A Multi-Phase Direct Collocation Program for Descent Trajectory Optimization

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December 2019

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1.1 The Need for Modernization

There's a need.

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2.1 Frame Definitions

There are three reference frames of importance that will be discussed in this section. The first is the inertial reference frame I that is assumed to be inertially fixed in space. The planet-fixed frame, PF, is related to the inertial frame I by a constant-angular-velocity rotation along the $\hat{I}_z$ vector. The DCM that represents the rotation from the inertial reference frame to the planet fixed frame is:

$$C_I^{PF} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$\theta = \Omega(t - t_0) \tag{2}$$

Where Ω represents the planetary rotation rate, t is the current time referenced from some epoch t_0, and t_0 is a reference epoch that represents when this rotation matrix was the identity matrix. In this document, the angular rotation vector is expressed as:

$$\vec{\Omega} = \vec{\Omega}^I_{PF/I} = \begin{bmatrix} 0 \\ 0 \\ \Omega \end{bmatrix} \tag{3}$$

Another important frame of reference is the Up-East-North frame, or the UEN frame, which is a frame that is rotationally locked with the planet-fixed frame and represents the orientation of the landing site. Referenced from the planet-fixed frame, this frame is a rotation along the planet-fixed Z axis of longitude ϕ, followed by a negative rotation along the new frame's Y axis of latitude λ:

$$C_{PF}^{UEN} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\lambda & 0 & \sin\lambda \\ 0 & 1 & 0 \\ -\sin\lambda & 0 & \cos\lambda \end{bmatrix} \tag{4}$$

For normalization purposes, it may be convenient to translate the UEN frame by the distance of the planetary radius R_{eq} (and optionally by a specified altitude h) along its X axis, to arrive at a frame that is fixed at or near the surface of the planetary body. In this new frame (designated the Landing Site frame, or LS frame), it is intuitive to construct targeted position and velocity vectors that allow the optimizer to target a position directly above (or anywhere around) a fixed point on the surface.

2.2 Equations of Motion

This section describes the system dynamics of the problem.

A vehicle's current state can be expressed in cartesian, inertial coordinates in the following fashion:

$$\vec{r}^I_{v/c} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \tag{5}$$

$$\vec{v}^I_{v/c} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \tag{6}$$

Where $\vec{r}^I_{v/c}$ represents the position vector $\vec{r}$ of the vehicle v with respect to the planetary center c, expressed in the inertial frame I. When the superscript on a vector is omitted, it is implied that the vector is represented in the inertial frame. If the point of reference of a vector is omitted, it is implied that the point of reference of the vector is the planetary center c.

The vehicle's state changes are governed by the following system of equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{r}^I_{v/c}] = \dot{\vec{r}}^I_{v/c} = \vec{v}^I_{v/c} \tag{7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}[\vec{v}^I_{v/c}] = \dot{\vec{v}}^I_{v/c} = -\frac{\mu}{r^3}\vec{r} + \frac{T_m \eta}{m}\vec{1}_T \tag{8}$$

Where T_m is the maximum potential thrust of the vehicle, η is the current commanded throttle, m is the current vehicle mass, $\vec{1}_T$ is the unit thrust vector direction (also referred by in this document as $\hat{u}$), and μ is the gravitational parameter of the central body.

It is important to note here that the system of equations is impartial to the phase of the trajectory. During powered flight portions of the trajectory, in phase i, η is optimized such that the following constraint is enforced:

$$\eta_{lb_i} \leq \eta(t)_i \leq \eta_{ub_i} \tag{9}$$

where η_{lb_i} and η_{ub_i} are constant for a specific phase. It will be shown later that all partials pertaining to $\hat{u}$ when $\eta = 0$ will also be 0.

2.3 Objective Functions

Two common objective functions will be discussed throughout this document. The first one represents the minimum control problem, where the objective function

$$J_{mc} = \int_0^T \left[\frac{T_m \eta(\tau)}{m(\tau)} \right]^2 d\tau \tag{10}$$

seeks to minimize the square of acceleration over a period of time T. Note that the time of flight T is different from the maximum thrust T_m. This objective function can be expanded to the multi-phase problem objective function as follows:

$$J_{mc_{tot}} = \sum_{i=1}^{P} \kappa_i \int_0^{T_i} \left[\frac{T_{m_i} \eta(\tau)_i}{m(\tau)_i} \right]^2 d\tau \tag{11}$$

where P is the total number of flight phases of time T_i each, and κ_i is a weighting parameter for phase i. Note also that each phase i starts at relative time $t_i = 0$ and ends at time $t_i = T$.

The second objective function represents the minimum fuel problem, expressed as:

$$J_{mf} = \int_0^T \eta(\tau) d\tau \tag{12}$$

Expanded to multi-phase form:

$$J_{mf_{tot}} = \sum_{i=1}^{P} \kappa_i \int_0^{T_i} \eta(\tau)_i d\tau \tag{13}$$

The terms κ_i allow for a heavier weighting on propellant usage of a specific stage. In the special case of a two-phase descent where the main objective is only final mass to surface of the descent stage, the objective function would take the form of equation 13 with only $\kappa_P = 1$.

3 Direct Collocation Method

3.1 Collocation Overview

The decision vector for this type of multi-phase collocation is large, but not untraceable. For sake of clarity, a P-phase collocation method example will be shown with n collocation points in each phase (note: the amount of collocation points can vary per phase). It is also important to note that this matrix can be rearranged in any fashion, as some forms may bode well for NLP solving schemes that require specific gradient formats.

$$\vec{F} = \begin{bmatrix} J \\ \vec{R}_0 \\ \vec{V}_0 \\ m_{0_{[0...P]}} \\ \vec{R}_f \\ \vec{V}_f \\ \vec{R}_{m_{[0\to P]}} \\ \vec{V}_{m_{[0\to P]}} \\ \vec{R}_{c_{[0...P]}} \\ \vec{V}_{c_{[0...P]}} \\ \vec{m}_{c_{[0...P]}} \\ \vec{U}_{mag_{[0...P]}} \\ m_{f_{[0...P]}} \\ \vec{\eta}_{lb_{[0...P]}} \\ \vec{\eta}_{ub_{[0...P]}} \\ \nu_{lb_0} \\ \nu_{ub_0} \\ T_{lb_{[0...P]}} \\ T_{ub_{[0...P]}} \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 3 \\ 3 \\ P? \\ 3 \\ 3 \\ 3(P-1) \\ 3(P-1) \\ 3(n_0-1)+...+3(n_P-1) \\ 3(n_0-1)+...+3(n_P-1) \\ (n_0-1)+...+(n_P-1) \\ n_0+...+n_p \\ P \\ n_0+...+n_p \\ n_0+...+n_p \\ 1 \\ 1 \\ P \\ P \end{bmatrix} \tag{15}$$

$$\vec{X}_{dv} = \begin{bmatrix} \vec{R}_{[x...z]_{[0...P]}} \\ \vec{V}_{[x...z]_{[0...P]}} \\ \vec{U}_{[x...z]_{[0...P]}} \\ \vec{m}_{[0...P]} \\ \vec{\eta}_{[0...P]} \\ \nu_0 \\ T_{[0...P]} \end{bmatrix} \rightarrow \begin{bmatrix} 3n_0+...+3n_p \\ 3n_0+...+3n_p \\ 3n_0+...+3n_p \\ n_0+...+n_p \\ n_0+...+n_p \\ 1 \\ P \end{bmatrix} \tag{14}$$

Assume an example of a 3-phase trajectory with each phase containing a total of 15 collocation points. This would lead to a decision vector with a length of — and a fitness/constraint vector of length —. The constraint vector components will be described throughout the subsequent sections. The true form of the constraints may vary at mechanization depending on the NLP solver selected.

3.2 Path Constraints

Under trapezoidal collocation transcription, the path constraints of N collocation points will take on the following form within any arbitrary phase of the trajectory:

$$\vec{R}_{k+1} - \vec{R}_k = \frac{1}{2}\Delta t(\vec{V}_{k+1} + \vec{V}_k) \qquad k \in \{1, ..., N\} \tag{16}$$

$$\vec{V}_{k+1} - \vec{V}_k = \frac{1}{2}\Delta t\left[\frac{\mu}{R_{k+1}^3}\vec{R}_{k+1} + \frac{\mu}{R_k^3}\vec{R}_k + \frac{T_m\eta_k}{m_k}\hat{U}_k + \frac{T_m\eta_{k+1}}{m_{k+1}}\hat{U}_{k+1}\right] \qquad k \in \{1, ..., N\} \tag{17}$$

$$m_k - m_{k+1} = \frac{1}{2}\Delta t\frac{T_m}{g_0 I_{sp}}(\eta_{k+1} + \eta_k) \qquad k \in \{1, ..., N\} \tag{18}$$

3.3 Boundary Constraints

3.3.1 Initial and Final Trajectory States

It is required in this nonlinear program that the final targeted position and velocity remain constant in the planet-fixed frame. This requirement transcribes itself into a nonlinear equality constraint on the last collocation point of the last phase of the trajectory. In phase P of P with N collocation points:

$$\vec{R}^I_{des_f} = \vec{R}^I_{t/c} = C^I_{UEN}\vec{R}^{UEN}_{t/c} \tag{19}$$

$$\vec{V}^I_{des_f} = \vec{V}^I_{t/c} + \vec{\Omega} \times \vec{R}^I_{t/c} = C^I_{UEN}\vec{V}^{UEN}_{t/c} + \vec{\Omega} \times \vec{R}^I_{t/c} \tag{20}$$

Where $\vec{R}^{UEN}_{t/c}$ represents the target position vector expressed in the UEN frame (which is a fixed offset from the LS frame).

Also, for this problem, it is assumed that the initial orbit of the vehicle is fixed - that is, of the 6 necessary parameters to fully constrain an orbital state expressed in Kelperian elements, only the initial true anomaly of the orbit is allowed to be optimized. This allows for simpler derivation of partials, but requires that the vehicle's initial state always begin in a reference orbital plane. Choosing true anomaly ν as a decision vector parameter is the simplest way to approach this for both constraint and partial development. Given a certain right-ascension of the ascending node Ω, inclination i, eccentricity e, semi-major axis a and argument of periapsis ω, the starting position and velocity constraint can be expressed as a function of ν through the following equations:

$$\vec{R}_{des_0} = R_z(-\Omega)R_x(-i)R_z(-\omega)\vec{o} \tag{21}$$

$$\vec{R}_{des_0} = R_z(-\Omega)R_x(-i)R_z(-\omega)\dot{\vec{o}} \tag{22}$$

Where:

$$E = 2\arctan\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2}\right) \tag{23}$$

$$r_c = a(1 - e\cos E) \tag{24}$$

$$\vec{o} = r_c \begin{bmatrix} \cos\nu \\ \sin\nu \\ 0 \end{bmatrix} \qquad \dot{\vec{o}} = \frac{\sqrt{\mu a}}{r_c}\begin{bmatrix} -\sin E \\ \sqrt{1-e^2}\cos E \\ 0 \end{bmatrix} \tag{25}$$

3.3.2 Phase Boundary Constraints

At the boundaries of each phase, a constraint is set on the state of the vehicle. In other words, for phase i with n segments and phase $i+1$ as the subsequent phase, the following constraints must be met:

$$\vec{R}_{i_N} = \vec{R}_{i+1_1} \tag{26}$$

$$\vec{V}_{i_N} = \vec{V}_{i+1_1} \tag{27}$$

For mass, there are two forms of constraints that can be employed. The first is representative of mass continuity, when a phase break does not represent a staging of the vehicle:

$$m_{i_N} = m_{i+1_1} + \Delta m_{i\rightarrow i+1} \tag{28}$$

where $\Delta m_{i\rightarrow i+1}$ is a *fixed* mass drop that is allowed to occur from phase i to phase $i+1$. This phase delta could provide use if a fixed-mass component of the vehicle must be jettisoned at a phase boundary. Another form that is more synonymous to a staging of a vehicle requires an equality constraint on the initial mass of phase $i+1$:

$$mp_{i+1_1} = mp_{init_{i+1}} \tag{29}$$

Lastly, it is clear that an inequality constraint is required to ensure the vehicle does not spend more propellant in a specific stage than it actually has. In phase i with N collocation points, this represents itself as:

$$mp_{i_N} >= 0 \tag{30}$$

3.3.3 Other Constraints

An altitude constraint is optional, because in most cases, the most optimal trajectory for a descent is not a trajectory that intersects the planet before necessary. However, one can impose an altitude restriction on certain portions of flight using the following equation:

$$R_x^2 + R_y^2 + R_z^2 - R_{min}^2 >= 0 \tag{31}$$

Where R_{min} can vary each phase. This constraint is not covered in this derivation. Also required are throttling constraints:

$$\eta_{lb} <= \eta_k \tag{32}$$

3.4 Gradient

The gradient is practically always the most desirable key to a nonlinear program that usually speeds up a solver by orders of magnitude. Luckily, for the problem described in the sections above, the partials can be solved analytically with a healthy amount of dedication and patience. The gradient for this problem, i.e. the partial of the fitness vector (the objective function and all constraints) with respect to the decision vector, is outlined below.

$$\frac{\partial \vec{X}}{\partial \vec{F}} = \begin{bmatrix}
[0] & [0] & [0] & \frac{\partial J}{\partial \vec{m}_{[1\ldots P]}} & \frac{\partial J}{\partial \vec{\eta}_{[1\ldots P]}} & [0] & \frac{\partial J}{\partial T_{[1\ldots P]}} \\
\frac{\partial \vec{R}_0}{\partial \vec{R}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] & \frac{\partial \vec{R}_0}{\partial \nu_0} & [0] \\
[0] & \frac{\partial \vec{V}_0}{\partial \vec{V}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & \frac{\partial \vec{V}_0}{\partial \nu_0} & [0] \\
[0] & [0] & [0] & \frac{\partial m_{0_{[1\ldots P]}}}{\partial \vec{m}_{[1\ldots P]}} & [0] & [0] & [0] \\
\frac{\partial \vec{R}_f}{\partial \vec{R}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] & [0] & \frac{\partial \vec{R}_f}{\partial T_{[1\ldots P]}} \\
[0] & \frac{\partial \vec{V}_f}{\partial \vec{V}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] & \frac{\partial \vec{V}_f}{\partial T_{[1\ldots P]}} \\
\frac{\partial \vec{R}_{m_{[1\to 2\ldots P-1\to P]}}}{\partial \vec{R}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] & [0] & [0] \\
[0] & \frac{\partial \vec{V}_{m_{[1\to 2\ldots P-1\to P]}}}{\partial \vec{V}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] & [0] \\
\frac{\partial \vec{R}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{R}_{[xyz]_{[1\ldots P]}}} & \frac{\partial \vec{R}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{V}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] & \frac{\partial \vec{R}_{c_{[xyz]_{[1\ldots P]}}}}{\partial T_{[1\ldots P]}} \\
\frac{\partial \vec{V}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{R}_{[xyz]_{[1\ldots P]}}} & \frac{\partial \vec{V}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{V}_{[xyz]_{[1\ldots P]}}} & \frac{\partial \vec{V}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{U}_{[xyz]_{[1\ldots P]}}} & \frac{\partial \vec{V}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{m}_{[1\ldots P]}} & \frac{\partial \vec{V}_{c_{[xyz]_{[1\ldots P]}}}}{\partial \vec{\eta}_{[1\ldots P]}} & [0] & \frac{\partial \vec{V}_{c_{[xyz]_{[1\ldots P]}}}}{\partial T_{[1\ldots P]}} \\
[0] & [0] & [0] & \frac{\partial \vec{m}_{c_{[1\ldots P]}}}{\partial \vec{m}_{[1\ldots P]}} & \frac{\partial \vec{m}_{c_{[1\ldots P]}}}{\partial \vec{\eta}_{[1\ldots P]}} & [0] & \frac{\partial \vec{m}_{c_{[1\ldots P]}}}{\partial T_{[1\ldots P]}} \\
[0] & [0] & \frac{\partial \vec{U}_{mag_{[1\ldots P]}}}{\partial \vec{U}_{[xyz]_{[1\ldots P]}}} & [0] & [0] & [0] & [0] \\
[0] & [0] & [0] & \frac{\partial m_{f_{[1\ldots P]}}}{\partial \vec{m}_{[1\ldots P]}} & [0] & [0] & [0] \\
[0] & [0] & [0] & [0] & \frac{\partial \vec{\eta}_{lb_{[1\ldots P]}}}{\partial \vec{\eta}_{[1\ldots P]}} & [0] & [0] \\
[0] & [0] & [0] & [0] & \frac{\partial \vec{\eta}_{ub_{[1\ldots P]}}}{\partial \vec{\eta}_{[1\ldots P]}} & [0] & [0] \\
[0] & [0] & [0] & [0] & [0] & \frac{\partial T_{lb_{[1\ldots P]}}}{\partial T_{[1\ldots P]}} & [0] \\
[0] & [0] & [0] & [0] & [0] & \frac{\partial T_{ub_{[1\ldots P]}}}{\partial T_{[1\ldots P]}} & [0] \\
[0] & [0] & [0] & [0] & [0] & [0] & \frac{\partial \nu_{lb_0}}{\partial \nu_0} \\
[0] & [0] & [0] & [0] & [0] & [0] & \frac{\partial \nu_{ub_0}}{\partial \nu_0}
\end{bmatrix} \tag{33}$$

The equation subscripts in the next section often take on a standardized form for readibility. For example, take the following partial:

$$\frac{\partial R_{c_{X_{1_1}}}}{\partial R_{X_{1_2}}} \tag{34}$$

This represents partial of the path constraint for R_X for phase 1 and for collocation point 1, with respect to the R_X collocation point 2 in phase 1. Also, when not specified, the term Δt for a specific phase i represents:

$$\Delta t_i = T_i / N_i \tag{35}$$

Also, when not specified, for phase i, $T_m = T_{m_i}$ and $n = n_i$.

3.4.1 Objective Function - Minimum Control

This is for the minimum control problem of the entire trajectory for all phases with equal weighting, i.e.:

$$J = \sum_{i=1}^{P} \sum_{k=1}^{n_i - 1} \frac{1}{2} \Delta t_i \left[\left(\frac{T_{m_i} \eta_{i_k}}{m_{i_k}} \right)^2 + \left(\frac{T_{m_i} \eta_{i_{k+1}}}{m_{i_{k+1}}} \right)^2 \right] \tag{36}$$

Calculating required partials:

$$\frac{\partial J}{\partial \vec{m}_{[1 \ldots P]}} = \underset{(1) \times (n_1 + \ldots + n_P)}{\left[\frac{\partial J}{\partial \vec{m}_1} \quad \frac{\partial J}{\partial \vec{m}_2} \quad \cdots \quad \frac{\partial J}{\partial \vec{m}_P} \right]} \tag{37}$$

$$\frac{\partial J}{\partial \vec{m}_1} = \underset{(1) \times (n_1)}{\left[-\Delta t_1 \frac{T_{m_1}^2 \eta_{1_1}^2}{m_{1_1}^3} \quad -2\Delta t_1 \frac{T_{m_1}^2 \eta_{1_2}^2}{m_{1_2}^3} \quad \ldots \quad -2\Delta t_1 \frac{T_{m_1}^2 \eta_{1_{n-1}}^2}{m_{1_{n-1}}^3} \quad -\Delta t_1 \frac{T_{m_1}^2 \eta_{1_n}^2}{m_{1_n}^3} \right]} \tag{38}$$

This pattern is repeated for $\frac{\partial J}{\partial \vec{m}_2} \ldots \frac{\partial J}{\partial \vec{m}_P}$.

$$\frac{\partial J}{\partial \vec{\eta}_{[1 \ldots P]}} = \underset{(1) \times (n_1 + \ldots + n_P)}{\left[\frac{\partial J}{\partial \vec{\eta}_1} \quad \frac{\partial J}{\partial \vec{\eta}_2} \quad \cdots \quad \frac{\partial J}{\partial \vec{\eta}_P} \right]} \tag{39}$$

$$\frac{\partial J}{\partial \vec{\eta}_1} = \underset{(1) \times (n_1)}{\left[\Delta t_1 \frac{T_{m_1}^2 \eta_{1_1}}{m_{1_1}^2} \quad 2\Delta t_1 \frac{T_{m_1}^2 \eta_{1_2}}{m_{1_2}^2} \quad \ldots \quad 2\Delta t_1 \frac{T_{m_1}^2 \eta_{1_{n-1}}}{m_{1_{n\_1}}^2} \quad \Delta t_1 \frac{T_{m_1}^2 \eta_{1_n}}{m_{1_n}^2} \right]} \tag{40}$$

This pattern is repeated for $\frac{\partial J}{\partial \vec{\eta}_2} \ldots \frac{\partial J}{\partial \vec{\eta}_P}$.

$$\frac{\partial J}{\partial T_{[1 \ldots P]}} = \underset{(1) \times (P)}{\left[\frac{J_1}{T_1} \quad \frac{J_2}{T_2} \quad \cdots \quad \frac{J_n}{T_n} \right]} \tag{41}$$

Where

$$J_1 = \left\{ \sum_{k=1}^{n_i - 1} \frac{1}{2} \Delta t_i \left[\left(\frac{T_{m_i} \eta_{i_k}}{m_{i_k}} \right)^2 + \left(\frac{T_{m_i} \eta_{i_{k+1}}}{m_{i_{k+1}}} \right)^2 \right] \right\}_{i=1} \tag{42}$$

3.4.2 Initial Position Equality Constraint

The constraint takes on the form:

$$\vec{R}_0 = \vec{R}_{1_1} - \vec{R}_{des_0} \tag{43}$$

$$\frac{\partial \vec{R}_0}{\partial \vec{R}_{[XYZ]_{[1\ldots P]}}} = \underbrace{\begin{bmatrix} \frac{\partial \vec{R}_0}{\partial \vec{R}_{X_{[1\ldots P]}}} & \frac{\partial \vec{R}_0}{\partial \vec{R}_{Y_{[1\ldots P]}}} & \frac{\partial \vec{R}_0}{\partial \vec{R}_{Z_{[1\ldots P]}}} \end{bmatrix}}_{(3)\times(3n_1+\ldots+3n_P)} \tag{44}$$

$$\frac{\partial \vec{R}_0}{\partial \vec{R}_{X_{[1\ldots P]}}} = \underbrace{\begin{bmatrix} \frac{\partial \vec{R}_0}{\partial \vec{R}_{X_1}} & \frac{\partial \vec{R}_0}{\partial \vec{R}_{X_2}} & \ldots & \frac{\partial \vec{R}_0}{\partial \vec{R}_{X_P}} \end{bmatrix}}_{(3)\times(n_1+\ldots+n_p)} = \begin{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{(3)\times(n_1)} & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{(3)\times(n_2)} & \ldots & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{(3)\times(n_P)} \end{bmatrix} \tag{45}$$

Similarly,

$$\frac{\partial \vec{R}_0}{\partial \vec{R}_{Y_{[1\ldots P]}}} = \begin{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \end{bmatrix}}_{(3)\times(n_1)} & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{(3)\times(n_2)} & \ldots & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{(3)\times(n_P)} \end{bmatrix} \tag{46}$$

$$\frac{\partial \vec{R}_0}{\partial \vec{R}_{Z_{[1\ldots P]}}} = \begin{bmatrix} \underbrace{\begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{bmatrix}}_{(3)\times(n_1)} & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{(3)\times(n_2)} & \ldots & \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{(3)\times(n_P)} \end{bmatrix} \tag{47}$$

$$\frac{\partial \vec{R}_0}{\partial \nu_0} = -\frac{\partial \vec{R}_{des_0}}{\partial \nu_0} = -R_z(-\Omega)R_x(-i)R_z(-\omega)\frac{\partial}{\partial \nu}\left[\vec{o}\right] \tag{48}$$

$$\frac{\partial}{\partial \nu}\left[\vec{o}\right] = \frac{\partial}{\partial \nu}\left\{ r_c \begin{bmatrix} \cos\nu \\ \sin\nu \\ 0 \end{bmatrix} \right\} = \frac{\partial r_c}{\partial \nu}\begin{bmatrix} \cos\nu \\ \sin\nu \\ 0 \end{bmatrix} + r_c \begin{bmatrix} -\sin\nu \\ \cos\nu \\ 0 \end{bmatrix} \tag{49}$$

$$\frac{\partial r_c}{\partial \nu} = e\sin(E)\frac{\partial E}{\partial \nu} \tag{50}$$

$$\frac{\partial E}{\partial \nu} = \frac{2}{1+B^2}\frac{\partial B}{\partial \nu} \qquad B = \sqrt{\frac{1-e}{1+e}}\tan\frac{\nu}{2} \tag{51}$$

$$\frac{\partial B}{\partial \nu} = \frac{1}{2}\sqrt{\frac{1-e}{1+e}}\sec^2\frac{\nu}{2} \tag{52}$$

3.4.3 Initial Velocity Constraint

The constraint takes on the form:

$$\vec{V}_0 = \vec{V}_{1_1} - \vec{V}_{des_0} \tag{53}$$

$$\frac{\partial \vec{V}_0}{\partial \vec{V}_{[xyz]_{[1\ldots P]}}} = \frac{\partial \vec{R}_0}{\partial \vec{R}_{[XYZ]_{[1\ldots P]}}} \tag{54}$$

$$\frac{\partial \vec{V}_0}{\partial \nu_0} = -\frac{\partial \vec{V}_{des_0}}{\partial \nu_0} = -R_z(-\Omega)R_x(-i)R_z(-\omega)\frac{\partial}{\partial \nu}\left[\dot{\vec{o}}\right] \tag{55}$$

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\left[\dot{\vec{o}}\right] = \sqrt{\mu a}\frac{\mathrm{d}}{\mathrm{d}\nu}\left\{\begin{bmatrix} \frac{-\sin E}{r_c} \\ \frac{\sqrt{1-e^2}\cos E}{r_c} \\ 0 \end{bmatrix}\right\} \tag{56}$$

3.4.4 Initial Mass Constraints

For the special case of initialization mass:

$$m_{0_1} = m_{1_1} - m_{tot} \tag{57}$$

$$\frac{\partial m_{0_1}}{\partial \vec{m}_{[1\ldots P]}} \tag{58}$$