Multi-Model Communication Project
University of Wisconsin-Madison

Gershgorin-Harlim-Majda 2010 System Simulator

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Contents

1. Introduction

This code is an implementation of the Gershgorin-Harlim-Majda 2010 system (see Subsection 1.1). It inlcudes a Forward Euler, Backward Euler, adaptive Forward Euler, adaptive Backward Euler, and an analytic statistics solver (using the trapezoidal method for required integrals). It was originally intended for use in the multi-model communication project (working title).

This document is intended to be supplemental documentation for the `README.md` files included throughout the code repository.

1.1 The Gershgorin-Harlim-Majda 2010 System

Although this is not the official name for this test system, it was first proposed by Gershgorin, Harlim, and Majda in [3]. Specifically, the exactly solvable test model is

$$\frac{d\,u(t)}{dt} = (-\gamma(t) + i\,\omega)\,u(t) + b(t) + f(t) + \sigma\,\dot{W}(t), \tag{1.1.1}$$

$$\frac{d\,b(t)}{dt} = \left(-\gamma_b + i\,\omega_b\right)\left(b(t) + \hat{b}\right) + \sigma_b\,\dot{W}_b(t), \tag{1.1.2}$$

$$\frac{d\,\gamma(t)}{dt} = -d_\gamma\,\left(\gamma(t) + \hat{\gamma}\right) + \sigma_\gamma\,\dot{W}_\gamma(t), \tag{1.1.3}$$

where $u(t)$ and $b(t)$ are complex-valued and $\gamma(t)$ is real-valued. The terms $b(t)$ and $\gamma(t)$ represent additive and multiplicative bias corrections terms. Also, ω is the oscillation frequency of $u(t)$, $f(t)$ is external forcing, and σ characterizes the strength of the white noise forcing $\dot{W}(t)$. The parameters γ_b and d_γ represent the damping and parameters σ_b and σ_γ represent the strength of the white noise forcing of the additive and multiplicative bias correction terms, respectively. The parameters $\hat{b}$ and $\hat{\gamma}$ are the stationary mean bias correction values of $b(t)$ and $\gamma(t)$, correspondingly, and ω_b is the frequency of the additive noise. Note that the white noise $\dot{W}_\gamma(t)$ is real-valued while the white noises $\dot{W}(t)$ and $\dot{W}_b(t)$ are complex-valueds and their real and imaginary parts are independent real-valued white noises.

In [3], they consider an oscillatory external forcing

$$f(t) = A_f\,e^{i\,\omega_f\,t} \tag{1.1.4}$$

with A_f the strength of the external forcing and ω_f its frequency. We utilize a forcing of this form in the code.

The pathwise solutions and exact statistics (up to second order) are included in Appendices A and B, respectively.

2. Compiling and running

This code utilizes CMake as a build system, and was written on Ubuntu 20.04.1.[1] In particular, the requirements for compiling and running the code are:

- CMake (at least version 3.16).
- LAPACK (at least version 3.9.0).
- MPI (at least version 4.0).
- netCDF-Fortran (at least version 4.8.0).

The minimum required versions are not "hard" minimums; older versions may work, but the code was developed using these versions. Once you have these, enter the standard CMake commands

```
mkdir build
cd build
cmake ..
cmake --build .
```

This will compile the code, build the headers and `NAMELIST`, and create the executable `gershgorin_harlim_majda_2010`. To run the code, enter the command

```
mpirun -np X gershgorin_harlim_majda_10
```

(where `X` is the number of processors) and several output files `outXXX.nc` will be generated.

3. Code structure

In the most overview sense, the simulation code contains a single driver subroutine that calls subroutines to perform the following tasks:

1. Initialize the simulation, including reading the `NAMELIST`.
2. Execute the simulation.
3. Finalize the simulation.

Within executing the simulation, we have five different time-stepping schemes (called `timeStepScheme` in the code):

[1] In actuality, I utilized the Windows Subsystem for Linux on my Windows machine, which I find to be very useful for coding in Fortran!

(0) Forward Euler(-Maruyama): This scheme is defined in Chapter 3.2 of [4]. Due to the various values of $\gamma(t)$, this method is often unstable (in the sense of absolute stability).

(1) Backward (Drift-Implicit) Euler: This scheme is likewise defined in Chapter 3.2 of [4]. Due to the various values of $\gamma(t)$, this method is often unstable (in the sense of absolute stability), although notably less often than the Forward Euler method.

(2) Adaptive Forward Euler(-Maruyama): This scheme is based on the forward Euler(-Maruyama) scheme, although it adapts the time-step size to ensure absolute stability. Due to the bounded region of absolute stability for the forward Euler scheme, these time-steps tend to be small.

(3) Adaptive Backward (Drift-Implicit) Euler: This scheme is based on the forward Euler(-Maruyama) scheme, although it adapts the time-step size to ensure absolute stability. Due to the unbounded region of absolute stability for the backward Euler scheme, the original time-step may not need to be adjusted.

(4) Analytic Statistics with Trapezoidal Integration: This scheme is based on the equations for the first- and second-order statistics of the system (Appendix B). At each time-step k, the state of the system is entirely determined, so each of $b\left(t_k\right) := b_k$, $\gamma\left(t_k\right) := \gamma_k$, and $u\left(t_k\right) := u_k$ may be represented as Dirac-delta distributions. We may then find the mean, variances, and covariances associated with each of b_{k+1}, γ_{k+1}, and u_{k+1} and select them randomly. The implmentation of this scheme is incomplete, and the current progress on implementing it has shown that it may have stability issues for large time-step sizes.

None of these schemes have been statisically verified to be accurate as of writing this.

4. Future work

Due to needing to move on to more relevant work, this sub-project was ended prematurely. Hence, there is a notably long list of work that still needs to be done. I will separate these lists based on general category: **S**imulation **C**ode, **A**nalysis **S**cripts, and **S**ystem **E**quations.

The simulation code feels fairly complete to me, although there are additions and improvements that could be made.

SC–1) Possibly reformat the output files to be nicer to analyze. Not necessarily one run per file, but maybe a maximum of 100 runs per file.

SC–2) It is currently difficult to calculate the statistics for the runs that use adaptive time-stepping, as the step size may be adjusted differently for each run. It would be nice if each method ensure consistent time-steps between runs (e.g., use different time-step sizes to ensure information is output at the user-input time-step size).

SC–3) The scheme that uses the analytic statistics with the trapezoidal method for numeric integration seems very unstable. This may be due to the unboundedness of the variances and covariances of the system, but it may be an error in the code.

SC–4) The code has not yet been statistically analyzed, which is necessary if this code is to be used for other purposes.

The evaluation scripts are currently written in Python, and automatically read in data from the output files `outXXX.nc` in the `build` subcdirectory in which the code is compiled and linked (as opposed the the `build` subdirectory for this documentation.

AS–1) These scripts are incomplete as in the do not compare all of the statistics of the output of the simulations against the analytic formulas. The rest of the analytic formulas need to be implemented in an efficient way, e.g., calculate $\mathrm{Var}\left(u\left(t_{k+1}\right)\right)$ using $\mathrm{Var}\left(u\left(t_k\right)\right)$ instead of $\mathrm{Var}\left(u\left(t_0\right)\right)$.

AS–2) It would be convenient if the scripts could read in the `NAMELIST` to obtain the simulation parameters, or at least have an option to do so (possibly with command-line arguments).

AS–3) A written apology at the top of each script for their general sloppiness, including (but not limited to) no standard order to input arguments into the many functions used to calculate the analytic statistics.

Although we have analytic equations for the statistics of the system, there may be better ways to write them.

SE–1) There are many equations in Appendix B which are specific cases of more general equations, e.g., $\mathrm{Var}(\boldsymbol{J}(s,\ t))$ (Eq. B.4.7) and $\mathrm{Cov}(\boldsymbol{J}(s,\ t),\ \boldsymbol{J}(r,\ t))$ (Eqs. B.9.19 and B.9.20). It would be nice to eliminate the redundant special cases, or at least their derivations. Although it is nice to have separate derivations of the simpler forms as a way to check the more general forms.

SE–2) It might be useful to define variables/functions for more common terms in the equations, to cut down on how long they can be.

SE–3) It would be useful to know which statistics depend on each other, for when someone wants to make a plot of the analytic statistics and needs the statistics at the previous time-step to get the statistics at the current time-step.

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References

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A. Pathwise solutions of the Gershgorin-Harlim-Majda 2010 system

In this appendix, we will be deriving the pathwise solutions for the Gershgorin-Harlim-Majda 2010 system. The first subsection will provide various identities used throughout the derivations for the pathwise solutions, while subsequent subsections will include the derivations for individual pathwise solutions. We assume that the initial values of $b(t)$, $\gamma(t)$ and $u(t)$ are given.

For ease of reference, we give the equation numbers for each of the pathwise solutions below:

$b(t)$ – Eq. A.2.5.

$\gamma(t)$ – Eq. A.3.1.

$u(t)$ – Eq. A.4.4.

A.1 Identities used in pathwise solution derivations

The first identity we will use is for the derivative with respect to t of an integral whose bounds depend on t and integrand depends on t and another argument s, where s is the variable of integration.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{a_0(t)}^{a_1(t)} h(t,\ s)\ ds\right] = h\left(t,\ a_1(t)\right)\ \frac{\mathrm{d}a_1(t)}{\mathrm{d}t} - h\left(t,\ a_0(t)\right)\ \frac{\mathrm{d}a_0(t)}{\mathrm{d}t} + \int_{a_0(t)}^{a_1(t)} \frac{\partial}{\partial t}\left[h(t,\ s)\right]\ ds. \tag{A.1.1}$$

A.2 Pathwise solution for $b(t)$

The pathwise solution for $b(t)$ may be obtained in a relatively straightforward manner, utilizing an integrating factor of the form $e^{g(t)}$. We later find the exact form of $g(t)$, which leads to the pathwise solution for $b(t)$. For notational convenience, we define $\lambda_b := -\gamma_b + i\,\omega_b$ and $b_0 := b\left(t_0\right)$. We proceed as follows:

$$
\begin{aligned}
&& \frac{\mathrm{d}b(t)}{\mathrm{d}t} &= \lambda_b \left(b(t) - \widehat{b}\right) + \sigma_b \dot{W}_b(t) \\
\Rightarrow && \frac{\mathrm{d}b(t)}{\mathrm{d}t} - \lambda_b\, b(t) &= -\lambda_b \widehat{b} + \sigma_b \dot{W}_b(t) \\
\Rightarrow && \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{g(t)}\, b(t)\right] &= -\lambda_b \widehat{b}\, e^{g(t)} + \sigma_b\, e^{g(t)} \dot{W}_b(t) \\
\Rightarrow && \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\left[e^{g(s)}\, b(s)\right] ds &= -\lambda_b \widehat{b} \int_{t_0}^{t} e^{g(s)}\, ds + \sigma_b \int_{t_0}^{t} e^{g(s)} \frac{\mathrm{d}W_b(s)}{\mathrm{d}s}\, ds \\
\Rightarrow && e^{g(t)}\, b(t) &= e^{g\left(t_0\right)}\, b_0 - \lambda_b \widehat{b} \int_{t_0}^{t} e^{g(s)}\, ds + \sigma_b \int_{t_0}^{t} e^{g(s)}\, dW_b(s) \\
\Rightarrow && b(t) &= e^{g\left(t_0\right) - g(t)}\, b_0 - \lambda_b \widehat{b} \int_{t_0}^{t} e^{g(s) - g(t)}\, ds + \sigma_b \int_{t_0}^{t} e^{g(s) - g(t)}\, dW_b(s)
\end{aligned}
\tag{A.2.1}
$$

At this point, it is convenient to find the exact for of $g(t)$ so that we may evaluate the second integral in the final line above. In particular, $g(t)$ must satisfy

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{g(t)}\, b(t)\right] = e^{g(t)} \frac{\mathrm{d}b(t)}{\mathrm{d}t} + \frac{\mathrm{d}g(t)}{\mathrm{d}t} e^{g(t)}\, b(t) = e^{g(t)} \frac{\mathrm{d}b(t)}{\mathrm{d}t} - \lambda_b\, e^{g(t)}\, b(t)\,. \tag{A.2.2}
$$

Therefore,

$$
\begin{aligned}
&& \frac{\mathrm{d}g(t)}{\mathrm{d}t} &= -\lambda_b \\
\Rightarrow && \int_{t}^{s} \frac{\mathrm{d}g(t')}{\mathrm{d}t'}\, dt' &= -\lambda_b \int_{t}^{s} dt' \\
\Rightarrow && g(s) - g(t) &= -\lambda_b\, (s - t) = \lambda_b\, (t - s)\,.
\end{aligned}
\tag{A.2.3}
$$

With this, we may continue our derivation of the path-wise solution for $b(t)$

$$
b(t) = e^{\lambda_b\left(t - t_0\right)}\, b_0 - \lambda_b \widehat{b} \int_{t_0}^{t} e^{\lambda_b (t-s)}\, ds + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)
$$

$$\Rightarrow \qquad b(t) = e^{\lambda_b(t-t_0)}\, b_0 + \hat{b}\left(1 - e^{\lambda_b(t-t_0)}\right) + \sigma_b \int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s) \tag{A.2.4}$$

from which we obtain

$$b(t) = \hat{b} + \left(b_0 - \hat{b}\right) e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\,. \tag{A.2.5}$$

To verify this result, we may simply differentiate it with respect to t. To do this, we utilize the identity Eq. A.1.1

$$\begin{aligned}
\frac{\mathrm{d}b(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{b} + \left(b_0 - \hat{b}\right) e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\right] \\
&= \lambda_b \left(b_0 - \hat{b}\right) e^{\lambda_b(t-t_0)} + \sigma_b \left(\dot{W}_b(t) + \lambda_b \int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\right) \\
&= \lambda_b \left(\left(b_0 - \hat{b}\right) e^{\lambda_b(t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b(t-s)}\, dW_b(s)\right) + \sigma_b\, \dot{W}_b(t) \\
&= \lambda_b \left(b(t) - \hat{b}\right) + \sigma_b\, \dot{W}_b(t)
\end{aligned} \tag{A.2.6}$$

which matches the original differential equation for $b(t)$ (Eq. 1.1.1).

A.3 Pathwise solution for $\gamma(t)$

Following a very similar procedure to finding the pathwise solution for $b(t)$ (see Appendix A.2), we find that the path-wise solution for $\gamma(t)$ is

$$\gamma(t) = \hat{\gamma} + \left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma(t-s)}\, dW_\gamma(s) \tag{A.3.1}$$

where we have defined $\gamma_0 := \gamma\left(t_0\right)$. We verify this path-wise solution by differentiating it with respect to t (again utilizing the identity Eq. A.1.1).

$$\begin{aligned}
\frac{\mathrm{d}\gamma(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\hat{\gamma} + \left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma(t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma(t-s)}\, dW_\gamma(s)\right] \\
&= -d_\gamma \left(\gamma_0 - \hat{\gamma}\right) e^{-d_\gamma(t-t_0)} + \sigma_\gamma \left(\dot{W}_\gamma(t) - d_\gamma \int_{t_0}^{t} e^{-d_\gamma(t-s)}\, dW_\gamma(s)\right)
\end{aligned}$$

$$
\begin{aligned}
&= -d_\gamma \left(\left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)} \, dW_\gamma (s) \right) + \sigma_\gamma \dot{W}_\gamma (t) \\
&= -d_\gamma \left(\gamma (t) - \widehat{\gamma}\right) + \sigma_\gamma \dot{W}_\gamma (t)
\end{aligned} \tag{A.3.2}
$$

which matches the original differential equation for $\gamma(t)$ (Eq. 1.1.2).

A.4 Pathwise solution for $u(t)$

We again follow a very similar procedure to finding the pathwise solution for $b(t)$ (see Appendix A.2).

$$
\begin{aligned}
& & \frac{\mathrm{d}u(t)}{\mathrm{d}t} &= (-\gamma(t) + i\omega)\, u(t) + b(t) + f(t) + \sigma \dot{W}(t) \\
\Rightarrow & & \frac{\mathrm{d}u(t)}{\mathrm{d}t} - (-\gamma(t) + i\omega)\, u(t) &= b(t) + f(t) + \sigma \dot{W}(t) \\
\Rightarrow & & \frac{\mathrm{d}}{\mathrm{d}t}\left[e^{g(t)} u(t)\right] &= (b(t) + f(t))\, e^{g(t)} + \sigma e^{g(t)} \dot{W}(t) \\
\Rightarrow & & \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}s}\left[e^{g(s)} u(s)\right] ds &= \int_{t_0}^{t} (b(s) + f(s))\, e^{g(s)} \, ds + \sigma \int_{t_0}^{t} e^{g(s)} \frac{\mathrm{d}W(s)}{\mathrm{d}s} \, ds \\
\Rightarrow & & e^{g(t)} u(t) - e^{g(t_0)} u_0 &= \int_{t_0}^{t} (b(s) + f(s))\, e^{g(s)} \, ds + \sigma \int_{t_0}^{t} e^{g(s)} \, dW(s) \\
\Rightarrow & & u(t) &= e^{g(t_0) - g(t)} u_0 + \int_{t_0}^{t} (b(s) + f(s))\, e^{g(s) - g(t)} \, ds \\
& & &\quad + \sigma \int_{t_0}^{t} e^{g(s) - g(t)} \, dW(s)
\end{aligned} \tag{A.4.1}
$$

where $u_0 := u\left(t_0\right)$. To find $g(t)$, we note that it must satisfy

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{g(t)} u(t)\right] = e^{g(t)} \frac{\mathrm{d}u(t)}{\mathrm{d}t} + \frac{\mathrm{d}g(t)}{\mathrm{d}t} e^{g(t)} u(t) = e^{g(t)} \frac{\mathrm{d}u(t)}{\mathrm{d}t} - (-\gamma(t) + i\omega)\, e^{g(t)} u(t). \tag{A.4.2}
$$

Therefore, we may find $g(t)$ through integration

$$
\begin{aligned}
& & \frac{\mathrm{d}g(t)}{\mathrm{d}t} &= -(-\gamma(t) + i\omega) \\
\Rightarrow & & \frac{\mathrm{d}g(t)}{\mathrm{d}t} &= -\left(-\gamma(t) + \widehat{\gamma} - \widehat{\gamma} + i\omega\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \qquad & \int_t^s \frac{\mathrm{d}g\left(t'\right)}{\mathrm{d}t'}\,dt' = -\int_t^s \left(-\gamma\left(t'\right)+\widehat{\gamma}\right)\,dt' - \left(-\widehat{\gamma}+i\,\omega\right)\int_t^s dt' \\
\Rightarrow \qquad & g\left(s\right)-g\left(t\right) = -\int_s^t \left(\gamma\left(t'\right)-\widehat{\gamma}\right)\,dt' - \left(-\widehat{\gamma}+i\,\omega\right)\left(s-t\right) \\
\Rightarrow \qquad & g\left(s\right)-g\left(t\right) = -J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)
\end{aligned} \tag{A.4.3}
$$

where we have defined $J\left(s,\ t\right) := \int_s^t \left(\gamma\left(t'\right)-\widehat{\gamma}\right)\,dt'$ and $\widehat{\lambda} := -\widehat{\gamma}+i\,\omega$. Therefore, our pathwise solution for $u\left(t\right)$ is

$$
u\left(t\right) = e^{-J\left(t_0,\ t\right)+\widehat{\lambda}\left(t-t_0\right)}\,u_0 + \int_{t_0}^t \left(b\left(s\right)+f\left(s\right)\right)\,e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\,ds + \sigma\int_{t_0}^t e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\,dW\left(s\right). \tag{A.4.4}
$$

To verify this solution, we first differentiate each term on the right-hand side individually

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{-J\left(t_0,\ t\right)+\widehat{\lambda}\left(t-t_0\right)}\,u_0\right] &= e^{-J\left(t_0,\ t\right)+\widehat{\lambda}\left(t-t_0\right)}\,u_0\left(-\frac{\mathrm{d}}{\mathrm{d}t}\left[J\left(t_0,\ t\right)\right]+\widehat{\lambda}\right) \\
&= e^{-J\left(t_0,\ t\right)+\widehat{\lambda}\left(t-t_0\right)}\,u_0\left(-\left(\gamma\left(t\right)-\widehat{\gamma}\right)+\left(-\widehat{\gamma}+i\,\omega\right)\right) \\
&= e^{-J\left(t_0,\ t\right)+\widehat{\lambda}\left(t-t_0\right)}\,u_0\left(-\gamma\left(t\right)+i\,\omega\right),
\end{aligned} \tag{A.4.5}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^t \left(b\left(s\right)\ +f\left(s\right)\right)\,e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\,ds\right] \\
&\qquad = \left(b\left(t\right)+f\left(t\right)\right)+\int_{t_0}^t \left(b\left(s\right)+f\left(s\right)\right)\frac{\partial}{\partial t}\left[e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\right]\,ds \\
&\qquad = \left(b\left(t\right)+f\left(t\right)\right) \\
&\qquad\quad + \int_{t_0}^t \left(b\left(s\right)+f\left(s\right)\right)\,e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\left(-\gamma\left(t\right)+i\,\omega\right)\,ds \\
&\qquad = \left(b\left(t\right)+f\left(t\right)\right) \\
&\qquad\quad + \left(-\gamma\left(t\right)+i\,\omega\right)\int_{t_0}^t \left(b\left(s\right)+f\left(s\right)\right)\,e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\,ds,
\end{aligned} \tag{A.4.6}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[\sigma\int_{t_0}^t e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\,dW\left(s\right)\right] &= \sigma\,\dot{W}\left(t\right)+\sigma\int_{t_0}^t \frac{\partial}{\partial t}\left[e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\right]\,dW\left(s\right) \\
&= \sigma\,\dot{W}\left(t\right)+\left(-\gamma\left(t\right)+i\,\omega\right)\sigma\int_{t_0}^t e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\,dW\left(s\right).
\end{aligned} \tag{A.4.7}
$$

Adding these together, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}u(t)}{\mathrm{d}t} &= \left((-\gamma(t) + i\,\omega)\, e^{-J\left(t_0,\, t\right)+\hat{\lambda}\left(t-t_0\right)}\, u_0\right) \\
&\quad + \left((b(t) + f(t)) + (-\gamma(t) + i\,\omega) \int_{t_0}^{t} (b(s) + f(s))\, e^{-J(s,\, t)+\hat{\lambda}(t-s)}\, ds\right) \\
&\quad + \left(\sigma\, \dot{W}(t) + (-\gamma(t) + i\,\omega)\, \sigma \int_{t_0}^{t} e^{-J(s,\, t)+\hat{\lambda}(t-s)}\, dW(s)\right) \\
&= (-\gamma(t) + i\,\omega) \left(e^{-J\left(t_0,\, t\right)+\hat{\lambda}\left(t-t_0\right)}\, u_0 + \int_{t_0}^{t} (b(s) + f(s))\, e^{-J(s,\, t)+\hat{\lambda}(t-s)}\, ds\right. \\
&\qquad\qquad \left. + \,\sigma \int_{t_0}^{t} e^{-J(s,\, t)+\hat{\lambda}(t-s)}\, dW(s)\right) + b(t) + f(t) + \sigma\, \dot{W}(t) \\
&= (-\gamma(t) + i\,\omega)\, u(t) + b(t) + f(t) + \sigma\, \dot{W}(t) \qquad \text{(A.4.8)}
\end{aligned}
$$

which matches the original differential equation for $u(t)$ (Eq. 1.1.3).

B. Exact statistics of the Gershgorin-Harlim-Majda 2010 system

In this appendix, we will be deriving the exact statistics for the Gershgorin-Harlim-Majda 2010 system. The first subsection will provide various identities used throughout the derivations for the exact statistics, while subsequent subsections will include the derivations for individual statistics. We assume that the initial statistics are given.

Below we list the statistics and their equation number in this document, for ease of navigation. Note that $\langle x \rangle$ is the expected value of x, $\mathrm{Var}(x)$ to denote the variance of x, and $\mathrm{Cov}(x,\, y)$ to denote the covariance of x and y.

$\langle b(t) \rangle$ – Eq. B.2.1.

$\langle \gamma(t) \rangle$ – Eq. B.3.1.

$\langle u(t) \rangle$ – Eq. B.4.1.

$\mathrm{Var}(b(t))$ – Eq. B.5.2.

$\mathrm{Cov}\left(b(t),\, \overline{b(t)}\right)$ – Eq. B.6.1.

$\mathrm{Var}(\gamma(t))$ – Eq. B.7.1.

$\mathrm{Cov}(b(t),\, \gamma(t))$ – Eq. B.8.1.

$\mathrm{Var}\left(u\left(t\right)\right)$ – Eq. B.9.5.

$\mathrm{Cov}\left(u\left(t\right),\ \overline{u\left(t\right)}\right)$ – Eq. B.10.3.

$\mathrm{Cov}\left(u\left(t\right),\ b\left(t\right)\right)$ – Eq. B.11.2.

$\mathrm{Cov}\left(u\left(t\right),\ \overline{b\left(t\right)}\right)$ – Eq. B.12.2.

$\mathrm{Cov}\left(u\left(t\right),\ \gamma\left(t\right)\right)$ – Eq. B.13.2.

B.1 Identities used in exact statistics derivations

The first identity is very common in stochastic processes: the expected value of the integral of a martingale $f\left(t\right)$ on the interval $\left[a,\ b\right]$ against a real-value white noise $dW\left(t\right)$ is zero.

$$\mathrm{E}\left[\int_a^b f\left(t\right)\ dW\left(t\right)\right] = 0, \tag{B.1.1}$$

where we have used the notation $\mathrm{E}\left[x\right]$ to denote the expected value of x. Throughout this document, we will use both this notation and $\langle x\rangle$ to denote the expected value of x, ideally to improve the readability of the equations.

Knowledge of the identity Eq. B.1.1 is so widely used throughout our derivations that we will not refer to this equation specifically and instead leave it to the reader to identify when it has been used.

Similarly fundamental to Eq. B.1.1 is the Itô isometry, which states that for stochastic processes X_t and Y_t in the space of square-integrable adapted processes $L^2_{\mathrm{ad}}\left(\left[a,\ b\right]\times\Omega\right)$, we have

$$\mathrm{E}\left[\left(\int_a^b X_s\,dW\left(s\right)\right)\left(\int_a^b Y_s\,dW\left(s\right)\right)\right] = E\left(\int_{t_0}^t X_s\,Y_s\,ds\right), \tag{B.1.2}$$

where $dW\left(s\right)$ is a real-value white noise. Due to its fundamentality to our derivations, we will also not refer to this equation specifically and instead leave it to the reader to identify when it has been used.

We will also encounter expressions of the form

$$\int_a^b \int_c^t f\left(s,\ t\right)\ dW\left(s\right)\ dt,$$

where $f\left(s,\ t\right)$ satistfies the requirements of the Itô isometry (Eq. B.1.2), $dW\left(s\right)$ is real-value white noise, and $c \leq a \leq b$. To utilize the Itô isometry, we must change the order of integration, which is somewhat straightforward once we recognize that the region of integration is trapezoidal

$$\int_a^b \int_c^t f(s,\, t)\; dW(s)\; dt = \int_c^a \int_a^b f(s,\, t)\; dt\, dW(s) + \int_a^b \int_s^b f(s,\, t)\; dt\, dW(s)\,. \tag{B.1.3}$$

For complex-value white noise, which we will encounter in several places throuhgout our derivations, we will utilize the identity given immediately after Lemma 2.2.8 of [2] which states that, for a complex stochastic process $Z(s) = X(s) + i\, Y(s)$ and complex-value white-noise $dW(s) = \frac{1}{\sqrt{2}}\,(dU(s) + i\, dV(s))$, we have

$$\begin{aligned} \int_{t_0}^{t} Z(s)\; dW(s) = \frac{1}{\sqrt{2}} \Bigg(& \int_{t_0}^{t} X(s)\; dU(s) - \int_{t_0}^{t} Y(s)\; dV(s) \\ & + i \left(\int_{t_0}^{t} X(s)\; dV(s) + \int_{t_0}^{t} Y(s)\; dU(s) \right) \Bigg)\,. \end{aligned} \tag{B.1.4}$$

Since complex-value white noise is a local martingale, we may utilize this equation along with Itô isometry to obtain the following identity

$$\begin{aligned} E\left(\left(\int_{t_0}^{t} Z(s)\; dW(s) \right) \left(\int_{t_0}^{t} \overline{Z(s)}\, \overline{dW(s)} \right) \right) &= E\left(\int_{t_0}^{t} \left((X(s))^2 + (Y(s))^2 \right)\; ds \right) \\ &= E\left(\int_{t_0}^{t} Z(s)\; \overline{Z(s)}\, ds \right) \end{aligned} \tag{B.1.5}$$

We will also require identities regarding functions of complex- and real-valued Gaussian variables. The most general of these identities is given by

$$\left\langle Z\, W\, e^{b\,X} \right\rangle = \left(\mathrm{Cov}\left(Z,\, \overline{W} \right) + \left(\langle Z \rangle + b\, \mathrm{Cov}\left(Z,\, X \right) \right) \left(\langle W \rangle + b\, \mathrm{Cov}\left(W,\, X \right) \right) \right) e^{b\,\langle X \rangle + \frac{b^2}{2}\,\mathrm{Var}(X)}, \tag{B.1.6}$$

where Z and W are complex-valued Gaussian variables, X is a real-valued Gaussian variable, and b is a real constant. To verify Eq. B.1.6, we begin by writing $Z = A + i\, B$, $W = U + i\, V$, and defining a vector $\vec{\mathbf{v}} = \begin{bmatrix} A & B & U & V & X \end{bmatrix}^T$. Note, $\vec{\mathbf{v}}$ is a five-dimensional Gaussian variable since each of A, B, U, V, and X are Gaussian. To calculate $\left\langle Z\, W\, e^{b\,X} \right\rangle$, we will calculate each term on the right-hand side of the following equation

$$\left\langle Z\, W\, e^{b\,X} \right\rangle = \left\langle A\, U\, e^{b\,X} \right\rangle - \left\langle B\, V\, e^{b\,X} \right\rangle + i\, \left(\left\langle A\, V\, e^{b\,X} \right\rangle + \left\langle B\, U\, e^{b\,X} \right\rangle \right).$$

We begin by considering the moment-generating function for $\vec{\mathbf{v}}$

$$M_{\vec{\mathbf{v}}}\left(\vec{\mathbf{t}}\right) = \int_{\mathbb{R}^5} e^{\vec{\mathbf{t}}^T \vec{\mathbf{v}}'} f_{\vec{\mathbf{v}}}\left(\vec{\mathbf{v}}'\right) \, d\vec{\mathbf{v}}'$$

where $f_{\vec{\mathbf{v}}}\left(\vec{\mathbf{v}}'\right)$ is the probability density function for $\vec{\mathbf{v}}$. For brevity, we denote the i^{th} entry of $\vec{\mathbf{t}}$ by t_i and the j^{th} entry of $\vec{\mathbf{v}}$ by v_j. We have

$$\frac{\partial^2 M_{\vec{\mathbf{v}}}\left(\vec{\mathbf{t}}\right)}{\partial t_j \partial t_i} = \int_{\mathbb{R}^5} v_i{}' \, v_j{}' \, e^{\vec{\mathbf{t}}^T \vec{\mathbf{v}}'} f_{\vec{\mathbf{v}}}\left(\vec{\mathbf{v}}'\right) \, d\vec{\mathbf{v}}.$$

Evaluating this at $\vec{\mathbf{t}} = \begin{bmatrix} 0 & 0 & 0 & 0 & b \end{bmatrix}^T$ for various i, j gives the desired expected values. Now, since $\vec{\mathbf{v}}$ is a five-dimensional Gaussian variable its moment-generating function is given by

$$M_{\vec{\mathbf{v}}}\left(\vec{\mathbf{t}}\right) = e^{\vec{\mathbf{t}}^T \left(\langle \vec{\mathbf{v}} \rangle + \frac{1}{2} \mathbf{\Sigma} \vec{\mathbf{t}}\right)}$$

where $\mathbf{\Sigma}$ is the covariance matrix of A, B, U, V, and X. Hence, after much calculus, we find that

$$\left.\frac{\partial^2 M_{\vec{\mathbf{v}}}\left(\vec{\mathbf{t}}\right)}{\partial t_j \partial t_i}\right|_{t_5=b} = \left(\operatorname{Cov}\left(v_i, \, v_j\right) + \left(\langle v_i \rangle + b \operatorname{Cov}\left(v_i, \, X\right)\right) \left(\left\langle v_j \right\rangle + b \operatorname{Cov}\left(v_j, \, X\right)\right)\right) \cdot e^{b \langle X \rangle + \frac{b^2}{2} \operatorname{Var}(X)},$$

where we have evaluated the partial derivative at $t_1 = \cdots = t_4 = 0$ as well. Using appropriate values of i and j, we obtain

$$\begin{aligned}\left\langle Z \, W \, e^{b X} \right\rangle &= \left\langle A \, U \, e^{b X} \right\rangle - \left\langle B \, V \, e^{b X} \right\rangle + i \, \left(\left\langle A \, V \, e^{b X} \right\rangle + \left\langle B \, U \, e^{b X} \right\rangle\right) \\ &= \left(\operatorname{Cov}\left(Z, \, \overline{W}\right) + \left(\langle Z \rangle + b \operatorname{Cov}(Z, \, X)\right) \left(\langle W \rangle + b \operatorname{Cov}(W, \, X)\right)\right) e^{b \langle X \rangle + \frac{b^2}{2} \operatorname{Var}(X)}\end{aligned}$$

as desired. We will also use a simplified form of Eq. B.1.6 in which $W = 1$

$$\left\langle Z \, e^{b X} \right\rangle = \left(\langle Z \rangle + b \operatorname{Cov}(Z, \, X)\right) e^{b \langle X \rangle + \frac{b^2}{2} \operatorname{Var}(X)}. \tag{B.1.7}$$

B.2 Mean of $b(t)$

The derivation of the mean of $b(t)$ is fairly straightforward. Using our pathwise solution for $b(t)$ (Eq. A.2.5), we obtain

$$\begin{aligned}
\langle b(t)\rangle &= \mathrm{E}\left[\widehat{b} + \left(b_0 - \widehat{b}\right) e^{\lambda_b (t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right] \\
&= \widehat{b} + \left(\langle b_0\rangle - \widehat{b}\right) e^{\lambda_b (t-t_0)} + \sigma_b\, \mathrm{E}\left[\int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right] \\
&= \widehat{b} + \left(\langle b_0\rangle - \widehat{b}\right) e^{\lambda_b (t-t_0)}.
\end{aligned} \tag{B.2.1}$$

B.3 Mean of $\gamma(t)$

The derivation of the mean of $\gamma(t)$ is similarly straightforward. Using our pathwise solution for $\gamma(t)$ (Eq. A.3.1), we obtain

$$\begin{aligned}
\langle \gamma(t)\rangle &= \mathrm{E}\left[\widehat{\gamma} + \left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s)\right] \\
&= \widehat{\gamma} + \left(\langle \gamma_0\rangle - \widehat{\gamma}\right) e^{-d_\gamma (t-t_0)} + \sigma_\gamma\, \mathrm{E}\left[\int_{t_0}^{t} e^{-d_\gamma (t-s)}\, dW_\gamma(s)\right] \\
&= \widehat{\gamma} + \left(\langle \gamma_0\rangle - \widehat{\gamma}\right) e^{-d_\gamma (t-t_0)}.
\end{aligned} \tag{B.3.1}$$

B.4 Mean of $u(t)$

We begin our derivation of the mean of $u(t)$ in a similar manner, by utilizing the pathwise solution for $u(t)$ (Eq. A.4.4)

$$\begin{aligned}
\langle u(t)\rangle = {} & \mathrm{E}\left[u_0\, e^{-J(t_0,\, t)+\widehat{\lambda}(t-t_0)}\right] + \mathrm{E}\left[\int_{t_0}^{t} \left(b(s) + f(s)\right) e^{-J(s,\, t)+\widehat{\lambda}(t-s)}\, ds\right] \\
& + \mathrm{E}\left[\sigma \int_{t_0}^{t} e^{-J(s,\, t)+\widehat{\lambda}(t-s)}\, dW(s)\right] \\
= {} & e^{\widehat{\lambda}(t-t_0)}\, \mathrm{E}\left[u_0\, e^{-J(t_0,\, t)}\right] + \int_{t_0}^{t} e^{\widehat{\lambda}(t-s)}\, \mathrm{E}\left[b(s)\, e^{-J(s,\, t)}\right] ds \\
& + \int_{t_0}^{t} f(s)\, e^{\widehat{\lambda}(t-s)}\, \mathrm{E}\left[e^{-J(s,\, t)}\right] ds
\end{aligned}$$

$$
\begin{aligned}
&= e^{\widehat{\lambda}(t-t_0)} \left(\langle u_0 \rangle - \mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)\right) e^{-\left\langle J\left(t_0,\ t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\ t\right)\right)} \\
&\quad + \int_{t_0}^{t} e^{\widehat{\lambda}(t-s)} \left(\langle b(s) \rangle - \mathrm{Cov}\left(b(s),\ J(s,\ t)\right)\right) e^{-\langle J(s,\ t)\rangle + \frac{1}{2}\mathrm{Var}(J(s,\ t))}\, ds \\
&\quad + \int_{t_0}^{t} f(s)\, e^{\widehat{\lambda}(t-s)}\, e^{-\langle J(s,\ t)\rangle + \frac{1}{2}\mathrm{Var}(J(s,\ t))}\, ds, \qquad \text{(B.4.1)}
\end{aligned}
$$

where we have utilized Eq. B.1.7 to evaluate both $\mathrm{E}\left[u_0\, e^{-J\left(t_0,\ t\right)}\right]$ and $\mathrm{E}\left[b(s)\ e^{-J(s,\ t)}\right]$. To calculate the value of this expression, it is necessary to find $\langle J(s,\ t)\rangle$, $\mathrm{Var}(J(s,\ t))$, $\mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)$, and $\mathrm{Cov}(b(s),\ J(s,\ t))$. For ease of navigation, we list the equation numbers for these staistics below.

$\langle J(s,\ t)\rangle$ – Eq. B.4.2.

$\mathrm{Var}(J(s,\ t))$ – Eq. B.4.7.

$\mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)$ – Eq. B.4.8.

$\mathrm{Cov}(b(s),\ J(s,\ t))$ – Eq. B.4.13.

Even with substituting these equations into Eq. B.4.1, the form of Eq. B.4.1 is not simplified, so we instead keep it in its current form.

B.4.1 Mean of $J(s,\ t)$

Finding the expected value of $J(s,\ t)$ is very straightforwad, as we already have calculated the mean of $\gamma(t)$ (Eq. B.3.1).

$$
\begin{aligned}
\langle J(s,\ t)\rangle &= \mathrm{E}\left[\int_s^t \left(\gamma\left(s'\right) - \widehat{\gamma}\right)\, ds'\right] \\
&= \int_s^t \left(\left\langle \gamma\left(s'\right)\right\rangle - \widehat{\gamma}\right)\, ds' \\
&= \int_s^t \left(\langle \gamma_0 \rangle - \widehat{\gamma}\right)\, e^{-d_\gamma\left(s'-t_0\right)}\, ds' \\
&= \frac{\langle \gamma_0 \rangle - \widehat{\gamma}}{d_\gamma}\left(e^{-d_\gamma(s-t_0)} - e^{-d_\gamma(t-t_0)}\right). \qquad \text{(B.4.2)}
\end{aligned}
$$

B.4.2 Variance of $J(s,\ t)$

To find $\mathrm{Var}(J(s,\ t))$, we first note that

$$\mathrm{Var}\left(\boldsymbol{J}\left(s,\ t\right)\right) = \left\langle \left(\boldsymbol{J}\left(s,\ t\right)\right)^2 \right\rangle - \left(\left\langle \boldsymbol{J}\left(s,\ t\right)\right\rangle\right)^2 . \tag{B.4.3}$$

We previously calculated $\langle \boldsymbol{J}(s,\ t)\rangle$ (Eq. B.4.2), and so it remains to find $\left\langle (\boldsymbol{J}(s,\ t))^2 \right\rangle$.

$$\begin{aligned}
\left\langle (\boldsymbol{J}(s,\ t))^2 \right\rangle &= \mathrm{E}\left[\left(\int_s^t \left(\gamma\left(s'\right) - \widehat{\gamma}\right) ds'\right)^2\right] \\
&= \mathrm{E}\left[\left(\int_s^t \left(\left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma\left(s'-t_0\right)} + \sigma_\gamma \int_{t_0}^{s'} e^{-d_\gamma\left(s'-t'\right)} dW_\gamma\left(t'\right)\right) ds'\right)^2\right] \\
&= \mathrm{E}\left[\left(\int_s^t \left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma\left(s'-t_0\right)} ds' + \sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\left(s'-t'\right)} dW_\gamma\left(t'\right)\, ds'\right)^2\right] \\
&= \mathrm{E}\left[\left(\int_s^t \left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma\left(s'-t_0\right)} ds'\right)^2\right] \\
&\quad + 2\,\mathrm{E}\left[\left(\int_s^t \left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma\left(s'-t_0\right)} ds'\right)\left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\left(s'-t'\right)} dW_\gamma\left(t'\right)\, ds'\right)\right] \\
&\quad + \mathrm{E}\left[\left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\left(s'-t'\right)} dW_\gamma\left(t'\right)\, ds'\right)^2\right].
\end{aligned} \tag{B.4.4}$$

To utilize Itô isometry, we call upon the identity in Eq. B.1.3 to find that

$$\int_s^t \int_{t_0}^{s'} e^{-d_\gamma\left(s'-t'\right)} dW_\gamma\left(t'\right)\, ds' = \int_{t_0}^{s} \int_s^t e^{-d_\gamma\left(s'-t'\right)} ds'\, dW_\gamma\left(t'\right) + \int_s^t \int_{t'}^t e^{-d_\gamma\left(s'-t'\right)} ds'\, dW_\gamma\left(t'\right), \tag{B.4.5}$$

and so

$$\begin{aligned}
\left\langle (\boldsymbol{J}(s,\ t))^2 \right\rangle = &\frac{\mathrm{E}\left[\left(\gamma_0 - \widehat{\gamma}\right)^2\right]}{{d_\gamma}^2} \left(e^{-d_\gamma\left(t-t_0\right)} - e^{-d_\gamma\left(s-t_0\right)}\right)^2 \\
&+ 2\,\sigma_\gamma\, \mathrm{E}\left[\int_s^t \left(\gamma_0 - \widehat{\gamma}\right) e^{-d_\gamma\left(s'-t_0\right)} ds'\right] \\
&\quad \cdot \mathrm{E}\left[\int_{t_0}^{s} \int_s^t e^{-d_\gamma\left(s'-t'\right)} ds'\, dW_\gamma\left(t'\right) + \int_s^t \int_{t'}^t e^{-d_\gamma\left(s'-t'\right)} ds'\, dW_\gamma\left(t'\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\left(\int_{t_0}^{s}\int_{s}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\,(t') + \int_{s}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\,(t')\right)^2\right]\\
&=\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2\\
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\left(\int_{t_0}^{s}\int_{s}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\,(t')\right)^2\right]\\
&+2\,\sigma_\gamma{}^2\,\mathrm{E}\left[\left(\int_{t_0}^{s}\int_{s}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\,(t')\right)\left(\int_{s}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\,(t')\right)\right]\\
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\left(\int_{s}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\,(t')\right)^2\right]\\
&=\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2\\
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\int_{t_0}^{s}\left(\int_{s}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\right)^2 dt'\right]\\
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\int_{s}^{t}\left(\int_{t'}^{t} e^{-d_\gamma\,(s'-t')}\,ds'\right)^2 dt'\right]\\
&=\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2\\
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\int_{t_0}^{s}\frac{1}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t')}-e^{-d_\gamma\,(s-t')}\right)^2 dt'\right]\\
&+\sigma_\gamma{}^2\,\mathrm{E}\left[\int_{s}^{t}\frac{1}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t')}-1\right)^2 dt'\right]\\
&=\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2\\
&+\frac{\sigma_\gamma{}^2}{d_\gamma{}^2}\,\mathrm{E}\left[\int_{t_0}^{s}\left(e^{-2\,d_\gamma\,(t-t')}-2\,e^{-d_\gamma\,(t+s-2\,t')}+e^{-2\,d_\gamma\,(s-t')}\right) dt'\right]\\
&+\frac{\sigma_\gamma{}^2}{d_\gamma{}^2}\,\mathrm{E}\left[\int_{s}^{t}\left(e^{-2\,d_\gamma\,(t-t')}-2\,e^{-d_\gamma\,(t-t')}+1\right) dt'\right]\\
&=\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma{}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^3}\ \Big(e^{-2\,d_\gamma\,(t-s)}-e^{-2\,d_\gamma\,\left(t-t_0\right)}+2\,e^{-d_\gamma\,\left(t+s-2\,t_0\right)}-2\,e^{-d_\gamma\,(t-s)}\\
&\quad +1-e^{-2\,d_\gamma\,\left(s-t_0\right)}\Big)\\
&+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^3}\ \left(1-e^{-2\,d_\gamma\,(t-s)}+4\,e^{-d_\gamma\,(t-s)}-4+2\,d_\gamma\ (t-s)\right)\\
=\ &\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\ \left(e^{-d_\gamma\,\left(t-t_0\right)}-e^{-d_\gamma\,\left(s-t_0\right)}\right)^2\\
&+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^3}\ \Big(-2+2\,d_\gamma\ (t-s)+2\,e^{-d_\gamma\,\left(t+s-2\,t_0\right)}-e^{-2\,d_\gamma\,\left(t-t_0\right)}\\
&\quad +2\,e^{-d_\gamma\,(t-s)}-e^{-2\,d_\gamma\,\left(s-t_0\right)}\Big)\\
=\ &\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\ \left(e^{-d_\gamma\,\left(t-t_0\right)}-e^{-d_\gamma\,\left(s-t_0\right)}\right)^2\\
&+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^3}\ \Big(-2+2\,d_\gamma\ (t-s)\\
&\qquad +2\,e^{-d_\gamma\,\left(t+s-2\,t_0\right)}\left[1-\frac{1}{2}\,e^{-d_\gamma\,(t-s)}+\ e^{2\,d_\gamma\,\left(s-t_0\right)}-\frac{1}{2}\,e^{d_\gamma\,(t-s)}\right]\Big)\\
=\ &\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\ \left(e^{-d_\gamma\,\left(t-t_0\right)}-e^{-d_\gamma\,\left(s-t_0\right)}\right)^2\\
&+\frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\ \left(-1+d_\gamma\ (t-s)+e^{-d_\gamma\,\left(t+s-2\,t_0\right)}\left[1+\ e^{2\,d_\gamma\,\left(s-t_0\right)}-\cosh\left(d_\gamma\ (t-s)\right)\right]\right).
\end{aligned}
\tag{B.4.6}
$$

We may now calculate $\mathrm{Var}\left(\boldsymbol{J}\,(s,\ t)\right)$

$$
\begin{aligned}
\mathrm{Var}\left(\boldsymbol{J}\,(s,\ t)\right)&=\left\langle\left(\boldsymbol{J}\,(s,\ t)\right)^2\right\rangle-\left\langle\boldsymbol{J}\,(s,\ t)\right\rangle^2\\
&=\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\ \left(e^{-d_\gamma\,\left(t-t_0\right)}-e^{-d_\gamma\,\left(s-t_0\right)}\right)^2\\
&\quad+\frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\ \left(-1+d_\gamma\ (t-s)+e^{-d_\gamma\,\left(t+s-2\,t_0\right)}\left[1+\ e^{2\,d_\gamma\,\left(s-t_0\right)}-\cosh\left(d_\gamma\ (t-s)\right)\right]\right)\\
&\quad-\frac{\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)^2}{{d_\gamma}^2}\ \left(e^{-d_\gamma\,\left(s-t_0\right)}-e^{-d_\gamma\,\left(t-t_0\right)}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]-\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)^2}{{d_\gamma}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2 \\
&\quad+\frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\left(-1+d_\gamma\,(t-s)+e^{-d_\gamma\,(t+s-2\,t_0)}\left[1+\,e^{2\,d_\gamma\,(s-t_0)}-\cosh\left(d_\gamma\,(t-s)\right)\right]\right). \\
&= \frac{\mathrm{Var}\left(\gamma_0\right)}{{d_\gamma}^2}\left(e^{-d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(s-t_0)}\right)^2 \\
&\quad+\frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\left(-1+d_\gamma\,(t-s)+e^{-d_\gamma\,(t+s-2\,t_0)}\left[1+\,e^{2\,d_\gamma\,(s-t_0)}-\cosh\left(d_\gamma\,(t-s)\right)\right]\right).
\end{aligned}
\tag{B.4.7}
$$

B.4.3 Covariance of u_0 and $J\left(t_0,\,t\right)$

Using the equation for $\left\langle J\left(t_0,\,t\right)\right\rangle$ (obtained from Eq. B.4.2 by evaluating it at $s=t_0$), we may quickly find $\mathrm{Cov}\left(u_0,\;J\left(t_0,\,t\right)\right)$.

$$
\begin{aligned}
\mathrm{Cov}\left(u_0,\;J\left(t_0,\,t\right)\right) &= \mathrm{E}\left[u_0\,J\left(t_0,t\right)\right]-\langle u_0\rangle\left\langle J\left(t_0,\,t\right)\right\rangle \\
&= \mathrm{E}\left[u_0\int_{t_0}^{t}\left(\gamma\left(s'\right)-\widehat{\gamma}\right)\,ds'\right]-\frac{\langle u_0\rangle\,\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)}{d_\gamma}\left(1-e^{-d_\gamma\,(t-t_0)}\right) \\
&= \mathrm{E}\left[u_0\int_{t_0}^{t}\left(\gamma_0-\widehat{\gamma}\right)\,e^{-d_\gamma\,(s'-t_0)}\,ds'\right] \\
&\quad+\mathrm{E}\left[u_0\,\sigma_\gamma\int_{t_0}^{t}\int_{t_0}^{s'}e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\,ds'\right] \\
&\quad-\frac{\langle u_0\rangle\,\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)}{d_\gamma}\left(1-e^{-d_\gamma\,(t-t_0)}\right) \\
&= \frac{\mathrm{E}\left[u_0\,\left(\gamma_0-\widehat{\gamma}\right)\right]}{d_\gamma}\left(1-e^{-d_\gamma\,(t-t_0)}\right) \\
&\quad+\sigma_\gamma\,\mathrm{E}\left[u_0\int_{t_0}^{t}\int_{t'}^{t}e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\left(t'\right)\right] \\
&\quad-\frac{\langle u_0\rangle\,\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)}{d_\gamma}\left(1-e^{-d_\gamma\,(t-t_0)}\right) \\
&= \frac{\mathrm{E}\left[u_0\,\left(\gamma_0-\widehat{\gamma}\right)\right]-\langle u_0\rangle\,\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)}{d_\gamma}\left(1-e^{-d_\gamma\,(t-t_0)}\right) \\
&\quad+\sigma_\gamma\,\langle u_0\rangle\,\mathrm{E}\left[\int_{t_0}^{t}\int_{t'}^{t}e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\left(t'\right)\right]
\end{aligned}
$$

$$= \frac{\mathrm{Cov}\left(u_0,\ \gamma_0\right)}{d_\gamma}\ \left(1 - e^{-d_\gamma\,(t-t_0)}\right). \tag{B.4.8}$$

B.4.4 Covariance of $b\,(s)$ and $J\,(s,\ t)$

The calculate $\mathrm{Cov}\,(b\,(s)\,,\ J\,(s,\ t))$, we first note that

$$\mathrm{Cov}\,(b\,(s)\,,\ J\,(s,\ t)) = \mathrm{E}\,[b\,(s)\ J\,(s,\ t)] - \langle b\,(s)\rangle\ \langle J\,(s,\ t)\rangle\,. \tag{B.4.9}$$

As we have the mean of $b\,(s)$ (Eq. B.2.1) and the mean of $J\,(s,\ t)$ (Eq. B.4.2), we need only calculate $\mathrm{E}\,[b\,(s)\ J\,(s,\ t)]$.

$$\begin{aligned}
\mathrm{E}\,[b\,(s)\ J\,(s,\ t)] &= \mathrm{E}\left[\left(\hat{b} + \left(b_0 - \hat{b}\right)\ e^{\lambda_b\,(s-t_0)} + \sigma_b \int_{t_0}^{s} e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\right.\\
&\qquad \left.\cdot\left(\int_s^t \left(\gamma_0 - \hat{\gamma}\right)\ e^{-d_\gamma\,(s'-t_0)}\,ds' + \sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right)\right]\\
&= \mathrm{E}\left[\hat{b}\ \left(\int_s^t \left(\gamma_0 - \hat{\gamma}\right)\ e^{-d_\gamma\,(s'-t_0)}\,ds'\right)\right]\\
&\quad + \mathrm{E}\left[\left(b_0 - \hat{b}\right)\ e^{\lambda_b\,(s-t_0)}\ \left(\int_s^t \left(\gamma_0 - \hat{\gamma}\right)\ e^{-d_\gamma\,(s'-t_0)}\,ds'\right)\right]\\
&\quad + \mathrm{E}\left[\left(\int_{t_0}^{s} e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\ \left(\int_s^t \left(\gamma_0 - \hat{\gamma}\right)\ e^{-d_\gamma\,(s'-t_0)}\,ds'\right)\right]\\
&\quad + \mathrm{E}\left[\hat{b}\ \left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right)\right]\\
&\quad + \mathrm{E}\left[\left(b_0 - \hat{b}\right)\ e^{\lambda_b\,(s-t_0)}\ \left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right)\right]\\
&\quad + \mathrm{E}\left[\left(\int_{t_0}^{s} e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\ \left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right)\right].
\end{aligned} \tag{B.4.10}$$

For the sake of brevity, I will not be showing how the final four terms are each zero. Instead, observe that the third, fourth, and fifth terms each contain a single real-value white noise motion term $dW\,(t)$ and thus have an expected value of zero. The two noises in the sixth term are not correlated, and therefore will have an expected value of zero as well. Therefore,

$$
\begin{aligned}
\mathrm{E}\left[b(s)\ \boldsymbol{J}(s,\ t)\right] &= \mathrm{E}\left[\widehat{b}\left(\int_s^t\left(\gamma_0-\widehat{\gamma}\right)\ e^{-d_\gamma\left(s'-t_0\right)}\,ds'\right)\right] \\
&\quad+\mathrm{E}\left[\left(b_0-\widehat{b}\right)\ e^{\lambda_b\left(s-t_0\right)}\ \left(\int_s^t\left(\gamma_0-\widehat{\gamma}\right)\ e^{-d_\gamma\left(s'-t_0\right)}\,ds'\right)\right] \\
&=\left(\frac{\widehat{b}\,\mathrm{E}\left[\gamma_0-\widehat{\gamma}\right]}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) \\
&\quad+\left(\frac{e^{\lambda_b\left(s-t_0\right)}\,\mathrm{E}\left[\left(b_0-\widehat{b}\right)\ \left(\gamma_0-\widehat{\gamma}\right)\right]}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right).
\end{aligned}
\tag{B.4.11}
$$

The product of the expected value of $b(s)$ and the expected value of $\boldsymbol{J}(s,\ t)$ is given by

$$
\begin{aligned}
\langle b(s)\rangle\ \langle \boldsymbol{J}(s,\ t)\rangle &= \left(\widehat{b}+\left(\langle b_0\rangle-\widehat{b}\right)\ e^{\lambda_b\left(t-t_0\right)}\right)\ \left(\frac{\langle\gamma_0\rangle-\widehat{\gamma}}{d_\gamma}\ \left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right)\right) \\
&=\left(\frac{\widehat{b}\,\mathrm{E}\left[\gamma_0-\widehat{\gamma}\right]}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) \\
&\quad+\left(\frac{e^{\lambda_b\left(s-t_0\right)}\ E\left(b_0-\widehat{b}\right)\ E\left(\gamma_0-\widehat{\gamma}\right)}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right).
\end{aligned}
\tag{B.4.12}
$$

Therefore

$$
\begin{aligned}
\mathrm{Cov}\left(b(s),\ \boldsymbol{J}(s,\ t)\right) &= \mathrm{E}\left[b(s)\ \boldsymbol{J}(s,\ t)\right]-\langle b(s)\rangle\ \langle \boldsymbol{J}(s,\ t)\rangle \\
&=\left(\frac{\widehat{b}\,\mathrm{E}\left[\gamma_0-\widehat{\gamma}\right]}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) \\
&\quad+\left(\frac{e^{\lambda_b\left(s-t_0\right)}\,\mathrm{E}\left[\left(b_0-\widehat{b}\right)\ \left(\gamma_0-\widehat{\gamma}\right)\right]}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) \\
&\quad-\left(\frac{\widehat{b}\,\mathrm{E}\left[\gamma_0-\widehat{\gamma}\right]}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) \\
&\quad-\left(\frac{e^{\lambda_b\left(s-t_0\right)}\,\mathrm{E}\left[b_0-\widehat{b}\right]\ E\left(\gamma_0-\widehat{\gamma}\right)}{d_\gamma}\right)\left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right)
\end{aligned}
$$

$$= \frac{\mathrm{Cov}\left(b_0,\ \gamma_0\right)}{d_\gamma}\, e^{\lambda_b\,(s-t_0)}\,\left(e^{-d_\gamma\,(s-t_0)} - e^{-d_\gamma\,(t-t_0)}\right). \tag{B.4.13}$$

B.5 Variance of $b\,(t)$

Using our equation for the expected value of $b\,(t)$ (Eq. B.2.1), we obtain

$$\begin{aligned}
\mathrm{Var}\,(b\,(t)) = \mathrm{Cov}\,(b\,(t)\,,\ b\,(t)) &= \mathrm{E}\left[\left|b\,(t) - \langle b\,(t)\rangle\right|^2\right] \\
&= \mathrm{E}\left[\left|\left(b_0 - \langle b_0\rangle\right)\, e^{\lambda_b\,(t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dW_b\,(s)\right|^2\right] \\
&= \mathrm{E}\left[\left|b_0 - \langle b_0\rangle\right|^2 e^{-2\,\gamma_b\,(t-t_0)}\right] \\
&\quad + \mathrm{E}\left[\sigma_b\,\left(b_0 - \langle b_0\rangle \int_{t_0}^{t} e^{\lambda_b\,(t-t_0)+\overline{\lambda_b}(t-s)}\,\overline{dW_b\,(s)}\right)\right] \\
&\quad + \mathrm{E}\left[\sigma_b\,\left(\overline{b_0} - \overline{\langle b_0\rangle} \int_{t_0}^{t} e^{\overline{\lambda_b}\,(t-t_0)+\lambda_b(t-s)}\, dW_b\,(s)\right)\right] \\
&\quad + \mathrm{E}\left[\sigma_b^2\,\left(\int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dW_b\,(s)\right)\left(\int_{t_0}^{t} e^{\overline{\lambda_b}\,(t-s)}\,\overline{dW_b\,(s)}\right)\right] \\
&= \mathrm{Cov}\left(b_0,\ b_0\right)\, e^{-2\,\gamma_b\,(t-t_0)} \\
&\quad + \sigma_b\,\mathrm{E}\left[b_0 - \langle b_0\rangle\right]\,\mathrm{E}\left[\int_{t_0}^{t} e^{\lambda_b\,(t-t_0)+\overline{\lambda_b}(t-s)}\,\overline{dW_b\,(s)}\right] \\
&\quad + \sigma_b\,\mathrm{E}\left[\overline{b_0} - \overline{\langle b_0\rangle}\right]\,\mathrm{E}\left[\int_{t_0}^{t} e^{\overline{\lambda_b}\,(t-t_0)+\lambda_b\,(t-s)}\, dW_b\,(s)\right] \\
&\quad + \sigma_b^2\,\mathrm{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dW_b\,(s)\right)\left(\int_{t_0}^{t} e^{\overline{\lambda_b}\,(t-s)}\,\overline{dW_b\,(s)}\right)\right] \\
&= \mathrm{Cov}\left(b_0,\ b_0\right)\, e^{-2\,\gamma_b\,(t-t_0)} \\
&\quad + \sigma_b^2\,\mathrm{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dW_b\,(s)\right)\left(\int_{t_0}^{t} e^{\overline{\lambda_b}\,(t-s)}\,\overline{dW_b\,(s)}\right)\right],
\end{aligned} \tag{B.5.1}$$

where we have utilized the fact that integrating a complex-value function against a complex-value white noise may be decomposed into a real and imaginary part (see Eq. B.1.4). Using the identity Eq. B.1.5, we may evaluate the final term and complete our calculation.

$$\begin{aligned}
\mathrm{Var}\,(b\,(t)) &= \mathrm{Var}\left(b_0\right)\, e^{-2\,\gamma_b\,(t-t_0)} \\
&\quad + \sigma_b^2\,\mathrm{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dW_b\,(s)\right)\left(\int_{t_0}^{t} e^{\overline{\lambda_b}\,(t-s)}\,\overline{dW_b\,(s)}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Var}\left(b_0\right)\, e^{-2\gamma_b (t-t_0)} \\
&\quad + \sigma_b^2\, \mathrm{E}\left[\int_{t_0}^{t} e^{-2\gamma_b (t-s)} \left(\cos^2\left(\omega_b\, (t-s)\right) + \sin^2\left(\omega_b\, (t-s)\right)\right)\, ds\right] \\
&= \mathrm{Var}\left(b_0\right)\, e^{-2\gamma_b (t-t_0)} + \frac{\sigma_b^2}{2\gamma_b}\left(1 - e^{-2\gamma_b (t-t_0)}\right).
\end{aligned}
\tag{B.5.2}
$$

B.6 Covariance of $b(t)$ and $\overline{b(t)}$

By decomposing the noise $dW_b(t) = \frac{1}{\sqrt{2}}\left(dU_b(t) + dV_b(t)\right)$ into its real and complex parts, we may find $\mathrm{Cov}\left(b(t),\, \overline{b(t)}\right)$ in a manner similar to how we calculated $\mathrm{Var}(b(t))$ including our use of the equation for the expected value of $b(t)$ (Eq. B.2.1). For more detail, see Appendix B.5.

$$
\begin{aligned}
\mathrm{Cov}\left(b(t),\, \overline{b(t)}\right) &= \mathrm{E}\left[\left(b(t) - \langle b(t)\rangle\right)^2\right] \\
&= \mathrm{E}\left[\left(\left(b_0 - \langle b_0\rangle\right)\, e^{\lambda_b (t-t_0)} + \sigma_b \int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right)^2\right] \\
&= \mathrm{E}\left[\left(b_0 - \langle b_0\rangle\right)^2\, e^{2\lambda_b (t-t_0)}\right] \\
&\quad + \mathrm{E}\left[2\sigma_b\, \left(b_0 - \langle b_0\rangle\right) \int_{t_0}^{t} e^{\lambda_b (2t-s-t_0)}\, dW_b(s)\right] \\
&\quad + \mathrm{E}\left[\sigma_b^2\, \left(\int_{t_0}^{t} e^{\lambda_b (t-s)}\, dW_b(s)\right)^2\right] \\
&= \mathrm{Cov}\left(b_0,\, \overline{b_0}\right)\, e^{2\lambda_b (t-t_0)} + \sigma_b^2\, \mathrm{E}\left[\int_{t_0}^{t} e^{2\lambda_b (t-s)}\, ds\right] \\
&= \mathrm{Cov}\left(b_0,\, \overline{b_0}\right)\, e^{2\lambda_b (t-t_0)} \\
&\quad + 2\sigma_b\, \mathrm{E}\left[b_0 - \langle b_0\rangle\right]\, \mathrm{E}\left[\int_{t_0}^{t} e^{\lambda_b (2t-s-t_0)}\, dW_b(s)\right] \\
&\quad + \frac{\sigma_b^2}{2}\, \mathrm{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b (t-s)}\, dU_b(s)\right)^2\right] \\
&\quad + i\,\sigma_b^2\, \mathrm{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b (t-s)}\, dU_b(s)\right)\left(\int_{t_0}^{t} e^{\lambda_b (t-s)}\, dV_b(s)\right)\right] \\
&\quad - \frac{\sigma_b^2}{2}\, \mathrm{E}\left[\left(\int_{t_0}^{t} e^{\lambda_b (t-s)}\, dV_b(s)\right)^2\right] \\
&= \mathrm{Cov}\left(b_0,\, \overline{b_0}\right)\, e^{2\lambda_b (t-t_0)}
\end{aligned}
$$

$$
\begin{aligned}
&+ i\,\sigma_b^2\, \mathrm{E}\left[\int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dU_b\,(s)\right]\, \mathrm{E}\left[\int_{t_0}^{t} e^{\lambda_b\,(t-s)}\, dV_b\,(s)\right] \\
&= \mathrm{Cov}\left(b_0,\ \overline{b_0}\right)\, e^{2\,\lambda_b\,(t-t_0)}. \qquad \text{(B.6.1)}
\end{aligned}
$$

B.7 Variance of $\gamma\,(t)$

The variance of $\gamma\,(t)$ is more straightforward to calculate than the variance of $b\,(t)$ (Appendix B.5), as it only involves a real-value white noise instead of a complex-value one. For our calculation here, we utilize the equation for the expected value of $\gamma\,(t)$ (Eq. B.3.1).

$$
\begin{aligned}
\mathrm{Var}\,(\gamma\,(t)) &= \mathrm{E}\left[(\gamma\,(t) - \langle\gamma\,(t)\rangle)^2\right] \\
&= \mathrm{E}\left[\left(\left(\gamma_0 - \langle\gamma_0\rangle\right)\, e^{-d_\gamma\,(t-t_0)} + \sigma_\gamma \int_{t_0}^{t} e^{-d_\gamma\,(t-s)}\, dW_\gamma\,(s)\right)^2\right] \\
&= \mathrm{E}\left[\left(\gamma_0 - \langle\gamma_0\rangle\right)^2\, e^{-2\,d_\gamma\,(t-t_0)}\right] \\
&\quad + \mathrm{E}\left[2\,\sigma_\gamma\, \left(\gamma_0 - \langle\gamma_0\rangle\right) \int_{t_0}^{t} e^{-d_\gamma\,(2\,t-s-t_0)}\, dW_\gamma\,(s)\right] \\
&\quad + \mathrm{E}\left[\sigma_\gamma^2\, \left(\int_{t_0}^{t} e^{-d_\gamma\,(t-s)}\, dW_\gamma\,(s)\right)^2\right] \\
&= \mathrm{Var}\left(\gamma_0\right)\, e^{-2\,d_\gamma\,(t-t_0)} + \sigma_\gamma^2\, \mathrm{E}\left[\int_{t_0}^{t} e^{-2\,d_\gamma\,(t-s)}\, ds\right] \\
&= \mathrm{Var}\left(\gamma_0\right)\, e^{-2\,d_\gamma\,(t-t_0)} + \frac{\sigma_\gamma^2}{2\,d_\gamma}\left(1 - e^{-2\,d_\gamma\,(t-t_0)}\right). \qquad \text{(B.7.1)}
\end{aligned}
$$

B.8 Covariance of $b\,(t)$ and $\gamma\,(t)$

To find the covariance of $b\,(t)$ and $\gamma\,(t)$, we utilize their pathwise solutions (Eqs. A.2.5 and A.3.1, respectively) as well as the equations for their expected values (Eqs. B.2.1 and B.3.1, respectively), and the fact that the white noises $dW_b\,(t)$ and $dW_\gamma\,(t)$ are uncorrelated. We begin by decomposing the covariance into a difference of expected values, and continue from there

$$
\begin{aligned}
\mathrm{Cov}\,(b\,(t)\,,\ \gamma\,(t)) &= \mathrm{E}\,[b\,(t)\ \gamma\,(t)] - \langle b\,(t)\rangle\ \langle\gamma\,(t)\rangle \\
&= \widehat{b}\ \langle\gamma\,(t)\rangle + \widehat{\gamma}\ \langle b\,(t)\rangle + \mathrm{E}\left[\left(b_0 - \widehat{b}\right)\, \left(\gamma_0 - \widehat{\gamma}\right)\right]\, e^{\left(\lambda_b - d_\gamma\right)\,(t-t_0)} \\
&\quad - \left(\widehat{b}\ \langle\gamma\,(t)\rangle + \widehat{\gamma}\ \langle b\,(t)\rangle + \mathrm{E}\left[b_0 - \widehat{b}\right]\, \mathrm{E}\left[\gamma_0 - \widehat{\gamma}\right]\, e^{\left(\lambda_b - d_\gamma\right)\,(t-t_0)}\right)
\end{aligned}
$$

$$= \mathrm{Cov}\left(b_0,\ \gamma_0\right)\ e^{\left(\lambda_b - d_\gamma\right)\left(t-t_0\right)}. \tag{B.8.1}$$

B.9 Variance of $u\,(t)$

To obtain the variance of $u\,(t)$, we will first split the pathwise solution of $u\,(t)$ (Eq. A.4.4) into three terms

$$A = e^{-J\left(t_0,\ t\right)+\widehat{\lambda}\left(t-t_0\right)}\,u_0 \tag{B.9.1}$$

$$B = \int_{t_0}^{t} \left(b\,(s) + f\,(s)\right)\ e^{-J(s,\ t)+\widehat{\lambda}(t-s)}\,ds \tag{B.9.2}$$

$$C = \sigma \int_{t_0}^{t} e^{-J(s,\ t)+\widehat{\lambda}(t-s)}\,dW\,(s)\,. \tag{B.9.3}$$

The variance of $u\,(t)$ is thus given by

$$\begin{aligned}
\mathrm{Var}\,(u\,(t)) &= \mathrm{Cov}\,(u\,(t)\,,\ u\,(t)) = \mathrm{E}\left[\left|u\,(t) - \langle u\,(t)\rangle\right|^2\right] \\
&= \mathrm{E}\left[|u\,(t)|^2\right] - |\mathrm{E}\,[u\,(t)]|^2 \\
&= \mathrm{E}\left[|A|^2\right] + \mathrm{E}\left[|B|^2\right] + \mathrm{E}\left[|C|^2\right] + \mathrm{E}\left[A\,\overline{B}\right] + \mathrm{E}\left[A\,\overline{C}\right] \\
&\quad + \mathrm{E}\left[B\,\overline{A}\right] + \mathrm{E}\left[B\,\overline{C}\right] + \mathrm{E}\left[C\,\overline{A}\right] + \mathrm{E}\left[C\,\overline{B}\right] - |\langle u\,(t)\rangle|^2 \\
&= \mathrm{E}\left[|A|^2\right] + \mathrm{E}\left[|B|^2\right] + \mathrm{E}\left[|C|^2\right] \\
&\quad + 2\left(\mathrm{Re}\left(\mathrm{E}\left[A\,\overline{B}\right]\right) + \mathrm{Re}\left(\mathrm{E}\left[A\,\overline{C}\right]\right) + \mathrm{Re}\left(\mathrm{E}\left[B\,\overline{C}\right]\right)\right) \\
&\quad - |\langle u\,(t)\rangle|^2.
\end{aligned} \tag{B.9.4}$$

As we will soon discover, several of these terms evaluate to zero (specifically $\mathrm{E}\left[A\,\overline{C}\right]$ and $\mathrm{E}\left[B\,\overline{C}\right]$, see Appendix B.9.5), and attempting further simplifications of the above equation utilizing our previously derived equation for the expected value of $u\,(t)$ (Eq. B.4.1) will be fruitless, an so we shall write the reported form of $\mathrm{Var}\,(u\,(t))$ and list below the equation numbers for each term.

$$\mathrm{Var}\,(u\,(t)) = \mathrm{E}\left[|A|^2\right] + \mathrm{E}\left[|B|^2\right] + \mathrm{E}\left[|C|^2\right] + 2\,\mathrm{Re}\left(\mathrm{E}\left[A\,\overline{B}\right]\right) - |\langle u\,(t)\rangle|^2. \tag{B.9.5}$$

The equation numbers for the terms of this equation are

$\mathrm{E}\left[|A|^2\right]$ – Eq. B.9.7.

$\mathrm{E}\left[|B|^2\right]$ – Eq. B.9.14.

$\mathrm{E}\left[|C|^2\right]$ – Eq. B.9.26.

$\mathrm{E}\left[A\,\overline{B}\right]$ – Eq. B.9.27.

$\langle u(t)\rangle$ – Eq. B.4.1.

B.9.1 Mean of $|A|^2$

The calculation of the expected value of $|A|^2$ is fairly straightforward.

$$\begin{aligned}\mathrm{E}\left[|A|^2\right] &= \mathrm{E}\left[\left(u_0\, e^{-J(t_0,\, t)+\hat{\lambda}(t-t_0)}\right)\left(\overline{u_0}\, e^{-J(t_0,\, t)+\overline{\hat{\lambda}}(t-t_0)}\right)\right]\\ &= \mathrm{E}\left[u_0\,\overline{u_0}\, e^{-2\,J(t_0,\, t)-2\,\hat{\gamma}(t-t_0)}\right].\end{aligned} \tag{B.9.6}$$

Using the identity Eq. B.1.6, we obtain

$$\begin{aligned}\mathrm{E}\left[|A|^2\right] &= \Big(\mathrm{Cov}\left(u_0,\, u_0\right) + \left(\langle u_0\rangle - 2\,\mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)\right)\left(\overline{\langle u_0\rangle} - 2\,\mathrm{Cov}\left(\overline{u_0},\, J\left(t_0,\, t\right)\right)\right)\Big)\\ &\quad\cdot e^{-2\left\langle J(t_0,\, t)\right\rangle + 2\,\mathrm{Var}(J(t_0,\, t))}\\ &= \left(\left|\langle u_0\rangle\right|^2 + \mathrm{Var}\left(u_0\right) - 4\,\mathrm{Re}\left(\overline{\langle u_0\rangle}\,\mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)\right) + 4\left|\mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)\right|^2\right)\\ &\quad\cdot e^{-2\left\langle J(t_0,\, t)\right\rangle + 2\,\mathrm{Var}(J(t_0,\, t)) - 2\,\hat{\gamma}(t-t_0)}.\end{aligned} \tag{B.9.7}$$

Attempting further simplifcation using our previously derived equations is fruitless, and we instead list the equation numbers here.

$\mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)$ – Eq. B.4.8.

$\left\langle J\left(t_0,\, t\right)\right\rangle$ – Eq. B.4.2 (substituting $s = t_0$).

$\mathrm{Var}\left(J\left(t_0,\, t\right)\right)$ – Eq. B.4.7 (substituting $s = t_0$).

B.9.2 Mean of $|B|^2$

The calculation of the expected value of $|B|^2$ is one of the longest and most involved derivations we will encounter. We begin with an initial calculation which we will divide into four terms, each of which we will compute individually.

$$
\begin{aligned}
\mathrm{E}\left[|B|^2\right] &= \mathrm{E}\left[\left(\int_{t_0}^{t} (b(s)+f(s))\, e^{-J(s,\,t)+\widehat{\lambda}(t-s)}\, ds\right)\left(\int_{t_0}^{t}\left(\overline{b(r)}+f(r)\right) e^{-J(r,\,t)+\overline{\widehat{\lambda}}(t-r)}\, dr\right)\right] \\
&= \int_{t_0}^{t}\int_{t_0}^{t}\left(\mathrm{E}\left[b(s)\,\overline{b(r)}\, e^{-(J(s,\,t)+J(r,\,t))}\right] + \overline{f(r)}\,\mathrm{E}\left[b(s)\, e^{-(J(s,\,t)+J(r,\,t))}\right]\right. \\
&\qquad\qquad \left. + f(s)\,\mathrm{E}\left[\overline{b(r)}\, e^{-(J(s,\,t)+J(r,\,t))}\right] + f(s)\,\overline{f(r)}\,\mathrm{E}\left[e^{-(J(s,\,t)+J(r,\,t))}\right]\right) \\
&\qquad\qquad \cdot e^{-\widehat{\gamma}(2t-s-r)+i\omega(s-r)}\, ds\, dr \\
&= \int_{t_0}^{t}\int_{t_0}^{t}\left(\alpha+\overline{f(r)}\,\beta+f(s)\,\delta+f(s)\,\overline{f(r)}\,\eta\right) e^{-\widehat{\gamma}(2t-s-r)+i\omega(s-r)}\, ds\, dr.
\end{aligned}
\tag{B.9.8}
$$

We now calculate each of α, β, δ, and η of the expression on the right, utilizing the identities Eq. B.1.6 and Eq. B.1.7 throughout.

$$
\begin{aligned}
\alpha &= \mathrm{E}\left[b(s)\,\overline{b(r)}\, e^{-(J(s,\,t)+J(r,\,t))}\right] \\
&= \Big(\mathrm{Cov}(b(s),\, b(r)) + (\langle b(s)\rangle - \mathrm{Cov}(b(s),\, J(r,\,t)) - \mathrm{Cov}(b(s),\, J(s,\,t))) \\
&\qquad \cdot \left(\overline{\langle b(r)\rangle} - \overline{\mathrm{Cov}(b(r),\, J(r,\,t))} - \overline{\mathrm{Cov}(b(r),\, J(s,\,t))}\right)\Big) \\
&\quad \cdot e^{-\langle J(r,\,t)\rangle-\langle J(s,\,t)\rangle+\frac{1}{2}\mathrm{Var}(J(r,\,t))+\frac{1}{2}\mathrm{Var}(J(s,\,t))+\mathrm{Cov}(J(r,\,t),\,J(s,\,t))},
\end{aligned}
\tag{B.9.9}
$$

$$
\begin{aligned}
\beta &= \mathrm{E}\left[b(s)\, e^{-(J(s,\,t)+J(r,\,t))}\right] \\
&= (\langle b(s)\rangle - \mathrm{Cov}(b(s),\, J(s,\,t)) - \mathrm{Cov}(b(s),\, J(r,\,t))) \\
&\quad \cdot e^{-\langle J(s,\,t)\rangle-\langle J(r,\,t)\rangle+\frac{1}{2}\mathrm{Var}(J(s,\,t))+\frac{1}{2}\mathrm{Var}(J(r,\,t))+\mathrm{Cov}(J(s,\,t),\,J(r,\,t))},
\end{aligned}
\tag{B.9.10}
$$

$$
\begin{aligned}
\delta &= \mathrm{E}\left[\overline{b(r)}\, e^{-(J(s,\,t)+J(r,\,t))}\right] \\
&= \left(\left\langle\overline{b(r)}\right\rangle - \overline{\mathrm{Cov}(b(r),\, J(s,\,t))} - \overline{\mathrm{Cov}(b(r),\, J(r,\,t))}\right) \\
&\quad \cdot e^{-\langle J(s,\,t)\rangle-\langle J(r,\,t)\rangle+\frac{1}{2}\mathrm{Var}(J(s,\,t))+\frac{1}{2}\mathrm{Var}(J(r,\,t))+\mathrm{Cov}(J(s,\,t),\,J(r,\,t))},
\end{aligned}
\tag{B.9.11}
$$

$$
\eta = \mathrm{E}\left[e^{-(J(s,\,t)+J(r,\,t))}\right]
$$

$$= e^{-\langle J(s,\ t)\rangle-\langle J(r,\ t)\rangle+\frac{1}{2}\operatorname{Var}(J(s,\ t))+\frac{1}{2}\operatorname{Var}(J(r,\ t))+\operatorname{Cov}(J(s,\ t),\ J(r,\ t))}. \tag{B.9.12}$$

We denote the integrand of Eq. B.9.8 by $b_{var}(r,\ s)$, and it is given by

$$\begin{aligned}
b_{var}(r,\ s) = {} & e^{-\langle J(s,\ t)\rangle-\langle J(r,\ t)\rangle+\frac{1}{2}\operatorname{Var}(J(s,\ t))+\frac{1}{2}\operatorname{Var}(J(r,\ t))+\operatorname{Cov}(J(s,\ t),\ J(r,\ t))-\hat{\gamma}(2t-s-r)+i\omega(s-r)} \\
& \cdot (\operatorname{Cov}(b(s),\ b(r)) \\
& \quad + (\langle b(s)\rangle - \operatorname{Cov}(b(s),\ J(r,\ t)) - \operatorname{Cov}(b(s),\ J(s,\ t))) \\
& \qquad \cdot \left(\overline{\langle b(r)\rangle} - \overline{\operatorname{Cov}(b(r),\ J(r,\ t))} - \overline{\operatorname{Cov}(b(r),\ J(s,\ t))}\right) \\
& \quad + \overline{f(r)}\,(\langle b(s)\rangle - \operatorname{Cov}(b(s),\ J(s,\ t)) - \operatorname{Cov}(b(s),\ J(r,\ t))) \\
& \quad + f(s)\left(\overline{\langle b(r)\rangle} - \overline{\operatorname{Cov}(b(r),\ J(s,\ t))} - \overline{\operatorname{Cov}(b(r),\ J(r,\ t))}\right) \\
& \quad + f(s)\,\overline{f(r)}\Big) \\
= {} & e^{-\langle J(s,\ t)\rangle-\langle J(r,\ t)\rangle+\frac{1}{2}\operatorname{Var}(J(s,\ t))+\frac{1}{2}\operatorname{Var}(J(r,\ t))+\operatorname{Cov}(J(s,\ t),\ J(r,\ t))-\hat{\gamma}(2t-s-r)+i\omega(s-r)} \\
& \cdot (\operatorname{Cov}(b(s),\ b(r)) \\
& \quad + (\langle b(s)\rangle + f(s) - \operatorname{Cov}(b(s),\ J(r,\ t)) - \operatorname{Cov}(b(s),\ J(s,\ t))) \\
& \qquad \cdot \left(\overline{\langle b(r)\rangle} + \overline{f(r)} - \overline{\operatorname{Cov}(b(r),\ J(r,\ t))} - \overline{\operatorname{Cov}(b(r),\ J(s,\ t))}\right)\Big).
\end{aligned} \tag{B.9.13}$$

Hence, the mean value of $|B|^2$ is given by

$$\operatorname{E}\left[|B|^2\right] = \int_{t_0}^{t}\int_{t_0}^{t} b_{var}(s,\ r)\, ds\, dr \tag{B.9.14}$$

where $b_{var}(s,\ r)$ is defined in Eq. B.9.13. We note that to evaluate $b_{var}(s,\ r)$, the following statistics are required:

$\operatorname{Cov}(J(s,\ t),\ J(r,\ t))$ – Eqs. B.9.19 and B.9.20.

$\operatorname{Cov}(b(s),\ b(r))$ – Eq. B.9.24.

$\langle b(s)\rangle$, $\langle b(r)\rangle$ – Eq. B.2.1 (substituting $t = s$ and $t = r$, respectively).

$\operatorname{Cov}(b(s),\ J(r,\ t))$ – Eq. B.9.25 (making appropriate argument substitutions).

B.9.2.1 Covariance of $J(s,\ t)$ and $J(r,\ t)$

There are two cases for the calculation of the covariance of $J(s,\ t)$ and $J(r,\ t)$: i) $t_0 \le r \le s \le t$, and ii) $t_0 \le s \le r \le t$. The derivations are similar, and boil down to swapping the places of s and r in the final equation. Hence, we will include the entire derivation for the first case and simply include the final result for the second. We begin with the usual decomposition of covariance into a difference of expected values.

$$\mathrm{Cov}\left(J(s,\ t),\ J(r,\ t)\right) = \mathrm{E}\left[J(s,\ t)\ J(r,\ t)\right] - \langle J(s,\ t)\rangle\ \langle J(r,\ t)\rangle\,. \tag{B.9.15}$$

Since the expected values in the second term on the right are given by Eq. B.4.2 (with appropriate arguments), we calculate $\mathrm{E}\left[J(s,\ t)\ J(r,\ t)\right]$ first and later subtract the product of the expected values.

$$\begin{aligned}
\mathrm{E}\left[J(s,\ t)\ J(r,\ t)\right] = \mathrm{E}\Bigg[&\left(\int_s^t \left(\gamma_0 - \widehat{\gamma}\right)\ e^{-d_\gamma\,(s'-t_0)}\,ds' + \sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right) \\
&\cdot\left(\int_r^t \left(\gamma_0 - \widehat{\gamma}\right)\ e^{-d_\gamma\,(r'-t_0)}\,dr' + \sigma_\gamma \int_r^t \int_{t_0}^{r'} e^{-d_\gamma\,(r'-t')}\,dW_\gamma\left(t'\right)\ dr'\right)\Bigg] \\
= \mathrm{E}&\left[\left(\gamma_0 - \widehat{\gamma}\right)^2 \int_s^t \int_r^t e^{-d_\gamma\,(s'+r'-2\,t_0)}\,dr'\,ds'\right] \\
+\,\mathrm{E}&\left[\left(\left(\gamma_0 - \widehat{\gamma}\right) \int_s^t e^{-d_\gamma\,(s'-t_0)}\,ds'\right)\left(\sigma_\gamma \int_r^t \int_{t_0}^{r'} e^{-d_\gamma\,(r'-t')}\,dW_\gamma\left(t'\right)\ dr'\right)\right] \\
+\,\mathrm{E}&\left[\left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right)\left(\left(\gamma_0 - \widehat{\gamma}\right) \int_r^t e^{-d_\gamma\,(r'-t_0)}\,dr'\right)\right] \\
+\,\mathrm{E}\Bigg[&\left(\sigma_\gamma \int_s^t \int_{t_0}^{s'} e^{-d_\gamma\,(s'-t')}\,dW_\gamma\left(t'\right)\ ds'\right) \\
&\cdot\left(\sigma_\gamma \int_r^t \int_{t_0}^{r'} e^{-d_\gamma\,(r'-t')}\,dW_\gamma\left(t'\right)\ dr'\right)\Bigg] \\
= \frac{\mathrm{E}\left[\left(\gamma_0 - \widehat{\gamma}\right)^2\right]}{d_\gamma}& \int_s^t \left(e^{-d_\gamma\,(s'+r-2\,t_0)} - e^{-d_\gamma\,(s'+t-2\,t_0)}\right)\,ds' + 0 + 0 \\
+\,\sigma_\gamma{}^2\,\mathrm{E}\Bigg[&\left(\int_{t_0}^s \int_s^t e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\left(t'\right) + \int_s^t \int_{t'}^t e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\left(t'\right)\right) \\
&\cdot\left(\int_{t_0}^r \int_r^t e^{-d_\gamma\,(r'-t')}\,dr'\,dW_\gamma\left(t'\right) + \int_r^t \int_{t'}^t e^{-d_\gamma\,(r'-t')}\,dr'\,dW_\gamma\left(t'\right)\right)\Bigg] \\
= \frac{\mathrm{E}\left[\left(\gamma_0 - \widehat{\gamma}\right)^2\right]}{d_\gamma{}^2}& \left(e^{-2\,d_\gamma\,(t-t_0)} + e^{-d_\gamma\,(s+r-2\,t_0)} - e^{-d_\gamma\,(t+r-2\,t_0)} - e^{-2\,d_\gamma\,(s+t-2\,t_0)}\right) \\
+\,\sigma_\gamma{}^2\,\mathrm{E}&\left[\left(\int_{t_0}^s \int_s^t e^{-d_\gamma\,(s'-t')}\,ds'\,dW_\gamma\left(t'\right)\right)\left(\int_{t_0}^r \int_r^t e^{-d_\gamma\,(r'-t')}\,dr'\,dW_\gamma\left(t'\right)\right)\right]
\end{aligned} \tag{B.9.16}$$

$$
\begin{aligned}
&+ {\sigma_\gamma}^2 \,\mathrm{E}\left[\left(\int_{t_0}^{s}\int_{s}^{t} e^{-d_\gamma\,(s'-t')}\, ds'\, dW_\gamma\,(t')\right)\left(\int_{r}^{t}\int_{t'}^{t} e^{-d_\gamma\,(r'-t')}\, dr'\, dW_\gamma\,(t')\right)\right] \\
&+ {\sigma_\gamma}^2 \,\mathrm{E}\left[\left(\int_{s}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'-t')}\, ds'\, dW_\gamma\,(t')\right)\left(\int_{t_0}^{r}\int_{r}^{t} e^{-d_\gamma\,(r'-t')}\, dr'\, dW_\gamma\,(t')\right)\right] \\
&+ {\sigma_\gamma}^2 \,\mathrm{E}\left[\left(\int_{s}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'-t')}\, ds'\, dW_\gamma\,(t')\right)\left(\int_{r}^{t}\int_{t'}^{t} e^{-d_\gamma\,(r'-t')}\, dr'\, dW_\gamma\,(t')\right)\right] \\
= &\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\left(e^{-2\,d_\gamma\,(t-t_0)} + e^{-d_\gamma\,(s+r-2\,t_0)} - e^{-d_\gamma\,(t+r-2\,t_0)} - e^{-2\,d_\gamma\,(s+t-2\,t_0)}\right) \\
&+ {\sigma_\gamma}^2 \,\mathrm{E}\left[\int_{t_0}^{r}\int_{s}^{t}\int_{r}^{t} e^{-d_\gamma\,(s'+r'-2\,t')}\, dr'\, ds'\, dt'\right] \\
&+ {\sigma_\gamma}^2 \,\mathrm{E}\left[\int_{r}^{s}\int_{s}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'+r'-2\,t')}\, dr'\, ds'\, dt'\right] + 0 \\
&+ {\sigma_\gamma}^2 \,\mathrm{E}\left[\int_{s}^{t}\int_{t'}^{t}\int_{t'}^{t} e^{-d_\gamma\,(s'+r'-2\,t')}\, dr'\, ds'\, dt'\right] \\
= &\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\left(e^{-2\,d_\gamma\,(t-t_0)} + e^{-d_\gamma\,(s+r-2\,t_0)} - e^{-d_\gamma\,(t+r-2\,t_0)} - e^{-2\,d_\gamma\,(s+t-2\,t_0)}\right) \\
&+ \frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\left(\frac{1}{2}\left(e^{-d_\gamma\,(s-r)} - e^{-d_\gamma\,(s+r-2\,t_0)}\right) - \frac{1}{2}\left(e^{-d_\gamma\,(t-r)} - e^{-d_\gamma\,(t+r-2\,t_0)}\right)\right. \\
&\quad\left. +\frac{1}{2}\left(e^{-2\,d_\gamma\,(t-r)} - e^{-2\,d_\gamma\,(t-t_0)}\right) - \frac{1}{2}\left(e^{-d_\gamma\,(s+t-2\,r)} - e^{-d_\gamma\,(s+t-2\,t_0)}\right)\right) \\
&+ \frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\left(\left(1-e^{-d_\gamma}\right) - \left(e^{-d_\gamma\,(t-s)} - e^{-d_\gamma\,(t-r)}\right)\right. \\
&\quad\left. +\frac{1}{2}\left(e^{-2\,d_\gamma\,(t-s)} - e^{-2\,d_\gamma\,(t-r)}\right) - \frac{1}{2}\left(e^{-d_\gamma\,(t-s)} - e^{-d_\gamma\,(s+t-2\,r)}\right)\right) \\
&+ \frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\left(d_\gamma\,(t-s) - \left(1-e^{-d_\gamma\,(t-s)}\right)\right. \\
&\quad\left. +\frac{1}{2}\left(1-e^{-2\,d_\gamma\,(t-s)}\right) - \left(1-e^{-d_\gamma\,(t-s)}\right)\right) \\
= &\frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{{d_\gamma}^2}\left(e^{-2\,d_\gamma\,(t-t_0)} + e^{-d_\gamma\,(s+r-2\,t_0)} - e^{-d_\gamma\,(t+r-2\,t_0)} - e^{-2\,d_\gamma\,(s+t-2\,t_0)}\right) \\
&+ \frac{{\sigma_\gamma}^2}{{d_\gamma}^3}\left(\frac{1}{2}\,e^{-d_\gamma\,(s-r)} - \frac{1}{2}\,e^{-d_\gamma\,(s+r-2\,t_0)} + \frac{1}{2}\,e^{-d_\gamma\,(t-r)} + \frac{1}{2}\,e^{-d_\gamma\,(t+r-2\,t_0)}\right. \\
&\quad\left. -\frac{1}{2}\,e^{-2\,d_\gamma\,(t-t_0)} + \frac{1}{2}\,e^{-d_\gamma\,(s+t-2\,t_0)} - \frac{1}{2} + \frac{1}{2}\,e^{-d_\gamma\,(t-s)} + d_\gamma\,(t-s)\right)
\end{aligned}
$$

$$= \frac{\mathrm{E}\left[\left(\gamma_0 - \widehat{\gamma}\right)^2\right]}{{d_\gamma}^2} \left(e^{-2\, d_\gamma\, (t-t_0)} + e^{-d_\gamma\, (s+r-2\, t_0)} - e^{-d_\gamma\, (t+r-2\, t_0)} - e^{-2\, d_\gamma\, (s+t-2\, t_0)}\right)$$
$$+ \frac{{\sigma_\gamma}^2}{2\, {d_\gamma}^3} \left(-\left(1 + e^{-d_\gamma\, (s-r)}\right) + 2\, d_\gamma\, (t-s)\right.$$
$$+ e^{-d_\gamma\, (s+t-2\, t_0)}$$
$$\left.\cdot\left(\left(1 + e^{d_\gamma\, (s-r)}\right) + \left(e^{2\, d_\gamma\, (s-t_0)} + e^{d_\gamma\, (s+r-2\, t_0)}\right) - \left(e^{d_\gamma\, (t-r)} + e^{-d_\gamma\, (t-s)}\right)\right)\right). \tag{B.9.17}$$

Now, the product of $\langle J\,(s,\ t)\rangle$ and $\langle J\,(r,\ t)\rangle$ is given by

$$\langle J\,(s,\ t)\rangle\ \langle J\,(r,\ t)\rangle = \frac{\left(\langle\gamma_0\rangle - \widehat{\gamma}\right)^2}{{d_\gamma}^2} \left(e^{-2\, d_\gamma\, (t-t_0)} + e^{-d_\gamma\, (s+r-2\, t_0)} - e^{-d_\gamma\, (t+r-2\, t_0)} - e^{-2\, d_\gamma\, (s+t-2\, t_0)}\right), \tag{B.9.18}$$

and so

$$\mathrm{Cov}\,(J\,(s,\ t)\,,\ J\,(r,\ t)) = \mathrm{E}\,[J\,(s,\ t)\ J\,(r,\ t)] - \langle J\,(s,\ t)\rangle\ \langle J\,(r,\ t)\rangle$$
$$= \frac{\mathrm{Var}\left(\gamma_0\right)}{{d_\gamma}^2} \left(e^{-d_\gamma\, (t-t_0)} - e^{-d_\gamma\, (s-t_0)}\right) \left(e^{-d_\gamma\, (t-t_0)} - e^{-d_\gamma\, (r-t_0)}\right)$$
$$+ \frac{{\sigma_\gamma}^2}{2\, {d_\gamma}^3} \left(-\left(1 + e^{-d_\gamma\, (s-r)}\right) + 2\, d_\gamma\, (t-s)\right.$$
$$+ e^{-d_\gamma\, (s+t-2\, t_0)}$$
$$\cdot\left(\left(1 + e^{d_\gamma\, (s-r)}\right) + \left(e^{2\, d_\gamma\, (s-t_0)} + e^{d_\gamma\, (s+r-2\, t_0)}\right)\right.$$
$$\left.\left.- \left(e^{d_\gamma\, (t-r)} + e^{-d_\gamma\, (t-s)}\right)\right)\right), \tag{B.9.19}$$

for $t_0 \le r \le s \le t$. By plugging in $r = s$, we may verify that this reduces to our previous equation for $\mathrm{Var}\,(J\,(s,\ t))$ (Eq. B.4.7). The case of $t_0 \le s \le r \le t$ is given by

$$\mathrm{Cov}\,(J\,(s,\ t)\,,\ J\,(r,\ t)) = \mathrm{E}\,[J\,(s,\ t)\ J\,(r,\ t)] - \langle J\,(s,\ t)\rangle\ \langle J\,(r,\ t)\rangle$$
$$= \frac{\mathrm{Var}\left(\gamma_0\right)}{{d_\gamma}^2} \left(e^{-d_\gamma\, (t-t_0)} - e^{-d_\gamma\, (r-t_0)}\right) \left(e^{-d_\gamma\, (t-t_0)} - e^{-d_\gamma\, (s-t_0)}\right)$$

$$+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^3}\,\left(-\left(1+e^{-d_\gamma\,(r-s)}\right)+2\,d_\gamma\,(t-r)\right.$$
$$+\,e^{-d_\gamma\,(r+t-2\,t_0)}$$
$$\left.\cdot\left(\left(1+e^{d_\gamma\,(r-s)}\right)+\left(e^{2\,d_\gamma\,(r-t_0)}+e^{d_\gamma\,(r+s-2\,t_0)}\right)-\left(e^{d_\gamma\,(t-s)}+e^{-d_\gamma\,(t-r)}\right)\right)\right). \tag{B.9.20}$$

B.9.2.2 Covariance of $b\,(s)$ and $b\,(r)$

We begin with the usual decomposition of covariance into a difference of expected values.

$$\mathrm{Cov}\,(b\,(s)\,,\;b\,(r))=\mathrm{E}\left[b\,(s)\;\overline{b\,(r)}\right]-\langle b\,(s)\rangle\;\overline{\langle b\,(r)\rangle}. \tag{B.9.21}$$

Since the expected values in the second term on the right are given by Eq. B.2.1 (with appropriate arguments), we calculate $\mathrm{E}\left[b\,(s)\;\overline{b\,(r)}\right]$ first and later subtract the product of the expected values.

$$\begin{aligned}
\mathrm{E}\left[b\,(s)\;\overline{b\,(r)}\right]&=\mathrm{E}\left[\left(\widehat{b}+\left(b_0-\widehat{b}\right)\,e^{\lambda_b\,(s-t_0)}+\sigma_b\int_{t_0}^{s}e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\right.\\
&\qquad\left.\cdot\left(\overline{\widehat{b}}+\left(\overline{b_0}-\overline{\widehat{b}}\right)\,e^{\overline{\lambda_b}\,(r-t_0)}+\sigma_b\int_{t_0}^{r}e^{\overline{\lambda_b}\,(r-t')}\,\overline{dW_b\,(t')}\right)\right]\\
&=\mathrm{E}\left[\widehat{b}\,\overline{\widehat{b}}\right]+\mathrm{E}\left[\widehat{b}\,\left(\overline{b_0}-\overline{\widehat{b}}\right)\,e^{\overline{\lambda_b}\,(r-t_0)}\right]+\mathrm{E}\left[\widehat{b}\,\left(\sigma_b\int_{t_0}^{r}e^{\overline{\lambda_b}\,(r-t')}\,\overline{dW_b\,(t')}\right)\right]\\
&\quad+\mathrm{E}\left[\left(b_0-\widehat{b}\right)\,e^{\lambda_b\,(s-t_0)}\,\overline{\widehat{b}}\right]+\mathrm{E}\left[\left(b_0-\widehat{b}\right)\,e^{\lambda_b\,(s-t_0)}\,\left(\overline{b_0}-\overline{\widehat{b}}\right)\,e^{\overline{\lambda_b}\,(r-t_0)}\right]\\
&\quad+\mathrm{E}\left[\left(b_0-\widehat{b}\right)\,e^{\lambda_b\,(s-t_0)}\,\left(\sigma_b\int_{t_0}^{r}e^{\overline{\lambda_b}\,(r-t')}\,\overline{dW_b\,(t')}\right)\right]\\
&\quad+\mathrm{E}\left[\left(\sigma_b\int_{t_0}^{s}e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\,\overline{\widehat{b}}\right]\\
&\quad+\mathrm{E}\left[\left(\sigma_b\int_{t_0}^{s}e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\,\left(\overline{b_0}-\overline{\widehat{b}}\right)\,e^{\overline{\lambda_b}\,(r-t_0)}\right]\\
&\quad+\mathrm{E}\left[\left(\sigma_b\int_{t_0}^{s}e^{\lambda_b\,(s-t')}\,dW_b\left(t'\right)\right)\,\left(\sigma_b\int_{t_0}^{r}e^{\overline{\lambda_b}\,(r-t')}\,\overline{dW_b\,(t')}\right)\right]\\
&=\widehat{b}\,\overline{\widehat{b}}+\left(\overline{\langle b_0\rangle}-\overline{\widehat{b}}\right)\,\widehat{b}\,e^{\overline{\lambda_b}\,(r-t_0)}+0+\left(\langle b_0\rangle-\widehat{b}\right)\,\overline{\widehat{b}}\,e^{\lambda_b\,(s-t_0)}\\
&\quad+\mathrm{E}\left[\left(b_0-\widehat{b}\right)\,\left(\overline{b_0}-\overline{\widehat{b}}\right)\right]\,e^{\lambda_b\,(s-t_0)}\,e^{\overline{\lambda_b}\,(r-t_0)}+0+0+0
\end{aligned}$$

$$
\begin{aligned}
&+ \sigma_b^2 \, \mathrm{E}\left[e^{-\gamma_b (s+r) + i\,\omega_b (s-r)} \int_{t_0}^{\min(s,\, r)} e^{2\,\gamma_b t'} \, dt' \right] \\
&= \widehat{b}\,\overline{\widehat{b}} + \left(\overline{\langle b_0 \rangle} - \overline{\widehat{b}} \right) \widehat{b}\, e^{\overline{\lambda_b} (r-t_0)} + \left(\langle b_0 \rangle - \widehat{b} \right) \overline{\widehat{b}}\, e^{\lambda_b (s-t_0)} \\
&+ \mathrm{E}\left[\left(b_0 - \widehat{b} \right) \left(\overline{b_0} - \overline{\widehat{b}} \right) \right] e^{\lambda_b (s-t_0)} \, e^{\overline{\lambda_b} (r-t_0)} \\
&+ \frac{\sigma_b^2}{2\,\gamma_b} \left(e^{-\gamma_b (s+r-2\min(s,\, r)) + i\,\omega_b (s-r)} - e^{-\gamma_b \left(s+r-2\,t_0\right) + i\,\omega_b (s-r)} \right).
\end{aligned}
\tag{B.9.22}
$$

We also have

$$
\begin{aligned}
\langle b(s) \rangle \, \overline{\langle b(r) \rangle} &= \left(\widehat{b} + \left(\langle b_0 \rangle - \widehat{b} \right) e^{\lambda_b (s-t_0)} \right) \left(\overline{\widehat{b}} + \left(\overline{\langle b_0 \rangle} - \overline{\widehat{b}} \right) e^{\overline{\lambda_b} (r-t_0)} \right) \\
&= \widehat{b}\,\overline{\widehat{b}} + \left(\overline{\langle b_0 \rangle} - \overline{\widehat{b}} \right) \widehat{b}\, e^{\overline{\lambda_b} (r-t_0)} + \left(\langle b_0 \rangle - \widehat{b} \right) \overline{\widehat{b}}\, e^{\lambda_b (s-t_0)} \\
&+ \left(\langle b_0 \rangle - \widehat{b} \right) \left(\overline{\langle b_0 \rangle} - \overline{\widehat{b}} \right) e^{\lambda_b (s-t_0)} \, e^{\overline{\lambda_b} (r-t_0)},
\end{aligned}
\tag{B.9.23}
$$

and so

$$
\begin{aligned}
\mathrm{Cov}\left(b(s),\ b(r)\right) &= \mathrm{E}\left[b(s) \, \overline{b(r)} \right] - \langle b(s) \rangle \, \overline{\langle b(r) \rangle} \\
&= \mathrm{Var}\left(b_0\right) e^{\lambda_b (s-t_0)} \, e^{\overline{\lambda_b} (r-t_0)} \\
&+ \frac{\sigma_b^2}{2\,\gamma_b} \left(e^{-\gamma_b (s+r-2\min(s,\, r)) + i\,\omega_b (s-r)} - e^{-\gamma_b \left(s+r-2\,t_0\right) + i\,\omega_b (s-r)} \right) \\
&= \mathrm{Var}\left(b_0\right) e^{-\gamma_b \left(s+r-2\,t_0\right) + i\,\omega_b (s-r)} \\
&+ \frac{\sigma_b^2}{2\,\gamma_b} \left(e^{-\gamma_b (s+r-2\min(s,\, r)) + i\,\omega_b (s-r)} - e^{-\gamma_b \left(s+r-2\,t_0\right) + i\,\omega_b (s-r)} \right) \\
&= e^{-\gamma_b \left(s+r-2\,t_0\right) + i\,\omega_b (s-r)} \left(\mathrm{Var}\left(b_0\right) + \frac{\sigma_b^2}{2\,\gamma_b} \left(e^{2\,\gamma_b \left(\min(s,\, r) - t_0\right)} - 1 \right) \right).
\end{aligned}
\tag{B.9.24}
$$

B.9.2.3 Covariance of $b(s)$ and $J(r,\ t)$

The equation for $\mathrm{Cov}\left(b(s),\ J(r,\ t)\right)$ is found in a manner similar to the equation for $\mathrm{Cov}\left(b(s),\ J(s,\ t)\right)$ (Eq. B.4.13), and is given by

$$
\mathrm{Cov}\left(b(s),\ J(r,\ t)\right) = \frac{\mathrm{Cov}\left(b_0,\ \gamma_0\right)}{d_\gamma} e^{\lambda_b (s-t_0)} \left(e^{-d_\gamma (r-t_0)} - e^{-d_\gamma (t-t_0)} \right).
\tag{B.9.25}
$$

B.9.3 Mean of $|C|^2$

Like the calculation of $\mathrm{E}\left[|A|^2\right]$, the calculation of $\mathrm{E}\left[|C|^2\right]$ is relatively straightforward

$$
\begin{aligned}
\mathrm{E}\left[|C|^2\right] &= \mathrm{E}\left[\left(\sigma \int_{t_0}^{t} e^{-J(s,\,t)+\hat{\lambda}(t-s)}\, dW(s)\right)\left(\sigma \int_{t_0}^{t} e^{-J(s,\,t)+\overline{\hat{\lambda}}(t-s)}\, \overline{dW(s)}\right)\right] \\
&= \sigma^2\, \mathrm{E}\left[\int_{t_0}^{t} e^{-2J(s,\,t)-2\hat{\gamma}(t-s)}\, ds\right] \\
&= \sigma^2 \int_{t_0}^{t} e^{2\left(\mathrm{Var}(J(s,\,t))-\langle J(s,\,t)\rangle-\hat{\gamma}(t-s)\right)}\, ds. \qquad \text{(B.9.26)}
\end{aligned}
$$

To evaluate this equation, we need the following statistics

$\left\langle J\left(t_0,\, t\right)\right\rangle$ – Eq. B.4.2 (substituting $s = t_0$).

$\mathrm{Var}\left(J\left(t_0,\, t\right)\right)$ – Eq. B.4.7 (substituting $s = t_0$).

Substituting these equations into Eq B.9.26 does not simplify it further, so we will not do so here.

B.9.4 Mean of $A\,\overline{B}$

The calculation of $\mathrm{E}\left[A\,\overline{B}\right]$ is similarly long to the calculation of $\mathrm{E}\left[|B|^2\right]$, including a long list of other statistics we will need to evaluate the equation we obtain for it.

$$
\begin{aligned}
\mathrm{E}\left[A\,\overline{B}\right] &= \mathrm{E}\left[\left(e^{-J\left(t_0,\,t\right)+\hat{\lambda}\left(t-t_0\right)}\, u_0\right)\left(\int_{t_0}^{t}\left(\overline{b(s)}+\overline{f(s)}\right) e^{-J(s,\,t)+\overline{\hat{\lambda}}(t-s)}\, ds\right)\right] \\
&= \mathrm{E}\left[\int_{t_0}^{t} u_0\,\overline{b(s)}\, e^{-\left(J\left(t_0,\,t\right)+J(s,\,t)\right)-\hat{\gamma}\left(2t-s-t_0\right)+i\omega\left(s-t_0\right)}\, ds\right] \\
&\quad + \mathrm{E}\left[\int_{t_0}^{t} u_0\,\overline{f(s)}\, e^{-\left(J\left(t_0,\,t\right)+J(s,\,t)\right)-\hat{\gamma}\left(2t-s-t_0\right)+i\omega\left(s-t_0\right)}\, ds\right] \\
&= \int_{t_0}^{t} \Big(\mathrm{Cov}\left(u_0,\, b(s)\right) \\
&\qquad + \left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right) - \mathrm{Cov}\left(u_0,\, J(s,\, t)\right)\right) \\
&\qquad\quad \cdot\left(\overline{\langle b(s)\rangle}+\overline{f(s)}-\overline{\mathrm{Cov}\left(b(s),\, J\left(t_0,\, t\right)\right)}-\overline{\mathrm{Cov}\left(b(s),\, J(s,\, t)\right)}\right)\Big)
\end{aligned}
$$

$$\cdot\, e^{-\left\langle J\left(t_0,\, t\right)\right\rangle-\langle J(s,\, t)\rangle+\frac{1}{2}\,\mathrm{Var}\left(J\left(t_0,\, t\right)\right)+\frac{1}{2}\,\mathrm{Var}(J(s,\, t))+\mathrm{Cov}\left(J\left(t_0,\, t\right),\, J(s,\, t)\right)-\widehat{\gamma}\left(2\,t-s-t_0\right)+i\,\omega\left(s-t_0\right)}\, d\,s. \tag{B.9.27}$$

To evaluate this equation, we require the following statistics

Cov $\left(u_0,\ b\,(s)\right)$ – Eq B.9.28.

Cov $\left(u_0,\ \boldsymbol{J}\,\left(t_0,\ t\right)\right)$ – Eq. B.4.8.

Cov $\left(u_0,\ \boldsymbol{J}\,(s,\ t)\right)$ – Eq. B.9.29.

Cov $(b\,(s)\,,\ \boldsymbol{J}\,(s,\ t))$ – Eq. B.4.13 (with appropriate arguments).

$\langle \boldsymbol{J}\,(s,\ t)\rangle$ – Eq. B.4.2 (with appropriate arguments).

Var $(\boldsymbol{J}\,(s,\ t))$ – Eq. B.4.7 (with appropriate arguments).

Cov $(\boldsymbol{J}\,(s,\ t)\,,\ \boldsymbol{J}\,(r,\ t))$ – Eqs. B.9.19 and B.9.20 (with appropriate arguments).

B.9.4.1 Covariance of u_0 and $b\,(s)$

The calculation of the covariance of u_0 and $b\,(s)$ is relatively straightforward, and so we only give the final result

$$\mathrm{Cov}\left(u_0,\ b\,(s)\right)=\mathrm{Cov}\left(u_0,\ b_0\right)\ e^{\overline{\lambda_b}\,(s-t_0)}. \tag{B.9.28}$$

B.9.4.2 Covariance of u_0 and $J\,(s,\ t)$

The calculation of the covariance of u_0 and $\boldsymbol{J}\,(s,\ t)$ is as straightforward as the calculation of Cov $\left(u_0,\ \boldsymbol{J}\,\left(t_0,\ t\right)\right)$ (Eq. B.4.8), and so we only give the final result

$$\mathrm{Cov}\left(u_0,\ \boldsymbol{J}\,(s,\ t)\right)=\mathrm{Cov}\left(u_0,\ \gamma_0\right)\ \left(e^{-d_\gamma\,(s-t_0)}-e^{-d_\gamma\,(t-t_0)}\right). \tag{B.9.29}$$

In fact, by plugging in $s=t_0$ we may verify that this equation reduces to that of Cov $\left(u_0,\ \boldsymbol{J}\,(s,\ t)\right)$ (Eq. B.9.29).

B.9.5 Means of $A\,\overline{C}$ and $B\,\overline{C}$

Since $d\,W\,(s)$, $d\,W_\gamma\,(s)$, and $d\,W_b\,(s)$ are all uncorrelated, we may see that $\mathrm{E}\left[A\,\overline{C}\right]=\mathrm{E}\left[B\,\overline{C}\right]=0$.

B.10 Covariance of $u(t)$ an $\overline{u(t)}$

We now wish to calculate the covariance of $u(t)$ with its complex conjugate. Using the same splitting of $u(t)$ into the terms A, B, and C given in equations Eq. B.9.1, B.9.2, and B.9.3, we may see that this statistic is given by

$$\operatorname{Cov}\left(u(t),\ \overline{u(t)}\right) = \mathrm{E}\left[(u(t))^2\right] - (\langle u(t)\rangle)^2 \tag{B.10.1}$$

$$= \mathrm{E}\left[A^2\right] + \mathrm{E}\left[B^2\right] + \mathrm{E}\left[C^2\right] + 2\left(\mathrm{E}[A\,B] + \mathrm{E}[A\,C] + \mathrm{E}[B\,C]\right) - (\langle u(t)\rangle)^2 . \tag{B.10.2}$$

As we will soon discover, several of these terms evaluate to zero (specifically $\mathrm{E}[A\,C]$ and $\mathrm{E}[B\,C]$, see Appendix B.10.5), and attempiting further simplifications of the above equation utilizing our previously derived equation for the expected value of $u(t)$ (Eq B.4.1) will be fruitless, and so we shall write the report form of $\operatorname{Cov}\left(u(t),\ \overline{u(t)}\right)$ and list below the equation numbers for each term.

$$\operatorname{Cov}\left(u(t),\ \overline{u(t)}\right) = \mathrm{E}\left[A^2\right] + \mathrm{E}\left[B^2\right] + \mathrm{E}\left[C^2\right] + 2\,\mathrm{E}[A\,B] - (\langle u(t)\rangle)^2 . \tag{B.10.3}$$

The equation numbers for the terms of this equation are

- $\mathrm{E}\left[A^2\right]$ – Eq. B.10.4.
- $\mathrm{E}\left[B^2\right]$ – Eq. B.10.5.
- $\mathrm{E}\left[C^2\right]$ – Eq. B.10.8.
- $\mathrm{E}[AB]$ – Eq. B.10.9.
- $\mathrm{E}[u(t)]$ – Eq. B.4.1.

B.10.1 Mean of A^2

The calculation of the expected value of A^2 is fairly straightforward, and is very similar to the derivation of the expected value of $|A|^2$ (see Appendix B.9.1). We utilize the identity Eq. B.1.6 to obtain

$$\mathrm{E}\left[A^2\right] = \mathrm{E}\left[{u_0}^2\, e^{2\left(\hat{\lambda}(t-t_0) - J(t_0,\, t)\right)}\right]$$

$$= \left(\operatorname{Cov}\left(u_0,\ \overline{u_0}\right) + \left(\langle u_0\rangle - 2\operatorname{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)\right)^2\right) e^{2\left(\operatorname{Var}(J(t_0,\, t)) + \hat{\lambda}(t-t_0) - \langle J(t_0,\, t)\rangle\right)} . \tag{B.10.4}$$

Attempting further simplification using our previously derived equations is fruitless, and we instead list the equation numbers here:

$\mathrm{Cov}\left(u_0,\ \boldsymbol{J}\left(t_0,\ t\right)\right)$ – Eq. B.4.8.

$\mathrm{Var}\left(\boldsymbol{J}\left(t_0,\ t\right)\right)$ – Eq. B.4.7 (substituting $s = t_0$).

$\left\langle \boldsymbol{J}\left(t_0,\ t\right)\right\rangle$ – Eq. B.4.2 (substituting $s = t_0$).

B.10.2 Mean of B^2

The expected value of $\boldsymbol{B}^2$ is found in a very similar manner the the expected value of $|\boldsymbol{B}|^2$ (see Appendix B.9.2), and is given by

$$\left\langle \boldsymbol{B}^2 \right\rangle = \int_{t_0}^{t}\int_{t_0}^{t} b_{covar}(s,\ r)\ ds\, dr, \tag{B.10.5}$$

$$\begin{aligned} b_{covar}(s,\ r) = {} & e^{-\langle J(s,\ t)\rangle - \langle J(r,\ t)\rangle + \frac{1}{2}\mathrm{Var}(J(s,\ t)) + \frac{1}{2}\mathrm{Var}(J(r,\ t)) + \mathrm{Cov}(J(s,\ t),\ J(r,\ t)) + \widehat{\lambda}(2t - s - r)} \\ & \cdot \left(\mathrm{Cov}\left(b(s),\ \overline{b(r)}\right) + (\langle b(s)\rangle + f(s) - \mathrm{Cov}(b(s),\ \boldsymbol{J}(r,\ t)) - \mathrm{Cov}(b(s),\ \boldsymbol{J}(s,\ t)))\right. \\ & \left. \cdot (\langle b(r)\rangle + f(r) - \mathrm{Cov}(b(r),\ \boldsymbol{J}(r,\ t)) - \mathrm{Cov}(b(r),\ \boldsymbol{J}(s,\ t)))\right). \end{aligned} \tag{B.10.6}$$

To evaluate $b_{covar}(s,\ r)$, the following statistics are required

$\langle \boldsymbol{J}(s,\ t)\rangle$, $\langle \boldsymbol{J}(r,\ t)\rangle$ – Eq. B.4.2 (with appropriate arguments).

$\mathrm{Var}(\boldsymbol{J}(s,\ t))$, $\mathrm{Var}(\boldsymbol{J}(r,\ t))$ – Eq. B.4.7 (with appropraite arguments).

$\mathrm{Cov}(\boldsymbol{J}(s,\ t),\ \boldsymbol{J}(r,\ t))$ – Eqs. B.9.19 and B.9.20.

$\mathrm{Cov}\left(b(s),\ \overline{b(r)}\right)$ – Eq. B.10.7.

$\langle b(s)\rangle$, $\langle b(r)\rangle$ – Eq. B.2.1 (substituting $t = s$ and $t = r$, respectively).

$\mathrm{Cov}(b(s),\ \boldsymbol{J}(r,\ t))$, $\mathrm{Cov}(b(s),\ \boldsymbol{J}(s,\ t))$, $\mathrm{Cov}(b(r),\ \boldsymbol{J}(r,\ t))$, $\mathrm{Cov}(b(r),\ \boldsymbol{J}(s,\ t))$ – Eq. B.9.25.

B.10.2.1 Covariance of $b(s)$ and $\overline{b(r)}$

The covariance of $b(s)$ and $\overline{b(r)}$ is found in a similar manner to the covariance of $b(s)$ and $b(r)$ (see Appendix B.9.2.2), and is given by

$$\mathrm{Cov}\left(b(s),\ \overline{b(r)}\right) = \mathrm{Cov}\left(b_0,\ \overline{b_0}\right)\, e^{\lambda_b (s + r - 2t_0)}. \tag{B.10.7}$$

B.10.3 Mean of C^2

The expected value of C^2 is obtained in a straightforward manner, similarly to the expected value of $|C|^s$ (see Appendix B.9.3)

$$\begin{aligned}\left\langle C^2 \right\rangle &= \mathrm{E}\left[\sigma^2 \left(\int_{t_0}^{t} e^{-J\left(t_0,\, t\right)+\hat{\lambda}(t-s)}\, dW(s)\right)^2\right] \\ &= \sigma^2\, \mathrm{E}\left[\int_{t_0}^{t} e^{2\left(\hat{\lambda}(t-s)-J(s,\, t)\right)}\, ds\right] \\ &= \sigma^2 \int_{t_0}^{t} e^{2\left(\mathrm{Var}(J(s,\, t))+\hat{\lambda}(t-s)-\langle J(s,\, t)\rangle\right)}\, ds. \end{aligned} \tag{B.10.8}$$

To evaluate this equation, we need the following statistics

$\left\langle J\left(t_0,\, t\right)\right\rangle$ – Eq. B.4.2 (substituting $s = t_0$).

$\mathrm{Var}\left(J\left(t_0,\, t\right)\right)$ – Eq. B.4.7 (substituting $s = t_0$).

B.10.4 Mean of $A\,B$

The expected value of $A\,B$ is found in a manner similar to the expected value of $A\,\overline{B}$ (see Appendix B.9.4), and is given by

$$\begin{aligned}\mathrm{E}[A\,B] = \int_{t_0}^{t} \Big(&\overline{\mathrm{Cov}\left(u_0,\, b(s)\right)} + \left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right) - \mathrm{Cov}\left(u_0,\, J(s,\, t)\right)\right) \\ &\cdot \left(\langle b(s)\rangle + f(s) - \mathrm{Cov}\left(b(s),\, J\left(t_0,\, t\right)\right) - \mathrm{Cov}\left(b(s),\, J(s,\, t)\right)\right)\Big) \\ &\cdot e^{-\left\langle J\left(t_0,\, t\right)\right\rangle - \langle J(s,\, t)\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\, t\right)\right) + \frac{1}{2}\mathrm{Var}(J(s,\, t)) + \mathrm{Cov}\left(J\left(t_0,\, t\right),\, J(s,\, t)\right) + \hat{\lambda}\left(2t-s-t_0\right)}\, ds. \end{aligned} \tag{B.10.9}$$

To evaluate this equation, we require the following statistics:

$\mathrm{Cov}\left(u_0,\, b(s)\right)$ – Eq B.9.28.

$\mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)$ – Eq. B.4.8.

$\mathrm{Cov}\left(u_0,\, J(s,\, t)\right)$ – Eq. B.9.29.

$\mathrm{Cov}\left(b(s),\, J(s,\, t)\right)$ – Eq. B.4.13 (with appropriate arguments).

$\langle J(s,\ t)\rangle$ – Eq. B.4.2 (with appropriate arguments).

$\mathrm{Var}(J(s,\ t))$ – Eq. B.4.7 (with appropriate arguments).

$\mathrm{Cov}(J(s,\ t),\ J(r,\ t))$ – Eqs. B.9.19 and B.9.20 (with appropriate arguments).

B.10.5 Means of AC and BC

Since $dW(s)$, $dW_\gamma(s)$, and $dW_b(s)$ are all uncorrelated, we may see that $\mathrm{E}[AC] = \mathrm{E}[BC] = 0$.

B.11 Covariance of $u(t)$ and $b(t)$

To obtain the covariance of $u(t)$ and $b(t)$, we first utilize the decomposition of the covariance into the difference of expectations

$$\mathrm{Cov}(u(t),\ b(t)) = \mathrm{Cov}(A,\ b(t)) + \mathrm{Cov}(B,\ b(t)) + \mathrm{Cov}(C,\ b(t)), \tag{B.11.1}$$

where we have also utilized the decomposition of the pathwise solution of $u(t)$ into A, B, and C (Eqs. B.9.1, B.9.2, and B.9.3). As we will soon discover, the last of the terms on the right-hand side of the above equation is actually zero (see Appendix B.11.3), and so we report the covariance of $u(t)$ and $b(t)$ as

$$\mathrm{Cov}(u(t),\ b(t)) = \mathrm{Cov}(A,\ b(t)) + \mathrm{Cov}(B,\ b(t)), \tag{B.11.2}$$

The equation numbers for the terms of this equation are

$\mathrm{Cov}(A,\ b(t))$ – Eq. B.11.6.

$\mathrm{Cov}(B,\ b(t))$ – Eq. B.11.9.

B.11.1 Covariance of A and $b(t)$

To calculate the covariance of A (Eq. B.9.1) and $b(t)$, we decompose the covariance into the difference of expectations

$$\mathrm{Cov}(A,\ b(t)) = \mathrm{E}\left[A\,\overline{b(t)}\right] - \langle A\rangle\,\overline{\langle b(t)\rangle}. \tag{B.11.3}$$

To calculate the first of these terms on the right-hand side, we utilize the identity Eq. B.1.6 to obtain

$$
\begin{aligned}
\mathrm{E}\left[A\,\overline{b(t)}\right] &= \mathrm{E}\left[u_0\,\overline{\hat{b}}\,e^{-J(t_0,\,t)+\hat{\lambda}(t-t_0)}\right] + \mathrm{E}\left[u_0\,\left(\overline{b_0}-\overline{\hat{b}}\right)\,e^{-J(t_0,\,t)+\left(\hat{\lambda}-\overline{\lambda_b}\right)(t-t_0)}\right] \\
&\quad + \mathrm{E}\left[u_0\,\sigma_b\int_{t_0}^{t} e^{-J(t_0,\,t)+\hat{\lambda}(t-t_0)+\overline{\lambda_b}(s-t_0)}\,\overline{dW_b(t)}\right] \\
&= \overline{\hat{b}}\,\langle A\rangle \\
&\quad + \Big(\mathrm{Cov}\left(u_0,\,b_0\right) + \left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\,J\left(t_0,\,t\right)\right)\right)\left(\overline{\langle b_0\rangle}-\overline{\hat{b}}-\overline{\mathrm{Cov}\left(b_0,\,J\left(t_0,\,t\right)\right)}\right)\Big) \\
&\qquad \cdot e^{-\left\langle J\left(t_0,\,t\right)\right\rangle+\frac{1}{2}\mathrm{Var}\left(J\left(t_0,\,t\right)\right)+\left(\hat{\lambda}+\overline{\lambda_b}\right)(t-t_0)}.
\end{aligned}
\tag{B.11.4}
$$

To calculate the second of these terms on the right-hand side, we utilize the identity Eq. B.1.7 to obtain along with our previous equation for the expected value of $b(t)$ (Eq. B.2.1)

$$
\langle A\rangle\,\overline{\langle b(t)\rangle} = \overline{\hat{b}}\,\langle A\rangle+\left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\,J\left(t_0,\,t\right)\right)\right)\left(\overline{\langle b_0\rangle}-\overline{\hat{b}}\right)\,e^{-\left\langle J\left(t_0,\,t\right)\right\rangle+\frac{1}{2}\mathrm{Var}\left(J\left(t_0,\,t\right)\right)+\left(\hat{\lambda}+\overline{\lambda_b}\right)(t-t_0)}.
\tag{B.11.5}
$$

The difference of these terms gives us the desired covariance

$$
\begin{aligned}
\mathrm{Cov}\left(A,\,b(t)\right) &= \mathrm{E}\left[A\,\overline{b(t)}\right] - \langle A\rangle\,\overline{\langle b(t)\rangle} \\
&= \Big(\mathrm{Cov}\left(u_0,\,b_0\right) - \overline{\mathrm{Cov}\left(b_0,\,J\left(t_0,\,t\right)\right)}\,\left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\,J\left(t_0,\,t\right)\right)\right)\Big) \\
&\qquad \cdot e^{-\left\langle J\left(t_0,\,t\right)\right\rangle+\frac{1}{2}\mathrm{Var}\left(J\left(t_0,\,t\right)\right)+\left(\hat{\lambda}+\overline{\lambda_b}\right)(t-t_0)}.
\end{aligned}
\tag{B.11.6}
$$

The statistics required to calculate this value are

- $\mathrm{Cov}\left(b_0,\,J\left(t_0,\,t\right)\right)$ – Eq. B.4.13 (with $s = t_0$).
- $\mathrm{Cov}\left(u_0,\,J\left(t_0,\,t\right)\right)$ – Eq. B.4.8.
- $\left\langle J\left(t_0,\,t\right)\right\rangle$ – Eq. B.4.2 (with $s = t_0$).
- $\mathrm{Var}\left(J\left(t_0,\,t\right)\right)$ – Eq. B.4.7 (with $s = t_0$).

B.11.2 Covariance of B and $b(t)$

To calculate the covariance of B (Eq. B.9.2) and $b(t)$, similarly to the covariance of A and $b(t)$ (Appendix B.11.1), we decompose the covariance into the difference of expectations

$$\mathrm{Cov}\left(B,\ b\left(t\right)\right) = \mathrm{E}\left[B\,\overline{b\left(t\right)}\right] - \left\langle B\right\rangle\,\overline{\left\langle b\left(t\right)\right\rangle}. \tag{B.11.7}$$

To calculate the first of these terms on the right-hand side, we utilize the identity Eq. B.1.6 to obtain

$$\begin{aligned}
\mathrm{E}\left[B\,\overline{b\left(t\right)}\right] &= \int_{t_0}^{t} \mathrm{E}\left[\left(b\left(s\right) + f\left(s\right)\right)\,\overline{b\left(t\right)}\,e^{-J\left(s,\ t\right)+\widehat{\lambda}\left(t-s\right)}\right]\,ds \\
&= \int_{t_0}^{t} \left(\mathrm{Cov}\left(b\left(s\right),\ b\left(t\right)\right) + \left(\left\langle b\left(s\right)\right\rangle + f\left(s\right) - \mathrm{Cov}\left(b\left(s\right),\ J\left(s,\ t\right)\right)\right)\left(\overline{\left\langle b\left(t\right)\right\rangle} - \overline{\mathrm{Cov}\left(b\left(t\right),\ J\left(s,\ t\right)\right)}\right)\right) \\
&\qquad \cdot e^{-\left\langle J\left(s,\ t\right)\right\rangle+\frac{1}{2}\mathrm{Var}\left(J\left(s,\ t\right)\right)+\widehat{\lambda}\left(t-s\right)}\,ds.
\end{aligned} \tag{B.11.8}$$

To calculate the second of these terms on the right-hand side, we utilize the identity Eq. B.1.7 to obtain along with our previous equation for the expected value of $b\left(t\right)$ (Eq. B.2.1)

$$\begin{aligned}
\left\langle B\right\rangle\,\overline{\left\langle b\left(t\right)\right\rangle} = \int_{t_0}^{t} &\overline{\left\langle b\left(t\right)\right\rangle}\,\left(\left\langle b\left(s\right)\right\rangle + f\left(s\right) - \mathrm{Cov}\left(b\left(s\right),\ J\left(s,\ t\right)\right)\right) \\
&\cdot e^{-\left\langle J\left(s,\ t\right)\right\rangle+\frac{1}{2}\mathrm{Var}\left(J\left(s,\ t\right)\right)+\widehat{\lambda}\left(t-s\right)}\,ds.
\end{aligned}$$

The difference of these terms gives us the desired covariance

$$\begin{aligned}
\mathrm{Cov}\left(B,\ b\left(t\right)\right) &= \mathrm{E}\left[B\,\overline{b\left(t\right)}\right] - \left\langle B\right\rangle\,\overline{\left\langle b\left(t\right)\right\rangle} \\
&= \int_{t_0}^{t} \left(\mathrm{Cov}\left(b\left(s\right),\ b\left(t\right)\right) - \overline{\mathrm{Cov}\left(b\left(t\right),\ J\left(s,\ t\right)\right)}\,\left(\left\langle b\left(s\right)\right\rangle + f\left(s\right) - \mathrm{Cov}\left(b\left(s\right),\ J\left(s,\ t\right)\right)\right)\right) \\
&\qquad \cdot e^{-\left\langle J\left(s,\ t\right)\right\rangle+\frac{1}{2}\mathrm{Var}\left(J\left(s,\ t\right)\right)+\widehat{\lambda}\left(t-s\right)}\,ds.
\end{aligned} \tag{B.11.9}$$

The statistics required to calculate this value are

$\mathrm{Cov}\left(b\left(s\right),\ b\left(t\right)\right)$ – Eq. B.9.24 (with $r = t$).

$\mathrm{Cov}\left(b\left(t\right),\ J\left(s,\ t\right)\right)$, $\mathrm{Cov}\left(b\left(s\right),\ J\left(s,\ t\right)\right)$ – Eq. B.9.25 (with appropriate arguments).

$\left\langle b\left(s\right)\right\rangle$ – Eq. B.2.1 (with $t = s$).

$\left\langle J\left(s,\ t\right)\right\rangle$ – Eq. B.4.2.

$\mathrm{Var}\left(J\left(s,\ t\right)\right)$ – Eq. B.4.7.

B.11.3 Covariance of C and $b(t)$

The covariance of C (Eq. B.9.3) and $b(t)$ is zero, as the white-noises $dW(t)$ and $dW_b(t)$ are independent, and thus uncorrelated.

B.12 Covariance of $u(t)$ and $\overline{b(t)}$

To obtain the covariance of $u(t)$ and the conjugate of $b(t)$, we follow the method we used in the derivation of the covariance of $u(t)$ and $b(t)$ (Appendix B.11). Specifically, we first utilize the decomposition of the covariance into the difference of expectations

$$\mathrm{Cov}\left(u(t),\, \overline{b(t)}\right) = \mathrm{Cov}\left(A,\, \overline{b(t)}\right) + \mathrm{Cov}\left(B,\, \overline{b(t)}\right) + \mathrm{Cov}\left(C,\, \overline{b(t)}\right), \tag{B.12.1}$$

where we have also utilized the decomposition of the pathwise solution of $u(t)$ into A, B, and C (Eqs. B.9.1, B.9.2, and B.9.3). As we will soon discover, the last of the terms on the right-hand side of the above equation is actually zero (see Appendix B.12.3), and so we report the covariance of $u(t)$ and the conjugate of $b(t)$ as

$$\mathrm{Cov}\left(u(t),\, \overline{b(t)}\right) = \mathrm{Cov}\left(A,\, \overline{b(t)}\right) + \mathrm{Cov}\left(B,\, \overline{b(t)}\right), \tag{B.12.2}$$

The equation numbers for the terms of this equation are

$\mathrm{Cov}\left(A,\, \overline{b(t)}\right)$ – Eq. B.12.6.

$\mathrm{Cov}\left(B,\, \overline{b(t)}\right)$ – Eq. B.12.9.

B.12.1 Covariance of A and $\overline{b(t)}$

We calculate the covariance of A (Eq. B.9.1) and the conjugate of $b(t)$ in a very similar manner to how we calculated $\mathrm{Cov}(A,\, b(t))$ (Appendix B.11.1). We decompose the covariance into the difference of expectations

$$\mathrm{Cov}\left(A,\, \overline{b(t)}\right) = \mathrm{E}\left[A\, b(t)\right] - \langle A\rangle\, \langle b(t)\rangle\,. \tag{B.12.3}$$

To calculate the first of these terms on the right-hand side, we utilize the identity Eq. B.1.6 to obtain

$$\mathrm{E}\left[A\, b(t)\right] = \mathrm{E}\left[u_0\, \hat{b}\, e^{-J(t_0,\, t)+\hat{\lambda}(t-t_0)}\right] + \mathrm{E}\left[u_0\, \left(b_0 - \hat{b}\right)\, e^{-J(t_0,\, t)+\left(\hat{\lambda}-\lambda_b\right)(t-t_0)}\right]$$

$$
\begin{aligned}
&+ \mathrm{E}\left[u_0\, \sigma_b \int_{t_0}^{t} e^{-J(t_0,\, t)+\hat{\lambda}(t-t_0)+\lambda_b(s-t_0)}\, dW_b(t)\right] \\
&= \hat{b}\, \langle A\rangle \\
&\quad + \left(\mathrm{Cov}\left(u_0,\, \overline{b_0}\right) + \left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)\right)\left(\langle b_0\rangle - \hat{b} - \mathrm{Cov}\left(b_0,\, J\left(t_0,\, t\right)\right)\right)\right) \\
&\qquad \cdot e^{-\left\langle J\left(t_0,\, t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\, t\right)\right)+\left(\hat{\lambda}+\lambda_b\right)(t-t_0)}.
\end{aligned} \tag{B.12.4}
$$

To calculate the second of these terms on the right-hand side, we utilize the identity Eq. B.1.7 to obtain along with our previous equation for the expected value of $b(t)$ (Eq. B.2.1)

$$
\langle A\rangle\, \langle b(t)\rangle = \hat{b}\, \langle A\rangle + \left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)\right)\left(\langle b_0\rangle - \hat{b}\right) e^{-\left\langle J\left(t_0,\, t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\, t\right)\right)+\left(\hat{\lambda}+\lambda_b\right)(t-t_0)}. \tag{B.12.5}
$$

The difference of these terms gives us the desired covariance

$$
\begin{aligned}
\mathrm{Cov}\left(A,\, \overline{b(t)}\right) &= \mathrm{E}\left[A\, b(t)\right] - \langle A\rangle\, \langle b(t)\rangle \\
&= \left(\mathrm{Cov}\left(u_0,\, \overline{b_0}\right) - \mathrm{Cov}\left(b_0,\, J\left(t_0,\, t\right)\right)\left(\langle u_0\rangle - \mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)\right)\right) \\
&\qquad \cdot e^{-\left\langle J\left(t_0,\, t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\, t\right)\right)+\left(\hat{\lambda}+\lambda_b\right)(t-t_0)}.
\end{aligned} \tag{B.12.6}
$$

The statistics required to calculate this value are

$\mathrm{Cov}\left(b_0,\, J\left(t_0,\, t\right)\right)$ – Eq. B.4.13 (with $s = t_0$).

$\mathrm{Cov}\left(u_0,\, J\left(t_0,\, t\right)\right)$ – Eq. B.4.8.

$\left\langle J\left(t_0,\, t\right)\right\rangle$ – Eq. B.4.2 (with $s = t_0$).

$\mathrm{Var}\left(J\left(t_0,\, t\right)\right)$ – Eq. B.4.7 (with $s = t_0$).

B.12.2 Covariance of B and $\overline{b(t)}$

We calculate the covariance of B (Eq. B.9.2) and the conjugate of $b(t)$ in a manner very similar to the covariance of B and $b(t)$ (Appendix B.11.2), we decompose the covariance into the difference of expectations

$$
\mathrm{Cov}\left(B,\, \overline{b(t)}\right) = \mathrm{E}\left[B\, b(t)\right] - \langle B\rangle\, \langle b(t)\rangle\, . \tag{B.12.7}
$$

To calculate the first of these terms on the right-hand side, we utilize the identity Eq. B.1.6 to obtain

$$\begin{aligned}
\mathrm{E}\left[B\, b(t)\right] &= \int_{t_0}^{t} \mathrm{E}\left[\left(b(s) + f(s)\right)\, b(t)\, e^{-J(s,\, t) + \widehat{\lambda}(t-s)}\right] ds \\
&= \int_{t_0}^{t} \left(\mathrm{Cov}\left(b(s),\, \overline{b(t)}\right) + \left(\langle b(s)\rangle + f(s) - \mathrm{Cov}\left(b(s),\, J(s,\, t)\right)\right)\left(\langle b(t)\rangle - \mathrm{Cov}\left(b(t),\, J(s,\, t)\right)\right)\right) \\
&\qquad \cdot e^{-\langle J(s,\, t)\rangle + \frac{1}{2}\mathrm{Var}(J(s,\, t)) + \widehat{\lambda}(t-s)}\, ds. \qquad \text{(B.12.8)}
\end{aligned}$$

To calculate the second of these terms on the right-hand side, we utilize the identity Eq. B.1.7 to obtain along with our previous equation for the expected value of $b(t)$ (Eq. B.2.1)

$$\begin{aligned}
\langle B\rangle\, \langle b(t)\rangle = \int_{t_0}^{t} &\langle b(t)\rangle \left(\langle b(s)\rangle + f(s) - \mathrm{Cov}\left(b(s),\, J(s,\, t)\right)\right) \\
&\cdot e^{-\langle J(s,\, t)\rangle + \frac{1}{2}\mathrm{Var}(J(s,\, t)) + \widehat{\lambda}(t-s)}\, ds.
\end{aligned}$$

The difference of these terms gives us the desired covariance

$$\begin{aligned}
\mathrm{Cov}\left(B,\, \overline{b(t)}\right) &= \mathrm{E}\left[B\, b(t)\right] - \langle B\rangle\, \langle b(t)\rangle \\
&= \int_{t_0}^{t} \left(\mathrm{Cov}\left(b(s),\, \overline{b(t)}\right) - \mathrm{Cov}\left(b(t),\, J(s,\, t)\right)\left(\langle b(s)\rangle + f(s) - \mathrm{Cov}\left(b(s),\, J(s,\, t)\right)\right)\right) \\
&\qquad \cdot e^{-\langle J(s,\, t)\rangle + \frac{1}{2}\mathrm{Var}(J(s,\, t)) + \widehat{\lambda}(t-s)}\, ds. \qquad \text{(B.12.9)}
\end{aligned}$$

The statistics required to calculate this value are

- $\mathrm{Cov}\left(b(s),\, \overline{b(t)}\right)$ – Eq. B.10.7 (with $r = t$).
- $\mathrm{Cov}\left(b(t),\, J(s,\, t)\right)$, $\mathrm{Cov}\left(b(s),\, J(s,\, t)\right)$ – Eq. B.9.25 (with appropriate arguments).
- $\langle b(s)\rangle$ – Eq. B.2.1 (with $t = s$).
- $\langle J(s,\, t)\rangle$ – Eq. B.4.2.
- $\mathrm{Var}\left(J(s,\, t)\right)$ – Eq. B.4.7.

B.12.3 Covariance of C and $\overline{b(t)}$

The covariance of C (Eq. B.9.3) and the conjugate of $b(t)$ is zero, as the white-noises $dW(t)$ and $dW_b(t)$ are independent, and thus uncorrelated.

B.13 Covariance of $u(t)$ and $\gamma(t)$

To obtain the covariance of $u(t)$ and $\gamma(t)$, similarly to the covariance of $u(t)$ and $b(t)$ (Appendix B.11), we first utilize the decomposition of the covariance into the difference of expectations

$$\mathrm{Cov}\left(u(t),\ \gamma(t)\right) = \mathrm{Cov}\left(A,\ \gamma(t)\right) + \mathrm{Cov}\left(B,\ \gamma(t)\right) + \mathrm{Cov}\left(C,\ \gamma(t)\right), \tag{B.13.1}$$

where we have also utilized the decomposition of the pathwise solution of $u(t)$ into A, B, and C (Eqs. B.9.1, B.9.2, and B.9.3). As we will soon discover, the last of the terms on the right-hand side of the above equation is actually zero (see Appendix B.13.3), and so we report the covariance of $u(t)$ and $\gamma(t)$ as

$$\mathrm{Cov}\left(u(t),\ \gamma(t)\right) = \mathrm{Cov}\left(A,\ \gamma(t)\right) + \mathrm{Cov}\left(B,\ \gamma(t)\right), \tag{B.13.2}$$

The equation numbers for the terms of this equation are

$\mathrm{Cov}\left(A,\ \gamma(t)\right)$ – Eq. B.13.6.

$\mathrm{Cov}\left(B,\ \gamma(t)\right)$ – Eq. B.13.10.

B.13.1 Covariance of A and $\gamma(t)$

We calculate the covariance of A (Eq. B.9.1) and $\gamma(t)$ in a very similar manner to how we calculated $\mathrm{Cov}\left(A,\ b(t)\right)$ (Appendix B.11.1). We decompose the covariance into the difference of expectations

$$\mathrm{Cov}\left(A,\ \gamma(t)\right) = \mathrm{E}\left[A\,\gamma(t)\right] - \langle A\rangle\ \langle\gamma(t)\rangle\,. \tag{B.13.3}$$

To calculate the first of these terms on the right-hand side, we utilize the identity Eq. B.1.6 to obtain

$$\begin{aligned}\mathrm{E}\left[A\,\gamma(t)\right] = {} & \mathrm{E}\left[u_0\,\hat{\gamma}\,e^{-J(t_0,\,t)+\hat{\lambda}(t-t_0)}\right] + \mathrm{E}\left[u_0\,\left(\gamma_0-\hat{\gamma}\right)\,e^{-J(t_0,\,t)+\left(\hat{\lambda}-d_\gamma\right)(t-t_0)}\right]\\ & + \mathrm{E}\left[u_0\,\sigma_\gamma\int_{t_0}^{t} e^{-J(t_0,\,t)+\hat{\lambda}(t-t_0)+\lambda_b(s-t_0)}\,dW_\gamma(t)\right]\end{aligned}$$

$$
\begin{aligned}
&= \hat{\gamma}\ \langle A \rangle \\
&\quad + \left(\mathrm{Cov}\left(u_0,\ \gamma_0\right) + \left(\langle u_0 \rangle - \mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)\right)\ \left(\langle \gamma_0 \rangle - \hat{\gamma} - \mathrm{Cov}\left(b_0,\ J\left(t_0,\ t\right)\right)\right)\right) \\
&\qquad \cdot e^{-\left\langle J\left(t_0,\ t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\ t\right)\right) + \left(\hat{\lambda} - d_\gamma\right)\left(t - t_0\right)}.
\end{aligned}
\tag{B.13.4}
$$

To calculate the second of these terms on the right-hand side, we utilize the identity Eq. B.1.7 to obtain along with our previous equation for the expected value of $\gamma\left(t\right)$ (Eq. B.3.1)

$$
\langle A \rangle\ \langle \gamma\left(t\right) \rangle = \hat{\gamma}\ \langle A \rangle + \left(\langle u_0 \rangle - \mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)\right)\ \left(\langle \gamma_0 \rangle - \hat{\gamma}\right)\ e^{-\left\langle J\left(t_0,\ t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\ t\right)\right) + \left(\hat{\lambda} + d_\gamma\right)\left(t - t_0\right)}.
\tag{B.13.5}
$$

The difference of these terms gives us the desired covariance

$$
\begin{aligned}
\mathrm{Cov}\left(A,\ \gamma\left(t\right)\right) &= \mathrm{E}\left[A\, \gamma\left(t\right)\right] - \langle A \rangle\ \langle \gamma\left(t\right) \rangle \\
&= \left(\mathrm{Cov}\left(u_0,\ \gamma_0\right) - \mathrm{Cov}\left(\gamma_0,\ J\left(t_0,\ t\right)\right)\ \left(\langle u_0 \rangle - \mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)\right)\right) \\
&\qquad \cdot e^{-\left\langle J\left(t_0,\ t\right)\right\rangle + \frac{1}{2}\mathrm{Var}\left(J\left(t_0,\ t\right)\right) + \left(\hat{\lambda} - d_\gamma\right)\left(t - t_0\right)}.
\end{aligned}
\tag{B.13.6}
$$

The statistics required to calculate this value are

$\mathrm{Cov}\left(\gamma_0,\ J\left(t_0,\ t\right)\right)$ – Eq. B.13.7.

$\mathrm{Cov}\left(u_0,\ J\left(t_0,\ t\right)\right)$ – Eq. B.4.8.

$\left\langle J\left(t_0,\ t\right)\right\rangle$ – Eq. B.4.2 (with $s = t_0$).

$\mathrm{Var}\left(J\left(t_0,\ t\right)\right)$ – Eq. B.4.7 (with $s = t_0$).

B.13.1.1 Covaraince of γ_0 and $J\left(t_0,\ t\right)$

Following the techniques in derivation for the covariance of u_0 and $J\left(t_0,\ t\right)$ (Eq. B.4.8), we obtain

$$
\begin{aligned}
\mathrm{Cov}\left(\gamma_0,\ J\left(t_0,\ t\right)\right) &= \mathrm{E}\left[\gamma_0\, J\left(t_0,\ t\right)\right] - \langle \gamma_0 \rangle\ \left\langle J\left(t_0,\ t\right)\right\rangle \\
&= \frac{\mathrm{Var}\left(\gamma_0\right)}{d_\gamma}\ \left(1 - e^{-d_\gamma\left(t - t_0\right)}\right).
\end{aligned}
\tag{B.13.7}
$$

B.13.2 Covariance of B and $\gamma(t)$

We calculate the covariance of B (Eq. B.9.2) and $\gamma(t)$ in a manner very similar to the covariance of B and $b(t)$ (Appendix B.11.2), we decompose the covariance into the difference of expectations

$$\mathrm{Cov}(B,\ \gamma(t)) = \mathrm{E}[B\,\gamma(t)] - \langle B\rangle\ \langle\gamma(t)\rangle\,. \tag{B.13.8}$$

To calculate the first of these terms on the right-hand side, we utilize the identity Eq. B.1.6 to obtain

$$\begin{aligned}\mathrm{E}[B\,\gamma(t)] &= \int_{t_0}^{t} \mathrm{E}\left[(b(s)+f(s))\,\gamma(t)\ e^{-J(s,\ t)+\hat{\lambda}(t-s)}\right] ds\\ &= \int_{t_0}^{t} (\mathrm{Cov}(b(s),\ \gamma(t)) + (\langle b(s)\rangle + f(s) - \mathrm{Cov}(b(s),\ J(s,\ t)))\ (\langle\gamma(t)\rangle - \mathrm{Cov}(\gamma(t),\ J(s,\ t))))\\ &\qquad \cdot e^{-\langle J(s,\ t)\rangle+\frac{1}{2}\mathrm{Var}(J(s,\ t))+\hat{\lambda}(t-s)}\,ds.\end{aligned} \tag{B.13.9}$$

To calculate the second of these terms on the right-hand side, we utilize the identity Eq. B.1.7 to obtain along with our previous equation for the expected value of $\gamma(t)$ (Eq. B.3.1)

$$\begin{aligned}\langle B\rangle\ \langle\gamma(t)\rangle = \int_{t_0}^{t} &\langle\gamma(t)\rangle\ (\langle b(s)\rangle + f(s) - \mathrm{Cov}(b(s),\ J(s,\ t)))\\ &\cdot e^{-\langle J(s,\ t)\rangle+\frac{1}{2}\mathrm{Var}(J(s,\ t))+\hat{\lambda}(t-s)}\,ds.\end{aligned}$$

The difference of these terms gives us the desired covariance

$$\begin{aligned}\mathrm{Cov}(B,\ b(t)) &= \mathrm{E}[B\,\gamma(t)] - \langle B\rangle\ \langle\gamma(t)\rangle\\ &= \int_{t_0}^{t} (\mathrm{Cov}(b(s),\ \gamma(t)) - \mathrm{Cov}(\gamma(t),\ J(s,\ t))\ (\langle b(s)\rangle + f(s) - \mathrm{Cov}(b(s),\ J(s,\ t))))\\ &\qquad \cdot e^{-\langle J(s,\ t)\rangle+\frac{1}{2}\mathrm{Var}(J(s,\ t))+\hat{\lambda}(t-s)}\,ds.\end{aligned} \tag{B.13.10}$$

The statistics required to calculate this value are

$\mathrm{Cov}(b(s),\ \gamma(t))$ – Eq. B.13.11.

$\mathrm{Cov}(\gamma(t),\ J(s,\ t))$ – Eq. B.13.15.

$\mathrm{Cov}\left(b\left(s\right),\ J\left(s,\ t\right)\right)$ – Eq. B.9.25 (with appropriate arguments).

$\left\langle b\left(s\right)\right\rangle$ – Eq. B.2.1 (with $t = s$).

$\left\langle J\left(s,\ t\right)\right\rangle$ – Eq. B.4.2.

$\mathrm{Var}\left(J\left(s,\ t\right)\right)$ – Eq. B.4.7.

B.13.2.1 Covariance of $b\left(s\right)$ and $\gamma\left(t\right)$

Following a similar procedure to the derivation of the covariance of $b\left(t\right)$ and $\gamma\left(t\right)$ (Eq. B.8.1), we obtain

$$\mathrm{Cov}\left(b\left(s\right),\ \gamma\left(t\right)\right) = \mathrm{Cov}\left(b_0,\ \gamma_0\right)\ e^{\lambda_b\left(s-t_0\right)-d_\gamma\left(t-t_0\right)}. \tag{B.13.11}$$

B.13.2.2 Covariance of $\gamma\left(t\right)$ and $J\left(s,\ t\right)$

To begin our calculation of the covariance of $\gamma\left(t\right)$ and $J\left(s,\ t\right)$, we decompose the covariance into a difference of expected values.

$$\mathrm{Cov}\left(\gamma\left(t\right),\ J\left(s,\ t\right)\right) = \mathrm{E}\left[\gamma\left(t\right)\ J\left(s,\ t\right)\right] - \left\langle\gamma\left(t\right)\right\rangle\ \left\langle J\left(s,\ t\right)\right\rangle. \tag{B.13.12}$$

Since we have the expected value of $\gamma\left(t\right)$ (Eq. B.3.1) and $J\left(s,\ t\right)$ (Eq. B.4.2), it remains to find the expected value of their product. We begin by utilizing the path-wise solution for $\gamma\left(t\right)$ (Eq. A.3.1) and proceed from there.

$$\begin{aligned}
\mathrm{E}\left[\gamma\left(t\right)\ J\left(s,\ t\right)\right] &= \int_s^t \mathrm{E}\left[\gamma\left(t\right)\ \left(\gamma\left(t'\right)-\widehat{\gamma}\right)\right]\ dt' \\
&= \int_s^t \left(\widehat{\gamma}\ \left(\left\langle\gamma_0\right\rangle-\widehat{\gamma}\right)\ e^{-d_\gamma\left(t'-t_0\right)} + \mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]\ e^{-d_\gamma\left(t+t'-2\,t_0\right)}\right. \\
&\qquad \left. +{\sigma_\gamma}^2\,\mathrm{E}\left[\left(\int_{t_0}^t e^{-d_\gamma\left(t-s'\right)}\,dW_\gamma\left(s'\right)\right)\left(\int_{t_0}^{t'} e^{-d_\gamma\left(t'-s'\right)}\,dW_\gamma\left(s'\right)\right)\right]\right)\,dt' \\
&= \frac{\widehat{\gamma}\ \left(\left\langle\gamma_0\right\rangle-\widehat{\gamma}\right)}{d_\gamma}\ \left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) + \frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma}\ \left(e^{-d_\gamma\left(t+s-2\,t_0\right)}-e^{-2\,d_\gamma\left(t-t_0\right)}\right) \\
&\quad + {\sigma_\gamma}^2 \int_s^t \int_{t_0}^{t'} e^{-d_\gamma\left(t+t'-2\,s'\right)}\,ds'\,dt' \\
&= \frac{\widehat{\gamma}\ \left(\left\langle\gamma_0\right\rangle-\widehat{\gamma}\right)}{d_\gamma}\ \left(e^{-d_\gamma\left(s-t_0\right)}-e^{-d_\gamma\left(t-t_0\right)}\right) + \frac{\mathrm{E}\left[\left(\gamma_0-\widehat{\gamma}\right)^2\right]}{d_\gamma}\ \left(e^{-d_\gamma\left(t+s-2\,t_0\right)}-e^{-2\,d_\gamma\left(t-t_0\right)}\right)
\end{aligned}$$

$$+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^2}\left(1-e^{-d_\gamma\,(t-s)}+e^{-2\,d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(t+s-2\,t_0)}\right). \tag{B.13.13}$$

The product of the expected value of $\gamma\,(t)$ and the expected value of $\boldsymbol{J}\,(s,\ t)$ is given by

$$\langle\gamma\,(t)\rangle\ \langle\boldsymbol{J}\,(s,\ t)\rangle=\frac{\widehat{\gamma}\ \left(\langle\gamma_0\rangle-\widehat{\gamma}\right)}{d_\gamma}\left(e^{-d_\gamma\,(s-t_0)}-e^{-d_\gamma\,(t-t_0)}\right)+\frac{\left(\langle\gamma_0\rangle-\widehat{\gamma}\right)^2}{d_\gamma}\left(e^{-d_\gamma\,(t+s-2\,t_0)}-e^{-2\,d_\gamma\,(t-t_0)}\right), \tag{B.13.14}$$

and therefore

$$\begin{aligned}\mathrm{Cov}\,(\gamma\,(t),\ \boldsymbol{J}\,(s,\ t))&=\mathrm{E}\,[\gamma\,(t)\ \boldsymbol{J}\,(s,\ t)]-\langle\gamma\,(t)\rangle\ \langle\boldsymbol{J}\,(s,\ t)\rangle\\&=\frac{\mathrm{Var}\left(\gamma_0\right)}{d_\gamma}\left(e^{-d_\gamma\,(t+s-2\,t_0)}-e^{-2\,d_\gamma\,(t-t_0)}\right)\\&\quad+\frac{{\sigma_\gamma}^2}{2\,{d_\gamma}^2}\left(1-e^{-d_\gamma\,(t-s)}+e^{-2\,d_\gamma\,(t-t_0)}-e^{-d_\gamma\,(t+s-2\,t_0)}\right).\end{aligned} \tag{B.13.15}$$

B.13.3 Covariance of C and $\gamma\,(t)$

The covariance of C (Eq. B.9.3) and $\gamma\,(t)$ is zero, as the white-noises $dW\,(t)$ and $dW_\gamma\,(t)$ are independent, and thus uncorrelated.