MA2104

AY23/24 Sem 2

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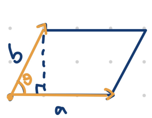
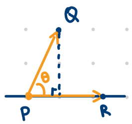
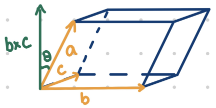
01. Vectors, Lines, Planes

- **Dot Product** - $a \cdot b = ||a||||b|| \cos\theta$
  - $a \cdot b = b \cdot a \qquad a \cdot (b + c) = a \cdot b + a \cdot c$
  - $a \cdot b = 0 \leftrightarrow a \perp b$
- **Projection** - $\text{proj}_a b = \frac{a \cdot b}{a \cdot a} a$
  - $\text{comp}_a b = ||\text{proj}_a b|| = \frac{a \cdot b}{||a||}$
- **Cross Product** - $a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \langle a_2 b_3 - a_3 b_2, -(a_1 b_3 - b_1 a_3), a_1 b_2 - a_2 b_1 \rangle$
  - $a \times b \perp a$ and $\perp b \qquad a \times b = -b \times a$
  - $||a \times b|| = ||a||||b|| \sin\theta$ Direction: Right hand rule
  - $A = ||a \times b|| \qquad ||PQ|| \sin\theta = \frac{||PQ \times PR||}{||PR||}$





- **Scalar Triple Product** - $a \cdot (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$
  - Result is a scalar value
  - $A_{\text{Base}} = ||b \times c|| \qquad V = Ah = a \cdot (b \times c)$



- **Line** - $\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle > + \langle a, b, c \rangle t$
  - 2D: Either parallel or intersecting
  - 3D: Either parallel, intersecting, or skew
- **Plane** - $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle$ where $\langle a, b, c \rangle$ is perpendicular to plane
- **Tangent Vector** - Given $r(t) = \langle f(t), g(t), h(t) \rangle$:

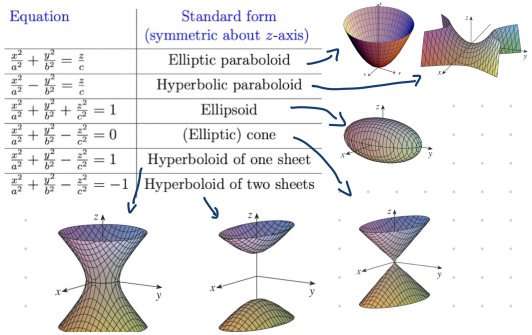
$$r'(a) = \lim_{\triangle t \to 0} \frac{r(a + \triangle t) - r(a)}{\triangle t} = \langle f'(a), g'(a), h'(a) \rangle$$

- $\frac{d}{dt}(r(t) + s(t)) = \frac{d}{dt} r(t) + \frac{d}{dt} s(t)$
- $\frac{d}{dt}(r(t)s(t)) = r'(t)s(t) + r(t)s'(t)$
- $\frac{d}{dt}(r(t) \cdot s(t)) = r'(t) \cdot s(t) + r(t) \cdot s'(t)$
- $\frac{d}{dt}(r(t) \times s(t)) = r'(t) \times s(t) + r(t) \times s'(t)$
- **Arc Length** - Given smooth $r(t) = \langle f(t), g(t), h(t) \rangle$:

$$S = \int_a^b ||r'(t)|| dt$$

02. Functions of 2 Variables

- **Surface** - $z = f(x, y)$
- **Horizontal Trace** - (Level curve) Intersects with horizontal plane (i.e. $f(x, y) = k$)
- **Level Surface** - $f(x, y, z) = k$
- **Vertical Trace** - Intersections with vertical plane
- **Contour Plot** - $f(x, y) = k$ with lots of k's
- **Quadric Surfaces** - $Ax^2 + By^2 + Cz^2 + J = 0$ or $Ax^2 + By^2 + Iz = 0$
  - **Cylinder** - There exists plane such that all planes parallel to plane intersect surface in some curve



- **Limit** - $\lim_{(x,y) \to (a,b))} f(x, y) = L$
  - To show limit DNE: Show 2 paths with different limits
  - To show limit exists:
    - Deduce from properties of limits or continuity
      - $\lim(\ldots \pm \ldots) = \lim \ldots \pm \lim \ldots$
      - $\lim(\ldots)(\ldots) = \lim(\ldots) \lim(\ldots)$
      - $\lim \frac{(\ldots)}{(\ldots)} = \frac{\lim(\ldots)}{\lim(\ldots)}$ where denom. $\neq 0$
    - **Squeeze Theorem** - $|f(x, y) - L| \leq g(x, y)$ and $\lim_{(x,y) \to (a,b)} g(x, y) = 0 \to \lim_{(x,y) \to (a,b) f(x,y) = L}$
- **Continuity** - $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$
  - If f and g are continuous, then $f \pm g$, fg, $\frac{f}{g}$, $f \circ g$ are all continuous
  - Polynomial, trigonometry, exponential, rational functions are all continuous, but not necessarily defined

03. Derivative

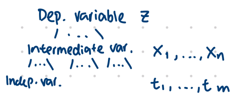
- **Partial Derivative** - Treat other variables as constants
  - $f_x = \frac{\partial f}{\partial x} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$
  - Intuition: Slope in direction of x, y, ...
  - Clairaut's Theorem: $f_{xy} = f_{yx}$
- **Tangent Plane** - Given surface $z = f(x, y)$:
  - $n = \langle 0, 1, f_y \rangle \times \langle 1, 0, f_x \rangle = \langle f_x(a, b), f_y(a, b), -1 \rangle$

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

- **Differentiability** - f_x and f_y are continuous $\to f$ is differentiable
  - f is differentiable $\to f_x$ and f_y exists
  - f is differentiable $\to f$ is continuous
  - **Increment** of $z = f(x, y)$ at (a, b) - $\triangle z = f(a + \triangle x, b + \triangle y) - f(a, b)$
  - Formal definition: Can write $\triangle z = f_x(a, b) \triangle x + f_y(a, b) \triangle y + \epsilon_1 \triangle x + \epsilon_2 \triangle y$ where ϵ_1 and ϵ_2 are functions of $\triangle x$ and $\triangle y$ respectively that both approach 0 as $(\triangle x, \triangle y) \to (0, 0)$
    - $f_x \triangle x + f_y \triangle y$: Change in tangent plane
- **Linear Approximation** - Given $z = f(x, y)$ is differentiable at (a, b):
  - Let $\triangle x$, $\triangle y$ be small increments in x, y from (a, b)
  - $\triangle z \approx f_x(a, b) \triangle x + f_y(a, b) \triangle y$

$$f(a + \triangle x, b + \triangle y) \approx f(a, b) + f_x(a, b) \triangle x + f_y(a, b) \triangle y$$

- **Chain Rule** - $\frac{\partial z}{\partial t_i} = \sum_{j=1}^{n} \frac{\partial z}{\partial x_j} \frac{\partial x_j}{\partial t_i}$



- **Implicit Differentiation** - Given $F(x, y, z) = 0$, z is implicitly defined by x and y

$$z_x = -\frac{F_x}{F_z} \qquad z_y = -\frac{F_y}{F_z}$$

- **Directional Derivative** - $D_u f(x, y) = \langle f_x, f_y \rangle \cdot u$ where u is a unit vector
  - Which direction yields min/max. directional derivative? Max: ∇f, Min: $-\nabla f$

04. Gradient Vector

- **Gradient Vector** - $\nabla f(x, y) = \langle f_x, f_y \rangle$
  - $\nabla f(x_0, y_0)$ is normal to level curve $f(x, y) = k$ at (x_0, y_0)
  - $\nabla f(x_0, y_0, z_0)$ is normal to level surface $f(x, y, z) = k$ at (x_0, y_0, z_0)
  - Tangent plane to level surface: $\nabla f(x_0, y_0, z_0) \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$
- **Extrema** - Point larger/smaller than surrounding points
  - f has local min/max. at (a, b) and $f_x(a, b)$, $f_y(a, b)$ exist $\to f_x(a, b) = f_y(a, b) = 0$
    - Converse: Not necessarily true (Saddle point)
- **Critical Point** - (a, b) where $f_x(a, b) = f_y(a, b) = 0$
- **Extreme Value Theorem** - $f(x, y)$ is continuous on closed and bounded set $D \subseteq \mathbb{R}^2 \to$ There exists absolute min/max.

05. Double Integral

06. Triple Integral

07. Change of Variables

08. Line Integral

09. Surface Integral

10. Divergence and Curl

- **Divergence** - Scalar measure of net outflow of vector field
  - $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$
  - 3D: div$F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot F$
  - 2D: div$F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$
- **Gauss' Theorem** - Let E be solid region where boundary surface S is piecewise smooth with **positive** orientation. Let $F(x, y, z)$ be vector field whose component functions have **continuous partial derivatives** on an **open region** with E:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div} \mathbf{F} dV$$

- **Curl** - Vector field measuring curling effect/circulation of underlying vector field
  - curl$F = \langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle = \nabla \times F$
- **Stokes' Theorem** - Let C be simple closed boundary curve of surface S with unit normals n. Suppose that C is **positively oriented with respect to** $\mathbf{n}$. Let F be vector field whose components have continuous partial derivatives on open region that contains S:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}$$

- Positively oriented with respect to $\mathbf{n}$: Right hand rule (Thumb follows $\mathbf{n}$)
- Stokes' Theorem is 3D version of Green's Theorem. Suppose S is flat and lies in xy-plane with upward orientation $\mathbf{k}$:
  - $\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot \mathbf{k} dA = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA$

11. Others

$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \quad \cos^2\theta = \frac{1 + \cos 2\theta}{2}$