ECO358 A2

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0 Introduction

The time horizon of the 3 US-based firms I'm looking at is 5 years (2018-03-13 to 2023-03-07). This means an appropriate risk-free rate r_f can be the annual bond yield of a **USA 5-year bond**, which is **3.687%** (Data obtained on March 13th). This is appropriate because the issuing country of the bond is the country location of all firms, and both time horizons are 5 years. Throughout the assignment, we will assume there are 252 trading days in a year, and there are 1254 data points throughout the 5-year interval.

I've chosen **Apple Inc. (APPL), Costco Wholesale Corporation (COST), and United Health Group Incorporated (UNH)** in the US stock market. This is because the three firms are in very different subsections of the US market, so the correlation between the individual stocks should be lower to diversify risks. Also, they have shown decent growth in the past 5 years. For this question, we assume that **S&P 500 Index** represents the market itself since it's a composite index of 500 top US firms.

1 Q1

A

Historical Data in the past 5 years for Apple Inc. (APPL), Costco Wholesale Corporation (COST), and United Health Group Incorporated (UNH), and S&P 500 Index, are downloaded from Yahoo Finance and located in the spreadsheet called "data" in the excel file, from 2018-03-13, with 1254 data points for each stock.

Below shows the first few lines:

Date	APPL	COST	UNH	S&P 500
2018-03-13	44.9925	187.460007	226.940002	2765.31
2018-03-14	44.610001	184.630005	225.380005	2749.48
2018-03-15	44.662498	185.509995	229.479996	2747.33
2018-03-16	44.505001	185.869995	227.860001	2752.01
2018-03-19	43.825001	184.149994	225.050003	2712.92
2018-03-20	43.810001	187.350006	226.779999	2716.94
2018-03-21	42.817501	186.070007	222.820007	2711.93
2018-03-22	42.212502	182.639999	215.210007	2643.69
2018-03-23	41.235001	180.839996	212.550003	2588.26
2018-03-26	43.192501	187.220001	219.070007	2658.55
2018-03-27	42.084999	183.149994	217.960007	2612.62

B

For each of the stock and the index, the return for each day i can be calculated via

$$R_{stock,t} = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1}$$

Below shows the first few lines:

Date	APPL	COST	UNH	S&P 500	Return APPL	Return COST	Return UNH	Return Index (S&P)
2018-03-13	44.9925	187.460007	226.940002	2765.31				
2018-03-14	44.610001	184.630005	225.380005	2749.48	-0.85%	-1.51%	-0.69%	-0.57%
2018-03-15	44.662498	185.509995	229.479996	2747.33	0.12%	0.48%	1.82%	-0.08%
2018-03-16	44.505001	185.869995	227.860001	2752.01	-0.35%	0.19%	-0.71%	0.17%
2018-03-19	43.825001	184.149994	225.050003	2712.92	-1.53%	-0.93%	-1.23%	-1.42%
2018-03-20	43.810001	187.350006	226.779999	2716.94	-0.03%	1.74%	0.77%	0.15%
2018-03-21	42.817501	186.070007	222.820007	2711.93	-2.27%	-0.68%	-1.75%	-0.18%
2018-03-22	42.212502	182.639999	215.210007	2643.69	-1.41%	-1.84%	-3.42%	-2.52%
2018-03-23	41.235001	180.839996	212.550003	2588.26	-2.32%	-0.99%	-1.24%	-2.10%
2018-03-26	43.192501	187.220001	219.070007	2658.55	4.75%	3.53%	3.07%	2.72%
2018-03-27	42.084999	183.149994	217.960007	2612.62	-2.56%	-2.17%	-0.51%	-1.73%
2018-03-28	41.619999	183.610001	218.5	2605	-1.10%	0.25%	0.25%	-0.29%

This explains the return of each stock per day in terms of percentage.

C

To calculate the beta and CAPM, we will need to compute some values first. We need to get the covariance of each stock w.r.t. the S&P index, the expected return of each stock in APR, and the volatility of each stock in APR. Below shows multiple steps of walking through the steps to find the betas (and CAPM in Questions 1e). Note: the resulting data of each step is shown in an image at the end of Question 1C.

Covariance w.r.t. index

Let n be the number of total days in the data, then $\forall i, j \in$ each valid day, Let $X = \{x_1, x_2, ..., x_n\}$ be the daily return of the stock, $Y = \{y_1, y_2, ..., y_n\}$ be the daily return of the index. Then we have:

$$Cov(X, Y) = \frac{1}{n-1} \sum (x_i - \bar{X})(y_j - \bar{Y}) \tag{2}$$

where $\bar{X}$ and $\bar{Y}$ are averages.

Note here we are using sample covariance instead of population covariance since it's based on Historical Data (explained in Lecture 5 page 20)

Expected Return

We want to average out the daily expected return of the stock and make it into APR. Since there are 252 trading days in a year, we will set $yr = 252$.

$$R_{stock} = \frac{\sum r_i}{n} * yr = \frac{\sum r_i}{1254} * 252 \tag{3}$$

for each day i.

This is the annual expected return of each stock in terms of percentage.

Standard Deviation

We want to first compute the daily variance and convert it to yearly, then square root it to get the standard deviation. Again, we will use sample variance in this case.

$$SD_{stock,annual} = [VAR_{stock,annual}]^{0.5} = [VAR_{stock,daily} * 252]^{0.5} = [(\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{X})^2) * 252]^{0.5} \tag{4}$$

This is the annual volatility of a stock, a measure of how much the stock's price is expected to fluctuate over a year.

Beta

Beta can be calculated via

$$\beta_{stock} = \frac{Cov(R_{stock}, R_{index})}{Var(R_{index})} = \frac{Cov(R_{stock}, R_{index})}{Cov(R_{index}, R_{index})} \tag{5}$$

Since we can calculate the covariance of all stocks w.r.t. index in the above step in Equation 3, the betas can be computed.

Here, beta represents the expected percentage change in the excess return of security for a 1% change in the excess return of the market portfolio.

With the betas being 1.23, 0.73, 0.90 for APPL, COST, and UNH, respectively, where a beta of more than 1 means the stock fluctuates more than the market (more market risks), while a beta of less than 1 means the stock fluctuates less than the market (less market risks).

Result

		APPL-S&P	COST-S&P	UNH-S&P	S&P-S&P
c)	Covariance-S&P	0.023%	0.014%	0.017%	0.0189%
	Expected Returns	30.05%	22.11%	19.22%	9.74%
	StDev	33.55%	24.44%	29.70%	21.80%
	Beta	1.23	0.73	0.90	1.00
e)	CAPM	11.15%	8.09%	9.13%	

D

This is shown all together in the image in Question 4 A. I have plotted volatility vs. expected return for APPL, COST, UNH, and S&P, all in APR.

E

We now have $r_f = 3.687\%$ from US 5-year bond yield, the expected return of the index as $E[R_{Mkt}] = 9.74\%$, and beta for all stocks.

So we can find the CAPM of each stock using

$$E[R_{stock}] = r_f + \beta_{stock} \times (E[R_{Mkt}] - r_f) \tag{6}$$

Since r_f and $E[R_{Mkt}]$ are computed as annualized terms, and beta is only a coefficient, we have $E[R_{stock}]$ expressed in APR without extra conversion. The CAPM for each stock is shown in the figure in Question 1C.

The CAPM captures the effect of the beta of each stock w.r.t. the market. Because all three stocks have positive betas, it means they tend to fluctuate with the market, where higher betas equate more exposure to market risk, thus having a higher cost of capital measured by CAPM.

F

The intuition behind the CAPM is that investors can earn a risk-free rate of return by investing in assets such as the five-year US bond, which are considered to have a low level of risk, and assumed to be no risk here. However, investors who are willing to take on additional risk by investing in stocks expect to earn a higher return to compensate for the added risk. It means the higher-than-market CAPM of my stocks means the existence of systematic risk and thus should be rewarded more.

Compared to DDM and DFCFM, the CAPM model focuses more on the asset's beta coefficient as a measure of risk. The DDM estimates the stock based on the present value of its future dividend payments, while the DFCFM estimates the stock based on the present value of its future cash flows.

One advantage of the CAPM over DDM and DFCFM is that it takes into account the systematic risk of a stock, which is not captured by the DDM or DFCFM. On the contrary, it does not count in idiosyncratic risks and makes assumptions about convenient lending and borrowing at risk-free rates, which typically don't hold in reality.

2 Q2

	USA 5 year bond	3.69%		
		Returns APPL	Returns COST	Returns UNH
	Expected daily returns	0.12%	0.09%	0.08%
a)	Covariance	Returns APPL	Returns COST	Returns UNH
	Returns APPL	0.000446588	0.000183983	0.000190677
	Returns COST	0.000183983	0.000237063	0.000126897
	Returns UNH	0.000190677	0.000126897	0.000349921
b)	Portfolio daily expected returns	0.0883%		
	Portfolio daily StDev	1.4999%		
	Portfolio Annual Expected returns	22.24%		
	Portfolio Annual StDev	23.81%		
c)	Portfolio Beta	0.915		
	Sharpe Ratio	77.94%		

Name	Weights
APPL	20%
COST	30%
UNH	50%
Sum	100%

A

For the covariance matrix, the covariance between two stocks can be computed using the same formula illustrated in Question 1c, in Equation 2, where X is stock 1, and Y is stock 2.

B

Let d be the number of stocks we are evaluating, in my case, $d = 3$. We will use matrix multiplication to calculate expected return and volatility conveniently.

Let $\vec{r}$ be a $d \times 1$ vector of stock returns. Let $\vec{w}$ be a $d \times 1$ vector of weights of stock in the portfolio. Let A be the $d \times d$ covariance matrix.

Note there all values are in terms of daily values, e.g. daily return and daily covariance.

Expected Return

Using daily expected return computed from Question 1c by Equation 3, we can get vector $\vec{r}$, and we set weights $\vec{w}$ arbitrarily for this question.

Then the expected return of the portfolio is

$$R_{p,daily} = \sum_{i=1}^{d} (r_i)(w_i) = \vec{r} \cdot \vec{w} = \vec{r}^T \times \vec{w} \tag{7}$$

Then we can convert the return to the annual rate

$$R_{p,annual} = R_{p,daily} * 252 = (\vec{r}^T \times \vec{w}) * 252 \tag{8}$$

This shows the annual expected return of the portfolio with weights $\vec{w}$ in terms of percentage.

Volatility

We can also compute the volatility of the portfolio easily using matrix multiplication

$$SD_{p,daily} = [Var_{p,daily}]^{0.5} = [\sum_{i=1}^{d}\sum_{j=1}^{d}(w_i)(w_j)(A_{i,j})]^{0.5} = [\vec{w}^T \times A \times \vec{w}]^{0.5} \tag{9}$$

Then we can convert it into an annual volatility rate:

$$SD_{p,annual} = SD_{p,daily} * \sqrt{252} = [\vec{w}^T \times A \times \vec{w}]^{0.5} * \sqrt{252} \tag{10}$$

This shows the annual volatility of the portfolio with weights $\vec{w}$ in terms of percentage.

C

We have computed Beta for each stock in Question 1c by Equation 5. Let $\vec{\beta}$ be a $d \times 1$ vector be betas for the portfolio.

Then beta can be computed by weighted product via matrix multiplication.

$$\beta_p = \sum_{i=1}^{d} w_i \beta_i = \vec{w}^T \times \vec{\beta} \tag{11}$$

This shows the beta coefficient of the portfolio with weights $\vec{w}$, where $\beta_p = 0.915$.

D

This is shown altogether in the image in Question 4 A. I have plotted volatility vs. expected return (in APR) for this portfolio, called Portfolio 1, with weights of 20%, 30%, and 50% for APPL, COST, and UNH, respectively.

E

By combining them in the portfolio, the overall beta is now 0.915, which is less than individual stocks such as Apple with a beta of 1.23. This indicates that the portfolio is less sensitive to market movements than a single stock, which means diversification has been successful. In addition, the combination of these stocks also reduces idiosyncratic risk of any individual stocks. While idiosyncratic risks are not captured by CAPM, having more stocks in a portfolio is safer albeit with systematic risks.

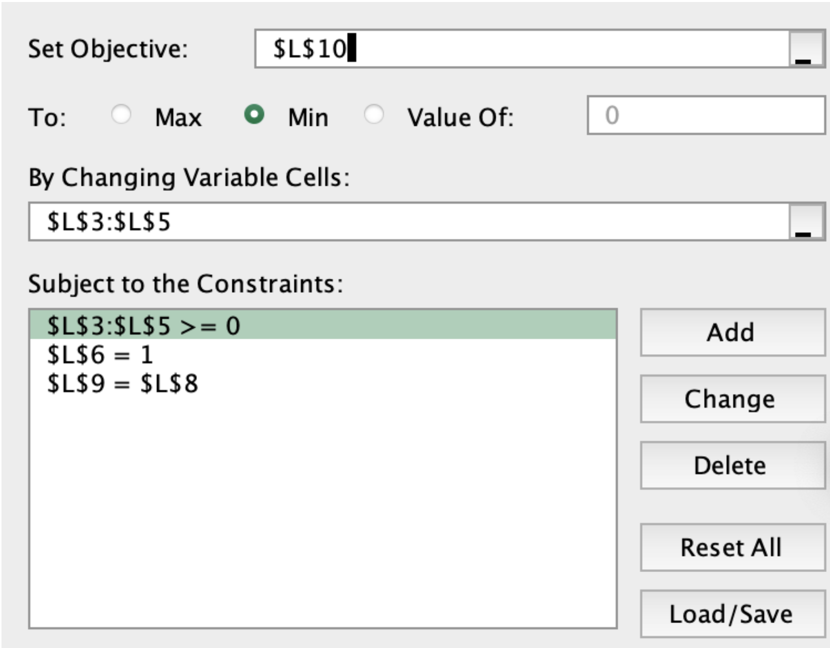
However, historical data may not accurately predict future performance since the relationship between individual stocks and the overall market may change over time. In addition, the beta of a stock can be affected by factors such as changes in company strategy, management, and market conditions in the near future. Therefore, investors should use historical beta values as a guide and but not the only tool to make investment decisions.

F

Among all stocks, the lowest expected return is UNH with 19.22 % and highest expected return is APPL with 30.05 %. This means the portfolio expected return should be in within this range in order to be feasible. I tested the 11 values of expected return = {20%, 21%, ..., 30%}. The solver is created to achieve this by setting the expected return and minimize volatility. Constraints are weights summing to one, each weights non-negative to ensure no short selling, and actual return equals the desired return.

I

I've created a solver that repeated 11 times to get minimized volatility for expected return = {20%, 21%, ..., 30%}. Below is an illustration:



Here is the result of optimal weights, where the highlighted volatility is the portfolio allocation with the lowest volatility:

Name	Weights	Name	Weights	Name	Weights	Name	Weights	Name	Weights
APPL	0.00%	APPL	0.00%	APPL	9.17%	APPL	19.48%	APPL	29.80%
COST	27.08%	COST	61.68%	COST	62.03%	COST	58.08%	COST	54.14%
UNH	72.92%	UNH	38.32%	UNH	28.80%	UNH	22.43%	UNH	16.06%
Sum	100%	Sum	100%	Sum	100%	Sum	100%	Sum	100%
Desired Return	20.00%	Desired Return	21.00%	Desired Return	22.00%	Desired Return	23.00%	Desired Return	24.00%
Outcome Returns	20.00%	Outcome Returns	21.00%	Outcome Returns	22.00%	Outcome Returns	23.00%	Outcome Returns	24.00%
Outcome StDev	25.28%	Outcome StDev	22.54%	Outcome StDev	22.47%	Outcome StDev	22.78%	Outcome StDev	23.43%
Portfolio Beta	0.853	Portfolio Beta	0.793	Portfolio Beta	0.823	Portfolio Beta	0.864	Portfolio Beta	0.906
Sharpe Ratio	64.53%	Sharpe Ratio	76.82%	Sharpe Ratio	81.50%	Sharpe Ratio	84.78%	Sharpe Ratio	86.69%
Name	Weights	Name	Weights	Name	Weights	Name	Weights	Name	Weights
APPL	40.11%	APPL	50.43%	APPL	61.86%	APPL	74.50%	APPL	87.15%
COST	50.20%	COST	46.25%	COST	38.14%	COST	25.50%	COST	12.85%
UNH	9.69%	UNH	3.32%	UNH	0.00%	UNH	0.00%	UNH	0.00%
Sum	100%	Sum	100%	Sum	100%	Sum	100%	Sum	100%
Desired Return	25.00%	Desired Return	26.00%	Desired Return	27.00%	Desired Return	28.00%	Desired Return	29.00%
Outcome Returns	25.00%	Outcome Returns	26.00%	Outcome Returns	27.00%	Outcome Returns	28.00%	Outcome Returns	29.00%
Outcome StDev	24.39%	Outcome StDev	25.64%	Outcome StDev	27.14%	Outcome StDev	28.98%	Outcome StDev	31.12%
Portfolio Beta	0.947	Portfolio Beta	0.988	Portfolio Beta	1.041	Portfolio Beta	1.105	Portfolio Beta	1.169
Sharpe Ratio	87.37%	Sharpe Ratio	87.04%	Sharpe Ratio	85.91%	Sharpe Ratio	83.90%	Sharpe Ratio	81.34%
Name	Weights								
APPL	99.79%								
COST	0.21%								
UNH	0.00%								
Sum	100%								
Desired Return	30.00%								
Outcome Returns	30.00%								
Outcome StDev	33.51%								
Portfolio Beta	1.233								
Sharpe Ratio	78.53%								

Each data point here is used to construct the efficient frontier for Question 4A

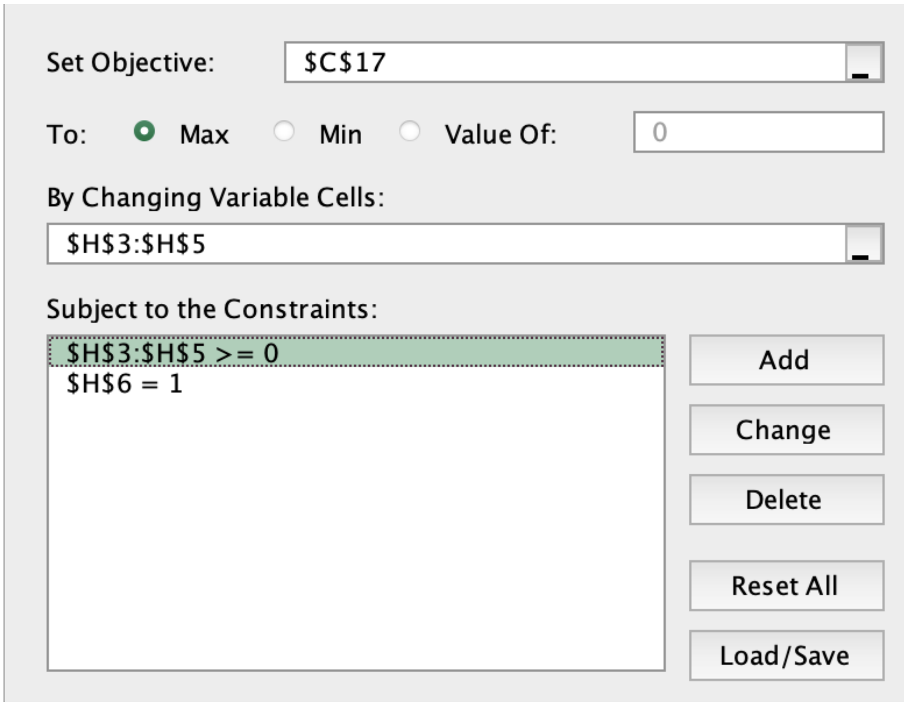
II

Using the portfolio allocation with the lowest volatility, we name this Bonus Portfolio and plot it in the image in Question 4 A, with weights of 9.17%, 62.03%, and 38.32% for APPL, COST, and UNH, respectively that yields 22% return and 22.47% volatility, in terms of APR.

3 Q3

A

Here, I used a solver to maximize the Sharpe ratio, with the constraints that weights sum up to 1, and each weight cannot be negative, where short selling is not permitted.



The maximized Sharpe Ratio is 87.38%, which shows the maximized reward-to-volatility ratio provided by a portfolio, shown below:

	USA 5 year bond	3.69%		
		Returns APPL	Returns COST	Returns UNH
	Expected daily returns	0.12%	0.09%	0.08%
a)	Covariance	Returns APPL	Returns COST	Returns UNH
	Returns APPL	0.000446588	0.000183983	0.000190677
	Returns COST	0.000183983	0.000237063	0.000126897
	Returns UNH	0.000190677	0.000126897	0.000349921
b)	Portfolio daily expected returns	0.0997%		
	Portfolio daily StDev	1.5459%		
	Portfolio Annual Expected returns	25.13%		
	Portfolio Annual StDev	24.54%		
c)	Portfolio Beta	0.952		
	Sharpe Ratio	87.38%		

Name	Weights
APPL	41.46%
COST	49.68%
UNH	8.86%
Sum	100%

B

To calculate the beta of the tangent portfolio, we will use the same method as in Question 2, shown in Equation 11, using a weighted average.

$$\beta_p = \vec{w}^T \times \vec{\beta} = [0.4146, 0.4968, 0.0886] \times [1.23, 0.73, 0.90]^T = 0.952 \tag{12}$$

We can see that the tangent portfolio's beta is slightly larger than my portfolio's beta in question 2, meaning more systematic risk. However, the tangent portfolio offers 3% more return, which achieves the optimal return-to-risk trade-off by the maximal Sharpe ratio.

C

This is shown altogether in the image in Question 4 A. I have plotted volatility vs. expected return for the tangent portfolio.

Looking at the figure in Question 4 A, the index portfolio (S&P) has a return of 9.74% and volatility of 21.80%, and the tangent portfolio has a return of 25.13% and volatility of 24.54%. We can see that the tangent portfolio has around 3% higher volatility than the S&P, but it offers around 15% more return than S&P, which it very attractive. In fact, we should always purchase at a combination of the risk-free investment and the tangent portfolio. This is because if we are risk-averse and want to take the risk no more than S&P, we can purchase a combination such that we get have much higher return (around 22% rather than 9.74% in S&P) with the volatility in S&P. This is the point on the orange line in Question Q4A where the x-value equals S&P volatility = 21.80%, and we get an expected return of around 22%, which is much higher than the S&P return of 9.74%.

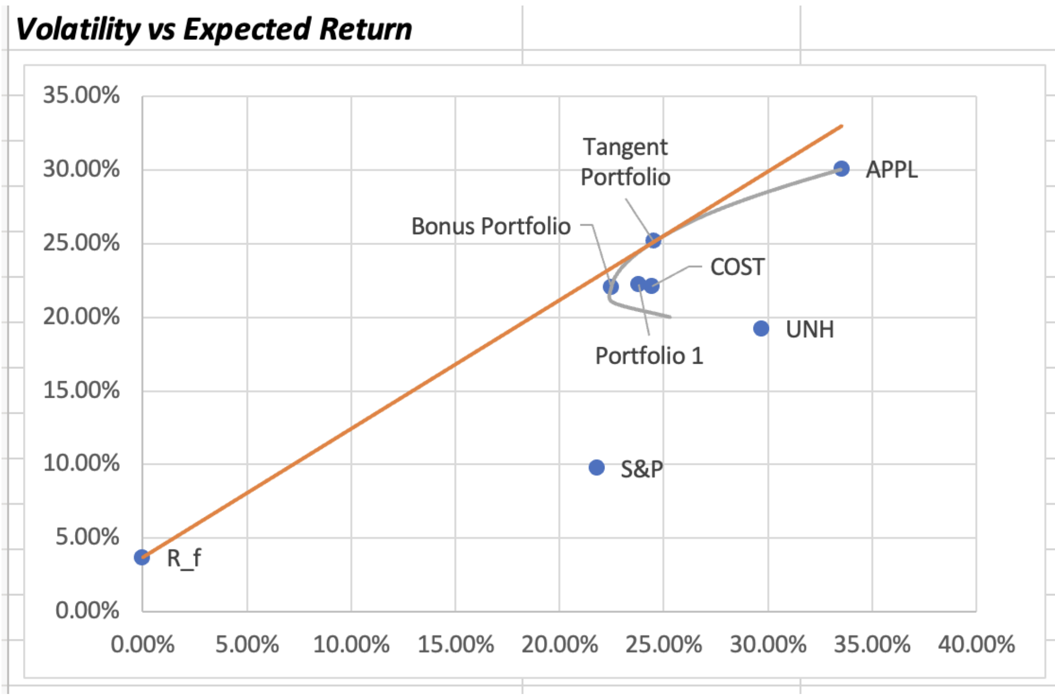
D

Because S&P represents the market with hundreds of firms, it is well diversified and therefore has the least volatility, suggesting the least risk. On the other hand, my portfolio from Q2 offers a much higher return, but at the same time has higher volatility. The tangent portfolio with the highest Sharpe ratio offers the best return-to-risk trade-off. While it has around 0.5% more volatility than my portfolio, it offers 3% more return. Therefore, purchasing stocks at a combination of risk-free investment and the tangent portfolio is the way to go (anywhere in the orange line) because, at any level of volatility, the combination offers more expected return than any existing portfolio.

4 Q4

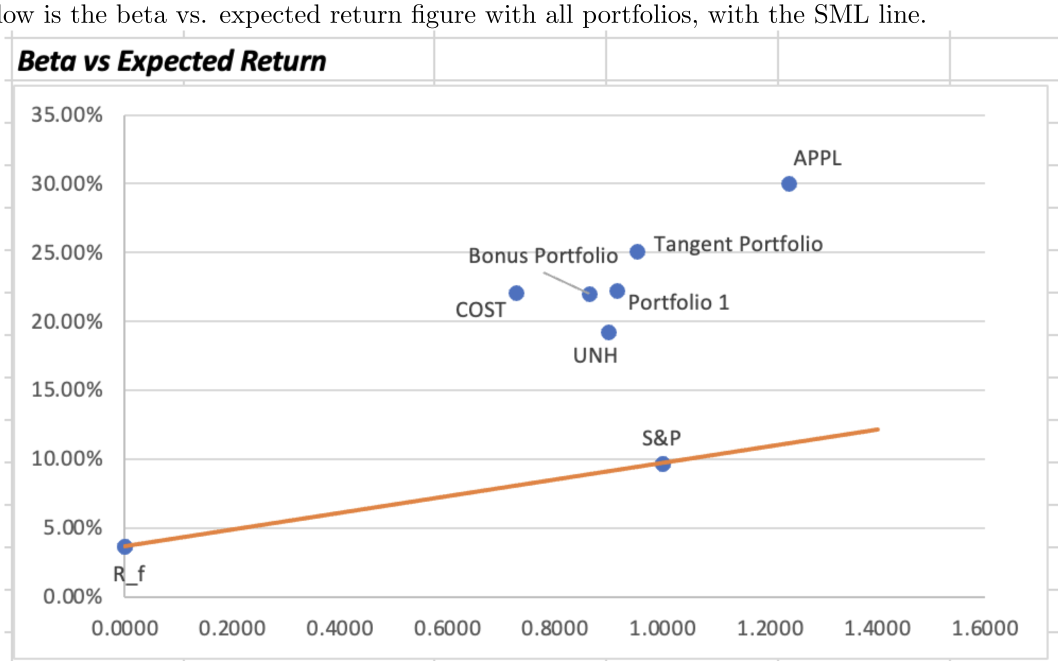
A

Below is the volatility vs. expected return figure with all portfolios, the line between risk free rate and tangent portfolio (Capital Market Line), and the efficient frontier.



B

Below is the beta vs. expected return figure with all portfolios, with the SML line.



C

They don't fall on the line. They seem to fit another parallel line to SML that has a higher y-intercept. This means the assumption of CAPM presumably doesn't hold in the real situation. This is because CAPM assumes $\alpha = 0$, which captures a risk-adjusted performance measure for the historical returns. In reality, all three stocks I've chosen seem to perform better than the prediction than the CAPM, meaning $\alpha \approx 15\% > 0$ in reality, according to the figure.

D

The SML (Security Market Line) is based on several assumptions and empirical justifications in my calculation.

Time Horizon

The SML assumes that the required return on a stock is based on the stock's expected cash flows over a long-term time horizon. To estimate beta, common practice is to use at least two years of weekly return data or five years of monthly return data. This assumption is based on the idea that short-term market fluctuations and noise can distort the true risk-return relationship. My calculation has included daily data for the past five years, so the requirement is satisfied.

Market Proxy

The SML assumes that the market proxy used to estimate beta accurately represents the overall market. In practice, S&P 500 is used as the US market proxy. It's important to note that different market proxies may have different betas and risk-return relationships, and proxies may not fully represent the actual market.

Beta Variation and Extrapolation

The SML assumes that beta is stable over time, but in reality, betas vary over time. As a result, many practitioners prefer to use average industry betas rather than individual stock betas. Average industry betas were not used in my experiment, which increases the risk of skewed results.

Outliers

The SML assumes that beta estimates obtained from linear regression are not affected by outliers, which are returns of unusually large magnitude. However, in practice, it is uncertain whether outliers are present in my data since no data cleaning was performed, which may skew the results.

Alpha α

The SML based on CAPM assumes alpha is 0, which is clearly not the case when it comes to strong stocks like APPL, COST, and UNH. It is better to not assume alpha as 0 to make a more flexible prediction on the portfolio