MTH 437 Project
Section 5.6 Multistep Methods

Group 9:
Jaeheok Kwak, Woosung Lee, Yiling Chen

Section 5.6 Multistep Methods:Introduction

An ***m*-step multistep method** for solving the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha, \tag{5.23}$$

has a difference equation for finding the approximation w_{i+1} at the mesh point t_{i+1} represented by the following equation, where m is an integer greater than 1:

$$\begin{aligned} \boxed{w_{i+1}} = {} & a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ & + h[\boxed{b_m}\boxed{f(t_{i+1}, w_{i+1})} + b_{m-1}f(t_i, w_i) \\ & + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})], \end{aligned} \tag{5.24}$$

for $i = m-1, m, \ldots, N-1$, where $h = (b-a)/N$, the $a_0, a_1, \ldots, a_{m-1}$ and $b_0, b_1, \ldots, b_m$ are constants, and the starting values

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad \ldots, \quad w_{m-1} = \alpha_{m-1}$$

are specified. ■

Section 5.6 Multistep Methods: Linear interpolating polynomial

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

$$\int_{t_i}^{t_{i+1}} y' = \int_{t_i}^{t_{i+1}} f(t, y)$$

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f(t_i, y(t_i))\, dt$$

$$y(t_{i+1}) = y(t_i) + f(t_i, y(t_i))\underbrace{(t_{i+1} - t_i)}_{\to h}$$

$$= y(t_i) + h \cdot P(t)$$

$* \; f(t_i, y(t_i)) = f_i$

Linear interpolating polynomial on interval $[t_{i-1}, t_i]$

$$y(t_{i+1}) = y(t_i) + \int_{t_i}^{t_{i+1}} \left[f_i + \left(\frac{f_i - f_{i-1}}{h}\right)(t - t_i)\right] dt$$

$$= y(t_i) + \left| f_i + \left(\frac{f_i - f_{i-1}}{h}\right)\frac{(t - t_i)^2}{2} \right|_{t_i}^{t_{i+1}}$$

$$= y(t_i) + hf_i + \left(\frac{f_i - f_{i-1}}{\cancel{h}}\right)\frac{h^{\cancel{2}}}{2}$$

$$= y(t_i) + \frac{h}{2}(3f_i - f_{i-1})$$

$$\therefore \; w_{i+1} = w_i + \frac{h}{2}\left[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})\right]$$

Section 5.6 Multistep Methods:Local Truncation Error

If $y(t)$ is the solution to the initial-value problem

$$y' = f(t, y), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and

$$\begin{aligned} w_{i+1} &= a_{m-1}w_i + a_{m-2}w_{i-1} + \cdots + a_0 w_{i+1-m} \\ &\quad + h[b_m f(t_{i+1}, w_{i+1}) + b_{m-1} f(t_i, w_i) + \cdots + b_0 f(t_{i+1-m}, w_{i+1-m})] \end{aligned}$$

is the $(i+1)$st step in a multistep method, the **local truncation error** at this step is

$$\begin{aligned} \tau_{i+1}(h) &= \frac{y(t_{i+1}) - a_{m-1}y(t_i) - \cdots - a_0 y(t_{i+1-m})}{h} \\ &\quad - [b_m f(t_{i+1}, y(t_{i+1})) + \cdots + b_0 f(t_{i+1-m}, y(t_{i+1-m}))], \end{aligned} \tag{5.32}$$

for each $i = m-1, m, \ldots, N-1$. ■

Section 5.6 Multistep Methods: Calculation process

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$

$$w_{i+1} = w_i + \frac{h}{24}[\boxed{55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2})} - \boxed{9f(t_{i-3}, w_{i-3})}], \tag{5.25}$$

for each $i = 3, 4, \ldots, N-1$, define an *explicit* four-step method known as the **fourth-order Adams-Bashforth technique.** The equations

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})], \tag{5.26}$$

for each $i = 2, 3, \ldots, N-1$, define an *implicit* three-step method known as the **fourth-order Adams-Moulton technique.**

Section 5.6 Multistep Methods: Adams-Bashforth methods

Adams-Bashforth Explicit Methods

Some of the explicit multistep methods, together with their required starting values and local truncation errors, are as follows. The derivation of these techniques is similar to the procedure in Examples 2 and 3.

Adams-Bashforth Two-Step Explicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{2}[3f(t_i, w_i) - f(t_{i-1}, w_{i-1})], \tag{5.33}$$

where $i = 1, 2, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = \frac{5}{12}y'''(\mu_i)h^2$, for some $\mu_i \in (t_{i-1}, t_{i+1})$.

Adams-Bashforth Three-Step Explicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{12}[23f(t_i, w_i) - 16f(t_{i-1}, w_{i-1}) + 5f(t_{i-2}, w_{i-2})], \tag{5.34}$$

where $i = 2, 3, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = \frac{3}{8}y^{(4)}(\mu_i)h^3$, for some $\mu_i \in (t_{i-2}, t_{i+1})$.

Section 5.6 Multistep Methods: Adams-Bashforth methods

Adams-Bashforth Four-Step Explicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$

$$w_{i+1} = w_i + \frac{h}{24}[55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})], \tag{5.35}$$

where $i = 3, 4, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = \frac{251}{720}y^{(5)}(\mu_i)h^4$, for some $\mu_i \in (t_{i-3}, t_{i+1})$.

Adams-Bashforth Five-Step Explicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3, \quad w_4 = \alpha_4,$$

$$\begin{aligned} w_{i+1} = w_i + \frac{h}{720}[&1901f(t_i, w_i) - 2774f(t_{i-1}, w_{i-1}) \\ &+ 2616f(t_{i-2}, w_{i-2}) - 1274f(t_{i-3}, w_{i-3}) + 251f(t_{i-4}, w_{i-4})], \end{aligned} \tag{5.36}$$

where $i = 4, 5, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = \frac{95}{288}y^{(6)}(\mu_i)h^5$, for some $\mu_i \in (t_{i-4}, t_{i+1})$.

Section 5.6 Multistep Methods: Adams-Moulton methods

Adams-Moulton Implicit Methods

Implicit methods are derived by using $(t_{i+1}, f(t_{i+1}, y(t_{i+1})))$ as an additional interpolation node in the approximation of the integral

$$\int_{t_i}^{t_{i+1}} f(t, y(t))\, dt.$$

Adams-Moulton Two-Step Implicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1,$$

$$w_{i+1} = w_i + \frac{h}{12}[5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})], \tag{5.37}$$

where $i = 1, 2, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = -\frac{1}{24}y^{(4)}(\mu_i)h^3$, for some $\mu_i \in (t_{i-1}, t_{i+1})$.

Adams-Moulton Three-Step Implicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2,$$

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})], \tag{5.38}$$

where $i = 2, 3, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = -\frac{19}{720}y^{(5)}(\mu_i)h^4$, for some $\mu_i \in (t_{i-2}, t_{i+1})$.

Section 5.6 Multistep Methods: Adams-Moulton methods

Adams-Moulton Four-Step Implicit Method

$$w_0 = \alpha, \quad w_1 = \alpha_1, \quad w_2 = \alpha_2, \quad w_3 = \alpha_3,$$

$$w_{i+1} = w_i + \frac{h}{720}[251 f(t_{i+1}, w_{i+1}) + 646 f(t_i, w_i) - 264 f(t_{i-1}, w_{i-1})$$

$$+ 106 f(t_{i-2}, w_{i-2}) - 19 f(t_{i-3}, w_{i-3})], \tag{5.39}$$

where $i = 3, 4, \ldots, N-1$. The local truncation error is $\tau_{i+1}(h) = -\frac{3}{160} y^{(6)}(\mu_i) h^5$, for some $\mu_i \in (t_{i-3}, t_{i+1})$.

Section 5.6 Multistep Methods: Adam-Bashforth example

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$ h=0.2

$y(0) = w_0 = 0.5, \quad y(0.2) \approx w_1 = 0.8292933, \quad y(0.4) \approx w_2 = 1.2140762,$ and $\quad y(0.6) \approx w_3 = 1.6489220.$

Solution For the fourth-order Adams-Bashforth method, we have

$$\begin{aligned}
y(0.8) \approx w_4 &= w_3 + \frac{0.2}{24}(55f(0.6, w_3) - 59f(0.4, w_2) + 37f(0.2, w_1) - 9f(0, w_0)) \\
&= 1.6489220 + \frac{0.2}{24}(55f(0.6, 1.6489220) - 59f(0.4, 1.2140762) \\
&\quad + 37f(0.2, 0.8292933) - 9f(0, 0.5)) \\
&= 1.6489220 + 0.0083333(55(2.2889220) - 59(2.0540762) \\
&\quad + 37(1.7892933) - 9(1.5)) \\
&= 2.1272892
\end{aligned}$$

t_i	Exact	Adams-Bashforth w_i	Error
0.8	2.1272295	2.1273124	0.0000828

Section 5.6 Multistep Methods: Adams-Moulton example

$$w_{i+1} = w_i + \frac{h}{24}[9f(t_{i+1}, w_{i+1}) + 19f(t_i, w_i) - 5f(t_{i-1}, w_{i-1}) + f(t_{i-2}, w_{i-2})],$$

for $i = 2, 3, \ldots, 9$. This reduces to

$$w_{i+1} = \frac{1}{24}[1.8w_{i+1} + 27.8w_i - w_{i-1} + 0.2w_{i-2} - 0.192i^2 - 0.192i + 4.736].$$

To use this method explicitly, we meed to solve the equation explicitly solve for w_{i+1}. This gives

$$w_{i+1} = \frac{1}{22.2}[27.8w_i - w_{i-1} + 0.2w_{i-2} - 0.192i^2 - 0.192i + 4.736],$$

for $i = 2, 3, \ldots, 9$.

$y=(0.8) \approx W_4 = \frac{1}{22}[27.8W_3 - W_2 + 0.2W_1 - 0.192(3)^2 - 0.192(3) - 0.192(3) + 4.736$

t_i	Exact	Adams-Bashforth w_i	Error	Adams-Moulton w_i	Error
0.8	2.1272295	2.1273124	0.0000828	2.1272136	0.0000160

Section 5.6 Multistep Methods: Adams Fourth-Order Predictor-Corrector

ALGORITHM 5.4

Adams Fourth-Order Predictor-Corrector

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N + 1)$ values of t.

Step 1 Set $h = (b - a)/N$;
$t_0 = a$;
$w_0 = \alpha$;
OUTPUT (t_0, w_0).

Step 2 For $i = 1, 2, 3$, do Steps 3–5.
(*Compute starting values using Runge-Kutta method.*)

Step 3 Set $K_1 = hf(t_{i-1}, w_{i-1})$;
$K_2 = hf(t_{i-1} + h/2, w_{i-1} + K_1/2)$;
$K_3 = hf(t_{i-1} + h/2, w_{i-1} + K_2/2)$;
$K_4 = hf(t_{i-1} + h, w_{i-1} + K_3)$.

Step 4 Set $w_i = w_{i-1} + (K_1 + 2K_2 + 2K_3 + K_4)/6$;
$t_i = a + ih$.

```
def adamsForthOrderPC(a,b,N,alpha):
```

```
t0,w0 = a,alpha
h = (b-a)/N
```

```
for i in range(1,4):

    k1 = f(t,w)*h
    k2 = h*f(t+(h/2),w+(k1/2))
    k3 = h*f(t+(h/2),w+(k2/2))
    k4 = h*f(t+h,w+k3)

    w = w + (k1+2*k2+2*k3+k4)/6
    t = a + i*h
```

Section 5.6 Multistep Methods: Adams Fourth-Order Predictor-Corrector

Step 5 OUTPUT (t_i, w_i).

Step 6 For $i = 4, \ldots, N$ do Steps 7–10.

Step 7 Set $t = a + ih$;

$$w = w_3 + h[55f(t_3, w_3) - 59f(t_2, w_2) + 37f(t_1, w_1) - 9f(t_0, w_0)]/24; \quad (\textit{Predict } w_i.)$$

$$w = w_3 + h[9f(t, w) + 19f(t_3, w_3) - 5f(t_2, w_2) + f(t_1, w_1)]/24. \quad (\textit{Correct } w_i.)$$

Step 8 OUTPUT (t, w).

Step 9 For $j = 0, 1, 2$

set $t_j = t_{j+1}$; (*Prepare for next iteration.*)

$w_j = w_{j+1}$.

Step 10 Set $t_3 = t$;

$w_3 = w$.

Step 11 STOP.

```
for i in range(4,N+1):

    t = a + i*h
    w_p = arr_w[3] + h*(55*f(arr_t[3],arr_w[3])-59*f(arr_t[2],arr_w[2])+37*f(arr_t[1],arr_w[1])-9*f(arr_t[0],arr_w[0]))/24
    w = arr_w[3] + h*(9*f(t,w_p)+19*f(arr_t[3],arr_w[3])-5*f(arr_t[2],arr_w[2])+f(arr_t[1],arr_w[1]))/24
```

```
for j in range(0,3):
    arr_t[j]=arr_t[j+1]
    arr_w[j]=arr_w[j+1]

arr_t[3]=t
arr_w[3]=w
```

Section 5.6 Multistep Methods: Adams Fourth-Order Predictor-Corrector

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

t_i	$y_i = y(t_i)$	w_i	Error $\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272056	0.0000239
1.0	2.6408591	2.6408286	0.0000305
1.2	3.1799415	3.1799026	0.0000389
1.4	3.7324000	3.7323505	0.0000495
1.6	4.2834838	4.2834208	0.0000630
1.8	4.8151763	4.8150964	0.0000799
2.0	5.3054720	5.3053707	0.0001013

```
t,w=adamsForthOrderPC(0,2,10,0.5)
w
```

```
[0.5,
 0.8292933333333334,
 1.2140762106666667,
 1.6489220170416001,
 2.1272056324187787,
 2.6408285969636,
 3.1799026354038826,
 3.7323504816223303,
 4.2834208235015,
 4.815096355330386,
 5.305370671515845]
```