

A Derivation of Probabilistic Mechanics
A Theory of Stochastic Processes and Matter Waves
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Classical Mechanics:

Given 2 dots and a utility function, the critical paths of the utility function through those 2 dots stabilize the utility function

$$S(x_2, t_2; x_1, t_1) = \int_{x_1, t_1}^{x_2, t_2} L(x, t, dx, dt) dt$$

$$\delta S = 0$$

Probabilistic Mechanics:

Given 2 times, and probability distributions for 1 dot at each time, the form of the utility function, and the fact that the utility function has increased by 1 unit (\hbar), the critical strategy of the average utility function stabilizes the average utility function.

$$\begin{aligned} \text{Avg}[S](t_2, t_1) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1(x_1; t_1) S(x_2, t_2; x_1, t_1) P_2(x_2; t_2) dx_1 dx_2 \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 S P_2 dx_1 dx_2 \end{aligned}$$

$$\delta \text{Avg}[S] = 0$$

$$S = \hbar$$

Note: I am going against modern quantum convention by using $S = \hbar$ instead of $S = -i\hbar$. This has the effect of making electrons, protons, etc. which have no trajectories out to have imaginary mass and gives real trajectories to real masses.

The above framework includes classical mechanics setups as well as new probabilistic setups.

Feynman Picture

Vary with respect to t

$$\begin{aligned}
0 &= \delta Avg[S](t_2, t_1) = \delta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 S P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(P_1 S P_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left(\frac{\overleftarrow{d}}{dt} S \delta t_1 + \delta \int_{t_1}^{t_2} (L dt) + \delta t_2 S \frac{\overrightarrow{d}}{dt} \right) P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left(\frac{\overleftarrow{d}}{dt} S \delta t_1 + \int_{t_1}^{t_2} \delta(L dt) + \delta t_2 S \frac{\overrightarrow{d}}{dt} \right) P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left(\frac{\overleftarrow{d}}{dt} S \delta t_1 + [L \delta t]_{t_1}^{t_2} + \delta t_2 S \frac{\overrightarrow{d}}{dt} \right) P_2 dx_1 dx_2
\end{aligned}$$

Conclusions

$$\begin{aligned}
P_1 L &= \hbar \frac{dP_1}{dt} \\
L P_2 &= -\hbar \frac{dP_2}{dt}
\end{aligned}$$

Solutions

$$\begin{aligned}
P_1(x, t) &= \frac{1}{Z} \int_{t_1}^t e^{\frac{1}{\hbar} \int_{x_1, t_1}^{x, t} L dt} P_1(x_1, t_1) Dx(t) \\
P_2(x, t) &= \frac{1}{Z} \int_{t_2}^t e^{-\frac{1}{\hbar} \int_{x_2, t_2}^{x, t} L dt} P_2(x_2, t_2) Dx(t)
\end{aligned}$$

Schroedinger and Heisenberg Pictures
Vary with respect to $(x(t), t)$

$$\begin{aligned}
0 = \delta \text{Avg}[S](t_2, t_1) &= \delta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 S P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(P_1 S P_2) dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left[\left(\overleftarrow{\frac{\partial}{\partial x_2}} \delta x_2 + \overleftarrow{\frac{\partial}{\partial t_2}} \delta t_2 \right) S + \delta \int_{x_1, t_1}^{x_2, t_2} (L dt) + S \left(\delta x_1 \overrightarrow{\frac{\partial}{\partial x_1}} + \delta t_1 \overrightarrow{\frac{\partial}{\partial t_1}} \right) \right] P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left[\left(\overleftarrow{\frac{\partial}{\partial x_2}} \delta x_2 + \overleftarrow{\frac{\partial}{\partial t_2}} \delta t_2 \right) S + \int_{x_1, t_1}^{x_2, t_2} (\delta L dt) + S \left(\delta x_1 \overrightarrow{\frac{\partial}{\partial x_1}} + \delta t_1 \overrightarrow{\frac{\partial}{\partial t_1}} \right) \right] P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left[\left(\overleftarrow{\frac{\partial}{\partial x_2}} \delta x_2 + \overleftarrow{\frac{\partial}{\partial t_2}} \delta t_2 \right) S + \int_{x_1, t_1}^{x_2, t_2} \left[\frac{\partial L dt}{\partial x} \delta x + \frac{\partial L dt}{\partial dx} \delta dx + \frac{\partial L dt}{\partial t} \delta t + \frac{\partial L dt}{\partial dt} \delta dt \right] \right] P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left[\left(\overleftarrow{\frac{\partial}{\partial x_2}} \delta x_2 + \overleftarrow{\frac{\partial}{\partial t_2}} \delta t_2 \right) S + \int_{x_1, t_1}^{x_2, t_2} \left[\frac{\partial L dt}{\partial x} \delta x - \frac{d \partial L dt}{\partial dx} \delta x + d \left(\frac{\partial L dt}{\partial dx} \delta x \right) + \frac{\partial L dt}{\partial t} \delta t \right] \right] P_2 dx_1 dx_2
\end{aligned}$$

Name

$$F dt := \frac{\partial L dt}{\partial x}$$

$$p := \frac{\partial L dt}{\partial dx}$$

$$Q dt := - \frac{\partial L dt}{\partial t}$$

$$E := - \frac{\partial L dt}{\partial dt}$$

$$\begin{aligned}
0 = \delta \text{Avg}[S](t_2, t_1) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left[\left(\overleftarrow{\frac{\partial}{\partial x_2}} \delta x_2 + \overleftarrow{\frac{\partial}{\partial t_2}} \delta t_2 \right) S + \int_{x_1, t_1}^{x_2, t_2} [(F dt - dp) \delta x + d(p \delta x) - (Q dt - dE) \delta t] \right] P_2 dx_1 dx_2 \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} P_1 \left[\left(\overleftarrow{\frac{\partial}{\partial x_2}} \delta x_2 + \overleftarrow{\frac{\partial}{\partial t_2}} \delta t_2 \right) S + \int_{x_1, t_1}^{x_2, t_2} [(F dt - dp) \delta x - (Q dt - dE) \delta t] + [p \delta x - E \delta t]_{\tau_1}^{\tau_2} + S(\delta x_1 \overrightarrow{\frac{\partial}{\partial x_1}} + \delta t_1 \overrightarrow{\frac{\partial}{\partial t_1}}) \right] P_2 dx_1 dx_2
\end{aligned}$$

Conclusions

$$F dt = dp$$

$$Q dt = dE$$

$$p = - \overleftarrow{\frac{\partial}{\partial x}} S = S \overrightarrow{\frac{\partial}{\partial x}} = - \hbar \overleftarrow{\frac{\partial}{\partial x}} = \hbar \overrightarrow{\frac{\partial}{\partial x}}$$

$$E = \overleftarrow{\frac{\partial}{\partial t}} S = - S \overrightarrow{\frac{\partial}{\partial t}} = \hbar \overleftarrow{\frac{\partial}{\partial t}} = - \hbar \overrightarrow{\frac{\partial}{\partial t}}$$

This gives us $[p, x] = \hbar$

The Hamiltonian

$$\begin{aligned}
d^2 S &= d(L dt) = \frac{\partial L dt}{\partial x} dx + \frac{\partial L dt}{\partial dx} d^2 x + \frac{\partial L dt}{\partial t} dt + \frac{\partial L dt}{\partial dt} d^2 t \\
&= dp dx + p d^2 x - dE dt - E d^2 t \\
&= d(p dx - E dt)
\end{aligned}$$

$$dS = Ldt = p dx - E dt$$

Let $H(x, p, t) = p \frac{dx}{dt} - L$ be the Legendre transform of L . Note that $H = E$.

Schroedinger Picture

$$H P_2 = \hbar \frac{\partial P_2}{\partial t}$$

$$P_1 H = -\hbar \frac{\partial P_1}{\partial t}$$

Solutions

$$P_1(x, t) = e^{-\int_{t_1}^t \frac{H dt}{\hbar}} P_1(x, t_1)$$

$$P_2(x, t) = P_2(x, t_2) e^{\int_{t_2}^t \frac{H dt}{\hbar}}$$

Density

$$\rho(x; t) = P_2(x; t) P_1(x; t)$$

$$\rho(x, t) = [P_2(x, t_2) e^{-\int_{t_2}^t \frac{H dt}{\hbar}}] [e^{\int_{t_1}^t \frac{H dt}{\hbar}} P_1(x, t_1)]$$

Time Independent Case

$$\frac{\partial V}{\partial t} = 0$$

$H(x, p, t)$ is not a function of time.

Consider eigenfunctions of $H(x, p)$.

$$H P_1 = E P_1$$

$$P_2 H = P_2 E$$

So we have $-\hbar \frac{\partial P_1}{\partial t} = E P_1$

Inclusion of Quantum Mechanics

If energy, momentum, and mass are all imaginary, then quantum mechanics follows.

Free Flight

$$L = \frac{1}{2} m \frac{dx^2}{dt^2}$$

Schroedinger Picture

$$H = \frac{p^2}{2m}$$

Positive Mass $m > 0$

$$\begin{aligned}-\hbar \frac{\partial P_1}{\partial t} &= \frac{p^2}{2m} P_1 = \frac{\hbar^2}{2m} \frac{\partial^2 P_1}{\partial x^2} \\ \hbar \frac{\partial P_2}{\partial t} &= P_2 \frac{p^2}{2m} = \frac{\hbar^2}{2m} \frac{\partial^2 P_2}{\partial x^2}\end{aligned}$$

Given $P_1(x; t_1) = \delta(x - x_1)$ and $P_2(x; t_2) = \delta(x - x_2)$

$$\begin{aligned}P_1 p &= -\hbar \frac{\partial^2 P_1}{\partial x^2} \\ p P_2 &= \hbar \frac{\partial^2 P_2}{\partial x^2}\end{aligned}$$

Take the Bilateral Laplace transform to get $P_1(p; t_1)$ and $P_2(p; t_2)$

$$\begin{aligned}P_2(p; t_2) &= \int_{-\infty}^{+\infty} e^{-px/\hbar} \delta(x - x_2) dx = e^{-px_2/\hbar} \\ P_1(p, t_1) &= \int_{-\infty}^{+\infty} e^{px/\hbar} \delta(x - x_1) dx = e^{px_1/\hbar}\end{aligned}$$

Solve these equations

$$\begin{aligned}-\hbar \frac{\partial P_1(p, t)}{\partial t} &= \frac{p^2}{2m} P_1(p, t) \\ \hbar \frac{\partial P_2(p, t)}{\partial t} &= \frac{p^2}{2m} P_2(p, t)\end{aligned}$$

With

$$\begin{aligned}P_1(p, t) &= e^{-\frac{p^2}{2m\hbar}(t-t_1)} P_1(p, t_1) = e^{-\frac{p^2}{2m\hbar}(t-t_1) + \frac{px_1}{\hbar}} \\ P_2(p, t) &= e^{\frac{p^2}{2m\hbar}(t-t_2)} P_2(p, t) = e^{\frac{p^2}{2m\hbar}(t-t_2) - \frac{px_2}{\hbar}}\end{aligned}$$

Reverse the Laplace Transform to get $P_1(x, t)$ and $P_2(x, t)$

$$\begin{aligned}P_1(x, t) &= \int_{-\infty}^{\infty} e^{-\frac{p^2}{2m\hbar}(t-t_1) - \frac{p}{\hbar}(x-x_1)} dp \\ P_2(x, t) &= \int_{-\infty}^{\infty} e^{\frac{p^2}{2m\hbar}(t-t_2) + \frac{p}{\hbar}(x-x_2)} dp\end{aligned}$$

$$\begin{aligned}\rho(x, t) &= P_1(x, t) P_2(x, t) \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [e^{-\frac{p_1^2}{2m\hbar}(t-t_1) - \frac{p_1}{\hbar}(x-x_1)}] [e^{\frac{p_2^2}{2m\hbar}(t-t_2) + \frac{p_2}{\hbar}(x-x_2)}] dp_1 dp_2\end{aligned}$$

Since we started with a classical situation of $P_1(x; t_1)$ and $P_2(x; t_2)$ being dirac delta spikes, we stay in a classical situation. In free-flight, we already know the answer is a delta spike that travels at constant velocity between $(x_1; t_1)$ and $(x_2; t_2)$

$$\rho(x, t) = \delta(x - [(\frac{x_2 - x_1}{t_2 - t_1})(t - t_1) + x_1])$$

$$\text{Left to show: } \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} [e^{-\frac{p_1^2}{2m\hbar}(t-t_1) - \frac{p_1}{\hbar}(x-x_1)}] [e^{\frac{p_2^2}{2m\hbar}(t-t_2) + \frac{p_2}{\hbar}(x-x_2)}] dp_1 dp_2 = \delta(x - [(\frac{x_2 - x_1}{t_2 - t_1})(t - t_1) + x_1])$$

Matter-Wave Free Flight (Quantum Mechanics: Imaginary Mass and Real Energy)