherwise, do not serve the agent. In other words, the agent has a critical value of $\hat{v}^\star$ and the outcome is identical to that from posting a take-it-or-leave-it price of $\hat{v}^\star$.

Definition 3.7 For an agent with value v drawn from distribution F and virtual value function ϕ, the *monopoly price* $\hat{v}^\star = \phi^{-1}(0)$ is the posted price that obtains the highest expected virtual surplus.

Now consider a single-item auction environment and the virtual surplus maximization mechanism for the profile of virtual value functions $\boldsymbol{\phi}$. The mechanism will serve the agent with the highest positive virtual value, or nobody if all virtual values are negative. To see what the critical value of an agent i in this auction is we can write out the condition that must hold for the agent to win. In particular, $\phi_i(v_i) \geq \max(\phi_j(v_j), 0)$ for all $j \neq i$, so i's critical value is

$$\hat{v}_i = \max(\phi_i^{-1}(\phi_j(v_j)), \phi_i^{-1}(0)) \tag{3.4}$$

for j with the highest virtual value of the other agents. Notice that the auction depends on the precise details of the virtual value functions (see Example 3.11 below). Notice that the second term in the maximization is the monopoly price $\hat{v}_i^{\star} = \phi_i^{-1}(0)$. If the other agents are not competitive, i.e., all agents j have $\phi_j(v_j) < 0$, then the optimization problem reduces to the single-agent case and agent i should see a reserve price of $\hat{v}_i^{\star}$.

Corollary 3.10 *For single-item environments and monotone virtual value functions, the auction that allocates to the agent with the highest non-negative virtual value maximizes virtual surplus in dominant strategy equilibrium.*

Example 3.11 Consider a two-agent single-item environment with agent 1's (Alice) value from $U[0,1]$ (as in Example 3.5) and agent 2's (Bob) value from $U[0,2]$ (with distribution function $F_2(z) = z/2$). The virtual values for revenue from equation (3.2) are $\phi_1(v_1) = 2v_1 - 1$ and $\phi_2(v_2) = 2v_2 - 2$. The virtual surplus maximization mechanism serves Alice whenever $\phi_1(v_1) > \max(\phi_2(v_2), 0)$, i.e., when $v_1 > \max(v_2 - 1/2, 1/2)$. Note that in this revenue-optimal auction Alice may have a lower value than Bob and still win.

Now suppose the virtual value functions are monotone, strictly increasing, identical, and denoted by ϕ. This happens when the agents are independent and identically distributed and, as discussed above, the function $v - \frac{1-F(v)}{f(v)}$ is strictly monotone. In such a scenario, $\phi_i^{-1}(\phi_j(v_j)) = \phi^{-1}(\phi(v_j)) = v_j$, and equation (3.4) for agent i's critical value simplifies to $\hat{v}_i = \max(v_j, \hat{v}^{\star})$ where j is the highest valued of the other agents. The virtual surplus maximizing auction thus serves the agent with the highest value that is at least $\hat{v}^{\star} = \phi^{-1}(0)$, a.k.a., the monopoly price. What auction has this equilibrium outcome? The second-price auction with monopoly reserve $\hat{v}^{\star}$.

Definition 3.8 The *second-price auction with reservation price* $\hat{v}$, sells the item if any agent bids above $\hat{v}$. The price the winning agent pays the maximum of the second highest bid and $\hat{v}$. The *monopoly-reserve auction* sets $\hat{v} = \hat{v}^\star$.

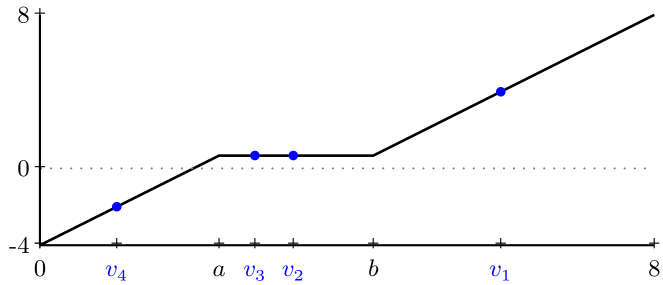
Corollary 3.12 *In single-item environments with identical strictly-increasing virtual value function ϕ, the virtual surplus maximizing mechanism is the second-price auction with monopoly reserve $\hat{v}^\star = \phi^{-1}(0)$.*

Example 3.13 Consider a two-agent single-item environment with i.i.d. uniform agents (as in Example 3.5). As we have calculated, $\phi(v) = 2v - 1$ is monotone and strictly increasing, the monopoly price is $\hat{v}^\star = \phi^{-1}(0) = 1/2$, and the revenue-optimal auction is the second-price auction with reserve price $1/2$. Our calculation at the introduction of this chapter showed its expected revenue to be 5/12. Now we see that this revenue is optimal among all mechanisms for this scenario.

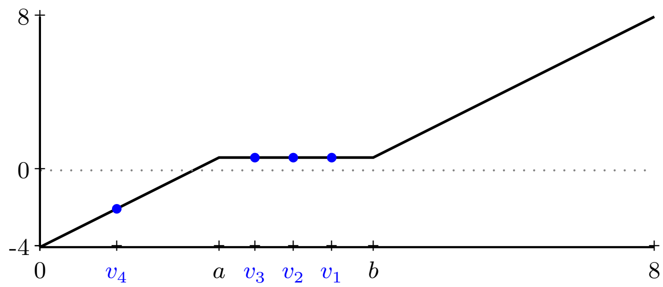
Notice that the optimal reserve price is not a function of the number of agents. For more intuition for why the reserve price is invariant to the number of agents, notice the following. Either the other agents are competitive and the reserve is irrelevant or the other agents are irrelevant and the designer faces the same revenue tradeoffs as in the single-agent example. This single-agent tradeoff is optimized by a reserve equal to the monopoly price. Furthermore, the result can easily be extended to single-item multi-unit auctions where the optimal reserve price is also not a function of the number of units that are for sale (and beyond, see Proposition 4.24 in Chapter 4).

We conclude this section by considering the case of symmetric virtual value functions that are increasing but not strictly so. Notice that, with strictly increasing virtual value functions and values drawn from a continuous distribution, ties in virtual value are a measure zero event, i.e., for any two agents i and j, $\mathbf{Pr}\big[\phi_i(v_i) = \phi_j(v_j)\big] = 0$. On the other hand, when virtual value functions are constant on an interval $[a, b]$ and the distribution assigns some non-zero probability to values in this interval, there is a measurable, i.e., non-zero, probability of ties. The virtual surplus maximization mechanism can break these ties arbitrarily or randomly. Especially in symmetric environments we will prefer the symmetric tie-breaking rule by, e.g., for single-unit environments, choosing the winner of the tie uniformly at random.

It is instructive to see exactly what the virtual surplus maximization mechanism does when there are ties in virtual values. Figure 3.2 depicts



(a) Unique highest virtual value.

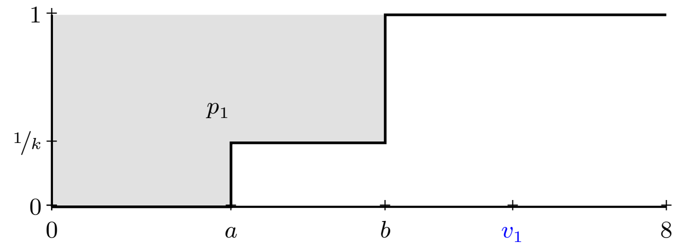


(b) Non-unique highest virtual value.

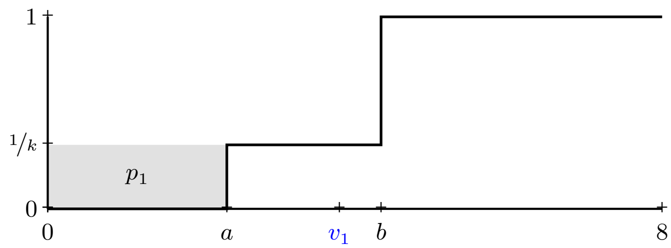
Figure 3.2 The weakly monotone virtual valuation function $\phi(v)$ under two realizations of four agent values depicting both the case where the highest virtual value is unique and the case where it is not unique.

such a virtual valuation function (which corresponds to the ironed virtual value for revenue for the bimodal agent that will be derived subsequently in Section 3.3.5). Instantiating the agents' values corresponds to picking points on the horizontal axis. The agents' virtual valuations can then be read off the plot. The optimal auction assigns the item to the agent with the highest virtual value. If there is a tie, it picks a random tied agent to win.

Figure 3.2(a) depicts a realization of values for $n = 4$ agents where the highest virtual value is unique. What does the virtual surplus maximization do here? It allocates the item to the highest-valued agent, i.e., agent 1 in the figure. Figure 3.2(b) depicts a second realization of values where the highest virtual value is not unique. With uniform random tie breaking, a random tied agent is selected as the winner, i.e., one of agents 1, 2, and 3 in the figure. In general if the highest virtual value has a k-agent tie then each of these tied agents wins with probability $1/k$.



(a) $b < v_1$



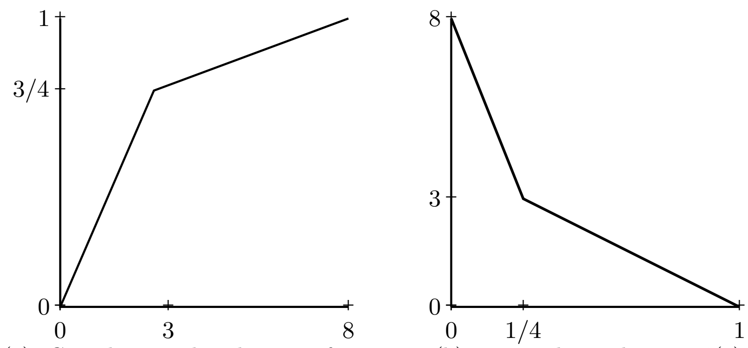
(b) $a \leq v_1 \leq b$

Figure 3.3 The allocation (black line) and payment rule (gray region) for agent 1 given fixed $\boldsymbol{v}_{-1}$ with $k-1$ of the other agents tied for having the highest virtual value, i.e., with values in $[a, b]$ (e.g., from virtual valuation function of Figure 3.2). For $v_1 \in [a, b]$, agent 1 would be in a k-agent tie for the highest virtual value; for $v_1 > b$ agent 1 would win outright.

The payment an agent must make in expectation over the random tie-breaking rule can be calculated as follows. Consider the case where there is a unique highest virtual value. The agent with this virtual value wins, assume it is agent 1 (Alice). To calculate her payment we need to consider her allocation rule for fixed values $\boldsymbol{v}_{-1}$ of the other agents. This allocation rule is

$$x_1(z, \boldsymbol{v}_{-1}) = \begin{cases} 1 & \text{if } z > b \\ 1/k & \text{if } z \in [a, b] \\ 0 & \text{if } z < a. \end{cases}$$

when $\boldsymbol{v}_{-1}$ has a $k-1$ agents in interval $[a, b]$. The $1/k$ probability of winning for $z \in [a, b]$ arises from our analysis of what happens in a k-agent tie. When Alice has the unique highest virtual value, i.e., $v_1 > b$, then $p_1 = b - {}^{b-a}/k$, see Figure 3.3(a). On the other hand, when Alice is tied for the highest virtual value with $k-1$ other agents with values in interval $[a, b]$, as depicted in Figure 3.3(b), her expected payment is $p_1 = a/k$. Of course, $x_1 = 1/k$ so such an expected payment can be



(a) Cumulative distribution function $F(v)$. (b) Inverse demand curve $V(q)$.

Figure 3.4 Depicted are the cumulative distribution function and inverse demand curve corresponding to the bimodal agent of Example 3.6. The inverse demand curve is obtained from the cumulative distribution function by rotating it 90 degrees counterclockwise.

implemented by charging a to the tied agent that wins and zero to the losers.

3.3.4 Quantile Space, Price-posting Revenue, and Derivation of Virtual Values

In this section we give an economically intuitive derivation of virtual value functions for revenue maximization.

Consider an agent Alice with a single-dimensional linear preference (Definition 2.6). Alice's preference is described by her value v which is drawn from distribution F. There is a one-to-one mapping between Alice's value and her strength relative to the distribution. For instance, Alice with value $v = 0.9$ drawn from $U[0, 1]$ is stronger than 90% and weaker than 10% of values drawn from the same distribution. Denote by *quantile* quantile q the relative strength of of a value where $q = 0$ is the strongest and $q = 1$ is the weakest, and by $V(\cdot)$ the *inverse demand curve* that maps quantiles to values. Importantly, the distribution of an agent's quantile is always $U[0, 1]$ as the probability that an agents quantile q is below a given $\hat{q}$ is exactly $\hat{q}$.

Definition 3.9 The *quantile* of a single-dimensional agent with value $v \sim F$ is the measure with respect to F of stronger values, i.e., $q =$

$1 - F(v)$; the *inverse demand curve* maps an agent's quantile to her value, i.e., $V(q) = F^{-1}(1-q)$.

Example 3.14 For the example of a uniform agent (Example 3.5) where $F(z) = z$, the inverse demand curve is $V(q) = 1 - q$; for the example of a bimodal agent (Example 3.6), the inverse demand curve is depicted in Figure 3.4.

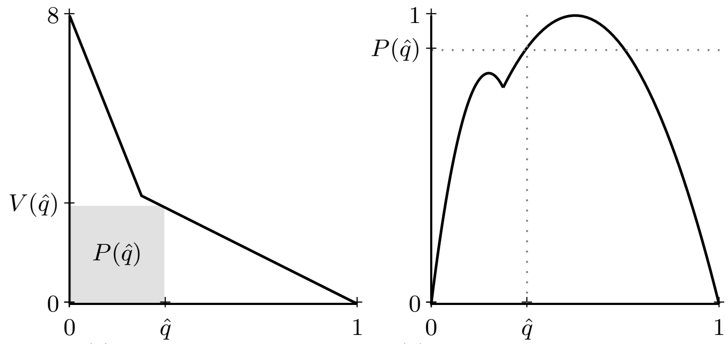
In Section 2.4 we defined the allocation rule for an agent as a function of her value as $x(\cdot)$ and characterized the allocation rules that can arise in Bayes-Nash equilibrium as the class of monotone non-decreasing functions (of value). The *allocation rule in quantile space* is denoted $y(q) = x(V(q))$. Since quantile and value are indexed in the opposite direction, $y(\cdot)$ will be monotone non-increasing in quantile.

Consider posting a take-it-or-leave-it price of $V(\hat{q})$ for some quantile $\hat{q}$. By the definition of the inverse demand curve $V(\cdot)$, such a price is accepted with probability $\hat{q}$. In other words, the ex ante sale probability of posting price $V(\hat{q})$ is $\hat{q}$. Notice that the allocation rule of this price-posting mechanism is simply the reverse step function that starts at one and steps from one to zero at $\hat{q}$. We can define a revenue curve by considering the revenue from this *price-posting* approach as a function of the ex ante service probability $\hat{q}$. For the uniform example, the price-posting revenue curve is $P(\hat{q}) = \hat{q} - \hat{q}^2$; for the bimodal example, it is depicted in Figure 3.5(b).

Definition 3.10 The *price-posting revenue curve* of a single-dimensional linear agent specified by inverse demand curve $V(\cdot)$ is $P(\hat{q}) = \hat{q} \cdot V(\hat{q})$ for any $\hat{q} \in [0, 1]$.

We can use revenue equivalence (via the payment identity) to express the revenue of any allocation rule in terms of the price-posting revenue curve. The main idea is the following. By revenue equivalence, any two mechanisms with the same allocation rule have the same revenue. Given an allocation rule y we can construct a mechanism with that allocation rule by taking the appropriate convex combination of price-posting mechanisms. Below we walk through this approach in detail.

An allocation rule y is a monotone non-increasing function from $[0, 1]$ to $[0, 1]$. The allocation rules for price postings are reverse step functions. The class of reverse step functions are a basis for the class of monotone non-increasing functions from $[0, 1]$ to $[0, 1]$: any such monotone non-increasing function can be expressed as a convex combination



(a) Inverse demand curve. (b) Price-posting revenue curve.

Figure 3.5 Depicted are the inverse demand curve and revenue curve corresponding to the bimodal agent of Example 3.6. The price-posting revenue curve is given by $P(\hat{q}) = \hat{q} \cdot V(\hat{q})$, i.e., the area of the rectangle of width $\hat{q}$ and height $V(\hat{q})$ that fits under the inverse demand curve.

of (a.k.a., distribution over) reverse step functions. Consider the distribution $G^y(z) = 1-y(z)$ and the mechanism that draws $\hat{q} \sim G^y$ and posts price $V(\hat{q})$. Notice, that the probability that Alice with fixed quantile q and value $V(q)$ is allocated by this mechanism is:

$$\mathbf{Pr}_{\hat{q}\sim G^y}[V(\hat{q}) < V(q)] = \mathbf{Pr}_{\hat{q}\sim G^y}[\hat{q} > q] = 1 - G^y(q) = y(q).$$

The mechanism resulting from the above convex combination of price postings has allocation rule exactly $y(\cdot)$ and Alice's expected payment (i.e., the expected revenue) is equal to the same convex combination of revenues $P(\hat{q})$ from posting price $V(\hat{q})$ with $\hat{q} \sim G^y$. This revenue is as follows, via a change of variables from $\hat{q} \sim G^y$ to $q \sim U[0,1]$ according to G^y's density function $g^y(z) = \frac{\mathrm{d}}{\mathrm{d}z}G^y(z) = \frac{\mathrm{d}}{\mathrm{d}z}(1 - y(z)) = -y'(z)$, integration by parts, and the assumption that $P(0) = P(1) = 0$ (there is no revenue from always selling or never selling; see Mathematical Note on page 60).

$$\begin{aligned}\mathbf{E}_{\hat{q}\sim G^y}[P(\hat{q})] &= \mathbf{E}_{q\sim U[0,1]}\left[-y'(q) \cdot P(q)\right] \\ &= \mathbf{E}_{q\sim U[0,1]}\left[P'(q) \cdot y(q)\right],\end{aligned}$$

where $P'(q) = \frac{\mathrm{d}}{\mathrm{d}q}P(q)$ is the marginal increase in price-posting revenue for an increase in ex ante allocation probability, a.k.a., the *marginal price-posting revenue* at q. Notice that the calculation of Alice's expected payment for allocation rule y above is implicitly taking the expectation over Alice's quantile $q \sim U[0,1]$ via the definition of the price-posting revenue curve $P(\cdot)$. Of course, by revenue equivalence (Theorem 2.2), any mechanism with the same allocation rule generates the same revenue.

Proposition 3.15 *A single-agent mechanism with allocation rule y has expected revenue equal to the allocated marginal price-posting revenue $\mathbf{E}_q\big[P'(q) \cdot y(q)\big]$.*

The above rephrasing of the expected revenue in terms of marginal revenue is an amortized analysis. Notice that if we serve Alice with quantile q with some probability then, were her quantile lower (i.e., stronger), she would be served with no lower a probability. Therefore, the contribution to the revenue from all quantiles above quantile q can be credited to the change in service probability at q. The marginal price-posting revenue is precisely this reamortizing of revenues across the different agent quantiles.

The marginal price-posting revenues are exactly the virtual values described previously by equation (3.2).

$$P'(q) = \tfrac{\mathrm{d}}{\mathrm{d}q}(q \cdot V(q)) = V(q) + qV'(q) = v - \tfrac{1-F(v)}{f(v)}, \qquad (3.5)$$

where the first equality follows from the definition of price-posting revenue (Definition 3.10) and the last equality follows from the definition of the inverse demand curve $V(\cdot)$ whereby $v = V(q)$ satisfies $F(v) = 1 - q$ and $1/f(v) = -\tfrac{\mathrm{d}}{\mathrm{d}q}V(q) = -V'(q)$. Recall that a distribution is regular if $v - \tfrac{1-F(v)}{f(v)}$ is monotone non-decreasing or, equivalently, the marginal price-posting revenue is monotone non-increasing, or equivalently the price-posting revenue curve is concave.

Proposition 3.16 *A distribution F is regular if and only if its corresponding price-posting revenue curve is concave.*

Proposition 3.15 shows the expected revenue of a mechanism is equal to its allocated marginal price-posting revenue. For regular distributions, the marginal price posting revenue derived above is monotone; therefore, we can conclude that the virtual surplus maximization mechanism with virtual value function defined by the marginal price-posting revenue curve (Definition 3.5) is dominant strategy incentive compatible and profit optimal (Corollary 3.9).

Theorem 3.17 *For agents with values drawn from regular distributions the marginal price-posting revenue curves are virtual value functions for revenue and the virtual surplus maximization mechanism optimizes expected profit in dominant strategy equilibrium.*

The price-posting revenue curve $P(\hat{q})$ is defined by the revenue obtained be posting a price that is accepted with probability $\hat{q}$. Consider instead the single-agent optimization of optimizing revenue subject to an ex ante constraint $\hat{q}$. This optimization problem is not generally solved by a price posting; however, for regular distributions it is. Subsequently in Section 3.4 we will consider this more general problem and define from it (optimal) revenue curves. For regular distributions price-posting revenue curves and (optimal) revenue curves are equal.

3.3.5 Virtual Surplus Maximization Subject to Monotonicity

We now turn our attention to the case where the non-game-theoretic problem of optimization of marginal price-posting revenue is not itself inherently monotone. An *irregular* distribution is one for which the price-posting revenue curve is non-concave (in quantile). The marginal price-posting revenue curves (and virtual value functions defined from them) are non-monotone; therefore, a higher value might result in a lower virtual value. As $\mathrm{OPT}(\boldsymbol{\phi}(\cdot))$ is non-monotone for such a virtual value function, there is no payment rule with which its outcome is incentive compatible (by the only-if direction of Corollary 2.12). We must instead optimize this virtual surplus subject to monotonicity.

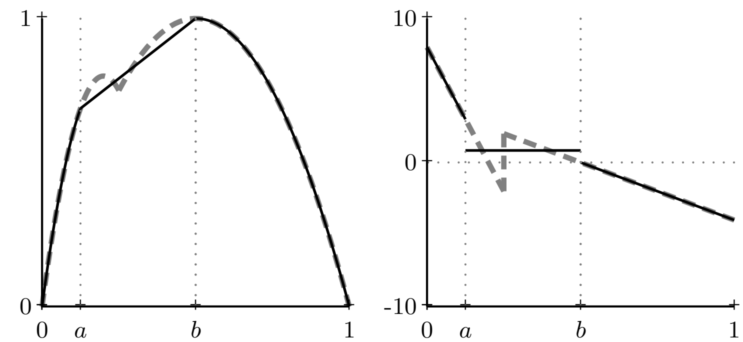
Recall that virtual values, e.g., $v - \frac{1-F(v)}{f(v)}$, correspond to an amortized analysis where we "charge" the value v if it is served for the lower price its service implies for higher values. When this direct approach to an amortized analysis gives a non-monotone virtual value function, the following generic *ironing procedure* gives an ironed virtual value function which is monotone and for which pointwise optimization is equivalent to the optimization of expected virtual surplus subject to monotonicity of the allocation rule.

There are two key ideas to this ironing procedure. First, if there is some interval $[a, b]$ of quantiles that all receive the same allocation probability, then the virtual values of these quantiles can be reamortized arbitrarily and the expected virtual value of the allocation rule is unchanged. Second, if we reamortize by simple averaging then we get "ironed" virtual values that are constant on the $[a, b]$ interval and optimization of the ironed virtual surplus will give the same allocation probability to quantiles within the interval. Therefore, the approach of the second part implies the assumption of the first part. Moreover, in terms of fixing non-monotonicities, after ironing the virtual value are constant (and therefore weakly monotone) on the interval $[a, b]$.

As in previous sections, the geometry of this reamortization is more transparent in quantile space rather than value space. This is because quantiles are drawn from a uniform distribution so reamortizing by moving virtual value from one quantile to another is balanced with respect to the distribution. If we were to do such a shift of virtual value in value space then we would need to normalize by the density function of the distribution. We therefore proceed by considering a virtual value function $\phi(\cdot)$ in quantile space. We denote the cumulative virtual value for quantiles at most $\hat{q}$ as $\Phi(\hat{q}) = \int_0^{\hat{q}} \phi(q)\,\mathrm{d}q$. For profit maximization, the virtual value functions correspond to marginal price-posting revenue curves and cumulative virtual value functions correspond to price-posting revenue curves, i.e., $\phi(q) = P'(q)$ and $\Phi(q) = P(q)$. The ironing procedure we will describe, however, can be applied to any non-monotone virtual value function.

The goal of ironing is arrive at a monotone (ironed) virtual value function, equivalently, a concave cumulative virtual value function, without any loss in virtual surplus for monotone allocation rules. We now investigate the consequences of the ironing procedure proposed above on the virtual value and cumulative virtual value functions. The averaging of virtual value over an interval $[a, b]$ in quantile space replaces the function on that interval with a constant equal to the original function's average. We can then integrate to see what the effect on the cumulative virtual value is. Notice that on $q \in [0, a]$ and $q \in [b, 1]$ this integral is identically $\Phi(q)$; while for $q \in [a, b]$ it is the integral of a constant function and therefore linearly connects $(a, \Phi(a))$ to $(b, \Phi(b))$ with a line segment. For the bimodal agent of Example 3.6 these quantities are depicted in Figure 3.6 with an arbitrary choice of a and b.

If we iron the virtual value functions and then optimize with ironed virtual values as virtual values, then the revenue is again the virtual surplus (by the correctness of ironing construction, e.g., as proven by Theorem 3.18, below). It remains to choose the appropriate intervals on which to iron so that the ironed virtual value functions are monotone (equivalently, the ironed revenue curve is concave) and the optimization of ironed virtual surplus also optimizes the virtual surplus. Intuitively, higher revenue curves produce higher revenues. As the ironing procedure operates on the cumulative virtual value functions by replacing an interval with a line segment, we can construct the concave hull, i.e., the smallest concave upper-bound, of the cumulative virtual value function



(a) Ironed cumulative virtual values. (b) Ironed virtual values.

Figure 3.6 Consider the bimodal agent of Example 3.6 and virtual value function equal to the marginal price-posting revenue curve. The cumulative virtual value and virtual value functions in quantile space are are depicted (thick, gray, dashed lines) in the left and right diagram, respectively. After ironing on an arbitrarily selected interval $[a, b]$, the resulting cumulative virtual value and virtual value functions are depicted (thin, black, solid lines).

by ironing. Notice that this ironed cumulative virtual value function has two advantages over the original cumulative virtual value function: it is pointwise higher and it is concave.

Definition 3.11 The *ironing procedure* for (non-monotone) virtual value function ϕ (in quantile space)[7] is:

(i) Define the *cumulative virtual value* function as $\Phi(\hat{q}) = \int_0^{\hat{q}} \phi(q)\,\mathrm{d}q$.
(ii) Define *ironed cumulative virtual value* function as $\bar{\Phi}(\cdot)$ as the concave hull of $\Phi(\cdot)$.
(iii) Define the *ironed virtual value* function as $\bar{\phi}(q) = \frac{\mathrm{d}}{\mathrm{d}q}\bar{\Phi}(q) = \bar{\Phi}'(q)$.

Theorem 3.18 *For any monotone allocation rule $y(\cdot)$ and any virtual value function $\phi(\cdot)$, the expected virtual surplus of an agent is upper-bounded by her expected ironed virtual surplus, i.e.,*

$$\mathbf{E}[\phi(q) \cdot y(q)] \leq \mathbf{E}[\bar{\phi}(q) \cdot y(q)]\,.$$

Furthermore, this inequality holds with equality if the allocation rule y satisfies $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$.

[7] The ironing procedure can also be expressed in value space by first mapping values to quantiles via the cumulative distribution function or inverse demand curve, executing the ironing procedure in quantile space, and then mapping ironed virtual value functions back into value space.

Proof By integration by parts for any virtual value function $\phi^\dagger(\cdot)$ and monotone allocation rule $y(\cdot)$ (see Mathematical Note on page 60),

$$\mathbf{E}[\phi^\dagger(q) \cdot y(q)] = \mathbf{E}[-y'(q) \cdot \Phi^\dagger(q)] \,. \tag{3.6}$$

Notice that the (non-increasing) monotonicity of the allocation rule $y(\cdot)$ implies the non-negativity of $-y'(q)$. With the left-hand side of equation (3.6) as the expected virtual surplus, it is clear that a higher cumulative virtual value implies no lower expected virtual surplus. By definition of $\bar{\Phi}(\cdot)$ as the concave hull of $\Phi(\cdot)$, $\bar{\Phi}(q) \geq \Phi(q)$ and, therefore, for any monotone allocation rule, in expectation, the ironed virtual surplus is at least the virtual surplus. I.e., $\mathbf{E}\big[-y(q) \cdot \bar{\Phi}(q)\big] \geq \mathbf{E}[-y(q) \cdot \Phi(q)]$.

To see the equality under the assumption that $y'(q) = 0$ for all q where $\bar{\Phi}(q) > \Phi(q)$, rewrite the difference between the ironed virtual surplus and the virtual surplus via equation (3.6) as,

$$\mathbf{E}\big[\bar{\phi}(q) \cdot y(q)\big] - \mathbf{E}[\phi(q) \cdot y(q)] = \mathbf{E}\big[-y'(q) \cdot \big[\bar{\Phi}(q) - \Phi(q)\big]\big] \,.$$

The assumption implies the term inside the expectation on the left-hand side is zero for all q. □

Corollary 3.19 *For any virtual value function $\phi(\cdot)$ with ironed virtual value $\bar{\phi}(\cdot)$ from the ironing procedure (Definition 3.11), the optimization of virtual surplus subject to monotonicity of the allocation rule is equivalent to optimization of ironed virtual surplus pointwise.*

We now conclude this section by summarizing the consequences of ironing for virtual surplus maximization. First, we can define the *ironed virtual surplus maximization* mechanism for virtual value functions $\boldsymbol{\phi}$ as the virtual surplus maximization mechanism applied to the ironed virtual value functions $\bar{\boldsymbol{\phi}}$. This profile $\bar{\boldsymbol{\phi}}$ of ironed virtual value functions is constructed from the profile $\boldsymbol{\phi}$ of virtual value functions by applying the ironing procedure individually to each virtual value function.

Theorem 3.20 *For any (non-monotone) virtual value functions $\boldsymbol{\phi}$, the ironed virtual surplus maximization mechanism maximizes expected virtual surplus in dominant strategy equilibrium.*

Corollary 3.21 *For (irregular) single-dimensional linear agents, the ironed marginal price-posting revenue curves are virtual value functions for revenue and the virtual surplus maximization mechanism optimizes expected profit in dominant strategy equilibrium.*

The ironing procedure above results in virtual value functions that are not strictly monotone. See Section 3.3.3 for a discussion of the virtual surplus maximization mechanism with non-strictly monotone virtual value functions in single-item environments.

3.4 Multi- to Single-agent Reduction

While the previous sections gave a complete approach to profit maximization for single-dimensional linear agents, here we give an alternative derivation that comes to the same conclusion but provides more conceptual understanding, especially for irregular distributions. The approach will be to reduce the problem of solving a multi-agent mechanism design problem to that of solving a collection of simple single-agent pricing problems. It observes and makes use of a *revenue-linearity* property that is satisfied by single-dimensional agents with linear utility. In Chapter 7 this reduction is extended to multi-dimensional non-linear agents.

A mechanism for a single agent is simply a menu of outcomes where, after the agent realizes her value from the distribution, she chooses the outcome she most prefers. This observation is known as the *taxation principle* and is a simple consequence of the revelation principle (Theorem 2.11). It can be seen as follows: The agent's actions in the mechanism induce a set of (possibly randomized) outcomes; for a fully rational agent, these probabilistic outcomes may as well be listed on a menu from which the agent just chooses her favorite. Each of these probabilistic outcomes can be summarized by its allocation probability and expected payment (as far as the preferences of a single-dimensional linear agent is concerned). We call such a probabilistic allocation a lottery, and the menu of lotteries and their accompanying prices a *lottery pricing*. The allocation and payment rules $(x(\cdot), p(\cdot))$ described in Section 3.1 precisely define such a menu where the outcomes are indexed so that the agent with value v prefers outcome $(x(v), p(v))$ over all other outcomes.

Below we will look at two optimization problems. The first will be an *ex ante pricing* problem where we look for the lottery pricing with the optimal revenue subject to a constraint on the ex ante service probability $\mathbf{E}_v[x(v)]$. The revenue of the optimal ex ante pricings induce a concave *revenue curve*. We will then look at an *interim pricing* problem where we have a constraint on the allocation rule $x(\cdot)$ and we again wish to optimize revenue subject to that constraint. The main conclusion will be that we can express the optimal interim pricing as a convex combination of optimal ex ante pricings. The decomposition will enable the expected payments to be expressed in terms of a monotone *marginal revenue curve* (cf. Section 3.3.4). Pointwise optimization of the allocated marginal revenue then gives the optimal revenue.

3.4.1 Revenue Curves

It will be more economically intuitive to study lottery pricings in quantile space. Alice has her quantile q drawn from the uniform distribution $U[0, 1]$ and value $V(\cdot)$ according to the inverse demand curve. Upon realizing her quantile, she will choose her preferred outcome from a lottery pricing. This two step process induces an allocation rule $y(q) = x(V(q))$ and an ex ante probability $\mathbf{E}_q[y(q)]$ that Alice is served. Recall that the allocation rule is taken in expectation with respect to the randomization in the outcome of the lottery that Alice buys, and the ex ante service probability is taken additionally in expectation with respect to the randomization of Alice's quantile.

Definition 3.12 With equality constraint $\hat{q}$ on the ex ante allocation probability, the single-agent *ex ante pricing problem* is to find the revenue-optimal lottery pricing. The optimal ex ante revenue, as a function of $\hat{q}$, is denoted by the *revenue curve* $R(\hat{q})$.

It will be important to contrast the revenue-optimal lottery pricing for an ex ante constraint $\hat{q}$ with the price posting that satisfies the same constraint. The revenues of these two pricings are given by the revenue curve $R(\hat{q})$ and price-posting revenue curve $P(\hat{q})$ (from Section 3.3.4). First, recall that the difficulty with deriving optimal mechanisms directly from the price-posting revenue curve $P(\cdot)$ is that it may not be concave. On the other hand the revenue curve $R(\cdot)$ is always concave.[8] Second, notice that the allocation rule for price posting, which serves all values that are at least $V(\hat{q})$, is the strongest allocation rule with ex ante service probability $\hat{q}$ in the following sense. Any other allocation rule can shift allocation probability from stronger (lower) quantiles to weaker (higher) quantiles but cannot allocate with any greater probability to

[8] This observation follows from the fact that the space of lottery pricings is convex: randomizing between two lottery pricings gives a lottery pricing that corresponds to the lotteries' convex combination and gives ex ante allocation probability and expected revenue according to the same convex combination. In contrast, the space of price postings is not convex: the convex combination of two price postings cannot be expressed as a price posting. Consequently and as we have already observed, the price-posting revenue curve is not generally concave.

the strongest $\hat{q}$ measure of quantiles. Therefore, for the ex ante probability $\hat{q}$, the allocation rule of the optimal ex ante pricing is no stronger than that of price posting. Third, the optimal ex ante pricing for constraint $\hat{q}$ obtains at least the revenue of price posting. This observation is immediate from the fact that it is optimizing over lottery pricings that include the posting price $V(\hat{q})$. We summarize these observations as the following proposition which, with Proposition 3.15 (essentially, revenue equivalence), will be sufficient for proving the optimality of marginal revenue maximization; we defer precise characterization of the optimal ex ante lottery pricing to later in this section.

Proposition 3.22 *The optimal ex ante pricing problems induce a concave revenue curve and, for any ex ante service probability, the optimal lottery has no stronger an allocation rule and no lower a revenue than price posting.*

3.4.2 Optimal and Marginal Revenue

We now formulate an interim lottery pricing problem that takes an allocation rule as a constraint and asks for the optimal lottery pricing with an allocation rule that is no stronger than the one given. To do so we must first generalize the definition of strength (as discussed previously when comparing price posting with optimal lotteries). Recall that with the same ex ante allocation probability the difference between the price posting and an optimal lottery is that the optimal lottery may have service probability shifted from strong (low) quantiles to weak (high) quantiles. This condition generalizes naturally.

The ex ante probability that allocation rule $y(\cdot)$ allocates to the strongest $\hat{q}$ measure of quantiles is $Y(\hat{q}) = \int_0^{\hat{q}} y(q)\,dq$; we refer to $Y(\cdot)$ as the *cumulative allocation rule* for $y(\cdot)$. The (non-increasing) monotonicity of allocation rules implies that cumulative allocation rules are concave. As follows, we can view an allocation rule $\hat{y}(\cdot)$ as a constraint via its cumulative allocation rule $\hat{Y}$.

Definition 3.13 Given an allocation constraint $\hat{y}$ with cumulative constraint $\hat{Y}$, the allocation rule y with cumulative allocation rule Y is *weaker* (resp. $\hat{y}$ is *stronger*) if and only if it satisfies $Y(\hat{q}) \leq \hat{Y}(\hat{q})$ for all $\hat{q}$; denote this relationship by $y \preceq \hat{y}$.

A strong allocation rule as a constraint corresponds to a weak constraint as it permits the most flexibility in allocation rules that satisfy it. The ex ante pricing problem for constraint $\hat{q}$ is a special case of the interim pricing problem. The strongest allocation rule that serves with probability $\hat{q}$ is the reverse step function that steps from one to zero at $\hat{q}$; therefore, the allocation constraint $\hat{y}^{\hat{q}}$ is the weakest constraint that allows service probability at most $\hat{q}$. In comparison, a general allocation constraint $\hat{y}$ (e.g., with total allocation probability $\mathbf{E}[\hat{y}(q)] = \hat{q}$) allows more fine-grained control by giving a constraint, for all $\hat{q}^{\dagger}$, on the cumulative service probability of any $[0, \hat{q}^{\dagger}]$ measure of quantiles by $\hat{Y}(\hat{q}^{\dagger})$. Of course, given an allocation constraint $\hat{y}$, the strongest allocation rule that satisfies the constraint is the constraint itself, i.e., $y = \hat{y}$. From this notion of strength we can take an allocation rule as a constraint and consider the optimization question of finding an allocation rule that is no stronger and with the highest possible revenue.

Definition 3.14 The optimal revenue subject to an allocation constraint $\hat{y}(\cdot)$ is $\mathbf{Rev}[\hat{y}]$ and it is attained by the *optimal interim pricing* for $\hat{y}$.

An important property of this definition of the strength of an allocation rule is that it closed under convex combination, i.e., if $\hat{y} = \hat{y}^{\dagger} + \hat{y}^{\ddagger}$, $y^{\dagger} \preceq \hat{y}^{\dagger}$, and $y^{\ddagger} \preceq \hat{y}^{\ddagger}$ then $y \preceq \hat{y}$ for $y = y^{\dagger} + y^{\ddagger}$. This means that one approach to construct an allocation rule y that satisfies the allocation constraint $\hat{y}$ is to express y as a convex combination of ex ante constraints, and to implement each with the optimal ex ante pricing. Relative to the construction of Proposition 3.15, using optimal lottery pricings improves on price postings in that for each $\hat{q}$ the optimal ex ante revenue $R(\hat{q})$ may exceed the price-posting revenue $P(\hat{q})$. Consider the mechanism that draws $\hat{q}$ from the distribution $G^{\hat{y}}(z) = 1 - \hat{y}(z)$ and offers Alice the optimal ex ante pricing for $\hat{q}$. The optimal revenue for allocation constraint $\hat{y}$ must be at least the revenue of this mechanism. By the Mathematical Note on page 60, we have:

$$\begin{aligned}\mathbf{Rev}[\hat{y}] &\geq \mathbf{E}_{\hat{q}\sim G^{\hat{y}}}[R(\hat{q})] \\ &= \mathbf{E}_q\big[-\hat{y}'(q) \cdot R(q)\big] \\ &= \mathbf{E}_q\big[R'(q) \cdot \hat{y}(q)\big]\,,\end{aligned}$$

where $R'(q) = \frac{\mathrm{d}}{\mathrm{d}q}R(q)$ is the *marginal revenue* at q.

Definition 3.15 The *allocated marginal revenue* of an allocation constraint $\hat{y}$ is $\mathbf{MargRev}[\hat{y}] = \mathbf{E}_q\big[R'(q) \cdot \hat{y}(q)\big]$.

3.4.3 Downward Closure and Pricing

We now make a brief aside to discuss downward closure of the environment and its relationship to the previously defined single-agent lottery pricing problems. Recall that a downward closure environment is one where from any feasible outcome it is always feasible to additionally reject and agent who was previously being served. Our definition of the optimal ex ante pricing problem is not downward closed as we required that the ex ante constraint be met with equality. On the other hand, our definition of the optimal interim pricing problem was downward closed as it was allowed that $Y(1) < \hat{Y}(1)$. These definitions were given above as they are the most informative.

It is possible to consider a downward-closed variant of the ex ante pricing problem where a lottery pricing is sought with ex ante probability at most $\hat{q}$. Obviously, adding downward closure results in a revenue curve that is monotone non-decreasing. From the non-downward-closed revenue curve, the downward-closed revenue curve is given as a function of $\hat{q}$ by $\max_{q \leq \hat{q}} R(q)$. Thus, the downward-closed revenue curve after the monopoly quantile is constant. Importantly, the downward-closed marginal revenue curve is always non-negative. It is similarly possible to consider a non-downward-closed variant of the interim pricing problem where it is additionally required that $Y(1) = \hat{Y}(1)$.

In our discussion of revenue linearity in the subsequent section, it will be important not to mix-and-match with respect to downward closure.

3.4.4 Revenue Linearity

The above derivation says the allocated marginal revenue of an allocation constraint is a lower bound on its optimal revenue. A central dichotomy in optimal mechanism design is given by the partitioning of single-agent problems into those for which this inequality is tight and those when it is not. Notice that linearity of the revenue operator $\mathbf{Rev}[\cdot]$ implies by the above derivation that for any allocation constraint the optimal revenue and allocated marginal revenue are equal.

Definition 3.16 A agent (with implicit utility function, type space, and distribution over types) is *revenue linear* if $\mathbf{Rev}[\cdot]$ is linear, i.e., if when $\hat{y} = \hat{y}^{\dagger} + \hat{y}^{\ddagger}$ then $\mathbf{Rev}[\hat{y}] = \mathbf{Rev}[\hat{y}^{\dagger}] + \mathbf{Rev}[\hat{y}^{\ddagger}]$.[9]

[9] It is assumed that the ex ante and interim problem are consistent with respect to downward closure, see Section 3.4.3.

Proposition 3.23 *For a revenue-linear agent and any allocation constraint $\hat{y}$, the optimal revenue is equal to the allocated marginal revenue, i.e.,* $\mathbf{Rev}[\hat{y}] = \mathbf{MargRev}[\hat{y}]$.

We now show that single-dimensional linear agents are revenue linear. This result is a consequence of three main ingredients: the concavity of the revenue curve $R(\cdot)$, that the optimal ex ante pricings which define the revenue curve gives more revenue with a weaker allocation rule than the price postings which define price-posting revenue curves (Proposition 3.22), and that revenue equivalence allows revenue to be expressed in terms of price-posting revenue curves (Proposition 3.15). Optimal revenue equaling allocated marginal revenue for single-dimensional linear agents, then, is an immediate corollary of this revenue linearity and Proposition 3.23.

Theorem 3.24 *A single-dimensional linear agent is revenue linear.*

Proof Before we begin, notice that for any revenue curve $R(\cdot)$ and allocation rule $y(\cdot)$ the allocated marginal revenue $\mathbf{MargRev}[y]$ can be equivalently expressed as

$$\mathbf{E}_q\big[-y'(q)R(q)\big] = \mathbf{E}_q\big[R'(q)y(q)\big] = \mathbf{E}_q\big[-R''(q)Y(q)\big] + R'(1)Y(1)$$

via integration by parts (with $R(1) = R(0) = Y(0) = 0$; see Mathematical Note on page 60). The same equations also govern the allocated marginal price-posting revenue in terms of revenue curve $P(\cdot)$. Two observations:

(i) The left-hand side shows that a pointwise higher revenue curve gives a no lower revenue (as $-y'(\cdot)$ is non-negative). In particular, the allocated marginal revenue exceeds the allocated marginal price-posting revenue as $R(q) \geq P(q)$ for all q (by Proposition 3.22).

(ii) The right-hand side shows that for concave revenue curves, i.e., where $-R''(\cdot)$ is non-negative, e.g., $R(\cdot)$ not $P(\cdot)$; a stronger allocation rule gives higher revenue. In particular, the allocation rule y obtained by optimizing for $\hat{y}$ has no higher allocated marginal revenue than does $\hat{y}$.[10]

[10] Consistency with respect to downward-closure (see Section 3.4.3) implies the inequality on the $R'(1)Y(1)$ term. For the downward-closed case: the marginal revenue $R'(1)$ is non-negative and thus $R'(1)\hat{Y}(1) \geq R'(1)Y(1)$. For the non-downward-closed case: it is required that $\hat{Y}(1) = Y(1)$ and thus $R'(1)\hat{Y}(1) = R'(1)Y(1)$.

We have already concluded that the allocated marginal revenue lower bounds the optimal revenue; so to prove the theorem it suffices to upper bound the optimal revenue by the allocated marginal revenue. Suppose we optimize for $\hat{y}$ and get some weaker allocation rule y, then y is a fixed point of $\mathbf{Rev}[\cdot]$ (optimizing with y as an allocation constraint gives back allocation rule y); therefore,

$$\mathbf{Rev}[\hat{y}] = \mathbf{Rev}[y].$$

By revenue equivalence (Proposition 3.15), the revenue of any allocation rule is equal to its allocated marginal price-posting revenue, so

$$\mathbf{Rev}[y] = \mathbf{E}\big[P'(q) \cdot y(q)\big] .$$

By observation (i), for allocation rule y, the allocated marginal revenue is at least the allocated marginal price-posting revenue,

$$\mathbf{E}\big[-y'(q) \cdot P(q)\big] \leq \mathbf{E}\big[-y'(q) \cdot R(q)\big] .$$

By observation (ii), the allocated marginal revenue for $\hat{y}$ is at least that of y,

$$\mathbf{E}\big[-R''(q) \cdot Y(q)\big] \leq \mathbf{E}[-R''(q) \cdot \hat{Y}(q)] = \mathbf{MargRev}[\hat{y}].$$

The above sequence of inequalities implies that the allocated marginal revenue is at least the optimal revenue for $\hat{y}$,

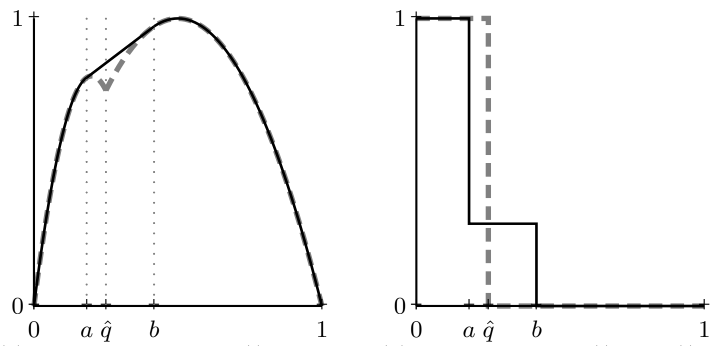
$$\mathbf{Rev}[\hat{y}] \leq \mathbf{MargRev}[\hat{y}]. \qquad \square$$

Corollary 3.25 *For an agent with single-dimensional, linear utility, the optimal revenue equals the marginal revenue, i.e.,*

$$\mathbf{Rev}[\hat{y}] = \mathbf{MargRev}[\hat{y}] = \mathbf{E}\big[R'(q)\hat{y}(q)\big] .$$

Observe that Corollary 3.25 implies that the marginal revenue curve is a virtual value function for revenue. The virtual surplus maximization mechanism for these virtual values maximizes expected profit.

Theorem 3.26 *For linear single-dimensional agents, the marginal revenue curves are a virtual value functions for revenue and the virtual surplus maximization mechanism optimizes expected profit in dominant strategy equilibrium.*



(a) Revenue curves $R(\cdot)$ and $P(\cdot)$.

(b) Allocation rules $\hat{y}(\cdot)$ and $y(\cdot)$.

Figure 3.7 Depicted are the revenue curve, price-posting revenue curve, and their allocation rules corresponding to ex ante allocation constraint $\hat{q}$ for the bimodal agent of Example 3.6. For this agent the revenue curve $R(\cdot)$ (thin, black, solid line) is obtained from the price-posting revenue curve $P(\cdot)$ (thick, grey, striped line) by replacing the curve on interval $[a, b]$ with a line segment. The allocation rule for posting price $V(\hat{q})$ is the reverse step function at $\hat{q}$ (thick, grey, striped line). For $\hat{q} \in [a, b]$ as depicted, the allocation rule (thin, black, solid line) for the $\hat{q}$ optimal ex ante pricing is the appropriate convex combination of the reverse step functions at a and b. Notice that the area under both allocation rules is equal to the ex ante service probability $\hat{q}$.

3.4.5 Optimal Ex Ante Pricings, Revisited

We now return to the question of characterizing the optimal ex ante pricings that define the revenue curve (Definition 3.12). Given an ex ante constraint $\hat{q}$, what is the optimal lottery pricing? We saw previously that price posting $V(\hat{q})$ is a simple way to serve an agent with ex ante probability $\hat{q}$. When the distribution is regular, it is easy to see that price posting is optimal. By monotonicity of the marginal price-posting revenue curve, the $\hat{q}$ measure of types with the highest marginal revenues is precisely those with quantile in $[0, \hat{q}]$. The mechanism that serves only these types is the $V(\hat{q})$ price posting. Therefore, for regular distributions $R(\cdot) = P(\cdot)$. The following is a restatement of Proposition 3.15 in terms of the revenue curve for the regular case.

Corollary 3.27 *For regular single-agent environments, allocation rule y has expected revenue equal to the allocated marginal revenue* $\mathbf{E}_q\left[R'(q) \cdot y(q)\right]$.

To solving the ex ante pricing problem for irregular distributions we will define a very natural class of lottery pricings which directly resolve the problematic non-convexity of the price-posting revenue curves. Suppose the price-posting revenue is non-concave at some $\hat{q}$, instead of posting price $V(\hat{q})$ another method for serving with ex ante probability $\hat{q}$ would be to pick any interval $[a, b]$ that contains $\hat{q}$ and take the appropriate convex combination of posting prices $V(a)$, which serves with probability $a < \hat{q}$, and $V(b)$, which serves with probability $b > \hat{q}$, so that the combined service probability is exactly $\hat{q}$. The revenue from this convex combination is the same convex combination of the revenues; the allocation rule is given by the same convex combination of the two reverse step functions. Figure 3.7(b) depicts these allocation rules. Formulaically,

$$y^{\hat{q}}(q) = \begin{cases} 1 & \text{if } q < a, \\ \frac{\hat{q}-a}{b-a} & \text{if } q \in [a, b], \text{ and} \\ 0 & \text{if } b < q. \end{cases}$$

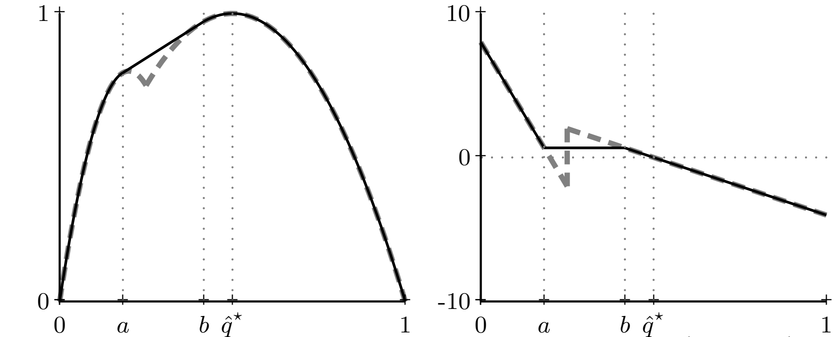
It is easy to see that via *two-price lotteries* of this form we can obtain an ex ante revenue for every $\hat{q}$ that corresponds to the convex hull of $P(\cdot)$. See Figure 3.7(a).

This class of two-price lotteries satisfies all the conditions that the optimal pricings satisfies with respect to Proposition 3.22. Optimal two-price lotteries (a) induce a concave revenue curve, (b) have at least the revenue of price posting, and (c) have allocation rules is no stronger than those of price posting. Consequently, via the exact same proof as Theorem 3.24 (and Corollary 3.25) the optimal revenue is given by convex combination of ex ante pricings from this class. Applying this revenue-optimality result to the allocation constraint $\hat{y}^{\hat{q}}(\cdot)$, for which the aforementioned convex combination places probability one on $\hat{q}$, we see that the optimal two-price lottery for ex ante constraint $\hat{q}$ is in fact optimal among all lottery pricings.

Theorem 3.28 *For a single-dimensional linear agent and ex ante constraint $\hat{q}$, the optimal ex ante pricing is a two-priced lottery and the optimal ex ante revenue $R(\hat{q})$ is given by the concave hull of the price-posting revenue curve $P(\cdot)$ at $\hat{q}$.*

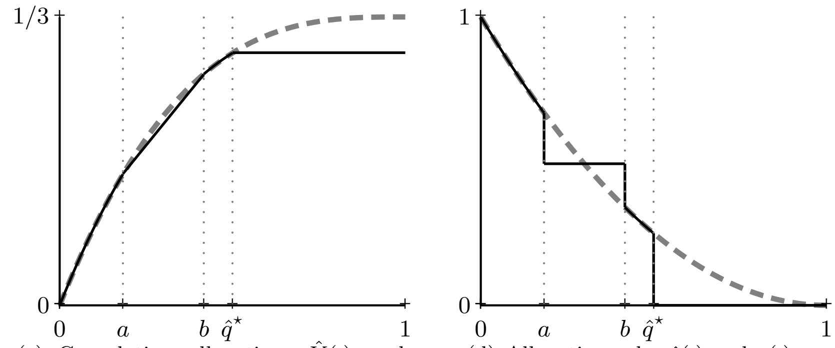
3.4.6 Optimal Interim Pricings, Revisited

We now reconsider the problem of finding the optimal interim pricing (with allocation rule y) for allocation constraint $\hat{y}$, i.e., solving $\mathbf{Rev}[\hat{y}]$.



(a) Revenue curves $R(\cdot)$ and $P(\cdot)$.

(b) Marginal revenues $R'(\cdot)$ and $P'(\cdot)$.



(c) Cumulative allocations $\hat{Y}(\cdot)$ and $Y(\cdot)$.

(d) Allocation rules $\hat{y}(\cdot)$ and $y(\cdot)$.

Figure 3.8 The optimal single-item auction is depicted for three bimodal agents (Example 3.6. The price-posting revenue curve $P(\cdot)$ is depicted by a thick, grey, dashed line in Figure 3.8(a). The revenue curve (thin, black, solid line) is its concave hull. The ironed interval (a, b) where $R(q) > P(q)$ is depicted. The allocation constraint $\hat{y}(q) = (1-q)^2$ (Figure 3.8(d), thick, grey, dashed line) corresponds to lowest-quantile-wins for three agents; the allocation rule $y(q)$ (thin, black, solid line) results from optimizing $\mathbf{Rev}[\hat{y}]$. Simply, ironing corresponds to a line-segment for revenue curves and cumulative allocation rules and to averaging for marginal revenues and allocation rules.

Recall that $\hat{y}$ is a constraint, but the allocation rule y of the optimal mechanism subject to $\hat{y}$ may be generally weaker than $\hat{y}$, i.e., $y \preceq \hat{y}$. Just as we can view the ironing of the price-posting revenue curve on interval I as averaging marginal price-posting revenue on this interval, we can so view the optimization of y subject to $\hat{y}$. To optimize a weakly monotone function $R'(\cdot)$ subject to $\hat{y}$ we should greedily assign low quantiles to high probabilities of service except on ironed intervals, i.e., $[a, b]$ where $q \in [a, b]$ satisfies $R''(q) = 0$. Quantiles on ironed intervals are assigned to the average probability of service for the ironed interval. One way to obtain such an allocation rule is via a *resampling transformation* σ that, for quantile q in some ironed interval $[a, b]$, resamples the quantile from this interval, i.e., as $y(q) = \mathbf{E}_{\sigma}[\hat{y}(\sigma(q))]$. The cumulative allocation rule Y is exactly equal to the cumulative allocation constraint $\hat{Y}$ except every ironed interval is replaced with a line segment. In other words, the revenue optimization of $\mathbf{Rev}[\cdot]$ can be effectively solved by superimposing the revenue curve and the allocation constraint on the same quantile axis and then ironing the allocation constraint where the revenue curve is ironed. Figure 3.8 illustrates this construction.

We will typically be in environments that are downward-closed where optimizing revenue allows the exclusion of any agent with negative virtual value. Thus, the optimal allocation rule y drops to zero after the quantile $\hat{q}^{\star}$ of the monopoly price; equivalently Y is flat after $\hat{q}^{\star}$. For non-downward-closed environments the definition of $\mathbf{Rev}[\cdot]$ can be modified so that the total ex ante allocation probability of the constraint is met with equality, i.e., $\hat{Y}(1) = Y(1)$. See Section 3.4.3.

3.5 Social Surplus with a Balanced Budget

In this section we explore the role that the designer's budget constraint plays on mechanism design for the objective of social surplus. Assume that the mechanism designer would like to maximize social surplus, but cannot subsidize the transaction, i.e., she is constrained to mechanisms with non-negative profit. Notice that such a constraint introduces a non-linearity into the designer's objective; however, this particular non-linearity instead can be instead represented as a constraint on total payments which, because revenue is linear (Theorem 3.24), is a linear constraint.

Recall that with outcome $(\boldsymbol{x}, \boldsymbol{p})$ the social surplus of a mechanism is $\sum_i v_i x_i - c(\boldsymbol{x})$ and its profit is $\sum_i p_i - c(\boldsymbol{x})$. There are two standard environments where budget balance is a crucial issue. First, in an *exchange* the mechanism designer is the mediator between a buyer and seller. The feasibility constraint is *all or none* in that either the trade occurs, in which case both agents are "served," or the trade does not occur, in which case neither agent is served. Second, in a *non-excludable public project* there is a fixed cost for producing a public good, e.g., for building a bridge, and if the good is produced then all agents can make use of the good. Again, the feasibility constraint is all or none.

The surplus maximization mechanism (Definition 3.3) has a deficit, i.e., negative profit, in non-trivial all-or-none environments. For instance, to maximize surplus in an exchange, the good should be traded when the buyer's value exceeds the seller's value for the good. The critical value for the buyer is the seller's value; the critical value for the seller is the buyer's value. When the good is sold the buyer pays the seller's value, the seller is paid the buyer's value, and the mechanism has a deficit of the difference between the two values. This difference is positive as otherwise the trade would not have occurred.

Here we address the question of maximizing social surplus subject to budget balance (taking both quantities in expectation). As with profit maximization, there is no mechanism that optimizes surplus subject to budget balance pointwise. E.g., in an exchange, if the values were known then the buyer and seller would be happy to trade at any price between their values; this is budget balanced. This approach, however, requires knowledge of a price that is between the buyer and seller's values, and this knowledge is not generally available in Bayesian mechanism design.

Our objective is surplus:

$$\text{Surplus}(\boldsymbol{v}, \boldsymbol{x}) = \sum\nolimits_i v_i x_i - c(\boldsymbol{x});$$

in addition to the feasibility constraint (which is given by $c(\cdot)$), incentive constraints (i.e., monotonicity of each agent's allocation rule), and individual rationality constraints we have a budget-balance constraint

$$\text{Profit}(\boldsymbol{p}, \boldsymbol{x}) = \sum\nolimits_i p_i - c(\boldsymbol{x}) \geq 0.$$

To optimize this objective in expectation subject to budget balanced in expectation we obtain the mathematical program

$$\begin{aligned} \max_{\boldsymbol{x}(\cdot), \boldsymbol{p}(\cdot)} \ & \mathbf{E}_{\boldsymbol{v}}\left[\sum\nolimits_i v_i x_i(\boldsymbol{v}) - c(\boldsymbol{x}(\boldsymbol{v}))\right] \\ \text{s.t. } & \boldsymbol{x}(\cdot) \text{ and } \boldsymbol{p}(\cdot) \text{ are IC and IR} \\ & \mathbf{E}_{\boldsymbol{v}}\left[\sum\nolimits_i p_i - c(\boldsymbol{x})\right] \geq 0 \end{aligned} \tag{3.7}$$

where expectations are simply integrals with respect to the density function of the valuation profile.

3.5.1 Lagrangian Relaxation

We will make two transformations of mathematical program (3.7) so as to be able to describe its solution. First, we will employ Proposition 3.15 to write expected payments in terms of the allocation rule (and the marginal price-posting revenue curve). Second, we will employ the method of Lagrangian relaxation on the budget-balance constraint to move it into the objective. Intuitively, Lagrangian relaxation allows the constraint to be violated but places a linear cost on violating the constraint. This cost is parameterized by the Lagrangian parameter λ, for high values of λ there is a high cost for violating the constraint (and a high benefit for slack in the constraint, i.e., the margin by which the constraint is satisfied), for low values of λ there is a low cost for violating the constraint. E.g., $\lambda = 0$ the optimization is the original problem without the budget-balance constraint; with $\lambda = \infty$ the optimization is entirely one of maximizing the slack in the constraint. In our case the slack in the constraint is the profit of the mechanism. Therefore, the $\lambda = \infty$ case is to maximize profit and the $\lambda = 0$ case is to maximize social surplus (without budget balance). Adjusting the Lagrangian parameter λ traces out the *Pareto frontier* between the two objectives of social surplus and profit (see Figure 3.9(a)). From this Pareto frontier we can see how to optimize social surplus subject to a constraint on profit (such as budget balance) or optimize profit subject to a constraint on social surplus. Notice that when the constraint that is Lagrangian relaxed is met with equality then it drops from the objective entirely and the objective value obtained is the optimal value of the original program.

In quantile space with payments expressed in terms of the allocation rule, the Lagrangian relaxation of our program is as follows.

$$\begin{aligned} \max_{\hat{\boldsymbol{y}}(\cdot)} \; & \mathbf{E}_{\boldsymbol{q}}\Big[\sum\nolimits_i V_i(q_i)\hat{y}_i(\boldsymbol{q}) - c(\hat{\boldsymbol{y}}(\boldsymbol{q}))\Big] \\ & \quad + \lambda \mathbf{E}_{\boldsymbol{q}}\Big[\sum\nolimits_i P'(q_i)\hat{y}_i(\boldsymbol{v}) - c(\hat{\boldsymbol{y}}(\boldsymbol{q}))\Big] \\ \text{s.t. } & \boldsymbol{y}(\cdot) \text{ is monotone.} \end{aligned} \tag{3.8}$$

Simplifying the objective with the identity (3.5) of $P'(q) = \frac{\mathrm{d}}{\mathrm{d}q}(q \cdot V(q)) = V(q) - q \cdot V'(q)$, we have

$$\mathbf{E}_{\boldsymbol{q}}\Big[\sum\nolimits_i \big[(1+\lambda) \cdot V_i(q_i) + \lambda q \cdot V_i'(q_i)\big] \cdot \hat{y}_i(\boldsymbol{q}) - (1+\lambda) \cdot c(\boldsymbol{y}(\boldsymbol{q}))\Big].$$

This is simply a (Lagrangian) virtual surplus optimization where agent

i's virtual value is

$$\phi_i^\lambda(q) = (1+\lambda)\cdot V_i(q_i) + \lambda q \cdot V_i'(q_i). \tag{3.9}$$

and with (Lagrangian) cost $(1+\lambda)c(\cdot)$, subject to monotonicity of each agent's the allocation rule.

If our original non-game-theoretic problem (without incentive and budget-balance constraints) is solvable, the same solution can be applied to solve this Lagrangian optimization. First, we can normalize the objective by dividing by $(1+\lambda)$, the result is a virtual surplus optimization with the same cost function as the original problem. Second, the budget-balance constrained optimization problem be effectively solved to an arbitrary degree of precision, e.g., by binary searching for the Lagrangian parameter λ for which solutions to the Lagrangian optimization are just barely budget balanced. The details of this search are described below.

3.5.2 Monotone Lagrangian Virtual Values

For any Lagrangian parameter λ, the optimal mechanism for the Lagrangian objective is the one that maximizes Lagrangian virtual surplus subject to monotonicity of each agent's the allocation rule. When the Lagrangian virtual value $\phi_i^\lambda(\cdot)$ is monotone non-increasing in q_i for each i the virtual surplus maximization mechanism for these Lagrangian virtual values and Lagrangian cost optimizes the Lagrangian objective in dominant strategy equilibrium (Corollary 3.9).

Lemma 3.29 *For a regular distribution (Definition 3.4, page 64) given by inverse demand function $V(\cdot)$ and any non-negative Lagrangian parameter λ, the Lagrangian virtual value function $\phi^\lambda(q) = (1+\lambda)\cdot V(q) + \lambda q \cdot V'(q)$ is monotonically decreasing.*

Proof The Lagrangian virtual value function of equation (3.9) is a convex combination of the inverse demand curve $V(\cdot)$ and the marginal price-posting revenue curve $P'(q) = V(q) - q \cdot V'(q)$, i.e., virtual values for revenue. The inverse demand curve is strictly decreasing by definition (Definition 3.9) and the marginal price-posting revenue curve is non-increasing by the regularity assumption (Proposition 3.16). The convex combination of two monotone functions is monotone; if one of the functions is strictly monotone then so is any non-trivial convex combination of them. The lemma follows. $\square$

To optimize expected social surplus subject to budget balance we need to tune the Lagrangian parameter so that the budget-balance constraint is met with equality. So tuned, the mechanism's expected profit will be zero and the expected Lagrangian objective will be equal to the true objective (expected social surplus). Expected profit is, as described above, a monotone function of the Lagrangian parameter. When expected profit is continuous in the Lagrangian parameter λ, this tuning of λ is straightforward. Recall that for surplus maximization subject to budget balance, the slack in the Lagrangian constraint is equal to the expected profit.

Lemma 3.30 *For Lagrangian virtual value functions that are continuous in the Lagrangian parameter, the slack in the Lagrangian constraint for expected Lagrangian virtual surplus maximization is continuously non-decreasing in the Lagrangian parameter.*

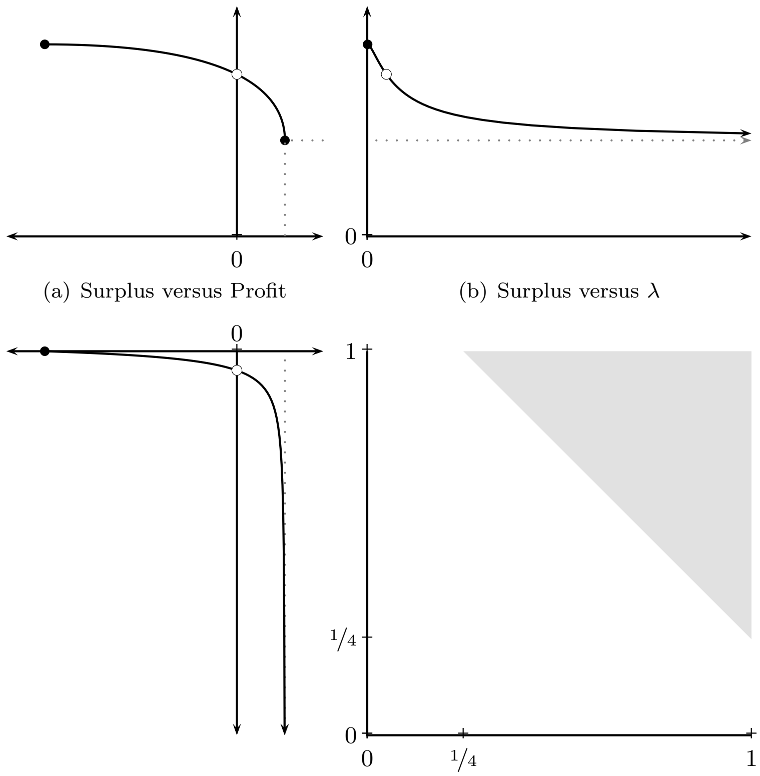
Proof The distribution of quantiles and a fixed Lagrangian parameter induce a distribution on profiles of Lagrangian virtual values. Continuity of Lagrangian virtual values with respect to the Lagrangian parameter implies that the joint density function on profiles of Lagrangian virtual values is continuous in the Lagrangian parameter. For any fixed profile of Lagrangian virtual values, Lagrangian virtual surplus maximization finds a (deterministic) pointwise optimal solution, the slack of this solution is also fixed and deterministic. As the distribution over these profiles is continuous in the Lagrangian parameter so is the expected slack. □

Theorem 3.31 *For regular general-costs environments, an Lagrangian virtual values from equation* (3.9)*, there exists a Lagrangian parameter for which the virtual surplus maximization mechanism has zero expected profit and with this parameter the mechanism maximizes expected social surplus subject to budget balance in dominant strategy equilibrium.*

Example 3.32 Consider two agents with uniformly distributed values and a non-excludable public project with cost one, i.e.,

$$c(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \boldsymbol{x} = (1,1), \\ 0 & \text{if } \boldsymbol{x} = (0,0), \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

The Lagrangian virtual values in value space are $\phi(v) = (2\lambda + 1) \cdot v - \lambda$. The Lagrangian virtual surplus mechanism serves both agents when $(2\lambda+1)(v_1+v_2)-2\lambda > 1+\lambda$ (for allocation $\boldsymbol{x} = (1,1)$, the left-hand side



(a) Surplus versus Profit (b) Surplus versus λ

(c) Profit versus λ (rotated 90°) (d) Optimal Mechanism, v_2 versus v_1

Figure 3.9 Depiction of the Pareto frontier for surplus (vertical axis) and profit (horizontal axis). On the Pareto frontier, the surplus maximizing point is profit minimizing (with negative profit) and the profit maximizing point is surplus minimizing. The surplus optimal point subject to budget balance is denoted by "$\circ$". The surplus and profit versus the Lagrangian parameter λ are depicted along with their asymptote (grey, dotted line) as $\lambda \to \infty$. The profit versus λ plot has been rotated 90° clockwise so as to line up with the profit axis of the Pareto frontier plot. The optimal mechanism is depicted by plotting v_2 versus v_1 where the region of valuation profiles for which the project is provided is shaded.

is the Lagrangian virtual surplus, the right-hand side is the Lagrangian cost), i.e., when

$$v_1 + v_2 \geq \tfrac{3\lambda+1}{2\lambda+1}. \tag{3.10}$$

For $\lambda = 0$ we serve if $v_1 + v_2 \geq 1$ (clearly this maximizes surplus) and for

$\lambda = \infty$ we serve if $v_1 + v_2 \geq 3/2$ (this maximizes profit). In equation (3.10) we see that (for the uniform distribution), for any Lagrangian parameter λ, the form of the optimal mechanism is a threshold rule on the sum of the agent values. It is easy then to solve for the threshold satisfies the budget-balance constraint with equality. The optimal threshold is $5/4$, the optimal Lagrangian parameter is $\lambda^\star = 1/2$, and the social surplus is $9/64 \approx 0.14$. This example is depicted in Figure 3.9.

3.5.3 Non-monotone Lagrangian Virtual Values and Partial Ironing

When the Lagrangian virtual value functions are non-monotone then the ironing procedure (Definition 3.11) can be applied and the virtual surplus maximization mechanism with the resulting ironed Lagrangian ironed virtual values is optimal for the Lagrangian objective. After ironing, however, the slack in the Lagrangian constraint, e.g., expected profit, is generally discontinuous in the Lagrangian parameter. In such case there is a point $\lambda^\star$ such that for $\lambda < \lambda^\star$ the expected profit of any solution is negative and for $\lambda > \lambda^\star$ the expected profit of any solution is positive. At $\lambda = \lambda^\star$ there are multiple solutions to the Lagrangian objective. These solutions vary in the contribution to the relaxed objective from the original objective and from the slack in the Lagrangian constraint (which is part of the relaxed objective); the expected profits of these solution span the gap between the negative profit solutions and the positive profit solutions. In particular, a convex combination of the supremum (with respect to expected profit) of solutions with negative profit with infimum of solutions with positive profit will optimize ironed Lagrangian virtual surplus and meet the budget-balance constraint with equality.

This convex combination of mechanisms can be interpreted as an ironed virtual surplus optimizer with a non-standard tie-breaking rule. Consider virtual value function $\phi(\cdot)$ and ironed virtual value function $\bar{\phi}(\cdot)$ constructed for $\phi(\cdot)$ for distribution F via the ironing procedure (Definition 3.11). By the definition of the ironing procedure, the cumulative ironed virtual value function $\bar{\Phi}(\cdot)$ is the smallest concave upper bound on the cumulative virtual value function $\Phi(\cdot)$. Define $[a, b]$ to be an *ironed interval* if $\bar{\Phi}(q) > \Phi(q)$ for $q \in (a, b)$ and $\bar{\Phi}(q) = \Phi(q)$ for $q \in \{a, b\}$. The ironing procedure gives ironed virtual values that are equal to virtual values in expectation under the assumption that all quantiles within the same ironed interval have the same allocation probability (Theorem 3.18). Such an outcome is always obtained for outcomes selected solely based on ironed virtual values (ignoring actual values).

For Lagrangian ironed virtual value functions, it may be that two adjacent ironed intervals have the same ironed virtual value. In such a case outcomes selected solely based on ironed virtual values will produce the same allocation probability for quantiles in the union of the adjacent ironed intervals. Notice that the equality of ironed virtual values across adjacent ironed intervals is sensitive to small changes in the Lagrangian parameter. With a slightly higher Lagrangian parameter these ironed intervals will be strictly merged; with a slightly lower Lagrangian parameter these ironed intervals will be strictly distinct. Thus, infimum mechanism is the one that tie-breaks to merge adjacent ironed intervals with the same ironed virtual value and the supremum mechanism is the one that tie-breaks to keep adjacent ironed intervals distinct. We refer to the mixing over two tie-breaking rule for maximizing ironed virtual surplus as *partial ironing*.

Theorem 3.33 *For general-cost environments, and Lagrangian virtual values from equation* (3.9)*, there exists a Lagrangian parameter and partial-ironing parameter for which the partially-ironed Lagrangian virtual surplus maximization mechanism optimizes social surplus subject to budget balance in dominant strategy equilibrium.*

Exercises

3.1 In computer networks such as the Internet it is often not possible to use monetary payments to ensure the allocation of resources to those who value them the most. Computational payments, e.g., in the form of "proofs of work", however, are often possible. One important difference between monetary payments and computational payments is that computational payments can be used to align incentives but do not transfer utility from the agents to the seller. I.e., the seller has no direct value from an agent performing a proof-of-work computation. Define the *residual surplus* as the social surplus less the payments, i.e., $\sum_i (v_i \cdot x_i - p_i) - c(\boldsymbol{x})$. (For more details, see the discussion of non-monetary payments in Chapter 1.)

Describe the mechanism that maximizes residual surplus when the distribution on agents' values satisfy the *monotone hazard rate* assumption, i.e., $f(v)/1-F(v)$ is monotone non-decreasing. Your description should first include a description in terms of virtual values and then you should interpret the implication of the monotone hazard rate assumption to give a simple description of the optimal mechanism. In particular, consider monotone hazard rate distributions in the following environments:

(a) a single-item auction with i.i.d. values,
(b) a single-item auction with non-identical values, and
(c) an environment with general costs specified by $c(\cdot)$ and non-identical values.

3.2 Give a mechanism with first-price payment semantics that implements the social surplus maximizing outcome in equilibrium for any single-dimensional agent environment. Hint: Your mechanism may be parameterized by the distribution.

3.3 Derive equation (3.3),

$$\mathbf{E}_{v\sim F}[p(v)] = \mathbf{E}_{v\sim F}[\phi(v)\cdot x(v)] \tag{3.3}$$

by taking expectation of the payment identity (3.1),

$$p(v) = v\cdot x(v) - \int_0^v x(z)\,\mathrm{d}z, \tag{3.1}$$

for $v \sim F$ and simplifying.

3.4 Consider the non-downward closed environment of *public projects*: either every agent can be served or none of them. I.e., the cost structure satisfies:

$$c(\boldsymbol{x}) = \begin{cases} 0 & \text{if } \sum_i x_i = 0, \\ 0 & \text{if } \sum_i x_i = n, \text{ and} \\ \infty & \text{otherwise.} \end{cases}$$

(a) Describe the revenue-optimal mechanism for general distributions.
(b) Describe the revenue-optimal mechanism when agents' values are i.i.d. from $U[0,1]$.
(c) Give an asymptotic, in terms of the number n of agents, analysis of the expected revenue of the revenue-optimal public project mechanism when agents' values are i.i.d. from $U[0,1]$.

3.5 Consider a two unit auction to four agents and a virtual value function that is strictly monotone except for an interval $[a,b]$ where it is a positive constant (e.g., Figure 3.2 on 69). Suppose the valuation profile $\boldsymbol{v}$ satisfies $v_1 > b$, $v_2, v_3 \in [a, b]$, and $v_4 < a$. Calculate the probability of winning and expected payments of all agents (in terms of a and b).

3.6 Consider profit maximization with values drawn from a discrete distribution. Derive virtual values for revenue for discrete single-dimensional type spaces for agents with linear utility. Assume that $T = \{v^0, \ldots, v^N\}$ with the probability that an agent's value is $v \in T$ given by probability mass function $f(v)$. Assume $v^0 = 0$. Note: You must first solve Exercise 2.2 to characterize BNE equilibrium.

(a) Derive virtual values for the special case where the values are uniform, i.e., $v^j = j$ for all j.
(b) Derive virtual values for the special case where the probabilities are uniform, i.e., $f(v^j) = 1/N$ for all j.
(c) Give virtual values for the general case.

(Hint: You should end up with a very similar formulation to that for continuous type spaces.)

3.7 The text has focused on *forward auctions* where the auctioneer is a seller and the agents are buyers. The same theory can be applied to *reverse auctions* (or *procurement*) where the auctioneer is a buyer and the agents are sellers. It is possible to consider reverse auctions within the framework described in this chapter where an agents value for service is negative, i.e., in order to provide the service they must pay a cost. It is more intuitive, however, to think in terms of positive costs instead of negative values.

(a) Derive a notion analogous to revenue curves for an agent (as a seller) with private cost drawn from a distribution F.
(b) Derive a notion of *virtual cost functions* analogous to virtual value functions.
(c) Suppose the auctioneer has a value of v for procuring a service from one of several sellers with costs distributed i.i.d. and uniformly on $[0, 1]$. Describe the auction that optimizes the seller's profit (value for procurement less payments made to agents).

3.8 Consider a profit-maximizing broker mediating the exchange between a buyer and a seller. The broker's profit is the difference between payment made by the buyer and payment made to the seller. Use the derivation of virtual values for revenue (from Section 3.3.4) and virtual costs (from Exercise 3.7).

(a) Derive the optimal exchange mechanism for regular distributions for the buyer and seller.
(b) Solve for the optimal exchange mechanism in the special case where the buyer's and seller's values are both distributed uniformly on $[0, 1]$.

3.9 In Example 3.32 it was shown that for to agents with uniform values on interval $[0, 1]$ and a cost of one for serving both of them together, the surplus maximizing mechanism with a balanced budget in expectation serves the agents when the sum of their values is at least $4/3$. There is a natural dominant strategy "second-price" implementation of this mechanism; instead give a "first-price" (a.k.a., pay-your-bid) implementation. Your mechanism should solicit bids, decide based on the bids whether to serve the agents, and charge each agent her bid if they are served.

Chapter Notes

The surplus-optimal Vickrey-Clarke-Groves (VCG) mechanism is credited to Vickrey (1961), Clarke (1971), and Groves (1973).

The characterization of revenue-optimal single-item auctions as virtual value maximizers (for regular distributions) and ironed virtual value maximizers (for irregular distributions) was derived by Roger Myerson (1981). Its generalization to single-dimensional agent environments is an obvious extension. The relationship between revenue-optimal auctions, price-posting revenue curves, and marginal price-posting revenue (equivalent to virtual values) is due to Bulow and Roberts (1989). The revenue-linearity-based approach is from Alaei et al. (2013).

Myerson and Satterthwaite (1983) characterized mechanisms that maximize social surplus subject to budget balance via Lagrangian relaxation of the budget-balance constraint. The discussion of partial ironing for Lagrangian virtual surplus maximizers given here is from Devanur et al. (2013). This partial ironing suggests that the optimal mechanism is not deterministic, the problem of finding a deterministic mechanism to maximize social surplus subject to budget balance is much more complex as the space of deterministic mechanisms is not convex (Diakonikolas et al., 2012).

4
Bayesian Approximation

One of the most intriguing conclusions from the preceding chapter is that for i.i.d. regular single-item environments the second-price auction with a reservation price is revenue optimal. This result is compelling as the solution it proposes is quite simple, therefore, making it easy to prescribe. Furthermore, reserve-price-based auctions are often employed in practice so this theory of optimal auctions is also descriptive. Unfortunately, i.i.d. regular single-item environments are hardly representative of the scenarios in which we would like to design good mechanisms. Furthermore, if any of the assumptions are relaxed, reserve-price-based mechanisms are not optimal.

Another point of contention is that auctions, even simple ones like the second-price auction, can be a slow and inconvenient way to allocate resources. In many contexts posted pricings are preferred to auctions. As we have seen, posted pricings are not optimal unless there is only a single consumer. In addition to being preferred for their speed and simplicity, posted pricings also offer robustness to out-of-model phenomena such as collusion. Therefore, approximation results for posted pricings imply that good collusion resistant mechanisms exist.

In this chapter we address these deficiencies by showing that while posted pricings and reserve-price-based mechanisms are not generally optimal, they are approximately optimal in a wide range of environments. Furthermore, these approximately optimal mechanisms are more robust, less dependent on the details of the distribution, and sometimes provide more conceptual understanding than their optimal counterparts. The approximation factor obtained by most of these approximation mechanisms is two. Meaning, for the worst distributional assumptions, the mechanism's expected performance is within a factor two of the optimal mechanism. Of course, in any particular environment these mechanisms may perform better than this worst-case guarantee.

A number of properties of the environment will be crucial for enabling good approximation mechanisms. As in Chapter 3 these are: independence of the distribution of preferences for the agents, distributional regularity as implied by the concavity of the price-posting revenue curve, and downward closure of the designer's feasibility constraint. In addition, two new structural restrictions on the environment will be introduced.

A *matroid set system* is one that is downward closed and satisfies an additional "augmentation property." An important characterization of the matroid property is that the surplus maximizing allocation (subject to feasibility) is given by the *greedy-by-value* algorithm: sort the agents by value, then consider each agent in-turn and serve the agent if doing so is feasible given the set of agents already being served. The optimality of greedy-by-value implies that the order of the agents' values is important for finding the surplus maximizing outcome, but the relative magnitudes of their values are not.

The *monotone hazard rate* condition is a refinement of the regularity property of a distribution of values. Intuitively, the monotone hazard rate condition restricts how heavy the tail of the distribution is, i.e., how much probability mass is on very high values. An important consequence of the monotone hazard rate assumption is that the optimal revenue and optimal social surplus are within a factor of $e \approx 2.718$ of each other. This will enable mechanism that optimize social surplus to give good approximations to revenue.

Mathematical Note Subsequently we will consider using monopoly reserve prices for distributions where these prices are not unique. For these distributions we should always assume the worst tie-breaking rule as it is always possible to perturb the distribution slightly to make that worst monopoly price unique. For example, recall that a regular distribution can be equivalently specified by its distribution function or its revenue curve. For instance the *equal revenue distribution* has constant revenue curve, $R^{\text{EQR}}(q) = 1$, and therefore any price on $[1, \infty)$ is optimal. A sufficient perturbation to make the price of one the unique monopoly price is given by revenue curve $R(q) = 1 - \epsilon(1 - q)$ which is uniquely maximized at monopoly quantile $\hat{q}^\star = 1$ with corresponding monopoly price $\hat{v}^\star = V(1) = R(1) = 1$.

In the previous two chapters, with the characterization of Bayes-Nash equilibrium (Theorem 2.2) and the characterization of profit-optimal mechanisms (Corollary 3.21), we assumed that the values of the agents were drawn from continuous distributions. In this chapter, especially when describing examples that show that the assumptions of a theorem are necessary, it will sometimes more expedient to work with discrete distributions. A discrete distribution is specified by a set of values and probabilities for these values.

There are two ways to relate these discrete examples to the continuous environments we have heretofore been considering. First, we could rederive Theorem 2.2 and Corollary 3.21 (and their variants) for discrete distributions (see Exercise 2.2 and Exercise 3.6, respectively). Importantly, via such a rederivation, it is apparent that discrete and continuous environments are intuitively similar. Second, we could consider a continuous perturbation of the discrete distribution which will exhibit the same phenomena with respect to optimization and approximation. For example, one such perturbation is, for a sufficiently small ϵ, to replace any value v from the discrete distribution with a uniform value from $[v, v + \epsilon]$.

4.1 Monopoly Reserve Pricing

We start our discussion of simple mechanisms that are approximately optimal by showing that a natural generalization of the second-price auction with monopoly reserve continues to be approximately optimal for regular but asymmetric distributions. Recall that monopoly prices are a property of virtual value functions which are a property of the distributions from which agents' values are drawn (Definition 3.7). When the agents' values are drawn from distinct distributions their monopoly prices are generally distinct. The following definition generalizes the second-price auction with a single reserve price to one with *discriminatory*, i.e., agent-specific, reserve prices.

Definition 4.1 The *second-price auction with (discriminatory) reserves* $\hat{\boldsymbol{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$ is:

(i) reject each agent i with $v_i < \hat{v}_i$,
(ii) allocate the item to the highest valued agent remaining (or none if none exists), and
(iii) charge the winner her critical price.

With non-identical distributions the optimal single-item auction indeed needs the exact marginal revenue functions to determine the optimal allocation (see Example 4.1). This contrasts to the i.i.d. regular case where all we needed was a single number, the monopoly price for the distribution, and reserve pricing with this number is optimal. Figure 4.1 compares allocations of the (asymmetric) optimal auction with those of the second-price auction with (asymmetric) monopoly reserves.

Example 4.1 Consider a two-agent single-item auction where agent 1 (Alice) and agent 2 (Bob) have values distributed uniformly on $[0, 2]$ and $[0, 3]$, respectively. The virtual value functions are $\phi_1(v_1) = 2v_1 - 2$ and $\phi_2(v_2) = 2v_2 - 3$. Alice's monopoly price one; Bob's monopoly price is $3/2$. Alice has a higher virtual value than Bob when $v_1 > v_2 - 1/2$. The optimal auction is asymmetric. It serves an agent only if one is above their respective monopoly price. If both are above their respective monopoly reserves, it serves the highest valued agent with a penalty of $1/2$ against Bob (cf. Example 3.11, page 67). In contrast the monopoly-reserves auction is the same but with no penalty for Bob. See Figure 4.1.

In the remainder of this section we show that if the agents' values are drawn from regular distributions then the (single item) monopoly-reserves auction is a two approximation to the optimal revenue. We will then show that, except for the consideration of more general feasibility constraints, this result is tight. The approximation bound of two is tight: we show by example that there is a non-identical regular distribution where the ratio of the optimal to monopoly-reserves revenue is two. The regularity assumption is tight: for irregular distributions the approximation ratio of monopoly reserves can be as bad as linear (i.e., it grows with the number of agents). Thus, we conclude that this two-approximation result for regular distributions in single-item environments is essentially the right answer. Later in the chapter we will consider the extent to which this result generalizes beyond single-item environments.

4.1.1 Approximation for Regular Distributions

The main result of this section shows that, though distinct, the monopoly-reserves auction and the revenue-optimal auction have similar revenues.

Theorem 4.2 *For single-item environments and agents with values drawn independently from (non-identical) regular distributions, the second-price auction with (asymmetric) monopoly reserve prices obtains at least half the revenue of the (asymmetric) optimal auction.*

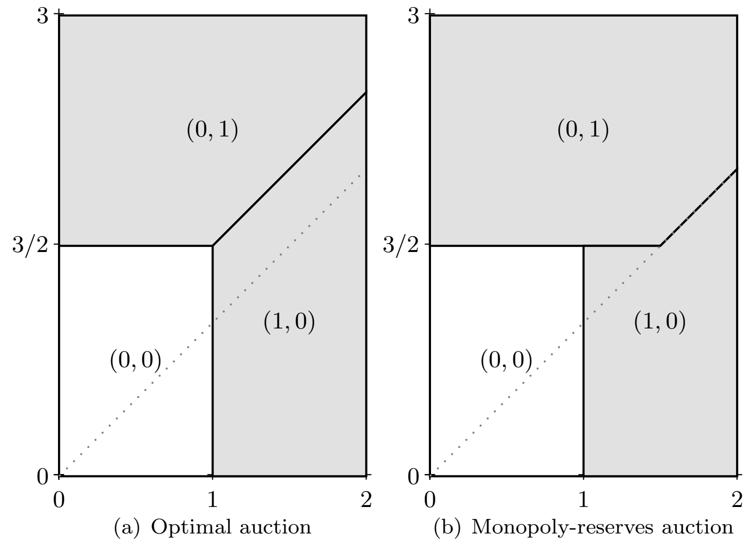


Figure 4.1 In Example 4.1 Agent 1 has value $v_1 \sim U[0,2]$; agent 2 has value $v_2 \sim U[0,3]$. In the space of valuation profiles $\boldsymbol{v} \in [0,2] \times [0,3]$, with agent 1's value on the horizontal axis and agent 2's value on the vertical axis, the allocation $\boldsymbol{x} = (x_1, x_2)$ for the (asymmetric) optimal auction and (asymmetric) monopoly-reserves auction are depicted.

The proof of Theorem 4.2 is enabled by the following three properties of regular distributions and virtual value functions. First, Corollary 3.27 shows that for a regular distribution, a monotone allocation rule, and virtual value given by the marginal revenue curve, the expected revenue is equal to the expected virtual surplus. The second and third properties are given by the two lemmas below.

Lemma 4.3 *For any virtual value function, the virtual values corresponding to values that exceed the monopoly price are non-negative.*

Proof The lemma follows immediately from the definition of virtual value functions which requires their monotonicity (Definition 3.6) . □

Lemma 4.4 *For any distribution, the value of an agent is at least her virtual value for revenue.*

Proof We prove the lemma for regular distributions (as is necessary for Theorem 4.2) and leave the general proof to Exercise 4.3. For regular distributions, where the virtual values for revenue are given by the formula $\phi(v) = v - \frac{1-F(v)}{f(v)}$, the lemma follows as both $1 - F(v)$ and $f(v)$ are non-negative. □

Our goal will be to show that the expected revenue of the monopoly-reserves auction is approximately an upper bound on the expected virtual surplus of the optimal auction (which is equal to its revenue). Consider running both auctions on the same random input. Notice that conditioned on the event that both auctions serve the same agent, both auctions obtain the same (conditional) expected virtual surplus. Notice also that conditioned on the event that the auctions serve distinct agents, the monopoly-reserves auction has higher expected payments than the optimal auction. It is not correct to bound revenue by combining conditional virtual values with conditional payments as the amortized analysis that defines virtual values is only correct under unconditional expectations. Therefore, for the second case we will instead relate the payment of monopoly reserves to the virtual value of the winner in the optimal auction (for which it gives an upper bound).

Proof of Theorem 4.2 Let REF denote the optimal auction and its expected revenue and APX denote the second-price auction with monopoly reserves and its expected revenue. Clearly, REF $\geq$ APX; our goal is to give an approximate inequality in the opposite direction by showing that $2\,\mathrm{APX} \geq \mathrm{REF}$. Let I be the winner of the optimal auction and J be the winner of the monopoly reserves auction. I and J are random variables. Notice that neither auctions sell the item if and only if all virtual values are negative; in this situation define $I = J = 0$. With these definitions and Corollary 3.27, $\mathrm{REF} = \mathbf{E}[\phi_I(v_I)]$ and $\mathrm{APX} = \mathbf{E}[\phi_J(v_J)]$.

We start by simply writing out the expected revenue of the optimal auction as its expected virtual surplus conditioned on $I = J$ and $I \neq J$.

$$\mathrm{REF} = \underbrace{\mathbf{E}[\phi_I(v_I) \mid I = J]\,\mathbf{Pr}[I = J]}_{\mathrm{REF}_=} + \underbrace{\mathbf{E}[\phi_I(v_I) \mid I \neq J]\,\mathbf{Pr}[I \neq J]}_{\mathrm{REF}_{\neq}}.$$

We will prove the theorem by showing that both the terms on the right-hand side are bounded from above by APX. Thus, $\mathrm{REF} \leq 2\,\mathrm{APX}$. For the first term:

$$\begin{aligned}
\mathrm{REF}_= &= \mathbf{E}[\phi_I(v_I) \mid I = J]\,\mathbf{Pr}[I = J] \\
&= \mathbf{E}[\phi_J(v_J) \mid I = J]\,\mathbf{Pr}[I = J] \\
&\leq \mathbf{E}[\phi_J(v_J) \mid I = J]\,\mathbf{Pr}[I = J] + \mathbf{E}[\phi_J(v_J) \mid I \neq J]\,\mathbf{Pr}[I \neq J] \\
&= \mathrm{APX}.
\end{aligned}$$

The inequality in the above calculation follows from Lemma 4.3 as even when $I \neq J$ the virtual value of J must be non-negative. Therefore, the term added is non-negative. For the second term:

$$\begin{aligned}
\mathrm{REF}_{\neq} &= \mathbf{E}[\phi_I(v_I) \mid I \neq J]\,\mathbf{Pr}[I \neq J] \\
&\leq \mathbf{E}[v_I \mid I \neq J]\,\mathbf{Pr}[I \neq J] \\
&\leq \mathbf{E}[p_J(\boldsymbol{v}) \mid I \neq J]\,\mathbf{Pr}[I \neq J] \\
&\leq \mathbf{E}[p_J(\boldsymbol{v}) \mid I \neq J]\,\mathbf{Pr}[I \neq J] + \mathbf{E}[p_J(\boldsymbol{v}) \mid I = J]\,\mathbf{Pr}[I = J] \\
&= \mathrm{APX}\,.
\end{aligned}$$

The first inequality in the above calculation follow from values upper bounding virtual values (Lemma 4.4). The second inequality follows because, among agents who meet their reserve, J is the highest valued agent and I is a lower valued agent. Therefore, as APX is a second-price auction, the winner J's payment is at least the loser I's value. The third inequality follows because payments are non-negative so the term added is non-negative. ☐

Theorem 4.2 shows that when agent values are non-identically distributed at least half of the revenue of the optimal asymmetric auction which is parameterized by complicated virtual value functions can be obtained by a simple auction which is parameterized by natural statistical quantities, namely, each distribution's monopoly price. The theorem holds for a broad class of distributions that satisfy the regularity property. While for specific distributions the approximation bound may be better than two, we will see subsequently, by example, that if the only assumption on the distribution is regularity then the approximation factor of two is tight.

Definition 4.2 The *equal-revenue distribution* has distribution function $F^{\mathrm{EQR}}(z) = 1 - 1/z$ and density function $f^{\mathrm{EQR}}(z) = 1/z^2$ on support $[1, \infty)$.

The equal-revenue distribution is so called because the revenue obtained from posting any price is the same. Consider posting price $\hat{v} > 1$. The expected revenue from such a price is $\hat{v} \cdot (1 - F^{\mathrm{EQR}}(\hat{v})) = 1$. As the price-posting revenue curve is the constant function $P^{\mathrm{EQR}}(\hat{q}) = 1$, the distribution is on the boundary between regularity and irregularity. As it is the boundary between regularity and irregularity, it often provides an extremal example for results that hold for regular distributions.

Lemma 4.5 *There is an (non-identical) regular two-agent single-item* *environment where the optimal auction obtains twice the revenue of the second-price auction with (discriminatory) monopoly reserves.*

Proof For any $\epsilon > 0$ we will give a distribution and show that there is an auction with expected revenue strictly greater than $2 - \epsilon$ but the revenue of the monopoly reserves auction is precisely one.

Consider the asymmetric two-agent single-item environment where agent 1 (Alice) has value (deterministically) one and agent 2 (Bob) has value distributed according to the equal-revenue distribution. The monopoly price for the equal-revenue distribution is ill-defined because every price is optimal, but a slight perturbation of the distribution has a unique monopoly price of $\hat{v}_2^\star = 1$ (see Mathematical Note on page 101). Thus the monopoly prices are $\hat{\boldsymbol{v}}^\star = (1, 1)$ and the expected revenue of the second-price auction with monopoly reserves is one.

Of course, for this distribution it is easy to see how we can do much better. Offer Bob a high price h. If he rejects this price then offer Alice a price of 1. Notice that by the definition of the equal-revenue distribution, Bob's expected payment is one, but still Bob rejects the offer with probability $1 - 1/h$ and the item can be sold to Alice. The expected revenue of the mechanism is $h \cdot 1/h + 1 \cdot (1 - 1/h) = 2 - 1/h$. Choosing $h > 1/\epsilon$ gives the claimed result. □

While the monopoly-reserves auction (parameterized by n monopoly prices) is significantly less complex than the optimal auction (parameterized by n virtual value functions), it is not often used in practice. In practice, even in asymmetric environments, auctions are often parameterized by a single *anonymous* reserve price. For regular, non-identical distributions anonymous reserve pricing continues to give a good approximation to the optimal auction. This and related results are discussed in Section 4.4.

4.1.2 Inapproximability Irregular Distributions

The second-price auction with monopoly reserve prices only guarantees a two approximation for regular distributions. The proof of Theorem 4.2 relied on regularity crucially when it invoked Corollary 3.27 to calculate revenue in terms of virtual surplus for all monotone allocation rules. Recall that for irregular distributions, revenue is only equal to virtual surplus for allocation rules that are constant where the virtual value functions are constant. For irregular distributions there are two challenges for that the monopoly-reserves auction must confront. First, even

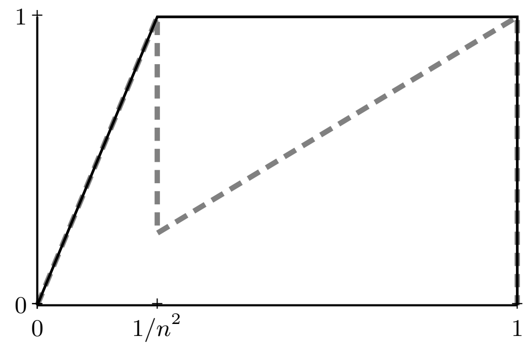


Figure 4.2 The revenue curve (thin, solid, black) and price-posting revenue curve (gray, thick, dashed) for the discrete two-point equal revenue distribution from the proof of Proposition 4.6 with $h = 2$. As usual for revenue curves, the horizontal axis is quantile.

if the distributions are identical, the optimal auction is not the second-price auction with monopoly reserves; it irons (see Section 3.3.3). Second, the distributions may not be identical. We show here that even for i.i.d. irregular distributions this trivial bound cannot be improved (Proposition 4.6), and that this lower bound is tight as the monopoly-reserves auction for (non-identical) irregular distributions is, trivially, an n approximation (Proposition 4.7).

Of course, irregular distributions that are "nearly regular" do not exhibit the above worst case behavior. For example, Exercise 4.6 formalizes a notion of near regularity under which reasonable approximation bounds can be proven.

Proposition 4.6 *For (irregular) i.i.d. n-agent single-item environments, the second-price auction with monopoly reserve is at best an n approximation.*

Proof Consider the discrete equal-revenue distribution on $\{1, h\}$, i.e., with $f(h) = 1/h$ and $f(1) = 1 - 1/h$, slightly perturbed so that the monopoly price is one (see Mathematical Note on page 101). With a monopoly reserve of $\hat{v}^\star = 1$ and all values at least one, the reserve is irrelevant for the second-price auction.

Consider the expected revenues of the second-price auction APX(h) and the optimal auction REF(h) as a function of h. We show the following limit result which implies the proposition.

$$\mathrm{APX} = \lim_{h\to\infty} \mathrm{APX}(h) = 1, \text{and} \tag{4.1}$$

$$\mathrm{REF} = \lim_{h\to\infty} \mathrm{REF}(h) = n. \tag{4.2}$$

An agent is high-valued with probability $1/h$ and low valued with probability $(1 - 1/h)$. The probability that there are exactly k high valued agents is:

$$\mathbf{Pr}[\text{exactly } k \text{ are high valued}] = \tbinom{n}{k} \cdot h^{-k} \cdot (1 - 1/h)^{n-k}.$$

For constant n and k and in the limit as h goes to infinity, the first term is constant and the last term is one. The middle term goes to zero at a rate of h^{-k}. Thus,

$$\lim_{h\to\infty} h^k \cdot \mathbf{Pr}[\text{exactly } k \text{ are high valued}] = \tbinom{n}{k}, \text{ and} \tag{4.3}$$

$$\lim_{h\to\infty} h^k \cdot \mathbf{Pr}[\text{at least } k \text{ are high valued}] = \tbinom{n}{k}. \tag{4.4}$$

For the discrete equal-revenue distribution, $\phi(1) = 0$ and $\phi(h) = h$ (see Figure 4.2 and Exercise 3.6). Now we can calculate $\mathrm{REF} = \lim_{h\to\infty} \mathrm{REF}(h)$ as $\phi(1)$ times the probability that there are no high-valued agents plus $\phi(h)$ times the probability that there are one or more high-valued agents. $\mathrm{REF} = 0 + \binom{n}{1} = n$.

We can similarly calculate $\mathrm{APX} = \lim_{h\to\infty} \mathrm{APX}(h)$ as one times the probability that there are one or fewer high-valued agents plus h times the probability that there are two or more high-valued agents. By equation (4.3) with $k = 0$ and 1, the first term is one; by equation (4.4) with $k = 2$, the second term is zero. Thus, $\mathrm{APX} = 1$. □

Proposition 4.7 *For (non-identical, irregular) n-agent single-item environments, the second-price auction with monopoly reserve is at worst an n approximation.*

Proof Let REF and APX and denote the monopoly-reserve auction and the optimal auction and their revenue, respectively, in an n-agent, single-item environment.

As usual for approximation bounds when the optimal mechanism REF is complex, we will formulate an upper bound that is simple. Denote by UB the optimal auction and its revenue for the n-agent, n-unit environment (a.k.a. a digital good). Clearly, $\mathrm{UB} \geq \mathrm{REF}$ as this auction could discard all but one unit and then simulate the outcome REF (the optimal single-unit auction). UB is also very simple. As there are n units and n

agents there is no competition between the agents and the optimization problem decomposes into n independent monopoly pricing problems. Denote by $\boldsymbol{R}^\star = (R_0^\star, \dots, R_n^\star)$ the profile of monopoly revenues. The revenue of the optimal n-unit auction is:

$$\mathrm{UB} = \sum\nolimits_i R_i^\star.$$

We now get a lower bound on the monopoly-reserves revenue APX. Consider the mechanism LB that chooses, before asking for agent reports, the agent $i^\star$ with the highest monopoly revenue and offers this agent her monopoly price $\hat{v}_{i^\star}^\star$. LB obtains revenue

$$\mathrm{LB} = \max\nolimits_i R_i^\star.$$

Moreover, APX $\geq$ LB as if $i^\star$ would accept price her monopoly price $\hat{v}_{i^\star}^\star$ then some agent in APX must accept a price of at least $\hat{v}_{i^\star}^\star$ (either agent $i^\star$ or an agent beating out agent $i^\star$).

Finally, we make the simple observation that $n{\cdot}\mathrm{LB} \geq \mathrm{UB}$ which proves the proposition. □

4.2 Oblivious Posted Pricings and the Prophet Inequality

Two problematic aspects of employing auctions to allocate resources is that (a) they require multiple rounds of communication (i.e., they are slow) and (b) they require all agents to be present at the time of the auction. Often both of these requirements are prohibitive. In routing in computer networks a packet needs to be routed, or not, quickly and, if the network is like the Internet, without state in the routers. Therefore, auctions are unrealistic for congestion control. In a supermarket where you go to buy lettuce, we should not hope to have all the lettuce buyers in the store at once. Finally, in selling goods on the Internet, eBay has found empirically that posted pricing via the “buy it now” option is more appropriate than a slow (days or weeks) ascending auction.

Posted pricings give very robust revenue guarantees. For instance, their revenue guarantees are impervious to many kinds of collusive behavior on the part of the agents. Moreover, the prices (to be posted) can also be used as reserve prices for the first- and second-price auctions and this only improves on the revenue from price posting.

In a posted pricing, distinct prices can be posted to the agents with first-come-first-served and while-supplies-last semantics. In this section we show that *oblivious posted pricing*, where agents arrive and consider their respective prices in any arbitrary order, gives a two approximation to the optimal auction. In the next section, we show that *sequential posted pricing*, where the mechanism chooses the order in which the agents are permitted to consider their respective posted prices, gives an improved approximation of $e/e-1 \approx 1.58$. Both results hold for objectives of revenue and social surplus and for any independent distribution on agent values (i.e., regularity is not assumed).

There are several challenges to the design and analysis of oblivious posted pricings. First, for any particular n-agent scenario, an oblivious posted pricing potentially requires optimization of n distinct prices. In high dimensions (i.e., large n) this optimization problem is computationally challenging. Moreover, it is not immediately clear that the resulting optimal prices would perform well in comparison to the optimal auction. To justify usage of posted pricings over auctions, we must be able to easily find good prices and these prices should give revenue that compares favorably to that of the optimal auction. The approach of this section is to solve both problems at once by identifying a class of easy-to-find posted pricings that perform well.

4.2.1 The Prophet Inequality

The oblivious posted pricing theorem we present is an application of a *prophet inequality* theorem from optimal stopping theory. Consider the following scenario. A gambler faces a series of n games, one on each of n days. Game i has prize v_i distributed independently according to distribution F_i. The order of the games and distribution of the prize values is fully known in advance to the gambler. On day i the gambler *realizes* the prize $v_i \sim F_i$ of game i and must decide whether to keep this prize and *stop* or to return the prize and *continue* playing. In other words, the gambler is only allowed to keep one prize and must decide whether or not to keep a given prize immediately on realizing the prize and before any future prizes are realized.

The gambler's optimal strategy can be calculated by *backwards induction*. On day n the gambler should stop with whatever prize is realized. This results in expected value $\mathbf{E}[v_n]$. On day $n-1$ the gambler should stop if the prize has greater value than $\hat{v}_{n-1} = \mathbf{E}[v_n]$, the expected value of the prize from the last day. On day $n-2$ the gambler should stop with if the prize has greater value than $\hat{v}_{n-2}$, the expected value of the strategy for the last two days. Proceeding in this manner the gambler can calculate a threshold $\hat{v}_i$ for each day where the optimal strategy is to stop with prize i if and only if $v_i \geq \hat{v}_i$.

This optimal strategy suffers from many of the drawbacks of optimal strategies. It is complicated: it takes n numbers to describe it. It is sensitive to small changes in the game, e.g., changing of the order of the games or making small changes to distribution i strictly above $\hat{v}_i$. It does not allow for much intuitive understanding of the properties of good strategies. Finally, it does not generalize well to give solutions to other similar kinds of games, e.g., that of our oblivious posted pricing problem.

Approximation gives a crisper picture. A *uniform threshold strategy* is given by a single threshold $\hat{v}$ and requires the gambler to accept the first prize i with $v_i \geq \hat{v}$. Threshold strategies are clearly suboptimal as even on day n if prize $v_n < \hat{v}$ the gambler will not stop and will, therefore, receive no prize. We refer to the prize selection procedure when multiple prizes are above the threshold as the *tie-breaking rule*. The tie-breaking rule implicit in the specification of the gambler's game is lexicographical, i.e., by "smallest i."

Theorem 4.8 *For any product distribution on prize values $\boldsymbol{F} = F_1 \times \cdots \times F_n$, there exists a uniform threshold strategy such that the expected prize of the gambler is at least half the expected value of the maximum prize; moreover, the bound is invariant with respect to the tie-breaking rule; moreover, for continuous distributions with non-negative support one such threshold strategy is the one where the probability that the gambler receives no prize is exactly $1/2$.*

Theorem 4.8 is a *prophet inequality*: it suggest that even though the gambler does not know the realizations of the prizes in advance, she can still do half as well as a prophet who does. While this result implies that the optimal (backwards induction) strategy satisfies the same performance guarantee, this guarantee was not at all clear from the original formulation of the optimal strategy.

Unlike the optimal (backwards induction) strategy this prophet inequality provides substantial conclusions. Most obviously, it is a very simple strategy. The result is clearly driven by trading off the probability of not stopping and receiving no prize with the probability of stopping early with a suboptimal prize. Notice that the order of the games makes no difference in the determination of the threshold, and if the distribution above or below the threshold changes, neither the bound nor suggested strategy is affected. Moreover, the invariance of the theorem to the tie-breaking rule suggests the bound can be applied to other related scenarios. The profit inequality is quite robust.

Proof of Theorem 4.8 Let REF denote prophet and her expected prize, i.e., the expected maximum prize, $\mathbf{E}[\max_i v_i]$, and APX denote a gambler with threshold strategy $\hat{v}$ and her expected prize. Define $\hat{q}_i = 1 - F_i(\hat{v}) = \mathbf{Pr}[v_i \geq \hat{v}]$ as the probability that prize i is above the threshold $\hat{v}$ and $\chi = \prod_i (1 - \hat{q}_i)$ as the probability that the gambler rejects all prizes. The proof follows in three steps. In terms of the threshold $\hat{v}$ and failure probability χ, we get an upper bound on the expected prophet's payoff. Likewise, we get a lower bound on expected gambler's payoff. Finally, we choose $\hat{v}$ so that $\chi = 1/2$ to obtain the bound. If there is no $\hat{v}$ with $\chi = 1/2$, which is possible if the distributions $\boldsymbol{F}$ are not continuous, we give a slightly more sophisticated method for choosing $\hat{v}$.

In the analysis below, the notation "$(v_i - \hat{v})^+$" is shorthand for "$\max(v_i - \hat{v}, 0)$." The prophet is allowed not to pick any prize, e.g., if all prizes have negative value, to denote this outcome we add a prize indexed 0 with value deterministically $v_0 = 0$; all summations are over prizes $i \in \{0, \ldots, n\}$.

(i) An upper bound on $\mathrm{REF} = \mathbf{E}[\max_i v_i]$:

The prophet's expected payoff is

$$\begin{aligned}
\mathrm{REF} = \mathbf{E}[\max_i v_i] &= \hat{v} + \mathbf{E}[\max_i (v_i - \hat{v})] \\
&\leq \hat{v} + \mathbf{E}[\max_i (v_i - \hat{v})^+] \\
&\leq \hat{v} + \sum\nolimits_i \mathbf{E}[(v_i - \hat{v})^+] \,. \qquad (4.5)
\end{aligned}$$

The last inequality follows because $(v_i - \hat{v})^+$ is non-negative.

(ii) A lower bound on $\mathrm{APX} = \mathbf{E}[\text{prize of gambler with threshold } \hat{v}]$:

We will split the gambler's payoff into two parts, the contribution from the first $\hat{v}$ units of the prize and the contribution, when prize i is selected, from the remaining $v_i - \hat{v}$ units of the prize. The first part is $\mathrm{APX}_1 = (1 - \chi) \cdot \hat{v}$. To get a lower bound on the second part we consider only the contribution from the no-tie case. For any i, let $\mathcal{E}_i$ be the event that all other prizes j are below the threshold $\hat{v}$ (but v_i is unconstrained). The bound is:

$$\begin{aligned}
\mathrm{APX}_2 &\geq \sum\nolimits_i \mathbf{E}[(v_i - \hat{v})^+ \mid \mathcal{E}_i] \, \mathbf{Pr}[\mathcal{E}_i] \\
&\geq \chi \cdot \sum\nolimits_i \mathbf{E}[(v_i - \hat{v})^+] \,.
\end{aligned}$$

The second line follows because $\mathbf{Pr}[\mathcal{E}_i] = \prod_{j \neq i}(1-\hat{q}_j) \geq \prod_j(1-\hat{q}_j) = \chi$ and because the conditioned variable $(v_i - \hat{v})^+$ is independent from the conditioning event $\mathcal{E}_i$. Therefore, the gambler's payoff is at least:

$$\text{APX} \geq (1-\chi) \cdot \hat{v} + \chi \cdot \sum\nolimits_i \mathbf{E}[(v_i - \hat{v})^+]\,. \tag{4.6}$$

(iii) Plug in $\hat{v}$ with $\chi = 1/2$:

From the upper and lower bounds of equations (4.5) and (4.6), if there is a non-negative $\hat{v}$ such that $\chi = 1/2$ then, for this $\hat{v}$, $\text{APX} \geq \text{REF}/2$.

For discontinuous distributions, e.g., ones with point masses, χ as a function of $\hat{v}$, denoted $\chi(\hat{v})$, may be discontinuous. Therefore, there may be no $\hat{v}$ with $\chi(\hat{v}) = 1/2$. For distributions that have negative values in their supports the $\hat{v}$ with $\chi(\hat{v}) = 1/2$ may be negative. For these cases there is another method for finding a suitable threshold $\hat{v}$. Observe that the two common terms of equations (4.5) and (4.6), namely $\hat{v}$ and $\sum_i \mathbf{E}[(v_i - \hat{v})^+]$ are continuous functions of $\hat{v}$. The former is strictly increasing from $\hat{v} = 0$, the latter strictly decreases to zero; therefore they must cross at some non-negative $\hat{v}^\dagger$. For $\hat{v}^\dagger$ satisfying $\hat{v}^\dagger = \sum_i \mathbf{E}[(v_i - \hat{v}^\dagger)^+]$, regardless of the corresponding $\chi \in [0,1]$, the right-hand side of equation (4.5) is exactly twice that of equation (4.6). For this $\hat{v}^\dagger$ the two-approximation bound holds. $\square$

The prophet inequality is tight in the sense that a better approximation bound cannot generally by obtained by a uniform threshold strategy (Exercise 4.9).

As alluded to above, the invariance to the tie-breaking rule implies that the prophet inequality gives approximation bounds in scenarios similar to the gambler's game. In an oblivious posted pricing agents arrive in a worst-case order and the first agent who desires to buy the item at her offered price does so. We now use the prophet inequality to show that there is are *oblivious posted pricings* that guarantee half the optimal surplus and half the optimal auction revenue, respectively.

4.2.2 Oblivious Posted Pricing

Consider attempting to allocate a resource to maximize the social surplus. We know from Corollary 1.4 that the second-price auction obtains the optimal surplus of $\max_i v_i$. Suppose we wish to instead us a simpler posted pricing mechanism. A uniform posted price corresponds to a uniform threshold in value space. In worst case arrival order, the agent with the lowest value above the posted price is the one who buys. This corresponds to a game like the gambler's with tie-breaking by smallest value v_i. The invariance of the prophet inequality to the tie-breaking rule allows the conclusion that posting an uniform (a.k.a. anonymous) price gives a two-approximation to the optimal social surplus.

Proposition 4.9 *In single-item environments there is an anonymous pricing whose expected social surplus under any order of agent arrival is at least half of that of the optimal social surplus.*

Not consider the objective of revenue. The revenue-optimal single-item auction select the winner with the highest (positive) virtual value (for revenue). To draw a connection between the auction problem and the gambler's problem, we note that the gambler's problem in prize space is similar to the auctioneer's problem in virtual-value space (with virtual value functions given by the marginal revenue curves of the agents' distributions). The gambler aims to maximize expected prize while the auctioneer aims to maximize expected virtual value. A uniform threshold in the gambler's prize space corresponds to a *uniform virtual price* in virtual-value space. Note, however, in value space uniform virtual prices correspond to non-uniform (a.k.a., discriminatory) prices.

Definition 4.3 A *virtual price* $\hat{\phi}$ corresponds to *uniform virtual pricing* $\hat{\boldsymbol{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$ satisfying $\phi_i(\hat{v}_i) = \hat{\phi}$ for all i.

Now compare uniform virtual pricing to the gambler's threshold strategy in the stopping game. The difference is the tie-breaking rule. For uniform virtual pricing, we obtain the worst revenue when the agents arrive in order of increasing price (in value space). Thus, the uniform virtual pricing revenue implicitly breaks ties by smallest posted price $\hat{v}_i$. The gambler's threshold strategy breaks ties by the ordering assumption on the games (i.e., lexicographically by smallest i). Recall, though, that irrespective of the tie-breaking rule the bound of the prophet inequality holds.

Theorem 4.10 *In single-item environments there is a uniform virtual pricing (for virtual values equal to marginal revenues) whose expected revenue under any order of agent arrival is at least half of that of the optimal auction.*

Proof A uniform virtual price $\hat{\phi}$ corresponds to non-uniform prices (in value space) $\hat{\boldsymbol{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$. The outcome of such a posted pricing, for the worst-case arrival order of agents, is as follows. When there is only one agent i with value v_i that exceeds her offered price $\hat{v}_i$, the revenue is precisely $\hat{v}_i$. When there are multiple agents S whose values exceed their offered prices, the one with the lowest price arrives first and pays her offered price of $\min_{i \in S} \hat{v}_i$. In other words, with respect to the gambler's game, the tie-breaking rule is by smallest $\hat{v}_i$.

To derive a bound on the revenue of is uniform virtual pricing with the worst-case arrival order we will relate its revenue to its virtual surplus. For the aforementioned outcome of a uniform virtual pricing (with virtual values as the marginal revenue) satisfies the conditions of Theorem 3.18. In particular, the induced allocation rule for each agent is constant wherever the marginal revenue is constant. Therefore, the expected revenue of a uniform virtual pricing is equal to its expected virtual surplus.

By the prophet inequality (Theorem 4.8), there is a uniform virtual price that obtains a virtual surplus of at least half the maximum virtual value (i.e., the optimal virtual surplus for single-item environments). Thus, the revenue of the corresponding price posting is at least half the optimal revenue. □

In Chapter 1 we saw that that an anonymous posted pricing can be a $e/e-1 \approx 1.58$ approximation to the optimal mechanism for social surplus for i.i.d. distributions (Theorem 1.5). This approximation factor also holds for revenue and i.i.d., regular distributions. In the next section we will give a more general result that shows that if the mechanism is allowed to order the agents (i.e., in the best-case order instead of the worst-case order as above) then this better $e/e-1$ bound can be had even for asymmetric distributions. In this context of best-case versus worst-case order, the i.i.d. special case is precisely the one where symmetry renders the ordering of agents irrelevant.

4.3 Sequential Posted Pricings and Correlation Gap

In this section we consider sequential posted pricings, i.e., where the mechanism posts prices to the agents in an order that it specifies. See Section 4.2 for additional motivation for posted pricings.

One of the main challenges in designing and analyzing simple approximation mechanisms is that the optimal mechanism is complex and, therefore, difficult to analyze. For single-item auctions, this complexity arises from virtual values which come from arbitrary monotone functions. The main approach for confronting this complexity is to derive a simple upper bound on the optimal auction and then exploit the structure suggested by this bound to construct an simple approximation mechanism.

4.3.1 The Ex Ante Relaxation

One method for obtaining a simple upper bound for an optimization problem is to relax some of the constraints in the problem. For example, ex post feasibility for a single-item auction requires that, in the outcome selected by the auction, at most a single agent is served. In other words, the feasibility constraint binds ex post. For Bayesian mechanism design problems, we can relax the feasibility constraint to bind ex ante. The corresponding ex ante constraint for a single-item environment is that the expected (over randomization in the mechanism and the agent types) number of agents served is at most one.

Definition 4.4 The *ex ante relaxation* of mechanism design problem is the optimization problem with the ex post feasibility constraint replaced with a constraint that holds in expectation over randomization of the mechanism and the agents' types. The solution to the ex ante relaxation is the *optimal ex ante mechanism.*

Proposition 4.11 *The optimal ex ante mechanism's performance upper bounds the optimal (ex post) mechanism's performance.*

To see what the optimal ex ante mechanism is, consider any mechanism and denote by $\hat{\boldsymbol{q}} = (\hat{q}_1, \ldots, \hat{q}_n)$ the ex ante probabilities that each of the agents is served by this mechanism. By linearity of expectation the expected number of agents served is $\sum_i \hat{q}_i$. For a single-item environment the ex ante feasibility constraint then requires that $\sum_i \hat{q}_i \leq 1$. Notice that as far as the ex ante constraint is concerned, the agents only impose externalities on each other via their ex ante allocation probability. If we fix attention to mechanisms for which agent i is allocated with ex ante probability $\hat{q}_i$ then the remaining allocation probability for the other agents is fixed to at most $1 - \hat{q}_i$. Any method of serving agent i with probability $\hat{q}_i$ can be combined with any other method for serving an expected $1 - \hat{q}_i$ number of the remaining agents. Thus, the relaxed optimization problem with an ex ante feasibility constraint decomposes across the agents.

Considering an agent i, one way to serve the agent with ex ante probability $\hat{q}_i$ is to use the ex ante optimal lottery pricing (Definition 3.12).

The expected payment of the agent is given by her revenue curve as $R_i(\hat{q}_i)$. Thus, for ex ante allocation probabilities $\hat{\boldsymbol{q}}$ the optimal revenue is $\sum_i R_i(\hat{q}_i)$. Recall that for regular distributions, this optimal pricing is simply to post the price $V_i(\hat{q}_i)$ which has probability $\hat{q}_i$ of being accepted by the agent. Therefore, for regular distributions the optimal ex ante mechanism is a posted pricing.

The optimal ex ante mechanism design problem is identical to the classical microeconomic problem of optimizing the amount of a unit supply of a good (e.g., grain) to fractionally allocate across each of several markets. Each market i has a concave revenue curve as a function of the faction of the supply allocated to it. Both of these optimization problem are given by the following convex program:

$$\begin{aligned} \max_{\hat{\boldsymbol{q}}} & \sum\nolimits_i R(\hat{q}_i) \\ \text{s.t.} & \sum\nolimits_i \hat{q}_i \leq 1. \end{aligned} \tag{4.7}$$

As described previously, the marginal revenue interpretation provides a simple method for solving this program. The optimal solution equates marginal revenues, i.e., $R_i'(\hat{q}_i) = R_j'(\hat{q}_j)$ for i and j with $\hat{q}_i$ and $\hat{q}_j$ strictly larger than zero. We conclude with the following proposition.

Proposition 4.12 *The optimal ex ante mechanism is a uniform virtual pricing (with virtual values defined as marginal revenues).*

Because, at least for regular distributions, the optimal ex ante mechanism is a price posting, it provides a convenient upper bound for determining the extent to which price posting (with the ex post constraint) approximates the optimal (ex post) auction. In particular, if we post the exact same prices then the difference between the ex ante and ex post posted pricing is in how violations of the ex post feasibility constraint are resolved. In the former, violations are ignored, in the latter they must be addressed. In the terminology of the previous section, we must address how ties, i.e., multiple agents desiring to buy at their respective prices, are to be resolved to respect the ex post feasibility constraint. Unlike the previous section where the oblivious ordering assumption required breaking ties in worst-case order, in this section we break ties in the mechanisms favor.

Consider the special-case where the distribution is regular and that the optimal ex ante revenue of $R_i(\hat{q}_i) = \hat{q}_i\hat{v}_i$ from agent i is obtained by posting price $\hat{v}_i = V_i(\hat{q}_i)$. The best order to break ties is in favor of higher prices, i.e., by larger $\hat{v}_i$. For general (possibly irregular distributions) this corresponds to ordering the agents by $R_i(\hat{q}_i)/\hat{q}_i$, i.e., the agent's bang-per-buck. The goal of this section is to prove an approximation bound on this sequential price posting.

4.3.2 The Correlation Gap

The sequential posted pricing theorem we present is an application of a *correlation gap* theorem from stochastic optimization. Consider a non-negative real-valued set function g over subsets S of an n element ground set $N = \{1, \ldots, n\}$ and a distribution over subsets given by $\mathcal{D}$. Let $\hat{q}_i$ be the ex ante[1] probability that element i is in the random set $S \sim \mathcal{D}$ and let $\mathcal{D}^I$ be the distribution over subsets induced by independently adding each element i to the set with probability equal to its ex ante probability $\hat{q}_i$. The *correlation gap* is then the ratio of the expected value of the set function for the (correlated) distribution $\mathcal{D}$, i.e., $\mathbf{E}_{S\sim\mathcal{D}}[g(S)]$, to the expected value of the set function for the independent distribution $\mathcal{D}^I$, i.e., $\mathbf{E}_{S\sim\mathcal{D}^I}[g(S)]$ A typical analysis of correlation gap will consider specific families of set functions g in worst case over distributions $\mathcal{D}$.

We show below that for any values $\hat{\boldsymbol{v}}$ the *maximum-weight-element* set function $g^{\mathrm{MWE}}(S) = \max_{i\in S} \hat{v}_i$ has a correlation gap of ${}^{e}/_{e-1}$.

Lemma 4.13 *The correlation gap for any maximum-weight-element set function and any distribution over sets is* ${}^{e}/_{e-1}$.

Proof This proof proceeds in three steps. First, we argue that it is without loss to consider distributions $\mathcal{D}$ over singleton sets. Second, we argue that it is without loss to consider set functions where the weights are uniform, i.e., the one-or-more set function. Third, we show that for distributions over singleton sets, the one-or-more set function has a correlation gap of ${}^{e}/_{e-1}$.

(i) We have a set function $g^{\mathrm{MWE}}(S) = \max_{i\in S} \hat{v}_i$. Add a dummy element 0 with weight $\hat{v}_0 = 0$; if $S = \emptyset$ then changing it to $\{0\}$ affects neither the correlated value nor the independent value. Moreover, the correlated value $\mathbf{E}_{S\sim\mathcal{D}}\left[g^{\mathrm{MWE}}(S)\right]$ is unaffected by changing the set to only ever include its highest weight element. This change to the distribution only (weakly) decreases the ex ante probabilities $\hat{\boldsymbol{q}} = (\hat{q}_1, \ldots, \hat{q}_n)$

[1] In probability theory, this probability is also known as the marginal probability of $i \in S$; however to avoid confusion with usage of the term "marginal" in economics, we will refer to it via its economic interpretation as an ex ante probability as if S was the feasible set output by a mechanism.

and the independent value $\mathbf{E}_{S\sim\mathcal{D}^I}\left[g^{\mathrm{MWE}}(S)\right]$ is monotone increasing in the ex ante probabilities. Therefore, this transformation only makes the correlation gap larger. We conclude that it is sufficient to bound the correlation gap for distributions $\mathcal{D}$ over singleton sets for which the ex ante probabilities sum to one, i.e., $\sum_i \hat{q}_i = 1$.

(ii) With set distribution $\mathcal{D}$ over singletons and a maximum-weight-element set function $g^{\mathrm{MWE}}(S) = \max_{i\in S} \hat{v}_i$, the correlated value simplifies to $\mathbf{E}_{S\sim\mathcal{D}}\left[g^{\mathrm{MWE}}(S)\right] = \sum_i \hat{q}_i\hat{v}_i$. Scaling the weights $\hat{\boldsymbol{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$ by the same factor has no effect on the correlation gap; therefore, it is without loss to normalize so that the correlated value is $\sum_i \hat{q}_i\hat{v}_i = 1$. We now argue that among all such normalized weights $\hat{\boldsymbol{v}}$, the ones that give the largest correlation gap are the uniform weights $\hat{v}_i = 1$ for all i. This special case of the maximum-weight-element set function is the one-or-more set function, $g^{\mathrm{OOM}}(S) = 1$ if $|S| \geq 1$ and otherwise $g^{\mathrm{OOM}}(S) = 0$.

Sort the elements by $\hat{v}_i$ and let $c_i = \prod_{j<i}(1-\hat{q}_j)$ denote the probability that no element with higher weight than i is in S and, therefore, i's contribution to the independent value is $c_i\hat{q}_i\hat{v}_i$. Let $\delta_i = \hat{q}_i \cdot (\hat{v}_i - 1)$ be the additional contribution in excess of one to the correlated value of i with value $\hat{v}_i$. Importantly, by our normalization assumption that $\sum_i \hat{q}_i\hat{v}_i = 1$, the sum of these excess contributions is zero, i.e., $\sum_i \delta_i = 0$. The expected independent value for the maximum-weight-element set function is

$$\sum\nolimits_i c_i\hat{q}_i\hat{v}_i = \sum\nolimits_i c_i \cdot (\hat{q}_i + \delta_i) \geq \sum\nolimits_i c_i\hat{q}_i. \tag{4.8}$$

where the inequality follows from monotonicity of c_i and the fact that $\sum_i \delta_i = 0$. The right-hand side of (4.8) is the expected independent value of the one-or-more set function. The correlated value is one for both (normalized) general weights and uniform weights, so uniform weights give no lower correlation gap.

(iii) The correlation gap of the one-or-more set function g^{OOM} on any distribution $\mathcal{D}$ over singletons can be bounded as follows. First, the expected correlated value is one. Second, the expected independent value is, for $S \sim \mathcal{D}^I$,

$$\begin{aligned}\mathbf{E}\left[g^{\mathrm{OOM}}(S)\right] &= \mathbf{Pr}[|S| \geq 1] = 1 - \mathbf{Pr}[|S| = 0] = 1 - \prod\nolimits_i (1 - \hat{q}_i)\\ &\geq 1 - (1 - {}^1\!/\!_n)^n \geq 1 - {}^1\!/\!_e,\end{aligned}$$

where the first inequality follows because $\sum_i \hat{q}_i = 1$ and because the product of a set of positive numbers with a fixed sum is maximized when the numbers are equal. The last inequality follows as $(1 - 1/n)^n$ is monotonically increasing in n and it is $1/e$ in the limit as n goes to infinity.[2] □

4.3.3 Sequential Posted Pricings

The correlation gap is central to the theory of approximation for sequential posted pricings. Contrast the revenue of the optimal ex ante mechanism (a price posting) with the revenue from sequentially posting the same prices. The optimal ex ante mechanism has total ex ante service probability $\sum_i \hat{q}_i \leq 1$ (by definition). If we could coordinate the randomization (by adding correlation to the randomization of agents' types and the mechanism) then we could obtain this optimal revenue and satisfy ex post feasibility. In a sequential posted pricing, of course, no such coordination is permitted. Instead, ex post feasibility is satisfied by serving the agent that arrives first in the specified sequence.

Given any $\hat{\boldsymbol{q}}$ with $\sum_i \hat{q}_i \leq 1$, consider the correlated distribution $\mathcal{D}$ that selects the singleton set $\{i\}$ with probability $\hat{q}_i$ and the empty set $\emptyset$ with probability $1 - \sum_i \hat{q}_i$. The induced ex ante probabilities of this correlated distribution are exactly $\hat{q}_i$ for each agent i. Assume for now that the distribution is regular and that the revenue of $R_i(\hat{q}_i) = \hat{q}_i \hat{v}_i$ is obtained by posting price $\hat{v}_i = V_i(\hat{q}_i)$. For the maximum-weight-element set function, i.e., $g^{\mathrm{MWE}}(S) = \max_{i \in S} \hat{v}_i$. For $S \sim \mathcal{D}$ the expected value of this set function is precisely the optimal ex ante revenue $\sum_i \hat{v}_i \hat{q}_i$.

On the other hand, consider sequentially posting prices $\hat{\boldsymbol{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$ to agents ordered by largest $\hat{v}_i$. Let S denote the set of agents whose values are at least their prices, i.e., $S = \{i : v_i \geq \hat{v}_i\}$. Each agent i is in S independently with probability $\hat{q}_i$. Importantly, S may have cardinality larger than one, but when it does, the ordering of agents by price implies that the agent $i \in S$ with the highest price wins. The revenue of the sequential posted pricing is given by the expected value of the maximum-weight-element set function $g^{\mathrm{MWE}}(S)$ on $S \sim \mathcal{D}^I$.

For regular distributions, the translation from the solution to the optimal ex ante mechanism which is given by $\hat{\boldsymbol{q}}$ to a sequential pricing is direct. As described above, the prices $\hat{v}_i = V_i(\hat{q}_i)$ are posted to agents in decreasing order of $\hat{v}_i$. For irregular distributions the $\hat{q}_i$ optimal lottery

[2] The last part of this analysis is identical to the proof of Theorem 1.5. Again, $(1 - 1/n)^n \leq 1/e$ is a standard observation that can be had by taking the natural logarithm and then applying L'Hopital's rule for evaluating the limit.

for agent i is not necessary a posted pricing. It may be, via Theorem 3.28, a lottery over two prices. These lottery pricings arise when $\hat{q}_i$ is in an interval where the revenue curve has been ironed and is therefore locally linear. The marginal revenue (i.e., virtual value) is constant on this interval. If we break ties in the optimization of program (4.7) lexicographically, then for the optimal ex ante probabilities $\hat{\boldsymbol{q}}$ at most one is contained strictly within an ironed interval. Recall that the marginal revenues of any agents who have non-zero ex ante allocation probability are equal. At this marginal revenue, the lexicographical tie breaking rule requires that we increase the allocation probability to the early agents before later agents. We stop when we run out of ex ante allocation probability and at this stopping point the ex ante allocation probabilities can be within at most one agents ironed interval.

By the above discussion, the suggested sequential pricing potentially has one agent receiving a lottery over two prices. The expected revenue of this pricing satisfies the approximation bound guaranteed by the correlation gap theorem. Of course, it cannot be the case that both the pricings in the support of the randomized pricing have revenue below the expected revenue of the lottery pricing. Therefore, the pricing with the higher revenue gives the desired approximation. Notice that the lexicographical ordering and derandomization steps may result in prices (in value space) that are discriminatory even in the case that the environment is symmetric (i.e., for i.i.d. distributions).

Theorem 4.14 *For any single-item environment, there is sequential posted pricing (ordered by price) with uniform virtual prices that obtains a revenue that is an* $e/e-1 \approx 1.58$ *approximation to the optimal auction revenue (and the optimal ex ante mechanism revenue).*

Proof By Proposition 4.11 the optimal ex ante revenue upper bounds the optimal auction revenue. The upper bound on the approximation ratio then follows directly from the correspondence between the revenues of the optimal ex ante mechanism and the sequential posted pricing revenue and the correlated and independent values for the maximum weight element set system (Lemma 4.13). The prices correspond to a uniform virtual pricing by the characterization of the optimal ex ante mechanism (Proposition 4.12). $\square$

The construction and analysis of Theorem 4.14 can similarly be applied to the objective of social surplus (see Exercise 4.10) to obtain an

$e/e-1$ by a sequential posted pricing that generalizes Theorem 1.5 to non-identical distributions.

4.4 Anonymous Reserves and Pricings

Thus far we have shown that simple posted pricings and reserve-price-based auctions approximate the optimal auction. Unfortunately, these prices are generally discriminatory and, thus, may be impractical for many scenarios, especially ones where agents could reasonably expect some degree of fairness of the auction protocol. We therefore consider the extent to which an *anonymous posted price* or an auction with an *anonymous reserve price*, i.e., the same for each agent, can approximate the revenue of the optimal, perhaps discriminatory, auction.

For instance, in the eBay auction the buyers are not identical. Some buyers have higher *ratings* and these ratings are public knowledge. The value distributions for agents with different ratings may generally be distinct and, therefore, the eBay auction may be suboptimal. Surely though, if the eBay auction was very far from optimal, eBay would have switched to a better auction. The theorem below gives some justification for eBay sticking with the second-price auction with anonymous reserve.

Our approach to approximation for (first- or second-price) auctions with anonymous reserve will be to show that anonymous price posting gives a good approximation and then to argue via the following proposition, that the auction revenue pointwise dominates the pricing revenue. While there is not a succinct close-form expression for the best anonymous reserve price for the second-price auction; the best anonymous posted price is precisely the monopoly price for the distribution of the maximum value. Notice that with distribution functions $F_1, \ldots, F_n$, the distribution of the maximum value has distribution function $F_{\max}(z) = \prod_i F_i(z)$. From this formula, the monopoly price can be directly calculated.

Proposition 4.15 *In any single-item environment, the revenues from the first- and second-price auctions with an anonymous reserve price is at least the revenue from the anonymous posted pricing with the same price.*

Proof Recall that a posted pricing of $\hat{v}$ obtains revenue $\hat{v}$ if and only if there is an agent with value at least $\hat{v}$. For the auction, the utility an agent receives for bidding strictly below $\hat{v}$ is zero, while individual

	regular auction	regular pricing	irregular
identical	1	$\approx {}^{e}/_{e-1}$	2
non-identical	$[2, 4]$	$[2, 4]$	n

Figure 4.3 Approximation bounds are given for the second-price auction with anonymous reserve and for anonymous posted pricing. If a number is given, then the bound is tight in worst case, if a range is given then the bound is not known to be tight. For irregular distributions, the auction and pricing bounds are the same. For i.i.d. regular distributions, the approximation ratio of anonymous pricing is upper bounded by ${}^{e}/_{e-1}$ for all n; for small n the bound can be improved, e.g., for $n = 1$ pricing is optimal, for $n = 2$ it is a ${}^{4}/_{3}$ approximation. A nearly matching lower bound is the subject of Exercise 4.12.

rationality implies that an agent with value $v \geq \hat{v}$ will have a non-negative utility from bidding on $[\hat{v}, v]$. Thus, the auction sells at a price of at least $\hat{v}$ if and only if there is an agent with value at least $\hat{v}$. □

4.4.1 Identical Distributions

We start with results for anonymous posted pricing and identical distributions; these bounds are summarized by the first row of Figure 4.3. For i.i.d. regular distributions the second-price auction with an anonymous reserve is optimal (Corollary 3.12). For anonymous posted pricing, Theorem 4.14 implies a ${}^{e}/_{e-1} \approx 1.58$ approximation for regular distributions and Theorem 4.10 implies a two approximation for irregular distributions. Notice that while Theorem 4.14 holds for irregular distributions, for identical irregular distributions the prices for which the result holds may not be anonymous (due to the derandomization step).

Corollary 4.16 *For i.i.d. regular single-item environments, anonymous posted pricing is an ${}^{e}/_{e-1}$ approximation to the optimal auction; this bound is nearly tight.*

Proof For i.i.d. distributions, the optimization problem of program (4.7) is symmetric and convex and, therefore, always admits a symmetric optimal solution. For regular distributions, this symmetric optimal solution corresponds to an anonymous posted pricing. Theorem 4.14 shows that this anonymous posted pricing is a ${}^{e}/_{e-1}$ approximation. For tightness, see Exercise 4.12. □

Corollary 4.17 *For i.i.d. (irregular) single-item environments, both anonymous posted pricing and the second-price auction with anonymous* *reserve are two approximations to the optimal auction revenue; these bounds are tight.*

Proof For any (possibly irregular) distribution, Theorem 4.10 shows that posting a uniform virtual price gives a two approximation to the revenue of the optimal auction. For i.i.d. distributions where the virtual value functions are identical, uniform virtual prices are anonymous. The price-posting result follows. By Proposition 4.15, using this anonymous price as a reserve price in the second-price auction only improves the revenue.

To see that this bound of two is tight, we give an i.i.d. irregular distribution for which the approximation ratio of anonymous reserve pricing for n agents is $2 - 1/n$. Consider the discrete distribution and $h \gg n$ where

$$v = \begin{cases} h \text{ (high valued)} & \text{w.p. } 1/h, \text{ and} \\ n \text{ (low valued)} & \text{otherwise.} \end{cases}$$

We then analyze the optimal auction revenue, REF, and the second-price auction with any reserve, APX, for n agents and in the limit as h goes to ∞. We show that REF $= 2n - 1$ and APX $= n$; the result follows. For any given value of h, the probability that there are k high-valued agents and $n - k$ low valued agents is the same as in the proof of Proposition 4.6; the analysis below makes use of equations (4.3) and (4.4) from its proof.

We start by analyzing REF. The virtual values are $\phi(h) = h$ and, as h goes to ∞, $\phi(n) = n - 1$. The optimal auction has virtual surplus $n - 1$ if there are no high-valued agents and virtual surplus h if there is one or more high-valued agents. The former case happens with probability that goes to one and so the expected virtual surplus is $n-1$; and in the limit, h times the probability of the latter case goes to n. Thus, REF $= 2n-1$.

We now analyze APX. We show that both a reserve of n and a reserve of h give the same revenue of n in the limit. For the first case: a reserve of n is never binding. The second-price auction has revenue h if there are two or more high-valued agents and a revenue of n if there are one or fewer. In the limit (as h goes to infinity) the contribution to the expected revenue of the first term is zero and that of the second term is n. For the second case: a reserve of h gives revenue of h when there is one or more high-valued agent, and otherwise zero. As above, the product of h and this probability is n in the limit. Thus, APX $= n$. □

4.4.2 Non-identical Distributions

We now turn to asymmetric distributions. For asymmetric distributions, the challenge with anonymous pricing comes from the asymmetry in the environment. For non-identical regular distributions, an anonymous posted pricing gives a constant approximation (implying the same for anonymous reserve pricing). For non-identical irregular distributions, anonymous posted and reserve pricing are n approximations. We begin with lower and upper bounds for regular distributions.

Lemma 4.18 *Anonymous reserve or posted pricing is at best a two approximation to the optimal revenue.*

Proof This lower bound is exhibited by an $n = 2$ agent example where agent 1's value is a point-mass at one and agent 2's value is drawn from the equal revenue distribution (Definition 4.2) on $[1, \infty)$, i.e., $F_2(z) = 1 - {}^1\!/z$. Recall that, for the equal revenue distribution, posting any price $\hat{v} \geq 1$ gives an expected revenue of one. For this asymmetric setting the revenue of the second-price auction with any anonymous reserve is exactly one. On the other hand, an auction could first offer the item to agent 2 at a very high price (for expected revenue of one), and if (with very high probability) agent 2 declines, then it could offer the item to agent 1 at a price of one. The expected revenue of this mechanism in the limit is two. □

Theorem 4.19 *For single-item environments and agents with values drawn independently from regular distributions, anonymous reserve and posted pricings give a four approximation to the revenue of the optimal auction. One such anonymous price is the monopoly price for the distribution of the maximum value.*

Proof This proof combines elements from the proof of the prophet inequality (Section 4.2.1, page 111) theorem with the upper bound on the optimal auction given by the ex ante relaxation (Section 4.3.1, page 117). Let REF $= \sum_i \hat{v}_i \hat{q}_i$ denote the optimal ex ante mechanism which posts prices $\hat{v}_i = V_i(\hat{q}_i)$ and, with out loss of generality, satisfies $\sum_i \hat{q}_i = 1$. Let APX denote the revenue from posting an anonymous price $\hat{v}$. A key part of the proof is to use regularity (i.e., convexity of the price-posting revenue curve) to derive a lower bound on the probability that an agent i with $\hat{v}_i$ (from the optimal ex ante mechanism, above) has value at least the anonymous price $\hat{v}$. The full proof is left to Exercise 4.13. □

We now give a tight inapproximation bound for anonymous reserves and pricings with irregular distributions. Recall the proof of Proposition 4.7 which implies that, for (non-identical) irregular distributions, posting an anonymous price that corresponds to the monopoly reserve price of the agent with the highest monopoly revenue gives an n approximation to the optimal auction. This is, in fact, the best bound guaranteed by the second-price auction with an anonymous reserve or an anonymous posted pricing.

Theorem 4.20 *For (non-identical, irregular) n-agent single-item environments the second-price auction with anonymous reserve and anonymous posted pricing are n approximations to the optimal auction revenue; these bounds are tight.*

Proof The upper bound can be seen by adapting the proof of Proposition 4.7 as per the above discussion. The lower bound can be seen by analyzing the optimal revenue and the revenue of the second-price auction with any anonymous reserve on the following discrete distribution in the limit as parameter h approaches infinity. Agent i's value is drawn as:

$$v_i = \begin{cases} h^i & \text{w.p. } h^{-i}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The details of this analysis are left to Exercise 4.15. □

4.5 Multi-unit Environments

The simplest environment we could consider generalizing approximation results to are multi-unit environments. In a multi-unit environment, there are multiple units of a single item for sale and each agent desires a single unit. Denote by k the number of units. For k-unit environments the surplus maximization mechanism is simply the $(k + 1)$st-price auction where the k agents with the highest bids win and are required to pay the $(k+1)$st bid. Except for the anonymous reserve pricing result for non-identical regular distributions, all of the single-item results extend to multi-unit environments.

Consider extending the results for monopoly reserve pricing to multi-unit environments. For regular (non-identical) k-unit environments, the $(k + 1)$st-price auction with monopoly reserves continues to be a two approximation to the revenue optimal auction. We defer the statement and proof this result to Section 4.6 where it is a special case of Theorem 4.28. For irregular distributions the tight approximation bound for single-unit environments of Proposition 4.6 and Proposition 4.7 generalize to k-unit environments where the approximation ratio of monopoly reserve pricing is ${}^{n}/_{k}$ (see Exercise 4.16).

It is possible to generalize and improve the prophet inequality to show that a gambler who is able to select k prizes can, with a uniform threshold, obtain a $(1 + \sqrt{{}^{8}/_{k} \ln k})$ approximation to the prophet (i.e., the expected maximum value of k prizes) for sufficiently large k. From this generalized prophet inequality, the same bound holds for oblivious posted pricing.

Proposition 4.21 *For k-unit environments with sufficiently large k, there is an oblivious posted pricing that is a $(1+\sqrt{{}^{8}/_{k} \ln k})$ approximation to the optimal auction.*

Sequential posted pricing bounds generalize to multi-unit environments and the bound obtained improves with k and asymptotically approach one, i.e., optimal. The proof of this generalization follows from considering the correlation gap of the k-maximum-weight-elements set function, reducing its correlation gap to that of the k-capped-cardinality set function $g(S) = \min(k, |S|)$ (the one-or-more set function is the 1-capped-cardinality), and showing that this set function's correlation gap in the limit as n approaches infinity is $(1 - ({}^{k}/_{e})^{k} \cdot {}^{1}/_{k!})^{-1}$ which, by Stirling's approximation[3] is $(1 - {}^{1}/_{\sqrt{2\pi k}})^{-1}$ (see Exercise 4.17).

Proposition 4.22 *For k-unit environments, there is a sequential posted pricing that is a $(1 - {}^{1}/_{\sqrt{2\pi k}})^{-1}$ approximation to the optimal auction.*

An anonymous reserve price continues to be revenue optimal for i.i.d. regular multi-unit environments. For i.i.d. regular multi-unit environments the correlation-gap-based sequential posted pricing result (Proposition 4.22, above) implies the same bound is attained by an anonymous pricing because for i.i.d. regular distributions, a uniform virtual pricing is an anonymous pricing (in value space). For i.i.d. irregular multi-unit environments the prophet-inequality-based oblivious posted pricing result (Proposition 4.21, above) implies the same bound by an anonymous pricing (and consequently for the $(k+1)$st price auction with an anony-

[3] Stirling's approximation is $k! = ({}^{k}/_{e})^{k}\sqrt{2\pi k}$. This approximation is obtained by approximating the natural logarithm as $\ln(k!) = \ln(1) + \ldots + \ln(k)$ by an integral instead of a sum.

mous reserve), because for i.i.d. distributions the uniform virtual pricing identified corresponds to an anonymous pricing (in value space).

The one result that does not generalize from single-item environments to multi-unit environments is the anonymous posted and reserve pricing for non-identical distributions. In fact, this lower bound holds more generally for any set system where where it is possible to serve k agents (see Lemma 4.23, below). For irregular, non-identical distributions the n-approximation bound of Theorem 4.20 for single-item environments generalizes and is tight.

Lemma 4.23 *For any (non-identical) regular environment where it is feasible to simultaneously serve k agents, anonymous pricing and anonymous reserve pricing are at best an $\mathcal{H}_k \approx \ln k$ approximation to the optimal mechanism revenue, where $\mathcal{H}_k$ is the kth* harmonic number $\mathcal{H}_k = \sum_{i=1}^{k} {}^1\!/i$.

Proof Fix a set of k agents that are feasible to simultaneously serve and reindex them without loss of generality to be $\{1, \ldots, k\}$. The value distribution that gives this bound is the one where F_i is a pointmass at ${}^1\!/i$ for agents $i \in \{1, \ldots, k\}$ and a pointmass at zero for agents $i > k$. For such a distribution, competition does not increase the price above the reserve, therefore anonymous reserve pricing is identical to anonymous posted pricing. For any $i \in \{1, \ldots, k\}$, anonymous pricing of ${}^1\!/i$ to all agents obtains revenue $i \cdot {}^1\!/i = 1$ as there are i agents with values that exceed ${}^1\!/i$. On the other hand, the optimal auction posts a discriminatory price to the top k agents of ${}^1\!/i$ for agent i; its revenue is the kth harmonic number $\sum_{i=1}^{k} {}^1\!/i = \mathcal{H}_k$. The kth harmonic number can be approximated by the integral $\int_1^k {}^1\!/i \,\mathrm{d}i$ and satisfies $\ln k - 1 \leq \mathcal{H}_k \leq \ln k$. ◻

To summarize the generalization of the single-item results to multi-unit environments: all approximation and inapproximation results generalize (and some improve) except for the anonymous pricing result for non-identical, regular distributions.

4.6 Ordinal Environments and Matroids

In Chapter 3 we saw that the second-price auction with the monopoly reserve was optimal for i.i.d. regular single-item environments. In the first section of this chapter we showed that the second-price auction with monopoly reserves is a two approximation for (non-identical) regular single-item environments. We now investigate to what extent the constraint on the environment to single-item feasibility can be relaxed while still preserving these approximation results. In this section we give equivalent algorithmic and combinatorial answers to this question. The algorithmic answer is "when the greedy-by-value algorithm works;" the combinatorial answer is "when the set system satisfies a augmentation property (i.e., matroids)."

Definition 4.5 The *greedy-by-value algorithm* is

(i) Sort the agents in decreasing order of value (and discard all agents with negative value).
(ii) $\boldsymbol{x} \leftarrow \mathbf{0}$ (the null assignment).
(iii) For each agent i (in sorted order),
  if $(1, \boldsymbol{x}_{-i})$ is feasible, $x_i \leftarrow 1$.
  (I.e., serve i if i can be served alongside previously served agents.)
(iv) Output $\boldsymbol{x}$.

Notice that the greedy-by-value algorithm is optimal for single-item environments. To optimize surplus in a single-item environment we wish to serve the agent with the highest value (when it is non-negative, and none otherwise). The greedy-by-value algorithm does just that. Notice also that the optimality of the greedy-by-value algorithm for all profiles of values implies that, for the purpose of selecting the optimal outcome, the relative magnitudes of the agents' values do not matter, only the order of the of the values (and zero) matters.

Definition 4.6 An environment is *ordinal* if for all valuation profiles, the greedy-by-value algorithm optimizes social surplus.

Recall the argument for i.i.d. regular single-item environments that showed that the optimal auction is the second-price auction with the monopoly reserve price (Corollary 3.12). An agent, Alice, had to satisfy two properties to win. She must have the highest virtual value and her virtual value must be non-negative. Having a non-negative virtual value is equivalent having a value of at least the monopoly price. Having the highest virtual value, by regularity and symmetry, is equivalent to having the highest value. Thus, Alice wins when she has the highest value and is at least the monopoly price. This auction is precisely the second-price auction with the monopoly reserve price. For general environments, the non-negativity of virtual value again suggests any agents who do not have values at least the monopoly reserve price should be rejected. For an ordinal environment with values drawn i.i.d. from a regular distribution, maximization of virtual surplus for the remaining agents gives the same outcome as maximizing the surplus of the remaining agents as symmetry and strictly increasing virtual value functions imply that the relative order values is identical to that of virtual values. We conclude with the following proposition.

Proposition 4.24 *For i.i.d. regular ordinal environments, surplus maximization with the monopoly reserve price optimizes expected revenue.*

We will see in the remainder of this section that ordinality is a sufficient condition on the feasibility constraint of the environment to permit the extension of several of the single-item results from the preceding sections. In particular, for regular (non-identical) distributions, surplus maximization with (discriminatory) monopoly reserves continues to be a two approximation. For general distributions a sequential posted pricing continues to be an $e/e-1$ approximation. Neither anonymous posted prices or reserve prices generalize (as they do not generalize even for the special case of multi-unit environments, see Section 4.5).

Definition 4.7 The surplus maximization mechanism with reserves $\hat{\boldsymbol{v}}$ is:

(i) filter out agents who do not meet their reserve price, $\boldsymbol{v}^{\dagger} \leftarrow \{\text{agents with } v_i \geq \hat{v}_i\}$
(ii) simulate the surplus maximization mechanism on the remaining agents, and

$$(\boldsymbol{x}, \boldsymbol{p}^{\dagger}) \leftarrow \mathrm{SM}(\boldsymbol{v}^{\dagger})$$

(iii) set prices $\boldsymbol{p}$ from critical values as:

$$p_i \leftarrow \begin{cases} \max(\hat{v}_i, p_i^{\dagger}) & \text{if } x_i = 1, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

where SM is the surplus maximization mechanism with no reserves.

4.6.1 Matroid Set Systems

As ordinal environments enable good approximation mechanisms, it is important to be able to understand and identify environments that are ordinal. For general feasibility environments (Definition 3.1) subsets of agents that can be simultaneously served are given by a set system.

We will see shortly, that set systems that correspond to ordinal environments, i.e., where the greedy-by-value algorithm optimizes social surplus, are matroid set systems. Checking ordinality of the environment then is equivalent to checking whether the matroid conditions hold.

Definition 4.8 A set system is $(N, \mathcal{I})$ where N is the *ground set* of elements and $\mathcal{I}$ is a set of feasible subsets of N.[4] A set system is a *matroid* if it satisfies:

- *downward closure*: subsets of feasible sets are feasible.
- *augmentation*: given two feasible sets, there is always an element from the larger whose union with the smaller is feasible.

$$\forall I, J \in \mathcal{I},\ |J| < |I| \Rightarrow \exists i \in I \setminus J,\ \{i\} \cup J \in \mathcal{I}.$$

The augmentation property trivially implies that all maximal feasible sets of a matroid have the same cardinality. These maximal feasible sets are referred to as *bases* of the matroid; the cardinality of the bases is the *rank* of the matroid. To get some more intuition for the role of the augmentation property, the following lemma shows that if the set system is not a matroid then the greedy-by-value algorithm is not always optimal.

Lemma 4.25 *The greedy-by-value algorithm selects the feasible set with largest surplus for all valuation profiles only if feasible sets are a matroid.*

Proof The lemma follows from showing for any non-matroid set system that there is a valuation profile $\boldsymbol{v}$ that gives a counterexample. First, we show that downward closure is necessary and then, for downward-closed set systems, that the augmentation property is necessary.

If the set system is not downward closed there are subsets $J \subset I$ with $I \in \mathcal{I}$ and $J \notin \mathcal{I}$. Consider the valuation profile $\boldsymbol{v}$ with

$$v_i = \begin{cases} 2 & \text{if } i \in J, \\ 1 & \text{if } i \in I \setminus J, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The optimal outcome is to select set I which is feasible and contains all the elements with positive value. The greedy-by-value algorithm will

[4] For matroid set systems the feasible sets are often referred to as *independent sets*. To avoid confusion with independent distributions and to promote the connection between the set system and a designer's feasibility constraint, we will prefer the former term.

start adding elements $i \in J$. As J is not feasible, it must fail to add at least one of these elements. This element is permanently discarded and, therefore, the set selected by greedy is not equal to I and, therefore, not optimal.

Now, assume that the set system is downward-closed but does not satisfy the augmentation property. In particular there exists sets $J, I \in \mathcal{I}$ with $|J| < |I|$ but there is no $i \in I \setminus J$ that can be added to J, i.e., such that $J \cup \{i\} \in \mathcal{I}$. Consider the valuation profile $\boldsymbol{v}$ with (for a ground set N of size n)

$$v_i = \begin{cases} n+1 & \text{if } i \in J, \\ n & \text{if } i \in I \setminus J, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The greedy-by-value algorithm first attempts to and succeeds at adding all the elements of J. As there are no elements in $I \setminus J$ that are feasible when added to J, the algorithm terminates selecting exactly the set J. Because I has at least one more element than J, the value of I exceeds the value of J, and the optimality of the algorithm is contradicted. $\square$

The following matroids will be of interest.

- In a *k-uniform matroid* all subsets of cardinality at most k are feasible. The 1-uniform matroid corresponds to a single-item auction; the k-uniform matroid corresponds to a k-unit auction.
- In a *transversal matroid* the ground set is the set of vertices of part A of the bipartite graph $G = (A, B, E)$ (where vertices A are adjacent to vertices B via edges E) and feasible sets are the subsets of A that can be simultaneously matched. E.g., if A is people, B is houses, and an edge from $a \in A$ to $b \in B$ suggests that b is acceptable to a; then the feasible sets are subsets of people that can simultaneously be assigned acceptable houses with no two people assigned the same house. Notice that k-uniform matroids are the special case where $|B| = k$ and all houses are acceptable to each person. Therefore, transversal matroids represent a generalization of k-unit auctions to a market environment where not all units are acceptable to every agent.
- In a *graphical matroid* the ground set is the set of edges E in graph $G = (V, E)$ and feasible sets are acyclic subgraphs (i.e., a *forest*). Maximal feasible sets in a connected graph are spanning trees. The greedy-by-value algorithm for graphical matroids is known as *Kruskal's algorithm.*

The matroid properties characterize the set systems for which the greedy-by-value algorithm optimizes social surplus. Typically the most succinct method for arguing that matroid/ordinal environments have good properties is by using the fact that the greedy-by-value algorithm is optimal. Typically the most succinct method for arguing that an environment is matroid/ordinal is by showing that it satisfies the augmentation property (and is downward closed).

Theorem 4.26 *The greedy-by-value algorithm selects the feasible set with largest surplus for all valuation profiles if and only if feasible sets are a matroid.*

Proof The "only if" direction was shown above by Lemma 4.25. The "if" direction is as follows. Let r be the *rank* of the matroid. Let $I = \{i_1, \ldots, i_r\}$ be the set of agents selected in the surplus maximizing assignment, and let $J = \{j_1, \ldots, j_r\}$ be the set of agents selected by greedy-by-value. The surplus from serving a subset S of the agents is $\sum_{i \in S} v_i$.

Assume for a contradiction that the surplus of set I is strictly more than the surplus of set J, i.e., greedy-by-value is not optimal. Index the agents of I and J in decreasing order of value. With respect to this ordering, there must exist a first index k such that $v_{i_k} > v_{j_k}$. Let $I_k = \{i_1, \ldots, i_k\}$ and let $J_{k-1} = \{j_1, \ldots, j_{k-1}\}$. Applying the augmentation property to sets I_k and J_{k-1} we see that there must exist some agent $i \in I_k \setminus J_{k-1}$ such that $J_{k-1} \cup \{i\}$ is feasible. Of course, by the ordering of I_k, $v_i \geq v_{i_k} > v_{j_k}$ which means that agent i was considered by greedy-by-value before it selected j_k. By downward closure and feasibility of $J_{k-1} \cup \{i\}$, when agent i was considered by greedy-by-value it was feasible. By definition of the algorithm, agent i should have been added; this is a contradiction. □

To verify that an environment is ordinal/matroid the most direct approach is to verify the augmentation property. As an example we show that constrained matching markets (a.k.a., the transversal matroid) are indeed a matroid.

Lemma 4.27 *For matching agents $N = \{1, \ldots, n\}$ to items $K = \{1, \ldots, k\}$ via bipartite graph $G = (N, K, E)$ where an agent $i \in N$ can be matched to an item $j \in K$ if edge $(i, j) \in E$, the subsets of agents N that correspond to matchings in G are the feasible sets of a matroid on ground set N.*

Proof Consider any two subsets $N^\dagger$ and $N^\ddagger$ of N that are feasible, i.e., that correspond to matching in G, with $|N^\dagger| < |N^\ddagger|$. We argue that there exists an $i \in N^\ddagger \setminus N^\dagger$ such that $N^\dagger \cup \{i\}$ is feasible.

A matching M corresponds to a subset of edges E such each vertex (either an agent in N or an item in K) in the induced subgraph (N, K, M) has degree (i.e., number of adjacent edges in M) at most one. Denote the matching that witnesses the feasibility of $N^\dagger$ by $M^\dagger$, and likewise, $M^\ddagger$ for $N^\ddagger$. Consider the induced subgraph $(N, K, M^\dagger \cup M^\ddagger)$. The vertices in this subgraph have degree at most two. A graph of degree at most two is a collection of paths and cycles.

There must be a path that starts at a vertex corresponding to an agent $i \in N^\ddagger \setminus N^\dagger$ and ends with a vertex corresponding to an item $j \in K$. This is because paths that start with agents $i \in N^\ddagger \setminus N^\dagger$ can only end at items or at agents $i \in N^\dagger \setminus N^\ddagger$. By the assumption $|N^\dagger| < |N^\ddagger|$, there are more agents in $N^\ddagger \setminus N^\dagger$ than $N^\dagger \setminus N^\ddagger$ and so a path ending in an item must exist.

This path that ends at an item must alternate between edges in $M^\ddagger$ and $M^\dagger$. This path has an odd number of edges as it starts with an agent and ends with an item. As it starts with an agent matched by $M^\ddagger$. It has one more edge from $M^\ddagger$ than $M^\dagger$. In matching theory and with respect to matching $M^\dagger$ this path is an *augmenting path* as swapping the edges between the matchings results in a new matching for $M^\dagger$ with one more matched edge, and consequently one more agent is matched. This additional matched agent is i. The existence of this new matching implies that $N^\dagger \cup \{i\}$ is feasible. Thus, the matroid augmentation property is satisfied. □

4.6.2 Monopoly Reserve Pricing

In matroid environments that are inherently asymmetric, the i.i.d. assumption is unnatural and therefore restrictive. As in single-item environments, the surplus maximization mechanism with (discriminatory) monopoly reserves continues to be a good approximation even when the agents' values are non-identically distributed.

Theorem 4.28 *In regular, matroid environments the revenue of the surplus maximization mechanism with monopoly reserves is a two approximation to the optimal mechanism revenue.*

There are two very useful facts about the surplus maximization mechanism in ordinal environments that enable the proof of Theorem 4.28.

The first shows that the critical value (which determine an agent's payment) for an agent is the value of the agent's "best replacement." The second shows that the surplus maximization mechanism is pointwise revenue monotone, i.e., if the values of any subset of agents increases the revenue of the mechanism does not decrease. These properties are summarized by Lemma 4.29 and Theorem 4.30, below. We will prove Lemma 4.29 and leave the formal proofs of Theorem 4.28 and Theorem 4.30 for Exercise 4.19 and Exercise 4.20, respectively.

Definition 4.9 If $I \cup \{i\} \in \mathcal{I}$ is surplus maximizing set containing i then the *best replacement* for i is $j = \operatorname{argmax}_{\{k \,:\, I\cup\{k\}\in\mathcal{I}\}} v_k$.

Definition 4.10 A mechanism is *revenue monotone* if for all valuation profiles $\boldsymbol{v} \geq \boldsymbol{v}^\dagger$ (i.e., for all i, $v_i \geq v_i^\dagger$), the revenue of the mechanism on $\boldsymbol{v}$ is at least its revenue on $\boldsymbol{v}^\dagger$.

Lemma 4.29 *In matroid environments, the surplus maximization mechanism on valuation profile $\boldsymbol{v}$ has the critical values $\hat{\boldsymbol{v}}$ satisfying, for each agent i, $\hat{v}_i = v_j$ where j is the* best replacement *for i.*

Proof The greedy-by-value algorithm is ordinal, therefore we can assume without loss of generality that the cumulative values of all subsets of agents are distinct. To see this, add a $U[0, \epsilon]$ random perturbation to each agent value, the event where two subsets sum to the same value has measure zero, and as $\epsilon \to 0$ the critical values for the perturbation approach the critical values for the original valuation profile, i.e., from equation (4.9) below.

To proceed with the proof, consider two alternative calculations of the critical value for player i. The first is from the proof of Lemma 3.1 where $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$ and $\mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i})$ are optimal surplus from agents other than i with i is not served and served, respectively.

$$\hat{v}_i = \mathrm{OPT}(0, \boldsymbol{v}_{-i}) - \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i}). \tag{4.9}$$

The second is from the greedy algorithm. Sort all agents except i by value, then consider placing agent i at any position in this ordering. Clearly, i is served when placed first. Let j be the first agent after which i would not be served. Then,

$$\hat{v}_i = v_j. \tag{4.10}$$

Now we compare these the two formulations of critical values given by equations (4.9) and (4.10). Consider i ordered immediately before and immediately after j and suppose that i is served in former order and not served in the later order. In the latter order, it must be that j is served as this is the only possible difference between the outcomes of the greedy algorithm for these two orderings up to the point that both i and j have been considered. Therefore, agent j must be served in the calculation of $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$. Let $J \cup \{j\}$ be the agents served in $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$ and let $I \cup \{i\}$ be the agents served in $\mathrm{OPT}(\infty, \boldsymbol{v}_{-i})$. We can deduce from equations (4.9) and (4.10) that,

$$\begin{aligned} v_j &= \hat{v}_i \\ &= \mathrm{OPT}(0, \boldsymbol{v}_{-i}) - \mathrm{OPT}_{-i}(\infty, \boldsymbol{v}_{-i}) \\ &= v_j + v(J) - v(I), \end{aligned}$$

where $v(S)$ denotes $\sum_{k \in S} v_k$. We conclude that $v(I) = v(J)$ which, by the assumption that the cumulative values of distinct subsets are distinct, implies that $I = J$. Meaning: j is a replacement for i; furthermore, by optimality of $J \cup \{j\}$ for $\mathrm{OPT}(0, \boldsymbol{v}_{-i})$, j must be the best, i.e., highest valued, replacement. □

Theorem 4.30 *In matroid environments, the surplus maximization mechanism is revenue monotone.*

4.6.3 Oblivious and Adaptive Posted Pricings

Recall that an oblivious posted pricing predetermines prices to offer each agent and its revenue must be guaranteed in worst case over the order that the agents arrive. It is conjectured that oblivious posted pricing is a constant approximation for any matroid environment. In contrast, an *adaptive posted pricing* is one that, for any arrival order of the agents, calculates the price to offer each agent when she arrives. The calculated price can be a function of the agents identity, the agents that have previously arrived and the agents that are currently being served by the mechanism. The proof of the following theorem is based on a *matroid prophet inequality* (that we will not cover in this text).

Theorem 4.31 *For (non-identical, irregular) matroid environments, there is an adaptive posted pricing that is a two approximation to the optimal mechanism revenue.*

4.6.4 Sequential Posted Pricings

The ${}^{e}\!/_{e-1}$ approximation for single-item sequential posted pricing and its proof via correlation gap extends to matroid environments. To present this extension, we first extend the definition of the optimal ex ante mechanism to matroids. We then relate the sequential posted pricing question to the optimal ex ante mechanism via the correlation gap. Finally, we conclude with a necessary extra step for adapting the pricing to irregular distributions.

Consider a matroid set system $(N, \mathcal{I})$. Previously we defined the rank of a matroid as the maximum cardinality of any feasible set. We can similarly define the rank of a not-necessarily-feasible subset S of the ground set N as the maximum cardinality of any feasible subset of it. In other words, it is the rank of the induced matroid on $(S, \mathcal{I})$. Let $\mathrm{rank}(S)$ denote this matroid rank function.

A profile of ex ante probabilities $\hat{\boldsymbol{q}} = (\hat{q}_1, \ldots, \hat{q}_n)$ is *ex ante feasible*, if there exists a distribution $\mathcal{D}$ over feasible sets $\mathcal{I}$ of the matroid that induces these ex ante probabilities. This definition is cumbersome; however, it is simplified by the following characterization. For any distribution $\mathcal{D}$ over feasible sets and any not-necessarily-feasible set S it must be that the expected number of agents served by $\mathcal{D}$ is at most the rank of that set. I.e., for all $S \subset N$,

$$\sum\nolimits_{i \in S} \hat{q}_i \leq \mathrm{rank}(S). \tag{4.11}$$

This inequality follows as the left-hand side is the expected number of agents in S that are served and the right hand side is the maximum number of agents in S that can be simultaneously served. It is impossible for this expected number to be higher than this maximum possible. In fact, this necessary condition is also sufficient.

Proposition 4.32 *For a matroid set system $(N, \mathcal{I})$, a profile of ex ante probabilities $\hat{\boldsymbol{q}}$ is ex ante feasible (i.e., there is a distribution $\mathcal{D}$ over feasible sets $\mathcal{I}$ that induces ex ante probabilities $\hat{\boldsymbol{q}}$) if and only if $\sum_{i \in S} \hat{q}_i \leq \mathrm{rank}(S)$ holds for all subsets S of N.*

From the above characterization of ex ante feasibility, we can write the optimal ex ante pricing program as follows.

$$\begin{aligned} &\max_{\hat{\boldsymbol{q}}} \sum\nolimits_i R(\hat{q}_i) && \qquad (4.12)\\ &\text{s.t.} \sum\nolimits_{i \in S} \hat{q}_i \leq \mathrm{rank}(S), && \forall S \subset N. \end{aligned}$$

If the objective were given by linear weights instead of concave revenue curves, this program would be optimized easily by the greedy-by-value algorithm (with values equal to weights).[5] With convex revenue curves,

[5] Readers familiar with convex optimization will note that the matroid rank

the marginal revenue approach enables this program to be optimized via a simple greedy-by-value based algorithm.[6]

Suppose for now that the distribution over agent values is regular. The revenue curve for an agent with inverse demand curve $V(\cdot)$ is consequently given by $R(\hat{q}) = \hat{q} \cdot \hat{v}$ for $\hat{v} = V(\hat{q})$ since, for a regular distribution, the $\hat{q}$ optimal ex ante pricing posts price $\hat{v}$. The optimal ex ante revenue from program (4.12) is thus $\sum_i \hat{q}_i \hat{v}_i$.

The ex ante optimal revenue can be interpreted as the correlated value of a set function as follows. Consider the *matroid weighted rank* function $\mathrm{rank}_{\hat{\boldsymbol{v}}}(\cdot)$ for weights $\hat{\boldsymbol{v}}$ defined for a feasible set $S \in \mathcal{I}$ as $\sum_{i \in S} \hat{v}_i$ and in general for not-necessarily-feasible set $S \subset N$ as that maximum over feasible subsets of S of the weighted rank of that subset. As $\hat{\boldsymbol{q}}$ is ex ante feasible, there exists a correlated distribution $\mathcal{D}$ over feasible sets which induces ex ante probabilities $\hat{\boldsymbol{q}}$. The correlated value of this distribution for the matroid weighted rank set function is exactly the optimal ex ante revenue.

Now consider the sequential posted pricing that orders the agents by decreasing price $\hat{v}_i$. When an agent i arrives in this order, if it is feasible to serve the agent along with the set of agents who have been previously served, then offer her price $\hat{v}_i$; otherwise, offer her a price of infinity (i.e., reject her). Consider the outcome of this process for valuation profile $\boldsymbol{v}$ where the set of agents willing to buy at their respective price is $S = \{i : v_i \geq \hat{v}_i\}$ (which may not be feasible). The revenue from this sequential posted pricing is given by the matroid weighted rank function as $\mathrm{rank}_{\hat{\boldsymbol{v}}}(S)$.

We conclude that the approximation factor of sequential posted pricing with respect to the optimal ex ante revenue (which upper bounds the optimal revenue for ex post feasibility) is given by the correlation gap of the matroid weighted rank set function. Thus, it remains to analyze the correlation gap of the matroid weighted rank set function. An approach, which we will discuss here to analyze the correlation gap of the matroid weighted rank set functions, is to observe that the matroid

function is submodular and therefore the constraint imposed by ex ante feasibility is that of a polymatroid.

[6] Discretize quantile space $[0, 1]$ into Q evenly sized pieces. Consider the Q-wise union of the matroid set system (the class of matroid set systems is closed under union). Calculate marginal revenues of each discretized quantile of each agent. Run the greedy-by-marginal-revenue algorithm. Calculate $\hat{q}_i$ as the total quantile of agent i that is served by algorithm, i.e., $1/Q$ times the number of i's discretized pieces that are served.

weighted rank function is *submodular* and that the correlation gap of any submodular function is $^{e}/_{e-1}$.

For ground set N, consider a real valued set function $g : 2^N \to \mathbb{R}$. Intuitively, *submodularity* corresponds to diminishing returns. Adding an element i to a large set increases the value of the set function less than it would for adding it to a smaller subset.

Definition 4.11 A set function g is *submodular* if for $S^\dagger \subset S^\ddagger$ and $i \notin S^\ddagger$,

$$g(S^\dagger \cup \{i\}) - g(S^\dagger) \geq g(S^\ddagger \cup \{i\}) - g(S^\ddagger).$$

Importantly, the matroid rank and weighted-rank functions are submodular (Definition 4.11). Therefore, the matroid structure imposes diminishing returns.

Theorem 4.33 *The matroid rank function is submodular; for any real valued weights, the matroid weighted-rank function is submodular.*

Proof We prove the special case of uniform weights (equivalently: that the matroid rank function is submodular; for the general case, see Exercise 4.21). Consider $S^\dagger \subset S^\ddagger$ and $i \notin S^\ddagger$ and the weights $\boldsymbol{v}_{-i}$ as

$$v_j = \begin{cases} 4 & \text{if } j \in S^\dagger, \\ 2 & \text{if } j \in S^\ddagger \setminus S^\dagger, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the case that $v_i = 1$ and $v_i = 3$. If i is added by greedy-by-value when $v_i = 1$ then i is certainly added by greedy-by-value when $v_i = 3$: moving i earlier in the greedy ordering only makes it more plausible that it is feasible to add i at the time i is considered. Therefore, difference in rank of $S^\dagger$ with and without i is at least the difference in rank of $S^\ddagger$ with and without i. Hence, the defining equation (Definition 4.11) for submodularity holds. □

We omit the proof of the following theorem and instead refer readers to the simpler proof that the maximum value element set function has correlation gap $^{e}/_{e-1}$ (see Lemma 4.13, Section 4.3).

Theorem 4.34 *The correlation gap for a submodular set function and any distribution over sets is* $^{e}/_{e-1}$.

For regular distributions and by the above discussion, the ex ante service probabilities from the ex ante program (4.12) corresponds to a sequential posted pricing that has approximation factor bounded by the correlation gap. The same bound can be obtained for irregular distributions as well (see Section 4.3 and Exercise 4.22).

Theorem 4.35 *For matroid environments, there is a sequential posted pricing with revenue that is a $e/e-1$ approximation to the optimal auction revenue.*

4.6.5 Anonymous Reserves

While Proposition 4.24 showed that anonymous reserves are optimal for i.i.d. regular matroid environments, this is the extent to which anonymous reserves give good approximation for matroid environments. Of course, all lower bounds for multi-unit environments extend to matroids (where the k-unit auction result generalizes to rank k matroids). In addition there two new lower bounds. For i.i.d. regular matroid environments, anonymous posted pricing does not give a constant approximation. For (irregular) i.i.d. matroid environments, neither anonymous reserve nor posted pricing gives a constant approximation (Exercise 4.23).

4.6.6 Beyond Ordinal Environments

Generalizing reserve and posted pricing approximation beyond ordinal environments is difficult because in general environments (even downward-closed ones) the optimal mechanism may choose to serve one agent over a set of other agents, or vice versa. For example, this would happen when the first agents virtual value exceeds the sum of the other agents' virtual values. Recall that the matroid property discussed previously guarantees that tradeoffs between serving agents is always done one for one (e.g., via Lemma 4.29). There are two, in fact opposite, effects we should be worried about when proceeding to general environments. First, in a general downward-closed environment one agent could potentially block many agents with each with comparable payments. Second, many agents with minimal payments could potentially block a few agents who would have made significant payments.

We illustrate the first effect with an impossibility result for posted pricing mechanisms.

Lemma 4.36 *For (i.i.d., regular) downward-closed environments the approximation ratio of posted pricing (oblivious or sequential) is at best $\Omega(\log n/ \log\log n)$.*

Proof Fix an integer h, set $n = h^{h+1}$, and partition the n agents into h^h parts of size h each. Consider the one-part-only feasibility constraint that forbids simultaneously serving agents in distinct parts, but allows and number of agents in the same part to be served. The agents' values are i.i.d. from the equal revenue distribution on $[1, h]$, i.e., with $F(z) = 1 - 1/z$ and a pointmass of $1/h$ at value h. Call an agent high-valued if her value is h and, otherwise, low-valued. We show that the approximation factor is at least $h/2 \cdot e-1/e$ and conclude that the approximation factor is $\Omega(h) = \Omega(\log n/\log\log n)$.[7]

To get a lower bound on the optimal revenue, REF, consider the mechanism that serves a part only if all agents in the part are high valued, charges each of the agents in the part h, and obtains a total revenue of h^2. As there are h^h parts and each part has probability h^{-h} of being all high valued, the probability that one or more of these parts is all high valued is given by the correlation gap of the one-or-more set function as $e-1/e$ (Lemma 4.13). Thus, the optimal revenue is at least $\text{REF} \geq h^2 \cdot e-1/e$.

To get an upper bound on the revenue of any posted pricing, notice that once one agent accepts a price, only agents in that same part as this agent can be simultaneously served. Since the distribution is equal revenue, the revenue from serving these remaining agents totals exactly $h-1$ (one from each of $h-1$ agents). The best revenue we can get from the first agent in the part is h. Thus, any posted pricing mechanism's revenue is upper bounded by $2h - 1$, and so $\text{APX} \leq 2h$. $\square$

Before we illustrate the second effect (many low-paying agents blocking a few high-paying agents), notice that the tradeoffs of optimizing virtual values (for revenue) can be much different from the tradeoffs of optimizing values (for social surplus). Therefore, the outcome from surplus maximization could be much different from that of virtual surplus maximization.

Example 4.37 The expected value the equal revenue distribution on $[1, h]$ is $\ln h - 1$ (for the unbounded equal revenue distribution it is infinite). This can be calculated from the formula $\mathbf{E}[v] = \int_0^\infty (1 - F^{\text{EQR}}(z))\,\mathrm{d}z$ with $F^{\text{EQR}}(z) = 1 - 1/z$. On the other hand, the monopoly revenue for the equal revenue distribution is one. Therefore, the optimal

[7] To see the asymptotic behavior of the approximation ratio in terms of n, notice that by definition $\log n = (h+1)\log h$, so (a) rearranging $h = \log n/\log h - 1$ and (b) taking the logarithm $\log\log n > \log(h+1) + \log\log h$. From (b), $\log\log n = \Theta(\log h)$ and plugging this into (a) $h = \Theta(\log n/\log\log n)$.

social surplus and optimal revenue for a regular single-agent environment can be arbitrarily separated.

Because of the difference between social surplus and potential revenue (i.e., virtual surplus) can be large, there may be a set of agents with high social surplus that collectively block another set of agents from whom a large revenue could be obtained. In the surplus maximization mechanism with reserves, the payment an agent makes is either her reserve price or the externality she imposes on the other agents. In the scenario under consideration it may be that none of the agents in the first set is individually responsible for other agents being rejected, consequently none impose any externality. Therefore, the revenue they contribute need not exceed the revenue that could have been obtained by serving the second set. We illustrate this phenomenon with an impossibility result for surplus maximization with monopoly reserves in regular downward-closed environments.

Lemma 4.38 *For (regular) downward-closed environments the approximation factor of the second-price auction with monopoly reserves is* $\Omega(\log n)$.

Proof Consider a one-versus-many set system on $n+1$ agents where it is feasible to serve agent 1 (Alice) or any subset of the remaining agents $2, \ldots, n+1$ (the Bobs). This set system is downward closed.

A sketch of the argument is as follows. The Bobs' values are distributed i.i.d. from an equal revenue distribution. If we decide to sell to the Bobs the best we can get is a revenue of n total (one from each). Of course, the social surplus of the Bobs is much bigger than the revenue that selling to them would generate (see Example 4.37, above). We then set Alice's value deterministically to a large value that is $\Theta(n \log n)$ but with high probability below the social surplus of the Bobs. The optimal auction could always sell to Alice at her high value; thus, REF is $\Theta(n \log n)$. Unfortunately, the monopoly reserves for the Bobs are one and, therefore, not binding. Surplus maximization with monopoly reserves will with high probability not serve Alice, and therefore derive most of its revenue from the Bobs. The maximum expected revenue obtainable from the Bobs is n; thus, APX $= \Theta(n)$. See Exercise 4.24 for the details. $\square$

In the next section we show; for a large class of important distributions that, intuitively, do not have tails that are too heavy; that virtual values and values are close. Consequently, maximizing surplus is similar enough to maximizing virtual surplus that monopoly reserve pricing gives a good approximation to the optimal mechanism.

4.7 Monotone-hazard-rate Distributions

An important property of electronic devices, such as light bulbs or computer chips, is how long they will operate before failing. If we model the lifetime of such a device as a random variable then the failure rate, a.k.a., *hazard rate*, for the distribution at a certain point in time is the conditional probability (actually: density) that the device will fail in the next instant given that it has survived thus far. Device failure is naturally modeled by a distribution with a monotone (non-decreasing) hazard rate, i.e., the longer the device has been running the more likely it is to fail in the next instant. The uniform, normal, and exponential distributions all have monotone hazard rate. The equal-revenue distribution (Definition 4.2) does not.

Definition 4.12 The *hazard rate* of distribution F (with density f) is $h(z) = \frac{f(z)}{1-F(z)}$. The distribution has *monotone hazard rate (MHR)* if $h(z)$ is monotone non-decreasing.

Intuitively distributions with monotone hazard rate are not *heavy tailed*. In fact, the exponential distribution, with $F^{\mathrm{EXP}}(z) = 1 - e^{-z}$, $f^{\mathrm{EXP}}(z) = e^{-z}$, and $h^{\mathrm{EXP}}(z) = 1$ is the boundary between monotone hazard rate and non; its hazard rate is constant. Hazard rates are clearly important for revenue-optimal auctions as the definition of virtual valuations (for revenue), expressed in terms of the hazard rate, is

$$\phi(v) = v - {}^{1}\!/_{h(v)}. \tag{4.13}$$

It is immediately clear from equation (4.13) that monotone hazard rate implies regularity (i.e., monotonicity of virtual value; Definition 3.4).

An important property of monotone hazard rate distributions that will enable approximation by the surplus maximization mechanism with monopoly reserves is that the optimal revenue is within a factor of $e \approx 2.718$ of the optimal surplus. We illustrate this bound with the exponential distribution (Example 4.39), prove it for the case of a single-agent environments, and defer general downward-closed environments to Exercise 4.25. Contrast these results to Example 4.37, above, which shows that for non-monotone-hazard-rate distributions, the ratio of surplus to revenue can be unbounded.

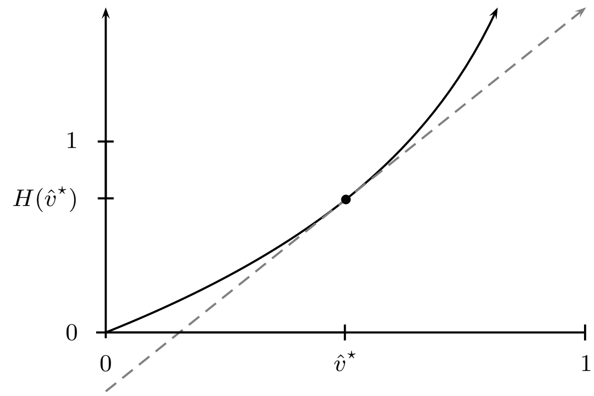


Figure 4.4 The cumulative hazard rate function (solid, black) for the uniform distribution is $H(v) = -\ln(1 - v)$ and it is lower bounded by its tangent (dashed, gray) at $\hat{v}^\star = 1/2$.

Example 4.39 The expected value the exponential distribution (with rate one) is one. This can be calculated from the formula $\mathbf{E}[v] = \int_0^\infty (1 - F^{\mathrm{EXP}}(z))\,\mathrm{d}z$ with $F^{\mathrm{EXP}}(z) = 1 - e^{-z}$. Since the exponential distribution has hazard rate $h^{\mathrm{EXP}}(z) = 1$, the virtual valuation formula for the exponential distribution is $\phi^{\mathrm{EXP}}(v) = v - 1$. The monopoly price is one. The probability that the agent accepts the monopoly price is $1 - F^{\mathrm{EXP}}(1) = 1/e$ so its expected revenue is $1/e$. The ratio of the expected surplus to expected revenue is e.

Theorem 4.40 *For any downward-closed, monotone-hazard-rate environment, the optimal expected revenue is an $e \approx 2.718$ approximation to the optimal expected surplus.*

Lemma 4.41 *For any monotone-hazard-rate distribution its expected value is at most e times more than the expected monopoly revenue.*

Proof Let REF $= \mathbf{E}[v]$ be the expected value and APX $= \hat{v}^\star \cdot (1 - F(\hat{v}^\star))$ be the expected monopoly revenue. Let $H(v) = \int_0^v h(z)\,\mathrm{d}z$ be the *cumulative hazard rate* of the distribution F. We can write

$$1 - F(v) = e^{-H(v)}, \tag{4.14}$$

an identity that can be easily verified by differentiating the natural logarithm of both sides of the equation.[8] Recall of course that the expectation

[8] We have $\frac{\mathrm{d}}{\mathrm{d}v}\ln(1 - F(v)) = \frac{-f(v)}{1-F(v)}$ and $\frac{\mathrm{d}}{\mathrm{d}v}\ln\left(e^{-H(v)}\right) = -h(v)$.

of $v \sim \boldsymbol{F}$ is $\int_0^\infty (1 - F(z))\,\mathrm{d}z$. To get an upper bound on this expectation we need to upper bound $e^{-H(v)}$ or equivalently lower bound $H(v)$.

The main difficulty is that the lower bound must be tight for the exponential distribution where optimal expected value is exactly e times more than the expected monopoly revenue. Notice that for the exponential distribution the hazard rate is constant; therefore, the cumulative hazard rate is linear. This observation suggests that perhaps we can get a good lower bound on the cumulative hazard rate with a linear function.

Let $\hat{v}^\star = \phi^{-1}(0)$ be the monopoly price. Since $H(v)$ is a convex function (it is the integral of a monotone function), we can get a lower bound $H(v)$ by the line tangent to it at $\hat{v}^\star$. See Figure 4.4. I.e.,

$$\begin{aligned} H(v) &\geq H(\hat{v}^\star) + h(\hat{v}^\star)(v - \hat{v}^\star) \\ &= H(\hat{v}^\star) + \tfrac{v - \hat{v}^\star}{\hat{v}^\star}. \end{aligned} \tag{4.15}$$

The second part follows because $\hat{v}^\star = 1/h(\hat{v}^\star)$ by the choice of monopoly price $\hat{v}^\star$ and equation (4.13). Now we use this bound to calculate a bound on the expectation.

$$\begin{aligned} \mathrm{REF} &= \int_0^\infty (1 - F(z))\,\mathrm{d}z = \int_0^\infty e^{-H(z)}\,\mathrm{d}z \\ &\leq \int_0^\infty e^{-H(\hat{v}^\star) - z/\hat{v}^\star + 1}\,\mathrm{d}z = e \cdot e^{-H(\hat{v}^\star)} \cdot \int_0^\infty e^{-z/\hat{v}^\star}\,\mathrm{d}z \\ &= e \cdot e^{-H(\hat{v}^\star)} \cdot \hat{v}^\star = e \cdot (1 - F(\hat{v}^\star)) \cdot \hat{v}^\star = e \cdot \mathrm{APX}\,. \end{aligned}$$

The first and last lines follow from equation (4.14); the inequality follows from equation (4.15). □

Shortly we will show that the surplus maximization mechanism with monopoly reserve prices is a two approximation to the optimal mechanism for monotone-hazard-rate downward-closed environments. This result is derived from the intuition that revenue and surplus are close. For revenue and surplus to be close, it must be that virtual values and values are close. Notice that the monotone-hazard-rate condition, via equation (4.13), implies that for higher values (which are more important for optimization) virtual value is even closer to value than for lower values (see Figure 4.5). The following lemma reformulates this intuition.

Lemma 4.42 *For any monotone-hazard-rate distribution F and $v \geq \hat{v}^\star$, $\phi(v) + \hat{v}^\star \geq v$.*

Proof Since $\hat{v}^\star = \phi^{-1}(0)$ it solves $\hat{v}^\star = 1/h(\hat{v}^\star)$. By MHR, $v \geq \hat{v}^\star$ implies

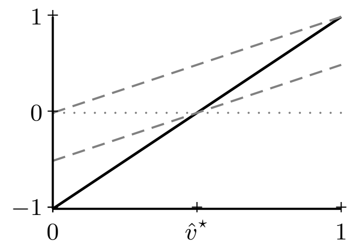


Figure 4.5 The virtual value for the uniform distribution is depicted. For $v \geq \hat{v}^\star$ the virtual value $\phi(v)$ (solid, black) is sandwiched between the value v (dashed, gray) and value less the monopoly price $v - \hat{v}^\star$ (dashed, gray).

$h(v) \geq h(\hat{v}^\star)$. Therefore,

$$\phi(v) + \hat{v}^\star = v - 1/h(v) + 1/h(\hat{v}^\star) \geq v. \qquad \square$$

Theorem 4.43 *For any monotone-hazard-rate downward-closed environment, the revenue of the surplus maximization mechanism with monopoly reserves is a two approximation to the optimal mechanism revenue.*

Proof Let APX denote the surplus maximization mechanism with monopoly reserves (and its expected revenue) and let REF denote the revenue-optimal mechanism (and its expected revenue). We start with two bounds on APX and then add them.

$$\begin{aligned} \mathrm{APX} &= \mathbf{E}[\text{APX's virtual surplus}]\,, \text{ and} \\ \mathrm{APX} &\geq \mathbf{E}[\text{APX's winners' reserve prices}]\,. \end{aligned}$$

Sum these two equations and let $\boldsymbol{x}(\boldsymbol{v})$ denote the allocation rule of APX,

$$\begin{aligned} 2 \cdot \mathrm{APX} &\geq \mathbf{E}[\text{APX's winners' virtual values + reserve prices}] \\ &= \mathbf{E}\Big[\sum\nolimits_i (\phi_i(v_i) + \hat{v}_i^\star) \cdot x_i(\boldsymbol{v})\Big] \\ &\geq \mathbf{E}\Big[\sum\nolimits_i v_i \cdot x_i(\boldsymbol{v})\Big] = \mathbf{E}[\text{APX's surplus}] \\ &\geq \mathbf{E}[\text{REF's surplus}] \geq \mathbf{E}[\text{REF's revenue}] = \mathrm{REF}\,. \end{aligned}$$

The second inequality follows from Lemma 4.42. By downward closure, neither REF nor APX sells to agents with negative virtual values. Of course, APX maximizes the surplus subject to not selling to agents with negative virtual values. Hence, the third inequality. The final inequality follows because the revenue of any mechanism is never more than its surplus. $\square$

We have seen in this section that, for monotone-hazard-rate distributions in downward closed environments, the optimal social surplus and optimal revenue are close. We then used this fact to show that a the monopoly-reserves auction is a good approximation to the optimal auction. Because surplus and revenue are close, the optimal surplus can be used as an upper bound on the optimal revenue. Finally, we showed that the monopoly-reserves auction has a revenue that approximates the optimal surplus. This approach of comparing revenue to surplus is somewhat brute-force, and there is thus a sense that these approximation bounds could be considered trivial.

Exercises

4.1 In Chapter 1 we saw that a lottery (Definition 1.2) was an n approximation to the optimal social surplus. At the time we claimed that this approximation guarantee was the best possible by a mechanism without transfers. Prove this claim.

4.2 Consider a two-agent single-item auction where agent 1 and agent 2 have values distributed uniformly on $[0, 2]$ and $[0, 3]$, respectively. Calculate and compare the expected revenue of the (asymmetric) revenue-optimal auction and the second-price auction with (asymmetric) monopoly reserves. In other words, calculate the expected revenues for the allocation rules of Example 3.11 which are depicted in Figure 4.1.

4.3 Finish the proof of Lemma 4.4 by showing that for any irregular distribution, the value of an agent is at least her virtual value for revenue. Hint: start by observing that with respect to the price-posting revenue curve $P(q) = q \cdot V(q)$, $V(q)$ is the slope of the line from the origin to the point $(q, P(q))$ on the curve, and that the lemma for the regular case implies that lines from the origin cross the curve only once.

4.4 Define a distribution to be *prepeak monotone* if its revenue curve is monotone non-decreasing on $[0, \hat{q}^\star]$, i.e., at values above the monopoly price. Notice that prepeak monotonicity is a weaker condition than regularity. First, it requires nothing of the distribution below the monopoly price. Second, above the monopoly price the price-posting revenue curve does not need to be concave. Reprove Theorem 4.2 with a weaker assumption that the agents' distributions are prepeak monotone.

4.5 Calculate the expected revenue of the optimal auction in an n-agent k-unit environment with values drawn i.i.d. from the equal revenue distribution (Definition 4.2; distribution function $F^{\text{EQR}}(z) = 1 - 1/z$). Express your answer in terms of n and k.

4.6 Show that the revenue from the single-item monopoly-reserves auction smoothly degrades as the distribution becomes more irregular. To show this you will need to formally define near regularity. One reasonable definition is as follows. A distribution F is *α-nearly regular* if there is a regular distribution $F^{\dagger}$ such that price-posting revenue curves of these distributions satisfy $P(q) \geq P^{\dagger}(q) \geq 1/\alpha P(q)$ for all q.

   (a) Explain why the definition above is a good definition for near regularity.
   (b) Prove an approximation bound the second-price auction with monopoly reserves in α-nearly regular environments.

4.7 Generalize the prophet inequality theorem to the case where both the prophet and the gambler face an ex ante constraint $\hat{q}$ on the probability that they accept any prize.

4.8 Show that another method for choosing the threshold in the prophet inequality is to set $\hat{v} = 1/2 \cdot \mathbf{E}[\max_i v_i]$. Hint: for this choice of $\hat{v}$, prove that $\hat{v} \leq \sum_i \mathbf{E}[(v_i - \hat{v})^+]$.

4.9 Show that the prophet inequality is tight in two senses.

   (a) Show that there is a distribution over prizes such that the expected prize of the optimal backwards induction strategy is half of the prophet's.
   (b) Show that there is a distribution over prizes such that the expected prize of any uniform threshold strategy is at most half of the optimal backwards induction strategy.

4.10 Adapt the statement and proof of Theorem 4.14 to the objective of social surplus. Be explicit about the prices and ordering of agents in the sequential posted pricing of your construction.

4.11 For two agents with values drawn from the uniform distribution, calculate and compare the price postings from:

   (a) the prophet inequality based oblivious posted pricing,
   (b) the correlation gap based sequential posted pricing, and
   (c) the optimal anonymous price posting.

4.12 For i.i.d. regular single-item environments, give a lower bound lower bound for the approximation ratio of anonymous pricing that

that nearly matches the upper bound. Hint: consider the regular distribution with revenue curve $R(q) = (1 - 1/n)q + 1//n$.

4.13 Prove Theorem 4.19 by adapting the analysis of the prophet inequality (Theorem 4.8) to show, for any (non-identical) regular single-item environment, that there exists an anonymous price (i.e., the same for each agent) such that price-posting obtains four approximation to the optimal ex ante mechanism revenue.

4.14 Show that there exists an i.i.d. distribution and a matroid for which the surplus maximization mechanism with an anonymous reserve is no better than an $\Omega(\log n / \log\log n)$ approximation to the optimal mechanism revenue.

4.15 Show that for (non-identical, irregular) n-agent single-item environments the second-price auction with anonymous reserve and anonymous posted pricing are at best n approximations to the optimal auction revenue (i.e., prove the lower bound of Theorem 4.20). To do so, analyze the revenue of the optimal auction and the second-price auction with any anonymous reserve when the agents values distributed as:

$$v_i = \begin{cases} h^i & \text{w.p. } h^{-i}\text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

and parameter h approaches infinity. Hint: the analysis of Proposition 4.6 is similar.

(a) Show that the optimal auction has an expected revenue of n in the limit of h.

(b) Show that posting anonymous price h^i (for $i \in \{1, \ldots, n\}$) has an expected revenue of one in the limit of h.

(c) Show that for the second-price auction and anonymous reserve price h^i (for $i \in \{1, \ldots, n\}$) has an expected revenue of one in the limit of h. Hint: notice that conditioned on their being exactly one agent with a positive value, anonymous reserve pricing and anonymous posted pricing give the same revenue.

(d) Combine the above three steps to prove the theorem.

4.16 Generalize Proposition 4.7 and Proposition 4.6 to show that for n-agent k-unit irregular environments the $(k+1)$st-price auction with monopoly reserves is a n/k approximation and give a matching lower bound, respectively.

4.17 Prove Proposition 4.22, i.e., for k-unit environments that there is a sequential posted pricing that is a $(1 - 1/\sqrt{2\pi k})^{-1}$ approximation to the optimal auction, by completing the following steps.

(a) Reduce the correlation gap of the k-maximum-weight-elements set function, i.e., for weights $\hat{\boldsymbol{v}} = (\hat{v}_1, \ldots, \hat{v}_n)$ the value of $g^{\mathrm{kMWE}}(S)$ for subset S is the sum of the k largest weight elements of S, and arbitrary correlated distributions to correlated distributions over sets of cardinality exactly k.

(b) Reduce the correlation gap of the k-maximum-weight-elements set function on correlated distributions over sets of cardinality k to the correlation gap of the k-capped-cardinality set function $g^{\mathrm{kCC}}(S) = \min(k, |S|)$ (over the same class of distributions).

(c) Show that the correlation gap of the k-capped-cardinality set function on correlated distributions over sets of cardinality k is $(1 - k/e)^k \cdot 1/k!)^{-1}$.

(d) Apply the correlation gap to obtain a bound on the approximation ratio of the revenue of a uniform virtual pricing for (non-identical, irregular) k-unit environments with respect to the optimal auction revenue. Explain exactly how to find an appropriate pricing.

4.18 Recall that a feasible set of a matroid is maximal if there is no element that can be added to it such that the union is feasible. It is easy to see that the augmentation property implies that all maximal feasible sets of a matroid have the same cardinality. Rederive this result directly from the fact that greedy-by-value is optimal.

4.19 Show that in regular, matroid environments the surplus maximization mechanism with monopoly reserves gives a two approximation to the optimal mechanism revenue, i.e., prove Theorem 4.28. Hint: This result can be proved using Lemma 4.29 and Theorem 4.30 and a similar argument to the proof of Theorem 4.2.

4.20 A mechanism $\mathcal{M}$ is *revenue monotone* if for all pairs of valuation profiles $\boldsymbol{v}$ and $\boldsymbol{v}^\dagger$ such that for all i, $v_i \geq v_i^\dagger$, the revenue of $\mathcal{M}$ on $\boldsymbol{v}$ is at least its revenue on $\boldsymbol{v}^\dagger$. It is easy to see that the second-price auction is revenue monotone.

(a) For single-dimensional linear agents, give a downward-closed environment for which the surplus maximization mechanism (Mechanism 3.3) is not revenue monotone.

(b) Prove that the surplus maximization mechanism is revenue monotone in matroid environments.

4.21 Prove, directly from the fact that greedy-by-value is optimal for matroid set systems, that the matroid rank function is submodular. I.e., complete the proof of Theorem 4.33.

4.22 Consider sequential posted pricings for irregular matroid environments.

(a) Show that there is a sequential posted pricing that is an $e/e-1$ approximation to the revenue optimal auction.

(b) Give an algorithm for finding such a sequential posted pricing. Assume you are given the ex ante service probabilities $\hat{\boldsymbol{q}}$ that optimizes program (4.12). Assume you are given oracle access to the single-agent optimal ex ante pricing problems for each agent, i.e., for any agent i and service probability $\hat{q}_i$ the oracle will tell you the revenue-optimal lottery pricing that this agent with ex ante probability $\hat{q}_i$. Finally, assume you have blackbox access to a procedure that for any sequential posted pricing $\hat{\boldsymbol{v}}$ will tell you the sequential posted pricing's expected revenue (assuming prices are offered to agents in decreasing order). Your algorithm should run in linear time in the number n of agents, i.e., it should have at most a linear number of basic computational steps and calls to any of the above oracles.

4.23 Show the following inapproximability results for anonymous reserve and posted pricing in i.i.d. matroid environments.

(a) For i.i.d. regular matroid environments, anonymous posted pricing does not give a constant approximation.

(b) For (irregular) i.i.d. matroid environments, neither anonymous reserve nor posted pricing gives a constant approximation.

4.24 Complete the proof of Lemma 4.38 by showing that there is a family of regular downward-closed environments that demonstrates that the surplus maximization mechanism with monopoly reserves is an $\Omega(\log n)$ approximation to the optimal revenue. Hint: to set the value of Alice such that with high probability the social surplus of the Bobs exceeds Alice's value you can truncate the equal revenue distribution to a finite value h and then employ a standard Chernoff-Hoeffding *concentration bound* that shows that the sum of i.i.d. random variables on $[0, h]$ is concentrated around its expectation. For a sum S of i.i.d. random variables on $[0, h]$:

$$\mathbf{Pr}[|S - \mathbf{E}[S]| \geq \delta] \leq 2e^{-2\delta^2/nh^2}.$$

4.25 Consider the following *surplus maximization mechanism with lazy* *monopoly reserves* where, intuitively, we run the surplus maximization mechanism SM and then reject any winner i whose value is below her monopoly price $\hat{v}_i^\star$:

(a) $(\boldsymbol{x}^\dagger, \boldsymbol{p}^\dagger) \leftarrow \mathrm{SM}(\boldsymbol{v})$,

(b) $x_i = \begin{cases} x_i^\dagger & \text{if } v_i \geq \hat{v}_i^\star \\ 0 & \text{otherwise, and} \end{cases}$

(c) $p_i = \max(\hat{v}_i^\star, p_i^\dagger)$.

Prove that the revenue of this mechanism is an e approximation to the optimal social surplus in any downward-closed, monotone-hazard-rate environment. Conclude Theorem 4.40 as a corollary.

Chapter Notes

For non-identical, regular, single-item environments, the proof that the second-price auction with monopoly reserves is a two approximation is from Chawla et al. (2007). The generalization of monopoly reserve pricing to general environments is from Hartline and Roughgarden (2009). They showed that it is a two approximation for regular matroid environments and for monotone-hazard-rate downward-closed environments. For single-item environments, the second-price auction with an anonymous reserve was shown to be between and two and four approximation by Hartline and Roughgarden (2009).

The prophet inequality theorem was proven by Samuel-Cahn (1984) and the connection between prophet inequalities and mechanism design was first made by Taghi-Hajiaghayi et al. (2007). Chawla et al. (2010) studied approximation of the optimal mechanism via oblivious and sequential posted pricings. They showed, via the prophet inequality, that a uniform virtual pricing is a two approximation for single-item environments. For k-unit environments, Taghi-Hajiaghayi et al. (2007) give a generalized prophet inequality with an upper bound of $(1+\sqrt{8/k \ln k})$ for sufficiently large k; an analogous approximation bound for uniform virtual pricing holds. Beyond single- and multi-unit environments, Chawla et al. (2010) showed that oblivious posted pricings give a three approximation for graphical matroid environments and upper bounded the approximation factor for general matroids of rank k as logarithmic in k. As of this writing, it is unknown whether there is an oblivious posted pricing give constant approximations for general matroids. On the other hand, Kleinberg and Weinberg (2012) show that there is an adaptive posted pricing that obtains a two approximation for any arrival order of the agents. This adaptive posted pricing determines the price to offer an agent when it arrives and this price can be based on the set of agents who have previously arrived and potentially been served.[9] See Alaei (2011) for a general framework for adaptive posted pricing.

The usage of the optimal ex ante mechanism as an upper bound on the optimal mechanism is from Chawla et al. (2007) and Alaei (2011). The approximation factor of sequential posted pricings were first studied by Chawla et al. (2010) they proved the $e/e-1$ approximation for single-item environments, a two approximation for matroid environments, and constant approximations for several other environments. The connection to correlation gap and the $e/e-1$ approximation for matroid environments was observed by Yan (2011) by way of the correlation gap theorem of Agrawal et al. (2010) for submodular set functions. Yan also gave the improved analysis for multi-unit auctions which shows that as the number k of available units increases the approximation factor from sequential posted pricing converges to one.

The non-game-theoretic analysis of the optimality of the greedy-by-value algorithm under matroid feasibility was initiated by Joseph Kruskal (1956) and there are books written solely on the structural properties of matroids, see e.g., Oxley (2006) or Welsh (2010). Mechanisms based on the greedy-by-value algorithm were first studied by Lehmann et al. (2002) who showed that even when these algorithms are not optimal, mechanisms derived from them are incentive compatible (cf. Chapter 8). The first comprehensive study of the revenue of the surplus maximizing mechanism in matroid environments was given by Talwar (2003); for instance, he proved critical values for matroid environments are given by the best replacement. The revenue monotonicity for matroid environments and non-monotonicity for non-matroids is discussed by Ausubel and Milgrom (2006), Day and Milgrom (2007), and Dughmi et al. (2009).

The amenability to approximation of environments with value distributions satisfying the monotone hazard rate as been observed several times, e.g., by Hartline et al. (2008), Hartline and Roughgarden (2009), and Bhattacharya et al. (2010). The structural comparison that shows that the optimal revenue is an $e \approx 2.718$ approximation to the optimal social surplus for for downward-closed, monotone-hazard-rate environments was given by Dhangwatnotai et al. (2010).

[9] Note that both the sequential posted pricings and oblivious posted pricings considered in this chapter fix the prices that each agent will receive before the mechanism is run.

5
Prior-independent Approximation

In the last two chapters we discussed mechanism that performed well for a given Bayesian prior distribution. Assuming the existence of such a Bayesian prior is natural when deriving mechanisms for games of incomplete information as the Bayes-Nash equilibrium concept requires a prior distribution that is common knowledge. In this chapter we will relax the assumption the designer has knowledge of the prior distribution and is able to tune the parameters of her mechanism with it. The goal of *prior-independent* mechanism design is to identify a single mechanism that has good performance for all distributions in a large family of relevant distributions, e.g., the family of i.i.d. regular distributions.

As is evident from our analysis of Bayesian optimal auctions, e.g., for profit maximization, for any auction that one might consider good for one prior, there is another prior for which another auction performs strictly better. This consequence is obvious because optimal auctions for distinct distributions are generally distinct. So, while no single auction is optimal for all value distributions, there may be a single auction that is approximately optimal across a wide range of distributions.

In this chapter we will take two approaches to prior-independent mechanism design. The first approach considers "resource" augmentation. We will show that in some environments the (prior-independent) surplus maximization mechanism with increasing competition, e.g., by recruiting more agents, earns more revenue than the revenue-optimal mechanism without the increased competition. The second approach is to design mechanisms that do a little market analysis on the fly. Via this second approach, we will show that for a large class of environments there is a single mechanism that approximates the revenue of the optimal mechanism.

5.1 Motivation

Since prior-independence is not without loss it is important to consider its motivation; however, before doing so recall the original justification for the common prior assumption (see Section 2.3). Auctions and mechanisms are games of incomplete information and in such games, in order to argue about strategic choice, we needed to formalize how players deal with uncertainty. We did this by assuming a Bayesian prior. In a *Stackelberg game*, instead of moving simultaneously, players make actions in a prespecified order. We can view mechanism design as a two stage Stackelberg game where the designer is a player who moves first and the agents are players who (simultaneously) move second. To analyze the Bayes-Nash equilibrium in such a Stackelberg game, the designer bases her strategy on the common prior. Without such prior knowledge, the problem of predicting the designer's strategy is ill posed. Thus, in so far as the theory of mechanism design should describe (or predict) the outcome of a game, within the standard equilibrium concept for games of incomplete information, a prior assumption is necessary.

As discussed in Chapter 1, in addition to being descriptive, the theory of mechanism design should be prescriptive. It should suggest to a designer how to solve a given mechanism design problem that she may confront. If the designer does not have prior information, then she cannot directly employ the suggestions of Bayesian mechanism design. The Bayesian theory of mechanism design is, thus, incomplete in so far as it would require the designer to acquire distribution information from "outside the system." In contrast, a prior-independent mechanism is required to solve both information acquisition and incentive problems and, therefore, must insure that loses due to inaccuracies in information acquisition the interplay between information acquisition and incentives are properly accounted for.

It is important to consider the incentives of information acquisition within the mechanism design problem; even if the designer has knowledge of a prior distribution, it may be problematic to employ this knowledge in a mechanism. Suppose the designer obtained her prior knowledge from previous market experience. The problem with designing the mechanism with this knowledge is that the earlier agents may strategize so that information about their preferences is not exploited by the designer later. For example, a monopolist who cannot commit not to lower the sale price in the future cannot sell at a high price now (see Exercise 5.1).

It is similarly important to consider the loses due to inaccuracies in information acquisition within the mechanism design problem. To learn the prior a designer could perform a *market analysis*, for example, by hiring a marketing firm to survey the market and provide distributional estimates of agent preferences. This mode of operation is quite reasonable in large markets. However, in large markets mechanism design is not such an interesting topic; each agent will have little impact on the others and therefore the designer may as well stick to posted-pricing mechanisms. Indeed, for commodity markets posted prices are standard in practice. Mechanisms, on the other hand, are most interesting in small, a.k.a., *thin*, markets. Contrast the large market for automobiles to the thin market for spacecrafts. There may be five organizations in the world in the market for spacecrafts; how would a designer optimize a mechanism for selling them? First, even if the agents' values do come from a distribution, the only way to sample from the distribution is to interview the agents themselves. Second, even if we did interview the agents, we could obtain at most five data points. This sample size is hardly enough for statistical approaches to be able to estimate the distribution of agent values. A motivating question this perspective raises, and one that is closely tied to prior-independent mechanism design, is: How many samples from a distribution are sufficient for the design of an approximately optimal mechanism?

There are other reasons to consider prior-independent mechanism design besides the questionable origin of prior information. Perhaps the most striking of which is the frequent inability of a designer to redesign a new mechanism for each scenario in which she wishes to run a mechanism. This is not just a concern; in many settings, it is a principle. Consider the standard Internet routing protocol IP. This is the protocol responsible for sending emails, browsing web pages, streaming video, etc. Notice that the workloads for each of these tasks is quite different. Emails are small and can be delivered with several minutes delay without issue. Web pages are small, but must be delivered immediately. Comparably, video streaming permits high latency but requires continuous bandwidth. It would be impractical to install new protocols in Internet routers each time a new network usage pattern arises. Instead, a protocol for computer networks, such as IP, should work pretty well in any setting, even ones well beyond the imaginations of the original designers.

5.2 "Resource" Augmentation

In this section we describe a classical result from auction theory that shows that a little more competition in a surplus maximizing mechanism revenue dominates the revenue maximizing mechanism without the increased competition. From an economic point of view this result questions the *exogenous-participation* assumption, i.e., that there a certain number of agents, say n, that will participate in the mechanism. If, for instance, agents only participate in the mechanism when their utility from doing so is large enough, i.e., with *endogenous participation*, then running an optimal mechanism may decrease participation and thus result in a lower revenue than the surplus maximizing mechanism.

On the other hand, the suggestion of this result, that slightly increasing competition can ensure good revenue, is inherently prior independent. The designer does not need to know the prior distribution to market her service so as to attract more agent participation.

5.2.1 Single-item Environments

The following theorem is due to Jeremy Bulow and Peter Klemperer and is known as the Bulow-Klemperer Theorem.

Theorem 5.1 *For i.i.d. regular single-item environments, the expected revenue of the second-price auction with $n+1$ agents is at least that of the optimal auction with n agents.*

Proof First consider the following question. What is the optimal single-item auction for $n+1$ agents that always sells the item? The requirement that the item always be sold implies that, even if all virtual values are negative, a winner must still be selected. Clearly the optimal such auction is the one that assigns the item to the agent with the highest virtual value (cf. Corollary 3.12). Since the distribution is i.i.d. and regular, the agent with the highest virtual value is the agent with the highest value. Therefore, this optimal auction that always sells the item is the second-price auction.

Now consider an $(n+1)$-agent mechanism LB that runs the optimal auction on agents $1, \ldots, n$ and if this auction fails to sell the item, it gives the item away for free to agent $n+1$. Obviously, LB's expected revenue is equal to the expected revenue of the optimal n-agent auction. It is, however, an $(n+1)$-agent auction that always sells the item. Therefore, its revenue is a lower bound on that of the optimal $(n+1)$-agent auction that always sells.

We conclude that the expected revenue of the second-price auction with $n+1$ agents is at least that of LB which is equal to that of the optimal auction for n agents. □

This resource augmentation result provides the beginning of a prior-independent theory for mechanism design. For instance, we can easily obtain a prior-independent approximation result as a corollary to Theorem 5.1 and Theorem 5.2, below.

Theorem 5.2 *For i.i.d. single-item environments the optimal $(n-1)$-agent auction is an $n/n-1$ approximation to the optimal n-agent auction.*

Proof See Exercise 5.2. □

Corollary 5.3 *For i.i.d. regular single-item environments with $n \geq 2$ agents, the second-price auction is an $n/n-1$ approximation to the optimal auction revenue.*

5.2.2 Multi-unit and Matroid Environments

Unfortunately, the "just add a single agent" result fails to generalize beyond single-item environments. Consider a multi-unit environment; is the revenue of the $(k+1)$st-price auction (i.e., the one that sells a unit to each of the k highest-valued agents at the $(k+1)$st highest value) with $n+1$ agents at least that of the optimal k-unit auction with n agents? No.

Example 5.4 For large n consider an n-unit environment and agents with uniformly distributed values on $[0, 1]$. With $n+1$ agents, the expected revenue of the $(n+1)$st-price auction on $n+1$ agents is about one as there are n winners and the $(n+1)$st value is $1/n+2 \approx 1/n$ in expectation.[1] On the other hand, the optimal auction with n agents will post a price of $1/2$ to each agent and achieve an expected revenue of $n/4$.

The resource augmentation result does extend, and in a very natural way, but more than a single agent must be recruited. For k-unit environments we have to recruit k additional agents. Notice that to extend the proof of Theorem 5.1 to a k-unit environment we can define the auction LB for $n+k$ agents to run the optimal n-agent auction on agents $1, \ldots, n$

[1] In expectation, uniform random variables evenly divide the interval they are over.

and to give any remaining units to agents $n+1, \ldots, n+k$. The desired conclusion follows. In fact, this argument can be extended to matroid environments. Of course matroid set systems are generally asymmetric, so we have to be specific as to the role with respect to the feasibility constraint of the added agents. The result is more intuitive when stated in terms of removing agents from the optimal mechanism instead of adding agents to surplus maximization mechanism, though the consequence is analogous. Recall from Section 4.6 that a base of a matroid is a feasible set of maximal cardinality.

Theorem 5.5 *For any i.i.d. regular matroid environment the expected revenue of the surplus maximization mechanism is at least that of the optimal mechanism in the environment obtained by removing any set of agents that corresponds to a base of the matroid.*

Recall that by the augmentation property of matroids, all bases are the same size. Notice that the theorem implies the aforementioned result for k-unit environments as any set of k agents forms a base of the k-uniform matroid. Similarly, for transversal matroids, which model constrained matching markets, recruiting a new base requires one additional agent for each of the items.

5.3 Single-sample Mechanisms

While the assumption that it is possible to recruit an additional agent seems not to be too severe, once we have to recruit k new agents in k-unit environments or a new base for matroid environments, the approach seems less actionable. In this section we will show that a single additional agent is enough to obtain a good approximation to the optimal auction revenue. We will not, however, just add this agent to the market; instead, we will use this agent for market analysis.

In the opening of this chapter we discussed the need to connect the size of the sample for market analysis with the size of the actual market. In this context, the assumption that the prior distribution is known is tantamount to assuming that an infinitely large sample is available for market analysis. In this section we show that this impossibly large sample can be approximated by a single sample from the distribution.

Definition 5.1 The *surplus maximization mechanism with lazy reserves* $\hat{\boldsymbol{v}}$ is the following:

(i) simulate the surplus maximization mechanism on the bids,

$$(\boldsymbol{x}^{\dagger}, \boldsymbol{p}^{\dagger}) \leftarrow \mathrm{SM}(\boldsymbol{v}),$$

(ii) serve the winners of the simulation who exceed their reserve prices,

$$x_i = \begin{cases} x_i^{\dagger} & \text{if } v_i \geq \hat{v}_i \\ 0 & \text{otherwise, and} \end{cases}$$

(iii) charge the winners (with $x_i = 1$) their critical values $p_i = \max(\hat{v}_i, p_i^{\dagger})$,

where SM denotes the surplus maximization mechanism.

The *lazy single-sample-reserve* mechanism sets $\hat{\boldsymbol{v}} = (\hat{v}, \ldots, \hat{v})$ for $\hat{v} \sim F$. The *lazy monopoly-reserve* mechanism sets $\hat{\boldsymbol{v}} = \hat{\boldsymbol{v}}^{\star}$.

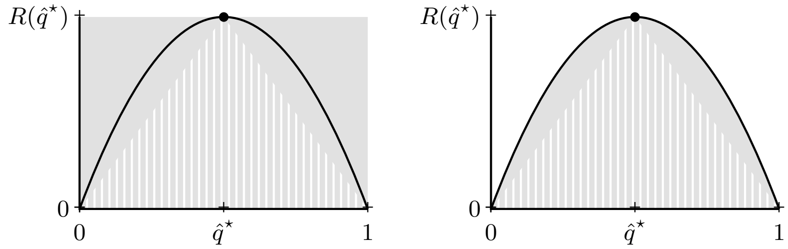
Proposition 5.6 *The surplus maximization mechanism with lazy reserves is dominant strategy incentive compatible.*

In comparison to the surplus maximization mechanism with reserve prices discussed in Chapter 4, where the reserve prices are used filter out low-valued agents before finding the surplus maximizing set (i.e., eagerly), lazy reserve prices filter out low-valued agents after finding the surplus maximizing set. It is relatively easy to find examples of downward-closed environments for which the order in which the reserve is applied affects the outcome (see Exercise 5.3). On the other hand, matroid environments, which include single-item and multi-unit environments, are distinct in that the order in which an anonymous reserve price is imposed does not change the auction outcome. Thus, for i.i.d. matroid environments we will not specify the order, i.e., lazy versus eager, of the reserve pricing.

5.3.1 The Geometric Interpretation

Consider a single-agent environment. The optimal auction in such an environment is simply to post the monopoly price as a take-it-or-leave-it offer. In comparison, the single-sample-reserve mechanism would post a random price that is drawn from the same distribution as the agent's value is drawn. We will give a geometric proof that shows that for regular distributions, the revenue from posting such a random price is within a factor of two of that of the (optimal) monopoly price.

This statement can be viewed as the $n = 1$ special case of the Theorem 5.1, i.e., that the two-agent second-price auction obtains at least the (one-agent) monopoly revenue. In a two-agent second-price auction each



(a) Revenue curve and REF (grey) (b) APX (gray) and inscribed triangle

Figure 5.1 The revenue curve (black line) for the uniform distribution is depicted. REF is the area of the rectangle (gray); by geometry the area of the inscribed triangle (white striped) is $1/2$ REF. APX is the area under the revenue curve (gray); by convexity it is lower bounded by the area of the inscribed triangle (white striped). Thus, REF $\geq$ APX $\geq$ $1/2$ REF.

agent is offered the a price equal to the value of the other, i.e., a random price from the distribution. Therefore, the two-agent second-price auction obtains twice the revenue of a single sample reserve. The result showing that the single-sample revenue is at least half of the monopoly revenue then implies that the two-agent second-price auction obtains at least the (one-agent) monopoly revenue.

Lemma 5.7 *For a regular single-agent environment, posting a random price from the agent's value distribution obtains at least half the revenue from posting the (optimal) monopoly price.*

Proof Let $R(\cdot)$ be the agent's revenue curve. Let $\hat{q}^\star$ be the quantile corresponding to the monopoly price, i.e., $\hat{q}^\star = \operatorname{argmax}_{\hat{q}} R(\hat{q})$. The expected revenue from (optimal) monopoly pricing is REF $= R(\hat{q}^\star)$; this revenue is represented in Figure 5.1(a) by the area of the rectangle (grey) of width one and height $R(\hat{q}^\star)$. Recall that drawing a random value from the distribution is equivalent to drawing a uniform quantile. The expected revenue from the corresponding random price is APX $= \mathbf{E}_{\hat{q}}[R(\hat{q})] = \int_0^1 R(\hat{q})\,\mathrm{d}\hat{q}$; this revenue is depicted in Figure 5.1(b) by the area below the revenue curve (grey). This area is convex because the revenue curve is concave; therefore, by geometry it contains an inscribed triangle with vertices corresponding to 0, $\hat{q}^\star$, and 1 on the revenue curve (Figure 5.1, white striped). This triangle has width one, height REF $= R(\hat{q}^\star)$, and therefore its area is equal to $1/2$ REF. Thus, APX $\geq 1/2$ REF. ☐

Example 5.8 For the uniform distribution where $R(\hat{q}) = \hat{q} - \hat{q}^2$, the quantities in the proof of Lemma 5.7 can be easily calculated:

$$\begin{aligned} \mathrm{REF} &= R(\hat{q}^\star) = 1/4 \\ &\geq \mathrm{APX} = \mathbf{E}_{\hat{q}\sim U[0,1]}[R(\hat{q})] = 1/6 \\ &\geq 1/2\,\mathrm{REF} = 1/8. \end{aligned}$$

5.3.2 Monopoly versus Single-sample Reserves

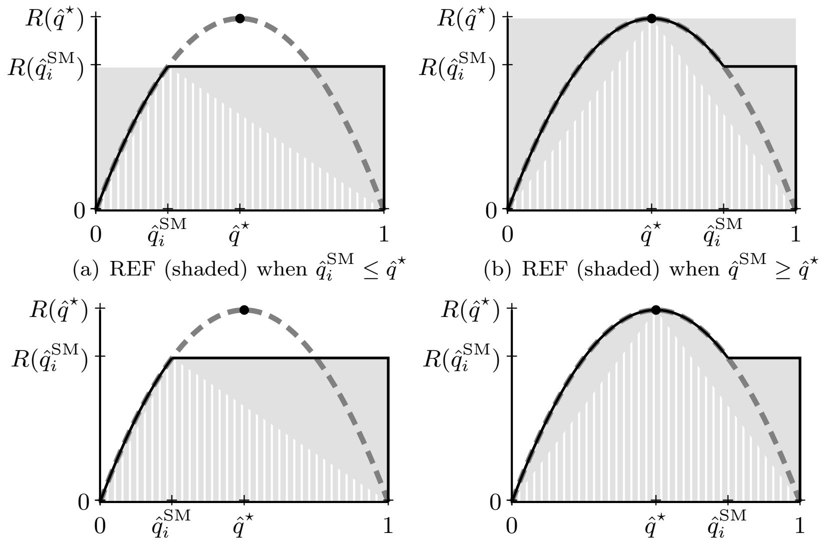
The geometric interpretation above is almost all that is necessary to show that the lazy single-sample-reserve mechanism is a good approximation to the optimal mechanism. We will show the result in two steps. First we will show that the lazy single-sample-reserve mechanism is a good approximation to the lazy monopoly-reserve mechanism. Then we argue that this lazy monopoly-reserve mechanism is approximately optimal.

Theorem 5.9 *For i.i.d. regular downward-closed environments, the expected revenue of the lazy single-sample-reserve mechanism is at least half of that of the lazy monopoly-reserve mechanism.*

Proof With the values $\boldsymbol{v}_{-i}$ of the other agents fixed, we will argue the stronger result that the contribution to the expected revenue from any agent i (Alice) in the lazy single-sample-reserve mechanism is at least half of that in the lazy monopoly-reserve mechanism (in expectation over her value and the sampled reserve). Let REF denote the lazy monopoly-reserve mechanism and Alice's contribution to its revenue, and let APX denote the lazy single-sample-reserve mechanism and her contribution to its revenue (again, both for fixed $\boldsymbol{v}_{-i}$).

Denote the monopoly quantile by $\hat{q}^\star$, denote the critical quantile for Alice in the surplus maximization mechanism with no reserve by $\hat{q}_i^{\mathrm{SM}}$, and denote the quantile of a lazy reserve by $\hat{q}$. Alice's wins in the surplus maximization mechanism with this lazy reserve when her quantile is below $\min(\hat{q}, \hat{q}_i^{\mathrm{SM}})$. For a fixed $\hat{q}_i^{\mathrm{SM}}$, the revenue from Alice, in expectation over her own quantile and as a function of the lazy reserve quantile $\hat{q}$, induces the revenue curve $R^\dagger(\hat{q}) = R(\min(\hat{q}, \hat{q}_i^{\mathrm{SM}}))$. Figure 5.2 depicts Alice's original revenue curve $R(\cdot)$ and this induced revenue curve $R^\dagger(\cdot)$ in the cases that $\hat{q}_i^{\mathrm{SM}} \leq \hat{q}^\star$ and $\hat{q}_i^{\mathrm{SM}} \geq \hat{q}^\star$.

Alice's expected payment in the lazy monopoly-reserve mechanism is $\mathrm{REF} = R^\dagger(\hat{q}^\star)$ which is geometrically the maximum height of the revenue curve $R^\dagger$; and her expected payment in the lazy single-sample-reserve mechanism, where $\hat{q} \sim U[0,1]$, is $\mathrm{APX} = \mathbf{E}_{\hat{q}}[R^\dagger(\hat{q})]$. We conclude with the same geometric argument as in Lemma 5.7 that relates REF



(a) REF (shaded) when $\hat{q}_i^{\mathrm{SM}} \leq \hat{q}^\star$ (b) REF (shaded) when $\hat{q}^{\mathrm{SM}} \geq \hat{q}^\star$

(c) APX (shaded) when $\hat{q}_i^{\mathrm{SM}} \leq \hat{q}^\star$ (d) APX (shaded) when $\hat{q}_i^{\mathrm{SM}} \geq \hat{q}^\star$

Figure 5.2 In each diagram, the revenue curve $R(\cdot)$ (thick, dashed, grey line) of the uniform distribution and the induced revenue curve $R^\dagger(\cdot) = R(\max(\cdot, \hat{q}_i^{\mathrm{SM}}))$ (thin, solid, black line). On the left is the case that $\hat{q}_i^{\mathrm{SM}} \leq \hat{q}^\star$; on the right is the case that $\hat{q}_i^{\mathrm{SM}} \geq \hat{q}^\star$. On the top the revenue of REF is shaded grey; on the bottom the revenue of APX is shaded in gray. The inscribed triangles (white striped) have area $1/2$ REF. Both on the left and on the right REF $\geq$ APX $\geq$ $1/2$ REF.

to a rectangle, APX to the area under the induced revenue curve, and $1/2$ REF to the area of an inscribed triangle (see Figure 5.2). □

5.3.3 Optimal versus Lazy Single-sample-reserve Mechanism

We have shown that lazy single-sample reserve pricing is almost as good as lazy monopoly reserve pricing. We now connect lazy monopoly reserve pricing to the revenue-optimal mechanism to show that the lazy single-sample mechanism is a good approximation to the optimal mechanism.

For i.i.d. matroid environments, as discussed above, lazy monopoly reserve pricing is identical to (eager) monopoly reserve pricing. Moreover, surplus maximization with the monopoly reserve is revenue optimal (Proposition 4.24). We conclude the following corollary. Recall that matroid environments include multi-unit environments as a special case.

Corollary 5.10 *For any i.i.d. regular matroid environment, the revenue of the single-sample-reserve mechanism is a two approximation to that of the revenue-optimal mechanism.*

Theorem 4.43 shows that for monotone-hazard-rate distributions the surplus maximization mechanism with (eager) monopoly reserves is a two approximation to the optimal mechanism; however, as in downward-closed environments eager and lazy reserve pricing are not identical (see Exercise 5.3), we have slightly more work to do. Recall Theorem 4.40 which states that for MHR distributions the optimal revenue and optimal social surplus are within an e factor of each other. One way to prove this theorem is, in fact, by showing that the revenue of the surplus maximization mechanism with lazy monopoly reserve prices is an e approximation to the optimal social surplus and hence so is the optimal revenue (see Exercise 4.25). Combining this observation with Theorem 5.9 it is evident that the lazy single-sample-reserve mechanism is a $2e$ approximation. The approximation bound can be improved to four via a more careful analysis that we omit.

Theorem 5.11 *For any i.i.d. monotone-hazard-rate downward-closed environment, the revenue of the lazy single-sample-reserve mechanism is a four approximation to that of the revenue-optimal mechanism.*

5.4 Prior-independent Mechanisms

We now turn to mechanisms that are completely prior independent. Unlike the mechanisms of the preceding section, these mechanisms will not require any distributional information, not even a single sample from the distribution. We will, however, still assume that there is a distribution.

Definition 5.2 A mechanism APX is a *prior-independent β approximation* if

$$\forall \boldsymbol{F}, \quad \mathbf{E}_{\boldsymbol{v}\sim\boldsymbol{F}}[\mathrm{APX}(\boldsymbol{v})] \geq \tfrac{1}{\beta}\mathbf{E}_{\boldsymbol{v}\sim\boldsymbol{F}}[\mathrm{REF}_{\boldsymbol{F}}(\boldsymbol{v})]$$

where $\mathrm{REF}_{\boldsymbol{F}}$ is the optimal mechanism for distribution $\boldsymbol{F}$ and "$\forall \boldsymbol{F}$" quantifies over all distributions in a given family.

The central idea behind the design of prior-independent mechanisms is that a small amount of market analysis can be done while the mechanism is being run. For example, the bids of some agents can be used as a market analysis to calculate the prices to be offered to other agents.

Consider the following k-unit auction:

(i) Solicit bids,
(ii) randomly reject an agent j, and
(iii) run the $(k+1)$st-price auction with reserve v_j on $\boldsymbol{v}_{-j}$.

This auction is clearly incentive compatible. Furthermore, it is easy to see that it is a $2n/n-1$ approximation for $n \geq 2$ agents with values drawn i.i.d. from a regular distribution. This follows from the fact that rejecting a random agent loses at most a $1/n$ fraction of the optimal revenue (Theorem 5.2), and from the previous single-sample-reserve result (Corollary 5.10). This approximation bound is clearly worst for $n = 2$ where it guarantees a four approximation. The same approach can be applied to matroid and downward-closed environments as well; instead, we will discuss a slightly more sophisticated approach.

5.4.1 Digital Good Environments

An important single-dimensional agent environment is that of a *digital good*, i.e., one where there is little or no cost for duplication. In terms of single-dimensional environments for mechanism design, the cost function for digital goods is $c(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}$; in other words, all outcomes are feasible. Digital goods can also be viewed as the special case of k-unit auctions where $k = n$. Therefore the mechanism above obtains a $2n/n-1$ approximation.

There are a number of approaches for improve this mechanism to remove the $n/n-1$ from the approximation factor. The following two approaches are natural.

Definition 5.3 For digital-good environments,

- the (digital good) *pairing auction* arbitrarily pairs agents and runs the second-price auction on each pair (assuming n is even), and
- the (digital good) *circuit auction* orders the agents arbitrarily (e.g., lexicographically) and offers each agent a price equal to the value of the preceding agent in the order (the first agent is offered the last agent's value).

The *random pairing auction* and the *random circuit auction* are the variants where the pairing or circuit is selected randomly.

Theorem 5.12 *For i.i.d. regular digital-good environments, any auction wherein each agent is offered the price of another random or arbitrary (but not value dependent) agent is a two approximation to the optimal auction revenue.*

The proof of this theorem follows directly from the geometric analysis of single-sample pricing (Lemma 5.7). Clearly, the pairing and circuit auctions satisfy the conditions of the above theorem. In conclusion, in i.i.d. environments it is relatively easy to obtain samples from the distribution while running a mechanism.

5.4.2 General Environments

We now adapt the results for digital goods to general environments. Consider the surplus maximizing mechanism with a lazy reserve price. First, the surplus maximizing set is found. Second, the agents that do not meet the reserve are rejected. We can view this second step as a digital good auction as, once we have selected a surplus maximizing feasible set, downward closure requires that any subset is feasible. The main idea of this section is to replace the lazy reserve part of the single-sample mechanism with any approximately optimal digital good auction (e.g., the circuit or pairing auction).

Consider the following definition of mechanism composition (cf. Exercise 5.9). Notice that the mechanisms we have been discussing can all be interpreted as calculating a critical value for each agent, serving each agent whose value exceeds her critical value, and charging each served agent her critical value. In fact, by Corollary 2.14, any randomization over deterministic dominant strategy incentive compatible mechanisms admits such an interpretation.

Definition 5.4 The *parallel composite* $\mathcal{M}$ of two (randomizations over) deterministic DSIC mechanisms, $\mathcal{M}^{\dagger}$ and $\mathcal{M}^{\ddagger}$ is as follows:

(i) Calculate the critical values $\hat{\boldsymbol{v}}^{\dagger}$ and $\hat{\boldsymbol{v}}^{\ddagger}$ of $\mathcal{M}^{\dagger}$ and $\mathcal{M}^{\ddagger}$, respectively.
(ii) The critical values of $\mathcal{M}$ are $\hat{v}_i = \max(\hat{v}_i^{\dagger}, \hat{v}_i^{\ddagger})$ for each agent i.
(iii) Allocation and payments are $x_i = x_i^{\dagger} x_i^{\ddagger}$ and $p_i = \hat{v}_i x_i$ for all i, respectively.

Notice that in the parallel composite, $\mathcal{M}$, the set of agents served is the intersection of those served by $\mathcal{M}^{\dagger}$ and $\mathcal{M}^{\ddagger}$. By downward closure, then, the outcome of the composition is feasible as long as the outcome of one of $\mathcal{M}^{\dagger}$ or $\mathcal{M}^{\ddagger}$ is feasible. The mechanism is dominant strategy incentive compatible by its definition via critical values and Corollary 2.14.

Proposition 5.13 *The parallel composite of two (randomizations over) deterministic dominant strategy incentive compatible mechanisms is dominant strategy incentive compatible and, if one of the mechanisms is feasible, feasible.*

Notice that the surplus maximization mechanism with a lazy reserve price is the composition, in the manner above, of the surplus maximization mechanism with a (digital good) uniform posted pricing. Consider composing the surplus maximization mechanism with either the pairing or circuit auctions. Both of the theorems below follow from analyses similar to that of the single-sample-reserve mechanism.

Definition 5.5 For downward-closed environments,

- the *pairing mechanism* is the parallel composite of the surplus maximization mechanism with the (digital goods) pairing auction, and
- the *circuit mechanism* is the parallel composite of the surplus maximization mechanism with the (digital goods) circuit auction.

Theorem 5.14 *For i.i.d. regular matroid environments, the revenues of the pairing and circuit mechanisms are two approximations to the optimal mechanism revenue.*

Theorem 5.15 *For i.i.d. monotone-hazard-rate downward-closed environments, the revenues of the pairing and circuit mechanisms are four approximations to the optimal mechanism revenue.*

The results presented in this chapter are representative of the techniques for the design and analysis of prior-independent approximation mechanisms; however, a number of extensions are possible. If we use more that one samples from the distribution, bounds for regular distributions can be improved and bounds for irregular distributions can be obtained. Both of these directions will be taken up during our discussion of prior-free mechanisms in Chapter 6. Finally, the i.i.d. assumption can be relaxed, either by assuming that agents are partitioned by demographic (see Exercise 5.10) or by an ordering assumption.

Exercises

5.1 Consider the sale of a magazine subscription over two periods to a single agent who has a linear uniform additive value for each period's issue of the magazine. Her value v is drawn from a regular distribution F and if x_1, x_2, p_1, and p_2 denote her allocation and payments in each period then her utility is $v(x_1 + x_2) - p_1 - p_2$. In each period, the designer publishes her mechanism and then the agent bids for receiving that period's issue of the magazine.

(a) Suppose that the designer can commit to the mechanism to be used in period two before the agent bids in period one, describe the revenue optimal mechanisms and the equilibrium behavior of the agent.

(b) Suppose that the designer cannot commit to the mechanism to be used in period two before the agent bids in period one, describe the revenue optimal mechanisms and the equilibrium behavior of the agent.

(c) Compare the revenues from the previous steps for the uniform distribution.

5.2 Prove Theorem 5.2: For i.i.d. single-item environments the optimal auction with $n-1$ agents auction is an ${}^{n}\!/\!{}_{n-1}$ approximation to the optimal auction with n agents.

5.3 Consider the surplus maximization mechanism with an anonymous reserve that is either lazy or eager.

(a) Find a valuation profile, downward-closed feasibility constraint, and anonymous reserve price such that different outcomes result from lazy and eager reserve pricing.

(b) Prove that for anonymous reserve pricing in matroid environments, lazy and eager reserve pricing give the same outcome.

5.4 Consider a regular single-agent environment. Show that posting the median price from the agent's value distribution obtains at least half the revenue from posting the monopoly price. The median price for an agent with inverse demand function $V(\cdot)$ is $\hat{v} = V({}^{1}\!/\!{}_{2})$.

5.5 In Example 5.8 it is apparent that the approximation bound of a sample reserve to the monopoly reserve for a uniform distribution is ${}^{3}\!/\!{}_{2}$. Use this bound to derive better bounds for the lazy single-sample-reserve mechanism versus the lazy monopoly-reserve mechanism. In particular, show that if the single-agent approximation of sample reserve to monopoly reserves is β then the the same bound holds in general for the lazy single-sample-reserve and lazy monopoly reserve mechanism.

5.6 Consider the surplus maximization mechanism with lazy monopoly reserve prices in downward-closed monotone-hazard-rate environments.

(a) Show that in a single-agent environment, that its expected surplus is at most twice its expected revenue.

(b) Show that in a downward-closed environment, that its expected surplus is at most twice its expected revenue.

5.7 Suppose we are in a non-identical environment, i.e., agent i's value is drawn from independently from distribution F_i, and suppose the mechanism can draw one sample from each agent's distribution.

(a) Give a constant approximation mechanism for regular, matroid environments (and give the constant).

(b) Give a constant approximation mechanism for monotone-hazard-rate, downward-closed environments (and give the constant).

5.8 This chapter has been mostly concerned with the profit objective. Suppose we wished to have a single mechanism that obtained good surplus and good profit.

(a) Show that surplus maximization with monopoly reserves is not generally a constant approximation to the optimal social surplus in regular, single-item environments.

(b) Show that the lazy single sample mechanism is a constant approximation to the optimal social surplus in i.i.d., regular, matroid environments.

(c) Investigate the Pareto frontier between prior-independent approximation of surplus and revenue. I.e., if a mechanism is an α approximation to the optimal surplus and a β approximation to the optimal revenue, plot it as point $(1/\alpha, 1/\beta)$ in the positive quadrant.

5.9 Define the *sequential composite* $\mathcal{M}$ of two mechanism $\mathcal{M}^\dagger$ and $\mathcal{M}^\ddagger$ as first simulating $\mathcal{M}^\dagger$, second simulating $\mathcal{M}^\ddagger$ on the winners of $\mathcal{M}^\dagger$, and serving the agents served by the second mechanism at the maximum of their prices in the two mechanisms.

(a) Give an example of deterministic DSIC mechanisms $\mathcal{M}^\dagger$ and $\mathcal{M}^\ddagger$ such that the sequential composite $\mathcal{M}$ is not DSIC.

(b) Show that if $\mathcal{M}^\dagger$ is the surplus maximizing mechanism (and $\mathcal{M}^\ddagger$ is any randomization over DSIC mechanisms) then the composition is DSIC.

(c) Describe a property of the surplus maximizing mechanism as $\mathcal{M}^{\dagger}$ that enables the incentive compatibility of the sequential composite $\mathcal{M}$.

5.10 Suppose the agents are divided into k markets where the value of agents in the same market are identically distributed, e.g., by demographic. Assume that the partitioning of agents into markets is known, but not the distributions of the markets. Assume there are at least two agents in each market. Unrelated to the markets, assume the environment has a downward-closed feasibility constraint.

(a) Give a prior-independent constant approximation to the revenue-optimal mechanism for regular matroid environments.

(b) Give a prior-independent constant approximation to the revenue-optimal mechanism for monotone-hazard-rate downward-closed environments.

Chapter Notes

The resource augmentation result that shows that recruiting one more agent to a single-item auction raises more revenue than setting the optimal reserve price is due to Bulow and Klemperer (1996). The proof of the Bulow-Klemperer Theorem that was presented in this text is due to René Kirkegaard (2006). A generalization of the Bulow-Klemperer Theorem to non-identical distributions was given by Hartline and Roughgarden (2009).

The single-sample mechanism and the geometric proof of the Bulow-Klemperer theorem are due to Dhangwatnotai et al. (2010). They also considered a relaxation of the i.i.d. assumption where there is a known partitioning of the agents into markets, e.g., by demographic or zip code, where there are at least two agents in each market. The pairing auction for digital good environments was proposed by Goldberg et al. (2001); however, in the possibly irregular environments that they considered it does not have good revenue guarantees.

Appendix
Mathematical Reference

Contained herein is reference to mathematical notations and conventions used throughout the text.

A.1 Big-oh Notation

We give asymptotic bounds using big-oh notation. Upper bounds are given with O, strict upper bounds are given with o, lower bounds are given with Ω, strict lower bounds are given with ω, and exact bounds are given with Θ. Formal definitions are given as follows:

Definition A.1 Function $f(n)$ is $O(g(n))$ if there exists a $c > 0$ and $n_0 > 0$ such that

$$\forall n > n_0,\ f(n) \leq c\, g(n).$$

Definition A.2 Function $f(n)$ is $\Omega(g(n))$ if there exists a $c > 0$ and $n_0 > 0$ such that

$$\forall n > n_0,\ f(n) \geq c\, g(n).$$

Definition A.3 Function $f(n)$ is $\Theta(g(n))$ if it is $O(g(n))$ and $\Omega(g(n))$.

Definition A.4 Function $f(n)$ is $o(g(n))$ if it is $O(g(n))$ but not $\Theta(g(n))$.

Definition A.5 Function $f(n)$ is $\omega(g(n))$ if it is $\Omega(g(n))$ but not $\Theta(g(n))$.

A.2 Common Probability Distributions

Common continuous probability distributions are *uniform* and *exponential*. Continuous distributions can be specified by their *cumulative distribution function*, denoted by F, or its derivative $f = F'$, the *probability density function*.

Definition A.6 The *uniform distribution* on support $[a, b]$, denoted $U[a, b]$, is defined as having a constant density function $f(z) = 1/(b-a)$ over $[a, b]$.

For example, the distribution $U[0, 1]$ has distribution $F(z) = z$ and density $f(z) = 1$. The expectation of the uniform distribution on $[a, b]$ is $\frac{a+b}{2}$. The monopoly price for the uniform distribution is $\max(b/2, a)$ (see Definition 3.7).

Definition A.7 The *exponential distribution* with rate λ has distribution $F(z) = 1 - e^{-\lambda z}$ and density $f(z) = \lambda e^{-\lambda z}$. The support of the exponential distribution is $[0, \infty)$.

The exponential distribution with rate λ has expectation $1/\lambda$ and monopoly price $1/\lambda$. The exponential distribution has constant *hazard rate* λ.

A.3 Expectation and Order Statistics

The *expectation* of a random variable $v \sim F$ is its "probability weighted average." For continuous random variables this expectation can be calculated as

$$\mathbf{E}[v] = \int_{-\infty}^{\infty} z\, f(z)\, \mathrm{d}z. \tag{A.1}$$

For continuous, non-negative random variables this expectation can be reformulated as

$$\mathbf{E}[v] = \int_{0}^{\infty} (1 - F(z))\, \mathrm{d}z \tag{A.2}$$

which follows from (A.1) and integration by parts.

An *order statistic* of a set of random variables is the value of the variable that is at a particular rank in the sorted order of the variables. For instance, when a valuation profile $\boldsymbol{v} = (v_1, \ldots, v_n)$ is drawn from a distribution then the ith largest value, which we have denoted $v_{(i)}$, is an order statistic. A fact that is useful for working out examples with the uniform distribution.

Fact A.1 *In expectation, i.i.d. random variables chosen uniformly from a given interval will evenly divide the interval.*

A.4 Integration by Parts

Integration by parts is the integration analog of the product rule for differentiation. We will denote the derivative of a function $\frac{d}{\mathrm{d}z}g(z)$ by $g'(z)$. The product rule for differentiation is:

$$[g(z)\,h(z)]' = g'(z)\,h(z) + g(z)\,h'(z). \tag{A.3}$$

The formula for integration by parts can be derived by integrating both sides of the equation and rearranging.

$$\int g'(z)\,h(z)\,\mathrm{d}z = g(z)\,h(z) - \int g(z)\,h'(z)\,\mathrm{d}z. \tag{A.4}$$

As an example we will derive (A.2) from (A.1). Plug $g(z) = 1 - F(z)$ and $h(z) = z$ into equation A.4.

$$\begin{aligned}
\mathbf{E}[v] &= \int_0^\infty z\,f(z)\,\mathrm{d}z \\
&= -\int_0^\infty h(z)\,g'(z)\,\mathrm{d}z \\
&= -\Big[h(z)\,g(z)\Big]_0^\infty + \int_0^\infty h'(z)\,g(z)\,\mathrm{d}z \\
&= -\Big[z\,(1 - F(z))\Big]_0^\infty + \int_0^\infty 1\,(1 - F(z))\,\mathrm{d}z \\
&= \int_0^\infty (1 - F(z))\,\mathrm{d}z.
\end{aligned}$$

The last equality follows because $z\,(1 - F(z))$ is zero at both zero and ∞.

A.5 Hazard Rates

The *hazard rate* of distribution F (with density f) is $h(z) = \frac{f(z)}{1-F(z)}$ (see Definition 4.12). The distribution has a *monotone hazard rate* (MHR) if $h(z)$ is monotone non-decreasing.

A distribution is completely specified by its hazard rate via the following formula.

$$F(z) = 1 - e^{-\int_{-\infty}^{z} h(z)\,\mathrm{d}z}.$$

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Index