Long Memory Overview

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What is Long Memory?

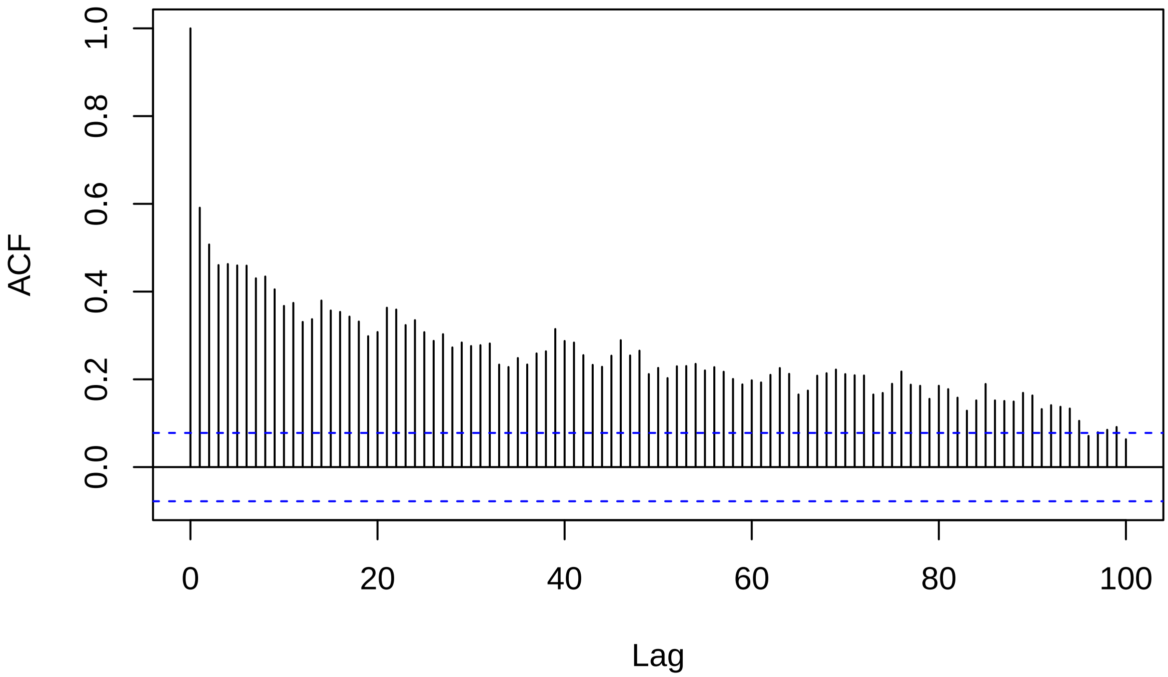
Long memory in time series is the property of a time series where the autocorrelation function (ACF) decays slowly, indicating a large amount of dependence between observations at distant time points. In layman's terms, it means that past values of the time series have a lasting impact on future values, beyond what might be expected from short-term fluctuations. It is a property of time series that comes up many disciplines ranging from hydrology to finance to telecommunications and many more. The concept of long memory, formalized through the Hurst exponent and fractional calculus, provides a powerful framework for analyzing and modeling time series data with memory-dependent properties.

Glaciers deposit yearly layers of sand and silt duringspring melting seasons, which can be reconstructed yearly over a period ranging from the time deglaciation began in New England (about 12,600 years ago) to the time it ended (about 6,000 years ago). These sedimentary deposits, called varves, can be used as indicators for paleoclimatic indicators, such as temperature due to the fact that in a warm year, more sand and silt are deposited from the glacier. The dataset *varve* has thicknesses of the yearly varves collected from one location in Massachusetts for 634 years, beginning 11,834 years ago. [2]

```
library(astsa)
```

```
## Warning: package 'astsa' was built under R version 4.2.3
```

```
lvarve=(log(varve)) #log transform the data
acf(lvarve,lag.max = 100,main="") #plot acf of data
```



This figure shows the acf of a log transformed time series of glacial varve data that exhibits typical long memory behavior of the acf decaying slowly over long periods of time.

Properties of Long Memory

A few properties of Long Memory are as follows

- **Slow Decay of Autocorrelation:** I would argue this is the the defining feature of long memory is the slow decay of autocorrelation. In long memory processes, correlations persists over a large number of lags, indicating that past values have a notable influence on future observations.
- **Non-stationarity:** Long memory processes are often non-stationary, in contrast to something like white noise that is stationary. Nonstationarity indicates that the statistical properties of the series are not constant and change with time. While the mean and variance may remain constant, the autocorrelation changes, reflecting the persistence of past influences. This non-stationarity calls for different techniques and modeling than traditional time series analysis to capture the long memory characteristics
- **Persistent Trends**: Long memory time series often show persistent trends. This means that trends usually continue for long periods of time. This persistence is a defining feature of long memory and indicates that past values have a strong influence on future values. It allows for predicting of future trends and values in many disciplines such as hydrology, markets, and biology

Mathematical Foundations of Long Memory

Fractional Noise

Fractional Gaussian Noise is a stochastic process characterized by its long memory characteristics, Gaussian distribution, and self-similarity. It is different from classic white noise because of the long memory characteristic which causes it to not be independent as white noise is typically assumed to be independent and identically distributed (i.i.d.).A useful tool to quantify long memory is the Hurst exponent. The Hurst exponent named after the hydrologist Harold Edwin Hurst, who first discovered and implemented it it in the 1950s while studying the long-term storage capacity of reservoirs. Interpretations for the Hurst exponent are as follows:

- $H = .5$ indicates random behavior and short term memory comparable to Gaussian white noise
- $H < .5$ where a decrease in the value is likely to be followed by an increase, and vice versa.
- $H > .5$ indicates persistent behavior and long term trends over time which is understood as long memory

Calculate the Hurst Exponent The Hurst exponent can be calculated using rescaled range analysis. Rescaled range looks at a data series and determines the persistence or mean-reverting tendencies within that data.[3] Its purpose is to provide an assessment of how the apparent variability of a series changes with the length of the time-period being considered.For a time series $X = X_1, X_2, ...X_n$ the method is as follows.[4]

(1) Caclulate mean value m

$$m = \frac{1}{n}\sum_{i=1}^{n} X_i$$

(2) Calculate mean adjusted series Y

$$Y_t = X_t - m$$
$$t = 1, 2, ..., n$$

(3) Calculate cumulative deviate series Z

$$Z_t = \sum_{i=1}^{t} Y_i$$
$$t = 1, 2, ..., n$$

(4) Calculate range series R

$$R_t = max(Z_1, Z_2, ..., Z_t) - min(Z_1, Z_2, ..., Z_t)$$
$$t = 1, 2, ..., n$$

(5) Calculate standard deviation series S

$$S_t = \sqrt{\frac{1}{t}\sum_{i=1}^{t}(X_i - u)^2}$$

$$t = 1, 2, ..., n$$

Here u is the mean value from X_1 to X_t

(6) Calculate rescaled range series (R/S)

$$(R/S)_t = R_t/S_t$$

$$t = 1, 2, ..., n$$

Hurst found that (R/S) scales by power-law as time increases, which indicates

$$(R/S)_t = c * t^H$$

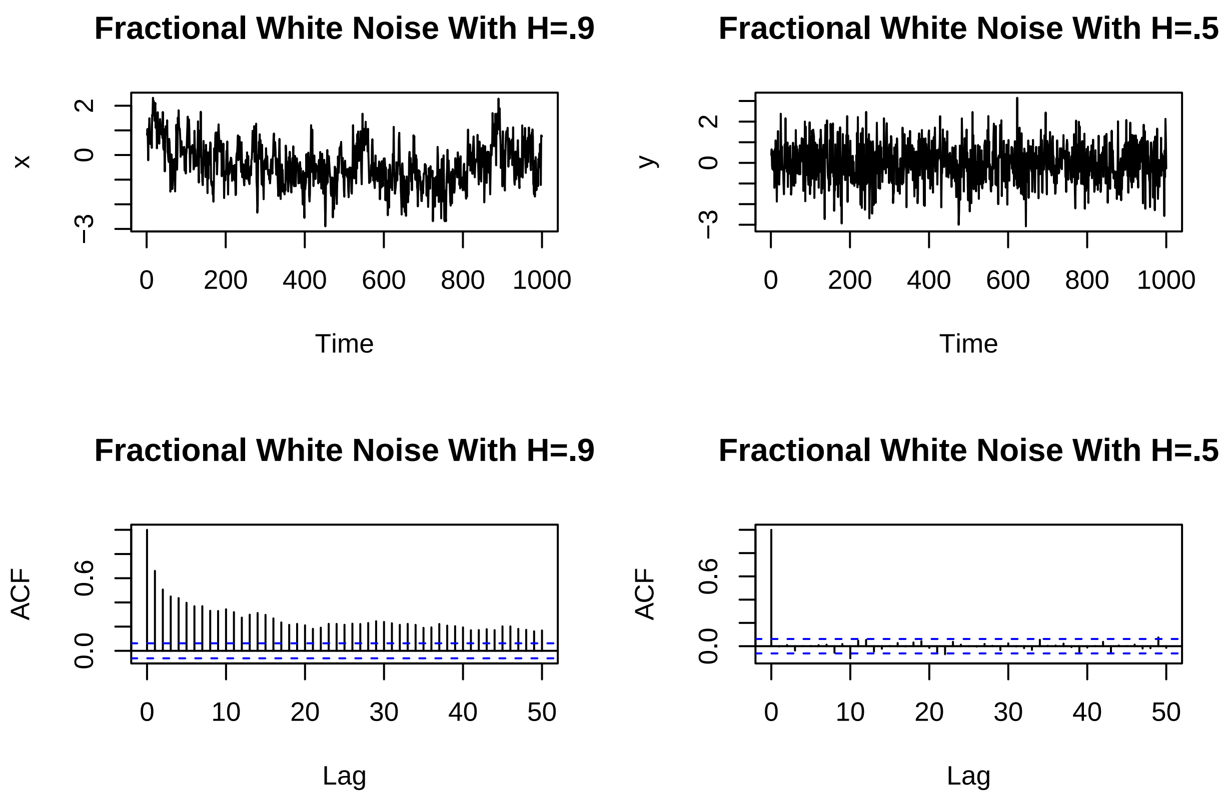
Here c is a constant and H is the Hurst exponent. To estimate the Hurst exponent, we plot (R/S) versus t in log-log axes. The slope of the regression line approximates the Hurst Exponent.

```
library(longmemo)
```

Simulating Fractional Noise

```
## Warning: package 'longmemo' was built under R version 4.2.3
```

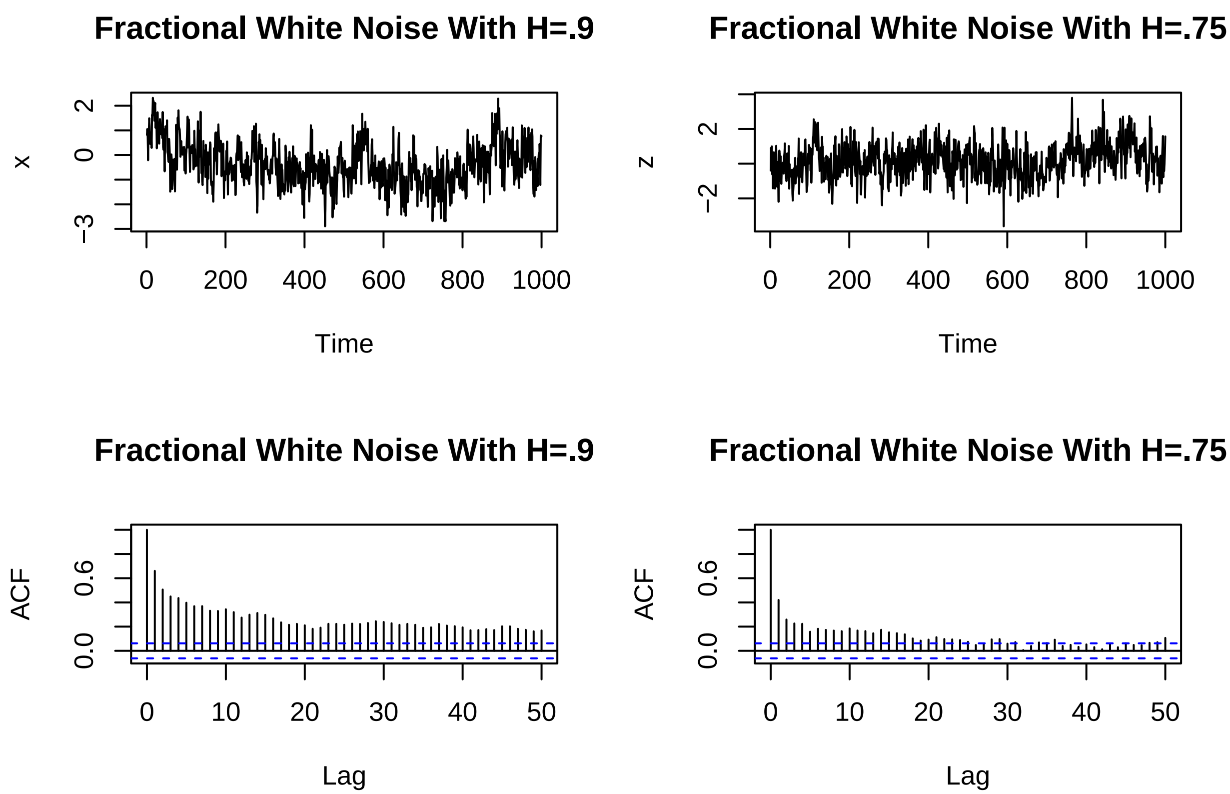
```
set.seed(451)
par(mfrow=c(2,2))
x=simFGN0(1000, .9) # Simulating the fractional noise
y=simFGN0(1000,.5)
plot.ts(x,main="Fractional White Noise With H=.9")
plot.ts(y, main="Fractional White Noise With H=.5")
acf(x,lag.max = 50,main="Fractional White Noise With H=.9")
acf(y,lag.max = 50, main = "Fractional White Noise With H=.5")
```



As we can see from the graphs the Fractional Noise generate with H= .9 shows trends in the graph and the ACF shows significant values up to lap 50 which shows that slowly decay indicative of long memory. On the contrary when we set H = .5 the noise looks the way we would expect it to as it looks completely random with no trends and we see that the acf is that of Gaussian white noise.

It will also be useful to compare two fractional Gaussian noise series with differing H values with both being > .5

```
set.seed(20)
library(longmemo)
par(mfrow=c(2,2))
z=simFGN0(1000,.75)
plot.ts(x,main="Fractional White Noise With H=.9")
plot.ts(z, main="Fractional White Noise With H=.75")
acf(x,lag.max = 50,main="Fractional White Noise With H=.9")
acf(z,lag.max = 50, main = "Fractional White Noise With H=.75")
```



Due to .9 > .75 the series with H = .9 shows more persistent behavior and long memory characteristics as the acf is decaying slowly and still has significant values up to lag 50. This is compared to the noise with H = .75 which does show some persistent behavior but dies out at around lag 20.

It should be noted that the function simFNG0 uses the Davies and Harte method of generating fractional Gaussian noise. There are many other methods such as Wavelet-based Methods, Fourier Filtering Method, The Hosking method, and several others. Each method has its own computational complexity, memory requirements, and accuracy. The choice of method depends on the specific application , computational resources, desired statistical properties of the generated noise, and the efficiency of the algorithm.

ARFIMA Models

A common method of analyzing Long Memory Data is the use of Autoregressive Fractionally Integrated Moving Average Models (ARFIMA) which are an expansion on ARIMA models that allow for the fractional differencing to account for long memory. This is in contrast to the traditional stationary ARMA process as they tend to have short memory. Due to the model implementing fractional differencing we assign d a non-integer value such as .5. This model is preferred to the typical ARIMA model as differencing with values or 1 or 2 can be too extreme on the data and cause it to lose its long memory characteristic that is very useful in forecasting and predicting future values. ARFIMA provides a solution to this and thismodel tends to more effectively model long memory time series and their characteristics. This can be given by the following model:[1]

$$\phi(B)(1-B)^d(x_t-\mu)=\theta(B)w_t$$

The model contains $\phi(B)$ and $\theta(B)$ which are the AR and MA components respectively This model contains a mean adjusted component which is common amongst ARFIMA models but the model can also be used without a mean adjusted component

$$\phi(B)(1-B)^d(x_t) = \theta(B)w_t$$

We can also write a simplier version of the model

$$\phi(B)\pi_d(B)(x_{t-\mu}) = \theta(B)w_t$$

In this model we can think of π_d as an operator that corresponds to differencing similar to how there are AR and MA operators

We can expand upon this using binomial expansion.[1]

$$(1-B)^d x_t = \sum_{j=0}^{\infty} \pi_j B^j x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$$

Where

$$\pi_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}$$

The ACF of x_t is given as

$$\rho(h) = \frac{\Gamma(h+d)\Gamma(1-d)}{\Gamma(h-d+1)\Gamma(d)} \sim h^{2d-1}$$

for large h, and from this we can gather that for 0 < d < .5

$$\sum_{h=-\infty}^{\infty} |\rho(h)| = \infty$$

This is where we can see the term long memory makes a lot of sense as since the ACF decays so slowly when we add all the values up the infinite sum equals infinity as opposed to an ACF with a cutoff value at an early lag.

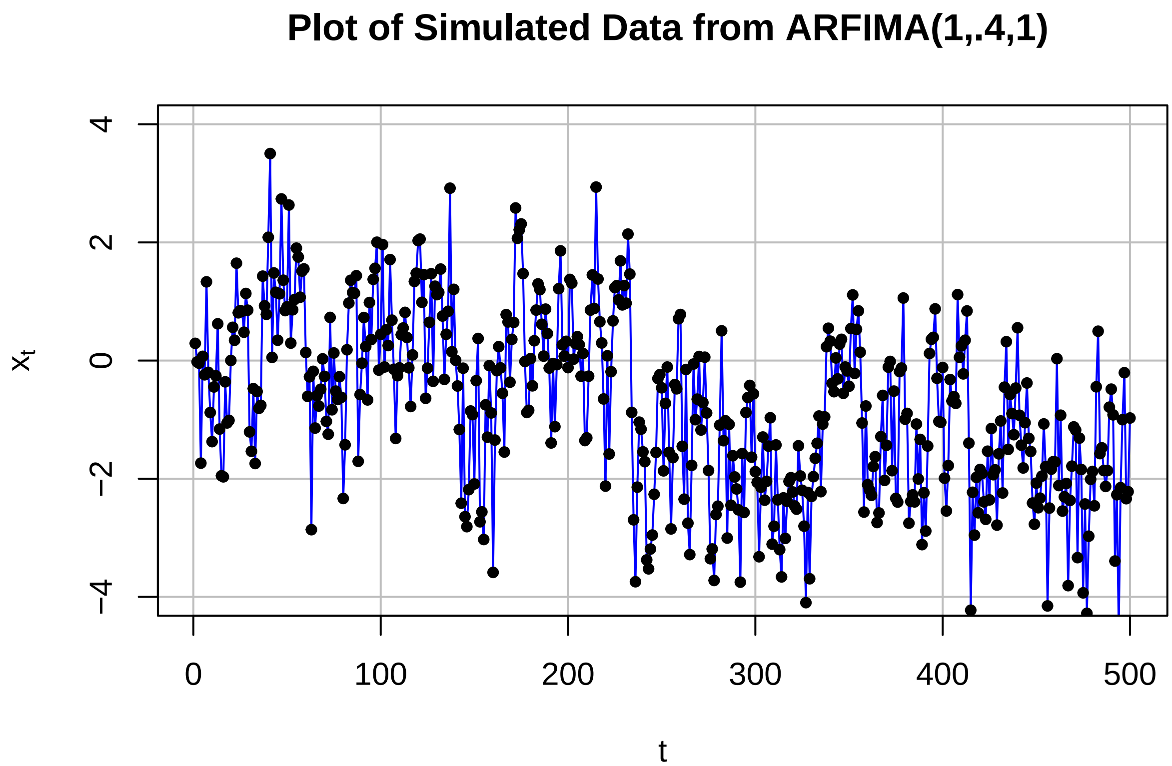
Simulating an ARFIMA model In r it is easy to simulate data from an ARFIMA model by using the $fracdiff$ package. We will simulate an ARFIMA(1,.35,0) with $\phi = .3$

```
set.seed(700)
library(fracdiff)
```

```
## Warning: package 'fracdiff' was built under R version 4.2.3
```

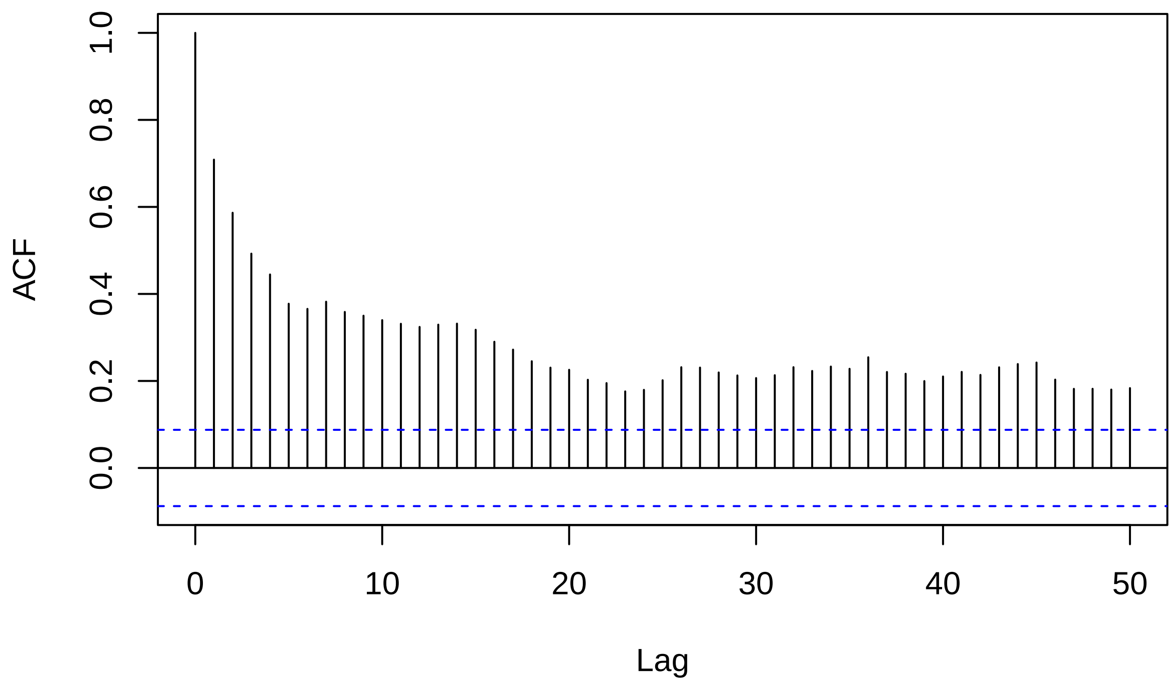
```
simarfima=fracdiff.sim(n=500,ar=.2,ma=NULL,d=.35,rand.gen = rnorm,sd = 1,mu=0)
x=simarfima$series

plot(x, ylab=expression(x[t]),xlab="t",type="l",col="blue",lwd=1, main= " Plot of Simulated Data from AR
points(x=x,pch=20,col="black")
```



The plot of the time series shows what appears to be an underlying oscillating pattern with parts that are decreasing and increasing.

```
acf(x,lag.max = 50,main="")
```



This dataset clearly exhibits long memory as we have significant positive ACF values up to Lag 50 with an overall slow decay of the ACF. It should also be noted that these values are not extremely high acf values but they are significant and positve so it shows long memory as long memory does not need to have large acf values just significant ones.

Examples of Long Memory in the real world

I wanted to end the report off with going a bit more in depth into examples of Long memory in the real world and how the characteristic is practical and useful in the respective topics

Economics:

- ***Economic Indicators:*** Economic indicators like inflation rates, GDP rates and unemployment rates often exhibit long memory behavior. Understanding long memory in economic time series is crucial for forecasting and policy-making.
- ***Stock Prices:*** Stock prices can exhibit long memory, showing clear long term trends as stock market graphs are one of the most recognizable examples of time series that non statisticians are exposed to. Traders often use that characteristic of long memory to predict about the future rise and fall of stock prices

Climate and Weather:

- ***Temperature***: Temperature time series often exhibit long memory due to many weather patterns persisting over long periods of time. Long memory models help in climate modeling, weather forecasting,

and analyzing long term climate change trends.
- ***Rainfall***: Rainfall patterns show long memory, where past rainfall influence and can be used to predict future precipitation. This helps in flood predictions and agricultural planning.

Telecommunications:

- ***Network Traffic***: Network traffic in telecommunications networks often displays long memory behavior. Understanding long memory in network traffic helps in optimization, server capacity managment, quality of service management.

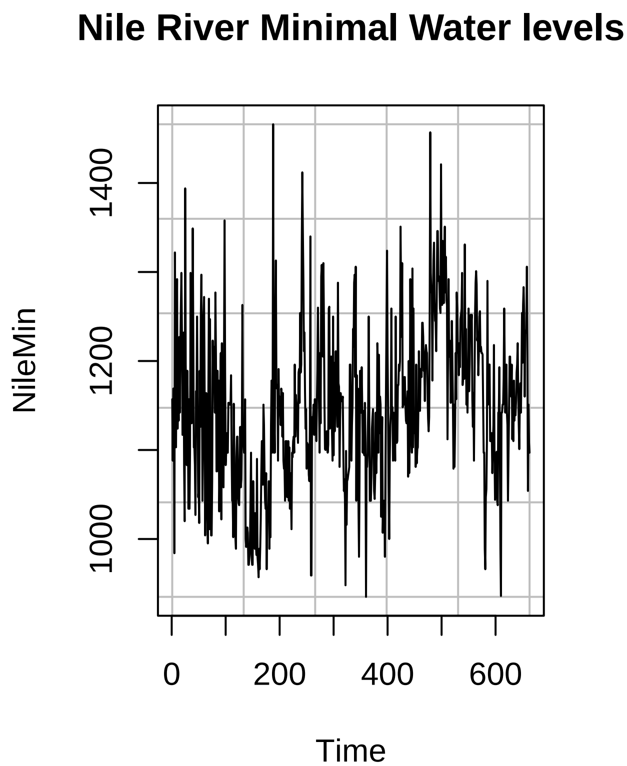
Medicine:

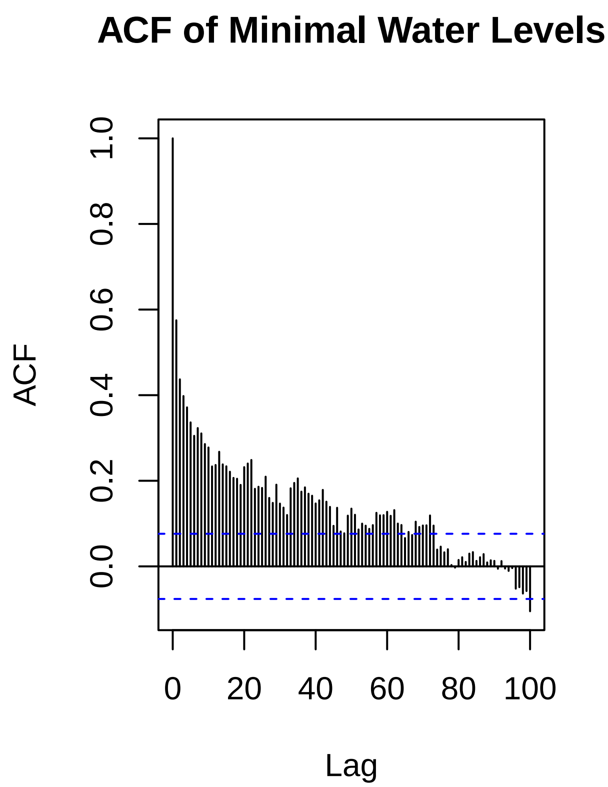
- ***Heart Rate Variability***: Heart rate variability, which is measure of the variation in time between indiviual heartbeats shows long memory properties. Analyzing long memory in heart rate variability helps in understanding cardiovascular health , stress levels , and responses .

Hydrology and Environmental Sciences:

- ***River Flow***: River flows often exhibit long memory behavior due to the persistence of hydrological processes. It was in river flow analysis that long memory was first discovered and explored by English hydrologist and statistician, Harold Edwin Hurst.

```r
data("NileMin")
par(mfrow=c(1,2))
plot(NileMin, main = "Nile River Minimal Water levels",panel.first = grid(col="gray",lty="solid"))
acf(NileMin,lag.max = 100,main="ACF of Minimal Water Levels")
```





Above gives minimal water levels in the Nile River from 622-1284, and its acf which shows clear long memory behavior as the acf slowly decays up to about lag 75.[5]

- ***Air Pollution Levels***: Time series of particulate matter or pollutant air pollution time series, display long memory properties influenced which can shine light on emission levels in certain areas that are causing the pollution or environmental policies that are or are not working.

It is also worth wile to note that with several of these topics and in the real world some may exhibit long memory properties but may not fully exhibit long memory in the classic sense. As is the case with many topics in statistics some portion of analysis is subjective as there is not a easy full proof way to say if a time series has long memory. It requires statistical tests, model selection, data quality, and model assumptions

Citations

1: Shumway, R. H., & Stoffer, D. S. (2017). Time Series Analysis and its applications: With R Examples. Springer.

2: Datasetsource: Tousson, O. (1925). Mémoire sur l'Histoire du Nil; Mémoire de l'Institut d'Egypte. Jan Beran (1994). Dataset no.1, p.20–22.

3: https://www.investopedia.com/terms/r/rescaled-rangeanalysis.asp#:~:text=Rescaled%20range%20analysis%20looks%20a

4: Qian, Bo; Rasheed, Khaled (2004). HURST EXPONENT AND FINANCIAL MARKET PREDICTABILITY. IASTED conference on Financial Engineering and Applications (FEA 2004). pp. 203–209.

5: Dataset sourced from Jones, R.H. (1984). Fitting multivariate models to unequally spaced data. In Time Series Analysis of Irregularly Observed Data, pp. 158-188. E. Parzen, ed. Lecture Notes in Statistics, 25, New York: Springer-Verlag.