Problem 1. Identify whether u-sub / Integration by Parts (IBP) / algebraic manipulation can be applied to rewrite the integral so that basic antiderivative rules can be used. If possible, rewrite the integral.

Note: Solutions fully solve the integral for reference.

(A) Integration by Parts

$$I_A = \int (1+x)e^x \, dx \overset{\text{IBP}}{=} (1+x)e^x - \int e^x \, dx = (1+x)e^x - e^x + C$$

$$\text{IBP} \begin{Bmatrix} u = 1+x & dv = e^x \, dx \\ du = dx & v = e^x \end{Bmatrix}$$

(B) Distribute denominator $(-x)$

$$I_B = \int \frac{4 - x + x^2}{-x} dx \overset{\text{distribute}}{=} \int \frac{4}{-x} + \frac{-x}{-x} + \frac{x^2}{-x} dx = \int (-4)\frac{1}{x} + 1 - x \, dx = -4\ln|x| + x - \frac{1}{2}x^2 + C$$

(C) U-substitution

$$I_C = \int te^{t^2} dt \overset{\text{u-sub}}{=} \int te^u \tfrac{1}{2}(t^{-1}) \, dt = \frac{1}{2}\int e^u \, du = \frac{1}{2}e^u + C = \frac{1}{2}e^{t^2} + C$$

$$\text{u-sub} \begin{Bmatrix} u = t^2, & du = 2t \, dt, \\ & dt = \frac{1}{2}t^{-1} du \end{Bmatrix}$$

(D) Integration by Parts

$$I_D = \int x\sin(x) \, dx \overset{\text{IBP}}{=} -x\cos(x) - \int (-\cos(x)) \, dx = -x\cos(x) + \sin(x) + C$$

$$\text{IBP} \begin{Bmatrix} u = x & dv = \sin(x) \, dx \\ du = dx & v = -\cos(x) \end{Bmatrix}$$

(E) U-substitution

$$I_E = \int \frac{1}{5}e^{-\frac{1}{5}t} dt \overset{\text{u-sub}}{=} \int \frac{1}{5}e^u (-5) \, du = \int (-1)e^u \, du = -e^u + C = -e^{-\frac{1}{5}t} + C$$

$$\text{u-sub} \begin{Bmatrix} u = -\frac{1}{5}t, & du = -\frac{1}{5} dt \\ & dt = -5 \, du \end{Bmatrix}$$

(F) U-Substitution

$$I_F = \int \frac{1}{t} - e^{-2t} dt = \ln|t| - \int e^{-2t} dt \overset{\text{u-sub}}{=} \ln|t| - \int e^u \left(-\tfrac{1}{2}\right) du = \ln|t| + \frac{1}{2}e^u + C = \ln|t| + \frac{1}{2}e^{-2t} + C$$

$$\text{u-sub} \begin{Bmatrix} u = -2t, & du = -2 \, dt \\ & dt = -\frac{1}{2} du \end{Bmatrix}$$

(G) Method 1. U-Substitution, then Integration by Parts

$$I_G = \int x^3 e^{-x^2}\,dx \overset{u\text{-sub}}{=} \int x^3 e^u \left(-\frac{1}{2x}\right) du = -\frac{1}{2}\int x^2 e^u\,du = -\frac{1}{2}\int (-u)e^u\,du$$

u-sub $\left\{ \begin{array}{l} u = -x^2,\ du = -2x\,dx \\ dx = -\frac{1}{2x}du,\ x^2 = -u \end{array} \right\}$

$$= \frac{1}{2}\int ue^u\,du \overset{IBP}{=} ue^u - \int e^u\,du = ue^u - e^u + C = -x^2e^{-x^2} - e^{-x^2} + C$$

IBP $\left\{ \begin{array}{ll} a = u & db = e^u\,du \\ da = du & b = e^u \end{array} \right\}$

Method 2. Integration by Parts with (internal) u-substitution.

$$I_G = \int x^2 e^{-x^2}\,dx \overset{IBP}{=} -\frac{1}{2}x^2e^{-x^2} + \int (\cancel{2}x)\left(+\frac{1}{\cancel{2}}e^{-x^2}\right)dx = -\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,dx$$

IBP $\left\{ \begin{array}{ll} u = x^2 & dv = xe^{-x^2}\,dx \leftarrow \text{solve using u-sub.} \\ du = 2x\,dx & \text{From ①: } v = -\frac{1}{2}e^{-x^2} \end{array} \right\}$

①: $v = \int xe^{-x^2}\,dx \overset{u\text{-sub}}{=} \int \left(-\frac{1}{2}\right) e^u\,du = -\frac{1}{2}e^u = -\frac{1}{2}e^{-x^2}$

u-sub $\left\{ \begin{array}{l} u = -x^2,\ du = -2x\,dx \\ x\,dx = -\frac{1}{2}du \end{array} \right\}$ (no +C)

$$I_G = -\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}\,dx \overset{①}{=} -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + C$$

(H) $I_H = \int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

We can't use U-sub / IBP here. To rewrite the integral we do Partial Fraction Decomposition (PFD).
↑ We'll learn this later.

(I) Integration by Parts TWICE

$$I_I = \int x^2e^x \overset{IBP①}{=} x^2e^x - \int 2xe^x\,dx \overset{IBP②}{=} x^2e^x + (-2x)e^x - \int (-2)e^x\,dx$$

IBP① $\left\{ \begin{array}{ll} u = x^2 & dv = e^x\,dx \\ du = 2x\,dx & v = e^x \end{array} \right\}$ IBP② $\left\{ \begin{array}{ll} u = -2x & dv = e^x\,dx \\ du = -2\,dx & v = e^x \end{array} \right\}$

$$= x^2e^x - 2xe^x + 2e^x + C$$

(J) U-Substitution

$$I_J = \int \frac{\ln(x)}{x}\,dx \overset{u\text{-sub}}{=} \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln x)^2 + C$$

u-sub $\left\{ u = \ln(x),\ du = \frac{1}{x}dx \right\}$