

Week 6 Recitation Worksheet.

Problem 1. Identify whether u-sub/Integration by Parts (IBP)/algebraic manipulation can be applied to rewrite the integral so that basic antiderivative rules can be used. If possible, rewrite the integral.

Note: Solutions fully solve the integral for reference.

(A) Integration by Parts

$$I_A = \int (1+x)e^x dx \stackrel{\text{IBP}}{=} (1+x)e^x - \int e^x dx = (1+x)e^x - e^x + C$$

$$\text{IBP} \left\{ \begin{array}{l} u = 1+x \\ du = dx \end{array} \quad \begin{array}{l} dv = e^x dx \\ v = e^x \end{array} \right\}$$

(B) Distribute denominator (-x)

$$I_B = \int \frac{4-x+x^2}{-x} dx \stackrel{\text{distribute}}{=} \int \frac{4}{-x} + \frac{-x}{-x} + \frac{x^2}{-x} dx = \int (-4)\frac{1}{x} + 1 - x dx = -4\ln|x| + x - \frac{1}{2}x^2 + C$$

(C) u-substitution

$$I_C = \int te^{t^2} dt \stackrel{u\text{-sub}}{=} \int te^u \frac{1}{2}(t^{-1}) dt = \frac{1}{2} \int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

$$u\text{-sub} \left\{ \begin{array}{l} u = t^2, \quad du = 2t dt \\ dt = \frac{1}{2}t^{-1} du \end{array} \right\}$$

(D) Integration by Parts

$$I_D = \int x \sin(x) dx \stackrel{\text{IBP}}{=} -x \cos(x) - \int (-\cos(x)) dx = -x \cos(x) + \sin(x) + C$$

$$\text{IBP} \left\{ \begin{array}{l} u = x \\ du = dx \end{array} \quad \begin{array}{l} dv = \sin(x) dx \\ v = -\cos(x) \end{array} \right\}$$

(E) u-substitution

$$I_E = \int \frac{1}{5} e^{-\frac{1}{5}t} dt \stackrel{u\text{-sub}}{=} \int \frac{1}{5} e^u (-5) du = \int (-1) e^u du = -e^u + C = -e^{-\frac{1}{5}t} + C$$

$$u\text{-sub} \left\{ \begin{array}{l} u = -\frac{1}{5}t, \quad du = -\frac{1}{5} dt \\ dt = -5 du \end{array} \right\}$$

(F) u-substitution

$$I_F = \int \frac{1}{t} - e^{-2t} dt = \ln|t| - \int e^{-2t} dt \stackrel{u\text{-sub}}{=} \ln|t| - \int e^u \left(-\frac{1}{2}\right) du = \ln|t| + \frac{1}{2}e^u + C = \ln|t| + \frac{1}{2}e^{-2t} + C$$

$$u\text{-sub} \left\{ \begin{array}{l} u = -2t, \quad du = -2 dt \\ dt = -\frac{1}{2} du \end{array} \right\}$$

(G) Method 1. U-substitution, then Integration by Parts

$$I_G = \int x^2 e^{-x^2} dx \stackrel{u\text{-sub}}{=} \int x^2 e^u \left(-\frac{1}{2x}\right) du = -\frac{1}{2} \int x^2 e^u du = -\frac{1}{2} \int (-u) e^u du$$

$$u\text{-sub} \left\{ \begin{array}{l} u = -x^2, du = -2x dx \\ dx = -\frac{1}{2x} du, x^2 = -u \end{array} \right\}$$

$$= \frac{1}{2} \int u e^u du \stackrel{IBP}{=} u e^u - \int e^u du = u e^u - e^u + C = -x^2 e^{-x^2} - e^{-x^2} + C$$

$$IBP \left\{ \begin{array}{l} a = u \\ da = du \end{array} \quad \begin{array}{l} db = e^u du \\ b = e^u \end{array} \right\}$$

Method 2. Integration by Parts with (internal) U-substitution.

$$I_G = \int x^2 e^{-x^2} dx \stackrel{IBP}{=} -\frac{1}{2} x^2 e^{-x^2} + \int (2x) \left(\frac{1}{2} e^{-x^2}\right) dx = -\frac{1}{2} x^2 e^{-x^2} + \int x e^{-x^2} dx$$

$$IBP \left\{ \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \quad \begin{array}{l} dv = x e^{-x^2} dx \leftarrow \text{solve using u-sub.} \\ \text{From ①: } v = -\frac{1}{2} e^{-x^2} \end{array} \right\}$$

$$\textcircled{1}: v = \int x e^{-x^2} dx \stackrel{u\text{-sub}}{=} \int \left(-\frac{1}{2}\right) e^u du = -\frac{1}{2} e^u = -\frac{1}{2} e^{-x^2}$$

$$u\text{-sub} \left\{ \begin{array}{l} u = -x^2, du = -2x dx \\ x dx = -\frac{1}{2} du \end{array} \right\} \quad (\text{no } +C)$$

$$I_G = -\frac{1}{2} x^2 e^{-x^2} + \int x e^{-x^2} dx \stackrel{\textcircled{1}}{=} -\frac{1}{2} x^2 e^{-x^2} - \frac{1}{2} e^{-x^2} + C$$

$$(H) I_H = \int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx$$

We can't use u-sub/IBP here. To rewrite the integral we do Partial Fraction Decomposition (PFD).
 ? We'll learn this later.

(I) Integration by Parts TWICE

$$I_I = \int x^2 e^x \stackrel{IBP\textcircled{1}}{=} x^2 e^x - \int 2x e^x dx \stackrel{IBP\textcircled{2}}{=} x^2 e^x + (-2x) e^x - \int (-2) e^x dx$$

$$IBP\textcircled{1} \left\{ \begin{array}{l} u = x^2 \\ du = 2x dx \end{array} \quad \begin{array}{l} dv = e^x dx \\ v = e^x \end{array} \right\}$$

$$IBP\textcircled{2} \left\{ \begin{array}{l} u = -2x \\ du = -2 dx \end{array} \quad \begin{array}{l} dv = e^x dx \\ v = e^x \end{array} \right\}$$

$$= x^2 e^x - 2x e^x + 2e^x + C$$

(J) U-substitution

$$I_J = \int \frac{\ln(x)}{x} dx \stackrel{u\text{-sub}}{=} \int u du = \frac{1}{2} u^2 + C = \frac{1}{2} (\ln x)^2 + C$$

$$u\text{-sub} \left\{ u = \ln(x), du = \frac{1}{x} dx \right\}$$