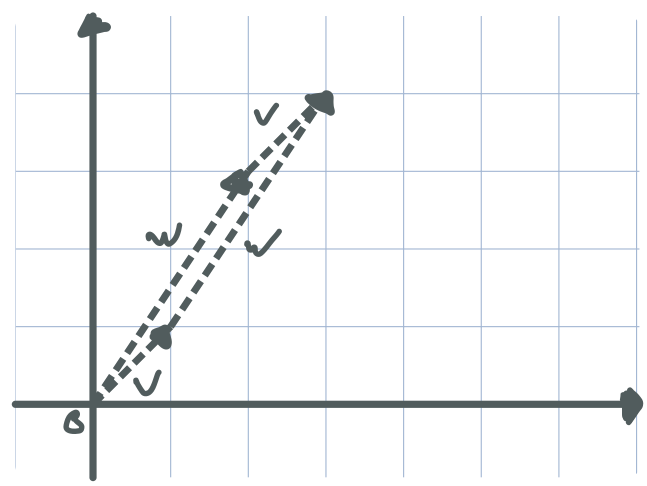
Linear Combinations:

$$cv + dw = c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}c+2d\\c+3d\end{bmatrix}$$

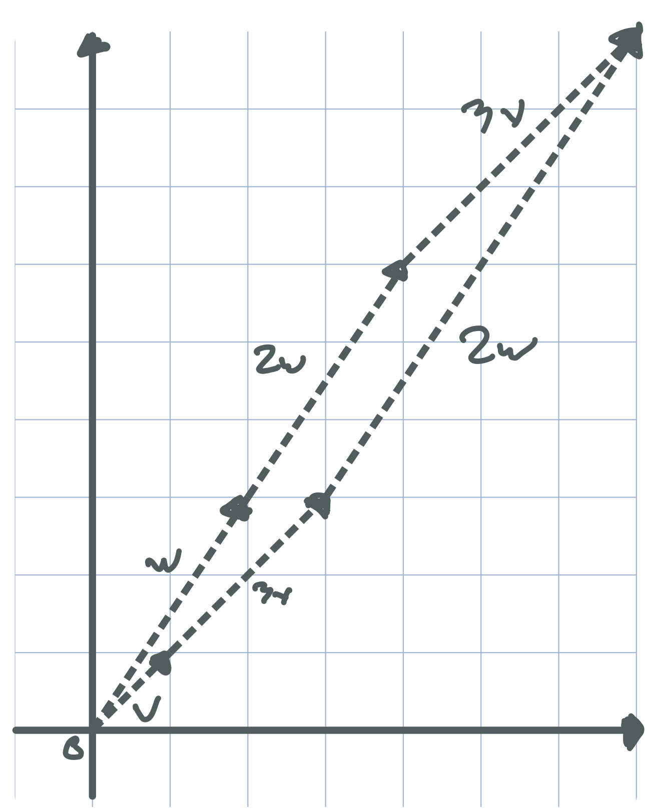
$$c = d = 1$$



$$c = 3, \quad d = 2$$

$$3\begin{bmatrix}1\\1\end{bmatrix} + 2\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} + \begin{bmatrix}4\\6\end{bmatrix}$$

$$= \begin{bmatrix}7\\9\end{bmatrix}$$



All linear combinations can fill the entire space or, if the vectors share angles, just a plane or line.

1.1 Vectors and Linear Combinations

$3v + 5w$ ⟸ linear combination
scale + add

$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + \begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$= \begin{bmatrix} 13 \\ 18 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ produces all of the $\begin{bmatrix} x \\ y \end{bmatrix}$ plane

$c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ fills a plane

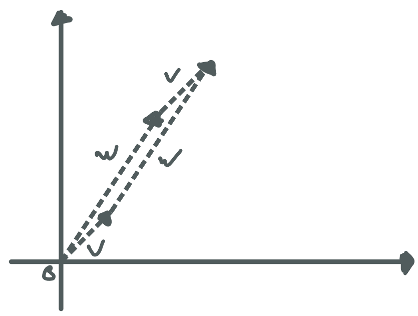
$$\begin{aligned} c + 2d &= 1 \\ c + 3d &= 0 \\ c + 4d &= 0 \end{aligned}$$

has no solution

$c = -3d \qquad -3d = -4d$

$c = -4d \qquad -3 = -4$

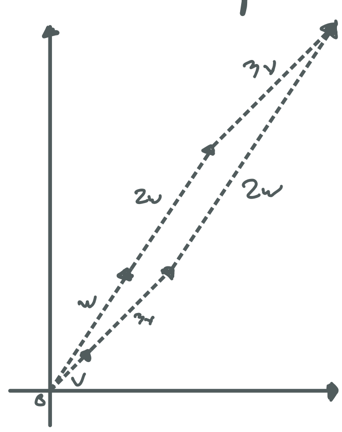
Vector Addition



$w + v = v + w$

just add components

Scalar Multiplication



$x v_{2\times 1} = \begin{bmatrix} x v_1 \\ x v_2 \end{bmatrix}$

just multiply components

Special Linear Combinations

$1v + 1w =$ sum

$1v - 1w =$ difference

$0v + 0w =$ zero vector

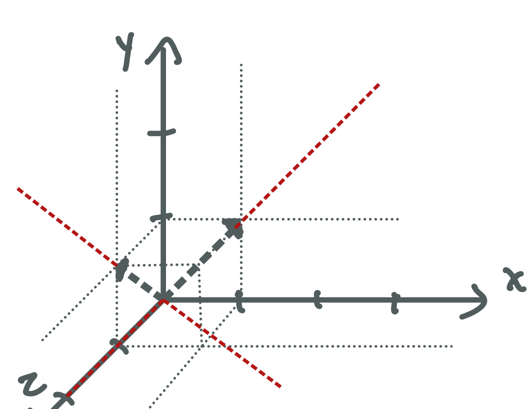
$cv + 0w = cv$

Zero vector is included in all spaces.

Worked Examples

1.1A

$v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$



$$cv + dw = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}c\\c\\0\end{bmatrix} + \begin{bmatrix}0\\d\\d\end{bmatrix}$$

$$= \begin{bmatrix}c\\c+d\\d\end{bmatrix}$$

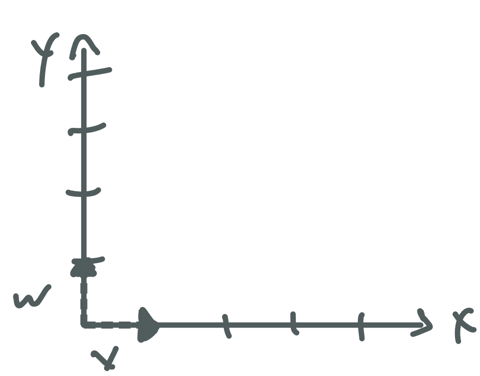
All linear combinations of v & w create a plane that intersects the xy plane at all magnitudes of v and the yz plane at all magnitudes of w.

vector $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is not on this plane.

Perpendicular vectors have 0 cross products

1.1 B

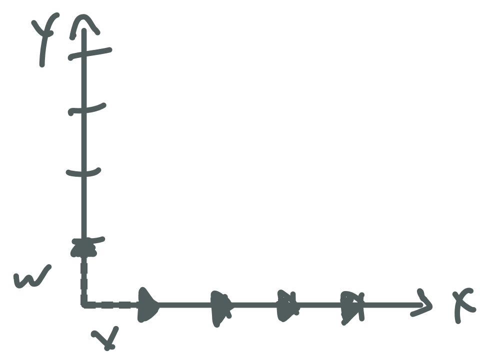
$v = \begin{bmatrix}1\\0\end{bmatrix}$ $w = \begin{bmatrix}0\\1\end{bmatrix}$



these are component vectors

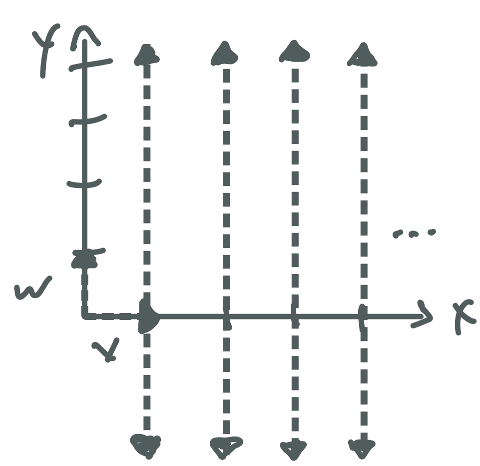
$\hat{j}$ & $\hat{k}$

(1) All positive whole number c are points that lie on the x-axis.

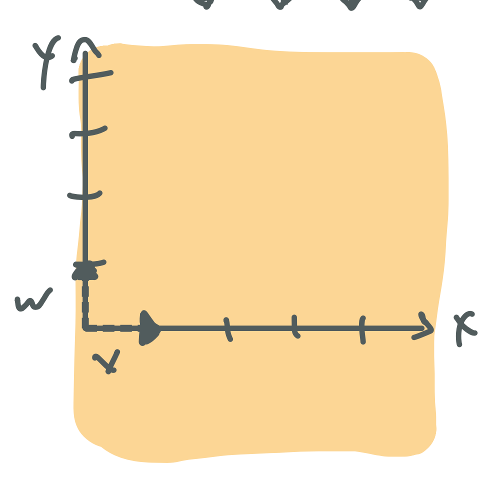


(2) This is the positive part of the x-axis.

(1') cv + dw represents vertical lines at positive whole numbers.



(2')



1.1 C

$$v = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$cv + dw = b$$

$$c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c \\ -c \end{bmatrix} + \begin{bmatrix} -d \\ 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2c - d \\ -c + 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$2c - d = 1$ $\qquad$ $-c + 2d = 0$

$2c = 1 + d$ $\qquad$ $c = 2d$

$4d = 1 + d$ $\qquad$ $c = 2(2c - 1)$

$3d = 1$ $\qquad$ $c = 4c - 2$

$d = 1/3$ $\qquad$ $2 = 3c$

$c = 2/3$

Problem Set 1.1

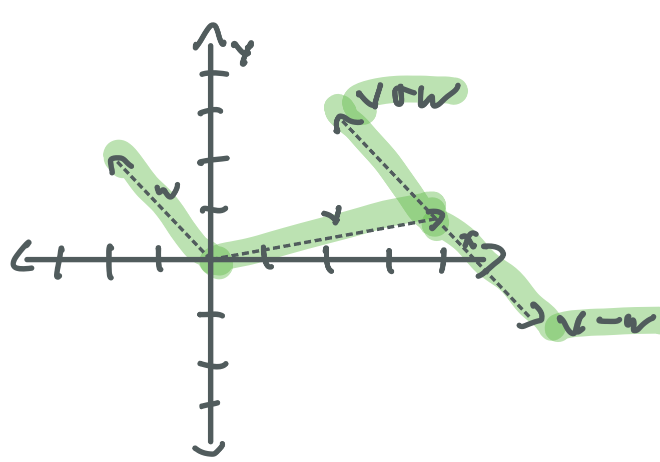
1. (a) $\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\6\\9\end{bmatrix}$ line

(b) $\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\2\\3\end{bmatrix}$ plane

(c) $\begin{bmatrix}2\\0\\d\end{bmatrix} + \begin{bmatrix}0\\2\\2\end{bmatrix} + \begin{bmatrix}2\\2\\3\end{bmatrix}$ R^3

2. $v = \begin{bmatrix}4\\1\end{bmatrix}$ $w = \begin{bmatrix}-2\\2\end{bmatrix}$ + $v + w$

+ $v - w$



3.

$v + w = \begin{bmatrix}5\\1\end{bmatrix}$ $v - w = \begin{bmatrix}1\\5\end{bmatrix}$

$w = \begin{bmatrix}5\\1\end{bmatrix} - v$ $-w = \begin{bmatrix}1\\5\end{bmatrix} - v$

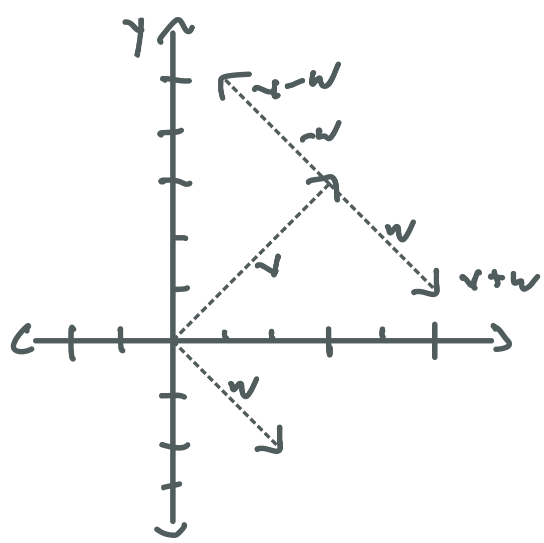
$w = \begin{bmatrix}-1\\-5\end{bmatrix} + v$

$$\begin{bmatrix}-1\\-5\end{bmatrix} + v = \begin{bmatrix}5\\1\end{bmatrix} - w \qquad w = \begin{bmatrix}5\\1\end{bmatrix} - \begin{bmatrix}3\\3\end{bmatrix}$$

$$2v = \begin{bmatrix}5\\1\end{bmatrix} - \begin{bmatrix}-1\\-5\end{bmatrix} \qquad w = \begin{bmatrix}2\\-2\end{bmatrix}$$

$$2v = \begin{bmatrix}6\\6\end{bmatrix}$$

$$v = \begin{bmatrix}3\\3\end{bmatrix}$$



4.

$$v = \begin{bmatrix}2\\1\end{bmatrix} \quad w = \begin{bmatrix}1\\2\end{bmatrix}$$

$$3v + w = 3\begin{bmatrix}2\\1\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}7\\5\end{bmatrix}$$

$$cv + dw = c\begin{bmatrix} 2 \\ 1 \end{bmatrix} + d\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2c \\ c \end{bmatrix} + \begin{bmatrix} d \\ 2d \end{bmatrix}$$

$$= \begin{bmatrix} 2c + d \\ c + 2d \end{bmatrix}$$

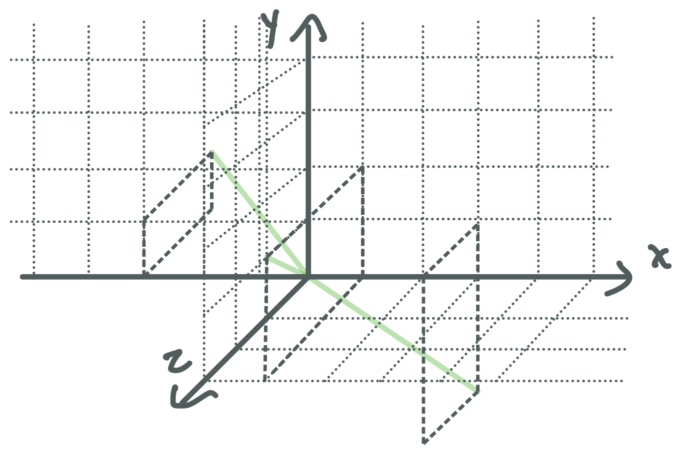
5. $u + v + w$ $\quad 2u + 2v + w$

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \quad w = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \\ -4 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$



$$w = cu + dv$$

$$\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + d\begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} c \\ 2c \\ 3c \end{bmatrix} + \begin{bmatrix} -3d \\ d \\ -2d \end{bmatrix}$$

$$2 = c - 3d$$
$$c = 2 + 3d$$

$$-3 = 2c + d$$
$$2c = -3 - d$$
$$c = \frac{-3-d}{2}$$

$$-1 = 3c - 2d$$
$$3c = -1 + 2d$$
$$c = \frac{-1+2d}{3}$$

$$2 + 3d = \frac{-3-d}{2}$$
$$4 + 6d = -3 - d$$
$$7d = -7$$
$$d = -1$$

$$c = 2 + 3(-1)$$
$$= 2 - 3$$
$$c = -1$$

6. $v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $w = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

zero

$$c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$$

$c = 3$

$d - 2c = 3$

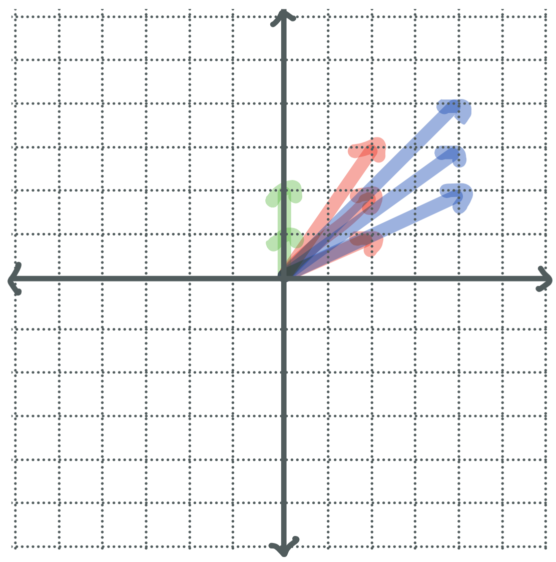
$d - 6 = 3$

$d = 9$

(3,3,6) is not a vector on the plane because its components do not add to 0.

All linear combinations of v and w produce vectors with component sums of 0.

7.



c	d	u	
0	0	(0, 0)	
0	1	(0, 1)	(y-axis)
0	2	(0, 2)	
1	0	(2, 1)	
1	1	(2, 2)	
1	2	(2, 3)	
2	0	(4, 2)	
2	1	(4, 3)	
2	2	(4, 4)	

8. $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$

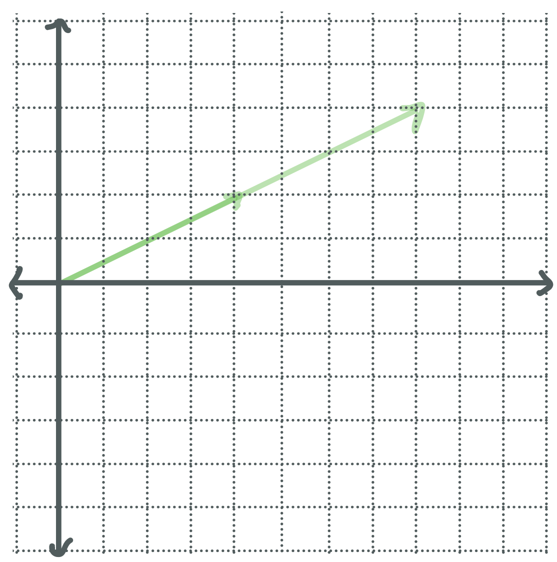
$$v - w = \begin{bmatrix} 4 \\ 2 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

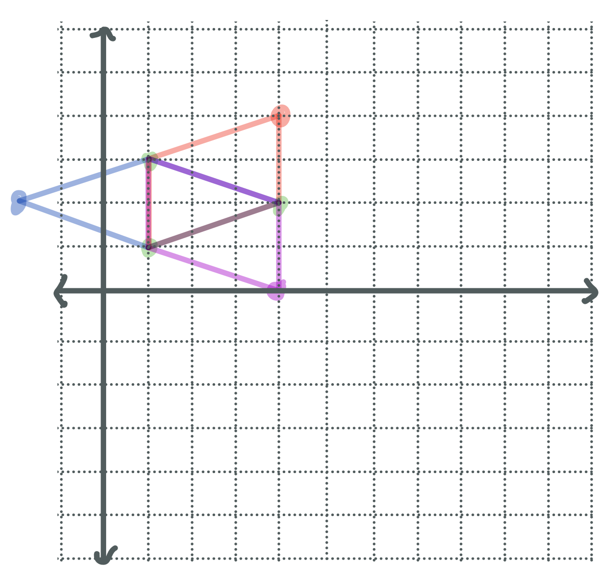
$(v+w)+(v-w)$

$v+w+v-w$

$2v$



9.



a. $(4,0)$

b. $(4,4)$

c. $(-2,2)$

10. $i+j=(1,1,0)$

$i+j+k=(1,1,1)$

All points in the unit cube have

$0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1$

11. (1, 1, 1)
(1, 1, 0)
(1, 0, 1)
(0, 1, 1)

(0.5, 0.5, 0.5)

(0.5, 0.5, 0)
(0.5, 0, 0.5)
(0, 0.5, 0.5)
(0.5, 0.5, 1)
(0.5, 1, 0.5)
(1, 0.5, 0.5)

12

12. The xy cartesian plane.

13. a. $V = (0, 0)$

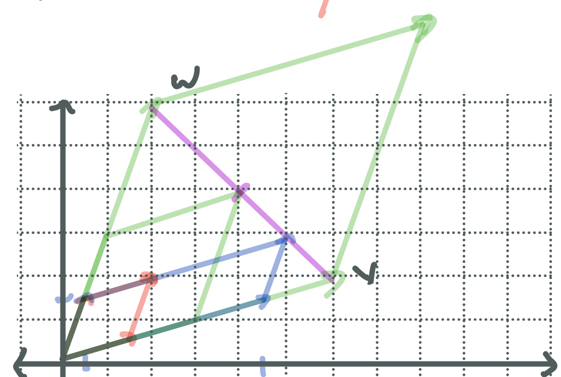
b. Because 8:00 is the inverse of 2:00

c. $\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right)$

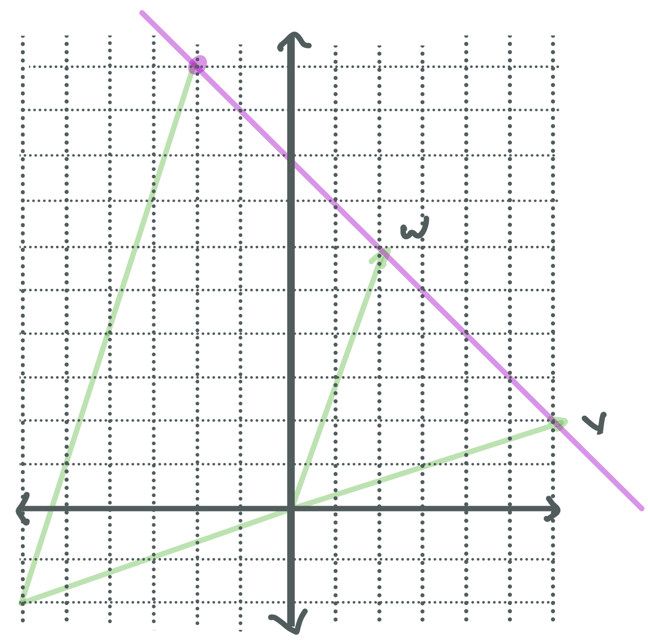
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

14. (0, 1) (0, 12)

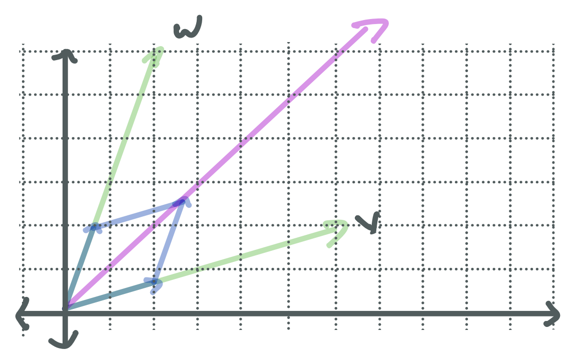
15.



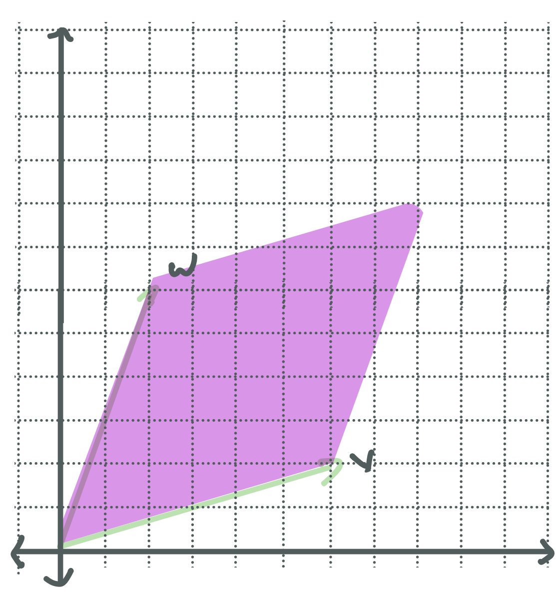
16.



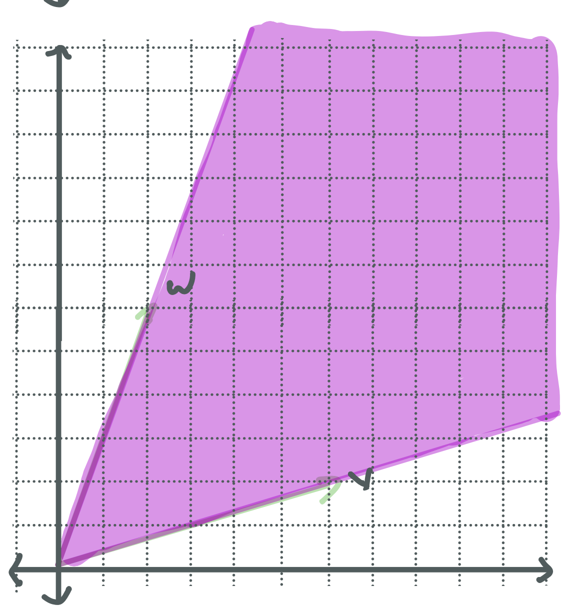
17.



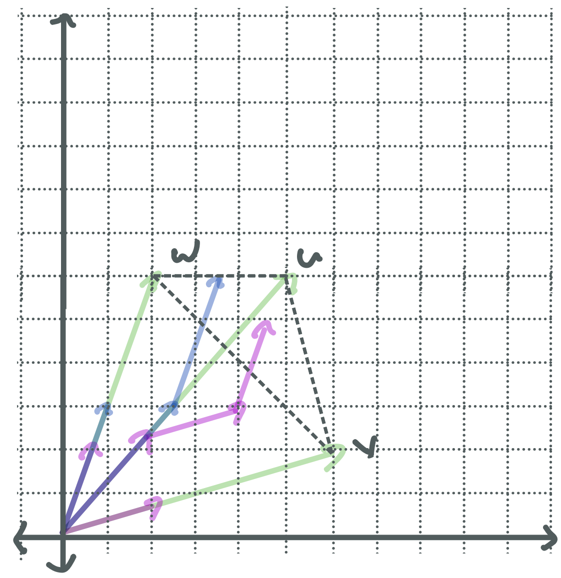
18.



19.



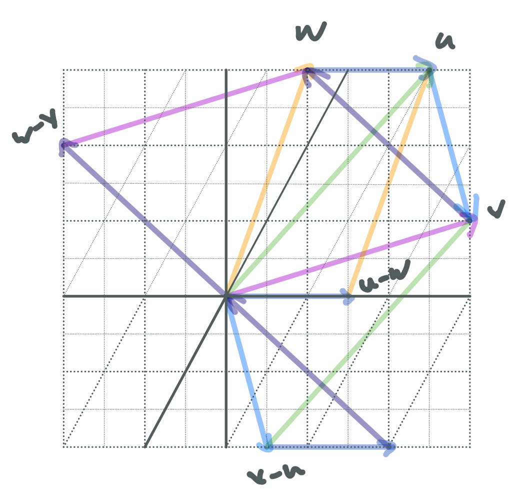
20.



$c+d+e=1$

$c \geq 0,\ d \geq 0,\ e \geq 0$

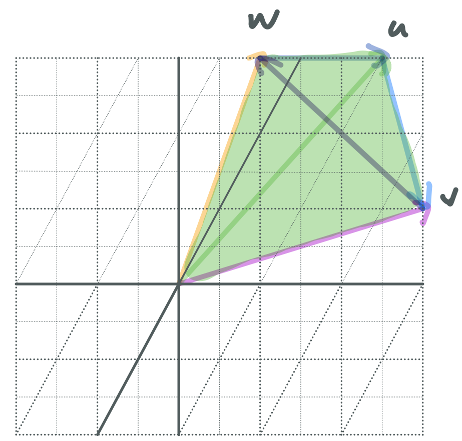
21.



Their sum is zero.

$\cancel{w}-\cancel{v}+\cancel{u}-\cancel{w}+\cancel{v}-\cancel{u}=\emptyset$

22.



$$\tfrac{1}{2}(u + v + w) \rightarrow c = \tfrac{1}{2}$$
$$d = \tfrac{1}{2}$$
$$e = \tfrac{1}{2}$$

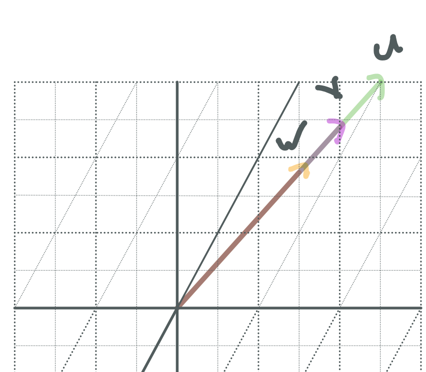
$$\tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} \neq 1$$

This vector is outside the pyramid.

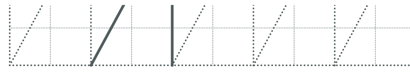
23. No, assuming they are not in the same plane. If they are then only R^2 can be reached.

24. The combinations of u and v fill a plane. v and w fill a different plane. The intersection of two planes is a line, cv is that line.

25.



u, v, and w can not have combinations that only fill a line unless they all point in the same direction.



To fill only a plane two vectors must point in the same direction and one must not.

26.

$$c\begin{bmatrix}1\\2\end{bmatrix} + d\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}14\\8\end{bmatrix}$$

$$\begin{bmatrix}c+3d\\2c+d\end{bmatrix} = \begin{bmatrix}14\\8\end{bmatrix}$$

$c = -3d + 14$

$2(-3d+14) + d = 8$

$-6d + 28 + d = 8$

$-5d = -20$

$d = 4$

$c = -3(4) + 14$

$= -12 + 14$

$c = 2$

27.

corners $\begin{cases} \text{square} \to 4 = 2^2 \\ \text{cube} \to 8 = 2^3 \\ \text{hypercube} \to 16 = 2^4 \end{cases}$

faces $\begin{cases} \text{square} \to 1 \to \\ \text{cube} \to 6 \to 2(n_{-1}) + (e_{-1}) \\ \text{hypercube} \to 24 \to 2(6) + (12) \end{cases}$

edge $\begin{cases} \text{square} \to 4 \to \\ \text{cube} \to 12 \to 2(n_{-1}) + (c_{-1}) \\ \text{hypercube} \to 32 \to 2(12) + (8) \end{cases}$

hypercube

28.

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} v_0 \\ v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} -w_0 \\ -w_1 \\ -w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$w_1 = 0$$
$$v_1 = 5$$

$$\begin{bmatrix} v_0 \\ 5 \\ v_2 \end{bmatrix} + \begin{bmatrix} w_0 \\ 0 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} v_0 \\ 5 \\ v_2 \end{bmatrix} + \begin{bmatrix} -w_0 \\ 0 \\ -w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$v_0 + w_0 = 4 \qquad v_0 + 1 = 4$$
$$v_0 = 4 - w_0 \qquad v_0 = 3$$

$$v_0 - w_0 = 2$$
$$v_0 = 2 + w_0$$

$$4 - w_0 = 2 + w_0$$
$$4 - 2 = 2w_0$$
$$2 = 2w_0$$
$$w_0 = 1$$

$$\begin{bmatrix} 3 \\ 5 \\ v_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ w_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 5 \\ v_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$v_2 + w_2 = 6 \qquad v_2 - w_2 = 8$$

$$v_2 = 6 - w_2 \qquad v_2 = 8 + w_2$$

$$8 + w_2 = 6 - w_2 \qquad v_2 = 8 - 1$$

$$2w_2 = 6 - 8 \qquad v_2 = 7$$

$$w_2 = -1$$

$$\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

$$v = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} \qquad w = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

six

2a.

$u = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad v = \begin{bmatrix} 2 \\ 7 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$cu + dv + ew = b$

$c\begin{bmatrix} 1 \\ 3 \end{bmatrix} + d\begin{bmatrix} 2 \\ 7 \end{bmatrix} + e\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$e = 0 \rightarrow c\begin{bmatrix} 1 \\ 3 \end{bmatrix} + d\begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c + 2d = 0$

$c = -2d$

$3c + 7d = 1$

$3c = 1 - 7d$

$c = \frac{1-7d}{3}$

$\frac{1-7d}{3} = -2d$

$\frac{1}{3} - \frac{7}{3}d = -2d$

$\frac{1}{3} = -2d + \frac{7}{3}d$

$= \frac{-6}{3}d + \frac{7}{3}d$

$\frac{1}{3} = \frac{1}{3}d$

$d = 1$

$c + 2 = 0$

$c = -2$

$d = 0 \rightarrow c\begin{bmatrix} 1 \\ 3 \end{bmatrix} + e\begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$c + e = 0$

$c = -e$

$c = -1/2$

$3c + 5e = 1$

$-3e + 5e = 1$

$2e = 1$

$e = 1/2$

No, if one vector is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Yes otherwise There are ∞ combinations to produce $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

20. v is pointing in the same direction as w, or one of v or w is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$$A_x = b$$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

21.

$$\begin{aligned} 2c - d &= 1 \\ -c + 2d - e &= 0 \\ -d + 2e &= 0 \end{aligned}$$

Yes, 3 unkowns, 3 independent equations.

$$2c - d = 1$$
$$2c = 1 + d$$
$$c = \frac{1+d}{2}$$

$$2c - \left(\frac{1}{2}\right) = 1$$
$$2c = \frac{3}{2}$$
$$c = \frac{3}{4}$$

$-c + 2d - e = 0$

$-\left(\frac{1+d}{2}\right) + 2d = e$

$-\frac{1}{2} - \frac{d}{2} + 2d = e$

$\boxed{e = \frac{3}{2}d - \frac{1}{2}}$

$e = \frac{3}{2}\left(\frac{1}{2}\right) - \frac{1}{2}$

$= \frac{3}{2} \cdot \frac{1}{2} - \frac{1}{2}$

$= \frac{3}{4} - \frac{2}{4}$

$\boxed{e = \frac{1}{4}}$

$-d + 2e = 0$

$-d + 2\left(\frac{3}{2}d - \frac{1}{2}\right) = 0$

$-d + 3d - 1 = 0$

$2d - 1 = 0$

$2d = 1$

$\boxed{d = \frac{1}{2}}$

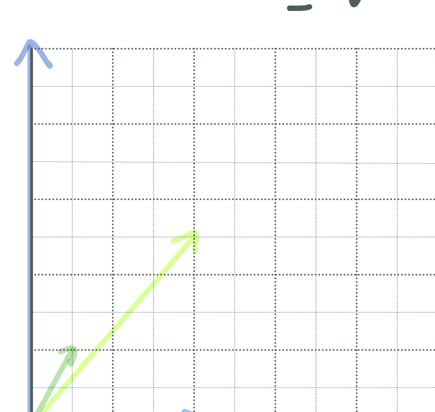
1.2 Lengths and Dot Products

1. Dot product turns two vectors into a scalar.

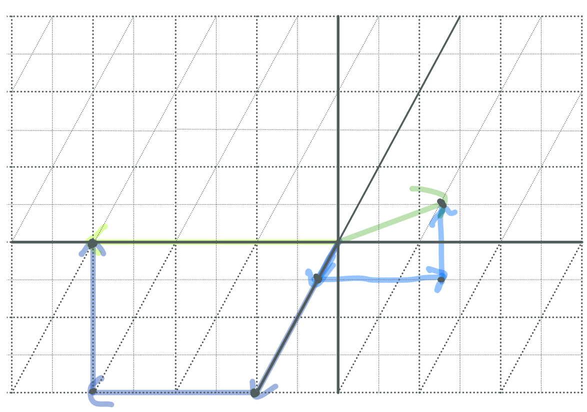
$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5$$

$$= 14$$



2. $v = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix}$



v and w are perpendicular because

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 0$$

$$4 - 12 + 8 = 0$$

3. The length of a vector is

$$\|v\| = \sqrt{v \cdot v}$$

$$v = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \|v\| = \sqrt{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}} = \sqrt{1 + 9 + 4}$$

$$\|v\| = \sqrt{14}$$

4. $u = \dfrac{v}{\|v\|} = \dfrac{v}{\sqrt{14}} = \dfrac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$u = \begin{bmatrix} \frac{}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix} \quad \|u\| = \sqrt{\begin{bmatrix} \frac{}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix} \cdot \begin{bmatrix} \frac{}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}}$$

$$\|u\| = \frac{1}{14} + \frac{9}{14} + \frac{4}{14}$$

$$= 1$$

5. The angle θ between v and w has

$$\cos\theta = \frac{v \cdot w}{\|v\| \, \|w\|}$$

6. The angle between $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ has

$$\cos\theta = \frac{1}{(1)(\sqrt{2})} = 45^\circ$$

7. All angles have $|\cos\theta| \leq 1$, so all vectors have $|$

$$\left| \frac{v \cdot w}{\|v\| \, \|w\|} \right| \leq 1$$

$$| v \cdot w | \leq \|v\| \, \|w\|$$

Example 1

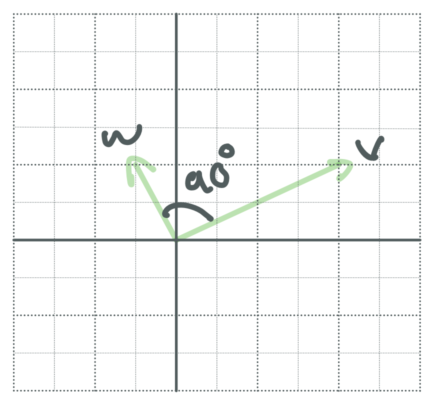
$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$v \cdot w = (4)(-1) + (2)(2)$$

$$= -4 + 4$$

$$= 0$$

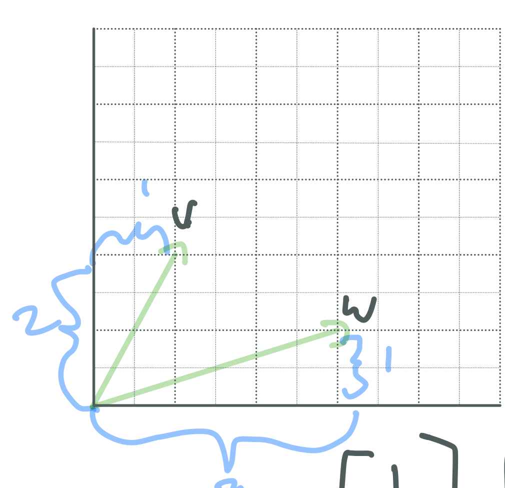
a zero dot product means these two vectors are perpendicular



$$i \cdot j = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= 0 + 0 = 0$$

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



$$v \cdot w = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

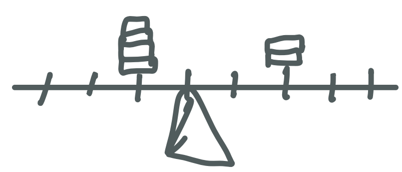
$$= 3 + 2 = 5$$

$$w \cdot v = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 3 + 2 = 5$$

$$\boxed{v \cdot w = w \cdot v}$$

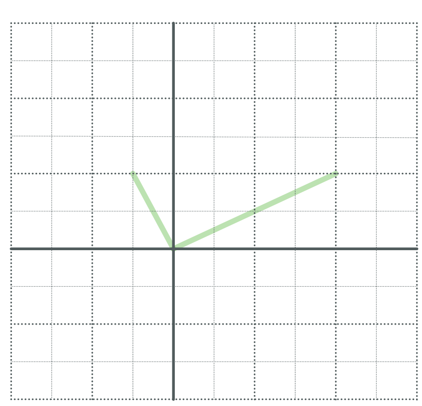
Example 2



$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

(weights) (distance)

$v \cdot v = 0 \to$ balance



Example 3

$$p = \begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix} \quad q = \begin{bmatrix} 5 \\ 8 \\ 3 \end{bmatrix}$$

(price) (quantity)

Income $= p \cdot q = 15 + 16 - 6 = 25$

Lengths and Unit Vectors

$$v \cdot v = \|v\|^2 = 14$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\|v\| = \sqrt{v \cdot v}$$
$$= \sqrt{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$
$$= \sqrt{1+4+9}$$
$$= \sqrt{14}$$

unit vector has a length of 1

$$u \cdot u = 1$$

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$u = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$ fills the unit circle

$$\theta = 0 \rightarrow u = i$$
$$\theta = \tfrac{1}{2}\pi \rightarrow u = j$$

$$u = \frac{v}{\|v\|}$$ Gives you the unit vector of v

Angle Between Two Vectors

$v \cdot w = 0$ when they are perpendicular

$v - w$ = vector pointing from w to v

$w - v$ = vector pointing from v to w

Proof

$$\|v\|^2 + \|w\|^2 = \|v-w\|^2$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$\left(\sqrt{v_1^2 + v_2^2}\right)^2 + \left(\sqrt{w_1^2 + w_2^2}\right)^2 = \|v-w\|^2$$

$$v_1^2 + v_2^2 + w_1^2 + w_2^2 = \left(\sqrt{(v_1-w_1)^2 + (v_2-w_2)^2}\right)^2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1-w_1 \\ v_2-w_2 \end{bmatrix}$$

$$\cancel{v_1^2} + \cancel{v_2^2} + \cancel{w_1^2} + \cancel{w_2^2} = (v_1-w_1)^2 + (v_2-w_2)^2$$

$$\ldots = v_1^2 - v_1w_1 - v_1w_1 + w_1^2 + v \ldots$$

$$= \cancel{v_1^2} - 2v_1w_1 + \cancel{w_1^2} + \cancel{v_2^2} - 2v_2w_2 + \cancel{w_2^2}$$

$$0 = -2v_1w_1 - 2v_2w_2$$

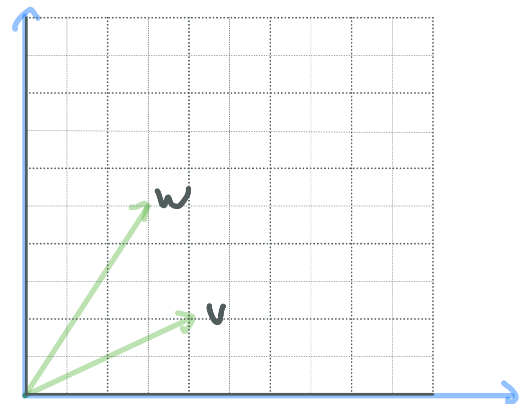
$$v_1w_1 + v_2w_2 = 0$$

The dot product $u \cdot U$ is the cosine of θ.

$$\frac{v \cdot w}{\|v\| \, \|w\|} = \cos\theta$$

Schwarz Inequality

$$|v \cdot w| \leq \|v\| \, \|w\|$$



$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad w = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{aligned} v \cdot w &= (4)(3) + (2)(5) \\ &= 12 + 10 \\ &= 22 \end{aligned}$$

$$\begin{aligned} \|v\| &= \sqrt{v \cdot v} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned} \qquad \begin{aligned} \|w\| &= \sqrt{w \cdot w} \\ &= \sqrt{9 + 25} \\ &= \sqrt{34} \end{aligned}$$

$$\|v\| \cdot \|w\| \approx 26.07$$

Triangle Inequality

$$\|v + w\| \leq \|v\| + \|w\|$$

Example 6

$\cos\theta$ $\qquad v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\frac{v \cdot w}{\|v\| \, \|w\|} = \cos\theta$$

$$v \cdot w = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$= 2 + 2$$

$$= 4$$

$$\|v\| = \sqrt{v \cdot v}$$

$$= \sqrt{4+1}$$

$$= \sqrt{5}$$

$$\|w\| = \sqrt{w \cdot w}$$

$$= \sqrt{1+4}$$

$$= \sqrt{5}$$

$$\cos\theta = \frac{4}{5}$$

Example 6

$$v = \begin{bmatrix} a \\ b \end{bmatrix}, w = \begin{bmatrix} b \\ a \end{bmatrix}$$

$$v \cdot w = 2ab$$

Worked Examples

1.2A $$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, w \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$|v \cdot w| \leq \|v\| \|w\|$$

$$v \cdot w = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$
$$= (3)(4) + (4)(3)$$
$$= 12 + 12$$
$$= 24$$

$$\|v\| = \sqrt{\begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}}$$
$$= \sqrt{3^2 + 4^2}$$
$$= \sqrt{9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

$$\|w\| = \sqrt{\begin{bmatrix} 4 \\ 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}}$$
$$= \sqrt{4^2 + 3^2}$$
$$= 5$$

$$24 \leq 25$$

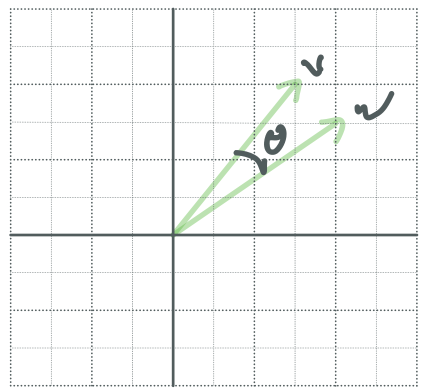
$$\|v + w\| \leq \|v\| + \|w\|$$

$$\|v + w\| = \left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\|$$
$$= \left\| \begin{bmatrix} 7 \\ 7 \end{bmatrix} \right\|$$
$$= \sqrt{49 + 49}$$
$$\approx 9.8995$$

$$9.8995 \leq 10$$

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

$$= \frac{24}{25}$$



$|v \cdot w| = \|v\|\|w\|$ and $\|v+w\| = \|v\| + \|w\|$

if both vectors point in the same direction.

1.2B

$$v = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad u = \frac{v}{\|v\|} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$$\|v\| = \sqrt{v \cdot v}$$

$$= \sqrt{9+16}$$

$$= 5$$

$$-\frac{5}{3}$$

2 perpendicular U

$$u \cdot U = 0$$

$$U = \begin{bmatrix} -5/3 \\ 5/4 \end{bmatrix} \text{ or } \begin{bmatrix} 5/3 \\ -5/4 \end{bmatrix}$$

$$\|U\| = \sqrt{(-5/3)^2 + (5/4)^2}$$

$\|u\| = \sqrt{(\ldots}$

$= \frac{25}{12}$

$$\frac{12}{25}\begin{bmatrix} -5/3 \\ 5/4 \end{bmatrix} = \boxed{\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix} \text{ or } \begin{bmatrix} 4/5 \\ -3/5 \end{bmatrix}}$$

1.2C

$$x = \begin{bmatrix} c \\ d \end{bmatrix} \qquad x \cdot r = 1$$

$$x \cdot s = 0$$

$$r = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$s = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$$

$$-c + 2d = 0$$

$$\begin{bmatrix} c \\ d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 1$$

$$2c - d = 1$$

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$-c = -2d$

$c = 2d$

$2(2d) - d = 1$

$3d = 1$

$d = 1/3$

$c = 2(1/3)$

$c = 2/3$

$$\boxed{\begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}}$$

Problem Set 1.2

1. $u = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \quad v = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$u \cdot v = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$
$$= -2.4 + 2.4$$
$$= 0$$

$$u \cdot w = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= -.6 + 1.6$$
$$= 1$$

$$u \cdot (v + w) = \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \cdot \left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$
$$= \begin{bmatrix} -.6 \\ .8 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$
$$= -3 + 4$$
$$= 1$$

$$w \cdot v = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \end{bmatrix} = 4 + 6 = 10$$

2.

$$\|u\| = \sqrt{u \cdot u}$$
$$= \sqrt{\begin{bmatrix} -.6 \\ .8 \end{bmatrix} \cdot \begin{bmatrix} -.6 \\ .8 \end{bmatrix}}$$
$$= \sqrt{(-.6)^2 + (.8)^2}$$
$$= \sqrt{.36 + .64}$$
$$= 1$$

$$\|v\| = \sqrt{v \cdot v} = \sqrt{16 + 9} = 5$$
$$\|w\| = \sqrt{w \cdot w} = \sqrt{1 + 4} = \sqrt{5}$$

$$|u \cdot v| \leq \|u\| \, \|v\|$$
$$0 \leq 5$$

$$|v \cdot w| \leq \|v\| \, \|w\|$$
$$10 \leq 5\sqrt{5}$$

3.

$$V = \frac{v}{\|v\|} = \frac{1}{5}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$$

$$W = \frac{w}{\|w\|} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

$$\frac{v \cdot w}{\|v\| \|w\|} = \cos\theta$$

$$\frac{10}{5\sqrt{5}} = \frac{10}{1} \cdot \frac{1}{5} \cdot \frac{1}{\sqrt{5}}$$
$$= \frac{2}{\sqrt{5}}$$

$$a = 2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\4\end{bmatrix}$$

$$\begin{bmatrix}1\\2\end{bmatrix} \cdot \begin{bmatrix}b_1\\b_2\end{bmatrix} = 0$$

$$b_1 + 2b_2 = 0$$

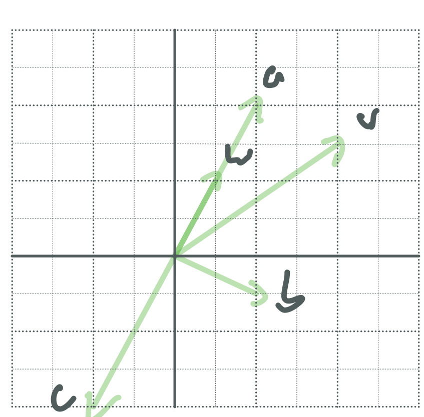
$$b_1 = -2b_2$$

$$b_1 = 2$$

$$b_2 = -1$$

$$b = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$c = -2\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}-2\\-4\end{bmatrix}$$



4.

$v = \uparrow \quad w = \circlearrowright$

$$v \cdot -v = \begin{bmatrix}1\\0\end{bmatrix} \cdot - \begin{bmatrix}1\\0\end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$= -1$$

$$(v+w)\cdot(v-w) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$(v-2w)\cdot(v+2w) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 - 4 = -3$$

6.

$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \qquad w = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$u_1 = \frac{v}{\|v\|} = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\|v\| = \sqrt{\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix}}$$

$$= \sqrt{1+9}$$

$$= \sqrt{10}$$

$$u_2 = \frac{w}{\|w\|} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$\|w\| = \sqrt{\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}}$$

$$= \sqrt{4+1+4}$$

$$= \sqrt{9}$$

$$= 3$$

$u_1 \cdot U_1 = 0$

$$\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} \cdot \begin{bmatrix} U_{1_1} \\ U_{1_2} \end{bmatrix} = 0$$

$$\frac{U_{1_1}}{\sqrt{10}} + \frac{3U_{1_2}}{\sqrt{10}} = 0$$

$$U_{1_1} + 3U_{1_2} = 0$$

$$U_{1_1} = 3$$

$$3 = -3U_{1_2}$$

$$1 = -U_{1_2}$$

$$U_{1_2} = -1$$

$$U_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$u_2 \cdot U_2 = 0$

$$\begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} \cdot \begin{bmatrix} U_{2_1} \\ U_{2_2} \\ U_{2_3} \end{bmatrix} = 0$$

$$\frac{2U_{2_1}}{3} + \frac{U_{2_2}}{3} + \frac{2U_{2_3}}{3} = 0$$

$$2U_{2_1} + U_{2_2} + 2U_{2_3} = 0$$

$$U_{2_1} = 1$$

$$U_{2_3} = 1$$

$$U_{2_2} = -4$$

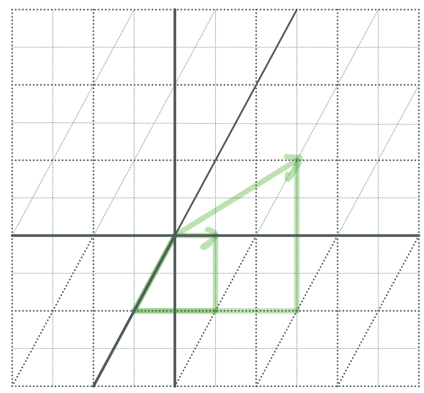
$$U_2 = \frac{1}{\sqrt{18}} \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix}$$

6. a. $\begin{bmatrix} 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0$ $\quad 2w_1 - w_2 = 0$

$2w_1 = w_2$

b. plane

c.



line

7. a.

$$\cos\theta = \frac{\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}1\\ 0\end{bmatrix}}{2} = \frac{1}{2}$$

$$\left\|\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}\right\| = \sqrt{\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}}$$

$$= \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3}$$

b.

$$\cos\theta = \frac{\begin{bmatrix}2\\ 2\\ -1\end{bmatrix}\cdot\begin{bmatrix}2\\ -1\\ 2\end{bmatrix}}{(3)(3)} = \frac{4-2-2}{9} = 0$$

$$\left\|\begin{bmatrix}2\\ 2\\ -1\end{bmatrix}\right\| = \sqrt{4+4+1}$$

$$= 3$$

$$\theta = \frac{\pi}{2}$$

$$\left\|\begin{bmatrix}2\\ -1\\ 2\end{bmatrix}\right\| = \sqrt{4+1+4}$$

$$= 3$$

c.

$$\cos\theta = \frac{\begin{bmatrix}1\\ \sqrt{3}\end{bmatrix}\cdot\begin{bmatrix}-1\\ \sqrt{3}\end{bmatrix}}{4} = \frac{-1+3}{4} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

d.

$$\cos\theta = \frac{\begin{bmatrix}3\\ 1\end{bmatrix}\cdot\begin{bmatrix}-1\\ -2\end{bmatrix}}{(\sqrt{10})(\sqrt{5})} = \frac{-3-2}{\ } = -\frac{1}{\ }$$

$5\sqrt{2}$ $\sqrt{2}$

$\left\| \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\| = \sqrt{9+1}$
$= \sqrt{10}$

$\theta = \dfrac{3\pi}{4}$

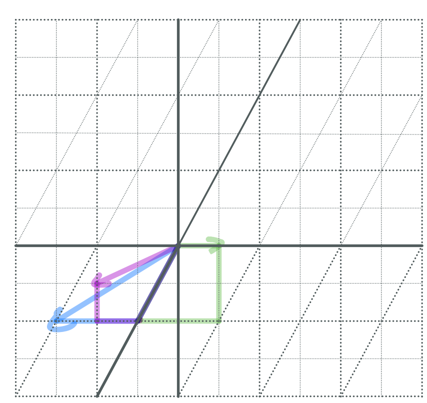
$\left\| \begin{bmatrix} -1 \\ -2 \end{bmatrix} \right\| = \sqrt{1+4}$
$= \sqrt{5}$

8. a) False, no vectors in 3 dimensions are parallel unless they are scalars of the same unit vector.

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot x = 0 \qquad v = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, w = \begin{bmatrix} 1 \\ -.5 \\ -.5 \end{bmatrix}$

$x_1 + x_2 + x_3 = 0$

All values of v & w reside on a plane.



b) True, in 2d space v & w reside on a line of vectors that are perpendicular. Thus you can add them and scale them and the result stays on this line. The same is true for 3d space, only with a plane of perpendicular vectors.

c) True, say $u = \hat{i}$ and $v = \hat{j}$, then

$\|u - v\| = \sqrt{2}$

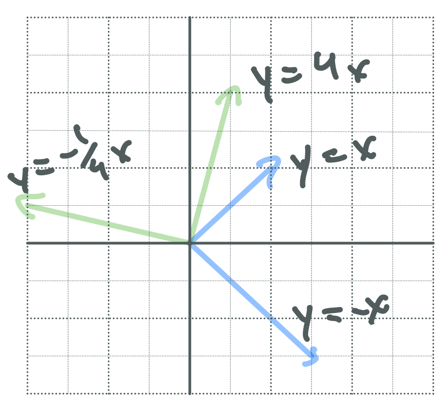
$$\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = \sqrt{2}$$

$$\left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\| = \sqrt{2}$$

$$\left\| \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\| = \sqrt{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \sqrt{1+1} = \sqrt{2}$$

All perpendicular unit vectors have a length of $\sqrt{2}$

9.



$$\frac{\text{rise}}{\text{run}} = \frac{v_2}{v_1} + \frac{w_2}{w_1}$$

$$\frac{v_2 w_2}{v_1 w_1} = -1$$

$v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ $w = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$

$$\frac{4 \cdot 1}{1 \cdot -4} = -1$$

$v = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $w = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

$$\frac{2 \cdot -3}{2 \cdot 3} = -1$$

$$v \cdot w = 0$$

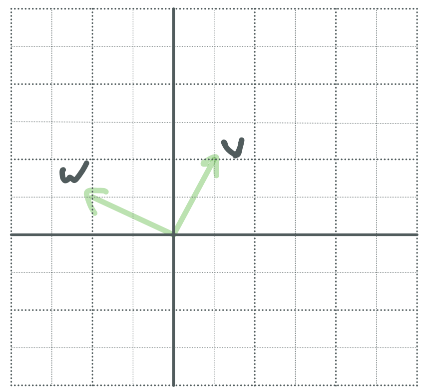
$$v_1 w_1 + v_2 w_2 = 0 \qquad \frac{v_2 w_2}{v_1 w_1} = -1$$

$$v_2 w_2 = -(v_1 w_1)$$

$$v_1 w_1 + (-v_1 w_1) = 0$$

$$0 = 0$$

10.



$$\frac{2}{1} \cdot \frac{1}{-2} = -1$$

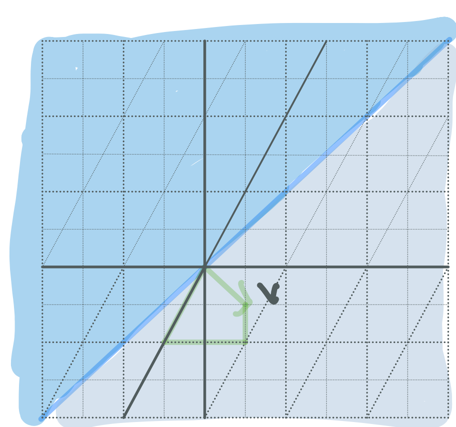
perpendicular

11.

$v \cdot w > 0$ acute angle $< 90°$

$v \cdot w = 0$ perpendicular $= 90°$

$v \cdot w < 0$ obtuse angle $> 90°$



$$v = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$v \cdot w < 0$

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} < 0$$

half of 3d space $\rightarrow 2w_1 + 2w_2 + w_3 < 0$

12.

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$w - cv \cdot v = 0$

$w_1 - c + w_2 - c = 0$

$w_1 + w_2 - 2c = 0$

$w_1 + w_2 = 2c$

$$c = \frac{w_1 + w_2}{2}$$

$$c = \frac{1+5}{2} = 3$$

13. $u = \begin{bmatrix}1\\0\\1\end{bmatrix}$

$u \cdot v = 0$

$u \cdot w = 0$

$v \cdot w = 0$

$v_1 + v_3 = 0$

$v_1 = -v_3$

$w_1 + w_3 = 0$

$w_1 = -w_3$

$v = \begin{bmatrix}0\\1\\0\end{bmatrix}$

$w = \begin{bmatrix}-1\\0\\1\end{bmatrix}$

$v_1w_1 + v_2w_2 + v_3w_3 = 0$

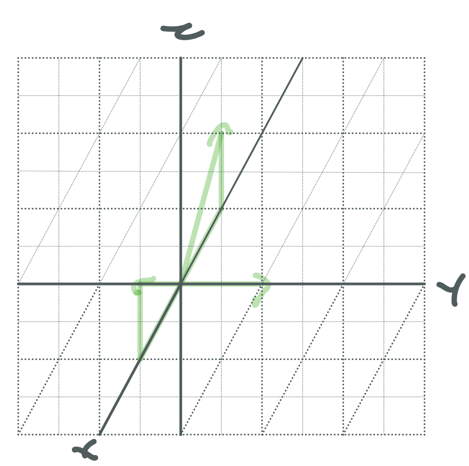
$(-v_3)(-w_3) + \dots = 0$

$2v_3w_3 = -v_2w_2$

$v_1w_1 + v_2w_2 + (-v_1)(-w_1)$

$2v_1w_1 = -v_2w_2$

$v_1w_1 = v_3w_3$



$\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} = 0$

$\begin{bmatrix}1\\0\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\1\end{bmatrix} = 0$

$\begin{bmatrix}0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\1\end{bmatrix} = 0$

14. $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-1\\-1\end{bmatrix} = 0$ $\quad$ $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\-1\\0\\0\end{bmatrix} = 0$

$$\begin{bmatrix} \vdots \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = 0$$

15. $\frac{1}{2}(2+8) = 5$

$$v = \begin{bmatrix} \sqrt{2} \\ \sqrt{8} \end{bmatrix} \quad w = \begin{bmatrix} \sqrt{8} \\ \sqrt{2} \end{bmatrix}$$

$$\cos\theta = \frac{v \cdot w}{\|v\|\,\|w\|}$$

$$v \cdot w = (\sqrt{2})(\sqrt{8}) + (\sqrt{8})(\sqrt{2})$$
$$= 2(\sqrt{2})(\sqrt{8})$$
$$= 2 \cdot 4 = 8$$

$$\|v\|\,\|w\| = \sqrt{2+8} \cdot \sqrt{8+2}$$
$$= 10$$

$$\cos\theta = \frac{4}{5}$$

16.

$$\|v\| = \sqrt{v \cdot v} = 3 \qquad u = \frac{v}{\|v\|}$$

$$u \cdot W = 0 \qquad u = (1/3, 1/3, \dots 1/3)$$

$$\begin{bmatrix} 1/3 \\ 1/3 \\ \vdots \\ 1/3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -3 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \qquad w = \frac{w}{\|w\|}$$

$$\|w\| = \sqrt{9+9} = 3\sqrt{2}$$

$$w = \left(1/\sqrt{2}, -1/\sqrt{2}, 0, \dots 0 \right)$$

17.

$$v = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

$$\cos\alpha = \frac{1+0+0}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}}$$

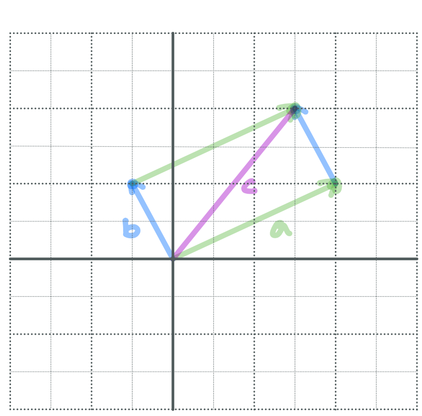
$$\|v\| = \sqrt{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} = \sqrt{2}$$

$$\cos\beta = 0$$

$$\cos\theta = \frac{-1}{\sqrt{2}\cdot 1} = \frac{-1}{\sqrt{2}}$$

$$\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

18.



$$\|a\| = \sqrt{\begin{bmatrix} 4 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} 4 \\ 2 \end{bmatrix}} = \sqrt{16+4} = 2\sqrt{5}$$

$$\|b\| = \sqrt{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\cdot\begin{bmatrix} -1 \\ 2 \end{bmatrix}} = \sqrt{1+4} = \sqrt{5}$$

$$\|c\| = 5$$

$$\|a\|^2 + \|b\|^2 = \|c\|^2$$

$$(2\sqrt{5})^2 + (\sqrt{5})^2 = (5)^2$$

$$20 + 5 = 25$$

19.

i.) $v \cdot w = w \cdot v$

2.) $u \cdot (v + w) = u \cdot v + u \cdot w$

3.) $(cv) \cdot w = c(v \cdot w)$

$u = v + w$

$$\|v+w\|^2 = v \cdot v + 2v \cdot w + w \cdot w$$
$$= (v+w)^2$$

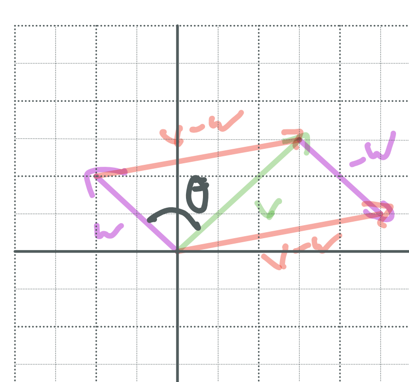
$$\|v+w\|^2 = (v+w)(v+w)$$

$$(v+w)^2 = (v+w)^2$$

20. $(v-w) \cdot (v-w) = v \cdot v - 2v \cdot w + w \cdot w$

$$\|v-w\|^2 = \|v\|^2 - 2\|v\|\|w\|\cos\theta + \|w\|^2$$

$$v \cdot v = \|v\|^2$$



21.

$$\|v+w\|^2 = \|v\|^2 + 2v \cdot w + \|w\|^2$$

$v \cdot w = v_1 w_1 + v_2 w_2$

$(v_1^2 + v_2^2)^{1/2} (w_1^2 + w_2^2)^{1/2}$

$$\|v\|\|w\| = (v_1 + v_2)(w_1 + w_2)$$
$$= (v_1 + v_2)(w_1 + w_2)$$
$$= v_1w_1 + v_1w_2 + v_2w_1 + v_2w_2$$

$$\|v+w\|^2 \le \|v\|^2 + 2\|v\|\|w\| + \|w\|^2$$
$$\le (\|v\| + \|w\|)^2$$
$$\|v+w\| \le \|v\| + \|w\|$$

22. $|v \cdot w| \le \|v\|\|w\|$

$$|v_1w_1 + v_2w_2| \le \sqrt{v_1^2 + v_2^2}\sqrt{w_1^2 + w_2^2}$$
$$(v_1w_1 + v_2w_2)^2 \le (v_1^2 + v_2^2)(w_1^2 + w_2^2)$$

$$(v_1w_1 + v_2w_2)(v_1w_1 + v_2w_2) \le \ldots$$
$$v_1^2w_1^2 + 2v_1v_2w_1w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$$
$$2v_1v_2w_1w_2 \le v_1^2w_2^2 + v_2^2w_1^2$$

$$(v_1w_2 - v_2w_1)^2 + 2v_1v_2w_1w_2 = v_1^2w_2^2 + v_2^2w_1^2$$
$$v_1^2v_2^2 - v_1v_2w_1w_2 - v_1v_2w_1w_2 + v_2^2v_1^2 + 2v_1v_2w_1w_2$$
$$v_1^2w_2^2 + v_2^2w_1^2 =$$

23. $\cos\beta = \dfrac{w_1}{\|w\|}$, $\sin\beta = \dfrac{w_2}{\|w\|}$

$$\cos\theta = \cos(\beta - \alpha)$$

$$= \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$= \frac{w_1}{\|w\|}\cdot\frac{v_1}{\|v\|} + \frac{w_2}{\|w\|}\cdot\frac{v_2}{\|v\|}$$

$$= \frac{v_1w_1 + v_2w_2}{\|w\|\,\|v\|}$$

$$\cos\theta = \frac{v\cdot w}{\|v\|\,\|w\|}$$

24.

$$|u\cdot U| \leq$$

$$|u_1||U_1| + |u_2||U_2| \leq$$

$$\frac{u_1^2 + U_1^2}{2} + \frac{u_2^2 + U_2^2}{2} = 1$$

$$u = \begin{bmatrix} .6 \\ .8 \end{bmatrix} \quad U = \begin{bmatrix} .8 \\ .6 \end{bmatrix}$$

$$|u\cdot U| = |.48 + .48| = .96$$

$$|u_1||U_1| + |u_2||U_2| = .96$$

$$\frac{u_1^2 + U_1^2}{2} + \frac{u_2^2 + U_2^2}{2} = \frac{.36 + .64}{2} + \frac{.64 + .36}{2}$$

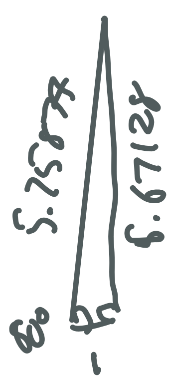
$$= 1$$

$$\cos\theta = \frac{u\cdot U}{\|u\|\,\|U\|} = u\cdot U = .96$$

25. cos θ can not be > 1 for unit vectors since the maximum value of the x component of the unit circle is 1.

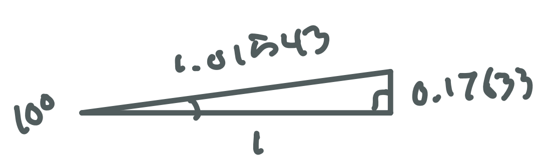
$$\cos\frac{\pi}{4} = \frac{adj}{hyp} = \frac{5\sqrt{2}}{5} = \sqrt{2}$$

For non-unit vectors:



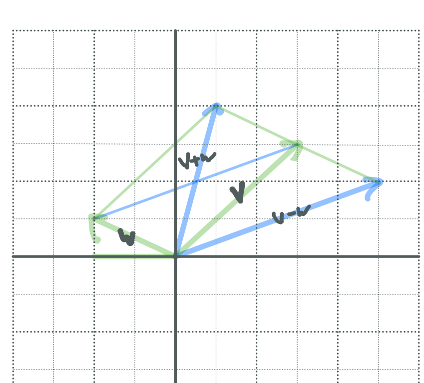
$$\cos 80° = \frac{1}{5.75877} \rightarrow 0$$

$$\cos 10° = \frac{1}{1.01543} \rightarrow 1$$



We are still limited to the range of −1 to +1.

26/27.



$$v = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$w = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\|v+w\|^2 + \|v-w\|^2 = 2\|v\|^2 + 2\|w\|^2$$

$$\left(\sqrt{\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix} \cdot \begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix}}\right)^2 \cdots =$$

$$(v_1+w_1)^2+(v_2+w_2)^2+(v_1-w_1)^2+(v_2-w_2)^2 =$$

$$v_1^2+2v_1w_1+w_1^2+v_2^2+2v_2w_2+w_2^2+v_1^2-2v_1w_1+w_1^2+v_2^2-2v_2w_2+w_2^2$$

$$2v_1^2+2v_2^2+2w_1^2+2w_2^2$$

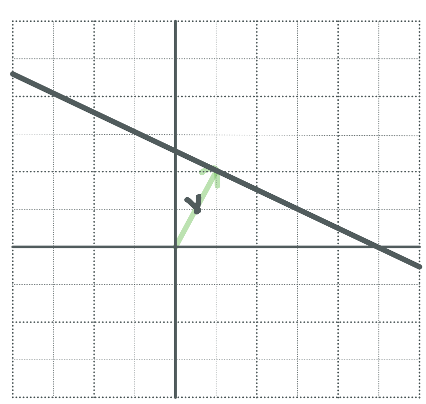
$$2(v_1^2+v_2^2+w_1^2+w_2^2) = 2(\|v\|^2+\|w\|^2)$$

$$v_1^2+v_2^2+w_1^2+w_2^2 = \left(\sqrt{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}\right)^2 \cdots$$

$$= v_1^2+v_2^2+w_1^2+w_2^2$$

28.

$$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$



$$x+2y=5$$
$$2y=-x+5$$
$$y=-\tfrac{1}{2}x+\tfrac{5}{2}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = x+2y$$

$$w_1+2w_2 = x+2y$$

$$2y=-x+w_1+2w_2$$

$$y=\frac{-1}{2}x+\frac{w_1}{2}+w_2$$

$$\frac{w_1}{2}+w_2=\frac{5}{2}$$

$$w_2=-\tfrac{1}{2}w_1+\tfrac{5}{2}$$

These lie on a line because the dot product between the two vectors is constant.

The shortest w is v.

29.

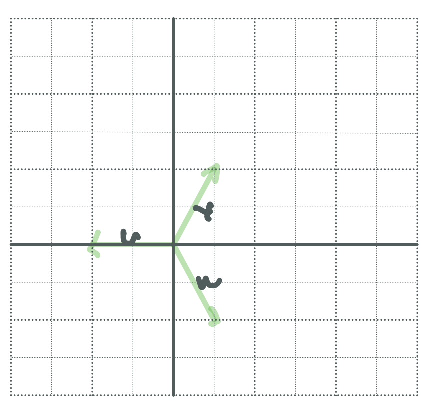
$\|v\| = 5$

$\|w\| = 3$

$\|v-w\|$ ranges from 2 to 8

$v \cdot w$ ranges from -15 to 15

30.



$u = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$u \cdot v = -2$

$v \cdot w = 1 - 4 = -3$

$u \cdot w = -2$

Yes

31.

$x + y + z = 0$

$x = 1$

$y = 2$

$z = -3$

$v = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \quad w = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

$$= \frac{-3+2-6}{1+4+9} = \frac{-7}{14} = -\frac{1}{2}$$

$$\boxed{\theta = \frac{2\pi}{3}}$$

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad w = \begin{bmatrix} z \\ x \\ y \end{bmatrix}$$

$$\cos\theta = \frac{v \cdot w}{\|v\|\|w\|}$$

$$\cos\theta = \frac{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} z \\ x \\ y \end{bmatrix}}{\sqrt{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}} \sqrt{\begin{bmatrix} z \\ x \\ y \end{bmatrix} \cdot \begin{bmatrix} z \\ x \\ y \end{bmatrix}}}$$

$$= \frac{xz + xy + yz}{\sqrt{(x^2 + y^2 + z^2)(z^2 + x^2 + y^2)}}$$

$$\cos\theta = \frac{xz + xy + yz}{x^2 + y^2 + z^2} = \frac{x(-x-y) + xy + y(-x-y)}{x^2 + y^2 + (-x-y)^2}$$

$$x + y + z = 0$$
$$z = -x - y$$

$$= \frac{-x^2 - xy + xy - xy - y^2}{x^2 + y^2 + x^2 + xy + xy + y^2}$$

$$= \frac{-1(x^2 + xy + y^2)}{2x^2 + 2xy + 2y^2}$$

$$\cos\theta = \frac{-1}{2}$$

$$\theta = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$

32. $(xyz)^{1/3} \le \frac{1}{3}(x+y+z)$

$$\sqrt{xy} \le \frac{x+y}{2}$$

$$xy \le \frac{(x+y)^2}{4}$$

$$4xy \le (x+y)^2$$

$$0 \le (x+y)^2 - 4xy$$

$$\le x^2 + xy + xy + y^2 - 4xy$$

$$\le x^2 - 2xy + y^2$$

$$0 \le (x-y)^2$$

$$\sqrt[3]{xyz} \le \frac{x+y+z}{3}$$

$$xyz \le \frac{(x+y+z)^3}{27}$$

$$27xyz \le (x+y+z)^3$$

$$0 \le (x+y+z)^3 - 27xyz$$

¯_(ツ)_/¯

33.

$$a = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad b = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$c = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \qquad d = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$a \cdot b = -\tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = 0$$
$$a \cdot c = \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} = 0$$
$$a \cdot d = \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} = 0$$
$$c \cdot b = -\tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} + \tfrac{1}{4} = 0$$
$$c \cdot d = \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4} = 0$$
$$b \cdot d = -\tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} - \tfrac{1}{4} = 0$$

34. See 1.2.ipynb

1.3 Matrices

1. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ is a 3×2 matrix

2. $Ax = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \end{bmatrix}$ is a <u>combination</u> of

$\begin{bmatrix} \cdot & \cdot \\ 5 & 6 \end{bmatrix} \begin{bmatrix} \cdot \\ x_2 \end{bmatrix}$ The columns

$$Ax = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

3. The 3 components of Ax are the dot products of the 3 rows of A with x.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 7 + 16 \\ 21 + 32 \\ 35 + 48 \end{bmatrix} = \begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}$$

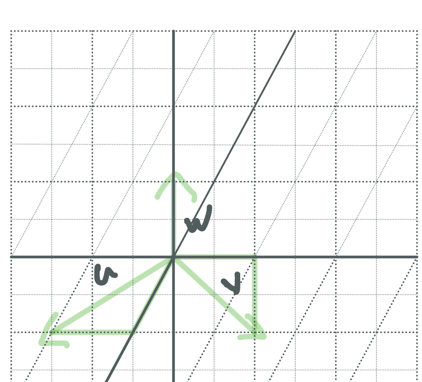
4. Equations in matrix form $Ax = b$

$\begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ replaces $\begin{matrix} 2x_1 + 5x_2 = b_1 \\ 3x_1 + 7x_2 = b_2 \end{matrix}$

5. The solution to $Ax = b$ can be written as

$x = A^{-1}b$. But not every A has a A^{-1}

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad w = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



$$x_1 \begin{bmatrix} 1 \\ -1 \\ \cdot \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \cdot \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ \cdot \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ \cdot \end{bmatrix}$$

$0 \qquad -1 \qquad 1 \qquad x_3 - x_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

A acts upon x

A is a difference matrix.

$$x = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \qquad Ax = \begin{bmatrix} 1+0+0 \\ -1+4+0 \\ 0-4+9 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = b$$

A is sorta instruction for transforming x into b.

Linear Equations

inverse problem

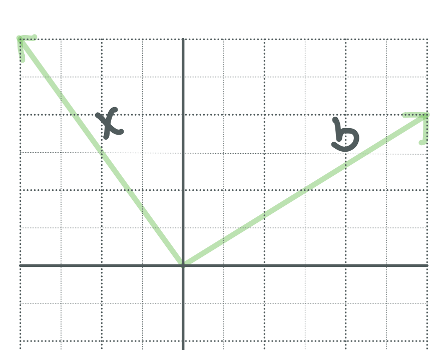
what about solving for A?

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\downarrow$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow A \begin{bmatrix} -2 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix}$$

This is a 90° rotation matrix!



Inverse of $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$?

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= b_1 \\ -x_1 + x_2 &= b_2 \\ -x_2 + x_3 &= b_3 \end{aligned} \quad \to \quad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \end{aligned}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

<u>Cyclic Differences</u>

9:05

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= b_1 + x_3 \\ x_2 &= b_2 + x_1 \\ &= b_2 + b_1 + x_3 \end{aligned}$$

$$\begin{bmatrix} x_1 & 0 & -x_3 \\ -x_1 & x_2 & 0 \\ 0 & -x_2 & x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$x_3 = b_3 + x_2$

$= b_3 + b_2 + b_1 + x_3$

$0 = b_3 + b_2 + b_1$

singular

$$\begin{bmatrix} x_1 - x_3 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\underbrace{x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

$x = -1$ $-y = 1$

$y - x = 0$ $y = -1$

$y = -1$

has a solution
all three vectors lie
on a plane

$$\underbrace{x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$x = 0$ $-y = 1$

$y - x = 0$ $0 \neq 1$

$y = 0$

has no solution
vectors fill R^3

Independence and Dependence

Independent Columns

- $Ax = 0$ has one solution, $x = 0$
- A has a A^{-1}
- column vectors fill the entire space
- A is an invertable matrix

Dependent Columns

[illegible] infinite solutions

- $Ax=0$ has multipl...
- A has no A^{-1}
- column vectors do not fill the space
  - some vectors provide no new information
- A is a singular matrix

Worked Examples

1.7A
$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1+x_2 \\ x_1-x_2+x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$x_1 = b_1$
$x_2 = b_1 + b_2$
$x_3 = b_2 + b_3$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ -1+2+0 \\ 1-2+3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$A^{-1}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0+0 \\ 1+1+0 \\ 0+1+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1.7B

$$E = \begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix} \quad E^{-1} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -l & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 20 \end{bmatrix} = \begin{bmatrix} 10 \\ -10l+20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix}\begin{bmatrix} 10 \\ -10l+20 \end{bmatrix} = \begin{bmatrix} 10 \\ 10l-10l+20 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

1.7C

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 - x_1 \\ -x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Problem Set 1.3

1. $3s_1 + 4s_2 + 5s_3 = b$

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad s_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad s_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = b = \begin{bmatrix} 3 \\ 3+4 \\ 3+4+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$$

$$\begin{aligned} x_1 &= b_1 \\ x_1 + x_2 &= b_2 \\ x_1 + x_2 + x_3 &= b_3 \end{aligned} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= -b_1 + b_2 \\ x_3 &= -b_2 + b_3 \end{aligned}$$

$$S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

$$\begin{aligned} y_1 &= 1 \\ y_2 &= 0 \\ y_3 &= 0 \end{aligned} \quad y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} y_1 &= 1 \\ y_2 &= -1+4 = 3 \\ y_3 &= -4+9 = 5 \end{aligned} \quad y = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$1+3+5+7+9=25$

3. $$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}$$

$$y_1=c_1$$
$$y_2=c_2-c_1$$
$$y_3=c_3-c_2$$

$$A^{-1}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}$$

$$A^{-1}c=y$$

independant

4. $$x_1w_1+x_2w_2+x_3w_3=0$$

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$1+4x_2+7x_3=0$$
$$4x_2=-7x_3-1$$
$$x_2=\frac{-1}{4}(7x_3+1)$$

$$2+5x_2+8x_3=0$$
$$5x_2=-8x_3-2$$
$$x_2=\frac{-2}{5}(4x_3+1)$$

$$3-12+9=0$$
True

$$\frac{1}{4}(7x_3+1)=\frac{2}{5}(4x_3+1)$$
$$7x_3+1=\frac{8}{5}(4x_3+1)$$
$$35x_3+5=32x_3+8$$
$$3x_3=3$$
$$x_3=1$$

$$x_2=\frac{-1}{4}(7(1)+1)$$
$$=\frac{-1}{4}(8)$$
$$x_2=-2$$

$$x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ dependent, plane

5.
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad x = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$x_1 + 2x_2 + 3x_3 = 0$$
$$4x_1 + 5x_2 + 6x_3 = 0$$
$$7x_1 + 8x_2 + 9x_3 = 0$$

$$x_1 = 1$$

$$1 + 2x_2 + 3x_3 = 0$$
$$4 + 5x_2 + 6x_3 = 0$$
$$7 + 8x_2 + 9x_3 = 0$$

$$x_2 = \frac{-1}{2}(3x_3 + 1) \qquad x_2 = -2$$
$$x_2 = \frac{-1}{5}(6x_3 + 4)$$
$$x_2 = \frac{-1}{8}(9x_3 + 7)$$

$$\frac{-1}{8}(9x_3 + 7) = \frac{-1}{2}(3x_3 + 1)$$
$$\frac{1}{4}(9x_3 + 7) = 3x_3 + 1$$
$$9x_3 + 7 = 12x_3 + 4$$

$$7-4 = 12x_3 - 9x_3$$
$$3 = 3x_3$$
$$x_3 = 1$$

6.

$$\begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ c \end{bmatrix}$$

$$x + y = 0$$
$$x = -y$$

$$3x + 2y = 1$$
$$-3y + 2y =$$
$$y = -1$$
$$x = 1$$

$$7 - 4 = c$$
$$\boxed{c = 3}$$

$$x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ 0 \\ 1 \end{bmatrix}$$

$$x = c$$
$$x + y = 0$$
$$y = 1$$
$$x = -1$$
$$\boxed{c = -1}$$

$$x\begin{bmatrix} c \\ 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} c \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} c \\ 5 \\ 6 \end{bmatrix}$$

$2x + y = 5$

$y = -2x + 5$

$6 + y = 5$

$y = -1$

$3x + 3y = 6$

$3x + 3(-2x + 5) = 6$

$3x - 6x + 15 = 6$

$-3x = -9$

$x = 3$

$3c - c = c$

$2c = c$

$c = 0$

7. If two vectors have a dot product of 0 then they are perpendicular to each other. Thus:

$$\begin{bmatrix} r_1 \cdot x \\ r_2 \cdot x \\ r_3 \cdot x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Means r_1, r_2, and r_3 are all

perpendicular to x.

8. $x_1 = b_1$

$x_2 = b_1 + b_2$

$x_3 = b_1 + b_2 + b_3$

$x_4 = b_1 + b_2 + b_3 + b_4$

$$x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} b$$

9.

$$C = \begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad Cx = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = x_4$ $\qquad$ $x_1 = x_2 = x_3 = x_4$

$x_2 = x_1$

$x_3 = x_2$

$x_4 = x_3$

10.

$$Az = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$z_1 = -b_1 - b_2 - b_3$$
$$z_2 = -b_2 - b_3$$
$$z_3 = -b_3$$

$$A^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

11. ?

12.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$x_1 = -b_2 - b_4$$
$$x_2 = b_1$$
$$x_3 = -b_4$$
$$x_4 = b_1 + b_3$$

$$C^{-1} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+2+0+0 \\ -5+0+4+0 \\ 0-2+0+1 \\ 0+0-4+0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ -1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0+1+0+4 \\ 2+0+0+0 \\ 0+0+0+4 \\ 2+0-1+0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \\ 1 \end{bmatrix}$$

13.

$$\begin{array}{c} x_2 \\ x_1 \\ ? \\ x_5 \\ x_4 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$$

$x_2 = b_1$

$x_1 = x_3 - b_2$

$x_4 = b_1 + b_3$

$x_5 = -x_3 + b_4$

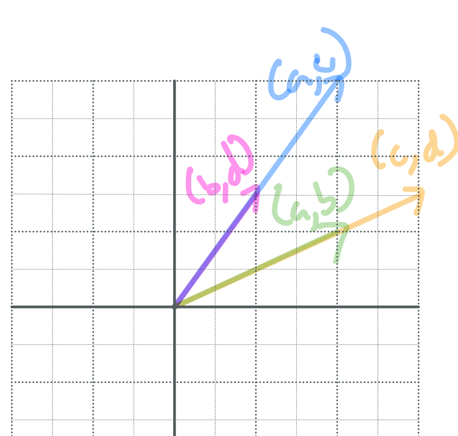
$x_4 = -b_5$

$b_1 + b_3 = -b_5$

$b_1 + b_3 + b_5 = 0$

singular

14.



$(a,b) = (4,2)$

$(c,d) = (6,3)$

$(a,c) = (4,6)$

$(b,d) = (2,3)$

$$x\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}c\\d\end{bmatrix} \longrightarrow y\begin{bmatrix}a\\c\end{bmatrix}=\begin{bmatrix}b\\d\end{bmatrix}$$

$ax = c$

$x = \frac{c}{a}$

$bx = d$

$\frac{bc}{a} = d$

$\underline{bc = ad}$

$ay = b$

$y = \frac{b}{a}$

$yc = d$

$\frac{bc}{a} = d$

$\underline{bc = ad}$

$$z\begin{bmatrix}d\\b\end{bmatrix}=\begin{bmatrix}c\\a\end{bmatrix} \qquad w\begin{bmatrix}d\\c\end{bmatrix}=\begin{bmatrix}b\\a\end{bmatrix}$$

$zd = c$

$z = \frac{c}{d}$

$zb = a$

$\underline{bc = ad}$

$wd = b$

$w = \frac{b}{d}$

$wc = a$

$\frac{bc}{a} = a$

$\underline{bc = ad}$