Tutorial: Foundations of Non-truthful Mechanism Design

Part I: Equilibrium Analysis

Tutor: Jason Hartline

Schedule:

Part Ia: 10-10:45am (http://ec20.sigecom.org/tech/tutorial)
Part Ib: 11-11:45am (http://ec20.sigecom.org/tech/tutorial)
Exercises: 12-1pm (http://ec20.sigecom.org/tech/tutorial-exercises)
(https://tinyurl.com/non-truthful-exercises)

Protocol:

During session, panelest will answer clarifying questions in chat.

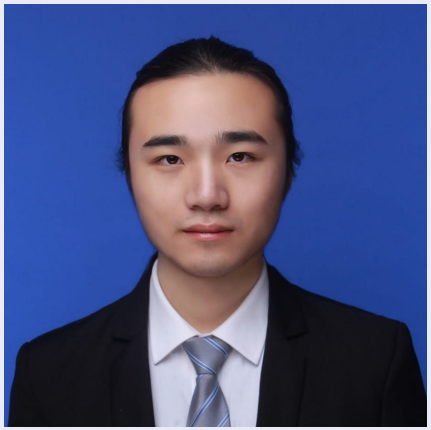
In post-session Q/A, "raise hand" to ask question.

Tutorial Cochairs

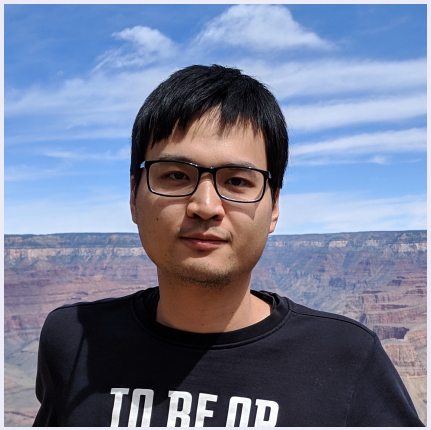
Brendan Lucier

Sigal Oren

Panelists



Yiding Feng



Yingkai Li

Foundations of Non-truthful Mechanism Design

http://jasonhartline.com/tutorial-non-truthful/

Jason Hartline

Northwestern University

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EC Tutorial 2020

Context: The Revelation Principle

Mechanism Design: identify mechanism that has good equilibrium.

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Goal: theory for non-truthful mechanism design.

Part I

Equilibrium Analysis

1. Warmup: Second-price and First-Price Auction Examples
2. Single-dimensional Environments
3. Revenue Equivalence and Applications
   - Characterizing Bayes-Nash equilbrium
   - Solving for Equilibrium
   - Uniqueness of Equilibrium
4. Robust Analysis of Equilibria

Warmup: Second-price Auction

Definition (Second-price Auction, SPA)

1. agents bid.
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Section 2

Single-dimensional Environments

References:

1. Hartline (202?) "Mechanism Design and Approximation" Chapter 2

Single-dimensional Linear Environments

Model

- agents: $\{1, \ldots, n\}$; values: $\mathbf{v} = (v_1, \ldots, v_n)$; bids: $\mathbf{b} = (b_1, \ldots, b_n)$

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- welfare: $\sum_i v_i \, \tilde{x}_i(\mathbf{b})$; revenue: $\sum_i p_i(\mathbf{b})$

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- winner-pays-bid: $\tilde{p}_i(\mathbf{b}) = b_i\, \tilde{x}_i(\mathbf{b})$

Encorporating Strategies and Prior Distribution

Three Stages of Mechanism Design

- ex ante: before values are drawn ($\mathbf{v}$ random)

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Compose ex post mechanism $(\tilde{x}, \tilde{p})$ and bid strategy b:

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Example: First-price Auction

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Example (two agents, uniform values, first-price auction)

equilibrium bids: $b_i(\mathsf{v}_i) = \mathsf{v}_i/2$; feasibility: $\mathscr{X} = \{\mathbf{x} \subset [0,1]^n : \sum_i \mathsf{x}_i \leq 1\}$

	ex post	interim
bid allocation rule		
value allocation rule		

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Section 3

Revenue Equivalence and Applications

References:

1. Myerson (1981) "Optimal Auction Design"
2. Chawla, Hartline (2013) "Auctions with unique equilibria"
3. Hartline (202?) "Mechanism Design and Approximation" Chapter 2

Bayes-Nash Equilibrium

Definition (Bayes-Nash equilibrium, BNE)

A strategy profile $\boldsymbol{b}$ such that for all i and v_i, bidding $b_i = b_i(v_i)$ is a best response when other agents bid $\boldsymbol{b}_{-i}(\mathbf{v}_{-i})$ with $\mathbf{v}_{-i} \sim \boldsymbol{F}_{-i}|_{v_i}$.

Notation

- value profile w.o. agent i's value: $\mathbf{v}_{-i} = (v_1, \ldots, v_{i-1}, ?, v_{i+1}, \ldots, v_n)$
- conditional distribution of $\boldsymbol{F}$ given v_i: $\boldsymbol{F}_{-i}|_{v_i}$ (if indep. $\boldsymbol{F}_{-i}|_{v_i} = \boldsymbol{F}_{-i}$)

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Example (Two agents, uniform values, first-price auction)

Strategies $\boldsymbol{b}$ as "$\forall i,\ b_i(v_i) = v_i/2$" is a Bayes-Nash equilibrium.

- values are $U[0,1]$
- bids under $\boldsymbol{b}$ are $U[0,1/2]$
- best response to bid $U[0,1/2]$ is $b_i(v_i) = v_i/2$

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Characterizing Bayes-Nash Equilibrium

Proposition

$(\boldsymbol{x}, \boldsymbol{p})$ are induced by BNE of some $\boldsymbol{b}$, $\boldsymbol{F}$, and $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{p}})$ if and only if:

$$\forall\, i,\ \mathsf{v}_i,\ \mathsf{z}: \quad \mathsf{v}_i\, x_i(\mathsf{v}_i) - p_i(\mathsf{v}_i) \geq \mathsf{v}_i\, x_i(\mathsf{z}) - p_i(\mathsf{z})$$

(and bids not in the range of $\boldsymbol{b}$ are weakly dominated.)

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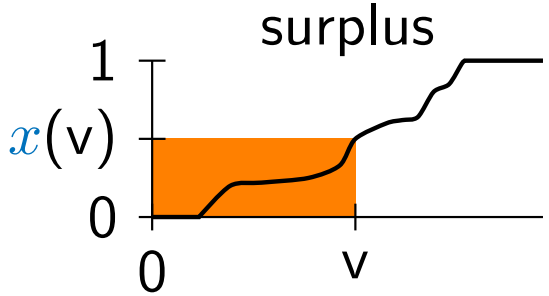

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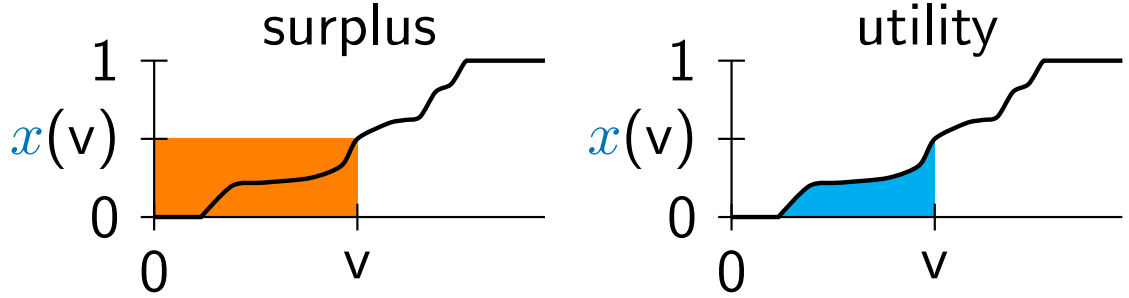

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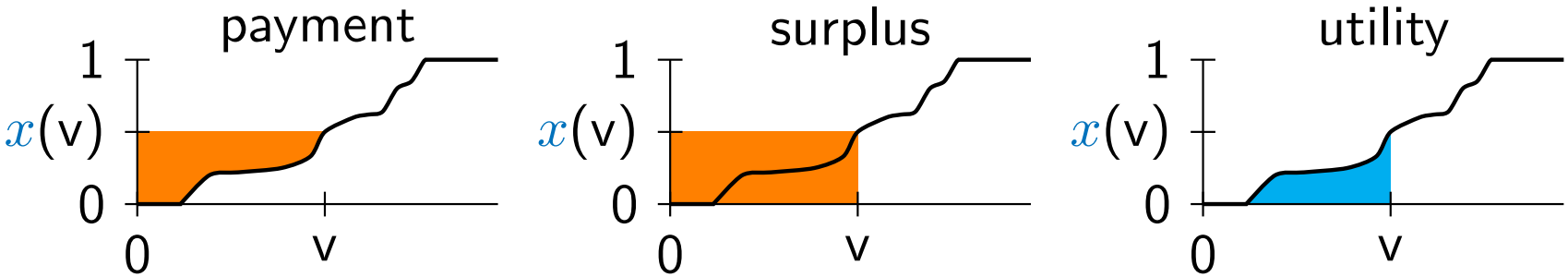

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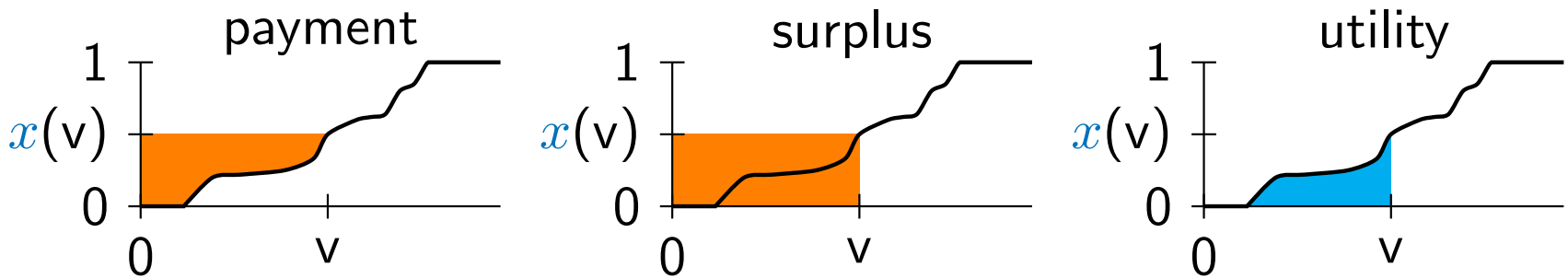

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Consequence: BNE of FPA is unique, symmetric, and welfare-maximal.

Solving for Bayes-Nash Equilibrium

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- check guess: $\mathbf{E}\left[v_{(2)} \mid v_{(2)} < v\right]$ is monotone in v □

Non-existence of Asymmetric BNE

Restriction for Lecture: single-item auction, continuous strategies

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- so all strategies are symmetric. □

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Revenue equivalence $\Rightarrow$ two formulas for agent's utility:

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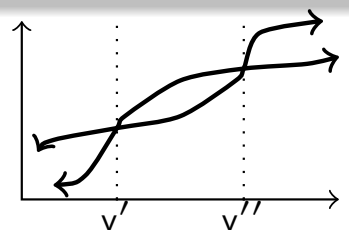

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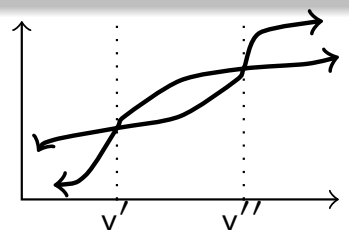

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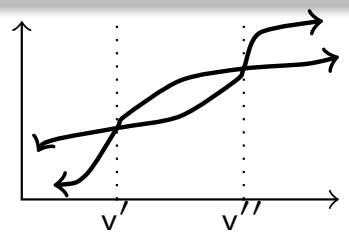

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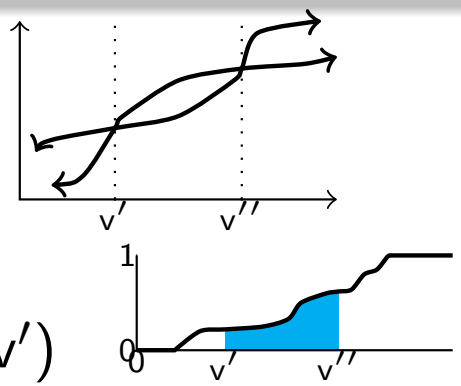

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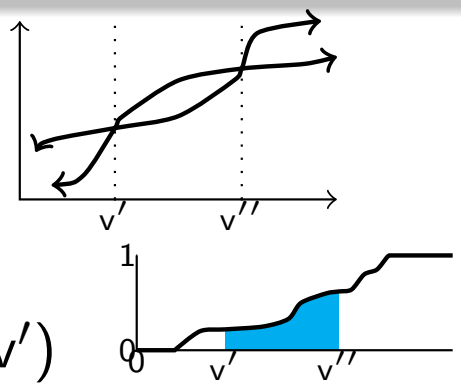

- but by Lemma and (1): $u_1(\mathsf{v}') = u_2(\mathsf{v}')$ and $u_1(\mathsf{v}'') = u_2(\mathsf{v}'')$ □

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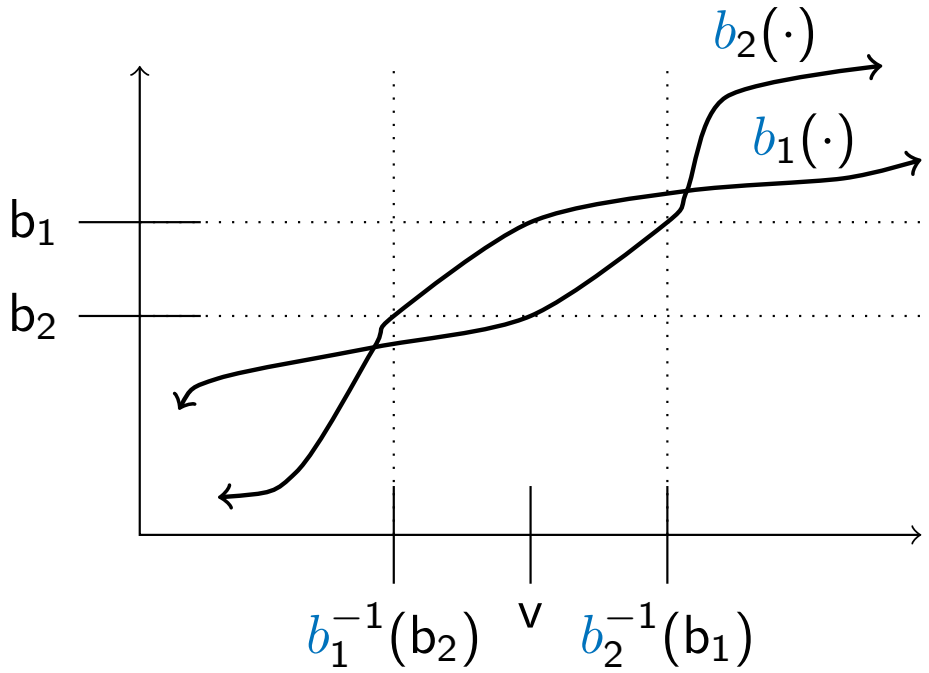

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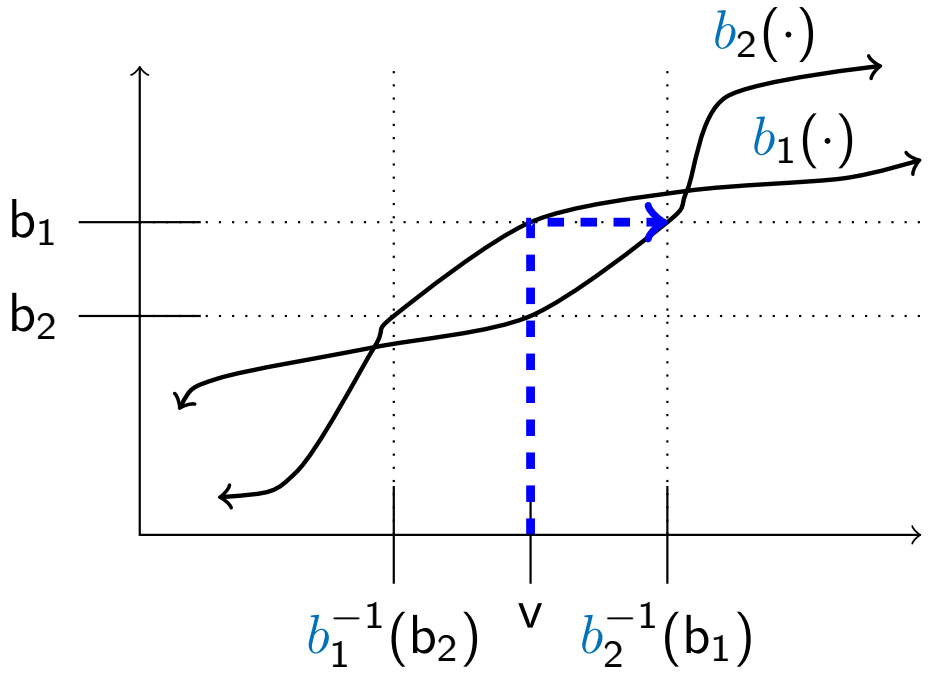

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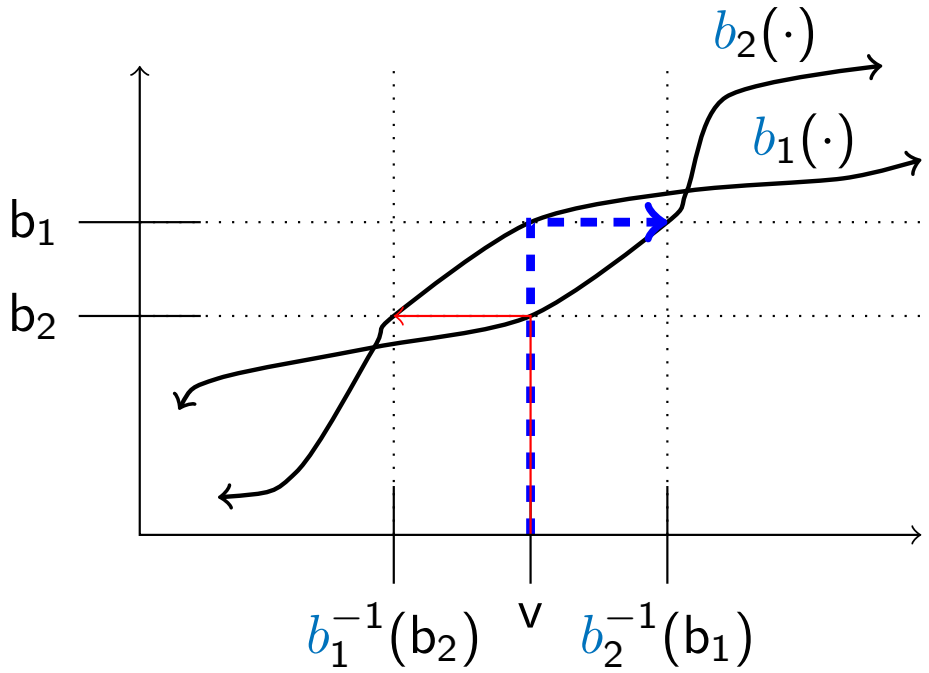

I.i.d. winner-pays-bid position auctions

Definition (Winner-pays-bid Position Auction [cf. Edelman, Ostrovsky, Schwarz '07]**)**

- n positions, allocation probabilities $\mathbf{w}$ with $w_1 \geq \ldots \geq w_n$,
- agents assigned to positions in order of bid,
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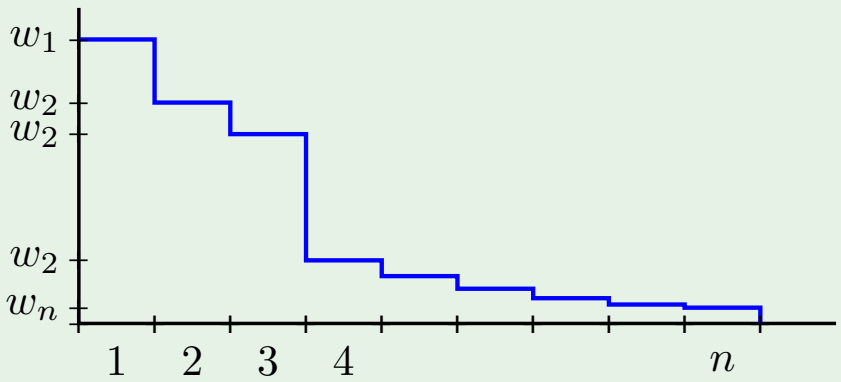

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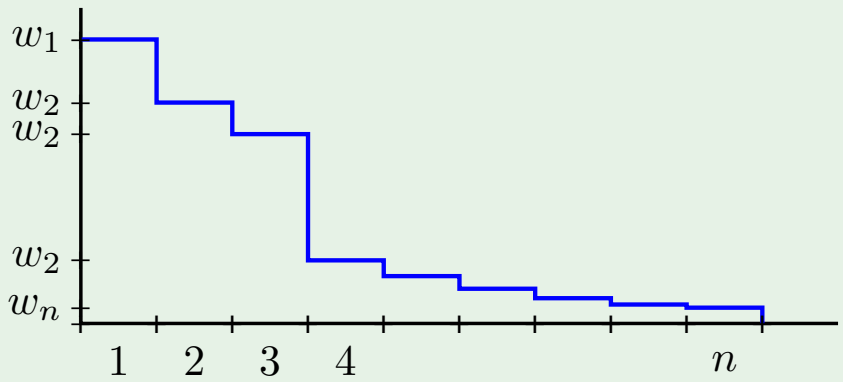

Theorem (Chawla, Hartline '13)

BNE of i.i.d. winner-pays-bid postion auction is unique, symmetric, and welfare-optimal.

Section 4

Robust Analysis of Equilibria

References:

1. Borodin, Lucier (2010) "Price of anarchy for greedy auctions"
2. Syrgkanis, Tardos (2013) "Composable and efficient mechanisms"
3. Hoy, Hartline, Taggart (2014) "Price of anarchy for auction revenue"
4. Dütting, Kesselheim (2015) "Algorithms against anarchy: Understanding non-truthful mechanisms"
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6. Hartline (202?) "Mechanism Design and Approximation" Chapter 6

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1. solicit bids.
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* approximate Nash equilibrium.

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- outcome: 101 wins at 100.01; welfare: 101; optimal welfare: 101.

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Robust Analysis of Welfare

Geometry of Best Response

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Geometry of Best Response

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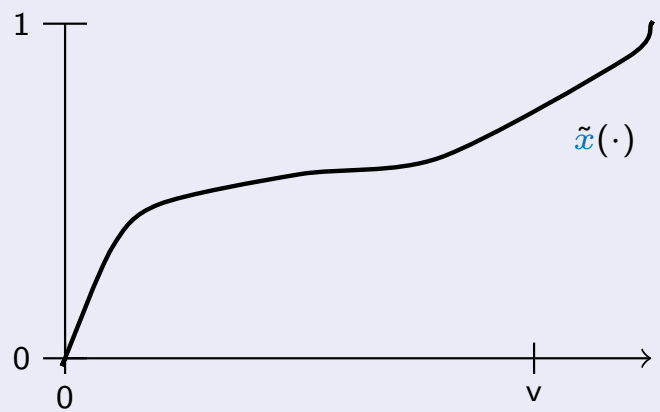

Robust Analysis of Welfare

Geometry of Best Response

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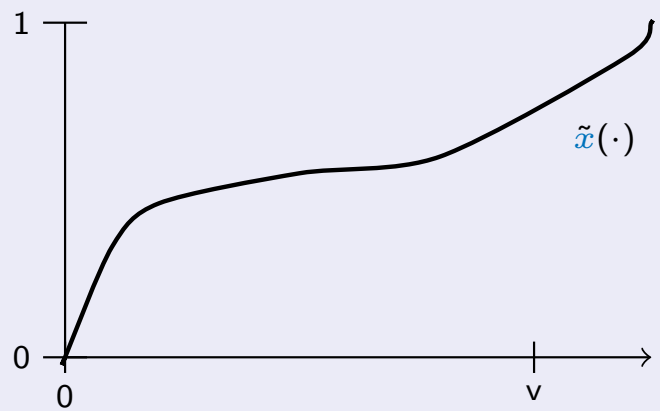

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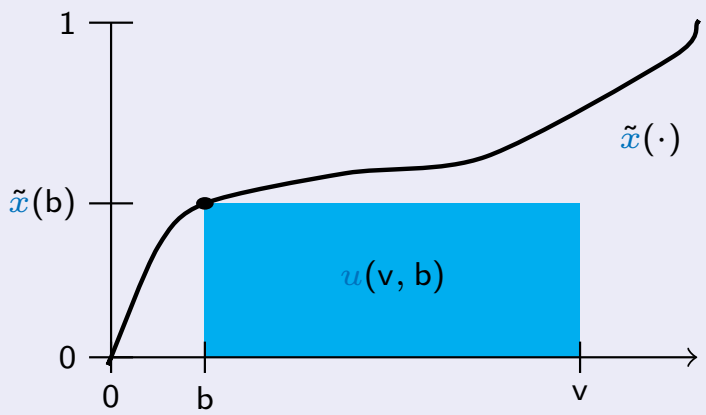

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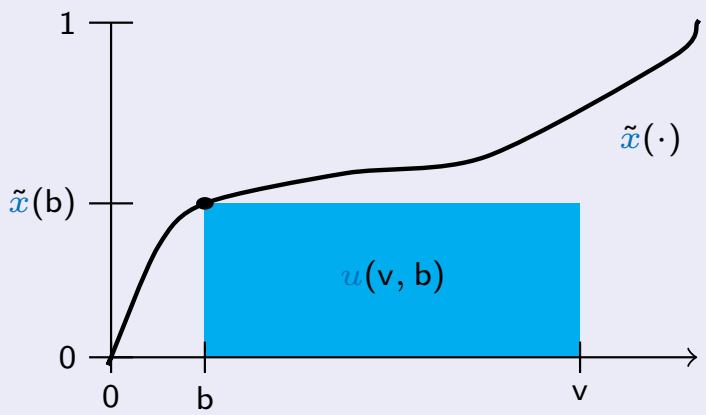

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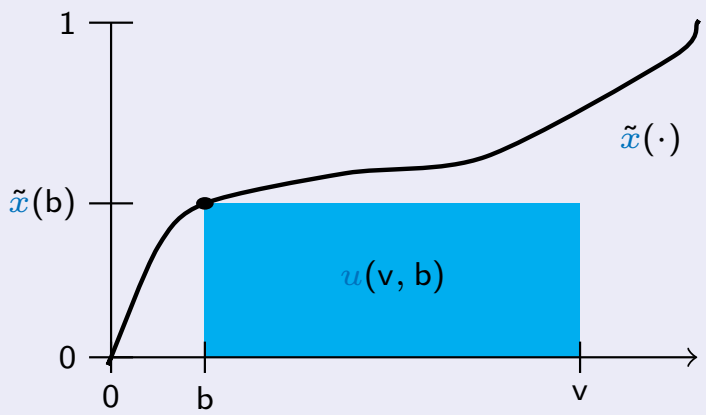

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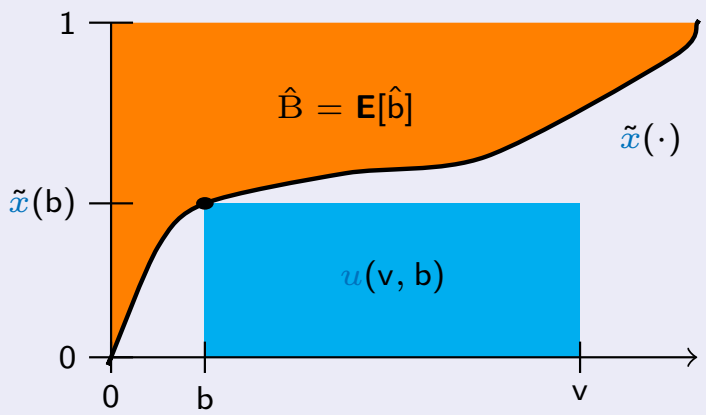

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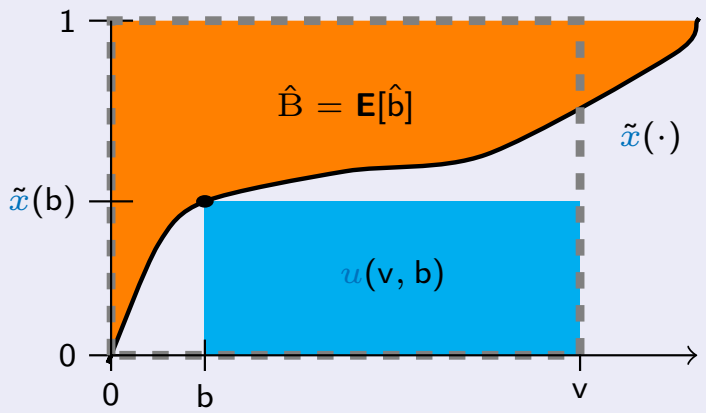

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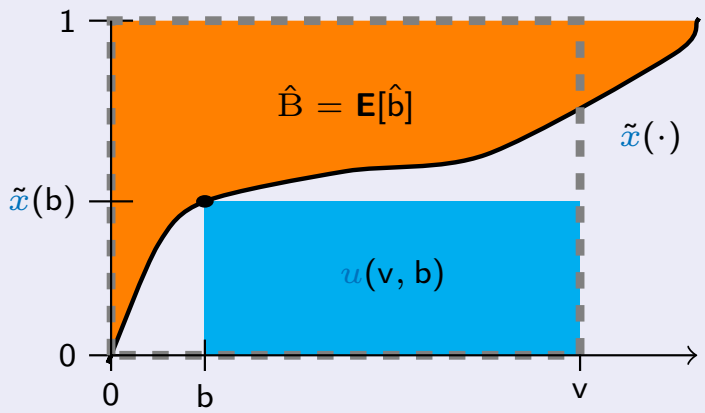

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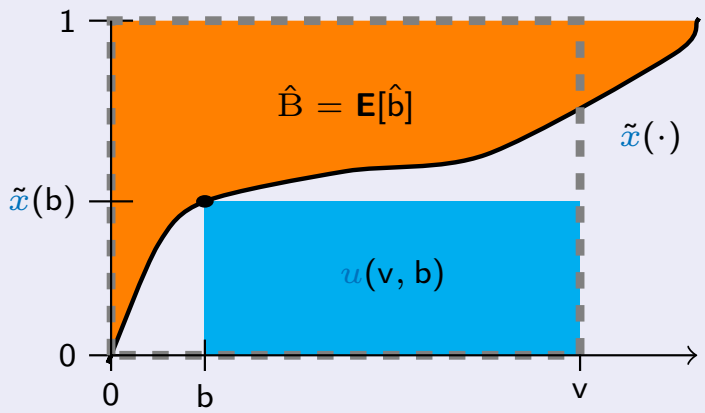

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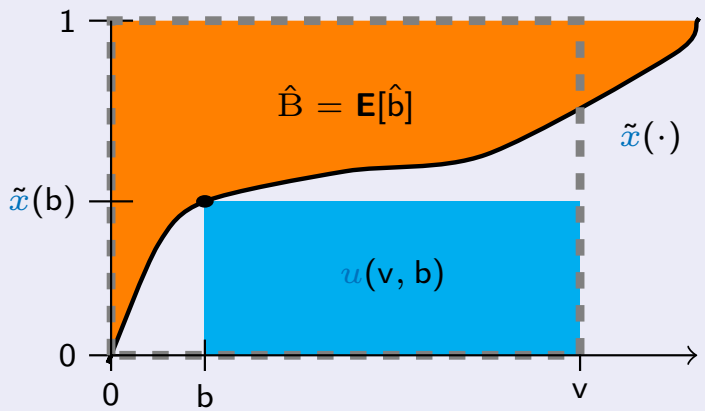

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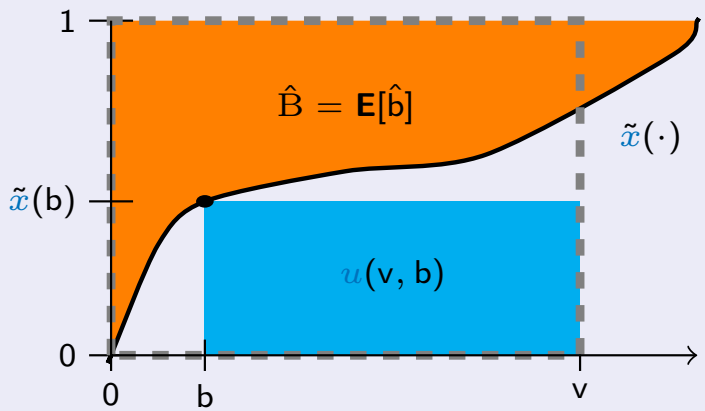

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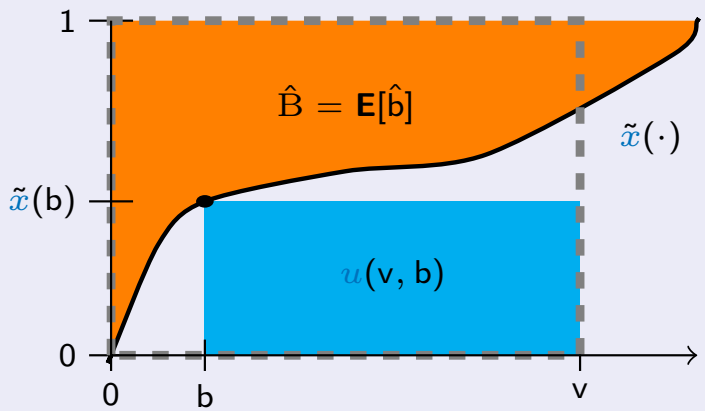

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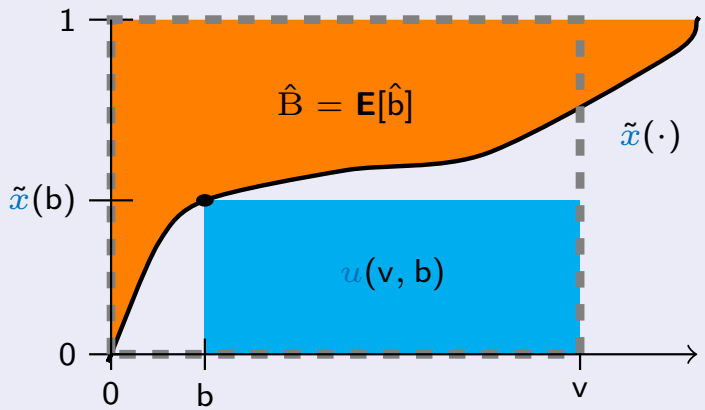

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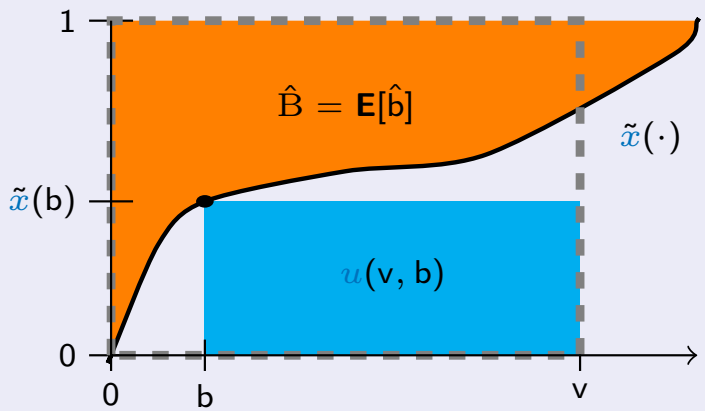

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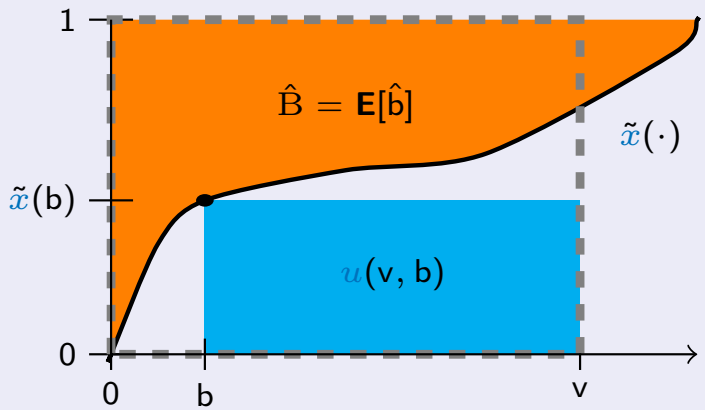

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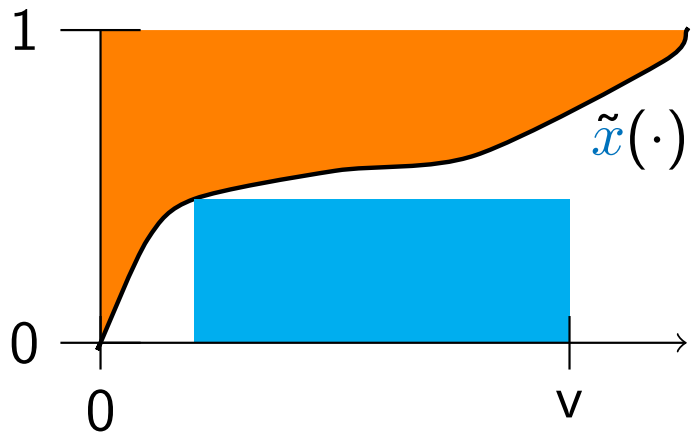

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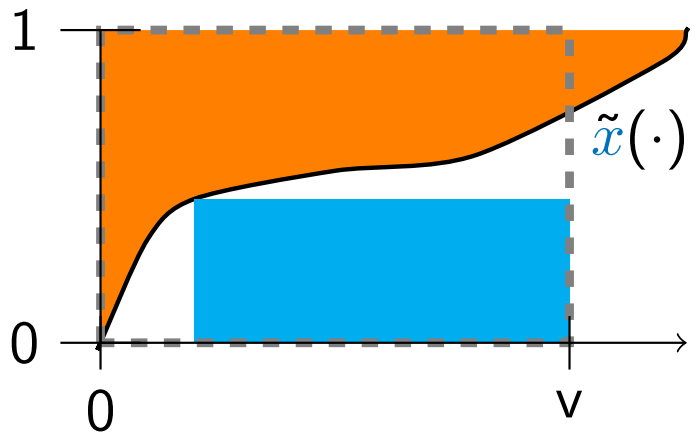

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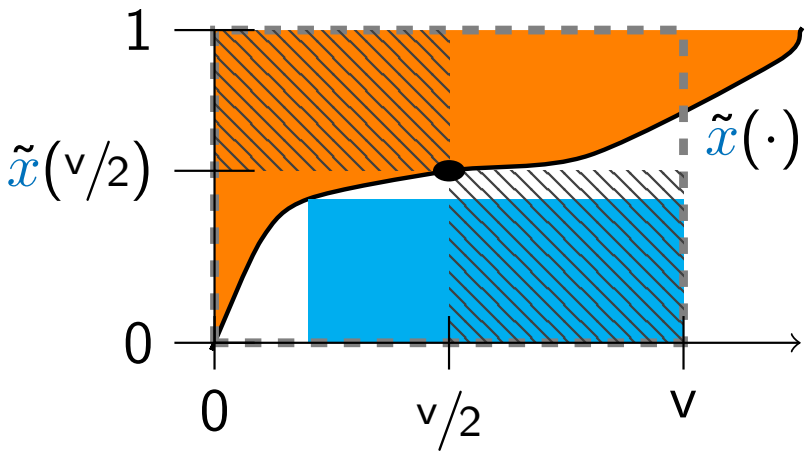

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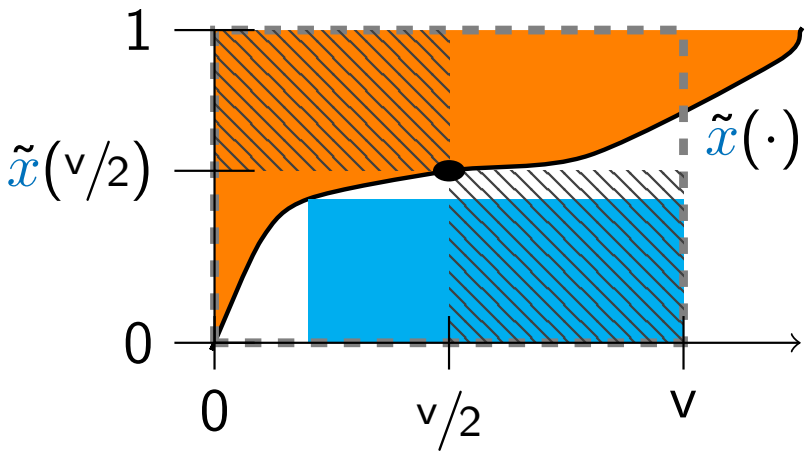

- More careful analysis gives ${}^{e-1}/_{e}$. □

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Properties of Conversion Ratio

- not an equilibrium property.
- tight in some environments.
- closed under simultaneous composition.
- closed under randomization.

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- m items; $m+2$ agents.
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- agents $h \in \{m+1, m+2\}$ bid $b_h = 1$ (one wins, one loses)
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- all agent utilities $= 0$ for bids ≤ 1.

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- agents $i \in \{1, \ldots, m\}$ bid $b_i = 0$ (all lose)
- all agent utilities $= 0$ for bids ≤ 1.

Nash welfare $= 1$; optimal welfare $= m$.

Example: Single-minded Combinatorial Auctions

Theorem (e.g., Lucier, Borodin '10)

winner-pays-bid highest-bids-win mechanisms can have very bad equilibria.

Example (Single-minded Combinatorial Auction)

Preferences:

- m items; $m+2$ agents.
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$\hat{B}_i = 1$; $\hat{B}_h = 1$; $\mathrm{OPT}(\hat{\mathbf{B}}) = m$; $\mathrm{Rev}(\mathbf{b}) = 1$.

Greedy Single-minded Combinatorial Auction

Definition (Greedy Winner-pays-bid Mechanism)

1. bidders bid,
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Qstn How can near optimal non-truthful mechaisms be designed?

Parts II and III

Part II: Non-truthful Sample Complexity

1. Counterfactual Estimation
2. I.i.d. Position Auctions
3. General Reduction to I.i.d. Position Auctions

Part III: Simplicity, Robustness, & the Revelation Gap

1. Revelation Gap
2. Implementation Theory