Quantum Mechanics is Imaginary.
A Derivation of Probalistic Mechanics
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Classical Mechanics:
Given 2 dots and a utility function, the critical paths of the utility function through those 2 dots stabilize the utility function

$$S(x_2, t_2; x_1, t_1) = \int_{x_1, t_1}^{x_2, t_2} L(x, t, dx, dt)dt$$

$$\delta S = 0$$

Probabilistic Mechanics:
Given 2 times, and probability distributions for 1 dot at each time, the form of the utility function, and the fact that the utility function has increased by 1 unit (k), the critical average dynamics of the average utility function stabilize the average utility function.

$$\begin{aligned} Avg[S](t_2, t_1) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_2(x_2; t_1) S(x_2, t_2; x_1, t_1) P_1(x_1; t_1) dx_1 dx_2 \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_2 S P_1 dx_1 dx_2 \end{aligned}$$

$$\begin{aligned} \delta Avg[S] &= 0 \\ S &= k \end{aligned}$$

Note: I am going against modern quantum convention by using $S = k > 0$. This has the effect of making electrons, protons, etc. which have no trajectories out to have imaginary mass and gives real trajectories to real masses.

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Feynman Picture
Vary with respect to t

$$0 = \delta Avg[S](t_2,t_1) = \delta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2 S P_1 dx_1 dx_2$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(P_2 S P_1) dx_1 dx_2$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2(\frac{\overleftarrow{d}}{dt} S\delta t_2 + \delta\int_{t_1}^{t_2}(Ldt) + \delta t_1 S\frac{\overrightarrow{d}}{dt})P_1 dx_1 dx_2$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2(\frac{\overleftarrow{d}}{dt} S\delta t_2 + \int_{t_1}^{t_2}\delta(Ldt) + \delta t_1 S\frac{\overrightarrow{d}}{dt})P_1 dx_1 dx_2$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2(\frac{\overleftarrow{d}}{dt} S\delta t_2 + [L\delta t]_{t_1}^{t_2} + \delta t_1 S\frac{\overrightarrow{d}}{dt})P_1 dx_1 dx_2$$

Conclusions
$L = -\frac{\overleftarrow{d}}{dt}S = S\frac{\overrightarrow{d}}{dt} = -k\frac{\overleftarrow{d}}{dt} = k\frac{\overrightarrow{d}}{dt}$
$LP_1 = k\frac{\overrightarrow{d}}{dt}P_1$
$P_2L = -P_2k\frac{\overleftarrow{d}}{dt}$
Solutions

$$P_1(x,t) = \frac{1}{Z}\int_{t_1}^{t} e^{\frac{1}{k}\int_{x_1,t_1}^{x,t} Ldt} P_1(x_1,t_1) Dx(t)$$
$$P_2(x,t) = \frac{1}{Z}\int_{t}^{t_2} P_2(x_2,t_2) e^{\frac{1}{k}\int_{x,t}^{x_2,t_2} Ldt} Dx(t)$$

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Schroedinger and Heisenberg Pictures
Vary with respect to $(x(t),t)$

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\begin{aligned}
0 = \delta Avg[S](t_2,t_1) &= \delta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2 S P_1 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(P_2 S P_1) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2[(\frac{\overleftarrow{\partial}}{\partial x}_2\delta x_2 + \frac{\overleftarrow{\partial}}{\partial t}_2\delta t_2)S + \delta\int_{x_1,t_1}^{x_2,t_2}(Ldt) + S(\delta x_1\frac{\overrightarrow{\partial}}{\partial x_1} + \delta t_1\frac{\overrightarrow{\partial}}{\partial t_1})]P_1 dx_1 dx \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2[(\frac{\overleftarrow{\partial}}{\partial x}_2\delta x_2 + \frac{\overleftarrow{\partial}}{\partial t}_2\delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}(\delta Ldt) + S(\delta x_1\frac{\overrightarrow{\partial}}{\partial x_1} + \delta t_1\frac{\overrightarrow{\partial}}{\partial t_1})]P_1 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2[(\frac{\overleftarrow{\partial}}{\partial x}_2\delta x_2 + \frac{\overleftarrow{\partial}}{\partial t}_2\delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[\frac{\partial Ldt}{\partial x}\delta x + \frac{\partial Ldt}{\partial dx}\delta dx + \frac{\partial Ldt}{\partial t}\delta t + \frac{\partial Ldt}{\partial dt} \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2[(\frac{\overleftarrow{\partial}}{\partial x}_2\delta x_2 + \frac{\overleftarrow{\partial}}{\partial t}_2\delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[\frac{\partial Ldt}{\partial x}\delta x - \frac{d\partial Ldt}{\partial dx}\delta x + d(\frac{\partial Ldt}{\partial dx}\delta x) +
\end{aligned}
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Name
$Fdt := \frac{\partial Ldt}{\partial x}$
$p := \frac{\partial Ldt}{\partial dx}$
$Qdt := -\frac{\partial Ldt}{\partial t}$
$E := -\frac{\partial Ldt}{\partial dt}$

$$
0 = \delta Avg[S](t_2,t_1) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2[(\frac{\overleftarrow{\partial}}{\partial x}_2\delta x_2 + \frac{\overleftarrow{\partial}}{\partial t}_2\delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[(Fdt - dp)\delta x + d(p\delta x) - (Qdt - dE)\delta t
$$

$$
= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} P_2[(\frac{\overleftarrow{\partial}}{\partial x}_2\delta x_2 + \frac{\overleftarrow{\partial}}{\partial t}_2\delta t_2)S + \int_{x_1,t_1}^{x_2,t_2}[(Fdt - dp)\delta x - (Qdt - dE)\delta t] + [p\delta x - E\delta t|_{\tau_1}^{\tau_2} + S(\delta x_1
$$

Conclusions
$Fdt = dp$
$Qdt = dE$
$p = -\frac{\overleftarrow{\partial}}{\partial x}S = S\frac{\overrightarrow{\partial}}{\partial x} = -k\frac{\overleftarrow{\partial}}{\partial x} = k\frac{\overrightarrow{\partial}}{\partial x}$
$E = \frac{\overleftarrow{\partial}}{\partial t}S = -S\frac{\overrightarrow{\partial}}{\partial t} = k\frac{\overleftarrow{\partial}}{\partial t} = -k\frac{\overrightarrow{\partial}}{\partial t}$
This gives us $[p,x] = k$
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The Hamiltonian

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\begin{aligned}
d^2S = d(Ldt) &= \frac{\partial Ldt}{\partial x}dx + \frac{\partial Ldt}{\partial dx}d^2x + \frac{\partial Ldt}{\partial t}dt + \frac{\partial Ldt}{\partial dt}d^2t \\
&= dpdx + pd^2x - dEdt - Ed^2t \\
&= d(pdx - Edt)
\end{aligned}
$$

$$Ldt = pdx - Edt$$

Let $H(x,p,t) = p\frac{dx}{dt} - L$ be the Legendre transform of L. Note that $H = E$.

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Schroedinger Picture

$HP_1 = -k\frac{\partial P_1}{\partial t}$
$P_2H = k\frac{\partial P_2}{\partial t}$

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Derivatives of the Action

From above

$$dS = pdx - Edt$$

So

$$\frac{\partial S}{\partial x} = p$$
$$\frac{\partial S}{\partial t} = -E$$

The Form of the Lagrangian

Assumption: As $F \to 0$ and $Q \to 0$, L goes to a constant. $d^2t = 0$.

$$d^2S = d(Ldt) = dLdt + Ld^2t = 0$$

$$0 = d(pdx - Edt) = dpdx + pd^2x + dEdt + Ed^2t$$

Since $dp = 0$, $dE = 0$, and $d^2t = 0$

$$0 = pd^2x$$

Since p is not necessarily 0, we have $d^2x = 0$.
The only possible form of L that gives $d^2x = 0$ is

$$L_{free} \propto \frac{dx^2}{dt^2}$$

So in general (could use more justification)

$$L = \frac{1}{2}m\frac{dx^2}{dt^2} - V(x,t)$$

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Free Flight

$$L = \frac{1}{2}m\frac{dx^2}{dt^2}$$

Schroedinger Picture

$$H = \frac{p^2}{2m}$$