Let $f(x)$ be a function continuous over the interval $[a,b]$.

A Riemann Sum $R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$ with $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

approximates the area under the curve of $y = f(x)$.

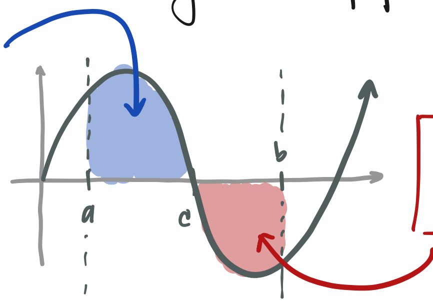
Key Idea: The approximation by R_n gets better as $n \to \infty$.

We'll define the integral of $f(x)$ over $[a,b]$ using this idea:

$$\boxed{\int_a^b f(x)\,dx := \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i)\,\Delta x\right)}$$

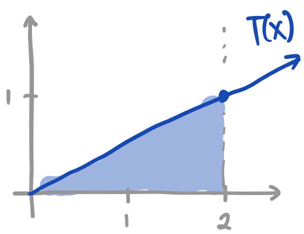
It can be proven that this produces the <u>signed</u> area of the area bounded by the curve of $f(x)$ and the x-axis.

$\boxed{\int_a^c f(x)\,dx \text{ is } (+)}$

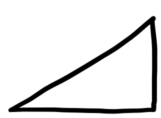


$\boxed{\int_c^b f(x)\,dx \text{ is } (-)}$

Example 1. Let $T(x) = \frac{1}{2}x$ and let $[a,b] = [0,2]$.



Geometrically, the area under the curve of $T(x)$ on $[0,2]$ is a triangle.



Note: $A_{\text{triangle}} = \frac{1}{2}(\text{base})(\text{height})$

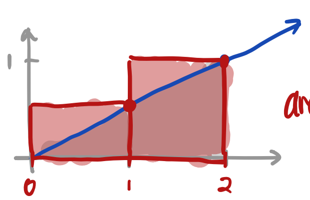
So, we should expect that $\int_0^2 T(x)\,dx = \frac{1}{2}(2)(1) = 1$.

Part (a). Approximate the area using R_n with $n = 2$.

$\Delta x = \frac{2-0}{2} = 1$; $x_i = a + i\Delta x$ so $x_1 = 1$ and $x_2 = 2$.

Then, $R_2 = \sum_{i=1}^{2} T(x_i)\,\Delta x = T(1)\cdot(1) + T(2)\cdot(2) = \left(\frac{1}{2}\right)(1) + (1)(1) = \boxed{\frac{3}{2}}$;

Illustrated:

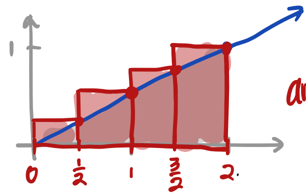


area: $\frac{3}{2}$.

Part (b). Approximate the area using R_n with $n=4$.

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}\ ; \quad x_i = 0 + i\left(\tfrac{1}{2}\right) : \quad x_1 = \tfrac{1}{2},\ x_2 = 1,\ x_3 = \tfrac{3}{2},\ x_4 = 2.$$

Then,
$$R_4 = \sum_{i=1}^{4} T(x_i)\Delta x = T(x_1)\Delta x + T(x_2)\Delta x + T(x_3)\Delta x + T(x_4)\Delta x$$
$$= \Delta x\left(T(\tfrac{1}{2}) + T(1) + T(\tfrac{3}{2}) + T(2)\right)$$
$$= \tfrac{1}{2}\left(\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{3}{4} + 1\right) = \tfrac{1}{2}\left(\tfrac{1}{4} + \tfrac{2}{4} + \tfrac{3}{4} + \tfrac{4}{4}\right) = \tfrac{1}{2}\left(\tfrac{10}{4}\right) = \boxed{\tfrac{5}{2}}\ ;$$



area: $\frac{5}{2}$

Part (c). Find a closed form expression of R_n for arbitrary $n \in \mathbb{N}$, i.e. no summation notation.

Note: $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{(n)(n+1)}{2}$;

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}, \quad x_i = a + i\Delta x = 0 + i\left(\tfrac{2}{n}\right) = \frac{2i}{n}\ ;$$

$$R_n = \sum_{i=1}^{n} T(x_i)\Delta x = \sum_{i=1}^{n} T\left(\frac{2i}{n}\right)\cdot\left(\frac{2}{n}\right) = \left(\frac{2}{n}\right)\sum_{i=1}^{n}\left(\frac{1}{2}\right)\left(\frac{2i}{n}\right)$$
$$= \left(\frac{\cancel{2}}{n}\right)\left(\frac{1}{\cancel{2}}\right)\left(\frac{2}{n}\right)\sum_{i=1}^{n}(i) = \frac{2}{n^2}\left[\frac{(n)(n+1)}{2}\right] = \frac{n^2+n}{n^2} = \boxed{1 + \frac{1}{n}}\ ;$$

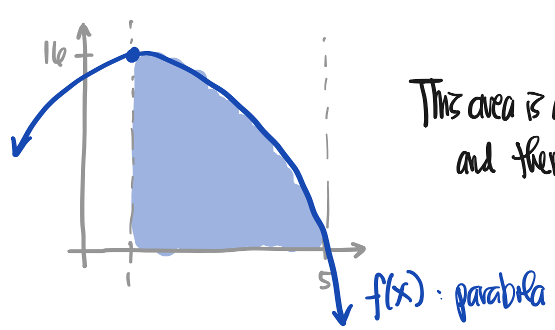
Observe that this matches with parts (a) and (b)
with $R_2 = \frac{3}{2} = 1 + \frac{1}{2}$ and $R_4 = \frac{5}{4} = 1 + \frac{1}{4}$

Part (d). Determine $\int_0^2 T(x)\,dx$ using the Limit Definition of the Integral.

$$\int_0^2 T(x)\,dx \overset{\text{defn}}{=} \lim_{n\to\infty} R_n \overset{\text{part (c)}}{=} \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1 + 0 = \boxed{1.}$$

Observe that this matches with the geometric area of a triangle.

Example 2. Let $f(x) = -(x-1)^2 + 16 = -x^2 + 2x + 15$ and $[a,b] = [1,5]$.



This area is a more complicated geometric shape and there isn't a nice basic formula for this.

Part (a). Approximate the area using R_n with $n=3$.

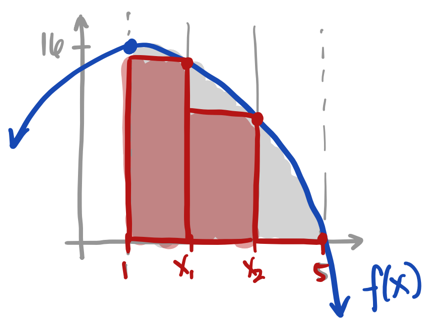
$\Delta x = \frac{5-1}{3} = \frac{4}{3}$; $x_i = a + i\Delta x = 1 + \frac{4}{3}(i)$

i	x_i	$f(x_i)$
1	$1+\frac{4}{3}(1) = 1+\frac{4}{3}$	$f(1+\frac{4}{3}) = -(1+\frac{4}{3}-1)^2+16 = \cdots = \frac{128}{9}$
2	$1+\frac{4}{3}(2) = 1+\frac{8}{3}$	$f(1+\frac{8}{3}) = -(1+\frac{8}{3}-1)^2+16 = \cdots = \frac{80}{9}$
3	$1+\frac{4}{3}(3) = 1+4 = 5$	$f(1+4) = -(1+4-1)^2+16 = \cdots = 0$

$$R_3 = \sum_{i=1}^{3} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$$

$$= \Delta x\left(\frac{128}{9} + \frac{80}{9} + 0\right) = \frac{4}{3}\left(\frac{208}{9}\right) = \boxed{\frac{832}{27}} \approx 30.82$$

Illustrated:



Part (b). Approximate the area using R_n with $n=4$.

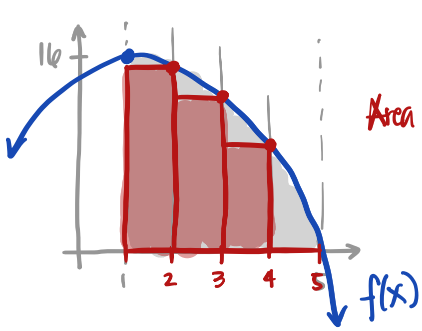
$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1$; $x_i = a + i\Delta x = 1 + i$;

i	x_i	$f(x_i) = -(x_i-1)^2+16$
1	$1+1=2$	$f(2) = -(2-1)^2+16 = -(1)^2+16 = 15$
2	$1+2=3$	$f(3) = -(3-1)^2+16 = -(2)^2+16 = -4+16 = 12$
3	$1+3=4$	$f(4) = \cdots = 7$
4	$1+4=5$	$f(5) = \cdots = 0$

$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= \Delta x(15+12+7+0) = (1)(34) = \boxed{34};$$

Illustrated:



Area of Red-Shaded Region: 34

Part (c). Find a closed form of R_n for arbitrary $n \in \mathbb{N}$.

Note: $1+2+\cdots+n = \sum_{k=1}^{n} k = \frac{1}{2}(n)(n+1)$ and

$$(1)^2 + (2)^2 + \cdots + (n)^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6};$$

$$\Delta x = \frac{5-1}{n} = \frac{4}{n}; \quad x_i = a + i\Delta x = 1 + \frac{4i}{n};$$

$$f(x_i) = f\left(1+\frac{4i}{n}\right) = -\left[\left(1-\frac{4i}{n}\right)-1\right]^2 + 16 = -\left(-\frac{4i}{n}\right)^2 + 16 = -\frac{16i^2}{n^2} + 16;$$

$$R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n}\left(-\frac{16i^2}{n^2}+16\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n}\left[-\frac{64i^2}{n^3}\right] + \sum_{i=1}^{n}\left[\frac{64}{n}\right]$$

$$= \left(-\frac{64}{n^3}\right)\sum_{i=1}^{n}(i^2) + \frac{64}{n}(n) = -\frac{64}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right) + 64$$

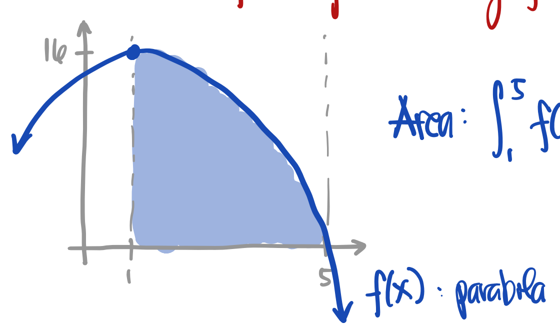
$$\boxed{= -64\left(\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}\right) + 64}; \leftarrow \text{This is good enough.}$$

Part (d). Calculate $\int_1^5 f(x)\,dx$ using the Limit Definition of the Integral.

$$\int_1^5 f(x)\,dx \overset{\text{defn}}{=} \lim_{n\to\infty} R_n \overset{\text{Part (c)}}{=} \lim_{n\to\infty}\left[-64\left(\frac{1}{3}+\cancel{\frac{1}{2n}}^{0}+\cancel{\frac{1}{6n^2}}^{0}\right)+64\right]$$

$$= -\frac{64}{3} + 64 = 64\left(\frac{2}{3}\right) = \boxed{\frac{128}{3}};$$

Part (e). Determine the area of the region bounded by $f(x)$ and the x-axis.



Area: $\int_1^5 f(x)\,dx \overset{\text{part (d)}}{=} \frac{128}{3}$;

Example 3. Let $c(x) = \sqrt{-x^2+36}$ and $[a,b] = [-6,6]$

Part (a). Determine $\int_{-6}^{6} c(x)\,dx = \int_{-6}^{6}\sqrt{-x^2+36}\,dx$ using geometry.

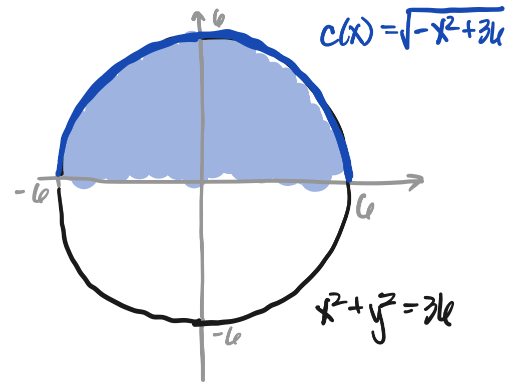
The graph of $c(x)$ is: $y = c(x) = \sqrt{-x^2+36}$

$$y^2 = -x^2 + 36;$$

$x^2 + y^2 = 36$; $\leftarrow$ This is the equation of a circle with radius $r=6$ centered at the origin $(0,0)$.

More generally, the equation of a circle centered at (h,k) with radius r is given by $(x-h)^2+(y-k)^2=r^2$

Therefore, the region under the curve of $c(x)$ is a semicircle with radius 6.



Since $A_{circle}=\pi r^2$,

$$\int_{-6}^{6} c(x)\,dx=\frac{1}{2}\pi r^2=\frac{1}{2}\pi(6)^2=\boxed{18\pi}\approx 56.55$$

Part (b). Approximate $\int_0^6 c(x)\,dx$ using R_n with $n=6$.

$\Delta x=\frac{6-(-6)}{6}=\frac{12}{6}=2$; $x_i=a+i\Delta x=-6+2i$;

i	x_i	$c(x_i)$
1	$-6+2(1)=-4$	$c(-4)=(-(-4)^2+36)^{\frac{1}{2}}=(-16+36)^{\frac{1}{2}}=\sqrt{20}=2\sqrt{5}$
2	$-6+2(2)=-2$	$c(-2)=(-(-2)^2+36)^{\frac{1}{2}}=(-4+36)^{\frac{1}{2}}=\sqrt{32}=4\sqrt{2}$
3	$-6+2(3)=0$	$c(0)=(-(0)^2+36)^{\frac{1}{2}}=(0+36)^{\frac{1}{2}}=\sqrt{36}=6$
4	$-6+2(4)=2$	$c(2)=\cdots=4\sqrt{2}$
5	$-6+2(5)=4$	$c(4)=\cdots=2\sqrt{5}$
6	$-6+2(6)=6$	$c(6)=(-(6)^2+36)^{\frac{1}{2}}=(0)^{\frac{1}{2}}=0$

$$R_6=\sum_{i=1}^{6}c(x_i)\,\Delta x=\Delta x\left(c(x_1)+c(x_2)+\cdots+c(x_6)\right)$$

$$=2\left(2\sqrt{5}+4\sqrt{2}+6+4\sqrt{2}+2\sqrt{5}+0\right)=2\left(4\sqrt{5}+8\sqrt{2}+6\right)$$

$$=\boxed{8\sqrt{5}+16\sqrt{2}+12}\approx 52.52$$

Part (c). Find a closed form expression for R_n with $n\in\mathbb{N}$.

$\Delta x=\frac{6-(-6)}{n}=\frac{12}{n}$; $x_i=a+i\Delta x=-6+\frac{12i}{n}$;

$$c(x_i)=\sqrt{-x_i^2+36}=\sqrt{(6-x_i)(6+x_i)}$$

$$=\left[\left(6-\left(-6+\frac{12i}{n}\right)\right)\left(6+\left(-6+\frac{12i}{n}\right)\right)\right]^{\frac{1}{2}}=\left[\left(12-\frac{12i}{n}\right)\left(\frac{12i}{n}\right)\right]^{\frac{1}{2}};$$

$$R_n=\sum_{i=1}^{n}c(x_i)\,\Delta x=\frac{12}{n}\sum_{i=1}^{n}\sqrt{\left(\frac{12i}{n}\right)\left(12-\frac{12i}{n}\right)}=\cdots$$

and we're stuck since $\sum\sqrt{\cdots}\neq\sqrt{\sum(\cdots)}$ and we don't have a nice formula.

Note that expanding $\sum_{i=1}^{n} f(x_i)$ is generally a very non-trivial problem.

However, we can skip this step using a very powerful result called the

Fundamental Theorem of Calculus

This relates integral calculus with differential calculus.