

Let $f(x)$ be a function continuous over the interval $[a, b]$.

A Riemann Sum $R_n = \sum_{i=1}^n f(x_i) \Delta x$ with $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$ approximates the area under the curve of $y = f(x)$.

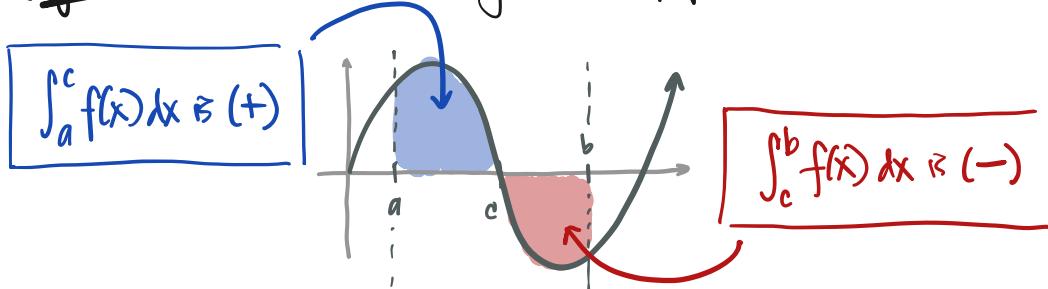
Key Idea: The approximation by R_n gets better as $n \rightarrow \infty$.

We'll define the integral of $f(x)$ over $[a, b]$ using this idea:

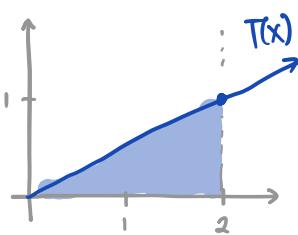
$$\int_a^b f(x) dx := \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n f(x_i) \Delta x \right)$$

It can be proven that this produces

the signed area of the area bounded by the curve of $f(x)$ and the x -axis.



Example 1. Let $T(x) = \frac{1}{2}x$ and let $[a, b] = [0, 2]$.



Geometrically, the area under the curve of $T(x)$ on $[0, 2]$ is a triangle.



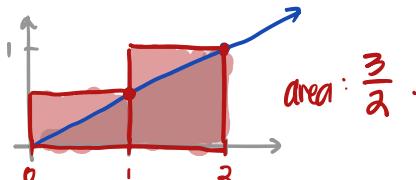
So, we should expect that $\int_0^2 T(x) dx = \frac{1}{2}(2)(1) = 1$.

Part (a). Approximate the area using R_n with $n = 2$.

$$\Delta x = \frac{2-0}{2} = 1 ; \quad x_i = a + i\Delta x \Rightarrow x_1 = 1 \text{ and } x_2 = 2.$$

$$\text{Then, } R_2 = \sum_{i=1}^2 T(x_i) \Delta x = T(1) \cdot (1) + T(2) \cdot (2) = \left(\frac{1}{2}\right)(1) + (1)(1) = \boxed{\frac{3}{2}};$$

Illustrated:



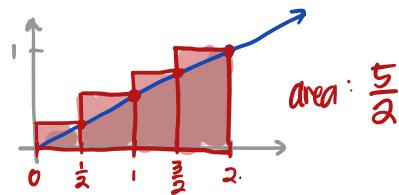
Part (b). Approximate the area using R_4 with $n=4$.

$$\Delta x = \frac{2-0}{4} = \frac{1}{2}; \quad x_i = 0 + i\left(\frac{1}{2}\right) \therefore x_1 = \frac{1}{2}, \quad x_2 = 1, \quad x_3 = \frac{3}{2}, \quad x_4 = 2.$$

$$\text{Then, } R_4 = \sum_{i=1}^4 T(x_i) \Delta x = T(x_1) \Delta x + T(x_2) \Delta x + T(x_3) \Delta x + T(x_4) \Delta x$$

$$= \Delta x \left(T\left(\frac{1}{2}\right) + T(1) + T\left(\frac{3}{2}\right) + T(2) \right)$$

$$= \frac{1}{2} \left(\frac{1}{4} + \frac{1}{2} + \frac{3}{4} + 1 \right) = \frac{1}{2} \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} \right) = \frac{1}{2} \left(\frac{10}{4} \right) = \boxed{\frac{5}{2}};$$



Part (c). Find a closed form expression of R_n for arbitrary $n \in \mathbb{N}$, i.e. no summation notation.

$$\text{Note: } \sum_{k=1}^n k = 1+2+\dots+n = \frac{(n)(n+1)}{2};$$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}, \quad x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n};$$

$$\begin{aligned} R_n &= \sum_{i=1}^n T(x_i) \Delta x = \sum_{i=1}^n T\left(\frac{2i}{n}\right) \cdot \left(\frac{2}{n}\right) = \left(\frac{2}{n}\right) \sum_{i=1}^n \left(\frac{1}{2}\right) \left(\frac{2i}{n}\right) \\ &= \left(\frac{2}{n}\right) \left(\frac{1}{2}\right) \left(\frac{2}{n}\right) \sum_{i=1}^n (i) = \frac{2}{n^2} \left[\frac{(n)(n+1)}{2} \right] = \frac{n^2+n}{n^2} = \boxed{1 + \frac{1}{n}}; \end{aligned}$$

Observe that this matches with parts (a) and (b)

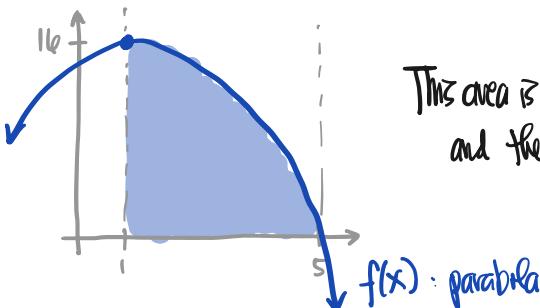
$$\text{with } R_2 = \frac{3}{2} = 1 + \frac{1}{2} \text{ and } R_4 = \frac{5}{4} = 1 + \frac{1}{4}$$

Part (d). Determine $\int_0^2 T(x) dx$ using the Limit Definition of the Integral.

$$\int_0^2 T(x) dx \stackrel{\text{defn}}{=} \lim_{n \rightarrow \infty} R_n \stackrel{\text{Part (c)}}{=} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right) = 1 + 0 = \boxed{1}.$$

Observe that this matches with the geometric area of a triangle.

Example 2. Let $f(x) = -(x-1)^2 + 16 = -x^2 + 2x + 15$ and $[a,b] = [1,5]$.



This area is a more complicated geometric shape and there isn't a nice basic formula for this.

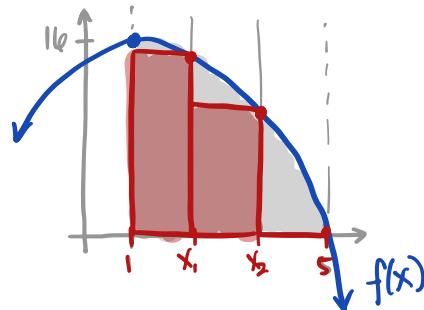
Part (a). Approximate the area using R_n with n=3.

$$\Delta x = \frac{5-1}{3} = \frac{4}{3}; \quad x_i = a + i\Delta x = 1 + \frac{4}{3}(i)$$

i	x_i	$f(x_i)$
1	$1 + \frac{4}{3}(1) = 1 + \frac{4}{3}$	$f(1 + \frac{4}{3}) = -(1 + \frac{4}{3} - 1)^2 + 16 = \dots = \frac{128}{9}$
2	$1 + \frac{4}{3}(2) = 1 + \frac{8}{3}$	$f(1 + \frac{8}{3}) = -(1 + \frac{8}{3} - 1)^2 + 16 = \dots = \frac{80}{9}$
3	$1 + \frac{4}{3}(3) = 1 + 4 = 5$	$f(1 + 4) = -(1 + 4 - 1)^2 + 16 = \dots = 0$

$$R_3 = \sum_{i=1}^3 f(x_i) \Delta x = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x \\ = \Delta x \left(\frac{128}{9} + \frac{80}{9} + 0 \right) = \frac{4}{3} \left(\frac{208}{9} \right) = \boxed{\frac{832}{27}} \approx 30.82$$

Illustrated:



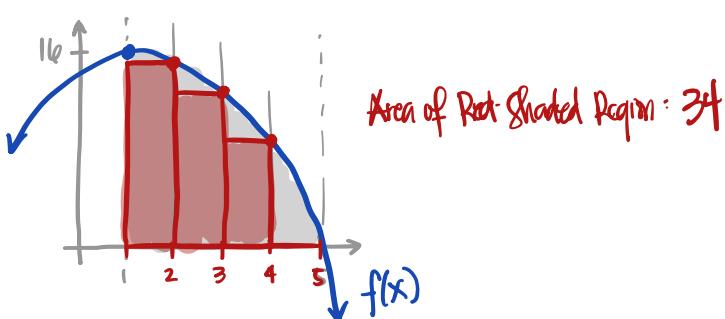
Part (b). Approximate the area using R_n with n=4.

$$\Delta x = \frac{5-1}{4} = \frac{4}{4} = 1; \quad x_i = a + i\Delta x = 1 + i;$$

i	x_i	$f(x_i) = -(x_i - 1)^2 + 16$
1	$1+1=2$	$f(2) = -(2-1)^2 + 16 = -(1)^2 + 16 = 15$
2	$1+2=3$	$f(3) = -(3-1)^2 + 16 = -(2)^2 + 16 = -4 + 16 = 12$
3	$1+3=4$	$f(4) = \dots = 7$
4	$1+4=5$	$f(5) = \dots = 0$

$$R_4 = \sum_{i=1}^4 f(x_i) \Delta x = f(x_1) \Delta x + f(x_2) \Delta x + f(x_3) \Delta x + f(x_4) \Delta x \\ = \Delta x (15 + 12 + 7 + 0) = (1)(34) = \boxed{34};$$

Illustrated:



Part(c). Find a closed form of R_n for arbitrary $n \in \mathbb{N}$.

Note: $1+2+\dots+n = \sum_{k=1}^n k = \frac{1}{2}(n)(n+1)$ and

$$(1)^2 + (2)^2 + \dots + (n)^2 = \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\Delta x = \frac{5-1}{n} = \frac{4}{n}; \quad x_i = a + i\Delta x = 1 + \frac{4i}{n};$$

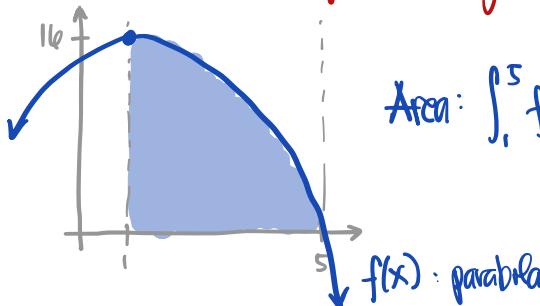
$$f(x_i) = f\left(1 + \frac{4i}{n}\right) = -\left[\left(1 - \frac{4i}{n}\right) - 1\right]^2 + 16 = -\left(-\frac{4i}{n}\right)^2 + 16 = -\frac{16i^2}{n^2} + 16;$$

$$\begin{aligned} R_n &= \sum_{i=1}^n f(x_i) \Delta x = \sum_{i=1}^n \left(-\frac{16i^2}{n^2} + 16\right) \left(\frac{4}{n}\right) = \sum_{i=1}^n \left[-\frac{64i^2}{n^3}\right] + \sum_{i=1}^n \left[\frac{64}{n}\right] \\ &= \left(-\frac{64}{n^3}\right) \sum_{i=1}^n (i^2) + \frac{64}{n} (n) = -\frac{64}{n^3} \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}\right) + 64 \\ &\boxed{= -64\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) + 64}; \quad \leftarrow \text{This is good enough.} \end{aligned}$$

Part(d). Calculate $\int_1^5 f(x) dx$ using the Limit Definition of the Integral.

$$\begin{aligned} \int_1^5 f(x) dx &\stackrel{\text{defn}}{=} \lim_{n \rightarrow \infty} R_n \stackrel{\text{Part (c)}}{=} \lim_{n \rightarrow \infty} \left[-64\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) + 64 \right] \\ &= -\frac{64}{3} + 64 = 64\left(\frac{2}{3}\right) = \boxed{\frac{128}{3}}; \end{aligned}$$

Part(e). Determine the area of the region bounded by $f(x)$ and the x -axis.



$$\text{Area: } \int_1^5 f(x) dx \stackrel{\text{Part (d)}}{=} \frac{128}{3};$$

Example 3. Let $c(x) = \sqrt{-x^2 + 36}$ and $[a,b] = [-6,6]$

Part(a). Determine $\int_{-6}^6 c(x) dx = \int_{-6}^6 \sqrt{-x^2 + 36} dx$ using geometry.

The graph of $c(x)$ is: $y = c(x) = \sqrt{-x^2 + 36}$

$$y^2 = -x^2 + 36;$$

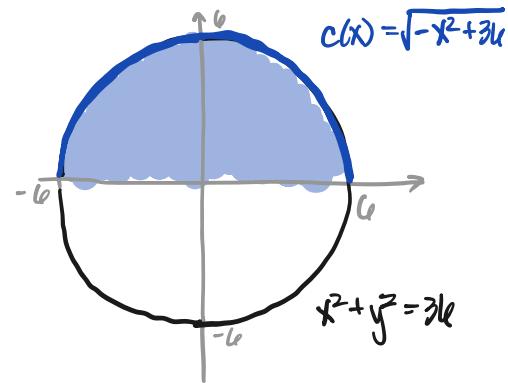
$x^2 + y^2 = 36$; \leftarrow This is the equation of a circle with radius $r = 6$ centered at the origin $(0,0)$.

More generally, the equation of a circle centered at (h, k) with radius r
 is given by $(x-h)^2 + (y-k)^2 = r^2$

Therefore, the region under the curve of $c(x)$
 is a semicircle with radius $6e$.

Since $A_{\text{circle}} = \pi r^2$,

$$\int_{-6e}^{6e} c(x) dx = \frac{1}{2} \pi r^2 = \frac{1}{2} \pi (6e)^2 = 18\pi \approx 56.55$$



Part(b). Approximate $\int_0^6 c(x) dx$ using R_n with $n=6e$.

$$\Delta x = \frac{6 - (-6)}{6} = \frac{12}{6} = 2; \quad x_i = a + i \Delta x = -6 + 2i;$$

i	x_i	$c(x_i)$
1	$-6 + 2(1) = -4$	$c(-4) = \sqrt{(-(-4)^2 + 36)} = \sqrt{-16 + 36} = \sqrt{20} = 2\sqrt{5}$
2	$-6 + 2(2) = -2$	$c(-2) = \sqrt{(-(2)^2 + 36)} = \sqrt{-4 + 36} = \sqrt{32} = 4\sqrt{2}$
3	$-6 + 2(3) = 0$	$c(0) = \sqrt{-(0)^2 + 36} = \sqrt{36} = 6$
4	$-6 + 2(4) = 2$	$c(2) = \dots = 4\sqrt{2}$
5	$-6 + 2(5) = 4$	$c(4) = \dots = 2\sqrt{5}$
6	$-6 + 2(6) = 6$	$c(6) = \sqrt{-(6)^2 + 36} = \sqrt{0} = 0$

$$\begin{aligned}
 R_6 &= \sum_{i=1}^{6e} c(x_i) \Delta x = \Delta x (c(x_1) + c(x_2) + \dots + c(x_6)) \\
 &= 2(2\sqrt{5} + 4\sqrt{2} + 6 + 4\sqrt{2} + 2\sqrt{5} + 0) = 2(4\sqrt{5} + 8\sqrt{2} + 6) \\
 &\boxed{= 8\sqrt{5} + 16\sqrt{2} + 12 \approx 52.52}
 \end{aligned}$$

Part(c). Find a closed form expression for R_n with $n \in \mathbb{N}$.

$$\Delta x = \frac{6 - (-6)}{n} = \frac{12}{n}; \quad x_i = a + i \Delta x = -6 + \frac{12i}{n};$$

$$\begin{aligned}
 c(x_i) &= \sqrt{-x_i^2 + 36} = \sqrt{(6 - x_i)(6 + x_i)} \\
 &= \sqrt{\left(6 - \left(-6 + \frac{12i}{n}\right)\right)\left(6 + \left(-6 + \frac{12i}{n}\right)\right)} = \sqrt{\left(12 - \frac{12i}{n}\right)\left(\frac{12i}{n}\right)} ;
 \end{aligned}$$

$$R_n = \sum_{i=1}^n c(x_i) \Delta x = \frac{12}{n} \sum_{i=1}^n \sqrt{\left(\frac{12i}{n}\right)\left(12 - \frac{12i}{n}\right)} = \dots \text{ and we're stuck since } \sum \sqrt{\dots} \neq \sqrt{\sum (\dots)}$$

and we don't have a nice formula.

Note that expanding $\sum_{i=1}^n f(x_i)$ is generally a very non-trivial problem.

However, we can skip this step using a very powerful result called the

Fundamental Theorem of Calculus

This relates integral calculus with differential calculus.