Chapter 2: Point Estimation

We split this chapter into two parts: Part I: Finding estimators and Estimator evaluation, and Part II: Best Unbiased Estimator (UMVUE).

1 INTRODUCTION

We will first study how to find an estimator by two estimation principles and how to assess the goodness of estimators in this note.

Point Estimation:

The idea of point estimation is so simple that we just use a statistic $T(\boldsymbol{x})$ to estimate the unknown paramete of interest, say $g(\theta)$, where $\boldsymbol{x} = (x_1, \dots, x_n)'$ is a realization of the random sample $\boldsymbol{X} = (X_1, \dots, X_n)'$ **or** $\{X_i : i = 1, \dots, n\}$ of size n from a population with a pdf $f(\cdot\,|\theta)$ or pmf $p(\cdot\,|\theta)$ and θ is in the parameter space Θ.

In some cases, there is an obvious or natural point estimator of an unknown parameter. For instance, sample mean of a random sample is a natural point estimator of the population mean. However, when we leave such a simple case, we need a more methodical estimation techniques(s) that will at least give us some reasonable candidates for consideration. In the following section, we will study two most commonly used estimation approaches in statistics. They are (i) method of moments estimation and (ii) maximum likelihood estimation.

Remark:

[Parameter of interest] Most often, the parameter(s) of our interest to be estimated (called estimand) is a function of the unknown distribution parameter(s) θ, say $g(\theta)$. For instance, we may be interested in μ^2, instead of μ, or σ/μ, instead of μ or σ only, etc.

[Estimator? Estimate?] An estimator is a funciton of the random sample $\boldsymbol{X}$, wheil an estimate is the realized value of the estimator that is obtained when a sample of data is actually taken.

[Why is called 'point' estimation?] Note that the statistic T indeed is a 'point' in R^k, where $k \geq 1$ represents the number of unknown parameters to be estimated. We use it to estimate $g(\theta)$, which is also a 'point' in R^d. So, that's why $T(\boldsymbol{x})$ is called a point estimate of $g(\theta)$.

Caution: We ONLY estimate an UNKNOWN parameter(s). For any KNOWN parameter, there is no point for us to estimate it!!!

2 METHODS OF FINDING ESTIMATORS

In practice, there are a lot of estimation techniques which can be used to estimate an unknown parameter(s), but we only detail two methods --- method of moments estimation and maximum likelihood estimation, partially because both of them are most popular in statistics and have some desirable properties, such as asymptotically unbiased and asymptotically normal.

2.1 METHOD OF MOMENTS ESTIMATION

The method of moments estimation is historically one of the oldest estimation methods in statistics, dating back at least to Karl Pearson in the late 1800s.

Karl Pearson (1857-1936)

Pearson's thinking underpins many of the 'classical' statistical methods which are still in common use today. Examples of his contributions are correlation coefficient, p-value, Pearson's goodness-of-fit test, and so on.

As its name suggests, this method is related to moments. The motivation of this method is that in some situations, the parameter of interest can be written as a function of the population moments about zero.

Basic idea of the method of moments estimation:

If the function can be specified, then we replace the population moments by their corresponding sample moments. The function of these sample moments is called ***the method of moments estimator*** (MME) for the parameter of interest.

We also use the abbreviation MME to stand for the method of moments estimate when we are talking of the realized value of the estimator.

In general, if there are k unknown parameters to be estimated, then the FIRST k or more population moments (about zero), i.e. $\mu_i' = E(X^i), for\ i = 1, \dots, k, \dots$, are required to involve.

More formally, we have the following definition of MME.

Definition (MME): Suppose that there are k unknown parameters $\theta_1, \dots, \theta_k$. If we can rewrite them in them of the first k or more moments, i.e.

$$\begin{cases} \theta_1 = g_1(\mu_1', \mu_2', \dots, \mu_k', \dots) \\ \theta_2 = g_2(\mu_1', \mu_2', \dots, \mu_k', \dots) \\ \vdots \\ \theta_k = g_k(\mu_1', \mu_2', \dots, \mu_k', \dots) \end{cases},$$

then, the method of moments estimator (MME), denoted by $(\tilde{\theta}_1, \tilde{\theta}_2, \dots, \tilde{\theta}_k)$, of $(\theta_1, \theta_2, \dots, \theta_k)$ is

$$\begin{cases} \tilde{\theta}_1 = g_1(\bar{X}, \overline{X^2}, \dots, \overline{X^k}, \dots) \\ \tilde{\theta}_2 = g_2(\bar{X}, \overline{X^2}, \dots, \overline{X^k}, \dots) \\ \vdots \\ \tilde{\theta}_k = g_k(\bar{X}, \overline{X^2}, \dots, \overline{X^k}, \dots) \end{cases}.$$

Remark:

- The method of moments estimation is "***quick and easy***", but MME obtained are often biased and it heavily relies on the existence of the required population moments.

- MME defined above ***may not be unique*** because the parameter can be written as different functions of moments, e.g. the parameter λ of a Poisson distribution is known to be the population mean μ_1' and population variance $\mu_2' - (\mu_1')^2$. ***One suggested way to fix this problem*** is to **use fewer or lower moments** to get MME. See the practice exercise for MME.

- **[Invariance property]** If $\tilde{\theta}_i$ is the MME for θ_i for $i = 1, \dots, k$, then $h(\tilde{\theta}_1, \tilde{\theta}_2, \dots, \tilde{\theta}_k)$ is the MME for $h(\theta_1, \theta_2, \dots, \theta_k)$, where h is a known function.

- A sequence of $\{\tilde{\theta} \in R^k : n = 1,2, \dots\}$ or simply $\tilde{\theta}$ is ***consistent*** and ***asymptotically unbiased*** for θ. It is also ***asymptotically normally distributed***. To be more precise, under certain assumptions like $E|X|^{2k} < \infty$, we have

$$\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N_k(\mathbf{0}, GHG'),$$

where G is a $k \times k$ matrix (suppose only the first k population moments are used in $g_1, \dots g_k$) with $\frac{\partial g_i}{\partial \mu_j'}$ as its $(i,j)^{th}$ entry and H is a $k \times k$ matrix with $\mu_{i+j}' - \mu_i'\mu_j'$ as its $(i,j)^{th}$ entry, for $i = 1, \dots, k$ and $j = 1, \dots, k$.

Example: Consider a r.s. of size n for X with $E|X|^4 < \infty$.

Take $\theta = \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} \mu_1' \\ \mu_2' - (\mu_1')^2 \end{pmatrix} = \begin{pmatrix} g_1(\mu_1', \mu_2') \\ g_2(\mu_1', \mu_2') \end{pmatrix}$.

Thus, we have

$$G = \begin{pmatrix} 1 & 0 \\ -2\mu_1' & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2\mu & 1 \end{pmatrix}$$

and

$$H = \begin{pmatrix} \mu_2' - \mu_1'\mu_1' & \mu_{1+2}' - \mu_1'\mu_2' \\ \mu_{2+1}' - \mu_2'\mu_1' & \mu_{2+2}' - \mu_2'\mu_2' \end{pmatrix} = \begin{pmatrix} \sigma^2 & \mu_3' - \mu\mu_2' \\ \mu_3' - \mu\mu_2' & \mu_4' - (\mu_2')^2 \end{pmatrix}.$$

After some algebra, we have

$$GHG' = \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}.$$

Note that

$$\tilde{\theta} = \begin{pmatrix} \bar{X} \\ \overline{X^2} - \bar{X}^2 \end{pmatrix} = \begin{pmatrix} \bar{X} \\ S_n^2 \end{pmatrix}.$$

Therefore, as n is large,

$$\sqrt{n}\left(\begin{pmatrix} \bar{X} \\ S_n^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}\right),$$

which implies that

$$\sqrt{n}(S_n^2 - \sigma^2) \xrightarrow{d} N(0, \mu_4 - \sigma^4).$$

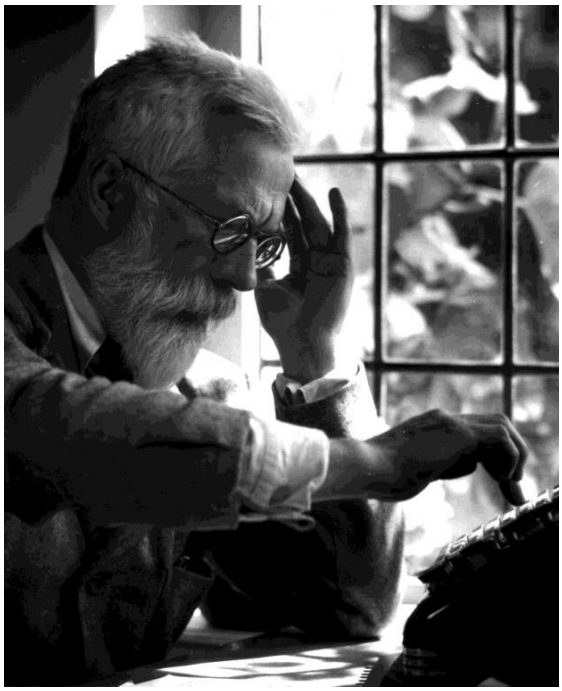
Using delta method with the above result and the condition that $\sigma^2 > 0$ yields

$$\sqrt{n}(S_n - \sigma) \xrightarrow{d} N\left(0, \frac{\mu_4 - \sigma^4}{4\sigma^2}\right).$$

Go to Practice Exercise for finding MME.

2.2 Maximum Likelihood Estimation

The method of maximum likelihood is, by far, the most popular technique for deriving estimators. It was popularized in mathematical statistics by Ronald Aylmer Fisher in 1922. Nowadays, there are still a lot of research studying the properties of this estimation method.



Ronald Aylmer Fisher (1890-1962)

Fisher is one of the most prominent statisticians of the 19th-20th century. Other examples of his contributions are sufficiency, consistency, efficiency, Fisher information, genetical statistics, etc.

More details about him can be found in the article "How Ronald Fisher became a mathematical statistician" by Stephen M. Stigler.

Before showing how to find this estimator, let's first understand what the 'likelihood' is.

What is 'likelihood'?

Consider a r.s. of size n from a population with a pdf $f(\cdot\,|\theta)$ or pmf $p(\cdot\,|\theta)$. After collection, we have the realization $\boldsymbol{x} = (x_1, \dots, x_n)'$. **The likelihood function is then defined by**

$L(\theta) = L(\theta_1, \theta_2, \dots, \theta_k|\boldsymbol{x}) = \prod_{i=1}^{n} f(x_i|\theta)$ for continuous cases and $L(\theta) = \prod_{i=1}^{n} p(x_i|\theta)$ for discrete cases. Note that the likelihood function can be used to quantify how the observed data is likely to occur.

Remark that $L(\theta)$ is a function of θ, with $\boldsymbol{x}$ held fixed. That is, the role of $\boldsymbol{x}$ and the parameter θ are interchanged between $L(\theta)$ and the joint pdf/pmf.

Basic idea of the maximum likelihood estimation:

It comes from **the statistical belief** that there is a high chance of getting our observed data. Thus, for each realization $\boldsymbol{x}$**, we want to find a value of** θ, denoted by $\hat{\theta}$, in Θ **at which** $L(\theta)$ **attains its maximum**. That is, we find a value --- ***maximum likelihood estimate*** (MLE) --- such that **our observed data is the most likely to occur**.

More formally, we have the following definition of MLE.

Definition (MLE): The maximum likelihood estimate is $\hat{\theta} = \underset{\theta \in \Theta}{\operatorname{argmax}} L(\theta)$, which means

$$L(\hat{\theta}) = \max_{\theta \in \Theta} L(\theta),$$

where $\max_{\theta \in \Theta}$ means the maximum over the parameter space Θ.

We also use the abbreviation MLE to stand for the maximum likelihood estimator when we are talking the random counterpart of the estimate.

In some cases, especially when differentiation is used, it is easier to work with a natural logarithm of $L(\theta)$, i.e. $l(\theta) = \log L(\theta)$, called ***log likelihood***, than it is to work with $L(\theta)$ directly. This is possible because the log function is strictly increasing, which implies that the maxima of $L(\theta)$ and $l(\theta)$ coincide.

Remark:

- MLE may be biased and it may not exist in Θ, especially when Θ is an open set, so for more general cases, we would define MLE in the closure $\overline{\Theta}$ of Θ. For instance, $\Theta = (0, 1)$ and $\overline{\Theta} = [0,1]$. However, the MLE taking a value outside Θ is not a reasonable estimator.

- MLE defined above may not be unique. See practice exercise for MLE.

- **[Invariance property]** If $\hat{\theta}_i$ is the MLE for θ_i for $i = 1, \dots, k$, then $h(\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_k)$ is the MLE for $h(\theta_1, \theta_2, \dots, \theta_k)$, where h is a known function.

- For $\theta \in R^k$, $\hat{\theta}$ is ***consistent***, ***asymptotically unbiased, asymptotically efficient*** and ***asymptotically normally distributed***. To be more precise, under regularity assumptions, we have

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N_k(\mathbf{0}, I_X^{-1}(\theta)),$$

where $I_X(\theta)$ is known as ***Fisher Information matrix*** (More details about this matrix will be discussed later) and it is a $k \times k$ matrix with the $(i,j)^{th}$ entry defined as

$$E\left[\left(\frac{\partial}{\partial \theta_i} \log f_X(X|\theta)\right)\left(\frac{\partial}{\partial \theta_j} \log f_X(X|\theta)\right)\right]$$

for $i = 1, \dots, k$ and $j = 1, \dots, k$.

There are three standard approaches to find MLE. **Our job is to find a global maximum!!!**

(i) If the parameter space Θ contains finitely many points, then an MLE can always be obtained by simply comparing finitely many value of (log) $L(\theta)$, for all $\theta \in \Theta$.

(ii) If $L(\theta)$ is differentiable on the interior of Θ, then one possible way of finding an MLE is to consider the values of $\theta = (\theta_1, \theta_2, \dots, \theta_k)'$ in the interior that solve the first-order/ likelihood/ log likelihood equations

$$\frac{\partial}{\partial \theta_i} L(\theta) = 0 \ \ or \ \frac{\partial}{\partial \theta_i} l(\theta) = 0, for\ i = 1, \dots, k.$$

However, this is just a necessary condition for a maximum (or minimum), not a sufficient condition. To be more precise, the solutions to the above equations are just the critical points, which may or may not be extrema. Furthermore, the zeros of the first derivative only locate the critical points in the interior of the domain of the (log) $L(\theta)$. If the maximum occur on the boundary, the first derivative may not be zero. Thus, the boundary must be checked separately for MLE.

[A special case] There is a case that we can get a global maximum easily. If there is a **unique critical point** and it has **a negative second derivative** of (log) $L(\theta)$, then it must be a global maximum. Note that for this case, we do not have to check any boundary point!!

Example: Consider a random sample of size n from $N(\theta, 1)$. Then,

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i-\theta)^2}{2}} = \frac{1}{(2\pi)^{n/2}} e^{-\frac{\sum(x_i-\theta)^2}{2}}.$$

The first derivative of $\log L(\theta)$ being 0 is

$$0 = \frac{d}{d\theta} l(\theta) = \frac{d}{d\theta}\left[\frac{-n}{2}\log(2\pi)\right] + \frac{d}{d\theta}\left[-\frac{\sum(x_i-\theta)^2}{2}\right] = \sum(x_i - \theta),$$

which yields the solution $\hat{\theta} = \bar{x}$. To verify that it is, in fact, a global maximum of $\log L(\theta)$ (or $L(\theta)$), we first note that it is the unique solution to the first-order equation. Second, we can check that

$$0 = l"(\theta) = \frac{d^2}{d\theta^2} l(\theta)\Big|_{\theta=\bar{x}} = -n < 0.$$

Therefore, $\hat{\theta} = \bar{x}$ is a global maximum --- MLE.

(iii) Another way to find an MLE is to abandon differentiation and proceed with a direct maximization. One general technique is to find a global upper bound on (log) $L(\theta)$ and then establish that there is a unique point for which the upper bound is attained.

Example (cont'): Instead of using calculus, we can also show that $\hat{\theta} = \bar{x}$ is MLE algebraically. Note that $\sum(x_i - \theta)^2 \geq \sum(x_i - \bar{x})^2$ for any θ, where they are equal if and only if $\theta = \bar{x}$. Thus, for any $\theta \in \Theta$,

$$L(\theta) \leq L(\bar{x})$$

with equality if and only if $\theta = \bar{x}$. Hence, the MLE for θ is $\bar{x}$.

Remark that the global maximum finding problem in the above case will be solved for large n situations when some regularity conditions are required.

For instance,

Corollary 3.8 *Under the assumptions of Theorem 3.7, if the likelihood equation has a unique root δ_n for each n and all $\mathbf{x}$, then $\{\delta_n\}$ is a consistent sequence of estimators of θ. If, in addition, the parameter space is an open interval $(\underline{\theta}, \bar{\theta})$ (not necessarily finite), then with probability tending to 1, δ_n maximizes the likelihood, that is, δ_n is the MLE, which is therefore consistent.*

More details can be found in the book "Theory of Point Estimation" written by E. L. Lehmann and George Casella.

Go to Practice Exercise for finding MLE.

Practice Exercises for finding MME and MLE

1. Consider a r.s. of size n from $N(\mu_0, \sigma^2)$, where $\sigma^2 \in (0, \infty)$ is unknown and μ_0 is known. Use the method of moments to estimate σ^2.

2. Consider a r.s. with size n of $X \sim Poisson(\lambda)$, where $\lambda \in (0, \infty)$. Find the MME of λ.

3. Consider a r.s. with size n of $X \sim Binomial(1, \theta)$, where $\theta \in [0,1]$. Find the MLE of θ, and then get the MLE when $n = 10$ and $\sum_{i=1}^{n} x_i = 4$.

4. Consider a r.s. of size n of X from $Gamma(\alpha, \beta)$, where $0 < \alpha < \infty$ and $0 < \beta < \infty$.
   a. Find MME when
      i. β is unknown but α is known, say α_0.
      ii. α is unknown but β is known, say β_0.
      iii. Both are unknown.
      iv. Both are unknown, but the mean is known to be K_0.
   b. Find MLE when
      i. β is unknown but α is known, say α_0.
      ii. α is unknown but β is known, say β_0.
      iii. Both are unknown.
      iv. Both are unknown, but the mean is known to be K_0.

5. Consider a r.s. of size n from $N(\mu, \sigma^2)$, where $\sigma^2 \in (0, \infty)$ is unknown and $\mu \in (-\infty, \infty)$ is unknown. Find the MME and MLE of $\theta = (\mu, \sigma^2)'$.

6. Suppose X is from $U[0, \theta]$, where $0 < \theta < \infty$. Find the MLE of θ if a r.s. with size n of X is considered.

7. Suppose X is from $U[\theta - 1, \theta + 1]$, where $-\infty < \theta < \infty$. Find the MLE of θ if a r.s. with size n of X is considered.

8. Consider a r.s. of size n from $N(\mu, 1)$, where $0 \leq \mu < \infty$ is unknown. Find the MME and MLE of μ.

R corner: Practical skill of finding MLE with R

Except for a few cases, typically we are only able to write down (log) $L(\theta)$ but cannot maximize it analytically because there are no explicit solutions to the likelihood equation. However, there is still some hope of maximizing it ***numerically*** by R or other statistical packages and, hence, finding MLE. Note that when this is done, there is still always the question of whether a local or global maximum is found.

Principle of the Numerical solution to likelihood equations

Example: Consider a r.s. with size n of $X \sim Cauchy(\theta)$. Find an MLE of θ.

First, try to get (log) $L(\theta)$. Since the pdf of X is $f_X(x|\theta) = \pi^{-1}[1 + (x-\theta)^2]^{-1}$, the likelihood is $L(\theta) = \pi^{-n}\prod_{i=1}^{n}[1 + (x_i - \theta)^2]^{-1}$ and

$$l(\theta) = -n\log\pi - \sum_{i=1}^{n}\log[1 + (x_i - \theta)^2].$$

Setting

$$l'(\theta) = \frac{d}{d\theta}l(\theta) = \sum_{i=1}^{n}\frac{2(x_i - \theta)}{1 + (x_i - \theta)^2} = 0$$

yields the MLE (Again, we then also have to check if it is a global maximum.) Note that the (solution) MLE cannot be solved explicitly in this case, but we can obtain/ approximate it by numerical method like ***Newton-Raphson Algorithm***.

According to Taylor, we have the following result:

$$0 = \frac{1}{n}l'(\hat{\theta}) \approx \frac{1}{n}l'(\theta) + (\hat{\theta} - \theta)\frac{1}{n}l"(\theta).$$

Thus,

$$\hat{\theta} \approx \theta - l'(\theta)[l"(\theta)]^{-1}.$$

Newton-Raphson Algorithm:

$$\theta_{j+1} \approx \theta_j - l'(\theta_j)\left[l"(\theta_j)\right]^{-1}, j = 0, 1, 2, \ldots,$$

R

We would use the R package ***maxLik*** to maximize (log) $L(\theta)$ in the following. Other R functions like ***optim*** can also be used.

[Case 1: One unknown parameter]

```
# https://stat.ethz.ch/R-manual/R-patched/library/stats/html/Cauchy.html
# Generate 46 data from Cauchy(\theta=10.5)
x = rcauchy(46, location = 10.5)
```

Define our own function for (log) L.

We first need to define our own R function for (log) likelihood.

```
llik = function(par)
{
  theta = par
  n = length(x)
  # log of the Cauchy likelihood
  ll = -n*log(pi)-sum(log(1+(x-theta)^2))
  # return the log likelihood to maximize
  return(ll)
}
```

```
##
## Please cite the 'maxLik' package as:
## Henningsen, Arne and Toomet, Ott (2011). maxLik: A package for maximum likelihood estimation in R. Computation
al Statistics 26(3), 443-458. DOI 10.1007/s00180-010-0217-1.
##
## If you have questions, suggestions, or comments regarding the 'maxLik' package, please use a forum or 'tracke
r' at maxLik's R-Forge site:
## https://r-forge.r-project.org/projects/maxlik/
```

The maxLik package can be used to find MLE. Like other R package, it must be installed and loaded before it can be used.

```
# install.packages("maxLik")
library("maxLik")
```

Now we can put everything together!

```
mle_cauchy = maxLik(logLik = llik, start = c(theta = 5), method = "NR")
summary(mle_cauchy)
```

```
## --------------------------------------------
## Maximum Likelihood estimation
## Newton-Raphson maximisation, 8 iterations
## Return code 1: gradient close to zero
## Log-Likelihood: -138.4513
## 1  free parameters
## Estimates:
##       Estimate Std. error t value Pr(> t)
## theta  10.8503     0.2319   46.78  <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## --------------------------------------------
```

[Case 2: Multi unknown parameters]

Next, we give an example of a normal distribution with two unknown parameters

```
# Generate 34 data from N(mu = 2, var = 9)
x = rnorm(34, mean = 2, sd = 3)
```

```
llik = function(par)
{
  mu = par[1]
  sigma = par[2]
  n = length(x)

  # log of the normal likelihood
  ll = -0.5*n*log(2*pi) - n*log(sigma) - sum(0.5*(x - mu)^2/sigma^2)
  # return the log likelihood to maximize
  return(ll)
}
```

Note that for this case *par* is defined to be a vector.

Thus, we have

```
mle_normal = maxLik(logLik = llik, start = c(mu = 0, sigma = 1), method = "NR")
summary(mle_normal)
```

```
## --------------------------------------------
## Maximum Likelihood estimation
## Newton-Raphson maximisation, 9 iterations
## Return code 1: gradient close to zero
## Log-Likelihood: -84.50297
## 2  free parameters
## Estimates:
##       Estimate Std. error t value Pr(> t)
## mu      2.2816     0.4969   4.592 4.4e-06 ***
## sigma   2.9050     0.3523   8.246 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
## --------------------------------------------
```

```
# If mu is known to be 2, then the MLE of sigma(mu=2) is
mle_normal_sigma = maxLik(logLik = llik, start=c(mu=2, sigma=1), fixed="mu", method = "NR")
summary(mle_normal_sigma)
```

```
## --------------------------------------------
## Maximum Likelihood estimation
## Newton-Raphson maximisation, 8 iterations
## Return code 1: gradient close to zero
## Log-Likelihood: -84.66201
## 1  free parameters
## Estimates:
##       Estimate Std. error t value Pr(> t)
## mu      2.0000     0.0000      NA      NA
## sigma   2.9186     0.3539   8.248  <2e-16 ***
```