Subproblem 1.

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Subproblem 2.

$$\Pr[X = k \wedge N = n] = \frac{e^{-\lambda}\lambda^n}{n!} \binom{n}{k} p^k (1-p)^{n-k}$$

Subproblem 4.

Here's a brief proof of the conclusion you used in your solution.

Proof Prove by calculation.

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} \Pr[N = n] \sum_{k=1}^{\infty} \Pr[X = k \big| N = n] \\
&= \sum_{n=1}^{\infty} \Pr[N = n] \sum_{k=1}^{\infty} Np \\
&= \lambda \cdot p
\end{aligned}$$

□

Bonus

$$\begin{aligned}
\Pr[X = k] &= \sum_{n=k}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!} \binom{n}{k} p^k (1-p)^{n-k} \\
&= e^{-\lambda} \sum_{n=k}^{\infty} \frac{\lambda^n}{n!} \cdot \frac{n!}{k!(n-k)!} \cdot p^k (1-p)^{n-k} \\
&= \frac{e^{-\lambda}}{k!} p^k (1-p)^{-k} \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^n}{(n-k)!} \\
&= \frac{e^{-\lambda}}{k!} p^k (1-p)^{-k} \cdot [\lambda(1-p)]^k \sum_{n=0}^{\infty} \frac{[\lambda(1-p)]^n}{n!} \\
&= \frac{e^{-\lambda}}{k!} p^k (1-p)^{-k} \cdot [\lambda(1-p)]^k \cdot e^{\lambda(-(p-1))} \\
&= \frac{\lambda^k \; p^k \; e^{-p\lambda}}{k!}
\end{aligned}$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$