MA2104

AY23/24 Sem 2

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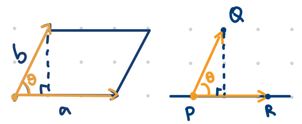
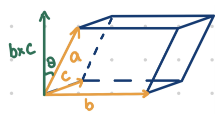
01. Vectors, Lines, Planes

- **Dot Product** - $a \cdot b = ||a||||b|| \cos\theta$
  - $a \cdot b = b \cdot a \qquad a \cdot (b+c) = a \cdot b + a \cdot c$
  - $a \cdot b = 0 \leftrightarrow a \perp b$
- **Projection** - $\text{proj}_a b = \frac{a \cdot b}{a \cdot a} a$
  - $\text{comp}_a b = ||\text{proj}_a b|| = \frac{a \cdot b}{||a||}$
- **Cross Product** - $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - b_1 a_3), a_1 b_2 - a_2 b_1 \rangle$
  - $a \times b \perp a$ and $\perp b \qquad a \times b = -b \times a$
  - $||a \times b|| = ||a||||b|| \sin\theta$ Direction: Right hand rule
  - $A = ||a \times b|| \qquad ||PQ|| \sin\theta = \frac{||PQ \times PR||}{||PR||}$



- **Scalar Triple Product** - $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
  - Result is a scalar value
  - $A_{\text{Base}} = ||b \times c|| \qquad V = Ah = a \cdot (b \times c)$



- **Line** - $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle > + \langle a, b, c \rangle t$
  - 2D: Either parallel or intersecting
  - 3D: Either parallel, intersecting, or skew
- **Plane** - $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$ where $\langle a, b, c \rangle$ is perpendicular to plane
- **Tangent Vector** - Given $r(t) = \langle f(t), g(t), h(t) \rangle$:

$$r'(a) = \lim_{\triangle t \to 0} \frac{r(a + \triangle t) - r(a)}{\triangle t} = \langle f'(a), g'(a), h'(a) \rangle$$

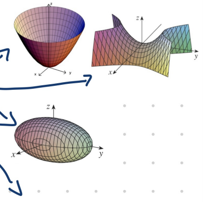
- $\frac{d}{dt}(r(t) + s(t)) = \frac{d}{dt} r(t) + \frac{d}{dt} s(t)$
- $\frac{d}{dt}(r(t)s(t)) = r'(t)s(t) + r(t)s'(t)$
- $\frac{d}{dt}(r(t) \cdot s(t)) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$
- $\frac{d}{dt}(r(t) \times s(t)) = r'(t) \times s(t) + r(t) \times s'(t)$
- **Arc Length** - Given smooth $r(t) = \langle f(t), g(t), h(t) \rangle$:

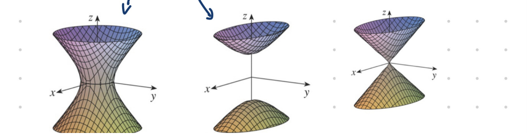
$$S = \int_a^b ||r'(t)||dt$$

02. Functions of 2 Variables

- **Surface** - $z = f(x, y)$
- **Horizontal Trace** - (Level curve) Intersects with horizontal plane (i.e. $f(x, y) = k$)
- **Level Surface** - $f(x, y, z) = k$
- **Vertical Trace** - Intersections with vertical plane
- **Contour Plot** - $f(x, y) = k$ with lots of k's
- **Quadric Surfaces** - $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$
  - **Cylinder** - There exists plane such that all planes parallel to plane intersect surface in some curve

Equation	Standard form (symmetric about z-axis)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	Elliptic paraboloid
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	Hyperbolic paraboloid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	Ellipsoid
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$	(Elliptic) cone
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Hyperboloid of one sheet
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$	Hyperboloid of two sheets





- **Limit** - $\lim_{(x,y) \to (a,b))} f(x, y) = L$
  - To show limit DNE: Show 2 paths with different limits
  - To show limit exists:
    - ∗ Deduce from properties of limits or continuity
      - · $\lim(\ldots \pm \ldots) = \lim \ldots \pm \lim \ldots$
      - · $\lim(\ldots)(\ldots) = \lim(\ldots) \lim(\ldots)$
      - · $\lim \frac{(\ldots)}{(\ldots)} = \frac{\lim(\ldots)}{\lim(\ldots)}$ where denom. $\neq 0$
    - ∗ **Squeeze Theorem** - $|f(x, y) - L| \leq g(x, y)$ and $\lim_{(x,y) \to (a,b)} g(x, y) = 0 \to \lim_{(x,y) \to (a,b) f(x,y) = L}$
- **Continuity** - $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$
  - If f and g are continuous, then $f \pm g$, fg, $\frac{f}{g}$, $f \circ g$ are all continuous
  - Polynomial, trigonometry, exponential, rational functions are all continuous, but not necessarily defined

03. Derivative

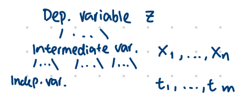
- **Partial Derivative** - Treat other variables as constants
  - $f_x = \frac{\partial f}{\partial x} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$
  - Intuition: Slope in direction of x, y, ...
  - **Clairaut's Theorem** - $f_{xy} = f_{yx}$
- **Tangent Plane** - Given surface $z = f(x, y)$:
  - $n = \langle 0, 1, f_y \rangle \times \langle 1, 0, f_x \rangle = \langle f_x(a, b), f_y(a, b), -1 \rangle$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- **Differentiability** - f_x and f_y are continuous $\to f$ is differentiable
  - f is differentiable $\to f_x$ and f_y exists
  - f is differentiable $\to f$ is continuous
  - **Increment** of $z = f(x, y)$ at (a, b) - $\triangle z = f(a + \triangle x, b + \triangle y) - f(a, b)$
  - Formal definition: Can write $\triangle z = f_x(a, b)\triangle x + f_y(a, b)\triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y$ where ϵ_1 and ϵ_2 are functions of $\triangle x$ and $\triangle y$ respectively that both approach 0 as $(\triangle x, \triangle y) \to (0, 0)$
    - ∗ $f_x \triangle x + f_y \triangle y$: Change in tangent plane
- **Linear Approximation** - Given $z = f(x, y)$ is differentiable at (a, b):
  - Let $\triangle x$, $\triangle y$ be small increments in x, y from (a, b)
  - $\triangle z \approx f_x(a, b)\triangle x + f_y(a, b)\triangle y$

$$f(a + \triangle x, b + \triangle y) \approx f(a, b) + f_x(a, b)\triangle x + f_y(a, b)\triangle y$$

- **Chain Rule** - $\frac{\partial z}{\partial t_i} = \sum_{j=1}^n \frac{\partial z}{\partial x_j} \frac{\partial x_j}{\partial t_i}$



- **Implicit Differentiation** - Given $F(x, y, z) = 0$, z is implicitly defined by x and y

$$z_x = -\frac{F_x}{F_z} \qquad z_y = -\frac{F_y}{F_z}$$

- **Directional Derivative** - $D_u f(x, y) = \langle f_x, f_y \rangle \cdot u$ where u is a unit vector
  - Which direction yields min/max. directional derivative? Min: $-\nabla f$, Max: ∇f

04. Gradient Vector

- **Gradient Vector** - $\nabla f(x, y) = \langle f_x, f_y \rangle$
  - $\nabla f(x_0, y_0)$ is normal to level curve $f(x, y) = k$ at (x_0, y_0)
  - $\nabla f(x_0, y_0, z_0)$ is normal to level surface $f(x, y, z) = k$ at (x_0, y_0, z_0)
  - Tangent plane to level surface: $\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$
- **Extrema** - Point larger/smaller than surrounding points
  - f has local min/max. at (a, b) and $f_x(a, b)$, $f_y(a, b)$ exist $\to f_x(a, b) = f_y(a, b) = 0$
    - ∗ Converse: Not necessarily true (Saddle point)
- **Critical Point** - (a, b) where $f_x(a, b) = f_y(a, b) = 0$
- **Extreme Value Theorem** - $f(x, y)$ is continuous on closed and bounded set $D \subseteq \mathbb{R}^2 \to$ There exists absolute min/max.

05. Double Integral

- **Fubini's Theorem** - $\int_a^b \int_c^d f dy dx = \int_c^d \int_a^b f dx dy$
- Type I: If $D = \{(x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$, then $\iint_D f dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f dy dx$
- Type II: If $D = \{(x, y) : c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$, then $\iint_D f dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f dx dy$
- Draw vertical/hor. arrows. Bounded area cannot split.
- $\iint_D f dA = \iint_{D_1} f dA + \cdots + \iint_{D_n} f dA$
- Area of plane region: $A(D) = \iint_D 1 dA$
- **Polar Coordinates** - (r, θ) where r is distance from origin to point and θ is angle from positive x-axis
  - $x = r\cos\theta \qquad y = r\sin\theta \qquad r = \sqrt{x^2 + y^2}$
  - $\theta = \tan^{-1} \frac{y}{x}$

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)(r) dr d\theta$$

06. Triple Integral

- Type I: If $E = \{(x, y, z) : (x, y \in D, u_1(x, y) \leq z \leq u_2(x, y))\}$ where D is projection of E onto xy-plane, then $\iint_E f dV = \iint_D (\int_{u_1(x,y)}^{u_2(x,y)} f dz) dA$
- Type II: If $E = \{(x, y, z) : (y, z \in D, u_1(y, z) \leq z \leq u_2(y, z))\}$ where D is projection of E onto yz-plane, then $\iint_E f dV = \iint_D (\int_{u_1(y,z)}^{u_2(y,z)} f dx) dA$
- Type III: If $E = \{(x, y, z) : (x, z \in D, u_1(x, z) \leq z \leq u_2(x, z))\}$ where D is projection of E onto xz-plane, then $\iint_E f dV = \iint_D (\int_{u_1(x,z)}^{u_2(x,z)} f dy) dA$
- Volume of solid: $V = \iint_E 1 dV$
- **Cylindrical Coordinates** - (r, θ, z) where z is distance from xy-plane to P

$$\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)(r) dz dr d\theta$$

- **Spherical Coordinates** - (ρ, θ, ϕ) where ρ is distance from origin to P and ϕ is angle from positive z-axis
  - $\rho \geq 0 \qquad 0 \leq \theta \leq 2\pi \qquad 0 \leq \phi \leq \pi$
  - $\rho^2 = x^2 + y^2 + z^2 \qquad x = \rho\sin\phi\cos\theta$
  - $y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$
  - Good for spheres and cones

$$\iiint_E f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)(\rho^2 \sin\phi) d\rho d\theta d\phi$$

07. Change of Variables

08. Line Integral

- **Scalar Field** - Scalar function $f(x,y)$ or $f(x,y,z)$
- **Vector Field** - Vector function $\mathbf{F}(x,y)$ or $\mathbf{F}(x,y,z)$
  - $\mathbf{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$
- **Line Integral over Scalar Field** - Suppose C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ where $a \le t \le b$:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))||\mathbf{r}'(t)||dt$$

  - Intuition: Area of curtain above C and under $f(x,y)$
  - Independent of orientation
  - Parameterization of line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$: $\mathbf{r}(t) = \mathbf{r}_0 + (\mathbf{r}_1 - \mathbf{r}_0)t$ where $0 \le t \le 1$
  - $\int_C f ds = \int_{C_1} f ds + \cdots + \int_{C_n} f ds$
  - 3D: If C is parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ where $a \le t \le b$, then $\int_C f(x,y,z)ds = \int_a^b f(x(t), y(t), z(t))||\mathbf{r}'(t)||dt$
- **Line Integral of Vector Field** - Let $\mathbf{F}$ be continuous vector field defined on smooth curve C parameterized by $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ where $a \le t \le b$:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot\mathbf{T}ds = \int_a^b \mathbf{F}(x(t), y(t), z(t))\cdot\mathbf{r}'(t)dt$$

  - Depends on orientation: $\int_{-C} \mathbf{F}\cdot d\mathbf{r} = -\int_C \mathbf{F}\cdot d\mathbf{r}$
  - Check if parameterization has same orientation!
  - Notation: $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b Px'(t)dt + \int_a^b Qy'(t)dt + \int_a^b Rz'(t)dt = \int_a^b Pdx + \int_a^b Qdy + \int_a^b Rdz$
- **Conservative Vector Field** - Vector field $\mathbf{F}$ that can be written as $\mathbf{F} = \nabla f$ for some scalar function f
  - **Potential Function of $\mathbf{F}$** - f
  - Test for Conservative Field in 2D Plane: Suppose $\mathbf{F}(x,y) = \langle P, Q\rangle$ is a vector field in an **open and simply-connected** (i.e. No holes) region D and both P and Q have continuous partial derivatives on D:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} \leftrightarrow \mathbf{F} \text{ is conservative on } D$$

  - Test for Conservative Field in 3D Space: Suppose $\mathbf{F}(x,y,z) = \langle P, Q, R\rangle$ is a vector field in an **open and simply-connected** region D and both P, Q, and R have continuous partial derivatives on D:

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$$

$$\leftrightarrow \mathbf{F} \text{ is conservative on } D$$

  - If assumptions not met, cannot use these tests
- **Fundamental Theorem for Line Integral** - Suppose $\mathbf{F}$ is a conservative vector field with potential function f and C is smooth curve from point A to B:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(B) - f(A)$$

  - $\exists$ 2 paths with same initial and terminal points with diff. line integrals $\rightarrow$ Vector field is not conservative
- **Green's Theorem** - Let C be **positively oriented**, piecewise-smooth, **simple closed** (i.e. No intersection with itself, except at start and end) and let D be region bounded by C. Let $\mathbf{F}(x,y) = \langle P, Q\rangle$. If P and Q have continuous partial derivatives on open region with D:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C Pdx + Qdy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dA$$

  - Positive orientation: Counterclockwise
  - ∂D: Positively oriented boundary of region D
  - Area of Plane Region: Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and let D be region bounded by C:

$$A = \int_C xdy = -\int_C ydx = \frac{1}{2}(\int_C cdy - ydx)$$

09. Surface Integral

- **Parametric Surface** - Vector function $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$ parametrizes surface S in xyz-space
  - How? $z = f(x,y)$, Cylindrical, Spherical Coord.
- **Surface Integral of Scalar Field** - Let S be parameterized by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, (u,v) \in D$:

$$\iint_S f(x,y,z)dS$$

$$= \iint_D f(x(u,v), y(u,v), z(u,v))||\mathbf{r}_u \times \mathbf{r}_v||dA$$

  - Tangent Plane of Surface: $\mathbf{r}_u(a,b) \times \mathbf{r}_v(a,b) \perp S$ at point $(x(a,b), y(a,b), z(a,b))$
  - Special case: If S is surface $z = g(x,y)$, then $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$ and:

$$\iint_S f(x,y,z)dS$$

$$= \iint_D f(x,y,g(x,y))(\sqrt{g_x^2 + g_y^2 + 1})dA$$

  - Surface Area: $A(S) = \iint_S 1dS = \iint_D ||\mathbf{r}_u \times \mathbf{r}_v||dA$
- **Oriented Surface** - Possible to define unit normal vector $\mathbf{n}$ at each point (x,y,z) not on boundary such that $\mathbf{n}$ is continuous function of (x,y,z)
  - All orientable surfaces have 2 orientations
  - **Open Surface**
  - **Closed Surface** - No boundary (e.g. Sphere, Donut)
    - * Pos. orientation: Outward, Neg. orien.: Inward
  - $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$ Opposite orientation: $-\mathbf{n}$
- **Surface Integral of Vector Field** - (aka **Flux** of $\mathbf{F}$ across S) Given 3D vector field $\mathbf{F}(x,y,z) = \langle P, Q, R\rangle$ and surface S with given orientation $\mathbf{n}$:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)dA$$

  - Check orientation $\mathbf{n}$ of S in qn. is given by $\frac{\mathbf{r}_u \times \mathbf{r}_v}{||\mathbf{r}_u \times \mathbf{r}_v||}$
  - Special case: If S is surface $z = g(x,y)$, then flux across S in upward orientation:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D (-Pg_x - Qg_y + R)dA$$

  - Special case: Downward orientation

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D (Pg_x + Qg_y - R)dA$$

  - Tips: Flat surfaces have constant $\mathbf{n}$, Not all components of $\mathbf{n}$ need to be computed, Plug in constraints when parameterizing $\mathbf{F}$ to simplify problem

10. Divergence and Curl

- **Divergence** - Scalar measure of net outflow of vector field
  - $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle$
  - 3D: $\text{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F$
  - 2D: $\text{div}F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$
- **Gauss' Theorem** - Let E be solid region where boundary surface S is piecewise smooth with **positive** orientation. Let $F(x,y,z)$ be vector field whose component functions have **continuous partial derivatives** on an **open region** with E:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \text{div}\mathbf{F}dV$$

- **Curl** - Vector field measuring curling effect/circulation of underlying vector field
  - $\text{curl}F = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\rangle = \nabla \times F$
- **Stokes' Theorem** - Let C be simple closed boundary curve of surface S with unit normals n. Suppose that C is **positively oriented with respect to $\mathbf{n}$**. Let F be vector field whose components have continuous partial derivatives on open region that contains S:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S}$$

- Positively oriented with respect to $\mathbf{n}$: Right hand rule (Thumb follows $\mathbf{n}$)
- Stokes' Theorem is 3D version of Green's Theorem. Suppose S is flat and lies in xy-plane with upward orientation $\mathbf{k}$:
  - $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot\mathbf{k}dA = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dA$

11. Others

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\cos^2\theta = \frac{1+\cos 2\theta}{2}$