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Entropy is the only Increasing Functional of Kac's One-dimensional Caricature of a Maxwellian Gas

By

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1. Introduction

Consider a functional $H[f] = \int h(f) = \int\limits_{R^1} h[f(b)]\,db$ converging for f bounded as in $c_1 \exp(-c_2 a^2) < f < c_3 \exp(-c_4 a^2)$ $(0 < c_1, c_2, c_3, c_4)$** and suppose that $H[p(t,.)]$ is an increasing function of t (>0) for solutions $p = p(t, b)$ of Boltzmann's problem for Kac's gas with initial data $p(0+,.) = f$ bounded as above. Then $h(b) = c_5 b + c_6 b\, lg\, b$ $(c_6 \leqq 0)$, i.e., besides $\int p (\equiv \int f,\ t > 0)$***, the entropy $H[p] = -\int p\, lg\, p$ is the only such functional of Kac's gas. I believe the same holds for a 3-dimensional gas with bounded scattering cross-section but cannot prove it except for a Maxwellian-like gas under an unattractive condition.

2. Gibbs' lemma and its dual, Boltzmann's equation, entropy

Gibbs' lemma states that among all non-negative functions $f = f(a)$ $(a \in R^1)$ subject to

(1a) $$\int f = 1$$

(1b) $$\int a^2 f = \sigma^2 < \infty,$$

the Gauss function $g \equiv (2\pi\sigma^2)^{-1/2} \exp(-a^2/2\sigma^2)$ makes the entropy

$$H[f] \equiv -\int f\, lg\, f$$

as big as possible; in fact, since $b\, lg\, b - b + 1 > 0\, (0 \leqq b \neq 1)$,

$$H[f] = H[g] - \int \left[\frac{f}{g}\, lg \frac{f}{g} - \frac{f}{g} + 1\right] g < H[g] = lg \sqrt{2\pi e}\,\sigma$$

unless $f \equiv g$.

Gibbs' lemma has the following entertaining dual: *if $h = h(b)$ $(b > 0)$ satisfies the conditions*

(2a) $$h(0+) = 0$$

(2b) $$h \in C^1(0, \infty)$$

(2c) $$|h'(b)| < b^{-1} |lg\, b|^{-3/2-\delta} \quad (b \downarrow 0,\ \delta > 0)$$

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** c_1, c_2, etc., denote constants.

*** $\int f$ means the integral of f over R^1.

and if, for each positive σ, $H[f] \equiv \int h(f) \leqq H[g]$ *for each non-negative function* f *subject to* (1) *and bounded as in*

$$c_1 \exp(-c_2 a^2) < f < c_3 \exp(-c_4 a^2) \quad (c_1, c_2, c_3, c_4 > 0); \tag{3}$$

then

$$h(b) = c_5 b + c_6 b \lg b \qquad (c_6 \leqq 0);$$

the proof is similar to the argument leading from (8) to (9b) below, so it is left to the reader. (2c) is imposed for technical reasons, but it should be appreciated that it makes the integral defining $H[f]$ converge like $\int\limits_1^\infty a^{-1-2\delta}$ for functions f bounded as in (3).

In KAC's one-dimensional caricature of a Maxwellian gas, the distribution $\int\limits_{a<b} p(t, a)\, da$ of molecular speeds at time $t > 0$ is determined by the solution $p = p(t, a)$ $(t > 0, a \in R^1)$ of BOLTZMANN's problem

$$\frac{\partial p}{\partial t} = \int\limits_{R^1 \times S^1} [p(\dot{a})\, p(\dot{b}) - p(a)\, p(b)]\, db\, do, \tag{4a}$$

$$\dot{a} = a \cos\theta - b \sin\theta, \quad \dot{b} = a \sin\theta + b \cos\theta$$

$$do = \text{the uniform distribution on the circle } S^1 : 0 \leqq \theta < 2\pi$$

subject to the initial conditions

$$\begin{gathered} p(0+, .) = f \geqq 0, \\ \int f = 1, \\ \int a^2 f = \sigma^2 < \infty. \end{gathered} \tag{4b}$$

E. WILD [*5*; see also *4*] has presented the solution in the following compact shape:

$$p = e^{-t} \sum_{n=1}^{\infty} (1 - e^{-t})^{n-1} p_n(f). \tag{5}$$

$p_n(f)$ stands for a convex combination of n-fold products of f with itself using as multiplication the commutative but non-associative convolution

$$f_1 \otimes f_2(a) = \int\limits_{R^1 \times S^1} f_1(\dot{a})\, f_2(\dot{b})\, db\, do$$

figuring in (4a).

Because of

$$\int f_1 \otimes f_2 = \int f_1 \int f_2,$$

$$\int a^2 f_1 \otimes f_2 = \tfrac{1}{2}(\sigma_1^2 + \sigma_2^2) \quad (\sigma_1^2 = \int a^2 f_1,\ \sigma_2^2 = \int a^2 f_2),$$

and

$$g \otimes g(a) = \int g(\dot{a})\, g(\dot{b})\, db\, do = \int g(a)\, g(b)\, db\, do = g(a),$$

it is immediate from (5) that $\int p = 1$, $\int a^2 p = \sigma^2$, and that p is bounded as in (3) for small times $0 < t \leqq t(f)$ if its initial data f is so bounded.

Bounds of this description are ample to justify BOLTZMANN's H-theorem:

$$\dot{H}[p] \geqq 0^{\star}$$

for the entropy $H[p] \equiv -\int p\, lg\, p$, since a bound $|\dot{p}| < c_3 \exp(-c_4 a^2)$ $(t \leqq t(f))$ permits us to differentiate under the integral sign, obtaining

$$\begin{aligned}
\dot{H}[p] &= -\int \dot{p}(lg\, p + 1) = -\int \dot{p}\, lg\, p \\
&= -\int_{R^2 \times S^1} (B - A)\, lg\, p(a)\, da\, db\, do \quad (A = p(a)\, p(b),\ B = p(\dot{a})\, p(\dot{b})) \\
&= -\int (B - A)\, lg\, p(b) = \int (B - A)\, lg\, p(\dot{a}) = \int (B - A)\, lg\, p(\dot{b}) \\
&= \frac{1}{4} \int (B - A)\, lg \frac{B}{A} \geqq 0.
\end{aligned}$$

Accordingly, *the entropy is an increasing functional of* KAC's *gas for* $0 < t \leqq t(f)$ *and* f *bounded as in* (3).

Given h as in (2), let $H[p] \equiv \int h(p)$ increase for small times $0 < t \leqq t(f)$ for solutions p of BOLTZMANN's problem with initial data f bounded as in (3). Supposing that a solution so bounded at time $t = 0$ continued between similar bounds at all later times while tending to the Gauss function g (Maxwellian distribution), it would follow that $H[p]$ increased to its upper bound $H[g]$ as $t \uparrow \infty$, whence $H[f] \leqq \lim_{t \uparrow \infty} H[p] = H[g]$ for f bounded as in (3), and an application of the dual of GIBBS' lemma would entail

(6) $$H[p] = c_5 + c_6 \int p\, lg\, p \qquad (c_6 \leqq 0);$$

in brief, *the entropy* $H[p] = -\int p\, lg\, p$ *would be the only increasing functional of* KAC*'s gas aside from* $H[p] = \int p = 1$.

CARLEMAN [*1*] obtained bounds

$$c_7 \exp(-c_8 a^{2+\delta}) < p < c_9 (1 + a^2)^{-6-\delta}$$

for radial solutions of BOLTZMANN's problem for a 3-dimensional gas of hard balls, but Gaussian bounds as needed above are not available at the present time. In spite of this, the statement that entropy is the only increasing functional of

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$\star$ stands for differentiation with regard to time, except in case of $\dot{a}$ and $\dot{b}$.

KAC's gas is still correct, and it is the purpose of this note to prove it using another method *.

BOLTZMANN's problem with its single increasing functional makes a striking contrast to the heat flow problem $\partial p/\partial t = \partial^2 p/\partial a^2$ or to BURGERS' one-dimensional caricature of the NAVIER-STOKES problem $\partial p/\partial t = \partial^2 p/\partial a^2 + p\,\partial p/\partial a$; in fact, if $h = h(b)$ $(b > 0)$ is a smooth concave function and if $p = p(t, a)$ is a positive solution of either problem with nice initial data, then the functional $H[p] \equiv \int h(p)$ satisfies

$$\dot{H}[p] = -\int h''(p)\left(\frac{\partial p}{\partial a}\right)^2 \geqq 0^{**}.$$

3. Proof that entropy is the only increasing functional of Kac's gas

Beginning the proof of (6) for increasing functionals of KAC's gas under the conditions (2) on h, since a solution p of BOLTZMANN's problem with initial data bounded as in (3) lies under similar bounds for small times, (4a) leads at once to

$$\dot{H}[p] = \int h'(p)\,\dot{p} = \int h'(p)\,[p \otimes p - p] \geqq 0 \quad \text{at} \quad t = 0, \tag{7}$$

and using $g \otimes g = g$, it appears that among functions f subject to (1) and bounded as in (3), the Gauss kernel makes the functional $\dot{H}[f] \equiv \int h'(f)\,[f \otimes f - f]$ smallest.

Choose

$$a \in R^1, \quad 0 < \varepsilon < 1, \quad t > 0, \quad l^2 = \sigma^{-2}[(1-\varepsilon)\,\sigma^2 + \varepsilon(t + a^2)], \quad \text{and}$$
$$k(b) = (2\pi t)^{-1/2} \exp(-(b-a)^2/2t)$$

so that the combination $f_\varepsilon(b) \equiv l[(1-\varepsilon)g + \varepsilon k]\,(lb)$ satisfies (1) and is bounded as in (3) as $\varepsilon \downarrow 0$. It follows that

$$0 \leqq \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \dot{H}[f_\varepsilon] = \int h'(g)[2g \otimes k - g - k] \tag{8}$$

(the justification is similar to that of (7)), and letting $t \downarrow 0$ gives

$$\begin{aligned} 0 &\leqq 2\int h'[g(a\cos\theta + b\sin\theta)]\,g(b)\,db\,do - \\ &\quad - \int h'(g)\,g - h'[g(a)] \equiv \\ &\equiv \Delta \end{aligned} \tag{9a}$$

(the computation is simplified on using $d\dot{a}\,d\dot{b} = da\,db$). But also

$$\begin{aligned} 0 &\leqq \int \Delta g \\ &= 2\int h'[g(a\cos\theta + b\sin\theta)]\,g(a)\,g(b)\,da\,db\,do - \\ &\quad - 2\int h'(g)\,g \\ &= 2\int h'[g(a)]\,g(\dot{a})\,g(\dot{b})\,d\dot{a}\,d\dot{b}\,do - 2\int h'(g)\,g \\ &= 2\int h'[g(a)]\,g(a)\,g(b)\,da\,db\,do - 2\int h'(g)\,g \\ &= 0 \end{aligned}$$

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* [3] contains an earlier attempt at such a result.

** $\int h'(p)\,p\,\dfrac{\partial p}{\partial a} = \int p\,\dfrac{\partial}{\partial a}\,h(p) = -\int h(p)\,dp = 0$ if p is smooth and $= 0$ at $\pm\infty$.

(use $d\dot{a}\,d\dot{b} = da\,db$ and $g(\dot{a})\,g(\dot{b}) = g(a)\,g(b)$), permitting the improvement of (9a) to

$$0 = 2\int h'[g(a\cos\theta + b\sin\theta)]\,g(b)\,db\,do - \qquad (9\text{b})$$
$$-\int h'(g)\,g - h'[g(a)],$$

and defining $h'(g) \equiv f$ and $h'(g)\,g \equiv e$, (9b) can be expressed as

$$\int_{R^1\times S^1} [f(\dot{a}) + f(\dot{b}) - f(a) - f(b)]\,g(b)\,db\,do \equiv 0 \qquad (10\text{a})$$

or as

$$e + \left(\int e\right) g = 2e \otimes g. \qquad (10\text{b})$$

(10b) is best suited to the present purpose.

Because of (2)

$$|e| < c_{10} \text{ near } 0, \qquad (11\text{a})$$

$$|e| < c_{11}\,|a|^{-3-2\delta} \text{ near } \infty \qquad (11\text{b})$$

so that $e \in L^1(R^1)$ permitting the use of Fourier transform in (10b) with the result that

$$\begin{aligned}
\hat{e}(\gamma) &+ \hat{e}(0)\exp(-\gamma^2/2)\\
&= 2\int e^{i\gamma a}\,da \int e(\dot{a})\,g(\dot{b})\,db\,do\\
&= 2\int e^{i\gamma \dot{a}}\,e(a)\,g(b)\,da\,db\,do\\
&= 2\int \hat{e}(\gamma\cos\theta)\exp(-\gamma^2\sin^2\theta/2)\,do,
\end{aligned}$$

or, what is the same and simpler to look at

$$\begin{aligned}
\bar{e}(\gamma) &\equiv \hat{e}(\gamma)\exp(\gamma^2/2) - \hat{e}(0)\\
&= 2\exp(\gamma^2/2)\int \hat{e}(\gamma\cos\theta)\exp(-\gamma^2\sin^2\theta/2)\,do - 2\hat{e}(0)\\
&= 2\int \bar{e}(\gamma\cos\theta)\,do.
\end{aligned} \qquad (12)$$

Because of (11b), $\bar{e}''$ is continuous, and differentiating (12) one obtains

$$\bar{e}'(0) = 2\int \bar{e}'(0)\cos\theta\,do = 0, \qquad (13\text{a})$$

$$\bar{e}''(\gamma) = 2\int \bar{e}''(\gamma\cos\theta)\cos^2\theta\,do. \qquad (13\text{b})$$

(13b) states that for each choice of γ $e''[\gamma\cos\theta_1\cos\theta_2\ldots\cos\theta_n]$ $(n \geqq 0)$ is a martingale. Here, θ_1, θ_2, etc. are independent with common law

$$P(\theta_1 \in d\theta) = 2\cos^2\theta\,do \quad \left(2\int\cos^2\theta\,do = 1\right),$$

and now the bound

$$\begin{aligned}
&P(|\cos\theta_1\ldots\cos\theta_n| > (4/5)^{n/2})\\
&< (\tfrac{5}{4}E(\cos^2\theta_1))^n\\
&= (\tfrac{5}{2}\int\cos^4\theta\,do)^n\\
&= (15/16)^n
\end{aligned}$$

combined with the BOREL-CANTELLI lemma implies

$$\bar{e}''(\gamma) = \lim_{n\uparrow\infty} E[\bar{e}''(\gamma\cos\theta_1\ldots\cos\theta_n)] = \bar{e}''(0):$$

in brief, $\bar{e}$ is a constant multiple of γ^2, and now inverting the Fourier transform leads at once to (6), completing the proof.