Instructions: Please work in small groups (3 or 4 students) on the following problems. You are expected to finish these problems even if there was not enough time in class to finish.

(1) Determine the partial fraction decomposition for the following rational function. You can leave the numerators as A, B, etc.

$$f(x) = \frac{2x^5 - 6x + 4}{x(x-2)^2(x^2+x+5)} = \frac{2x^5 - 6x + 4}{x^5 - 3x^4 + 5x^3 - 16x^2 + 20x}$$

Part (a). PFD requires that deg(numerator) < deg(denominator). We need to do long division.

$$\begin{array}{r|l} & \quad 2 \\ x^5 - 3x^4 + 5x^3 - 16x^2 + 20x & \cancel{2x^5} + 0x^4 + 0x^3 + 0x^2 - 6x + 4 \\ & +(\cancel{-2x^5} + 6x^4 - 10x^3 + 32x^2 - 40x) \\ \hline & \quad 6x^4 - 10x^3 + 32x^2 - 40x + 4 \end{array}$$

← Stop here.

Then, $f(x) = 2 + \dfrac{6x^4 - 10x^3 + 32x^2 - 40x + 4}{x^5 - 3x^4 + 5x^3 - 16x^2 + 20x}$

$= 2 + \dfrac{6x^4 - 10x^3 + 32x^2 - 40x + 4}{x(x-2)^2(x^2+x+5)}$

↑ Do PFD on this term.

Part (b). Form the PFD template.

Factors of denominator: (x) contributes $\frac{A}{x}$ (linear factor) to the PFD.

$(x-2)^2$ contributes $\frac{B}{x-2} + \frac{C}{(x-2)^2}$ (repeated linear factor) to the PFD.

(x^2+x+5) : we need to determine if this is an irreducible quadratic.

Use the discriminant $(b^2 - 4ac)$ in the quadratic formula.

$x^2 + x + 5 = 0$; $a = 1$, $b = 1$, $c = 5$.

$b^2 - 4ac = (1)^2 - 4(1)(5) = 1 - 20 = -19$;

Therefore, $x^2 + x + 5 = 0$ has imaginary solutions and $x^2 + x + 5$ is irreducible quadratic.

So, (x^2+x+5) contributes $\frac{Dx+E}{x^2+x+5}$ in the PFD.

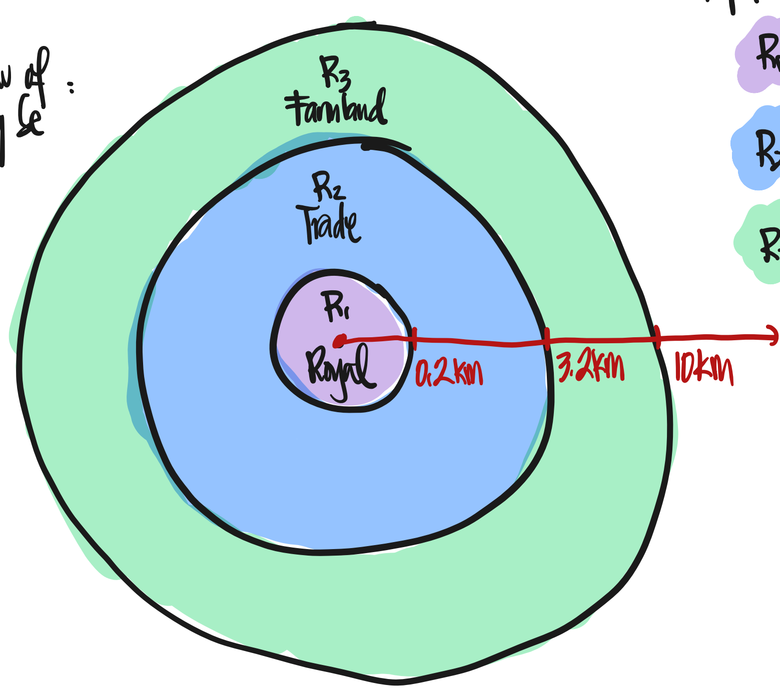
Therefore, $$\boxed{f(x) = 2 + \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+x+5}}$$

(2) (This is a radial density question). Ba Sing Se, The Impenetrable City, is laid out in three concentric circles.

The innermost circle (the royal palace) is **0.2km** in radius, and has a population density parameterized by $p_1(r) = \dfrac{\mathit{60}}{(\mathbf{0.2} - \mathit{0.8}r)}$ people per square kilometer, where r is the distance from the center of the innermost circle in kilometers. The second circle is populated by merchants and tradesfolk, and has a density of $p_2(r) = \mathit{2000}\sin\left(\dfrac{\pi(r - \mathbf{0.2})}{\mathbf{3}}\right)$ people per square kilometer, where r continues to be the distance from the center of the innermost circle in kilometers. The outermost circle is mostly farmland, and has a population density of $p_3(r) = \mathit{250}e^{-(r-\mathbf{3.2})}$ people per square kilometer, where r is still the distance from the center of the innermost circle in kilometers. The circular outermost wall is **10 kilometers** from the center of the innermost circle.

(a) Sketch a diagram of Ba Sing Se to make sure you understand how the parameters in the various equations correspond to the layout of the city. Note that the numbers **0.2**, **3**, **3.2**, and **10 kilometers** can help you determine the distance between the exact center of the city and each of the three walls. The numbers *60*, *0.8*, *2000*, and *500* affect the population density.

Top View of Ba Sing Se:



Population densities of Regions:

R_1: $p_1(r) = \dfrac{60}{0.2 - 0.8r}$

R_2: $p_2(r) = 2000\sin\left(\frac{\pi}{3}(r - 0.2)\right)$

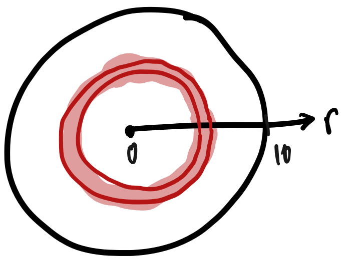
R_3: $p_3(r) = 250e^{-(r-3.2)}$

Note: The border of the 2nd circle having radius 3.2km is not explicitly stated in the problem.
This can be implied by population density needing to be positive.
In the case of R_2: $p_2(r) = 2000\sin\left(\frac{\pi}{3}(r-0.2)\right)$ is positive when $\frac{\pi}{3}(r-0.2) \in [0, \pi]$.
Then, $\frac{\pi}{3}(r-0.2) = \pi$; $r - 0.2 = 3$; $r = 3.2$km.

(b) Determine the total population of Ba Sing Se. You are welcome to use technology to help you solve any integrals you choose to set up.

population = (population density)(area).

For the integral, we do circular/concentric slices with respect to the radial axis r.



Bounds: $r \in [0, 10]$

Area of Slice: $dA = (\text{circumference})\, dr = 2\pi r\, dr$

Population in Slice: $dP = (\text{density})\, dA = 2\pi r\, (\text{density})\, dr$

$$\text{with density} = \begin{cases} p_1(r) & \text{if } r \in [0, 0.2) \\ p_2(r) & \text{if } r \in [0.2, 3.2) \\ p_3(r) & \text{if } r \in [3.2, 10) \end{cases}$$

$$\text{Then, } P = \int dP = 2\pi \left[\int_0^{0.2} r\left(\frac{60}{0.2 - 0.8r}\right) dr + \int_{0.2}^{3.2} (r) 2000 \sin\left(\tfrac{\pi}{3}(r - 0.2)\right) dr + \int_{3.2}^{10} (r) 250 e^{-(r-3.2)} dr \right]$$

Possible ways to solve: Use u-sub with $u = 0.2 - 0.8r$; IBP with $u = 2000r$, $dv = \sin(\cdots)\, dr$; IBP with $u = 250r$, $dv = e^{(\cdots)}\, dr$

$= \cdots (\text{solver}) \cdots = 2\pi(15.2 + 6493.5 + 1046.9) = \boxed{47\,473 \text{ people in Ba Sing Se}}$

(c) Determine the average population density of Ba Sing Se in people per square kilometer. Compare that to the maximum population density of the city.

Part (1). average population $= \dfrac{\text{total population}}{\text{total area}}$; From (b): total pop. $P = 47\,473$;

Total Area: Ba Sing Se is a circle with radius $r = 10$ km: $A = \pi r^2 = \pi(10)^2 = 100\pi$;

average population density $= \dfrac{47\,473}{100\pi} = \boxed{151 \text{ people/km}^2}$

Part (2). What is the maximum population density of city?

On R_1: max of $p_1(r) = \dfrac{60}{0.2 - 0.8r}$ on $r \in [0, 0.2]$ is $p_1(0.2) = 1500$;

On R_2: max of $p_2(r) = 2000 \sin\left(\frac{\pi}{3}(r - 0.2)\right)$ on $r \in [0.2, 3.2]$ is $p_2(1.7) = 2000 = 2000 \sin\left(\frac{\pi}{2}\right)$;

On R_3: max of $p_3(r) = 250 e^{-(r-3.2)}$ on $r \in [3.2, 10]$ is $p_3(3.2) = 250 e^0 = 250$;

max population density $= \boxed{2000 \text{ people/km}^2}$

Problem 2. Desmos Graph + Integration.

1. $p_1(r) = \dfrac{60}{0.2 - 0.8r} \{0 \le r \le 0.2\}$

2. $p_2(r) = 2000 \sin\left(\dfrac{\pi}{3}(r - 0.2)\right) \{0.2 \le r \le 3.2\}$

3. $p_3(r) = 250e^{-(r-3.2)} \{3.2 \le r \le 10\}$

4. $A_1 = \displaystyle\int_0^{0.2} r \cdot p_1(r)\, dr$

   $A_1 = 15.1769608581$

5. $A_2 = \displaystyle\int_{0.2}^{3.2} r \cdot p_2(r)\, dr$

   $A_2 = 6493.52167815$

6. $A_3 = \displaystyle\int_{3.2}^{10} r \cdot p_3(r)\, dr$

   $A_3 = 1046.93711834$

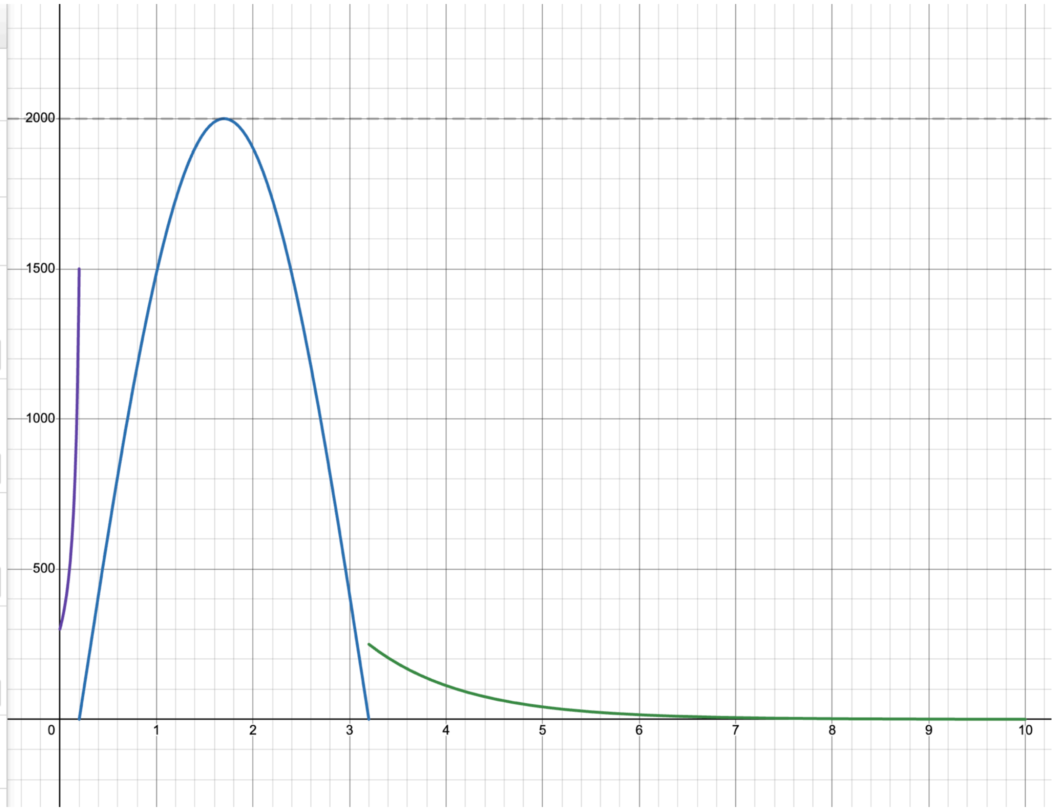
7. $P = \dfrac{2\pi(A_1 + A_2 + A_3)}{100\pi}$

   $P = 151.112715147$

8. $y = 2000$

   −10 2000

9.



(3) In your written homework, you were asked to compare $\int_1^\infty \frac{1}{x}dx$ to $\int_1^\infty \frac{1}{x^2}dx$. One of these integrals converges; the other does not. Your question today is: Can you determine the smallest value of p for which the improper integral $\int_1^\infty \frac{1}{x^p}dx$ converges?

See next page for calculation.

$\int_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p \in (1, \infty)$.

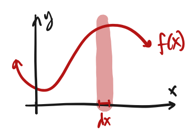
So, there does not exist a smallest p such that $\int_1^\infty \frac{1}{x^p}\,dx$ converges.

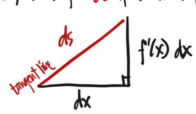
(4) Set up the following integrals:

(a) The integral that will tell you the arc length of $f(x) = \ln x$ from $x = 1$ to $x = 5$.

$f'(x) = \frac{1}{x}$ and $S = \int_{x=1}^{x=5} \sqrt{1 + (f'(x))^2}\, dx = \boxed{\int_1^5 \sqrt{1 + \frac{1}{x^2}}\, dx}$

Note. This comes from slicing the curve with respect to x and using the Pythagorean theorem. The curve of $f(x)$ on $[x, x+dx]$ is approximated by the tangent line of $f(x)$. So, the length ds of the curve of $f(x)$ on $x \in [x, x+dx]$ is:





and $ds = \sqrt{(dx)^2 + (f'(x)\,dx)^2} = \underline{\sqrt{1 + (f'(x))^2}\, dx}$

(b) The integral that will tell you the volume of a solid formed by rotating the part of $f(x) = e^{-x}$ that lies in the first quadrant around the x-axis. What kind of integral is this? Is this a finite volume? Hint: $(e^{-x})^2 = e^{-2x}$

See next page for details:

What kind of integral is this?

If using the Washer Method: $V = \int_0^\infty \pi(e^{-x})^2\, dx$: "Infinite Bound" integral

If using the Shells Method: $V = \int_0^1 2\pi(y)(-1)\ln(y)\, dy$: "Infinite Integrand" integral

Is this a finite volume? YES with $V = \frac{\pi}{2}$.

Problem 3. We examine 2 cases.

① Assume $p=1$. Then, $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln|x|\right]_1^b = \lim_{b\to\infty}\left(\ln|b| - \ln(1)\right) = \infty$. Div.

② Assume $p \neq 1$. Then, $\int_1^\infty \frac{1}{x^p}\,dx = \lim_{b\to\infty}\int_1^b x^{-p}\,dx = \lim_{b\to\infty}\left[\frac{x^{-p+1}}{-p+1}\right]_1^b = \left(\frac{1}{-p+1}\right)\lim_{b\to\infty}\left(b^{-p+1} - 1\right)$;

We examine 2 further cases.

ⓐ Assume $-p+1>0$ (i.e. positive). Then, $\lim_{b\to\infty} b^{-p+1} = \lim_{b\to\infty} b^{(\text{positive})} = \infty$ and $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

ⓑ Assume $-p+1<0$ (i.e. negative).

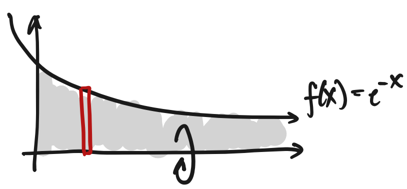
Then, $\lim_{b\to\infty} b^{-p+1} = \lim_{b\to\infty} \frac{1}{b^{(\text{positive})}} = 0$ and $\int_1^\infty \frac{1}{x^p}\,dx = \left(\frac{1}{-p+1}\right)\lim_{b\to\infty}\left(b^{-p+1}-1\right) = \frac{(-1)}{-p+1}$;

Therefore, $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $-p+1<0$ ← Solve this inequality for p.

$-p+1<0$; $-p<-1$; $p>1$ (inequality flips). That is, $p\in(1,\infty)$.

Problem 4(b). Solid: Region is bounded by $y=e^{-x}$ and the x-axis, rotated about the x-axis.

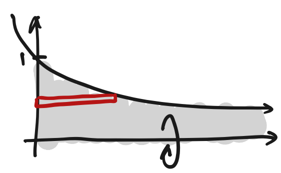
Method 1. Use the Washer Method.



bounds: $x\in[0,\infty)$, thickness: dx
outer radius: $r_{out} = y = e^{-x}$;
inner radius: $r_{in} = 0$.

$$V = \int_0^\infty \pi\left(e^{-x}\right)^2 dx = \lim_{b\to\infty}\int_0^b \pi e^{-2x}\,dx = \lim_{b\to\infty}\left[\frac{\pi}{-2}e^{-2x}\right]_0^b = \left(-\frac{\pi}{2}\right)\lim_{b\to\infty}\left[e^{-2b} - e^0\right] = \boxed{\frac{\pi}{2}}$$

Method 2. Use Cylindrical Shells



thickness: dy, bounds: $y\in[0,1]$
radius: y
height: $h = x$ with $y=e^{-x}$, $\ln(y) = -x$, $x = -\ln(y)$

$$V = \int_0^1 2\pi(y)(-1)\ln(y)\,dy = (-2\pi)\lim_{a\to 0}\int_a^1 y\ln(y)\,dy;$$

$$I = \int y\ln(y)\,dy = \frac{1}{2}y^2\ln(y) - \int \frac{1}{2}y^2(y^{-1})\,dy = \frac{1}{2}y^2\ln(y) - \int\frac{1}{2}y\,dy = \frac{1}{2}y^2\ln(y) - \frac{1}{4}y^2 + C;$$

IBP: $u=\ln(y)$, $dv = y\,dy$; $du = y^{-1}\,dy$, $v=\frac{1}{2}y^2$

$$V = (-2\pi)\lim_{a\to 0}\left[\frac{1}{2}y^2\ln(y) - \frac{1}{4}y^2\right]_a^1 = (-2\pi)\lim_{a\to 0}\left(0 - \frac{1}{4} - \frac{1}{2}a^2\ln(a) + \frac{1}{4}a^2\right)$$

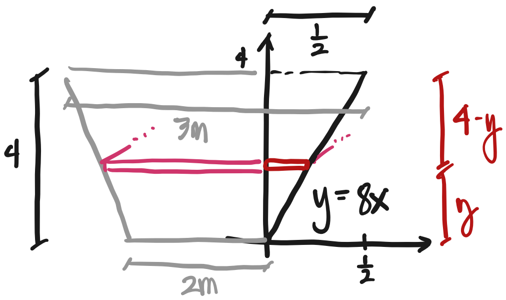
$$= (-2\pi)\left(-\frac{1}{4}\right) + \pi\lim_{a\to 0}\left(a^2\ln(a)\right) + 0 \overset{(*)}{=} \frac{\pi}{2} + 0 = \boxed{\frac{\pi}{2}};$$

(*): $\lim_{a\to 0}\left(a^2\ln(a)\right) \overset{\text{indefinite form}}{=} \lim_{a\to 0}\left(\frac{\ln(a)}{a^{-2}}\right) \overset{\text{L'H}}{=} \lim_{a\to 0}\frac{\frac{1}{a}}{-2a^{-3}} = \lim_{a\to 0}\left(\frac{1}{a}\right)\left(-\frac{1}{2}a^3\right) = \lim_{a\to 0}\left(-\frac{1}{2}a^2\right) = 0$

(5) A 10 foot long by 4 foot tall trough with a trapezoidal cross-section of 2 feet at the bottom and 3 feet at the top is full of water.

(a) How much work is required to pump all the water out of the trough to the level of the top of the trough? (Water weighs about 62 lbs/cubic foot).

Model the trapezoid using the line passing through $(0,0)$ and $(\frac{1}{2},4)$: $y=(4)(\frac{1}{2})^{-1}x = 8x$.



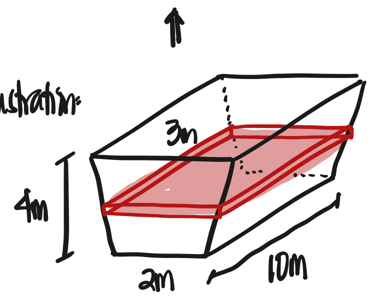
thickness: dy, bounds: $y\in[0,4]$

volume of slice: $dV = (2+2x)(10)\,dy$ with $y=8x$; $x=\frac{1}{8}y$;

$$dV = (2+2(\tfrac{1}{8}y))(10)\,dy = 10(2+\tfrac{1}{4}y)\,dy = \tfrac{10}{4}(8+y)\,dy$$

pumping distance: $4-y$

Illustration:



$$W = \int (\text{displacement})\,dV = 62\int_0^4 (4-y)(\tfrac{10}{4})(8+y)\,dy$$

$$= 62(\tfrac{10}{4})\int_0^4 32-4y-y^2\,dy = \frac{620}{4}\left[32y-\frac{4}{2}y^2-\frac{1}{3}y^3\right]_0^4$$

$$= \frac{620}{4}\left[32(4)-2(4)^2-\tfrac{1}{3}(4)^3-0\right] = \boxed{\frac{34720}{3} \approx 11573}.$$

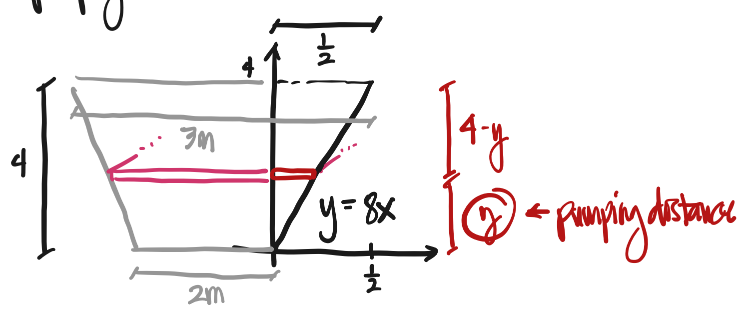
(b) If the trough has a tap at the bottom, how much work could the water do if instead of being pumped out the top, it was drained out the bottom?

We could use the same model as part (a) but change the pumping distance.

That is: thickness: dy, bounds: $y\in[0,4]$

volume of slice: $dV = \frac{10}{4}(y+8)\,dy$

pumping distance: y



$$\text{Then, } W = \int_0^4 62(y)(\tfrac{10}{4})(y+8)\,dy = \frac{620}{4}\int_0^4 y^2+8y\,dy = \frac{620}{4}\left[\frac{1}{3}y^3+\frac{8}{2}y^2\right]_0^4$$

$$= \frac{620}{4}\left[\tfrac{1}{3}(4)^3+4(4)^2-0\right] = (\tfrac{1}{3})(39680) \approx \underline{13227}.$$

(6) What questions do you have for your TA?