Chevalier & Chegg 1985 Galactic Wind Model

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1 Introduction

Galactic winds are astrophysical phenomena observed in galaxies, where high-velocity outflows of gas and dust are expelled from the galactic disks into the surrounding intergalactic medium. These winds play a crucial role in the evolution and dynamics of galaxies by regulating star formation, distributing metals, and influencing the intergalactic environment. Galactic winds are believed to be driven by the combined effects of supernovae explosions, active galactic nuclei, and cosmic rays. The exact mechanisms responsible for launching and accelerating these winds (in particular galaxies) are still under investigation. However, theoretical models, such as the pioneering work by Chevalier and Clegg (1985), have been developed to understand and describe the dynamics of galactic winds. These models make certain assumptions about the wind's geometry, energetics, and physical properties, providing valuable insights into the processes governing these powerful outflows. By studying galactic winds, researchers aim to unravel the complex interplay between galaxies, their interstellar medium, and the larger-scale cosmic environment, shedding light on the formation and evolution of galaxies throughout the universe.

2 Derivation

In their pioneering work, Chevalier and Clegg (1985) introduced a stationary energy-driven galactic wind model that makes certain assumptions. The model presumes a spherically symmetrical wind where gravitational forces are not significant. Here, the total mass and energy input are designated as $\dot{M}$ and $\dot{E}$, respectively. The dynamics of this model are governed by the fluid flow equations given below:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\rho u r^2\right) = q \tag{1}$$

$$\rho u \frac{\mathrm{d}u}{\mathrm{d}r} = -\frac{\mathrm{d}P}{\mathrm{d}r} - qu \tag{2}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left[\rho u r^2\left(\frac{1}{2}u^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho}\right)\right] = Q \tag{3}$$

In these equations, the variables $q = \dot{M}/V$ and $Q = \dot{E}/V$ hold for $r < R$, where $V = \frac{4}{3}\pi R^3$ represents the volume. For $r > R$, both q and Q become zero. Here, u denotes the wind velocity, r represents the radial coordinate, ρ is the density, P stands for the pressure, and $\gamma = 5/3$ is the adiabatic index. The mass and energy input are constrained by the radius R.

2.1 $r < R$ Case:

To streamline the analytical solution, the Mach number $M = u/c_s$ is introduced, where the speed of sound $c_s^2 = \gamma\frac{P}{\rho}$. This enables the momentum equation (Equation 2) to be rewritten as:

$$Mc_s\frac{\mathrm{d}(Mc_s)}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} - \frac{qu}{\rho}, \tag{4}$$

A subsequent simplification leads to:

$$\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r} = \frac{\mathrm{d}(P/\rho)}{\mathrm{d}r} + \frac{P}{\rho}\frac{\mathrm{dln}\rho}{\mathrm{d}r} = \frac{\mathrm{d}\left(c_s^2/\gamma\right)}{\mathrm{d}r} + \frac{c_s^2}{\gamma}\frac{\mathrm{dln}\rho}{\mathrm{d}r}. \tag{5}$$

This allows Equation 4 to be recast in terms of M and c_s as:

$$Mc_s\left[c_s\frac{\mathrm{d}M}{\mathrm{d}r}+M\frac{\mathrm{d}c_s}{\mathrm{d}r}\right]=-\frac{1}{\gamma}\frac{\mathrm{d}c_s^2}{\mathrm{d}r}-\frac{c_s^2}{\gamma}\frac{\mathrm{d}\ln\rho}{\mathrm{d}r}-\frac{qMc_s}{\rho}. \tag{6}$$

By integrating the mass equation (Equation 1) and the energy equation (Equation 3), and recasting them in terms of M and c_s, the following relationship is derived:

$$\frac{1}{2}M^2c_s^2+\frac{c_s^2}{\gamma-1}=\frac{Q}{q}. \tag{7}$$

This equation can be rearranged to express c_s^2 as a function of Q, q, and M:

$$c_s^2=\frac{Q/q}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}. \tag{8}$$

With Equation 8 as a foundation, we can determine the derivatives of c_s and c_s^2 with respect to r:

$$\frac{\mathrm{d}c_s^2}{\mathrm{d}r}=\frac{-(Q/q)M}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\frac{\mathrm{d}M}{\mathrm{d}r} \tag{9}$$

and

$$\frac{\mathrm{d}c_s}{\mathrm{d}r}=\frac{-(Q/q)^{1/2}(M/2)}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^{3/2}}\frac{\mathrm{d}M}{\mathrm{d}r}. \tag{10}$$

To solve for Equation 6, it is necessary to recast the density ρ in terms of M and c_s:

$$\dot{M}=q\frac{4}{3}\pi R^3=4\pi\rho uR^3/r, \tag{11}$$

which allows us to express ρ as:

$$\rho_{r<R}=\frac{qr}{3Mc_s}, \tag{12}$$

that is consistent with Equation A.10 in Zhang (2015). Hence the derivative of ρ with respect to r is,

$$\frac{\mathrm{d}\rho}{\mathrm{d}r}=\frac{q}{3}\left(\frac{1}{Mc_s}-\frac{r}{c_sM^2}\frac{\mathrm{d}M}{\mathrm{d}r}-\frac{r}{c_s^2M}\frac{\mathrm{d}c_s}{\mathrm{d}r}\right). \tag{13}$$

If we plug Equation 13 into Equation 6, we can obtain

$$Mc_s\left[c_s\frac{\mathrm{d}M}{\mathrm{d}r}+M\frac{\mathrm{d}c_s}{\mathrm{d}r}\right]=-\frac{1}{\gamma}\frac{\mathrm{d}c_s^2}{\mathrm{d}r}-\frac{c_s^2}{\gamma r}\left(1-\frac{r}{M}\frac{\mathrm{d}M}{\mathrm{d}r}-\frac{r}{c_s}\frac{\mathrm{d}c_s}{\mathrm{d}r}\right)-\frac{3M^2c_s^2}{r}. \tag{14}$$

Plugging in Equations 8 and 10, the LHS of Equation 14 can be rewritten as

$$\begin{aligned}Mc_s\left[c_s\frac{\mathrm{d}M}{\mathrm{d}r}+M\frac{\mathrm{d}c_s}{\mathrm{d}r}\right]&=M\frac{Q/q}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\frac{\mathrm{d}M}{\mathrm{d}r}-M^2\frac{(Q/q)(M/2)}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\frac{\mathrm{d}M}{\mathrm{d}r}\\&=\left[\frac{M(Q/q)\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)-M^2(M/2)(Q/q)}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\right]\frac{\mathrm{d}M}{\mathrm{d}r}\\&=\left[\frac{(Q/q)\frac{M}{\gamma-1}}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\right]\frac{\mathrm{d}M}{\mathrm{d}r}\end{aligned} \tag{15}$$

Similarly, plugging in Equations 8, 9, and 10, the RHS of Equation 14 can be rewritten as

$$\begin{aligned}&-\frac{1}{\gamma}\frac{\mathrm{d}c_s^2}{\mathrm{d}r}-\frac{c_s^2}{\gamma r}\left(1-\frac{r}{M}\frac{\mathrm{d}M}{\mathrm{d}r}-\frac{r}{c_s}\frac{\mathrm{d}c_s}{\mathrm{d}r}\right)-\frac{3M^2c_s^2}{r}\\&=\frac{M}{\gamma}\frac{Q/q}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\frac{\mathrm{d}M}{\mathrm{d}r}-\frac{1}{\gamma}\left[\frac{Q/q}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\right]\left[\frac{1}{r}-\frac{\mathrm{d}M/\mathrm{d}r}{M}+\frac{(M/2)(\mathrm{d}M/\mathrm{d}r)}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\right]-\frac{3M^2(Q/q)}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)r}\\&=\frac{(\mathrm{d}M/\mathrm{d}r)}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\left[\frac{M}{2\gamma}\frac{Q}{q}+\frac{Q}{q}\left(\frac{M}{2\gamma}+\frac{1}{M\gamma(\gamma-1)}\right)\right]-\frac{(Q/q)}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\left[\frac{1}{\gamma r}+\frac{3M^2}{r}\right].\end{aligned} \tag{16}$$

After equating the LHS and RHS of Equation 14, we would get

$$\left[\frac{(Q/q)\frac{M}{\gamma-1}}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\right]\frac{\mathrm{d}M}{\mathrm{d}r}=\frac{(\mathrm{d}M/\mathrm{d}r)}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\left[\frac{M}{2\gamma}\frac{Q}{q}+\frac{Q}{q}\left(\frac{M}{2\gamma}+\frac{1}{M\gamma(\gamma-1)}\right)\right]-\frac{(Q/q)}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\left[\frac{1}{\gamma r}+\frac{3M^2}{r}\right]. \tag{17}$$

Dividing the term Q/q from both sides of Equation 17, we can simplify this equation as follows:

$$\frac{\mathrm{d}M}{\mathrm{d}r}\frac{1-M^2}{M\gamma(\gamma-1)}=\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)\left(\frac{1}{\gamma}+3M^2\right)\frac{1}{r}. \tag{18}$$

If we rewrite every term in Equation 18 in terms of M^2 (i.e., $\frac{\mathrm{d}M^2}{\mathrm{d}r}=2M\frac{\mathrm{d}M}{\mathrm{d}r}$), we would obtain

$$\frac{\mathrm{d}M^2}{\mathrm{d}r}\frac{1-M^2}{2M^2\gamma(\gamma-1)}=\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)\left(\frac{1}{\gamma}+3M^2\right)\frac{1}{r}. \tag{19}$$

With further simplification, we can rewrite Equation 19 as

$$\left[\frac{1-M^2}{(1+3\gamma M^2)(1+\frac{1}{2}(\gamma-1)M^2)}\right]\frac{\mathrm{d}M^2}{M^2}=2\frac{\mathrm{d}r}{r}. \tag{20}$$

Given the boundary condition that $M(r=R)=1$ (i.e., ensure a smooth transition from subsonic flow at the center to supersonic flow at large r), we can use `Mathematica` to integrate the LHS of Equation 20 from $M^2=M_*$ to $M^2=1$,

$$\int_{M_*}^{1}\left[\frac{1-M_*'}{(1+3\gamma M_*')(1+\frac{1}{2}(\gamma-1)M_*')}\right]\frac{\mathrm{d}M_*'}{M_*'}=\log\left[\left(\frac{1+\gamma}{2+M_*(\gamma-1)}\right)^{\frac{1+\gamma}{1+5\gamma}}\left(\frac{1+3\gamma M_*}{1+3\gamma}\right)^{\frac{2+6\gamma}{1+5\gamma}}M_*^{-1}\right]. \tag{21}$$

Similarly, we need to integrate the RHS of Equation 20 from $\xi=r/R$ to $\xi=1$,

$$\int_{\xi}^{1}2\frac{\mathrm{d}\xi'}{\xi'}=-2\log\xi. \tag{22}$$

After equating Equations 21 and 22, we would obtain

$$\left(\frac{1+\gamma}{2+M_*(\gamma-1)}\right)^{\frac{1+\gamma}{1+5\gamma}}\left(\frac{1+3\gamma M_*}{1+3\gamma}\right)^{\frac{2+6\gamma}{1+5\gamma}}M_*^{-1}=\xi^{-2}. \tag{23}$$

Equation 23 can be simplified to

$$\begin{aligned}\left(\frac{(\gamma-1)+2/M_*}{1+\gamma}\right)^{\frac{1+\gamma}{2(1+5\gamma)}}\left(\frac{3\gamma+1/M_*}{1+3\gamma}\right)^{-\frac{1+3\gamma}{1+5\gamma}}M_*^{1/2}M_*^{\frac{1+\gamma}{2(1+5\gamma)}}M_*^{-\frac{1+3\gamma}{1+5\gamma}}&=\xi\\ \left(\frac{(\gamma-1)+2/M_*}{1+\gamma}\right)^{\frac{1+\gamma}{2(1+5\gamma)}}\left(\frac{3\gamma+1/M_*}{1+3\gamma}\right)^{-\frac{1+3\gamma}{1+5\gamma}}&=\xi\\ \left(\frac{(\gamma-1)+2/M^2}{1+\gamma}\right)^{\frac{1+\gamma}{2(1+5\gamma)}}\left(\frac{3\gamma+1/M^2}{1+3\gamma}\right)^{-\frac{1+3\gamma}{1+5\gamma}}&=\frac{r}{R},\end{aligned} \tag{24}$$

which is consistent with Equation 4 in Chevalier and Clegg (1985).

2.2 $r > R$ Case:

Since $q=Q=0$ for $r>R$, the momentum equation becomes

$$u\frac{\mathrm{d}u}{\mathrm{d}r}=\frac{-1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}, \tag{25}$$

or equivalently

$$Mc_s\left[c_s\frac{\mathrm{d}M}{\mathrm{d}r}+M\frac{\mathrm{d}c_s}{\mathrm{d}r}\right]=\frac{-1}{\gamma}\frac{\mathrm{d}c_s^2}{\mathrm{d}r}-\frac{c_s^2}{\gamma}\frac{\mathrm{dln}\rho}{\mathrm{d}r}. \tag{26}$$

In this case, the equation of ρ in terms of M and c_s is provided as

$$\rho_{r>R}=\frac{\dot{M}}{4\pi r^2u}=\frac{qR^3}{3r^2u}=\frac{qR^3}{3Mc_sr^2}, \tag{27}$$

which is consistent with Equation A.12 in Zhang (2015), but Equation 7 still valids for this regime (i.e., Equation 8 stays the same).

To simplify Equation 26, we need to find the derivative of ρ with respect to r, which is

$$\frac{\mathrm{d}\rho}{\mathrm{d}r} = \frac{qR^3}{3}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{Mc_s r^2}\right) = \frac{qR^3}{3}\left(\frac{-2}{Mc_s r^3} - \frac{1}{M^2 c_s r^2}\frac{\mathrm{d}M}{\mathrm{d}r} - \frac{1}{r^2 c_s^2 M}\frac{\mathrm{d}c_s}{\mathrm{d}r}\right). \tag{28}$$

If we plug Equation 28 into Equation 26, we then obtain

$$Mc_s\left[c_s\frac{\mathrm{d}M}{\mathrm{d}r} + M\frac{\mathrm{d}c_s}{\mathrm{d}r}\right] = \frac{-1}{\gamma}\frac{\mathrm{d}c_s^2}{\mathrm{d}r} - \frac{c_s^2}{\gamma}\left(\frac{-2}{r} - \frac{\mathrm{d}M/\mathrm{d}r}{M} - \frac{\mathrm{d}c_s/\mathrm{d}r}{c_s}\right). \tag{29}$$

The LHS of Equation 26 is the same as Equation 15. Similarly, plugging in Equations 8, 9, and 10, the RHS of Equation 26 can be rewritten as

$$\begin{aligned}\frac{-1}{\gamma}\frac{\mathrm{d}c_s^2}{\mathrm{d}r} - \frac{c_s^2}{\gamma}\frac{\mathrm{dln}\rho}{\mathrm{d}r} &= \frac{M}{\gamma}\frac{Q/q}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\frac{\mathrm{d}M}{\mathrm{d}r} - \frac{1}{\gamma}\left[\frac{Q/q}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\right]\left[\frac{-2}{r} - \frac{\mathrm{d}M/\mathrm{d}r}{M} - \frac{\mathrm{d}c_s/\mathrm{d}r}{c_s}\right] \\ &= \frac{M}{2\gamma}\frac{Q/q}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\frac{\mathrm{d}M}{\mathrm{d}r} + \frac{1}{\gamma M}\frac{(Q/q)}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\frac{\mathrm{d}M}{\mathrm{d}r} + \frac{2}{\gamma r}\frac{Q/q}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}.\end{aligned} \tag{30}$$

After equating the LHS and RHS of Equation 26, we would get

$$\left[\frac{(Q/q)\frac{M}{\gamma-1}}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\right]\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{M}{2\gamma}\frac{Q/q}{\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right)^2}\frac{\mathrm{d}M}{\mathrm{d}r} + \frac{1}{\gamma M}\frac{(Q/q)}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}\frac{\mathrm{d}M}{\mathrm{d}r} + \frac{2}{\gamma r}\frac{Q/q}{\frac{1}{2}M^2+\frac{1}{\gamma-1}}. \tag{31}$$

After further simplification, Equation 31 would become

$$\frac{\mathrm{d}M}{\mathrm{d}r}\frac{M^2-1}{M(\gamma-1)} = \frac{2}{r}\left(\frac{1}{2}M^2+\frac{1}{\gamma-1}\right). \tag{32}$$

Similarly, if we rewrite every term in Equation 32 in terms of M^2, we would obtain

$$\left[\frac{M^2-1}{1+\frac{1}{2}(\gamma-1)M^2}\right]\frac{\mathrm{d}M^2}{M^2} = 4\frac{\mathrm{d}r}{r}. \tag{33}$$

Similarly, given the boundary condition that $M(r=R)=1$, we can use `Mathematica` to integrate the LHS of Equation 33 from $M^2=1$ to $M^2=M_*$,

$$\int_1^{M_*}\frac{M_*'-1}{1+\frac{1}{2}(\gamma-1)M_*'}\frac{\mathrm{d}M_*'}{M_*'} = \log\left[\left(\frac{M_*(2+M_*(\gamma-1))}{1+\gamma}\right)^{\frac{1}{\gamma-1}}\left(\frac{2+M_*(\gamma-1)}{M_*(1+\gamma)}\right)^{\frac{\gamma}{\gamma-1}}\right]. \tag{34}$$

Then, we need to integrate the RHS of Equation 33 from $\xi=1$ to $\xi=r/R$,

$$\int_1^{\xi}4\frac{\mathrm{d}\xi'}{\xi'} = 4\log\xi. \tag{35}$$

After equating Equations 34 and 35, we would obtain

$$\begin{aligned}\left(\frac{M_*(2+M_*(\gamma-1))}{1+\gamma}\right)^{\frac{1}{\gamma-1}}\left(\frac{2+M_*(\gamma-1)}{M_*(1+\gamma)}\right)^{\frac{\gamma}{\gamma-1}} &= \xi^4 \\ M_*^{\frac{1}{\gamma-1}}M_*^{-\frac{\gamma}{\gamma-1}}M_*^{\frac{\gamma+1}{\gamma-1}}\left(\frac{\gamma-1+2/M_*}{1+\gamma}\right)^{\frac{\gamma+1}{\gamma-1}} &= \xi^4 \\ M^{\frac{2}{\gamma-1}}\left(\frac{\gamma-1+2/M^2}{1+\gamma}\right)^{\frac{\gamma+1}{2(\gamma-1)}} &= \left(\frac{r}{R}\right)^2,\end{aligned} \tag{36}$$

which is consistent with Equation 5 in Chevalier and Clegg (1985).

3 Results

Following the methodology outlined by Chevalier and Clegg (1985), we can derive the radial profiles of velocity, density, and pressure in their dimensionless form. These quantities can be expressed in terms of the mass and energy input as follows:

$$u = u^* \dot{M}^{-1/2} \dot{E}^{1/2} \tag{37}$$

$$\rho = \rho^* \dot{M}^{3/2} \dot{E}^{-1/2} R^{-2} \tag{38}$$

$$P = P^* \dot{M}^{1/2} \dot{E}^{1/2} R^{-2} \tag{39}$$

The dimensionless physical variables u^*, ρ^*, and P^* can subsequently be expressed as a function of the Mach number, yielding:

$$u_*^2 = \frac{M^2}{\frac{1}{2}M^2 + \frac{1}{\gamma-1}} \tag{40}$$

$$\rho_* = \begin{cases} \frac{\xi}{4\pi u_*}, & \text{for } \xi < 1 \\ \left(4\pi\xi^2 u_*\right)^{-1}, & \text{for } \xi > 1 \end{cases} \tag{41}$$

$$P_* = \frac{2\rho_*}{\gamma\left(M^2 + \frac{2}{\gamma-1}\right)}, \tag{42}$$

where Equations 40, 41, and 42 are consistent with Equations A.9 - A.12 in Zhang (2015).

The derivation of the temperature profile from the pressure and density profiles is an intriguing and essential step. Such a temperature profile is pivotal as it can be integrated into the radiative cooling term, should the need arise to incorporate it into the original fluid equations. Given that $n = \rho/\mu m$ (where μ represents the mean molecular weight and m the proton mass) and $T = P/nk_{\mathrm{B}}$ (with k_{B} denoting the Boltzmann constant), we can arrive at a corresponding expression for temperature as

$$T = \frac{\mu m}{k_{\mathrm{B}}} \frac{P}{\rho} = \frac{\mu m}{k_{\mathrm{B}}} \frac{P^* \dot{M}^{1/2} \dot{E}^{1/2} R^{-2}}{\rho^* \dot{M}^{3/2} \dot{E}^{-1/2} R^{-2}} = \frac{\mu m}{k_{\mathrm{B}}} \frac{\dot{E}}{\dot{M}} \frac{P^*}{\rho^*} = \frac{\mu m}{k_{\mathrm{B}}} \frac{Q}{q} \frac{P^*}{\rho^*} = \frac{\mu m}{k_{\mathrm{B}}} \frac{Q}{q} T^*, \tag{43}$$

where the dimensionless temperature profile $T^* = P^*/\rho^*$. The radial profile of each dimensionless physical variable is shown in Figure 1.

References

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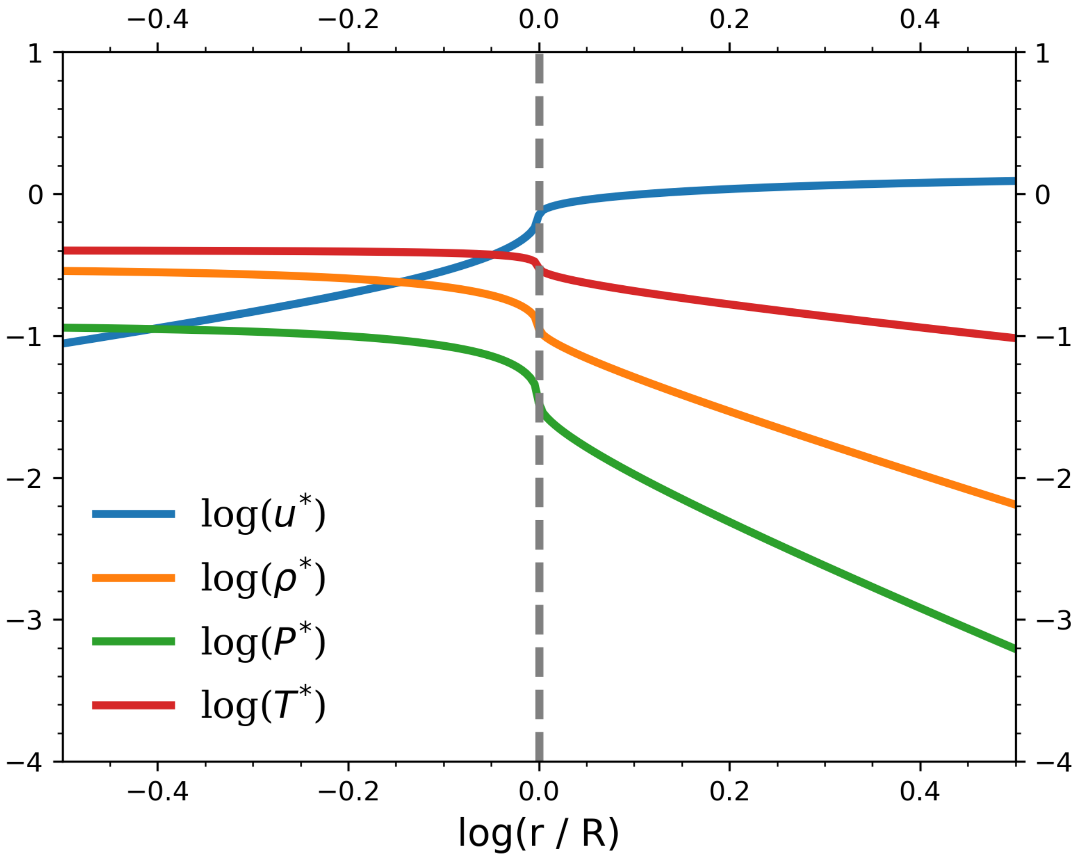

Figure 1: The Chevalier and Clegg (1985) stationary wind solution as a function of r/R, where R is the radius of the region of mass production $\dot{M}$ and energy production $\dot{E}$. The dimensionless variables are $u^* = u/\left(\dot{M}^{-1/2}\dot{E}^{1/2}\right)$, $\rho^* = \rho/\left(\dot{M}^{3/2}\dot{E}^{-1/2}R^{-2}\right)$, $P^* = P/\left(\dot{M}^{1/2}\dot{E}^{1/2}R^{-2}\right)$, and $T^* = P^*/\rho^* = T/\left(\frac{\mu m}{k_{\mathrm{B}}}\frac{Q}{q}\right)$.