EC 476 PART IV PROBLEM SET 2 MULTITASKING

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An agent Arthur (A) works for a principal Patty (P) who asks him to perform two tasks, 1 and 2, which can either success $\mathbf{Y}_i = 1$ or fail $\mathbf{Y}_i = 0$ for task outputs $\mathbf{Y}_1$ and $\mathbf{Y}_2$.

Both tasks are unpleasant for Arthur, but Patty will benefit from their being done. Arthur's cost of effort in the two tasks is given by the quadratic form

$$c(e_1, e_2) = \frac{1}{2}\big(e_1^2 + 2\gamma e_1 e_2 + e_2^2\big). \tag{1}$$

The probability that each task succeeds is exactly equal to his effort input,

$$\mathbb{P}\left[\tilde{y}_i = 1 \,\middle|\, e_i\right] = e_i \tag{2}$$

for $i \in \{1, 2\}$.

1. (a) Interpret the sign of γ.
   (b) Give examples of workers performing multiple tasks for which $\gamma > 0$ and for which $\gamma < 0$.

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2. Patty cannot observe the output of task two directly, but rather only a signal about, $\tilde{\sigma}_2$ correlated with it according to

$$\mathbb{P}\left[\tilde{\sigma}_2 = 1 \,\middle|\, \mathbf{Y}_2 = 1\right] = p > \underline{p} > \frac{1}{2} \tag{3}$$

and

$$\mathbb{P}\left[\tilde{\sigma}_2 = 1 \,\middle|\, \mathbf{Y}_2 = 0\right] = 1 - p \tag{4}$$

—so $1 - p$ is the probability of a false positive and a false negative.

Suppose $\underline{p}$ is large enough to permit the first-order approach/to induce positive wages for both actions.

(a) Are Type I and Type II errors usually complementary like this? Try to give examples of each or argue why you cannot.

(b) Notice that Patty receives output $y_2 \in \{0, 1\}$ but cannot contract on it directly. Why might this be the case? Give an example.

- Note that the classic Holmstrom moral hazard problem that we studied with Amrita has output directly as a noisy measure of effort, as would be the case here if $\tilde{\sigma}_2$ rather than $\tilde{y}_2$ accrued to Patty.

3. Patty offers Arthur a contract $b = (b_1, b_2)$, where

$$b_1 : y_1 \mapsto b(y_1) \tag{5}$$

and

$$b_2 : \sigma_2 \mapsto b(\sigma_2) \tag{6}$$

to maximize her expected profits subject to Arthur's incentive compatibility, limited liability, and participation constraints.

Suppose that Arthur's outside option is zero and Patty cannot fine him.

(a) Show that the expected signal from task 2 conditional on effort e_2 is

$$\mathbb{E}\left[\tilde{\sigma}_2 \,\middle|\, e_2\right] = pe_2 + (1-p)(1-e_2). \tag{7}$$

(b) Draw the game associated form and write Patty's problem for choosing b at the first node.

(c) Argue that $b_1(0) = b_2(0) = 0$. Thus we write $b_1 \equiv b_1(1)$ and $b_2 \equiv b_2(1)$ henceforth.

4. (a) Show that given the contract b, Arthur's incentive compatibility condition is

$$\big(\hat{e}_1, \hat{e}_2\big) = \operatorname{Arg\,max}\left\{e_1 b_1 + \big(pe_2 + (1-p)(1-e_2)\big)b_2 - c(e_1, e_2)\ ;\ e_1, e_2 \in [0,1]\right\}. \tag{8}$$

(b) The objective is concave, so compute the first-order conditions to find the optimum:

$$b_1 = \hat{e}_1 + \gamma \hat{e}_2 \tag{9}$$

and

$$b_2(2p-1) = \gamma \hat{e}_1 + \hat{e}_2. \tag{10}$$

(c) Notice that there is a one-to-one relationship between the variables b_1 and b_2 and the variables e_1 and e_2. This will be useful in what follows. Is it related to the Revelation Principle?[1]

5. Patty now has to maximize her expected profit

$$\Pi_{\mathrm{P}} = (1-b_1)e_1 + e_2 - \big(1-p+(2p-1)e_2\big)b_2 \tag{11}$$

over b_1 and b_2 subject $e_1 = \hat{e}_1$ and $e_2 = \hat{e}_2$.

[1] Hint: No.

(a) Argue that you can solve Patty's problem in (e_1, e_2) space equivalently to (b_1, b_2) space.

(b) Show that Patty's problem is thus to maximize

$$e_1 - e_1^2 - 2\gamma e_1 e_2 + e_2 - e_2^2 - \frac{(1-p)(\gamma e_1 + e_2)}{2p-1}. \qquad (12)$$

(c) Use the first-order approach to get

$$1 - 2e_1 - 2\gamma e_2 + \frac{\gamma(1-p)}{2p-1} = 0 \qquad (13)$$

and

$$1 - 2\gamma e_1 - 2\gamma e_2 - \frac{1-p}{2p-1} = 0. \qquad (14)$$

(d) Write (immediately) that

$$b_1 = \frac{1}{2}\left(1 - \frac{\gamma(1-p)}{2p-1}\right) \qquad (15)$$

and

$$b_2 = \frac{1}{2}\frac{1}{2p-1}\left(1 - \frac{1-p}{2p-1}\right). \qquad (16)$$

6. (a) Remember that the roles of p and γ and recall that $p > 1/2$ so

$$\frac{1-p}{2p-1} > 0. \qquad (17)$$

(b) Comment on what happens when $p \to 1$ and why. What about when $p \to 1/2$?

(c) Comment on why p enters the formula for b_1 even though the first task is measured perfectly.

(d) You should have noticed in the previous question that this interaction is all about γ. Comment on what happens as γ varies, noticing that

$$\frac{\partial b_1}{\partial \gamma} = \frac{1}{2}\frac{p-1}{2p-1} < 0. \tag{18}$$

(e) Plot $b_2(p)$ and comment.

(f) What is the smallest $p = \underline{p}$ that will induce positive wages for both tasks? Does this coincide with positive effort for all γ? Comment.