Multi-Model Communication Project
University of Wisconsin-Madison

Gershgorin-Majda 2010 System Simulator

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Contents

1. Introduction

This code is a test example for the multi-model forecast Kalman filter, as initially formulated in November 2020. In particular, we assume linear, Gaussian models and utilize some linear combination of our models to create a psuedo-truth signal for each model that that model observes. The purpose of this code, at least initially, is to be able to run the Gershgorin-Majda 2010 system, as well as the additive and multiplicative models.

This document is inteded to be supplemental documentation for the `README.md` files included throughout the code repository.

1.1 The Gershgorin-Majda 2010 System

Although this is not the official name for this test system, it was first proposed by Gershgorin and Majda in [1]. Specifically, the exactly solvable test model is

$$\frac{d\,u(t)}{dt} = (-\gamma(t) + i\,\omega)\,u(t) + b(t) + f(t) + \sigma\,\dot{W}(t), \tag{1.1.1}$$

$$\frac{d\,b(t)}{dt} = \left(-\gamma_b + i\,\omega_b\right)\left(b(t) + \hat{b}\right) + \sigma_b\,\dot{W}_b(t), \tag{1.1.2}$$

$$\frac{d\,\gamma(t)}{dt} = -d_\gamma\,\left(\gamma(t) + \hat{\gamma}\right) + \sigma_\gamma\,\dot{W}_\gamma(t), \tag{1.1.3}$$

where $u(t)$ and $b(t)$ are complex-valued and $\gamma(t)$ is real-valued. The terms $b(t)$ and $\gamma(t)$ represent additive and multiplicative bias corrections terms. Also, ω is the oscillation frequency of $u(t)$, $f(t)$ is external forcing, and σ characterizes the strength of the white noise forcing $\dot{W}(t)$. The parameters γ_b and d_γ represent the damping and parameters σ_b and σ_γ represent the strength of the white noise forcing of the additive and multiplicative bias correction terms, respectively. The parameters $\hat{b}$ and $\hat{\gamma}$ are the stationary mean bias correction values of $b(t)$ and $\gamma(t)$, correspondingly, and ω_b is the frequency of the additive noise. Note that the white noise $\dot{W}_\gamma(t)$ is real-valued while the white noises $\dot{W}(t)$ and $\dot{W}_b(t)$ are complex-valueds and their real and imaginary parts are independent real-valued white noises.

In [1], they consider an oscillatory external forcing

$$f(t) = A_f\,e^{i\,\omega_f\,t} \tag{1.1.4}$$

with A_f the strength og the external forcing and ω_f its frequency. For the purpose of testing some parameter estimation techniques, Gershgorin and Majda considered an additive model of the original Gershgorin-Majda 2010 system where there is only additive bias correction

$$\frac{d\,u\,(t)}{dt} = \left(-\overline{d} + i\,\omega\right)\,u\,(t) + b\,(t) + f\,(t) + \sigma\,\dot{W}\,(t)\,, \tag{1.1.5}$$

$$\frac{d\,b\,(t)}{dt} = \left(-\gamma_b + i\,\omega_b\right)\,\left(b\,(t) + \widehat{b}\right) + \sigma_b\,\dot{W}_b\,(t)\,, \tag{1.1.6}$$

where $\overline{d}$ is the mean value of the damping, as well as a multiplicative model where there is only multiplicative bias correction

$$\frac{d\,u\,(t)}{dt} = \left(-\gamma\,(t) + i\,\omega\right)\,u\,(t) + f\,(t) + \sigma\,\dot{W}\,(t)\,, \tag{1.1.7}$$

$$\frac{d\,\gamma\,(t)}{dt} = -d_\gamma\,\left(\gamma\,(t) + \widehat{\gamma}\right) + \sigma_\gamma\,\dot{W}_\gamma\,(t)\,. \tag{1.1.8}$$

2. Compiling and running

This code utilizes CMake as a build system, and was designed for compilation on Ubuntu 20.04.1. In particular, here are the requirements for compiling and running the code:

- CMake (at least version 3.16).
- netCDF-Fortran (at least version 4.8.0).

Once you have these, simply navigate to the `build` subdirectory and enter the standard CMake commands

```
cmake ..
cmake --build .
```

This will compile the code, build the headers and `NAMELIST`, and create the executable `gershgorin_majda_10`. To run the code, enter the command

```
./gershgorin_majda_10
```

and an output file `out.nc` will be generated.

3. Code structure

In the most overview sense, the simulation code contains a single driver subroutine that calls subroutines to perform the following tasks:

1. Initialize the simulation, including reading the `NAMELIST`.

2. Execute the simulation.

3. Finalize the simulation.

Acknowledgements

This project took advantage of netCDF software developed by UCAR/Unidata (http://doi.org/10.5065/D6H70CW6) and CMake software.

References

[1] B. Gershgorin, J. Harlim, and A. Majda, *Test models for improving filtering with model errors through stochastic parameter estimation*, Journal of Computational Physics, 229 (2020), pp. 1–31.

A. Pathwise Solutions of the Gershgorin-Harlim-Majda 2010 System

In this appendix, we will be deriving the pathwise solutions for the Gershgorin-Harlim-Majda 2010 system. The first subsection will provide various identities used throughout the derivations for the pathwise solutions, while subsequent subsections will include the derivations for individual pathwise solutions. We assume that the initial values of $b(t)$, $\gamma(t)$ and $u(t)$ are given.

For ease of reference, the pathwise solution for $b(t)$ is given in Eq. A.2.1, while that of $\gamma(t)$ is given in Eq. A.3.1, and that of $u(t)$ is given in Eq. **??**.

A.1 Identities Used in Pathwise Solution Derivations

The first identity we will use is for the derivative with respect to t of an integral whose bounds depend on t and integrand depends on t and another argument s, where s is the variable of integration.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{a_0(t)}^{a_1(t)} h(t,\ s)\ ds\right] = h\left(t,\ a_1(t)\right)\ \frac{\mathrm{d}a_1(t)}{\mathrm{d}t} - h\left(t,\ a_0(t)\right)\ \frac{\mathrm{d}a_0(t)}{\mathrm{d}t} + \int_{a_0(t)}^{a_1(t)} \frac{\partial}{\partial t}\left[h(t,\ s)\right]\ ds. \tag{A.1.1}$$

A.2 Pathwise solution for $b(t)$

The pathwise solution for $b(t)$ may be obtained in a relatively straightforward manner, utilizing an integrating factor of the form $e^{g(t)}$. We later find the exact form of $g(t)$, which leads to the pathwise solution for $b(t)$. For notational convenience, we define $\lambda_b := -\gamma_b + i\,\omega_b$ and $b_0 := b\left(t_0\right)$. We proceed as follows:

$$
\begin{aligned}
& & \frac{\mathrm{d}b(t)}{\mathrm{d}t} &= \lambda_b \left(b(t) - \widehat{b} \right) + \sigma_b \dot{W}_b(t) \\
&\Rightarrow & \frac{\mathrm{d}b(t)}{\mathrm{d}t} - \lambda_b b(t) &= -\lambda_b \widehat{b} + \sigma_b \dot{W}_b(t) \\
&\Rightarrow & \frac{\mathrm{d}}{\mathrm{d}t} \left[e^{g(t)} b(t) \right] &= -\lambda_b \widehat{b} e^{g(t)} + \sigma_b e^{g(t)} \dot{W}_b(t) \\
&\Rightarrow & \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s} \left[e^{g(s)} b(s) \right] ds &= -\lambda_b \widehat{b} \int_{t_0}^{t} e^{g(s)} ds + \sigma_b \int_{t_0}^{t} e^{g(s)} \frac{\mathrm{d}W_b(s)}{\mathrm{d}s} ds \\
&\Rightarrow & e^{g(t)} b(t) &= e^{g\left(t_0\right)} b_0 - \lambda_b \widehat{b} \int_{t_0}^{t} e^{g(s)} ds + \sigma_b \int_{t_0}^{t} e^{g(s)} dW_b(s) \\
&\Rightarrow & b(t) &= e^{g\left(t_0\right) - g(t)} b_0 - \lambda_b \widehat{b} \int_{t_0}^{t} e^{g(s) - g(t)} ds + \sigma_b \int_{t_0}^{t} e^{g(s) - g(t)} dW_b(s)
\end{aligned}
$$

At this point, it is convenient to find the exact for of $g(t)$ so that we may evaluate the second integral in the final line above. In particular, $g(t)$ must satisfy

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left[e^{g(t)} b(t) \right] = e^{g(t)} \frac{\mathrm{d}b(t)}{\mathrm{d}t} + \frac{\mathrm{d}g(t)}{\mathrm{d}t} e^{g(t)} b(t) = e^{g(t)} \frac{\mathrm{d}b(t)}{\mathrm{d}t} - \lambda_b e^{g(t)} b(t).
$$

Therefore,

$$
\begin{aligned}
& & \frac{\mathrm{d}g(t)}{\mathrm{d}t} &= -\lambda_b \\
&\Rightarrow & \int_{t}^{s} \frac{\mathrm{d}g(t')}{\mathrm{d}t'} dt' &= -\lambda_b \int_{t}^{s} dt' \\
&\Rightarrow & g(s) - g(t) &= -\lambda_b (s - t) = \lambda_b (t - s).
\end{aligned}
$$

With this, we may continue our derivation of the path-wise solution for $b(t)$

$$
\begin{aligned}
& & b(t) &= e^{\lambda_b \left(t - t_0\right)} b_0 - \lambda_b \widehat{b} \int_{t_0}^{t} e^{\lambda_b (t-s)} ds + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)} dW_b(s) \\
&\Rightarrow & b(t) &= e^{\lambda_b \left(t - t_0\right)} b_0 + \widehat{b} \left(1 - e^{\lambda_b \left(t - t_0\right)} \right) + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)} dW_b(s)
\end{aligned}
$$

from which we obtain

$$
b(t) = \widehat{b} + \left(b_0 - \widehat{b} \right) e^{\lambda_b \left(t - t_0\right)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)} dW_b(s). \tag{A.2.1}
$$

To verify this result, we may simply differentiate it with respect to t. To do this, we utilize the identity Eq. A.1.1

$$\begin{aligned}
\frac{\mathrm{d}b(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{b} + \left(b_0 - \hat{b}\right) e^{\lambda_b (t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right] \\
&= \lambda_b \left(b_0 - \hat{b}\right) e^{\lambda_b (t-t_0)} + \sigma_b \left(\dot{W}_b(t) + \lambda_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right) \\
&= \lambda_b \left(\left(b_0 - \hat{b}\right) e^{\lambda_b (t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right) + \sigma_b \dot{W}_b(t) \\
&= \lambda_b \left(b(t) - \hat{b}\right) + \sigma_b \dot{W}_b(t)
\end{aligned}$$

which matches the original differential equation for $b(t)$ (Eq. 1.1.1).

A.3 Pathwise solution for $\gamma(t)$

Following a very similar procedure to finding the pathwise solution for $b(t)$ (see Appendix A.2), we find that the path-wise solution for $\gamma(t)$ is

$$\gamma(t) = \hat{\gamma} + \left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s) \tag{A.3.1}$$

where we have defined $\gamma_0 := \gamma\left(t_0\right)$. We verify this path-wise solution by differentiating it with respect to t (again utilizing the identity Eq. A.1.1).

$$\begin{aligned}
\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\gamma} + \left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s)\right] \\
&= -d_\gamma \left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma \left(\dot{W}_\gamma(t) - d_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s)\right) \\
&= -d_\gamma \left(\left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s)\right) + \sigma_\gamma \dot{W}_\gamma(t) \\
&= -d_\gamma \left(\gamma(t) - \hat{\gamma}\right) + \sigma_\gamma \dot{W}_\gamma(t)
\end{aligned}$$

which matches the original differential equation for $\gamma(t)$ (Eq. 1.1.2).

A.4 Pathwise solution for $u(t)$

Hello world!

B. Exact Statistics of the Gershgorin-Harlim-Majda 2010 System

In this appendix, we will be deriving the exact statistics for the Gershgorin-Harlim-Majda 2010 system. The first subsection will provide various identities used throughout the derivations for the exact statistics, while subsequent subsections will incluse the derivations for individual statistics. We assume that the initial statistics are given.