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$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Example

In a packaging plant, a machine packs carton with jars. It is supposed that a new machine will pack faster on the average than the machine currently used. To test that hypothesis the time it takes each machine to pack ten cartons are recorded. The results seconds, are shown in the following table:

New Machine	Old machine
42.1, 41.3, 42.4, 43.2, 41.8, 41.0, 41.8, 42.8, 42.3, 42.7	42.7, 43.8, 42.5, 43.1, 44.0, 43.6, 43.3, 43.5, 41.7, 44.1

Do the data provide sufficient evidence to conclude that, on the average, the new machine packs faster? Perform the required hypothesis test at the 5% level of significance.

→ Solution:

Step 1: Setting up Null and Alternative Hypothesis

Let X_1 = Packing time of new machine (sec)

X_2 = Packing time of old machine (sec) (Machine currently in use).

Let μ_1 and μ_2 be the mean packing time of new & old machines respectively.

Null Hypothesis:

$H_0 : \mu_1 = \mu_2$ (There is no significant difference in the packing time of new and old machines)

Alternative Hypothesis:

$H_1 : \mu_1 < \mu_2$ (The mean packing time of new machine is significantly lesser than that of old machine)

Step 2: Choice of α for the test

α = level of significance of the test

= Prob{Type I error}

= 0.05.

Step 3: Test statistics.

The appropriate test statistics for the test is given by,

$$t = \frac{\bar{X_1} - \bar{X_2}}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

where,

S_p = pooled estimate of common SD. σ.

$$= \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}}$$

$\bar{X_1}$ = mean of first sample

$\bar{X_2}$ = mean of second sample

n_1 = size of first sample

n_2 = size of second sample

S_1 = S.D. of first sample

S_2 = SD of second sample

Probability distribution of t is students t-distribution with n_1+n_2-1 d.f.

$\alpha(1)$ 0.05

Step 4 : Critical t (Tabulated t)

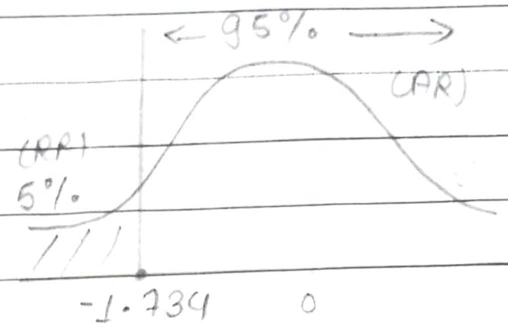
The test is left sided and $\alpha = 5\%$. 18 →

Degrees of freedom $= n_1+n_2-2$

$= 10+10-2 = 18$.

Reject	Accept Ho

-1.734



critical t from table

$t_c = t_{0.05}(18)$

$= -1.734$

AR : $t > -1.734$

RR : $t \le -1.734$.

Step 5: calculated t (observed t)

Now,

$\Sigma X_1 = 421.4$

$\Sigma X_1^2 = 17762$.

$\bar{X}_1 = \dfrac{\Sigma X_1}{n_1}$

$= \dfrac{421.4}{10}$

$= 42.14$

$\Sigma X_2 = 432.3$

$\Sigma X_2^2 = 18693.39$

$\bar{X}_2 = \dfrac{\Sigma X_2}{n_2}$

$= \dfrac{432.3}{10}$

$= 43.23$

Now,

$SD_1 = \sqrt{\dfrac{1}{n-1}\left\{\Sigma X_1^2 - n_1 \cdot \bar{X}_1^2\right\}}$

$SD_2 = \sqrt{\dfrac{1}{n-1}\left\{\Sigma X_2^2 - n_2 \cdot \bar{X}_2^2\right\}}$

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$$= \sqrt{\frac{1}{10-1}\left\{17762 - 10 \times 42.14^2\right\}} \qquad = \sqrt{\frac{1}{9}\left\{18693.39 - 10 \times 43.23^2\right\}}$$

$$= 0.68345 \qquad = 0.749888$$

Now,

$$S_p = \sqrt{\frac{(10-1)(0.6835)^2 + (10-1)(0.7499)^2}{10+10-2}}$$

$$= 0.71747$$

$$t = \frac{42.14 - 43.23}{0.7175\sqrt{\frac{1}{10} + \frac{1}{10}}}$$

$$= -3.39695$$

Step 6: Statistical Decision

Since cal $t = -3.397$ falls in the lower rejection region ($t \leq -1.734$), we strongly reject H_0 in favour of H_1.

Step 7: Conclusion.

The mean packing time of new machine is significantly lesser than that of old machine, i.e. the new machine is more efficient than the old machine.

5. Paired t-test

→

Function of the test

The paired samples t-test compares two means that are from the same individual, object or related units. The two means typically represent two diffⁿ times (eg: pre-test and post test with an intervention between the two time points) or two different but related condition or unit (eg: left & right ears, twins). The purpose of the test is to determine whether there is statistical evidence that the mean ~~dev~~ difference between paired observations on a particular outcome is significantly different from zero.

The test is known as Dependent sample t. Test or Repeated measures of t Test.

Test Assumptions

1. Dependent variable is continuous (interval or ratio scale data)
2. Sample must be related or dependent. This means that the subjects in the first group are also in the second group.
3. Random sample of data.
4. Normal distribution (approx.) of the difference between the paired values.
5. No outlines in the difference between the two related groups.

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Null and Alternative Hypothesis

Let Δ be the difference (Before-After) between two sets of observations in the population. If X_1 = measurement in the first occasion and X_2 = measurements in second occasion, then $\Delta = X_1 - X_2$. Let μ_Δ be the average diff^n of two sets of measurements, i.e. average of the difference Δ.

The null and alternative hypothesis are written as follows:-

Null Hypothesis	Alternative	No. of tails.
$H_0: \mu_\Delta = 0$ or $H_0: \mu_1 - \mu_2 = 0$ (the diff^n bet^n paired pop^n means is equal to 0)	$H_1: \mu_\Delta \neq 0$	Two tailed test.
$H_1: \mu_1 - \mu_2 \neq 0$ (the diff^n bet^n the paired pop^n means is not 0)		
$H_0: \mu_\Delta > 0$	$H_1: \mu_\Delta < 0$	Lower tailed test
$H_0: \mu_\Delta \leq 0$	$H_1: \mu_\Delta > 0$	Right tailed test

~~The~~ Test Statistics

The test statistics for the paired sample t-test is given by:

$$t = \frac{\bar{x}_d - \mu_d}{s_D/\sqrt{n}} = \frac{\bar{x}_d}{s_d/\sqrt{n}}$$

Under null hypothesis H_0, the value of pop^n mean diff^n μ_d is assumed to be zero.

where,

$\bar{d}$ or $\bar{x}_d$ = Sample mean of the differences

n = sample size (no. of observations)

S_d = Sample SD of the differences

$S_d/\sqrt{n}$ = Standard error of the sample differences d.

Decision Rule

We adopt the following decision rule:

Hypothesis

Case I (Two sided) Reject Ho if $|cal\ t| \geq t_{\alpha/2}(n-1)$

Case II (Left sided) Reject Ho if $|cal\ t| \leq t_{\alpha}(n-1)$

Case III (Right ") Reject Ho if $|cal\ t| \geq +t_{\alpha}(n-1)$

Two tailed

Reject	Donot reject	Reject Ho

$-t_{\alpha/2}(n-1)$ $\qquad$ $+t_{\alpha/2}(n-1)$

Left tailed

Reject Ho	Accept Ho.

$-t_{\alpha}(n-1)$

Right tailed

Accept	Reject Ho

$+t_{\alpha}(n-1)$

Problem

Fifteen secretaries of a certain corporation were sent for a two day training to increase their typing skills. The table below shows the typing speed of the secretaries in words per minute before and after the training?

Secretary	Before	After
1	75	75
2	60	65
3	54	59
4	67	66
5	60	65
6	85	86
7	60	70
8	74	71
9	69	68
10	85	86
11	82	80
12	68	70
13	70	75
14	58	57
15	72	80

Assuming the data to be normally distributed $\alpha = 0.05$ is there an evidence for the training inproved the typing speed of the secretaries.

→ Solution:

Step 1: Setting up Null and Alternative Hypothesis

Let X_1 = Typing speed of secretaries before training program (mins)

X_2 = Typing speed of secretaries after training program (mins).

μ_1 = Mean typing speed of secretaries before training program.

μ_2 = Mean typing speed of secretaries after.

Let Δ be the difference between scores (Before - After $= X_1 - X_2$) and μ_Δ be the mean of such differences.

Null Hypothesis

$H_0: \mu_\Delta = 0$ (Training program isn't effective)

Alternative Hypothesis

$H_1: \mu_\Delta < 0$ (Training program is effective)

Step 2: choice of α for the test

α = level of significance of the test

= Prob {Type I error}

= 0.05

Step 3: Test statistics

The test statistics for this test is,

$$t = \frac{\bar{d}}{s_d/\sqrt{n}}$$

where,

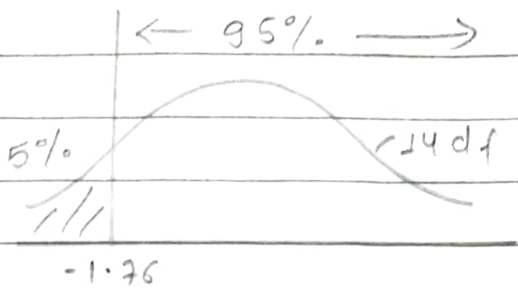
$\bar{d}$ = mean of sample difference $d = X_1 - X_2$

S_d = S.D. of sample difference d

n = No. of pairs of data = sample size.

The probability distribution of the test statistics is student t-distribution with n-1 d.f.

Step 4: Critical t (Tabulated t)
The test is left sided and $\alpha = 5\%$.
The rejection region lies in the left tail of t-curve.
From table



$d.f. = n-1$
$= 15-1 = 14$
$t_c = t_{0.05}(14) = -1.76$

Reject Ho	Accept Ho

-1.76

AR: $t > -1.76$
RR: $t \leq -1.76$

Step 5: calculated of t

SN	Typing speed		d=	
	Before (X_1)	After (X_2)	$d = X_1 - X_2$	d^2
1.	75	75	0	0
2.	60	65	-5	25
3.	54	59	-5	25
4.	67	66	1	1
5.	60	65	-5	25
6.	85	86	-1	1
7.	60	70	-10	100
8.	74	71	3	9
9.	69	68	1	1
10.	85	86	-1	1

11.	82	80	2	4
12.	68	70	-2	4
13.	70	75	-5	25
14.	58	57	1	1
15.	72	80	-8	64
			$\Sigma d = -34$	$\Sigma d^2 = 286$

Now,

$$\bar{d} = \frac{\Sigma d}{n} = \frac{-34}{15} = -2.24$$

$$Sd = \sqrt{\frac{1}{n-1}\left\{\Sigma d^2 - n.\bar{d}^2\right\}}$$

$$= \sqrt{\frac{1}{14}\left\{286 - 15 \times (-2.24)^2\right\}}$$

$$= 3.86.$$

$$t = \frac{-2.27}{3.86/\sqrt{15}} = -2.2776.$$

Step 6 : Statistical Decision

Since calt = -2.28 falls in rejection region ($t \leq 1.76$), we reject Ho in favour of H_1 at 5% level of significance.

Step 7 : Conclusion

secretaries are benefited by training program. Training program is effective. The secretaries have improved their typing speed and training has enhances their typing speed.

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6. Z-test for proportion (π), single sample case

→

Function of test:

The function of this test is to compare observed (or sample) proportion of success with the theoretical proportion of success i.e. to test whether population proportion p of success is significantly different from some hypothesized value.

Test Assumptions:

1. Variable of interest is categorical and having two categories.
2. Sample is drawn randomly from population.
3. Sample size is relatively large so that sampling distribution of sample proportion of success is approx normally distributed

Hypothesis to test

Let π be the true proportion of success in the population and let π_0 be the hypothesized value. Then null and alternative hypothesis for the test is given below:

Null Hypothesis	Alternative	No. of tails
$H_0: \pi = \pi_0$	$H_1: \pi \neq \pi_0$	Two
$H_0: \pi \geq \pi_0$	$H_1: \pi < \pi_0$	One
$H_0: \pi \leq \pi_0$	$H_1: \pi > \pi_0$	One

Test statistics

The approximate test statistics under H_0 (if H_0 is true) is given by,

$$Z = \frac{p - \pi_0}{\sqrt{\pi(1-\pi_0)/n}}$$

where,

p = sample proportion of success

n = sample size

π_0 = Hypothesized value of proportion of success

$1-\pi_0$ = Hypothesized value of propⁿ of failure

The quantity $\sqrt{\pi_0(1-\pi_0)/n}$ is called standard error of p.

Distribution of test statistics.

The test statistics is distributed as Z i.e. standard Normal Distribution with mean = 0 and variance = 1.

Decision Rule

We will adopt following decision rule.

Hypothesis	
Case I (Two sided)	Reject Ho if $\lvert cal\ Z \rvert \geq Z_{\alpha/2}$, i.e. Accept Ho. $-Z_{\alpha/2} < cal\ Z < +Z_{\alpha/2}$
Case II (Left ")	Reject Ho if $cal\ Z \leq -Z_0$
Case III (Right ")	Reject Ho if $cal\ Z \geq +Z_0$

The critical value of Z.

Significance level (α)	Two tailed	Left tailed	Right tailed
5%	1.96	-1.65	+1.65
1%	2.58	-2.33	+2.33

Numerical

An e-commerce research company claims that 60% or more graduate students have bought merchandise on-line. A consumer group is suspicious of the claim and thinks that the proportion is lower than 60%. A random sample of 80 graduate students show that only 22 students have ever done so. Is there enough evidence to show that the true proportion is lower than 60%? Conduct the test at 5% Type I error rate, and use the p-value and rejection region approaches.

→ Solution

Step 1: Setting up Null and Alternative Hypothesis

Population of graduates are distributed as follows:

- If 60% or more graduate us ecommerce for online purchase (success)
- if less than 60% of graduate use ecommerce for online purchases (failure)

Let π be the proportion of graduates who use e-commerce for online purchase.

Null Hypothesis

$H_0: \pi \geq 60\%$ (claim of research company)

Alternative Hypothesis

$H_1: \pi < 60\%$ (consumer's claim)

Step 2: Choice of α for the test

α = Level of significance
= prop{Type I error}
= 0.05

Step 3: Test statistics

The approximate test statistics for the test is given by,

$$z = \frac{p - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$$

$$= \frac{z}{\sqrt{\frac{P_0 Q_0}{n}}}$$

where,

p = Sample proportion of success

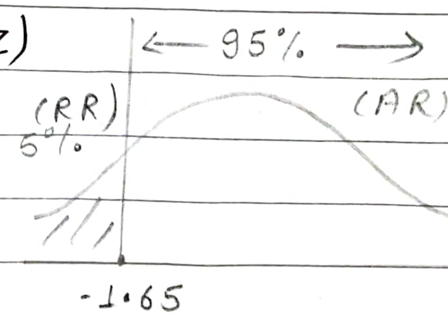
π_0 = Hypothesize value of π
(value of π assumed value H_0)

n = sample size

The probability distribution of test statistics is z with mean = 0 and SD = 1.

Step 4: Critical z (Tabulated z)

Test is left-sided and $\alpha = 5\%$



From table of z-distribution

z_c = critical z
$= z_{0.05}$
$= -1.65$

AR: $z > -1.65$

RR: $z \leq 1.65$

Reject H_0	Accept H_0

-1.65

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Step 5 : Calculated z (observed z)

p = sample proportion of success.

$= X/n$ X = No. of successes in sample

$= \frac{22}{80}$ = 22

n = sample size = 80

$= 0.275$

$$\therefore Z = \frac{p - \pi_0}{\sqrt{\frac{\pi_0(1-\pi_0)}{n}}}$$

$$= \frac{0.275 - 0.60}{\sqrt{\frac{0.60 \times 0.40}{80}}}$$

$$= -5.933$$

Step 6 : Statistical Decision

Since cal z = -5.933 << Tab z = -1.65, we strongly reject Ho at 5% level of significance, in favour of H1.

Step 7 : Conclusion

The claim of research company is not valid. Actually less than 60% graduates use e-commerce for online purchase.

7. Two samples z-test for diff^n of two proportions
→

Function of the test
This test allows us to compare two pop^n proportions of success to know if they are really different or not.

Test Assumption
1. Measurements are categorical (eg: Yes or No to the question "Do you smoke?") and taken from two distinct groups (eg: Female, Male)
2. Samples are drawn randomly from each pop^n using SRS.
3. Samples are independent.
4. Both samples are large, i.e. at least 30 observations are drawn in each sample, which guarantee the normal distribution of sampling distrib^n of diff^n of sample proportion of success.

Hypothesis to test
Let π_1 and π_2 be the proportion of success in pop^n I and II respectively. Then we have following hypothesis testing problems.

Two tailed test	Left tailed test	Right tailed test
$H_0: \pi_1 = \pi_2$	$H_0: \pi_1 \geq \pi_2$	$H_0: \pi_1 \leq \pi_2$
$H_1: \pi_1 \neq \pi_2$	$H_1: \pi_1 < \pi_2$	$H_1: \pi_1 > \pi_2$

Test statistics

The approximate formula for the test statistic comparing two proportions is given by:

$$Z = \frac{(\text{observed diff}^n \text{ in sample proportion}) - (\text{hypothesized diff}^n \text{ of props}^{ns})}{\text{standard error of diff}^n \text{ of proportions}}$$

$$= \frac{p_1 - p_2}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where,

p_1 = prop^n of success for sample 1

p_2 = prop^n of success for sample 2.

n_1 = No. of obs. in sample 1.

n_2 = No. of obs. in sample 2.

$\hat{\pi}$ = pooled estimate of common prop^n of success.

$1-\hat{\pi}$ = pooled estimate of common prop^n of failure

Numerical

A political consultant wondered if support for a candidate was significantly different between men and women. The candidate surveyed a random sample of voters. Here are the results.

Support candidates?	Men	Women
Yes	242424	484848
No	565656	525252
Total	808080	10 10 100

A political consultant ~~foondered~~ if ~~sypport~~ ~~for~~ a candidate ~~was~~ significantly

The consultant wants to test if these results suggests a significant difference in support between men and women. Can we conclude from these data that in the sampled proportions there is a difference in the proportions?

Let $\alpha = 0.05$.

→ Solution:

Step 1: Setting of Null and Alternative Hypothesis.

Variable = support for candidate (yes/NO)

Support for candidate (yes) - success.

No support for candidate (No) - failure.

Let π_1 be the proportion of voters who said 'yes' among male voters and π_2 be the proportion of voters who said 'yes' among female voters.

Null Hypothesis

$H_0: \pi_1 = \pi_2$ (The candidate is equally popular among the male and female voters).

Alternative Hypothesis

$H_1: \pi_1 \neq \pi_2$ (There is significant difference in support between Male and Female voters for the candidate)

Step 2: Choosing α for test

α = level of significance

= prop{Type I error}

= 0.05.

Step 3: Test statistics

The appropriate test statistics is given by;

$$Z = \frac{p_1 - p_2}{\sqrt{\hat{\pi}(1-\hat{\pi})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where,

p_1 = proportion of success in sample 1.

p_2 = proportion of success in sample 2.

n_1 = size of sample in sample 1.

n_2 = size of sample in sample 2.

$\hat{\pi}$ = Pooled estimate of common proportion of success π

$$= \frac{n_1 p_1 + n_2 p_2}{n_1 + n_2}$$

$$= \frac{X_1 + X_2}{n_1 + n_2}$$

X_1 = No. of successes in sample 1

X_2 = No. of successes in sample 2

The test statistics has SND with mean = 0 and S.D = 1.

Step 4: Critical Z (Tabulated Z)

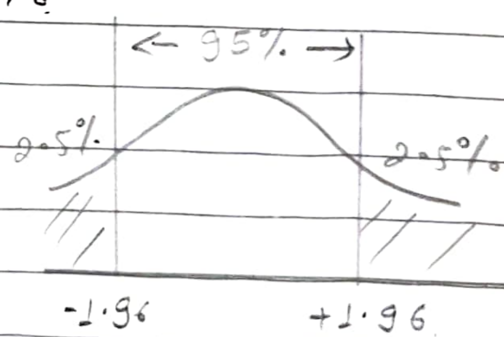
The test is two-sided and $\alpha = 5\%$.

From table of Z-distribution,

Z_c = critical Z.

$= Z_{0.025}$

$= 1.96$



Reject	Accept H_0	Reject

-1.96 +1.96.

AR: $-1.96 < Z < +1.96$

RR: $Z \geq +1.96$ OR $Z \leq -1.96$

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Step 5: calculated Z

p_1 = propn of 'yes' response among male candidate

$$= \frac{x_1}{n_1} = \frac{242424}{808080} = 0.30$$

p_2 = propn of 'yes' response among female "

$$= \frac{x_2}{n_2} = \frac{484848}{1010100} = 0.48$$

$$\therefore \hat{\pi} = \frac{x_1 + x_2}{n_1 + n_2} = \frac{242424 + 484848}{808080 + 1010100}$$

$$= 0.4$$

$$\therefore 1 - \hat{\pi} = 1 - 0.4 = 0.6$$

$$\therefore Z = \frac{0.30 - 0.48}{\sqrt{0.4 \times 0.6 \left(\frac{1}{808080} + \frac{1}{1010100} \right)}}$$

$$= -246.18$$

Step 6: Statistical Decision

Since cal Z = −246.18 ≪ lower critical Z = −1.96, we strongly reject H_0 at 5% level of significance in favour of H_1.

Step 7: Conclusion

There is significant difference in support between Male and female voters for the candidate.. Since $p_2 = 0.48$ and $p_1 = 0.30$, it demonstrates that the candidate is more popular among female candidate.

Till class test