A Derivation of Probalistic Mechanics
A Theory of Stochastic Processes and Matter Waves
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Classical Mechanics:

Given 2 dots and a utility function, the critical paths of the utility function through those 2 dots stabilize the utility function

$$S(x_2, t_2; x_1, t_1) = \int_{x_1,t_1}^{x_2,t_2} L(x, t, dx, dt)dt$$

$$\delta S = 0$$

Probabilistic Mechanics:

Given 2 times, and probability distributions for 1 dot at each time, the form of the utility function, and the fact that the utility function has increased by 1 unit ($\hbar$), the critical strategy of the average utility function stabilizes the average utility function.

$$\begin{aligned} Avg[S](t_2, t_1) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1(x_1; t_1)S(x_2, t_2; x_1, t_1)P_2(x_2; t_2)dx_1dx_2 \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 S P_2 dx_1 dx_2 \end{aligned}$$

$$\begin{aligned} \delta Avg[S] &= 0 \\ S &= \hbar \end{aligned}$$

Note: I am going against modern quantum convention by using $S = \hbar$ instead of $S = -i\hbar$. This has the effect of making electrons, protons, etc. which have no trajectories out to have imaginary mass and gives real trajectories to real masses.

The above framework includes classical mechanics setups as well as new probabilistic setups.

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**Feynman Picture**

Vary with respect to $t$

$$
\begin{aligned}
0 &= \delta Avg[S](t_2,t_1) = \delta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1 S P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(P_1 S P_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1(\frac{\overleftarrow{d}}{dt} S\delta t_1 + \delta \int_{t_1}^{t_2}(Ldt) + \delta t_2 S \frac{\overrightarrow{d}}{dt}) P_2 dx_1 dx_2 \\
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1(\frac{\overleftarrow{d}}{dt} S\delta t_1 + \int_{t_1}^{t_2}\delta(Ldt) + \delta t_2 S \frac{\overrightarrow{d}}{dt}) P_2 dx_1 dx_2 \\
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1(\frac{\overleftarrow{d}}{dt} S\delta t_1 + [L\delta t]_{t_1}^{t_2} + \delta t_2 S \frac{\overrightarrow{d}}{dt}) P_2 dx_1 dx_2
\end{aligned}
$$

Conclusions

$$
P_1 L = \hbar \frac{dP_1}{dt}
$$
$$
LP_2 = -\hbar \frac{dP_2}{dt}
$$

Solutions

$$
P_1(x,t) = \frac{1}{Z}\int_{t_1}^{t} e^{\frac{1}{\hbar}\int_{x_1,t_1}^{x,t} Ldt} P_1(x_1,t_1) Dx(t)
$$
$$
P_2(x,t) = \frac{1}{Z}\int_{t_2}^{t} e^{-\frac{1}{\hbar}\int_{x_2,t_2}^{x,t} Ldt} P_2(x_2,t_2) Dx(t)
$$

—

**Schroedinger and Heisenberg Pictures**

Vary with respect to $(x(t),t)$

$$
\begin{aligned}
0 = \delta Avg[S](t_2,t_1) &= \delta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1 S P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(P_1 S P_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1[(\overleftarrow{\frac{\partial}{\partial x}}_2 \delta x_2 + \overleftarrow{\frac{\partial}{\partial t}}_2 \delta t_2)S + \delta \int_{x_1,t_1}^{x_2,t_2}(Ldt) + S(\delta x_1 \overrightarrow{\frac{\partial}{\partial x_1}} + \delta t_1 \overrightarrow{\frac{\partial}{\partial t_1}})]P_2 dx_1 dx \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1[(\overleftarrow{\frac{\partial}{\partial x}}_2 \delta x_2 + \overleftarrow{\frac{\partial}{\partial t}}_2 \delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}(\delta Ldt) + S(\delta x_1 \overrightarrow{\frac{\partial}{\partial x_1}} + \delta t_1 \overrightarrow{\frac{\partial}{\partial t_1}})]P_2 dx_1 dx \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1[(\overleftarrow{\frac{\partial}{\partial x}}_2 \delta x_2 + \overleftarrow{\frac{\partial}{\partial t}}_2 \delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[\frac{\partial Ldt}{\partial x}\delta x + \frac{\partial Ldt}{\partial dx}\delta dx + \frac{\partial Ldt}{\partial t}\delta t + \frac{\partial Ldt}{\partial dt} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1[(\overleftarrow{\frac{\partial}{\partial x}}_2 \delta x_2 + \overleftarrow{\frac{\partial}{\partial t}}_2 \delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[\frac{\partial Ldt}{\partial x}\delta x - \frac{d\partial Ldt}{\partial dx}\delta x + d(\frac{\partial Ldt}{\partial dx}\delta x) +
\end{aligned}
$$

Name
$Fdt := \frac{\partial Ldt}{\partial x}$
$p := \frac{\partial Ldt}{\partial dx}$
$Qdt := -\frac{\partial Ldt}{\partial t}$
$E := -\frac{\partial Ldt}{\partial dt}$

$$
0 = \delta Avg[S](t_2,t_1) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1[(\overleftarrow{\frac{\partial}{\partial x}}_2 \delta x_2 + \overleftarrow{\frac{\partial}{\partial t}}_2 \delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[(Fdt - dp)\delta x + d(p\delta x) - (Qdt - dE)\delta t
$$

$$
= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_1[(\overleftarrow{\frac{\partial}{\partial x}}_2 \delta x_2 + \overleftarrow{\frac{\partial}{\partial t}}_2 \delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[(Fdt - dp)\delta x - (Qdt - dE)\delta t] + [p\delta x - E\delta t|_{\tau_1}^{\tau_2} + S(\delta x_1
$$

Conclusions
$Fdt = dp$
$Qdt = dE$
$p = -\overleftarrow{\frac{\partial}{\partial x}} S = S\overrightarrow{\frac{\partial}{\partial x}} = -\hbar\overleftarrow{\frac{\partial}{\partial x}} = \hbar\overrightarrow{\frac{\partial}{\partial x}}$
$E = \overleftarrow{\frac{\partial}{\partial t}} S = -S\overrightarrow{\frac{\partial}{\partial t}} = \hbar\overleftarrow{\frac{\partial}{\partial t}} = -\hbar\overrightarrow{\frac{\partial}{\partial t}}$
This gives us $[p,x] = \hbar$
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**The Hamiltonian**

$$
\begin{aligned}
d^2S = d(Ldt) &= \frac{\partial Ldt}{\partial x}dx + \frac{\partial Ldt}{\partial dx}d^2x + \frac{\partial Ldt}{\partial t}dt + \frac{\partial Ldt}{\partial dt}d^2t \\
&= dpdx + pd^2x - dEdt - Ed^2t \\
&= d(pdx - Edt)
\end{aligned}
$$

$$dS = Ldt = pdx - Edt$$

Let $H(x,p,t) = p\frac{dx}{dt} - L$ be the Legendre transform of $L$. Note that $H = E$.

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**Schroedinger Picture**

$HP_2 = \hbar\frac{\partial P_2}{\partial t}$

$P_1 H = -\hbar\frac{\partial P_1}{\partial t}$

**Solutions**

$$P_1(x,t) = e^{-\int_{t_1}^{t}\frac{Hdt}{\hbar}} P_1(x,t_1)$$
$$P_2(x,t) = P_2(x,t_2)e^{\int_{t_2}^{t}\frac{Hdt}{\hbar}}$$

**Density**

$\rho(x;t) = P_2(x;t)P_1(x;t)$

$$\rho(x,t) = [P_2(x,t_2)e^{\frac{-\int_t^{t_2} Hdt}{\hbar}}][e^{\frac{-\int_{t_1}^{t} Hdt}{\hbar}}P_1(x,t_1)]$$

**Time Independent Case**

$$\frac{\partial V}{\partial t} = 0$$

$H(x,p,t)$ is not a function of time.

Consider eigenfunctions of $H(x,p)$.

$$HP_1 = EP_1$$
$$P_2 H = P_2 E$$

So we have $-\hbar\frac{\partial P_1}{\partial t} = EP_1$

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**Inclusion of Quantum Mechanics**

If energy and momentum are real while mass, position, and velocity are imaginary, then quantum mechanics follows.

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**Free Flight**

$$L = \frac{1}{2}m\frac{dx^2}{dt^2}$$

**Schroedinger Picture**

$$H = \frac{p^2}{2m}$$

$$-\hbar\frac{\partial P_1}{\partial t} = \frac{p^2}{2m}P_1 = \frac{\hbar^2}{2m}\frac{\partial^2 P_1}{\partial x^2}$$
$$\hbar\frac{\partial P_2}{\partial t} = P_2\frac{p^2}{2m} = \frac{\hbar^2}{2m}\frac{\partial^2 P_2}{\partial x^2}$$

Given $P_1(x;t_1) = \delta(x-x_1)$ and $P_2(x;t_2) = \delta(x-x_2)$

$$P_1 p = -\hbar\frac{\partial P_1}{\partial x}$$
$$pP_2 = \hbar\frac{\partial P}{\partial x^2}$$

Take the Bilateral Laplace transform to get $P_1(p;t_1)$ and $P_2(p;t_2)$

$$P_2(p;t_2) = \int_{-\infty}^{+\infty} e^{-px/\hbar}\delta(x-x_2)dx = e^{-px_2/\hbar}$$
$$P_1(p,t_1) = \int_{-\infty}^{+\infty} e^{px/\hbar}\delta(x-x_1)dx = e^{px_1/\hbar}$$

Solve these equations

$$-\hbar\frac{\partial P_1(p,t)}{\partial t} = \frac{p^2}{2m}P_1(p,t)$$
$$\hbar\frac{\partial P_2(p,t)}{\partial t} = \frac{p^2}{2m}P_2(p,t)$$

With

$$P_1(p,t) = e^{-\frac{p^2}{2m\hbar}(t-t_1)}P_1(p,t_1) = e^{-\frac{p^2}{2m\hbar}(t-t_1)+\frac{px_1}{\hbar}}$$
$$P_2(p,t) = e^{\frac{p^2}{2m\hbar}(t-t_2)}P_2(p,t) = e^{\frac{p^2}{2m\hbar}(t-t_2)-\frac{px_2}{\hbar}}$$

Reverse the Laplace Transform to get $P_1(x,t)$ and $P_2(x,t)$

$$P_1(x,t) = \int_{-\infty}^{\infty} e^{-\frac{p^2}{2m\hbar}(t-t_1)-\frac{p}{\hbar}(x-x_1)}dp$$
$$P_2(x,t) = \int_{-\infty}^{\infty} e^{\frac{p^2}{2m\hbar}(t-t_2)+\frac{p}{\hbar}(x-x_2)}dp$$

$$\begin{aligned}\rho(x,t) &= P_1(x,t)P_2(x,t) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}[e^{-\frac{p_1^2}{2m\hbar}(t-t_1)-\frac{p_1}{\hbar}(x-x_1)}][e^{\frac{p_2^2}{2m\hbar}(t-t_2)+\frac{p_2}{\hbar}(x-x_2)}]dp_1dp_2\end{aligned}$$

Since we started with a classical situation of $P_1(x;t_1)$ and $P_2(x;t_2)$ being dirac delta spikes, we stay in a classical situation. In free-flight, we already know the answer is a delta spike that travels at constant velocity between $(x_1;t_1)$ and $(x_2;t_2)$

$$\rho(x,t) = \delta(x - [(\frac{x_2 - x_1}{t_2 - t_1})(t - t_1) + x_1])$$

Left to show: $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [e^{-\frac{p_1^2}{2m\hbar}(t-t_1) - \frac{p_1}{\hbar}(x-x_1)}][e^{\frac{p_2^2}{2m\hbar}(t-t_2) + \frac{p_2}{\hbar}(x-x_2)}]dp_1 dp_2 = \delta(x - [(\frac{x_2-x_1}{t_2-t_1})(t - t_1) + x_1])$

$$\rho(x,t_1) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [e^{-\frac{p_1}{\hbar}(x-x_1)}][e^{\frac{p_2^2}{2m\hbar}(t_1-t_2) + \frac{p_2}{\hbar}(x-x_2)}]dp_1 dp_2$$

which is $\delta(x - x_1)$

$$\rho(x,t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [e^{-\frac{p_1^2}{2m\hbar}(t_2-t_1) - \frac{p_1}{\hbar}(x-x_1)}][e^{\frac{p_2}{\hbar}(x-x_2)}]dp_1 dp_2$$

which is $\delta(x - x_2)$

Since the base of the exponential in the integral is linear in $(x,t)$, we must have the result

$$\rho(x,t) = \delta(x - [(\frac{x_2 - x_1}{t_2 - t_1})(t - t_1) + x_1])$$

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**Matter-Wave Free Flight (Quantum Mechanics: Imaginary Mass, Position, and Velocity and Real Energy and Momentum)**

Note that with this setup, we get Quantum Mechanical free flight