PROCYCLICAL PROMISES INSTIGATE INSTABILITY*

Jason Roderick Donaldson†
LSE

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DRAFT

Abstract

A model of borrowing constraints based on limited enforcement and capital diversion shows that entrepreneurs' procyclicality increases their debt capacity, leading to endogenous fluctuations in capital prices and expected aggregate output. Two mechanisms are at work: (1) Because creditors assume strong bargaining positions when capital prices are high, they are more willing to finance projects that pay off in booms. Hence (2) procyclical entrepreneurs stretch their endowments further, allocating more capital to productive projects and driving up the price of capital in the market. Even though the worst recessions occur after credit booms in which procyclical firms are highly levered, borrowing is inefficiently low from a second-best perspective.

1 Introduction

The economy is made of promises. To borrow to fund projects, entrepreneurs must make promises and creditors must trust in them. No promise is more important than corporate debt secured by productive capital. Given the assumptions that creditors enforce repayment only via the threat to seize capital and that debtors can scrap their projects early in the market, forgoing future cash flows but abandoning their debts, the model below shows, first, that procyclicality, defined as the comovement

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†Contact: jasonrdonaldson@gmail.com

between a project's success and the price of capital, increases debt capacity and, second, that fluctuations in capital prices and expected output arise endogenously as a result of financing frictions alone.

Entrepreneurs have no incentive to continue when their projects fail, so they divert capital, leaving debt holders empty-handed. Entrepreneurs who succeed keep their assets in place, but renegotiate their debts down to their lenders' outside options. Procyclical entrepreneurs succeed in booms, when outsiders' redeployment value of capital is high; exactly then, when they have incentive to stick around to pay off their debts, their creditors assume a strong bargaining position. Because no one can commit not to renegotiate or not to abscond, procyclical entrepreneurs give their creditors a valuable threat by which to induce repayment ex post. Procyclical entrepreneurs' promises are credible and hence valuable. Procyclicality increases debt capacity. This interaction between debtors' threats to renegotiate and to divert capital causes fluctuations in capital prices and expected output that are absent in the benchmark models with perfect enforcement, no borrowing, renegotiation without capital diversion, and capital diversion without renegotiation. Procyclicality is a valuable resource for an entrepreneur and for society from a second-best perspective, because it increases borrowing capacity, allowing capital to flow to those who deploy it most efficiently.

Procyclicality stretches entrepreneurs' endowments allowing them to increase output, so taxing counter-cyclical entrepreneurs to subsidize procyclical entrepreneurs increases expected total production and utilitarian welfare. Entrepreneurs' projects are risky. The worst crashes (lowest output) in the model occur after procyclical entrepreneurs borrow heavily and their investments fail, but their leverage is still inefficiently constrained ex ante.

The remainder of the introduction summarizes the model and main results—first generally (subsection 1.1) and then in detail (subsection 1.2)—and describes the paper's context in the literature and its incremental contribution relative to several papers (subsection 1.3). Section 2 sets up the formal model and section 3 solves it. Section 4 describes four benchmarks to contextualize the results. Section 5.1 describes a welfare-improving policy intervention. Section 6 states real-world analogues of model variables (subsection 6.1) and enumerates predictions on the signs of coefficients of linear regressions from correlations in the model (subsection 6.2) and a natural experiment viewed as a shock to the model (subsection 6.3).

1.1 Description of Model and Summary of Results

In the model, overlapping generations of entrepreneurs with risky, highly productive, constant-returns technologies borrow from long-lived investors with safe, decreasing returns technologies to buy capital and invest.

Creditors' ability to enforce contracts is limited. They anticipate two constraints to repayment, the seizure constraint and the continuation constraint. The seizure constraint results from entrepreneurs' inability to commit not to renegotiate; it says that they never repay more than their creditors' liquidation value of the capital collateralizing their loans. The continuation constraint results from entrepreneurs' inability to commit not to divert capital; it says that they never repay at all if their revenue from diverting capital—liquidating it early but forgoing future cash flows—exceeds their payoff from completing their projects net of repayments. High capital prices ex post give creditors strong bargaining positions. Low capital prices ex ante allow entrepreneurs to invest cheaply. Definition 2.7.1 defines a project's cyclicality as the ratio of the expected price of capital conditional on its success to the price of capital today. Proposition 3.4.1 and its corollaries say that an entrepreneur's debt capacity depends on only his project's cyclicality and, further, that the size of his balance sheet is a constant multiple of his equity endowment, where the scale factor—termed the "collateral multiplier"—is increasing in project cyclicality. The reason is that the price of capital in the event of success determines what he will repay and, therefore, what he can borrow, his debt capacity; dividing the size of his balance sheet by the capital price today gives his total capital investment. The higher the price in the event of success tomorrow the more he can borrow; the lower the price today the more he can buy. To be procyclical—to succeed and repay when capital prices are high relative to today—is to be able to promise high repayments credibly and to be able to buy capital cheaply, to be able to lever up. Because procyclical entrepreneurs can stretch their endowments further, prices are higher when procyclical entrepreneurs have high endowments than when countercyclical entrepreneurs do (proposition 3.6.3). Investors are deep-pocketed, so only the distribution of endowments, not the aggregate quantity, drives the result. While standard models (like the CAPM) suggest that countercyclicality is a valuable resource for firms because the insurance premium lowers their cost of capital, the limited-enforcement channel here inverts the result: procyclicality increases commitment power, which loosens financial constraints. Prices cycle because promises are procyclical; aggregate risk comes out of the promise-keeping mechanism, namely from the interaction, and only the interaction, between borrowers' capital diversion and debt renegotiation. All macroeconomic fluctuations in the real economy result from constraints on the right-hand side of entrepreneurs' balance sheets.

1.2 Model Specifics

At each date $t \in \mathbb{Z}$ the markov state ω_t assumes one of three equally likely values, a, b, or 0. Risk-neutral entrepreneurs are born at each date and die at the following date; they have one of two technologies, called α and β. In state a,

α-entrepreneurs have endowment w and β-entrepreneurs have endowment nil. In state b, β-entrepreneurs have endowment w and α entrepreneurs have endowment nil. In state 0 both types have endowment nil. Deep-pocketed, infinitely-lived, risk-neutral investors have a decreasing-returns deterministic technology. Entrepreneurs' technologies are always more productive than investors', but they are risky: α-entrepreneurs' technologies payoff only in state a and β-entrepreneurs' technologies payoff only in state 0, otherwise they are identical.

Entrepreneurs borrow from investors via secured debt contracts: creditors gain the right to seize capital k_t should they default. Entrepreneurs can divert capital ex post, after learning the state ω_{t+1}, in which case they liquidate at the market price p_{t+1}; they forfeit the proceeds of their investments but also renege on their debts and leave their creditors without a means to enforce repayment, since no capital is left to seize. Whenever entrepreneurs fail they divert all their capital. Assumption 2.6.1 asserts that discounting is high enough to bound prices above so that entrepreneurs never prefer diversion to repayment when they succeed (lemma 3.3.1). Creditors know that α-entrepreneurs repay if and only if $\omega_{t+1} = a$ and β-entrepreneurs repay if and only if $\omega_{t+1} = 0$.

If an entrepreneur continues his project, he renegotiates his debts by making his creditor a take-it-or-leave-it offer. Investors' redeployment value is determined by the market price of capital, so t-entrepreneurs never repay more than $p_{t+1}k_t$ (cf. lemma 3.2.1). Since creditors anticipate that entrepreneurs will divert their capital unless they succeed, investors lend α-entrepreneurs up to $p^a k_t/(3R)$ and β-entrepreneurs up to $p^0 k_t/(3R)$, where $p^{\omega_{t+1}} := p_{t+1}$ in state ω_{t+1}, R is the gross interest rate, and $1/3$ is the probability of success. The formulae reveal immediately that creditors lend more to finance projects that succeed when capital prices are high, the essence of the first main result: procyclicality increases debt capacity (3.4.1 and its corollaries).

When entrepreneurs borrow heavily, they drive up the price of capital since investors' marginal productivity increases as the residual supply of capital they hold decreases. Corollary 3.4.4 notes that entrepreneurs can borrow only if they have positive endowments—thanks to renegotiation, an entrepreneur with zero endowment can credibly promise to repay no more than tomorrow's value of the capital he buys with his loan, and an investor is better-off purchasing capital in the market today than lending to him. For this reason, investors hold all of the capital in state 0 (when no entrepreneur has an endowment), so their marginal productivity is low and thus capital is cheap. Because β-entrepreneurs succeed in state 0, when the price of capital is low, creditors assume a weak bargaining position when debtors have incentive to continue their projects. β-entrepreneurs' equilibrium repayments are thus relatively low and so, therefore, is their borrowing capacity when they have positive endowments, in state b. α-entrepreneurs succeed and repay in state a, when the next generation of α-entrepreneurs borrows and invests, driving up the price of capital. Unlike β-entrepreneurs, α-entrepreneurs put their creditors in a strong bargaining position when their projects succeed, making their capital good collateral and allowing them to borrow more freely when they have positive endowments, in state a. Since α-entrepreneurs stretch their endowments further than β-entrepreneurs, investors are left with more capital in b-states than in a-states, driving prices up when α-entrepreneurs invest and damping them when β-entrepreneurs do. Capital prices are low in state 0, higher in state b, and higher still in state a, $p^0 < p^b < p^a$. This is the second main result: price fluctuations result from changes in debt capacity (proposition 3.6.3).

Price fluctuations are endogenous in the sense that they result only from the interaction between the renegotiation and capital diversion frictions. Without the two frictions together, prices coincide in a- and b-states. Section 4 considers four benchmarks—the cases of prefect enforcement (subsection 4.1), no borrowing (subsection 4.2), renegotiation without capital diversion (subsection 4.3), and capital diversion without renegotiation (subsection 4.4)—and demonstrates that the price in state a equals the price in state b in each. With perfect enforcement, the most productive agent always holds all of the capital; since technologies are constant across states, so is the marginal productivity and therefore the price. With no borrowing, entrepreneurs always invest all of their endowment in capital; since their endowments are identical in a- and b-states so is the price of capital—without borrowing, debt capacity is irrelevant. With renegotiation but not capital diversion, entrepreneurs can borrow up to the discounted future value of the capital they hold—collateral is state independent; since all states are equally likely, they can pledge the same in a- and b-states and prices do not change. With capital diversion but not renegotiation, when entrepreneurs fail they divert capital, and when they succeed, they repay the face value of debt so long as it does not exceed their total cash flow less their value of diverted capital; since all technologies are equally productive, all entrepreneurs can pledge and hence borrow the same amount and capital prices are the same in a- and b-states.

With both frictions, α-entrepreneurs are endogenously procyclical. They borrow more and invest in the economy's most productive technology, boosting prices to generate fluctuations in expected output. Their procyclicality is a scarce resource. β-entrepreneurs would wish to pay to sacrifice their insurance technology for the procyclical technology. Further, proposition 5.2.1 says that, given conditions for price effects not to be too large, a social planner who anticipates the agents' enforcement problems and resulting borrowing constraints would tax β-entrepreneurs and subsidize α-entrepreneurs to increase expected output and utilitarian welfare. The intervention loosens the hold that limited enforcement has on the economy, injecting equity into the balance sheets of entrepreneurs whose constraints are least crippling and thereby allowing more capital to flow into the most productive technologies. The policy leads to deeper recessions and increased price volatility, because it makes prices and investment even higher in booms (a states) and output plummets after highly levered entrepreneurs fail. The welfare implications therefore cast some doubt on the benefits of implementing macroprudential policies to maintain financial stability.

1.3 Literature and Motivation

As in Shleifer and Vishny (1992), in my model general equilibrium asset liquidation values determine debt capacity. In both their model and mine, asset buyers' funding constraints and the wedge between the value of the assets in first- and second-best use, which they term "asset illiquidity", pin down liquidation values. They assume that debtors cannot reschedule their loans, so liquidation values do not matter when borrowers succeed and repay, but only matter when they fail and default, when creditors seize collateral and sell it to the highest bidder. They model two firms in an industry with correlated projects; they emphasize that when one is forced to liquidate the other is likely to be cash-strapped, its financial constraints preventing it from acquiring its competitor's old assets, leaving them to be redeployed inefficiently by an industry outsider. In my model, in contrast, liquidation values are most important when projects succeed because they determine outsiders' threat points in renegotiation, while, when entrepreneurs fail, they have incentive to divert capital, decreasing the quantity of liquidatable assets. Shleifer and Vishny conclude that, because cyclical assets are illiquid in downturns, "cyclical and growth assets are poor candidates for debt finance" (p. 1359); in my model, the interaction between renegotiation and capital diversion flips the result.

Hart and Moore (1998) also focuses on the interaction between these frictions. As in my model, creditors' right to foreclose on capital is the essential enforcement mechanism. In their three-date model, an entrepreneur requires a fixed capital investment to start a project comprising risky returns and asset liquidation values at the middle and final dates. The main results say, roughly, that when only the interim payoffs are risky, optimal debt contracts maximize financial slack, whereas when only the terminal return is risky, optimal debt contracts constitute entrepreneurs' "maximum equity participation". Depending on his project's specific risks, an entrepreneur either borrows to capacity to maintain a cushion of working capital or puts up all of his own money to take on as little debt as possible in order to minimize liquidation when the surplus lost from foreclosure is greatest. They do not analyze the comovement of liquidation and continuation values, the variable of primary interest for me. More specifically, after the entrepreneur gets his enterprise off the ground, he renegotiates his debts and scales up his project at the interim date, when he also potentially diverts capital but not assets in place. Liquidation—tantamount to withdrawal of the entrepreneur's specific capital—preempts the project's bearing fruit at the final date, when the entrepreneur will never repay anything. I weaken the assumption that assets in place cannot be diverted, supposing instead that a market exists where the entrepreneur can liquidate by himself. In my model the creditor faces a further constraint to repayment: the debtor absconds unless his project's terminal cash flows less repayments exceed his revenues from diverting capital early. Because projects are scaleable, entrepreneurs always borrow to capacity—or write the "fastest" debt contract in Hart and Moore's language—not because they wish to maintain financial slack, but, rather, because they want to buy more capital. My innovations with respect to this paper are, firstly, to show that the comovement between liquidation values and inside returns—procyclicality—is a valuable resource for financially constrained entrepreneurs and, secondly, to endogenize liquidation values in a dynamic general equilibrium framework.

Kiyotaki and Moore (1997) studies price and output fluctuations when a small, unanticipated technological shock hits the steady state of an infinite-horizon economy in which entrepreneurs must post assets to secure their loans—capital famously plays a dual role, it yields output and serves as collateral. The resulting price change is the same order of magnitude as the productivity change. Because prices represent the entire future productivity of assets, in the Arrow–Debreu world a momentary change in productivity leads "the price to experience a tiny blip" (p. 214), but, in Kiyotaki and Moore's model, since increased productivity loosens borrowing constraints allowing further asset purchases which, in turn, increase productivity and loosen borrowing constraints (repeat). The interplay between the two functions of capital converts the blip into a wallop. The feedback loop between slackened budget constraints and increased borrowing capacity works both within and between periods, effects which Kiyotaki and Moore refer to as the static and intertemporal multipliers. The collateral multiplier (section 3.4) in my model relies on the same spiralling back-and-forth, but, since the constrained agents—the entrepreneurs—are short-lived, the long-term consequences of immediate constraints are absent. But, because my model is stochastic and repayments depend on both entrepreneurs' success and the aggregate state tomorrow, the one-period-ahead effects are subtler; my analysis separates the changes in the price today from changes in the price tomorrow. Capital demand curves can slope upward in Kiyotaki and Moore, because more expensive capital means more valuable collateral which comes with increased borrowing capacity, output, and profits. My overlapping generations set-up renders cumulative wealth unimportant: prices are only forward-looking—the only state variable is the aggregate state. Demand curves have the vanilla downward slope in today's price, but they slope upward in the price when repayment occurs, namely in the event that entrepreneurs succeed tomorrow. Price changes, not price levels, matter in my model; specifically, the ratio of the expected price given success to the price today—entrepreneurs' cyclicality—determines demand today. Assuming that entrepreneurs live for only two dates allows me to respond to the challenge that Kiyotaki and Moore pose in their concluding remarks, "The pressing next step in the research is to construct a fully fledged stochastic model, in which a shock is not a zero probability event and is rationally anticipated" (p. 243), but my main contribution is to demonstrate that price and output fluctuations result endogenously from the collateral friction alone, even absent exogenous variation in productivity. In my model economic instability arises even when no blip at all shows up in the Arrow–Debreu archetype.

In a 2003 paper, Krishnamurthy builds a stripped-down version of Kiyotaki and Moore's 1997 model to analyze the hypothesis that state-contingent hedging contracts prevent the economy from amplifying shocks. He shows that, even if insurers require collateral to force entrepreneurs to repay, permitting hedging kills amplification. When limited enforcement is two-sided, however, and entrepreneurs also demand collateral from insurers to secure their hedges, the supply of collateral fails to stretch far enough to insure all risk and the amplification mechanism reemerges. Since in my model entrepreneurs can divert capital as well as cash flows, the optimal state-contingent contract yields the same transfers as standard debt after renegotiation or abscondence (cf. the discussion in section 2.6).

Lorenzoni (2008) uses the Krishnamurthy (2003) structure to assess the welfare consequences of leverage. In a three-date model, entrepreneurs first borrow from investors via state-contingent contracts and then, at the interim date, they receive perfectly correlated payoffs and invest in deterministic constant-returns projects. As in my model, deep-pocketed investors have a decreasing-returns technology; they are marginal since entrepreneurs are constrained. In both models, entrepreneurs' borrowing constraints lead to variations in the residual capital supply held by investors—and thus to changes in the marginal productivity of capital—that drive price fluctuations. In Lorenzoni's model, entrepreneurs sell poorly performing assets to pay their creditors. As they liquidate more capital, investors hold more, lowering marginal productivity and, therefore, prices, thereby forcing entrepreneurs to sell more capital to meet their debts. The more they borrow the more they must promise to repay and the more they must liquidate when returns are low. Because agents are price-takers, they fail to internalize the negative impact of heavy leverage on prices. The main result is that, because of this pecuniary externality, the competitive equilibrium is constrained suboptimal: because highly-levered entrepreneurs must liquidate assets to repay their debts at the middle date, it allocates too much capital to the unproductive investors in expectation. Capping borrowing leads to a pareto improvement. Lorenzoni warns against over-borrowing. In my model, entrepreneurs do not borrow enough. The inefficiency is more direct: repayment constraints prevent investors from lending to the agents who use it most productively and, as in Lorenzoni's low-return states, the economy never achieves the first-best allocation. Transferring wealth to productive agents with more balance sheet capacity, namely to procyclical entrepreneurs, increases welfare because they gear up to invest more. While my prescription is orthogonal to Lorenzoni's, it requires the caveat that price effects like those he focuses on must not be too big: as procyclical agents receive subsidies they drive up the capital price, reducing their ability to stretch their endowments and damping the benefits of the transfers (that entrepreneurs's initial wealth is not too large suffices for the result to hold at the margin, cf. proposition 5.2.1).

The theoretical framework of general equilibrium with endogenous contracts and collateral constraints that Geanakoplos built in his 1997 article "Promises, Promises" has lead him to argue, like Lorenzoni, for the regulation of excessive leverage, citing increased volatility and severe crashes as features of an economy in which the leverage cycle is left unmanaged. In their 2004 paper, he and Kubler use the equilibrium concept to demonstrate that a maturity mismatch can arise endogenously, causing inefficient liquidation when collateral prices fall. As in Lorenzoni's model, small borrowers do not take the collateral price effects into account when they borrow, leading to excessive leverage which a regulator should cap for a pareto improvement. The work is most important for my model in its conceptual underpinnings. I use Geanakoplos's definition of a contract as a promise-collateral pair determined in equilibrium; I also borrow the notion that some goods function as collateral (my capital) while others do not (my fruit)—property rights are effectively enforced only over capital goods.

Many models study the role of limited enforcement in dynamic financial contracting or in macroeconomics. Cooley, Marimon, and Quadrini (2004) does both. They use a general equilibrium framework to endogenize entrepreneurs' outside options from repudiation in an infinite-horizon model of capital diversion. They show that limited enforcement amplifies productivity shocks: when new projects are highly productive, investors must give them strong incentives not to abandon their commitments and search for new opportunities, loosening incentive constraints and allowing more efficient capital allocation. Like Kiyotaki and Moore (1997), they consider steady state equilibria and simulate their responses to exogenous shocks. Analogously, in my model, high capital prices increase entrepreneurs' incentive to abscond, but since they are short-lived with bang-bang technologies, capital price fluctuations do not determine equilibrium repudiation—they quit only when their own projects fail—but affect only renegotiated repayments. Contractual constraints make economies that rely on debt to allocate capital sensitive to productivity shocks. Kiyotaki and Moore (1997) and Cooley et al. (2004) each demonstrates a channel by which limited enforcement amplifies shocks, based on constraints to check renegotiation and capital diversion, respectively. My model shows that the two mechanisms do more when they interact: beyond aggravating extrinsic output changes, they generate fluctuations endogenously.

2 Model

2.1 Background Environment

Lying in the background is the probability space $(\mathfrak{S}, \mathsf{F}, \mathbb{P})$ with the set of states $\mathfrak{S} = \{(\omega_t)_{t\in\mathbb{Z}}\ ;\ \omega_t \in \{a, b, 0\}\}$, the natural filtration F of ω_t (viewed as a random process), and the probability $\mathbb{P}$ with $\mathbb{P}\{\omega_s \mid \mathcal{F}_t\} = 1/3$ for each $\mathcal{F}_t \in \mathsf{F}$ and any ω_s with $s > t$.

Overlaid is an extensive form game in which the refinement from $\mathcal{F}_{t-1}$ to $\mathcal{F}_t$ is nature acting at date t, termed "the realization of ω_t". The histories that include the realization of ω_t but not of ω_{t+1} constitute period t.

"Today" and "tomorrow" refer to $\mathcal{F}_t$- and $\mathcal{F}_{t+1}$-measurable variables from the point of view of period t.

2.2 Goods, Players, and Technologies

The numeraire is a perishable consumption good called fruit and measured in pounds. Capital is in supply K and produces fruit according to players' technologies; it does not depreciate. p_t denotes the price of capital at date t. A player with technology τ and capital k produces $\tau(k)(\omega)$ of fruit if ω realizes tomorrow.

A unit continuum of long-lived, deep-pocketed players called investors have technology $\tau = \gamma$, where $\gamma' > 0$, $\gamma'' < 0$, and $\gamma'(0) = A$. At each date they act to maximize the expected value of future consumption discounted at gross rate R,

$$U_t(c) = \mathbb{E}_t\left[\sum_{s=t}^{\infty} \frac{c_s}{R^{s-t}}\right], \tag{1}$$

over feasible consumption profiles $\{c_s\}_{s\geq t}$ (given beliefs about other players' action profiles).

Entrepreneurs are short-lived players with risky technologies. At each date a unit of α-entrepreneurs and a unit of β-entrepreneurs are born, where α-entrepreneurs have w-pound endowments if $\omega_t = a$ and nothing otherwise and, likewise, β-entrepreneurs have w-pound endowments if $w_t = b$ and nothing otherwise. An entrepreneur born at date t with technology τ is called a t- or τ-entrepreneur, depending on the context; at date t he is called young and at date $t+1$ (when he will die) he is called old.

α-entrepreneurs have technology

$$\alpha(k)(\omega_{t+1}) = \begin{cases} 3Ak & \text{if } \omega_{t+1} = \alpha, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

and β-entrepreneurs have technology

$$\beta(k)(\omega_{t+1}) = \begin{cases} 3Ak & \text{if } \omega_{t+1} = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

An entrepreneur born at date t acts to maximize his expected consumption at $t+1$.

A project is an entrepreneur's technology given his capital investment. Liquidation is the extraction of capital from a project before it bears fruit. A project is successful if $\tau(k)(\omega_{t+1}) \neq 0$.

e and i denote typical entrepreneurs and investors; α, β refer to types of entrepreneurs. Below, k_t^τ denotes the capital τ-entrepreneurs hold and and k_t^e denotes the capital entrepreneurs hold cumulatively, $k_t^e = k_t^\alpha + k_t^\beta$.

2.3 Contracts

A contract to borrow a pound is a pair $\mathsf{c} = \left(F \mid \bar{k}\right)$. It specifies that a creditor delivers one pound to the debtor today and the debtor pays F pounds to the creditor tomorrow. If he does not, the creditor has the right to seize k units of the debtor's capital; seizure destroys the successful projects' fruit. With all contracts comes the risk that the debtor will divert capital, denoted $\zeta = \mathrm{d}$, or, if he does not, $\zeta = \neg\mathrm{d}$, the risk that he will renegotiate to a repayment $F' < F$. Subsection 2.4 below describes the timing of the stage game that players play in each period, including the renegotiation protocol, which follows Hart and Moore (1998) and ascribes bargaining to power entrepreneurs. Denote a τ-entrepreneur's actual repayment at $t+1$ associated with contract c by the random variable $T_{t+1}(\mathsf{c}\,;\tau) \equiv T(\mathsf{c})$—viz. the contract c written at date t induces transfer $T(\mathsf{c})(\omega_{t+1})$ when ω_{t+1} realizes at date $t+1$. The value of the contract c to an investor is $\mathbb{E}_t\left[T(\mathsf{c})\right]/R$.

2.4 Stage Game

In each period $t \in \mathbb{Z}$, first the state realizes, revealing the payoffs of old entrepreneurs. Then, young entrepreneurs are born, determining the date-t price of capital and thus the liquidation values of old entrepreneurs' collateral. Old entrepreneurs either divert and liquidate their projects or wait for them to bear fruit, only to renegotiate their debts; then they consume and die. Meanwhile, young entrepreneurs borrow to fund their projects and buy capital in the market, determining the liquidation values for investors and the previous generation of entrepreneurs.

The following sequence of moves describes the extensive form of the stage game.

1. ω_t realizes. t-entrepreneurs are born.
2. Each old entrepreneur e either diverts capital $\zeta = \mathrm{d}$ or does not $\zeta = \neg\mathrm{d}$.
   - If $\zeta = \mathrm{d}$, e sells his capital k_{t-1} for $p_t k_{t-1}$; he makes no transfer to his creditor.
3. If $\zeta = \neg\mathrm{d}$, e's project pays off and he offers a repayment F' to his creditor.
   - If the creditor accepts the offer, $\xi = \mathrm{a}$, or if $F' \geq F$, then e makes him transfer F'; if the creditor rejects the offer, $\xi = \neg\mathrm{a}$, then he seizes the collateral backing his loan $\bar{k}_{t-1}$ to obtain $p_t \bar{k}_{t-1}$.
4. Each t-entrepreneur offers an (arbitrary) investor a pair (ℓ, c) to borrow ℓ pounds via the contract $\mathsf{c} = \big(F \,\big|\, \bar{k}_t\big)$.
   - Each investor accepts or rejects the offer.
5. Each player submits a demand for capital $k_t(p_t)$ (subject his budget constraint); the price p_t clears the capital market.
   - If an entrepreneur has pledged more capital than he holds (if $\bar{k}_t > k_t$), his creditor seizes his capital and invests it, leaving him with nil.
6. Old entrepreneurs and investors consume; young entrepreneurs and investors invest.

2.5 Solution Concept

The solution concept is markov equilibrium.

Since, therefore, p_t depends on only ω_t, henceforth use the following notation.

NOTATION 2.5.1. *Write*

$$p^{\omega_t} := p_t \tag{4}$$

and

$$\bar{p} := \mathbb{E}\left[p_{t+1}\right] = \frac{p^a + p^b + p^0}{3}. \tag{5}$$

And note that $\bar{p} \equiv \mathbb{E}_t\left[p_{t+1}\right]$.

2.6 Assumptions

Since entrepreneurs' technologies return nil given failure, scrapping unsuccessful projects is efficient and no inefficient liquidation will occur in equilibrium. As a result, nothing is lost in assuming that debt is non-contingent; equilibrium transfers remain unchanged if the aggregate state is contractible and contracts are optimal.

The assumption below that investors are relatively impatient ensures that prices are never so high (cf. lemma 3.3) that entrepreneurs prefer to divert capital and liquidate than to consume the fruit of a successful project tomorrow (lemma 3.3.1).

Assumption 2.6.1.

$$R > 4/3. \tag{6}$$

The assumption suffices to streamline proofs and ensure uniqueness of the equilibrium action profile and price system (equations (56)-(58)) by providing a uniform bound on prices.

To ensure that entrepreneurs' borrowing constraints bind—that they do not hold all capital (corollary 3.4.5)—assume further that the entrepreneurs' endowment is small relative to the supply of capital. Specifically, assume that entrepreneurs' endowments are always less than the present value of the economy's maximum expected output, i.e. the expected output obtained if entrepreneurs did invest all capital.

Assumption 2.6.2.

$$Rw \leq AK. \tag{7}$$

2.7 Notations

The outcome of their projects will determine old entrepreneurs' behaviour. The following definition gives a notation for the state in which projects succeed.

Notation 2.7.1. *$\sigma(\tau) = \omega_t$ if the project $\tau(k)$ succeeds in state ω_t, i.e. $\sigma(\alpha) = \alpha$ and $\sigma(\beta) = 0$.*

The price of capital given success will determine young entrepreneurs' borrowing capacity. Since projects succeed in only one state, the expected capital price given success is just the price in that state,

$$\mathbb{E}\left[p_{t+1} \,\middle|\, \omega_{t+1} = \sigma(\tau) \right] = p^{\sigma(\tau)}. \tag{8}$$

A special notation for this price is convenient.

Notation 2.7.2.

$$P^{\tau} := p^{\sigma(\tau)}. \tag{9}$$

This notation facilitates the notion of project cyclicality as the ratio of the value of capital given success to the value of capital today.

Definition 2.7.1. *The cyclicality χ of a project is*

$$\chi_t(\tau) := \frac{P^{\tau}}{p_t}. \tag{10}$$

A project is called procyclical if it succeeds when prices are increasing or $\chi \geq 1$ and called countercycical if it succeeds when prices are decreasing or $\chi < 1$. In equilibrium an increasing bijection will pair prices and expected output, so procyclciality will coincide with success when expected output increases.

3 Results

3.1 Investors' Indifference Condition and Price Bounds

An investor i who holds capital $k_t^i > 0$ at date t must be indifferent between consuming and buying capital. The condition that $\gamma'(0) = A$ ensures the identity holds even in the corner in which investors hold no capital, $k_t^i = 0$. Since investors are deep-pocketed and risk neutral and γ is concave, the pricing identity follows immediately from the first-order condition

$$\frac{\partial}{\partial k}\bigg|_{k=k_t^i} \left(-p_t k + \frac{1}{R}\Big(\gamma(k) + \mathbb{E}_t\left[p_{t+1}k\right]\Big)\right) = 0. \tag{11}$$

Lemma 3.1.1 below summarizes.

LEMMA 3.1.1.

$$p_t = \frac{\gamma'\big(k_t^i\big) + \mathbb{E}_t\left[p_{t+1}\right]}{R} \tag{12}$$

where k_t^i is the capital held by any investor i at date t.

This expression implies that the price of capital is bounded above by that of a perpetuity that pays A at each date when the gross interest rate is R.

LEMMA 3.1.2.

$$p_t \leq \frac{A}{R-1}. \tag{13}$$

Proof. The result follows from $\gamma'' < 0$:

$$\begin{aligned}
p_t &= \frac{1}{R}\big(\gamma'(k^i) + \mathbb{E}_t\left[p_{t+1}\right]\big) && (14)\\
&\leq \frac{1}{R}\big(\gamma'(0) + \mathbb{E}_t\left[p_{t+1}\right]\big) && (15)\\
&\leq \frac{\gamma'(0)}{R} + \frac{\gamma'(0)}{R^2} + \frac{\mathbb{E}_t\left[p_{t+2}\right]}{R} && (16)\\
&\leq \gamma'(0)\left(\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \cdots\right) && (17)\\
&= \frac{\gamma'(0)}{R-1} && (18)\\
&= \frac{A}{R-1}. && (19)
\end{aligned}$$

□

The investors' indifference condition and the restriction to markov equilibria provide a lower bound on prices.

Lemma 3.1.3. *For any* $\omega \in \{a, b, 0\}$,

$$3Rp_t > p^{\omega_t}. \tag{20}$$

Proof. Immediately from equation (12),

$$\begin{aligned}
3Rp_t &= 3\gamma'(k_t^i) + 3\mathbb{E}_t\left[p^{\omega_{t+1}}\right] && (21)\\
&= 3\gamma'(k_t^i) + p^a + p^b + p^0 && (22)\\
&> \max\left\{p^a, p^b, p^0\right\} && (23)\\
&\geq p^{\omega_t}. && (24)
\end{aligned}$$

□

Lemma constitutes a bound on cyclicality:

$$\begin{aligned}
\chi_t^\tau &= \frac{P^\tau}{p_t} && (25)\\
&\leq \frac{\max\left\{p^a, p^b, p^0\right\}}{p_t} && (26)\\
&\leq 3R. && (27)
\end{aligned}$$

3.2 Renegotiation

Lemma 3.3 and assumption 2.6.1 suffice to solve the stage game by backward induction. First: because the entrepreneur has the bargaining power, he repays at most his creditor's seizure value.

Lemma 3.2.1. *If* $\zeta = \neg\mathrm{d}$,

$$T(F \mid \bar{k}) = \min\left\{F\,,\, p_{t+1}\bar{k}\right\}. \tag{28}$$

Proof. See appendix A.2 for the standard argument.

□

3.3 Capital Diversion

A failing entrepreneur may divert capital and liquidate it, obtaining the value of his assets in place, forgoing his project's fruit but avoiding paying his debts. A successful entrepreneur repays as long as his payoff from continued production is sufficiently high relative to his anticipated repayment. Now, lemma 3.3.1 demonstrates that assumption 2.6.1 ensures that successful entrepreneurs continue their projects and thus make transfers to their creditors. The result emphasizes the importance of dynamic borrowing relationships; debtors repay their debts only because they anticipate future cash flows and must avoid early liquidation.

LEMMA 3.3.1. *A τ-entrepreneur plays $\zeta = \neg \mathrm{d}$ if and only if $\omega_t = \sigma(\tau)$.*

Proof. The proof is in appendix A.3. Sufficiency follows from noting that if $F > 0$ an entrepreneur with no cash flow always diverts because otherwise he would forfeit F. Necessity results from bounding prices relative to cash flows using lemma and assumption 2.6.1.

□

3.4 Collateral Multiplier

When an entrepreneur acquires investment capital, his stock of collateral expands, thus allowing him to borrow to acquire still more capital. This dual role of capital creates a multiplier effect whereby an increase in capital leads to a further increase in capital. The same dual role of capital generates the static multiplier in Kiyotaki and Moore (1997); the mechanism below is different because, in my stochastic model, a borrower's collateral multiplier is a function only of the cyclicality of his project.

An investor accepts a τ-entrepreneur's offer to borrow ℓ against the promise $\mathsf{c} = (\infty \,|\, k)$ whenever present value of the expected transfer—the probability of success times the value of collateral given success divided by investors' discount rate—exceeds the value of the loan or the borrowing constraint

$$\ell \leq \mathbb{E}_t \left[T(\mathsf{c}) \right] = \frac{P^\tau k}{3R} \tag{29}$$

is satisfied. With fruit w he can buy w/p_t units of capital which he can pledge to borrow $P^\tau w/(3Rp_t)$ pounds, with which he will buy an additional $P^\tau_{t+1} w/(3Rp_t^2)$ units of capital, which he can pledge to borrow.... Viz., with fruit endowment w an entrepreneur can acquire capital up to

$$\begin{aligned} k^\tau &= \frac{w}{p_t} + \left(\frac{P^\tau}{3Rp_t} \right) \frac{w}{p_t} + \left(\frac{P^\tau}{3R} \right)^2 \frac{w}{p_t} + \cdots && (30) \\ &= \frac{w}{p_t} \sum_{n=0}^{\infty} \left(\frac{P^\tau}{3Rp_t} \right)^n && (31) \\ &= \frac{3w}{3p_t - P^\tau/R}. && (32) \end{aligned}$$

Proposition 3.4.1 below demonstrates that entrepreneurs' balance sheets stretch by a multiplier that depends only on their cyclicality; the proof arrives at the same formula as the series above as a solution of the linear system of binding budget and borrowing constraints (and serves as a reminder of the connection between geometric series and fixed points—recall the school proof: $\sum x^n = 1 + x \sum x^n$).

PROPOSITION 3.4.1. *A τ-entrepreneur with endowment w can hold assets worth up to*

$$A^\tau(w) = S^{\chi_t(\tau)} w \tag{33}$$

where

$$S^{\chi_t(\tau)} := \frac{3R}{3R - \chi_t(\tau)}. \tag{34}$$

Proof. The maximum liability ℓ a τ-entrepreneur can secure with capital k is given by his binding borrowing constraint (inequality (29) holding with equality)

$$\ell = \frac{P^\tau k}{3R} \tag{35}$$

and the maximum capital an entrepreneur can obtain comes from his binding budget constraint given this loan,

$$p_t k = w + \ell. \tag{36}$$

Substituting k from equation (36) into equation (35) gives

$$\ell = \frac{P^\tau}{3Rp_t}(w + \ell) \tag{37}$$

$$= \frac{\chi_t(\tau)(w + \ell)}{3R}. \tag{38}$$

Rearranging gives

$$\ell = \frac{\chi_t(\tau) w}{3R - \chi_t(\tau)} \tag{39}$$

and the asset value is

$$A^\tau(w) = w + \mathrm{DC}^{\chi_t(\tau)} = \frac{3Rw}{3R - \chi_t(\tau)}. \tag{40}$$

□

The constant of proportionality S^χ, called the collateral multiplier, describes the gross maximum feasible leverage of an entrepreneur with cyclicality χ—his ability to lever up does not depend on his equity endowment w—

$$\frac{\text{assets}}{\text{equity}} \leq \frac{S^\chi w}{w} = \frac{3R}{3R - \chi}. \tag{41}$$

Figure 1 illustrates the maximal balance sheet expansion.

The expression for the maximum size of an entrepreneur's balance sheet immediately gives an expression for his maximal liability, or debt capacity DC^χ, which

A BALANCE SHEET STRETCHES

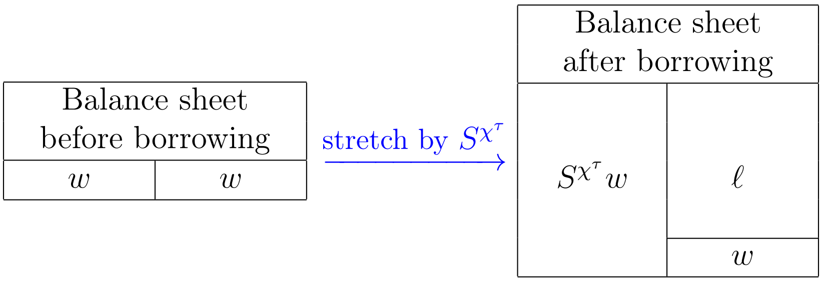

Figure 1: Entrepreneurs' balance sheets expand by up to the collateral multiplier S^χ.

is likewise proportional to his endowment by a multiplier which depends on only cyclicality, as now stated in corollary 3.4.1.

COROLLARY 3.4.1. *An entrepreneur with endowment w has debt capacity*

$$\mathrm{DC}^\chi(w) = \frac{\chi w}{3R - \chi}. \tag{42}$$

The formula for the collateral multiplier reveals that cyclicality is valuable to entrepreneurs, granting them commitment power: the procyclical entrepreneurs can borrow more and invest more, as corollary 3.4.2 now states.

COROLLARY 3.4.2. *The multiplier S^χ and the debt capacity DC^χ are increasing in entrepreneurs' cyclicality χ.*

Proof. Immediate from differentiation of S^χ and DC^χ.

□

A procyclical borrower can not only borrow more than a countercyclical borrower initially, but he can also buy more capital with his loan and thus reuse his initial liquidity to lever up even further. Thus, the sensitivity of debt capacity to cyclicality increases in cyclicality, as stated formally in corollary 3.4.3. The observation offers an insight outside of the model: more levered firms are more sensitive to cyclicality $\chi_t(\tau) = P^\tau/p_t$, and therefore must adjust their balance sheets more in response to fluctuations in the price p_t.

COROLLARY 3.4.3. *The multiplier S^χ and the debt capacity DC^χ are convex in entrepreneurs' cyclicality χ.*

Proof. Immediate from second differentiation of S^χ and DC^χ and the bound $\chi < 3R$ from inequality (27).

□

Now, corollary 3.4.4 states the immediate result that, since debt capacity is proportional to equity, penniless entrepreneurs have no way to raise funds.

Debt Capacity as a Function of Cyclicality

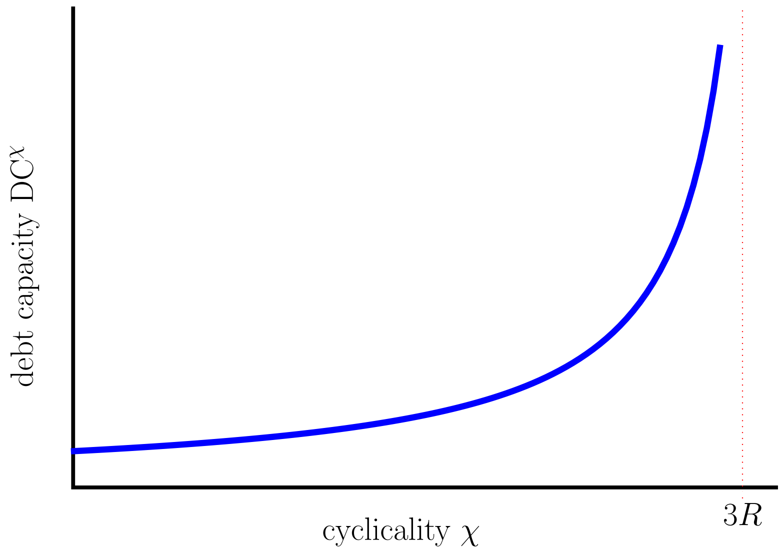

Figure 2: Debt capacity is an increasing, convex function of cyclicality.

Corollary 3.4.4. *Entrepreneurs with endowment zero do not invest, i.e.* $k_t^{\beta} = 0$ *if* $\omega_t \in \{a, 0\}$ *and* $k_t^{\alpha} = 0$ *if* $\omega_t \in \{b, 0\}$.

Finally, the upper bound on entrepreneurs' ability to borrow combines with assumption 2.6.2 (which says that entrepreneurs' endowments are not too large) to imply that entrepreneurs never hold all of the capital, ensuring an interior solution.

Corollary 3.4.5. *Entrepreneurs never hold all of the capital,* $k_t^e < K$.

Proof. The proof is in appendix A.4. It supposes that entrepreneurs do hold all the capital in one state and uses the markov assumption to tighten the lower bound on the price. It then combines the upper bound on balance sheet size (proposition 3.4.1) with assumption 2.6.2 for a contradiction. □

3.5 Entrepreneurs Borrow to Capacity

Entrepreneurs will always borrow to capacity. Since they consume only late, they borrow as much as they can so long as expected repayments are not prohibitively high relative to capital prices today. To strictly prefer to borrow, entrepreneurs must be infra-marginal; that they never hold all of the capital (corollary 3.4.5) will suffice.

Any investor to whom an entrepreneur offers (ℓ, c) accepts if and only if

$$\frac{\min\left\{F\,,\,P^{\tau}\bar{k}\right\}}{3R} \geq \ell, \tag{43}$$

since the debtor repays only one-third of the time, when he succeeds. Each t-entrepreneur thus determines $k, \bar{k}, F$, and ℓ to solve the programme of maximizing

$$\mathbb{E}_t\left[p_{t+1}k\right] + \frac{1}{3}\Big(3Ak - \min\left\{F\,,\,P^{\tau}\bar{k}\right\}\Big) \tag{44}$$

subject to

$$k \leq \bar{k}, \tag{45}$$

$$p_t k \leq w + \ell, \tag{46}$$

$$\ell \leq \frac{\min\left\{F\,,\,P^{\tau}k\right\}}{3R} \tag{47}$$

(having omitted the time subscripts and player superscripts on the choice variables). The expectation in the objective embeds the value of liquidation when the project succeeds as well as in both states when it fails.

Lemma 3.5.1. $F \geq P^{\tau}k$.

Proof. Since his objective is increasing in k, the entrepreneur's programme reduces to determining k and F to maximize

$$\bar{p}\,k + \frac{1}{3}\Big(3Ak - \min\left\{F\,,\,P^{\tau}k\right\}\Big) \tag{48}$$

subject to the borrowing constraint

$$p_t k \leq w + \frac{\min\left\{F\,,\,P^{\tau}k\right\}}{3R}. \tag{49}$$

Now suppose (in anticipation of a contradiction) $F < P^{\tau}k$. The objective is increasing in k and decreasing in F so the constraint

$$p_t k \leq w + \frac{F}{3R} \tag{50}$$

binds. The unconstrained objective is

$$\left(\frac{w}{p_t} + \frac{F}{3Rp_t}\right)\bar{p} + \left(\frac{w}{p_t} + \frac{F}{3Rp_t}\right)A - \frac{F}{3} = \frac{1}{p_t}\left[\left(\frac{A+\bar{p}}{R} - p_t\right)\frac{F}{3} + (A+\bar{p})w\right]. \tag{51}$$

Equation (12) and the assumption that $\gamma' < A$ imply

$$p_t = \frac{\gamma'(k_t^i) + \bar{p}}{R} \leq \frac{A+\bar{p}}{R}. \tag{52}$$

If the inequality is strict, then the objective is strictly increasing in F so the solution contradicts the assumption $F < P^{\tau}k$.

The inequality must bind:

$$\frac{\gamma'(k_t^i)+\bar{p}}{R}=\frac{A+\bar{p}}{R} \tag{53}$$

or $\gamma'(k_t^i)=A$, so $k_t^i=0$ and $k_t^e=K$, which contradicts corollary 3.4.5.

□

3.6 Prices

Lemma 3.5.1 says entrepreneurs always borrow to capacity and equation (32) says entrepreneurs hold maximal capital, so

$$k_t^e=\frac{3w}{3p_t-P^\tau/R}=S^\tau w/p_t. \tag{54}$$

Corollary 3.4.4 says that only α entrepreneurs invest in state a and only β entrepreneurs invest in state b, so if $\omega\in\{a,b\}$ then

$$k_t^i=K-S^\tau w/p_t, \tag{55}$$

and if $\omega=0$ then $k_t^e=0$ and $k_t^i=K$. The equilibrium price system now follows from equation (12), establishing proposition 3.6.1 below.

Proposition 3.6.1. *In equilibrium, the prices solve*

$$Rp^a=\bar{p}+\gamma'\big(K-S_t^\alpha w/p^a\big), \tag{56}$$
$$Rp^b=\bar{p}+\gamma'\big(K-S_t^\beta w/p^b\big), \tag{57}$$
$$Rp^0=\bar{p}+\gamma'(K). \tag{58}$$

Proposition 3.6.2. *The system (56)-(58) has a solution (a markov equilibrium exists).*

Proof. The proof is in appendix A.5. It recasts the system (56)-(58) as a fixed point problem in order to apply Brouwer's theorem after some massaging to ensure the image is compact despite the singularities in the denominator of S^τ. □

Analysis of the system in proposition 3.6.1 gives the next main result: when α-entrepreneurs have positive endowments prices are higher than when β-entrepreneurs have positive endowments. Procyclicality, not insurance, is the valuable resource in this economy.

Proposition 3.6.3.

$$p^0<p^b<p^a. \tag{59}$$

Proof. The proof is in two steps.

Step 1: Lemma 3.1 implies immediately that

$$\chi_t(\tau) < 3R \tag{60}$$

so $S^\chi > 0$, giving that

$$p^0 < \min\left\{p^a, p^b\right\} \tag{61}$$

by $\gamma' < 0$.

Step 2: Suppose (in anticipation of a contradiction) that $p^b \geq p^a$ so

$$Rp^b - Rp^a = \gamma'\left(K - \frac{3Rw}{3Rp^b - p^0}\right) - \gamma'\left(K - \frac{3Rw}{3Rp^a - p^a}\right) > 0, \tag{62}$$

having subtracted equation (56) from equation (57). Or, equivalently, by $\gamma' < 0$,

$$K - \frac{3Rw}{3Rp^b - p^0} < K - \frac{3Rw}{3Rp^a - p^a}. \tag{63}$$

Since the denominators are positive by lemma 3.1,

$$3Rp^a - p^a > 3Rp^b - p^0. \tag{64}$$

Rewrite to see that

$$3R(p^a - p^b) > p^a - p^0 > 0, \tag{65}$$

where the final inequality follows from step 1 and implies that $p^a > p^b$, a contradiction. □

4 Benchmarks

4.1 Complete Markets/Perfect Enforcement

Since agents are risk-neutral, with no enforcement problems the most productive agents hold all of the capital. The marginal return on capital is A in every state, because investors' technologies don't change. The following proposition is immediate.

PROPOSITION 4.1.1. *With perfect enforcement,*

$$p^a = p^b = p^0 = \frac{A}{R-1}. \tag{66}$$

There is no aggregate risk in the economy.

4.2 No Borrowing

When agents cannot borrow at all, entrepreneurs spend their endowments and only their endowments on capital. In states a and b their (binding) budget constraints read

$$p^{\omega} k^{e,\omega} = w, \tag{67}$$

so $k^{e,a} = 1/p^a$, $k^{e,b} = 1/p^b$, and $k^{e,0} = 0$. The pricing equation (12) implies

$$Rp^a = \bar{p} + \gamma' \Big(K - w/p^a\Big), \tag{68}$$

$$Rp^b = \bar{p} + \gamma' \left(K - w/p^b\right), \tag{69}$$

$$Rp^0 = \bar{p} + \gamma' (K). \tag{70}$$

PROPOSITION 4.2.1. *With no borrowing, $p^a = p^b$.*

Proof. Suppose (in anticipation of a contradiction) that $p^a > p^b$. Subtracting equation (69) from (68) implies

$$\gamma' \Big(K - w/p^a\Big) > \gamma' \left(K - w/p^b\right) > 0 \tag{71}$$

and, since γ' is decreasing, $p^b > p^a$, a contradiction. Thus $p^b \leq p^a$. Repeating the argument supposing $p^b > p^a$ gives the result. $\square$

4.3 Renegotiation without Capital Diversion

With renegotiation but not capital diversion borrowers repay the value of their capital in each state,

$$T = p_{t+1} k_t^e \tag{72}$$

so the binding borrowing constraint reads

$$\ell = \frac{\bar{p} k_t^e}{R} \tag{73}$$

and the budget constraint implies

$$p^{\omega} k^{e,\omega} = w + \ell = w + \frac{\bar{p} k^{e,\omega}}{R} \tag{74}$$

if $\omega \in \{a, b\}$ and $k^{e,0} = 0$. The price system is now

$$Rp^a = \bar{p} + \gamma' \left(K - \frac{Rw}{Rp^a - \bar{p}}\right), \tag{75}$$

$$Rp^b = \bar{p} + \gamma' \left(K - \frac{Rw}{Rp^b - \bar{p}}\right), \tag{76}$$

$$Rp^0 = \bar{p} + \gamma' (K). \tag{77}$$

Proposition 4.3.1. *Without capital diversion, $p^a = p^b$.*

Proof. The proof is in appendix A.6. It is almost identical to the proof of proposition 4.2.1. $\square$

4.4 Capital Diversion without Renegotiation

If borrowers divert capital when it is profitable but never renegotiate their debts, they repay only when they succeed, with repayments capped by incentive constraints, when they play $\zeta = \neg\mathrm{d}$ whenever

$$3Ak_t + p_{t+1}k_t - T \geq p_{t+1}k_t \tag{78}$$

or $T \leq 3Ak_t$. The proof of lemma 3.5.1, stating that entrepreneurs assume maximum leverage, implies here that entrepreneurs set the maximum face value that will induce repayment, or $F = 3Ak_t$. As in the full model, only entrepreneurs with positive endowments can borrow in equilibrium, but the result no longer follows from the formula (39) for entrepreneurs' debt capacity and requires a separate proof.

Lemma 4.4.1. *Without renegotiation, entrepreneurs with zero endowment do not borrow.*

Proof. The proof is in two steps. Step 1 demonstrates that if t prices are low, entrepreneurs are never constrained. Step 2 shows without constraints prices are high, a contradiction.

Step 1: A τ-entrepreneur with capital k repays nil when he fails and at most $3Ak$ when he succeeds, so his binding borrowing constraint gives his maximal liability,

$$\ell = \frac{Ak}{R}. \tag{79}$$

If his endowment is nil, his budget constraint reads

$$p_t k \leq \frac{Ak}{R}. \tag{80}$$

If $p_t \leq A/R$ he is unconstrained and if $p_t > A/R$ he cannot borrow.

Step 2: Suppose (in anticipation of a contradiction) that $p_t \leq A/R$. Call the state ω so $p_t = p^\omega$. Then entrepreneurs are unconstrained and the pricing equation (12) gives

$$\begin{aligned} p_t = p^\omega &= \frac{A + \bar{p}}{R} && (81)\\ &= \frac{3A + p^a + p^b + p^0}{3R} && (82)\\ &\geq \frac{3A + p^\omega}{3R} && (83) \end{aligned}$$

which combines with the hypothesis to give

$$p^{\omega} \geq \frac{3A}{3R-1} > \frac{A}{R} \geq p^{\omega}, \tag{84}$$

a contradiction.

Therefore $p_t < A/R$ and entrepreneurs without endowments cannot borrow.

□

The price system without renegotiation follows from entrepreneurs' borrowing to capacity: if $\omega \in \{a, b\}$ then

$$p^{\omega} k^{e,\omega} = w + \frac{Ak^{e,\omega}}{R} \tag{85}$$

or

$$k^{e,\omega} = \frac{Rw}{Rp^{\omega} - A} \tag{86}$$

and if $\omega = 0$ then $k^{e,0} = 0$.

$$Rp^a = \bar{p} + \gamma'\Big(K - \frac{Rw}{Rp^a - A}\Big), \tag{87}$$

$$Rp^b = \bar{p} + \gamma'\left(K - \frac{Rw}{Rp^b - A}\right), \tag{88}$$

$$Rp^0 = \bar{p} + \gamma'(K). \tag{89}$$

PROPOSITION 4.4.1. *Without renegotiation, $p^a = p^b$.*

Proof. The proof is in appendix A.7. It is almost identical to the proofs of propositions 4.2.1 and 4.3.1.

□

5 Welfare and Policy

5.1 Welfare

The date-t expectation of $(t+1)$-expected output (isomorphic to utilitarian welfare thanks to transferable utility) is

$$W := \mathbb{E}_t\left[\alpha\left(k^{\alpha}_{t+1}\right) + \beta\big(k^{\beta}_{t+1}\big) + \gamma\left(k^{\gamma}_{t+1}\right)\right] \tag{90}$$

$$= \frac{1}{3}\left(\frac{3ARw}{3Rp^a - p^a} + \gamma\left(K - \frac{3Rw}{3Rp^a - p^a}\right) + \right. \\ \left. + \frac{3ARw}{3Rp^b - p^0} + \gamma\left(K - \frac{3Rw}{3Rp^b - p^0}\right) + \gamma(K)\right). \tag{91}$$

If a t-entrepreneur is equally likely to be type-α or type-β, increases in output are ex ante pareto improvements—all unborn entrepreneurs are better off.

5.2 Taxes and Subsidies

Allocating more capital to entrepreneurs increases welfare because it allows the most productive agents to invest more. Reallocating wealth only among entrepreneurs may also lead to an ex ante pareto improvement (in the sense just described in section 5.1 above). A social planner who must break even in expectation can levy a tax ε on β-entrepreneurs in state b and subsidize α-entrepreneurs in state a, making welfare

$$W(\varepsilon) := \frac{1}{3}\left(\frac{3AR(w+\varepsilon)}{3Rp^a_\varepsilon - p^a_\varepsilon} + \gamma\left(K - \frac{3R(w+\varepsilon)}{3Rp^a_\varepsilon - p^a_\varepsilon}\right) + \right. \tag{92}$$

$$\left. + \frac{3AR(w-\varepsilon)}{3Rp^b_\varepsilon - p^0_\varepsilon} + \gamma\left(K - \frac{3R(w-\varepsilon)}{3Rp^b_\varepsilon - p^0_\varepsilon}\right) + \gamma(K)\right). \tag{93}$$

Subscripts now denote values of the transfer ε (and no longer time). A dot above a variable denotes the rate of change with respect to the tax level, $\dot{x} := dx/d\varepsilon$. The shorthands

$$\gamma'_a := \gamma'\left(K - \frac{3Rw}{(3R-1)p^a_0}\right), \qquad \gamma''_a := \gamma''\left(K - \frac{3Rw}{(3R-1)p^a_0}\right), \tag{94}$$

$$\gamma'_b := \gamma'\left(K - \frac{3Rw}{3Rp^b_0 - p^0_0}\right), \qquad \gamma''_b := \gamma'\left(K - \frac{3Rw}{3Rp^b_0 - p^0_0}\right) \tag{95}$$

save space below.

The next result, lemma 5.2.1, gives a necessary and sufficient condition for a transfer from β-entrepreneurs to α-entrepreneurs to increase welfare.

LEMMA 5.2.1. $\dot{W}(0) > 0$ *if and only if*

$$1 - w\,\frac{\dot{p}^a_0}{p^a_0} > -\frac{A-\gamma'_a}{A-\gamma'_b}\,\frac{(3R-1)p^a_0}{3Rp^b_0 - p^0_0}\left(1 + w\,\frac{3R\dot{p}^b_0 - \dot{p}^0_0}{3Rp^b - p^0}\right). \tag{96}$$

Proof. Differentiating W gives

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \frac{w+\varepsilon}{(3R-1)p^a_\varepsilon} > \frac{A-{\gamma_b}'}{A-{\gamma_a}'}\,\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \frac{w-\varepsilon}{3Rp^b_\varepsilon - p^0_\varepsilon}. \tag{97}$$

Applying the quotient rule and rearranging gives the result.

□

α-entrepreneurs borrow more efficiently than β-entrepreneurs, so transferring a pound from a β-entrepreneur to an α-entrepreneur increases efficient capital investment. This direct effect means that so long as the indirect price effects, which in turn determine changes in balance sheet capacity, are not too large, a social planner indeed wishes to transfer wealth to procyclical entrepreneurs in aggregate. A sufficient condition is that entrepreneurs' wealth is not too large, as stated in proposition 5.2.1 presently.

PROPOSITION 5.2.1. *If w is small, a marginal transfer from β-entrepreneurs to α-entrepreneurs increases welfare, i.e.* $\dot{W}(0) > 0$.

Proof. Since the coefficient on the right-hand side of inequality (96) is negative,

$$-\frac{A-\gamma'_a}{A-\gamma'_b}\frac{(3R-1)p_0^a}{3Rp_0^b - p_0^0} < 0, \tag{98}$$

as long as the ratios

$$\frac{\dot{p}_0^a}{p_0^a} \quad \text{and} \quad \frac{3R\dot{p}_0^b - \dot{p}_0^0}{3p_0^b - p_0^0} \tag{99}$$

making w small ensures the condition is satisfied. Since

$$\frac{\gamma'(K)}{R-1} \leq p^\omega \leq \frac{A}{R-1}, \tag{100}$$

it suffices to show that $\dot{p}_0^w$ is finite. Perturbing the price system (56)-(58) and differentiating with respect to ε about $\varepsilon = 0$ reveals that $(\dot{p}_0^a, \dot{p}_0^b, \dot{p}_0^0)$ solves the linear system

$$\begin{pmatrix} 3R-1-\frac{9R\gamma''_a w}{(3R-1)(p_0^a)^2} & -1 & -1 \\ -1 & 3R-1-\frac{27R^2\gamma''_b w}{(3Rp_0^b-p_0^0)^2} & \frac{9R\gamma''_b w}{(3Rp_0^b-p_0^0)^2}-1 \\ -1 & -1 & 3R-1 \end{pmatrix} \begin{pmatrix} \dot{p}_0^a \\ \dot{p}_0^b \\ \dot{p}_0^0 \end{pmatrix} = \begin{pmatrix} -\frac{9R\gamma''_a}{(3R-1)p_0^a} \\ \frac{9R\gamma''_b}{3Rp_0^b-p_0^0} \\ 0 \end{pmatrix}, \tag{101}$$

which is well-defined for any (p_0^a, p_0^b, p_0^0) satisfying the bounds (100) and any w.

□

6 Predictions

6.1 Framework and Definitions

This section recasts the model in terms of (theoretically) measurable quantities to state some testable implications. As emphasized, the interaction between the two kinds of limited enforcement—the inability to commit not to renegotiate debt and not to divert capital—effects all of the main results; therefore, the predictions below apply when enforcement frictions are very important, for example in developing countries in which creditor rights are weak and enforcement is costly.

Define the return on capital as the price ratio,

$$r_{t+1} := \frac{p_{t+1}}{p_t} \tag{102}$$

and call its one-step-ahead expectation the expected return,

$$\bar{r}_t := \mathbb{E}_t \left[r_{t+1} \right]. \tag{103}$$

The "beta" of an entrepreneur's project τ is its linear projection on to capital returns,

$$\text{beta}_t^\tau := \frac{\text{Cov}_t \left[r_{t+1}, \tau \right]}{\text{Var}_t \left[r_{t+1} \right]}. \tag{104}$$

Since the equilibrium is markov, the conditional variance of returns is constant. Define

$$\Sigma := \text{Var}_t \left[r_{t+1} \right] \tag{105}$$

to write

$$\text{beta}_t^\tau := \frac{\text{Cov}_t \left[r_{t+1}, \tau \right]}{\Sigma} \tag{106}$$

and compute the covariance:

$$\begin{aligned} \text{Cov}_t \left[r_{t+1}, \tau \right] &= \mathbb{E}_t \left[r_{t+1} \tau \right] - \mathbb{E}_t \left[r_{t+1} \right] \mathbb{E}_t \left[\tau \right] &(107) \\ &= \mathbb{E}_t \left[\frac{p_{t+1}}{p_t} \mathbb{1}_{\{\omega_{t+1} = \sigma(\tau)\}} \right] - \mathbb{E}_t \left[r_{t+1} \right] \mathbb{E}_t \left[\tau \right] &(108) \\ &= \frac{1}{3} \frac{P^\tau}{p_t} 3A - A \bar{r}_t &(109) \\ &= A \big(\chi_t^\tau - \bar{r}_t \big) &(110) \end{aligned}$$

(having made use of the success indicator notation 2.7.1). The next lemma summarizes the calculation and reveals that beta_t^τ proxies for cyclicality in the model.

Lemma 6.1.1.

$$\text{beta}_t^\tau = \frac{A(\chi_t^\tau - \bar{r}_t)}{\Sigma}. \tag{111}$$

Total expected output is

$$\mathbb{E}_t \left[\text{output}_{t+1} \right] = A k_t^e + \gamma (K - k_t^e) \tag{112}$$

so productivity (normalized by K) is

$$\text{productivity}_t := A + \gamma' (K - k_t^e). \tag{113}$$

Since capital is the only durable asset in the economy and an increasing bijection maps prices to expected output, use capital prices to proxy for the state of the economy,

$$\text{market}_t := \bar{K} p_t. \tag{114}$$

Now since p_t is high exactly when k^e is high (because $\gamma' < A$), expected output is high exactly when the market is high. Call date t a "boom" if market_t is high.

The asset value or size of an entrepreneur's enterprise is the sum of his equity endowment w and the present value of his debt ℓ_t^τ,

$$\text{size}_t^\tau := w + \ell_t^\tau. \tag{115}$$

A a τ-entrepreneur's (gross) leverage is his size divided by his equity,

$$\text{leverage}_t^\tau := \frac{\text{size}_t^\tau}{w}. \tag{116}$$

The next two definitions use the difference between entrepreneurs' and investors' values of capital to speak to book-to-market ratios. Call the market value of assets the value of capital to an entrepreneur discounted at the market rate,

$$\text{market value of assets}_t^\tau := \frac{\mathbb{E}_t\left[p_{t+1}k + \tau(k)\right]}{R} \tag{117}$$

$$= \frac{(A+\bar{p})k}{R}. \tag{118}$$

Note that this is not the value at which an entrepreneur can sell his assets in the market, but rather the value of the entrepreneurs' firm. An entrepreneur with capital k holds assets with book value

$$\text{book value of assets}_t^\tau := p_t k. \tag{119}$$

The book-to-market ratio is thus

$$\text{book-to-market}_t^\tau := \frac{\text{book value of assets}_t^\tau}{\text{market value of assets}_t^\tau} := \frac{Rp_t}{A+\bar{p}}. \tag{120}$$

6.2 Correlations

Since debt capacity is increasing in cyclicality and entrepreneurs are always maximally levered (lemma 3.5.1 and corollary 3.4.2), size and leverage are increasing in beta.

PREDICTION 6.2.1. *Size is increasing in cyclicality.*

In the cross-sectional regression

$$\text{size}_t^\tau = \beta\, \text{beta}_t^\tau + \varepsilon_t, \tag{121}$$

the estimate of the coefficient β is positive, $\hat{\beta} > 0$.

Prediction 6.2.2. *Leverage is increasing in cyclicality.*

In the cross-sectional regression

$$\text{leverage}_t^{\tau} = \beta\, \text{beta}_t^{\tau} + \varepsilon_t, \tag{122}$$

the estimate of the coefficient β *is positive,* $\hat{\beta} > 0$.

Booms occur when procyclical agents can borrow, giving the analogous predictions in the time-series.

Prediction 6.2.3. *Average size is high in booms.*

In the time-series regression

$$\overline{\text{size}}_t = \beta\, \text{market}_t + \varepsilon_t, \tag{123}$$

the estimate of the coefficient β *is positive,* $\hat{\beta} > 0$.

Prediction 6.2.4. *Average leverage is high in booms.*

In the time-series regression

$$\overline{\text{leverage}}_t = \beta\, \text{market}_t + \varepsilon_t, \tag{124}$$

the estimate of the coefficient β *is positive,* $\hat{\beta} > 0$.

The procyclicality of leverage is well-documented empirically. See, for example, Geanakoplos (2010), Gorton and Metrick (2012), and Adrian and Shin (2010).

From the proof of proposition 3.6.3, entrepreneurs hold more capital in a-states than in b-states and more capital in b-states than in 0-states:

$$K > K - \frac{3Rw}{3Rp^b - p^0} > K - \frac{3Rw}{(3R-1)p^a} \tag{125}$$

or

$$k^{e,0} < k^{e,b} < k^{e,a}. \tag{126}$$

Thus, immediately from the definition (equation (113)), productivity is high in booms.

Prediction 6.2.5. *Productivity is high in booms.*

In the time-series regression

$$\text{productivity}_t = \beta \text{market}_t + \varepsilon_t, \tag{127}$$

the estimate of the coefficient β *is positive,* $\hat{\beta} > 0$.

This prediction volunteers an explanation of the puzzle of procyclical productivity originating with Hall (1988). The explanation is only partial because the phenomenon occurs at the firm level and, formally, the model explains it only in aggregate. However, I think the explanation that productivity is higher because of capital allocation—that resources flow more efficiently to productive firms in booms than in busts due to increased debt capacity—may be new.

Now, the definitions of size and book-to-market (equations (115) and (120)) imply mechanically that the book-to-market ratio is high in booms.

PREDICTION 6.2.6. *The average book-to-market ratio is high in booms. In the time-series regression*

$$\overline{\text{book-to-market}_t} = \beta\, \text{market}_t + \varepsilon_t, \tag{128}$$

the estimate of the coefficient β is positive, $\hat{\beta} > 0$.

In the model, book-to-market ratio is the reciprocal of Tobin's q. The supply of capital is fixed, so aggregate q proxies for average q, which, in turn, proxies for marginal q due to decreasing returns to scale. The prediction now reads that the marginal productivity of capital is low in booms.

6.3 A Natural Experiment

Since the collateral multiplier S^{χ} is increasing in cyclicality (corollary 3.4.2), procyclical entrepreneurs' balance sheets are more sensitive to their endowments than are countercyclical entrepreneurs'. A shock to endowments, resulting, for example, from foreign capital flowing into a newly opened economy, provides a natural experiment for difference-in-differences analysis of the model's predictions.

Specifically, suppose that a uniform, unanticipated positive shock to endowments occurs at date t^* so that all endowments are $\overline{w}$ before or at t^* and are $\underline{w} < \overline{w}$ after t^*. Proposition 3.4.1 and lemma 3.5.1 say that

$$\text{size}_t^{\tau} = S^{\chi_t(\tau)} w = \frac{3Rw}{3R - \chi_t(\tau)}, \tag{129}$$

so if H and L denote entrepreneurs with cyclicality χ^H and χ^L respectively

$$\left(\text{size}_{t+1}^H - \text{size}_{t+1}^L\right) - \left(\text{size}_t^H - \text{size}_t^L\right) = \left(\frac{\chi^H}{3R - \chi^H} - \frac{\chi^L}{3R - \chi^L}\right)\left(\,\overline{w} - \underline{w}\,\right) > 0. \tag{130}$$

If some firms have cyclicality χ^H and the rest have cyclicality χ^L, then

$$\left(\text{debt}^H_{t^*+1} - \text{debt}^L_{t^*+1}\right) - \left(\text{debt}^H_{t^*} - \text{debt}^L_{t^*}\right) = \left(S^{\chi^H}\overline{w} - S^{\chi^L}\overline{w}\right) - \left(S^{\chi^H}\underline{w} - S^{\chi^L}\underline{w}\right) \tag{131}$$

$$= \left(\frac{3R}{3R-\chi^H} - \frac{3R}{3R-\chi^L}\right)\left(\overline{w} - \underline{w}\right) > 0, \tag{132}$$

which immediately gives the following predictions for the panel regressions of size and leverage against cyclicality.

PREDICTION 6.3.1. *Positive shocks to endowments increase the size of procyclical firms more than of countercyclical firms.*

If a positive shock to capital occurs at time t^*, then in the panel regression

$$\text{size}^\tau_t = \alpha + \beta \mathbb{1}_{\left\{\overline{\text{beta}}^i_t \geq \text{beta}^*\right\}} \mathbb{1}_{\left\{t \geq t^*\right\}} + \gamma \mathbb{1}_{\left\{\overline{\text{beta}}^i_t \geq \text{beta}^*\right\}} + \delta \mathbb{1}_{\left\{t \geq t^*\right\}}, \tag{133}$$

the estimate of the coefficient β is positive, $\hat{\beta} > 0$, for any beta*.

PREDICTION 6.3.2. *Positive shocks to endowments increase the leverage of procyclical firms more than of countercyclical ones.*

If a positive shock to capital occurs at time t^*, then in the panel regression

$$\text{leverage}^\tau_t = \alpha + \beta \mathbb{1}_{\left\{\overline{\text{beta}}^i_t > \text{beta}^*\right\}} \mathbb{1}_{\left\{t \geq t^*\right\}} + \gamma \mathbb{1}_{\left\{\overline{\text{beta}}^i_t > \text{beta}^*\right\}} + \delta \mathbb{1}_{\left\{t \geq t^*\right\}}, \tag{134}$$

the estimate of the coefficient β is positive, $\hat{\beta} > 0$, for any beta*.

7 Conclusions

Borrowers' inability to commit not to renegotiate loans makes collateral valuable to creditors even when their debtors are not near bankruptcy. Borrowers' incentive to divert capital in anticipation of default—to line their own pockets and avoid handing over good-quality assets to their creditors—decreases the quantity and quality of assets that creditors do liquidate when they repossess a firm. The threats of renegotiation and capital diversion interact. They make collateral relatively more valuable to creditors when debtors' projects succeed than when they fail. Since liquidation values vary with the aggregate economy and creditors value the comovement between liquidation values and borrower success, debtor procyclicality is a valuable resource for creditors ex post. It allows them to enforce repayment. Because creditors can enforce loans to procyclical borrowers ex post, they are willing to lend to them ex ante. Thus procyclicality is a valuable resource for borrowers, it grants them the power to commit and allows them to lever up.

Increased cyclicality loosens borrowing constraints and improves allocative efficiency. When productive agents are unconstrained, prices increase to reflect productivity of capital in its best use, while the marginal buyer may be rather unproductive when the most productive firms are constrained. Financing frictions matter for the macroeconomy. The interaction between the threats of renegotiation and capital diversion lead to fluctuations in capital prices and expected output, creating an endogenous component of the business cycle.

Policy makers must take firms' financing constraints into account. To increase efficiency in the private economy, governments must provide liquidity to firms that can stretch their collateral to allocate capital efficiently. When enforcement is limited, governments should subsidize procyclical firms. Since limits to lending result from borrowers' inability to commit to repay, subsidizing private lenders—injecting capital into the banking sector—may not increase lending or aid capital allocation. Direct corporate subsidies are necessary.

A Proofs

A.1 *Proof of Lemma 3.3*

The result follows from $\gamma'' < 0$:

$$\begin{aligned}
p_t &= \frac{1}{R}\big(\gamma'(k^i) + \mathbb{E}_t\left[p_{t+1}\right]\big) && (135)\\
&\leq \frac{1}{R}\big(\gamma'(0) + \mathbb{E}_t\left[p_{t+1}\right]\big) && (136)\\
&\leq \frac{\gamma'(0)}{R} + \frac{\gamma'(0)}{R^2} + \frac{\mathbb{E}_t\left[p_{t+2}\right]}{R} && (137)\\
&\leq \gamma'(0)\left(\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3} + \cdots\right) && (138)\\
&= \frac{\gamma'(0)}{R-1} && (139)\\
&= \frac{A}{R-1}. && (140)
\end{aligned}$$

A.2 *Proof of Lemma 3.2.1*

An entrepreneur never plays $F' > F$ because $F' = F$ induces the same action ($\xi = \mathrm{a}$) and gives him a higher payoff. Suppose $F' < F$.

If $F' < p_{t+1}\bar{k}$ the creditor plays $\xi = \neg\mathrm{a}$, leaving the entrepreneur with nil, so $F' \geq p_{t+1}\bar{k}$. If $F' > p_{t+1}\bar{k}$ then $\xi = \mathrm{a}$, but $F'' = (F' + p_{t+1}\bar{k})/2$ is superior for the debtor and $\xi = \mathrm{a}$ still, so $F' \leq p_{t+1}k$. Thus if $F' < F$ then $F' = p_{t+1}k$ and $F' = F$ otherwise, which is to say $F' = \min\{F\,,\, p_t k\}$.

A.3 *Proof of Lemma 3.3.1*

If the $\omega_t \neq \sigma(\tau)$, then $\zeta = \neg\mathrm{d}$ yields $p_{t+1}k_t - \min\left\{F\,,\, p_{t+1}\bar{k}_t\right\}$ and $\zeta = \mathrm{d}$ yields $p_{t+1}k_t$, but

$$p_{t+1}k_t - \min\left\{F\,,\, p_{t+1}\bar{k}_t\right\} \geq p_{t+1}k_t \tag{141}$$

only if $F \leq 0$ (which, due to lemma 3.2.1 above, implies he would have no debt), so a failing entrepreneur always plays $\zeta = \mathrm{d}$.

If $\omega_t = \sigma(\tau)$, $\zeta = \neg\mathrm{d}$ yields $3Ak_t + p_{t+1}k_t - \min\left\{F\,,\, p_{t+1}\bar{k}_t\right\}$ and $\zeta = \mathrm{d}$ yields $p_{t+1}k_t$; rearranging implies the entrepreneur does not abscond so long as

$$\min\left\{F\,,\, p_{t+1}\bar{k}_t\right\} < 3Ak_t \tag{142}$$

which holds since

$$\begin{aligned}
\min\left\{F\,,\,p_{t+1}\bar{k}_t\right\} &\leq p_{t+1}\bar{k}_t && (143)\\
&\leq p_{t+1}k_t && (144)\\
&\leq \frac{Ak_t}{R-1} && (145)\\
&< \frac{Ak_t}{4/3-1} && (146)\\
&= 3Ak_t, && (147)
\end{aligned}$$

where the last inequality follows from assumption 2.6.1.

A.4 Proof of Lemma 3.4.5

Suppose (in anticipation of a contradiction) that at $\omega \in \{a, b, 0\}$ $k_t^e = K$ and consequently $k_t^i = 0$. Equation (12) gives the price

$$p^\omega = \frac{\gamma'(0) + \bar{p}}{R} = \frac{A + \bar{p}}{R}. \tag{148}$$

For $\omega' \neq \omega$,

$$p^{\omega'} = \frac{\gamma'(k_t^i) + \bar{p}}{R} \geq \frac{\bar{p}}{R}. \tag{149}$$

Since, from above,

$$\begin{aligned}
3\bar{p} &= p^a + p^b + p^0 && (150)\\
&= \frac{A + \bar{p}}{R} + \frac{2\,\bar{p}}{R} && (151)\\
&= \frac{A + 3\bar{p}}{R}, && (152)
\end{aligned}$$

$$\bar{p} \geq \frac{A/3}{R-1} \tag{153}$$

and

$$p^{\omega'} \geq \frac{A/3}{R(R-1)}. \tag{154}$$

Combine this inequality with equation (32) above to compute:

$$k_t^e \leq \frac{3Rw}{3Rp^{\omega} - P^{\tau}} \tag{155}$$
$$= \frac{3Rw}{3(A+\bar{p}) - P^{\tau}} \tag{156}$$
$$= \frac{3Rw}{3A + p^a + p^b + p^0 - P^{\tau}} \tag{157}$$
$$\leq \frac{3Rw}{3A + 2\min\{p^a, p^b, p^0\}} \tag{158}$$
$$\leq \frac{3Rw}{3A + 2\frac{A/3}{R(R-1)}} \tag{159}$$
$$< \frac{Rw}{A} \tag{160}$$
$$\leq K, \tag{161}$$

by assumption 2.6.2, contradicting $k_t^e = K$.

A.5 Proof of Lemma 3.6.2

The proof recasts solutions of system (56)-(58) as fixed points of a continuous mapping from a closed ball to itself and applies Brouwer's theorem.

For simplicity, employ the convention that a real number divided by zero is infinity and that a real number minus infinity is minus infinity—$x/0 = \infty$ and $x - \infty = -\infty$.

First, extend γ to the extended real line by defining the function $\bar{\gamma} : \mathbb{R} \cup \{-\infty\} \to \mathbb{R}$ via

$$\bar{\gamma}'(k) := \begin{cases} \gamma'(0) & \text{if } k < 0, \\ \gamma'(k) & \text{if } k \in [0, K], \\ \gamma'(K) & \text{if } k > K. \end{cases} \tag{162}$$

$\bar{\gamma}'$ inherits monotonicity from γ.

Now define the compact domain

$$\Omega := \left\{ \left(p^a, p^b, p^0\right) \in \mathbb{R}^3 \,\middle|\, 0 \leq p^{\omega} \leq \frac{A}{R-1},\ p^{\omega} \leq 3Rp^{\omega'} \text{ for all } \omega, \omega' \right\} \tag{163}$$

and the function $\Gamma : \Omega \to \mathbb{R}^3$ by the action

$$\Gamma : \begin{pmatrix} p^a \\ p^b \\ p^0 \end{pmatrix} \mapsto \frac{1}{R} \begin{pmatrix} \dfrac{p^a + p^b + p^0}{3} + \bar{\gamma}'\left(K - \dfrac{3Rw}{3Rp^a - p^a}\right) \\ \dfrac{p^a + p^b + p^0}{3} + \bar{\gamma}'\left(K - \dfrac{3Rw}{3Rp^b - p^0}\right) \\ \dfrac{p^a + p^b + p^0}{3} + \bar{\gamma}'\left(K\right) \end{pmatrix}. \tag{164}$$

Away from the singular points of the argument of $\bar{\gamma}'$, continuity of Γ is immediate. In their neighbourhoods, namely as $p^a \searrow 0$ or $p^0 \nearrow 3Rp^b$, $\bar{\gamma}'$ is flat since the argument is negative, $\bar{\gamma}' \equiv \gamma'(0)$, giving continuity.

Now observe that $\Gamma(\Omega) \subset \Omega$ because $\bar{\gamma}'$ is decreasing. Since $\bar{\gamma}' \leq A$ and

$$\bar{p} \leq \max\left\{p^a, p^b, p^c\right\} \leq \frac{A}{R-1}, \tag{165}$$

for any $\omega \in \{a, b, 0\}$,

$$\begin{aligned} p^\omega &\leq \frac{A/(R-1) + \gamma'(0)}{R} && (166)\\ &= \frac{A}{R}\left(\frac{1}{R-1} + 1\right) && (167)\\ &= \frac{A}{R-1} && (168) \end{aligned}$$

and

$$p^\omega \geq \frac{\gamma'(K)}{R} \geq 0, \tag{169}$$

or $0 \leq p^\omega \leq A/(R-1)$. Finally, since $\bar{\gamma}' > 0$,

$$3Rp^\omega \geq p^a + p^b + p^0 \geq \max\left\{p^a, p^b, p^0\right\}, \tag{170}$$

thus $3Rp^\omega \geq p^{\omega'}$ for any ω and ω' and $\Gamma : \Omega \to \Omega$. Γ has a fixed point by Bouwer's theorem. The point solves (56)–(58)—in which γ' replaces $\bar{\gamma}'$—so long as γ is well-defined there, namely if entrepreneurs' capital is indeed nonnegative and not greater than the total supply. Positivity is immediate from $S^\chi \geq 0$ and corollary 3.4.5 (the proof of which depends only on the bounds on γ', which coincide with those on $\bar{\gamma}'$) implies $k^e < K$. A fixed point exists.

A.6 Proof of Proposition 4.3.1

Suppose (in anticipation of a contradiction) $p^a > p^b$. Subtracting equation (76) from equation (75) implies

$$\gamma'\left(K - \frac{Rw}{Rp^a - \bar{p}}\right) - \gamma'\left(K - \frac{Rw}{Rp^b - \bar{p}}\right) > 0 \tag{171}$$

and, since γ' is decreasing,

$$\frac{Rw}{Rp^a - \bar{p}} > \frac{Rw}{Rp^b - \bar{p}} \tag{172}$$

or $p^b > p^a$, a contradiction. Thus $p^b \leq p^a$. Repeating the argument supposing $p^b > p^a$ gives the result.

A.7 Proof of Proposition 4.4.1

Suppose (in anticipation of a contradiction) $p^a > p^b$. Subtracting equation (88) from equation (87) implies

$$\gamma'\left(K - \frac{Rw}{Rp^a - A}\right) - \gamma'\left(K - \frac{Rw}{Rp^b - A}\right) > 0 \tag{173}$$

and, since γ' is decreasing,

$$\frac{Rw}{Rp^a - A} > \frac{Rw}{Rp^b - A} \tag{174}$$

or $p^b > p^a$, a contradiction. Thus $p^b \leq p^a$. Repeating the argument supposing $p^b > p^a$ gives the result.

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