고윳값, 고유벡터

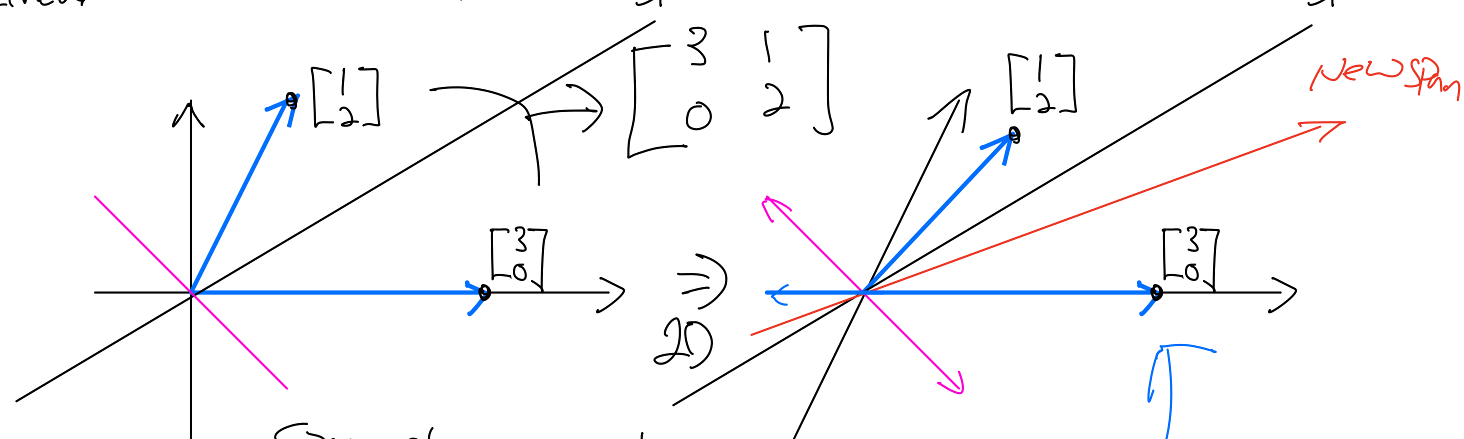
3Blue 1Brown

Eigen-things are not actually so bad

Many - Prerequisites

① Linear-Transformations
② Determinants
③ Linear Systems
④ Change of Basis

① Linear - Transformation



Span changes when Dimension changes

Some spans remain still

x-axis

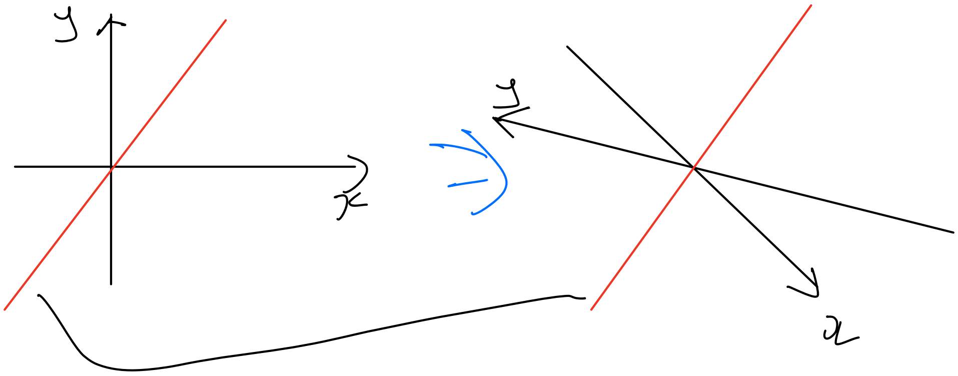
When linear transformation all the other vectors on x-axis ($\begin{bmatrix} 3 \\ 0 \end{bmatrix}$)
stays still, but only expands

Eigen-vector
with Eigen-Value 3

also, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ stays still but only expands
and vectors on it.

Eigen-Vectors, with Eigen-Value 2

Can Eigen-Values be negative? Yes, if we have eigen-value of -1.5 when vector is flipped and squished by a factor of 1.5



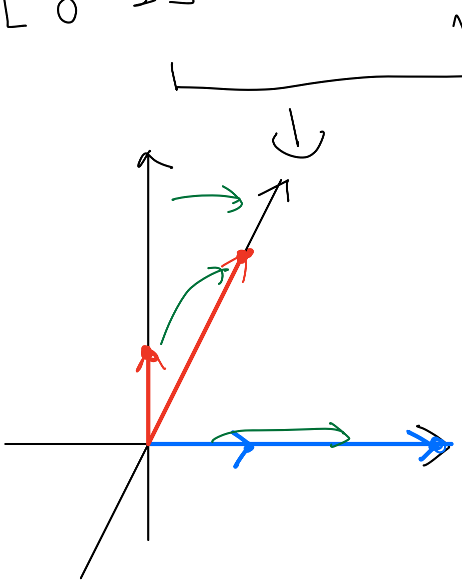
Eigen-Vector.

→ EigenValue = 1

In 3D rotation, Eigen Vector Becomes the axis of rotation

$$\begin{bmatrix} \cos(\theta)\cos(\phi) & -\sin(\phi) & \cos(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) & \cos(\theta) & \sin(\theta)\sin(\phi) \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$$

$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \Rightarrow$ Point $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and Point $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$$\underbrace{A}_{\text{matrix}}\vec{v} = \underbrace{\lambda}_{\text{corresponding Eigen-Value}}\vec{v}$$

Eigenvector

Matrix-Vector multiplication

=

Scalar multiplication

Solve λ for $\vec{v}$

Scaling by $\lambda \iff$ Matrix multiplication by

$$\underbrace{\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}}_{I}$$

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = (\lambda I)\vec{v}$$

$$A\vec{v} - (\lambda I)\vec{v} = 0$$

$$\underbrace{(A-\lambda I)}_{\text{new Matrix}}\vec{v} = 0$$

if $A = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \\ 2 & 6 & 5 \end{bmatrix}$

eg. $\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = 0$$

$$(A-\lambda I)^{-1} = \vec{v}$$

When $\lambda = 1$

matrix $A-\lambda I$ squishes space on to a line

↳ $A\vec{v} = \lambda\vec{v}$

if $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$

Seeking λ

$$\begin{bmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = \underbrace{(3-\lambda)(2-\lambda)}_{\text{quadratic polynomial in } \lambda} - 1\times 0$$

$\lambda = 2$ or $\lambda = 3$

if $\lambda = 2$

$$\begin{bmatrix} 3-2 & 1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There could be no eigenvector

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \overset{90^\circ}{\Rightarrow} \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = (-\lambda)(-\lambda) - (-1)(1)$$

$$= \lambda^2 + 1 = 0$$

$$\lambda = i \text{ or } \lambda = -i$$

A single Eigen value can have more than a line full of eigen vectors

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Eigen basis

What if both basis vectors are eigen vectors?

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$ any time when matrix has only 0 other than the diagonals, ⇒ diagonal matrix

ex)

$$\begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

if a transformation has alot of eigen vectors, we can choose a set that spans the full space

⇒ change coordinate system so that these eigen vectors are the basis vector

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

not all matrices can be diagonal