

01. Vectors, Lines, Planes

• **Dot Product** - $a \cdot b = ||a|| ||b|| \cos \theta$

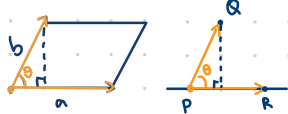
- $a \cdot b = b \cdot a$ $a \cdot (b + c) = a \cdot b + a \cdot c$
- $a \cdot b = 0 \Leftrightarrow a \perp b$

• **Projection** - $\text{proj}_a b = \frac{a \cdot b}{a \cdot a} a$

• $\text{comp}_a b = ||\text{proj}_a b|| = \frac{a \cdot b}{||a||}$

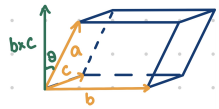
• **Cross Product** - $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - b_1 a_3), a_1 b_2 - a_2 b_1 \rangle$

- $a \times b \perp a$ and $\perp b$ $a \times b = -b \times a$
- $||a \times b|| = ||a|| ||b|| \sin \theta$ Direction: Right hand rule
- $A = ||a \times b||$ $||PQ|| \sin \theta = \frac{||PQ \times PR||}{||PR||}$



• **Scalar Triple Product** - $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

- Result is a scalar value
- $A_{\text{Base}} = ||b \times c||$ $V = Ah = a \cdot (b \times c)$



• **Line** - $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t$

- 2D: Either parallel or intersecting
- 3D: Either parallel, intersecting, or skew

• **Plane** - $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$ where $\langle a, b, c \rangle$ is perpendicular to plane

• **Tangent Vector** - Given $r(t) = \langle f(t), g(t), h(t) \rangle$:

$$r'(a) = \lim_{\Delta t \rightarrow 0} \frac{r(a + \Delta t) - r(a)}{\Delta t} = \langle f'(a), g'(a), h'(a) \rangle$$

- $\frac{d}{dt}(r(t) + s(t)) = r'(t) + s'(t)$
- $\frac{d}{dt}(r(t)s(t)) = r'(t)s(t) + r(t)s'(t)$
- $\frac{d}{dt}(r(t) \cdot s(t)) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$
- $\frac{d}{dt}(r(t) \times s(t)) = r'(t) \times s(t) + r(t) \times s'(t)$

• **Arc Length** - Given smooth $r(t) = \langle f(t), g(t), h(t) \rangle$:

$$S = \int_a^b ||r'(t)|| dt$$

02. Functions of 2 Variables

• **Surface** - $z = f(x, y)$

• **Horizontal Trace** - (Level curve) Intersects with horizontal plane (i.e. $f(x, y) = k$)

• **Level Surface** - $f(x, y, z) = k$

• **Vertical Trace** - Intersections with vertical plane

• **Contour Plot** - $f(x, y) = k$ with lots of k 's

• **Quadric Surfaces** - $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$

• **Cylinder** - There exists plane such that all planes parallel to plane intersect surface in some curve

Equation	Standard form (symmetric about z-axis)	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	Elliptic paraboloid	
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	Hyperbolic paraboloid	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	(Elliptic) cone	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of one sheet	
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	Hyperboloid of two sheets	

• **Limit** - $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$

- To show limit DNE: Show 2 paths with different limits
- To show limit exists:

* Deduce from properties of limits or continuity

$$\begin{aligned} \cdot \lim(\dots \pm \dots) &= \lim \dots \pm \lim \dots \\ \cdot \lim(\dots)(\dots) &= \lim(\dots) \lim(\dots) \\ \cdot \lim\left(\frac{\dots}{\dots}\right) &= \frac{\lim(\dots)}{\lim(\dots)} \text{ where denom. } \neq 0 \end{aligned}$$

* **Squeeze Theorem** - $|f(x, y) - L| \leq g(x, y)$ and $\lim_{(x,y) \rightarrow (a,b)} g(x, y) = 0 \rightarrow \lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$

• **Continuity** - $\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b)$

- If f and g are cont., then $f \pm g$, fg , $\frac{f}{g}$, $f \circ g$ are cont.
- Polynomial, trigonometry, exponential, rational functions are all continuous, but not necessarily defined

03. Derivative

• **Partial Derivative** - Treat other variables as constants

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_x = \frac{\partial f}{\partial x} \quad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

• Intuition: Slope in direction of x , y , ...

• **Tangent Plane** - Given surface $z = f(x, y)$:

$$\mathbf{n} = \langle 0, 1, f_y \rangle \times \langle 1, 0, f_x \rangle = \langle f_x(a, b), f_y(a, b), -1 \rangle$$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

• **Clairaut's Theorem** - $f_{xy} = f_{yx}$

• **Differentiability** - f can be approx. by a tangent plane

- f_x and f_y are continuous $\rightarrow f$ is differentiable
- f is differentiable $\rightarrow f_x$ and f_y exists
- f is differentiable $\rightarrow f$ is continuous

• **Increment** of $z = f(x, y)$ at (a, b) - $\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$

• Formal definition: Can write $\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$ where ϵ_1 and ϵ_2 are functions of Δx and Δy respectively that both approach 0 as $(\Delta x, \Delta y) \rightarrow (0, 0)$

* $f_x \Delta x + f_y \Delta y$: Change in tangent plane

• **Linear Approximation** - Given $z = f(x, y)$ is differentiable at (a, b) :

- Let $\Delta x, \Delta y$ be small increments in x, y from (a, b)
- $\Delta z \approx f_x(a, b)\Delta x + f_y(a, b)\Delta y$

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + f_x(a, b)\Delta x + f_y(a, b)\Delta y$$

• **Chain Rule** - $\frac{\partial z}{\partial t_i} = \sum_{j=1}^n \frac{\partial z}{\partial x_j} \frac{\partial x_j}{\partial t_i}$

Dep. variable z
Intermediate var. x_1, \dots, x_n
Indep. var. t_1, \dots, t_m

• **Implicit Differentiation** - Given $F(x, y, z) = 0$, z is implicitly defined by x and y

$$z_x = -\frac{F_x}{F_z} \quad z_y = -\frac{F_y}{F_z}$$

• **Directional Derivative** - $D_u f(x, y) = \langle f_x, f_y \rangle \cdot u$ where u is a unit vector

- Which direction yields min/max. directional derivative? Min: $-\nabla f$, Max: ∇f

04. Gradient Vector

• **Gradient Vector** - $\nabla f(x, y) = \langle f_x, f_y \rangle$

- $\nabla f(x_0, y_0)$ is normal to level curve $f(x, y) = k$ at (x_0, y_0)
- $\nabla f(x_0, y_0, z_0)$ is normal to level surface $f(x, y, z) = k$ at (x_0, y_0, z_0)
- Tangent plane to level surface: $\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

• **Extrema** - Point larger/smaller than surrounding points

- f has local min/max. at (a, b) and $f_x(a, b), f_y(a, b)$ exist $\rightarrow f_x(a, b) = f_y(a, b) = 0$

* Converse: Not necessarily true (Saddle point)

• **Critical Point** - (a, b) where $f_x(a, b) = f_y(a, b) = 0$

• **Extreme Value Theorem** - $f(x, y)$ is continuous on closed and bounded set $D \subseteq \mathbb{R}^2 \rightarrow$ There exists absolute min/max.

• To find absolute min/max.:

1. Find values of f at critical points of D
2. Find extreme values of f on boundary of D

• **Lagrange Multiplier** - Find extrema of f with constraint $g(x, y) = k$

• Suppose min/max. of f with constraint $g(x, y) = k$ occurs at (x_0, y_0) :

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$$

• Suppose min/max. of f with constraint $g(x, y, z) = k$ occurs at (x_0, y_0, z_0) :

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

• If there are 2 constraints $g(x, y, z) = c_1$ and $h(x, y, z) = c_2$ (i.e. Curve), then $\nabla f = \lambda \nabla g + \mu \nabla h$

05. Double Integral

• **Fubini's Theorem** - $\int_a^b \int_c^d f dy dx = \int_c^d \int_a^b f dx dy$

• Type I: If $D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then $\iint_D f dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f dy dx$

• Type II: If $D = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then $\iint_D f dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f dx dy$

• Draw vertical/hor. arrows. Bounded area cannot split.

$$\iint_D f dA = \iint_{D_1} f dA + \dots + \iint_{D_n} f dA$$

• Area of plane region: $A(D) = \iint_D 1 dA$

• **Polar Coordinates** - (r, θ) where r is distance from origin to point and θ is angle from positive x -axis

$$\begin{aligned} x &= r \cos \theta & y &= r \sin \theta & r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1} \frac{y}{x} \end{aligned}$$

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r \cos \theta, r \sin \theta) (r) dr d\theta$$

06. Triple Integral

• Type I: If $E = \{(x, y, z) : (x, y \in D, u_1(x, y) \leq z \leq u_2(x, y))\}$ where D is projection of E onto xy -plane, then $\iiint_E f dV = \iint_D (\int_{u_1(x, y)}^{u_2(x, y)} f dz) dA$

• Type II: If $E = \{(x, y, z) : (y, z \in D, u_1(y, z) \leq x \leq u_2(y, z))\}$ where D is projection of E onto yz -plane, then $\iiint_E f dV = \iint_D (\int_{u_1(y, z)}^{u_2(y, z)} f dx) dA$

• Type III: If $E = \{(x, y, z) : (x, z \in D, u_1(x, z) \leq y \leq u_2(x, z))\}$ where D is projection of E onto xz -plane, then $\iiint_E f dV = \iint_D (\int_{u_1(x, z)}^{u_2(x, z)} f dy) dA$

• Volume of solid: $V = \iiint_E 1 dV$

• **Cylindrical Coordinates** - (r, θ, z) where z is distance from xy -plane to P

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r \cos \theta, r \sin \theta)}^{u_2(r \cos \theta, r \sin \theta)} f(r \cos \theta, r \sin \theta, z) (r) dz dr d\theta$$

- **Spherical Coordinates** - (ρ, θ, ϕ) where ρ is distance from origin to P and ϕ is angle from positive z -axis
 - $\rho \geq 0 \quad 0 \leq \theta \leq 2\pi \quad 0 \leq \phi \leq \pi$
 - $\rho^2 = x^2 + y^2 + z^2 \quad x = \rho \sin \phi \cos \theta$
 - $y = \rho \sin \phi \sin \theta \quad z = \rho \cos \phi$
 - Good for spheres and cones

$$\iiint_E f(x,y,z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)(\rho^2 \sin \phi) d\rho d\theta d\phi$$

07. Change of Variables

- **Plane Transformation** - $T : (u, v) \mapsto (x, y)$ given by $x = x(u, v)$ and $y = y(u, v)$

- To get image R under T , apply T to boundary

- **Jacobian** of transformation T given by $x = x(u, v)$ and $y = y(u, v)$:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

- Change of Variable in Double Integral:

$$\iint_R f(x,y)dA = \iint_S f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

- dA is image of rectangle $dudv$ under T

- 3D:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint_R f(x,y,z)dA = \iiint_S f(x(u,v,w),y(u,v,w),z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dudvdw$$

- Tips:

- Choice of T important! See from graph or integral
 - * Circle: $x = r \cos \theta$ and $y = r \sin \theta$
 - * Ellipse: $x = ar \cos \theta$ and $y = br \sin \theta$
- Find new bounds after T and get Jacobian
- When finding Jacobian, if expressing x, y with u, v is difficult, can use $\frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(x,y)} = 1$

08. Line Integral

- **Scalar Field** - Scalar function $f(x, y)$ or $f(x, y, z)$
- **Vector Field** - Vector function $\mathbf{F}(x, y)$ or $\mathbf{F}(x, y, z)$
 - $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$
- **Line Integral over Scalar Field** - Suppose C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t) \rangle$ where $a \leq t \leq b$:

$$\int_C f(x,y)ds = \int_a^b f(x(t),y(t))||\mathbf{r}'(t)||dt$$

- Intuition: Area of curtain above C and under $f(x, y)$

- Independent of orientation
- Parameterization of line segment from \mathbf{r}_0 to \mathbf{r}_1 : $\mathbf{r}(t) = \mathbf{r}_0 + (\mathbf{r}_1 - \mathbf{r}_0)t$ where $0 \leq t \leq 1$
- $\int_C f ds = \int_{C_1} f ds + \dots + \int_{C_n} f ds$
- 3D: If C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $a \leq t \leq b$, then $\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t))||\mathbf{r}'(t)||dt$
- **Line Integral of Vector Field** - Let \mathbf{F} be continuous vector field defined on smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ where $a \leq t \leq b$:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(x(t), y(t), z(t)) \cdot \mathbf{r}'(t) dt$$

- Depends on orientation: $\int_{-C} \mathbf{F} \cdot d\mathbf{r} = - \int_C \mathbf{F} \cdot d\mathbf{r}$
- Check if parameterization has same orientation!
- Notation: $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b P x'(t) dt + \int_a^b Q y'(t) dt + \int_a^b R z'(t) dt = \int_a^b P dx + \int_a^b Q dy + \int_a^b R dz$

- **Conservative Vector Field** - Vector field \mathbf{F} that can be written as $\mathbf{F} = \nabla f$ for some scalar function f

- **Potential Function of \mathbf{F}** - f

- Test for Conservative Field in 2D Plane: Suppose $\mathbf{F}(x, y) = \langle P, Q \rangle$ is a vector field in an **open and simply-connected** (i.e. No holes) region D and both P and Q have continuous partial derivatives on D :

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \leftrightarrow \mathbf{F} \text{ is conservative on } D$$

- Test for Conservative Field in 3D Space: Suppose $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ is a vector field in an **open and simply-connected** region D and both P , Q , and R have continuous partial derivatives on D :

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \leftrightarrow \mathbf{F} \text{ is conservative on } D$$

- If assumptions not met, cannot use these tests

- **Fundamental Theorem for Line Integral** - Suppose \mathbf{F} is a conservative vector field with potential function f and C is smooth curve from point A to B :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(B) - f(A)$$

- \exists 2 paths with same initial and terminal points with diff. line integrals \rightarrow Vector field is not conservative

- **Green's Theorem** - Let C be **positively oriented**, piecewise-smooth, **simple closed** (i.e. No intersection with itself, except at start and end) and let D be region bounded by C . Let $\mathbf{F}(x, y) = \langle P, Q \rangle$. If P and Q have continuous partial derivatives on open region with D :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$$

- Positive orientation: Counterclockwise
- ∂D : Positively oriented boundary of region D
- Area of Plane Region: Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and let D be region bounded by C :

$$A = \int_C x dy = - \int_C y dx = \frac{1}{2} (\int_C x dy - y dx)$$

09. Surface Integral

- **Parametric Surface** - Vector function $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ parametrizes surface S in xyz -space

- How? $z = f(x, y)$, Cylindrical, Spherical Coord.

- **Surface Integral of Scalar Field** - Let S be parameterized by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle, (u, v) \in D$:

$$\iint_S f(x,y,z) dS$$

$$= \iint_D f(x(u,v),y(u,v),z(u,v))||\mathbf{r}_u \times \mathbf{r}_v||dA$$

- Tangent Plane of Surface: $\mathbf{r}_u(a, b) \times \mathbf{r}_v(a, b) \perp S$ at point $(x(a, b), y(a, b), z(a, b))$
- Special case: If S is surface $z = g(x, y)$, then $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1 \rangle$ and:

$$\iint_S f(x,y,z) dS$$

$$= \iint_D f(x,y,g(x,y))(\sqrt{g_x^2 + g_y^2 + 1})dA$$

- Surface Area: $A(S) = \iint_S 1 dS = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v||dA$

- **Oriented Surface** - Possible to define unit normal vector \mathbf{n} at each point (x, y, z) not on boundary such that \mathbf{n} is continuous function of (x, y, z)

- All orientable surfaces have 2 orientations

- **Open Surface**

- **Closed Surface** - No boundary (e.g. Sphere, Donut)
 - * Pos. orientation: Outward, Neg. orien.: Inward

- $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$ (Unit vec.) Opposite orientation: $-\mathbf{n}$

- **Surface Integral of Vector Field** - (aka **Flux** of \mathbf{F} across S) Given 3D vector field $\mathbf{F}(x, y, z) = \langle P, Q, R \rangle$ and surface S with given orientation \mathbf{n} :

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$$

- Check orientation \mathbf{n} of S in qn. is given by $\frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$
- Special case: If S is surface $z = g(x, y)$, then flux across S in upward orientation:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R) dA$$

- Special case: Downward orientation

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D (Pg_x + Qg_y - R) dA$$

- Tips: Flat surfaces have constant \mathbf{n} , Not all components of \mathbf{n} need to be computed, Plug in constraints when parameterizing \mathbf{F} to simplify problem

- * If using spherical coordinates $(r(\theta, \phi))$ to parametrize sphere of radius ϕ : $\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle \phi^2 \sin^2 \phi \cos \theta, \phi^2 \sin^2 \phi \sin \theta, \phi^2 \sin \phi \cos \phi \rangle$ (Outwards)

10. Divergence and Curl

- **Divergence** - Scalar measure of net outflow of vector field

$$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$$

$$\text{3D: } \operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F$$

$$\text{2D: } \operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

- **Gauss' Theorem** - Let E be solid region where boundary surface S is piecewise smooth with **positive** orientation. Let $F(x, y, z)$ be vector field whose component functions have **continuous partial derivatives** on an **open region** with E :

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} F dV$$

- **Curl** - Vector field measuring curling effect/circulation of underlying vector field

- $\operatorname{curl} F = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle = \nabla \times F$
- \mathbf{F} is conservative $\rightarrow \operatorname{curl} \mathbf{F} = \operatorname{curl} \nabla f = 0$
- $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$

- **Stokes' Theorem** - Let C be simple closed boundary curve of surface S with unit normals \mathbf{n} . Suppose that C is **positively oriented with respect to \mathbf{n}** . Let F be vector field whose components have continuous partial derivatives on open region that contains S :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

- Positively oriented with respect to \mathbf{n} : Right hand rule (Thumb follows \mathbf{n})
- Stokes' Theorem is 3D version of Green's Theorem. Suppose S is flat and lies in xy -plane with upward orientation \mathbf{k} :

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{k} dA = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$$

- If S_1 and S_2 are oriented surfaces with same oriented boundary curve C and both satisfy assumptions of Stokes' Theorem:

$$\iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

- Flux of $\operatorname{curl} \mathbf{F}$ over closed surface is 0

11. Others

- $\sin^2 \theta = \frac{1-\cos 2\theta}{2} \quad \cos^2 \theta = \frac{1+\cos 2\theta}{2}$
- $(\sin x)' = \cos x \quad (\csc x)' = -\csc x \cot x$
- $(\cos x)' = -\sin x \quad (\sec x)' = \sec x \tan x$
- $(\tan x)' = \sec^2 x \quad (\cot x)' = -\csc^2 x$
- $(e^x)' = e^x \quad (a^x)' = a^x \ln x \quad (\ln x)' = \frac{1}{x}$
- $(\log_a x)' = \frac{1}{x \ln a} \quad (\frac{f}{g})' = \frac{f'g - fg'}{g^2}$
- $(fg)' = f'g + fg' \quad \int u dv = uv - \int v du$
- $|a \cdot b| \leq ||a|| ||b|| \quad ||a + b|| \leq ||a|| + ||b||$