Notes on Chapter 5 Sequences and Linear Difference Equations.

A first order linear difference equation is a sequence (x_n) of the form $x_{n+1} = f(x_n)$ for some function $f: \mathbb{R} \to \mathbb{R}$.

Geometric: $x_{n+1} = r x_n$ for some real number r.

General Solution: $x_n = r^n x_0$;

$$\lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } |r|<1 \\ x_0 & \text{if } r=1 \\ \operatorname{sign}(x_0)\infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<-1 \end{cases};$$

Linear: $x_{n+1} = a_n x_n + b_n$ with a_n and b_n sequences of real numbers.

If $a_n = a$ and $b_n = b$ (i.e. a_n and b_n are constant sequences):

we have $x_{n+1} = a x_n + b$

with general solution
$$x_n = \begin{cases} x_0 + nb & \text{if } a=1 \\ a^n x_0 & \text{if } b=0 \\ \left(x_0 - \frac{b}{1-a}\right)a^n + \frac{b}{1-a} & \text{if } a\neq 1 \text{ and } b\neq 0 \end{cases};$$

with limits: $\lim_{n\to\infty}(x_0) = x_0$;

$$\lim_{n\to\infty}(x_0+nb) = \begin{cases} +\infty & \text{if } b>0 \\ -\infty & \text{if } b<0 \end{cases}$$

See geometric series for $\lim_{n\to\infty} a^n x_0$.

$$\lim_{n\to\infty}\left[\left(x_0 - \frac{b}{1-a}\right)a^n + \frac{b}{1-a}\right] = \begin{cases} \operatorname{sign}\left(x_0 - \frac{b}{1-a}\right)\infty & \text{if } a>1 \\ \frac{b}{1-a} & \text{if } |a|<1 \\ \text{DNE} & \text{if } a<-1 \end{cases}$$

Application to Pharmacokinetics.

For most drugs, the amount of drug remaining $x(t)$ in the body after t hours of introduction can be modeled using the exponential decay model.

That is, $x(t) = be^{-kt}$ with $k>0$ a drug-specific decay constant and b the initial dose.

In the situation where medication is given regularly with duration τ and dosage b, (Note: τ is a Greek letter Tau, lowercase)
the amount of drug remaining can be modeled using difference equations:

$$x_{n+1} = a x_n + b \quad \text{with } a = e^{-k\tau}.$$

Here, $x_0 = b$ is the initial dosage;

x_n represents the amount of drug in the body after the n^{th} dose (including leftovers from past dosages);

$a x_n$ is the amount left after a duration of τ from the n^{th} dose (i.e. after decay); and

b is the new dosage being given, i.e. the $(n+1)^{th}$ dose;

This equation has a general solution of $x_n = \frac{b}{1-e^{-k\tau}}\left(1 - e^{-k\tau(n+1)}\right)$ with $\lim_{n\to\infty}(x_n) = \frac{b}{1-e^{-k\tau}}$;

As such, we refer to the max amount of drug in the body to be $x_{max} = \frac{b}{1-e^{-k\tau}}$ or equivalently, $b = x_{max}(1-e^{-k\tau})$;

Note that τ is inversely proportional to x_{max}; i.e. if τ decreases, x_{max} increases and if τ increases, x_{max} decreases.
So, when calculating τ, the convention is to <u>round up</u>. e.g. $\tau = 4.6$ hours can be rounded up to 5 but not to 4.5.