Tutorial: Foundations of Non-truthful Mechanism Design

Part I: Equilibrium Analysis — Tutor: Jason Hartline

Schedule:

Part Ia: 10-10:45am (http://ec20.sigecom.org/tech/tutorial)
Part Ib: 11-11:45am (http://ec20.sigecom.org/tech/tutorial)
Exercises: 12-1pm (http://ec20.sigecom.org/tech/tutorial-exercises)
(https://tinyurl.com/non-truthful-exercises)

Protocol:

During session, panelest will answer clarifying questions in chat.

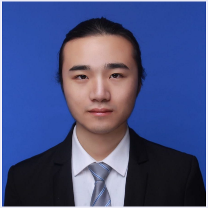
In post-session Q/A, “raise hand” to ask question.

Tutorial Cochairs

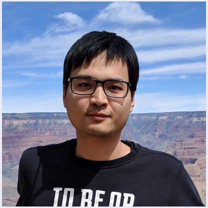
Brendan Lucier

Sigal Oren

Panelists



Yiding Feng



Yingkai Li

Foundations of Non-truthful Mechanism Design

http://jasonhartline.com/tutorial-non-truthful/

Jason Hartline

Northwestern University

hartline@northwestern.edu

EC Tutorial 2020

Context: The Revelation Principle

Mechanism Design: identify mechanism that has good equilibrium.

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- practical mechanisms are not truthful.
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Goal: theory for non-truthful mechanism design.

Part I

Equilibrium Analysis

Warmup: Second-price Auction

Definition (Second-price Auction, SPA)

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2. winner is highest bidder.
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- Expected welfare in equilibrium: $\mathbf{E}\left[\mathrm{v}_{(1)}\right] = 2/3$
- Expected revenue in equilibrium:

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- Expected welfare in equilibrium: $\mathbf{E}\left[v_{(1)}\right] = 2/3$
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Section 2

Single-dimensional Environments

References:

1. Hartline (202?) "Mechanism Design and Approximation" Chapter 2

Single-dimensional Linear Environments

Model

- agents: $\{1, \ldots, n\}$; values: $\mathbf{v} = (v_1, \ldots, v_n)$; bids: $\mathbf{b} = (b_1, \ldots, b_n)$

Single-dimensional Linear Environments

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- welfare: $\sum_i \mathrm{v}_i\, \tilde{x}_i(\mathbf{b})$; revenue: $\sum_i p_i(\mathbf{b})$

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Example (Single-item Environments; First-price Auction)

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Single-dimensional Linear Environments

Model

- agents: $\{1, \ldots, n\}$; values: $\mathbf{v} = (v_1, \ldots, v_n)$; bids: $\mathbf{b} = (b_1, \ldots, b_n)$
- linear utility: $v_i x_i - p_i$ for allocation $x_i \in [0, 1]$ and payment $p_i \in \mathbb{R}$
- feasibility constraint: $\mathscr{X} \subset [0, 1]^n$
- mechanism $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$:
  - ex post bid allocation rule: $\tilde{\boldsymbol{x}} : \mathbb{R}^n \to \mathscr{X}$
  - ex post bid payment rule: $\tilde{\boldsymbol{p}} : \mathbb{R}^n \to \mathbb{R}^n$
- welfare: $\sum_i v_i \, \tilde{x}_i(\mathbf{b})$; revenue: $\sum_i \tilde{p}_i(\mathbf{b})$

Example (Single-item Environments; First-price Auction)

- feasibility constraint: $\mathscr{X} = \{\mathbf{x} \subset [0, 1]^n : \sum_i x_i \leq 1\}$
- highest-bid-wins: $\tilde{\boldsymbol{x}}(\mathbf{b}) \in \text{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i b_i \, x_i$
- winner-pays-bid: $\tilde{p}_i(\mathbf{b}) = b_i \, \tilde{x}_i(\mathbf{b})$

Encorporating Strategies and Prior Distribution

Three Stages of Mechanism Design

- ex ante: before values are drawn ($\mathbf{v}$ random)

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- ex ante: before values are drawn ($\mathbf{v}$ random)
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Three Stages of Mechanism Design

- ex ante: before values are drawn ($\mathbf{v}$ random)
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- ex post: after bids and values are known ($\mathbf{v}$ known)

Encorporating Strategies and Prior Distribution

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- ex ante: before values are drawn ($\mathbf{v}$ random)
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Value Allocation and Payment Rules

Compose ex post mechanism $(\tilde{x}, \tilde{p})$ and bid strategy b:

- ex post allocation rule: $x(\mathbf{v}) = \tilde{x}(b(\mathbf{v}))$
- ex post payment rule: $p(\mathbf{v}) = \tilde{p}(b(\mathbf{v}))$

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Interim Allocation and Payment Rules

Encorporating Strategies and Prior Distribution

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Interim Allocation and Payment Rules

- $\tilde{x}_i(b_i) = \mathbf{E}_{\mathbf{v}}[\tilde{x}_i(b_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid v_i]$;

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Encorporating Strategies and Prior Distribution

Three Stages of Mechanism Design

- ex ante: before values are drawn ($\mathbf{v}$ random)
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- $\tilde{x}_i(\mathrm{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{x}_i(\mathrm{b}_i, b_{-i}(\mathbf{v}_{-i})) \mid \mathrm{v}_i]$; $\tilde{p}_i(\mathrm{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{p}_i(\mathrm{b}_i, b_{-i}(\mathbf{v}_{-i})) \mid \mathrm{v}_i]$
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Encorporating Strategies and Prior Distribution

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- ex ante: before values are drawn (**v** random)
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Note $\boldsymbol{x} : \mathbb{R}^n \to \mathscr{X}$ $\quad \boldsymbol{x} : (\mathbb{R} \to [0,1])^n$

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Note $x : \mathbb{R}^n \to \mathcal{X}$ $x : (\mathbb{R} \to [0,1])^n$ $p : \mathbb{R}^n \to \mathbb{R}^n$ $p : (\mathbb{R} \to \mathbb{R})^n$

Example: First-price Auction

Interim Allocation and Payment Rules

- $\tilde{x}_i(\mathsf{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{\mathcal{x}}_i(\mathsf{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathsf{v}_i]$; $\tilde{p}_i(\mathsf{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{\mathcal{p}}_i(\mathsf{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathsf{v}_i]$
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Example (two agents, uniform values, first-price auction)

equilibrium bids: $b_i(\mathrm{v}_i) = {}^{\mathrm{v}_i}/_2$; feasibility: $\mathscr{X} = \{\mathbf{x} \subset [0,1]^n : \sum_i \mathrm{x}_i \leq 1\}$

	ex post	interim
bid allocation rule		
value allocation rule		

Example: First-price Auction

Interim Allocation and Payment Rules

- $\tilde{x}_i(\mathsf{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{x}_i(\mathsf{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathsf{v}_i]$; $\tilde{p}_i(\mathsf{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{p}_i(\mathsf{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathsf{v}_i]$
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bid allocation rule	$\tilde{\boldsymbol{x}}(\mathbf{b}) \in \text{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i \mathsf{b}_i\, \mathsf{x}_i$	
value allocation rule		

Example: First-price Auction

Interim Allocation and Payment Rules

- $\tilde{x}_i(\mathrm{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{\mathscr{x}}_i(\mathrm{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathrm{v}_i]$; $\tilde{p}_i(\mathrm{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{\mathscr{p}}_i(\mathrm{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathrm{v}_i]$
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	ex post	interim
bid allocation rule	$\tilde{\boldsymbol{\mathscr{x}}}(\mathbf{b}) \in \mathrm{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i \mathrm{b}_i \, \mathrm{x}_i$	$\tilde{x}_i(\mathrm{b}_i) = 2\, \mathrm{b}_i$
value allocation rule		

Example: First-price Auction

Interim Allocation and Payment Rules

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bid allocation rule	$\tilde{\boldsymbol{\mathscr{x}}}(\mathbf{b}) \in \mathrm{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i \mathsf{b}_i\, \mathsf{x}_i$	$\tilde{x}_i(\mathsf{b}_i) = 2\,\mathsf{b}_i$
value allocation rule	$\boldsymbol{\mathscr{x}}(\mathbf{v}) \in \mathrm{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i {}^{\mathsf{v}_i}/_2\, \mathsf{x}_i$	

Example: First-price Auction

Interim Allocation and Payment Rules

- $\tilde{x}_i(\mathrm{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{\mathscr{x}}_i(\mathrm{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathrm{v}_i]$; $\tilde{p}_i(\mathrm{b}_i) = \mathbf{E}_{\mathbf{v}}[\tilde{\mathscr{p}}_i(\mathrm{b}_i, \boldsymbol{b}_{-i}(\mathbf{v}_{-i})) \mid \mathrm{v}_i]$
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Example (two agents, uniform values, first-price auction)

equilibrium bids: $b_i(\mathrm{v}_i) = \mathrm{v}_i/2$; feasibility: $\mathscr{X} = \{\mathbf{x} \subset [0,1]^n : \sum_i \mathrm{x}_i \leq 1\}$

	ex post	interim
bid allocation rule	$\tilde{\boldsymbol{\mathscr{x}}}(\mathbf{b}) \in \operatorname{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i \mathrm{b}_i \, \mathrm{x}_i$	$\tilde{x}_i(\mathrm{b}_i) = 2\,\mathrm{b}_i$
value allocation rule	$\boldsymbol{\mathscr{x}}(\mathbf{v}) \in \operatorname{argmax}_{\mathbf{x} \in \mathscr{X}} \sum_i \mathrm{v}_i/2 \, \mathrm{x}_i$	$x_i(\mathrm{v}_i) = \mathrm{v}_i$

Section 3

Revenue Equivalence and Applications

References:

1. Myerson (1981) "Optimal Auction Design"
2. Chawla, Hartline (2013) "Auctions with unique equilibria"
3. Hartline (202?) "Mechanism Design and Approximation" Chapter 2

Bayes-Nash Equilibrium

Definition (Bayes-Nash equilibrium, BNE)

A strategy profile $\boldsymbol{b}$ such that for all i and v_i, bidding $\mathsf{b}_i = b_i(\mathsf{v}_i)$ is a best response when other agents bid $\boldsymbol{b}_{-i}(\mathbf{v}_{-i})$ with $\mathbf{v}_{-i} \sim \boldsymbol{F}_{-i}|_{\mathsf{v}_i}$.

Notation

- value profile w.o. agent i's value: $\mathbf{v}_{-i} = (\mathsf{v}_1, \ldots, \mathsf{v}_{i-1}, ?, \mathsf{v}_{i+1}, \ldots, \mathsf{v}_n)$
- conditional distribution of $\boldsymbol{F}$ given v_i: $\boldsymbol{F}|_{\mathsf{v}_i}$ (if independent $\boldsymbol{F}|_{\mathsf{v}_i} = \boldsymbol{F}_{-i}$)

Bayes-Nash Equilibrium

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Example (Two agents, uniform values, first-price auction)

Strategies $\boldsymbol{b}$ as "$\forall i,\ b_i(\mathrm{v}_i) = \mathrm{v}_i/2$" is a Bayes-Nash equilibrium.

- values are $U[0,1]$
- bids under $\boldsymbol{b}$ are $U[0, 1/2]$
- best response to bid $U[0, 1/2]$ is $b_i(\mathrm{v}_i) = \mathrm{v}_i/2$

Notation

- value profile w.o. agent i's value: $\mathbf{v}_{-i} = (\mathrm{v}_1, \ldots, \mathrm{v}_{i-1}, ?, \mathrm{v}_{i+1}, \ldots, \mathrm{v}_n)$
- conditional distribution of $\boldsymbol{F}$ given v_i: $\boldsymbol{F}|_{\mathrm{v}_i}$ (if independent $\boldsymbol{F}|_{\mathrm{v}_i} = \boldsymbol{F}_{-i}$)

Characterizing Bayes-Nash Equilibrium

Proposition

$(\boldsymbol{x}, \boldsymbol{p})$ *are induced by BNE of some* $\boldsymbol{b}$, $\boldsymbol{F}$, *and* $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ *if and only if:*

$$\forall\, i,\ v_i,\ z: \quad v_i\, x_i(v_i) - p_i(v_i) \geq v_i\, x_i(z) - p_i(z)$$

(and bids not in the range of $\boldsymbol{b}$ *are weakly dominated.)*

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Theorem (Myerson '81)

$(\boldsymbol{x}, \boldsymbol{p})$ *are induced by BNE of some* $\boldsymbol{b}$, $\boldsymbol{F}$, *and* $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ *if and only if:*

a. *(monotonicity)* x_i *is monotonically non-decreasing*
b. *(payment identity)* $p_i(v_i) = v_i\, x_i(v_i) - \int_0^{v_i} x_i(z)\, dz + p_i(0)$.

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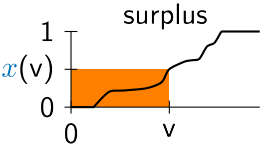

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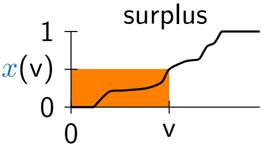

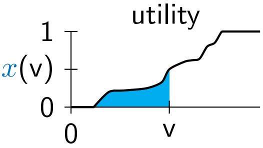

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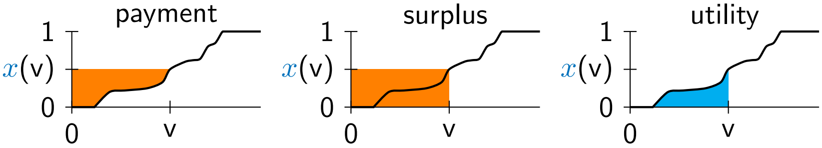

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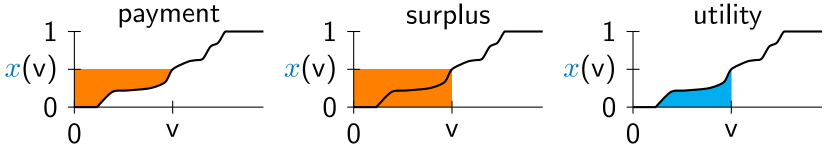
b. *(payment identity)* $p_i(v_i) = v_i\, x_i(v_i) - \int_0^{v_i} x_i(z)\, dz + p_i(0)$.

(and bids not in the range of $\boldsymbol{b}$ *are weakly dominated.)*



Cor: revenue equivalence: same BNE allocation $\Rightarrow$ same BNE revenue.

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Consequence: BNE of FPA is unique, symmetric, and welfare-maximal.

Solving for Bayes-Nash Equilibrium

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- check guess: $\mathbf{E}\left[\mathrm{v}_{(2)} \mid \mathrm{v}_{(2)} < \mathrm{v}\right]$ is monotone in v $\square$

Non-existence of Asymmetric BNE

Restriction for Lecture: single-item auction, continuous strategies

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- so all strategies are symmetric. □

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Revenue equivalence $\Rightarrow$ two formulas for agent's utility:

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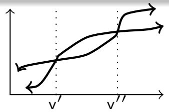

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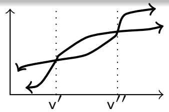

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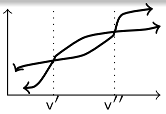

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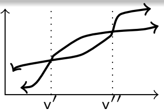

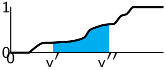

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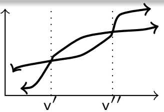
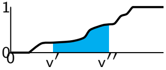
- but by Lemma and (1): $u_1(\mathsf{v}') = u_2(\mathsf{v}')$ and $u_1(\mathsf{v}'') = u_2(\mathsf{v}'')$ $\square$





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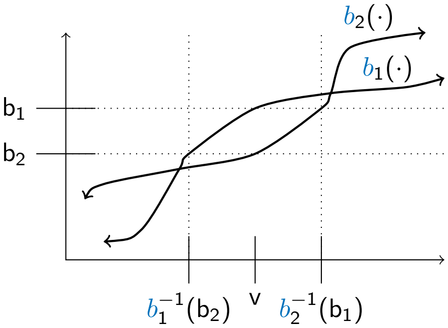

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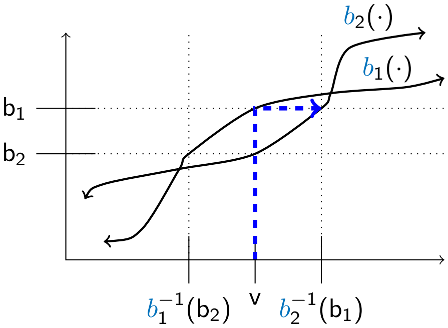

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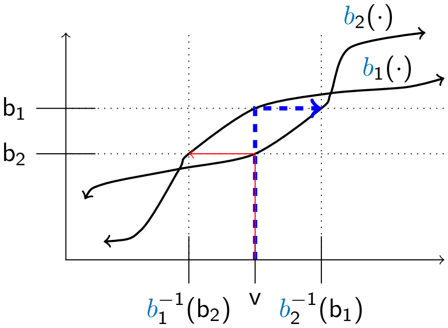

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I.i.d. winner-pays-bid position auctions

Definition (Winner-pays-bid Position Auction [cf. Edelman, Ostrovsky, Schwarz '07])

- n positions, allocation probabilities $\mathbf{w}$ with $\mathrm{w}_1 \geq \ldots \geq \mathrm{w}_n$,
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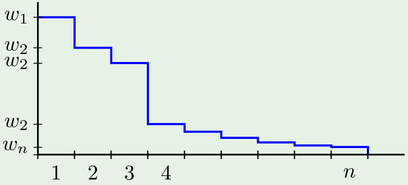

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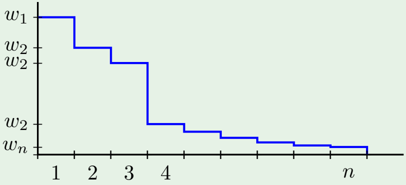

Theorem (Chawla, Hartline '13)

BNE of i.i.d. winner-pays-bid postion auction is unique, symmetric, and welfare-optimal.

Section 4

Robust Analysis of Equilibria

References:

1. Borodin, Lucier (2010) "Price of anarchy for greedy auctions"
2. Syrgkanis, Tardos (2013) "Composable and efficient mechanisms"
3. Hoy, Hartline, Taggart (2014) "Price of anarchy for auction revenue"
4. Dütting, Kesselheim (2015) "Algorithms against anarchy: Understanding non-truthful mechanisms"
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6. Hartline (202?) "Mechanism Design and Approximation" Chapter 6

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1. solicit bids.
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Robust Analysis of Welfare

Geometry of Best Response

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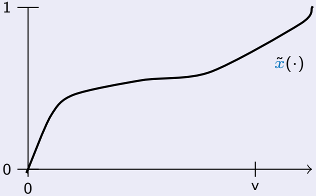
Geometry of Best Response

"utility or competition is high"



Robust Analysis of Welfare

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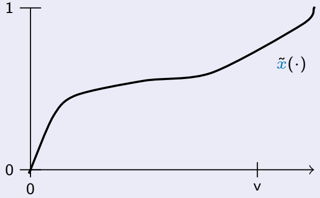

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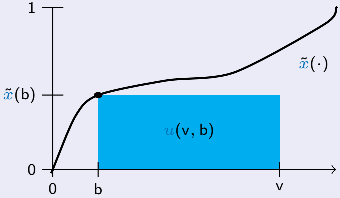

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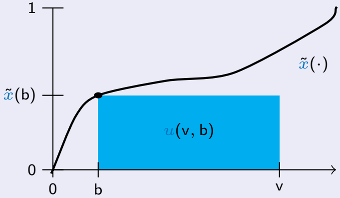

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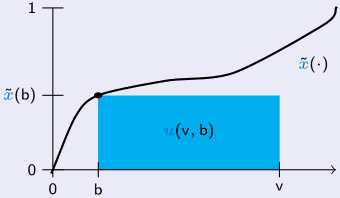

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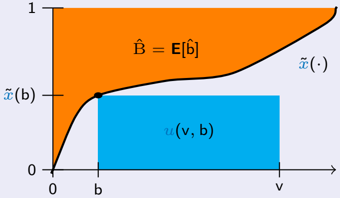

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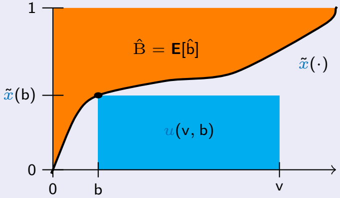

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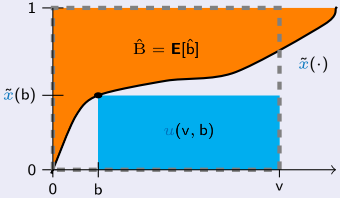

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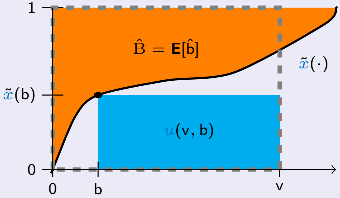

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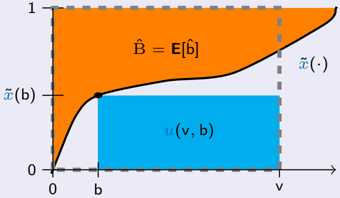

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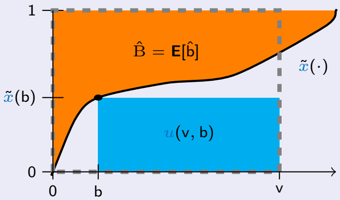

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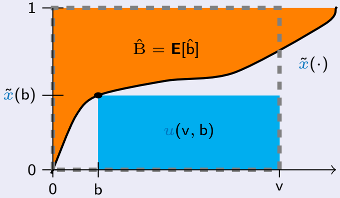

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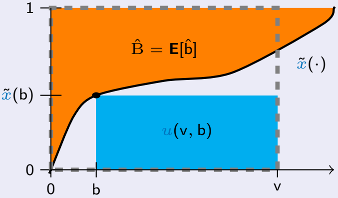

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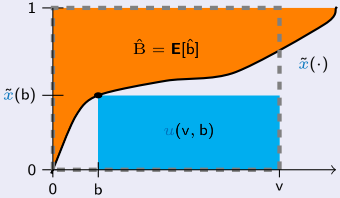

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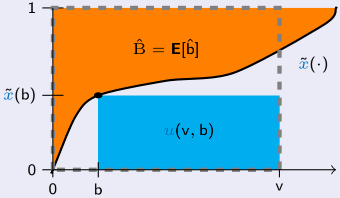

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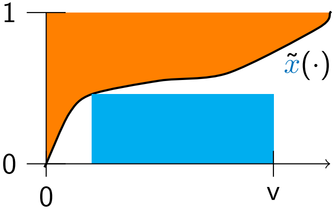

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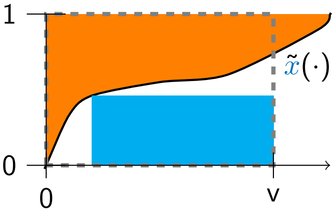

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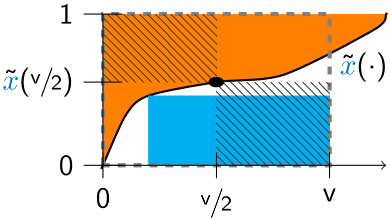

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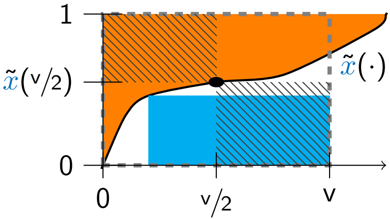

- More careful analysis gives ${}^{e-1}\!/_{e}$. $\square$

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Properties of Conversion Ratio

- not an equilibrium property.
- tight in some environments.
- closed under simultaneous composition.
- closed under randomization.

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Theorem (e.g., Lucier, Borodin '10)

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- m items; $m + 2$ agents.
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Nash welfare $= 1$; optimal welfare $= m$.
Conversation ratio is $\mu = m$:
$\hat{\mathrm{B}}_i = 1$; $\hat{\mathrm{B}}_h = 1$; $\mathrm{OPT}(\hat{\mathbf{B}}) = m$; $\mathrm{Rev}(\mathbf{b}) = 1$.

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Qstn How can near optimal non-truthful mechaisms be designed?

Parts II and III

Part II: Non-truthful Sample Complexity

1. Counterfactual Estimation
2. I.i.d. Position Auctions
3. General Reduction to I.i.d. Position Auctions

Part III: Simplicity, Robustness, & the Revelation Gap

1. Revelation Gap
2. Implementation Theory