EC476 PROBLEM SET 2 CLARIFICATION

Jason Roderick Donaldson*

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Based on Myerson 1993

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1 Set-up

Index the voters by $v \in [0, 1]$.

Candidate i's strategy is to offer each voter v a random amount $\mathbf{X}_i^v$ where $\mathbf{X}_i^v \sim F_i$, $\mathbb{E}\left[\mathbf{X}_i^v\right] = 1$. Assume that

$$\mathbf{X}_i^v \perp\!\!\!\perp \mathbf{X}_j^v \tag{1}$$

for candidates i and j.

The timing is as follows:

1. Candidates 1 and 2 simultaneously and independently offer x_1^v and x_2^v to each voter v, where x_i^v is a random realization of $\mathbf{X}_i^v \sim F_i$.
2. Voter v votes for Candidate i if $x_i^v > x_j^v$ and votes for Candidate j otherwise.
3. Candidate i wins if he receives more votes than Candidate j, namely if

$$\int_0^1 \mathbf{1}_{\left\{x_j^v \le x_i^v\right\}} \, dv > \frac{1}{2} \tag{2}$$

   and if the above holds with equality

$$\int_0^1 \mathbf{1}_{\left\{x_j^v \le x_i^v\right\}} \, dv = \frac{1}{2} \tag{3}$$

   then the candidates decide the election with a coin flip.
   Each Candidate obtains payoff 1 if he wins and zero otherwise.

*Finance Department, LSE, contact: `j.r.donaldson@lse.ac.uk`

Note that since the realizations x_i^v and x_j^v are drawn from an i.i.d. distribution,

$$\int_0^1 \mathbf{1}_{\left\{x_j^v \leq x_i^v\right\}} \, dv = \mathbb{P}\left\{\mathbf{X}_j^v \leq \mathbf{X}_i^v\right\} \tag{4}$$

for any v, by the law of large numbers.

2 Expected Utility

Thus, given Candidate 1's action F_1, Candidate 2 maximizes his expected utility over all feasible distributions F_2. That is he maximizes

$$\begin{aligned}
\mathbb{E}\left[(\text{prob. wins}) \cdot 1 + (\text{prob. loses}) \cdot 0\right] &= \mathbb{E}\left[\mathbb{P}\left\{\mathbf{X}_2^v \leq \mathbf{X}_1^v\right\}\right] &(5)\\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{P}\left\{\mathbf{X}_1^v \leq x_2 \middle| \mathbf{X}_2^v = x_2\right\}\right]\right] &(6)\\
&= \mathbb{E}\left[\mathbb{E}\left[F_1(x_2) \middle| \mathbf{X}_2^v = x_2\right]\right] &(7)\\
&= \mathbb{E}\left[F_1(\mathbf{X}_2^v)\right] &(8)\\
&= \int_0^\infty F_1(x_2) dF_2(x_2). &(9)
\end{aligned}$$

3 Equilibrium

We now prove that

$$F_1^*(x) = F_2^*(x) = \begin{cases} x/2 & \text{if } x \in [0, 2], \\ 1 & \text{if } x > 2 \end{cases} \tag{10}$$

is an equilibrium, namely that

$$\int_0^\infty F_2^*(x_1) dF_1^*(x_1) \geq \int_0^\infty F_2^*(x_1) dF_1(x_1) \tag{11}$$

and

$$\int_0^\infty F_1^*(x_2) dF_2^*(x_2) \geq \int_0^\infty F_1^*(x_2) dF_2(x_2) \tag{12}$$

for all distributions F_1 and F_2 with mean 1.

Find Candidate 2's best-response correspondence by observing that his expected payoff given F_1^* is

$$\int_0^\infty F_1^*(x_2)dF_2(x_2) = \int_0^2 \frac{x_2}{2}dF_2(x_2) + \int_2^\infty dF_2(x_2) \tag{13}$$
$$\leq \int_0^2 \frac{x_2}{2}dF_2(x_2) + \int_2^\infty \frac{x_2}{2}dF_2(x_2) \tag{14}$$
$$= \frac{1}{2}\int_0^\infty x_2 dF_2(x_2) \tag{15}$$
$$= \frac{1}{2}\mathbb{E}\left[\mathbf{X}_2^v\right] \tag{16}$$
$$= \frac{1}{2}. \tag{17}$$

Now note that if $\text{supp}(F_2) = [0,2]$ then

$$\int_0^2 F_1^*(x_2)dF_2(x_2) = \frac{1}{2}, \tag{18}$$

that is to say: no action F_2 can yield expected utility higher than 1/2 given F_1^*, but any action supported on [0,2] attains this maximum. Therefore, Candidate 2's best response correspondence to F_1^* is

$$\text{BR}_2\big(\text{U}[0,2]\big) = \left\{ F_2 \in \triangle \;\middle|\; \int_0^\infty x_2\, dF_2(x_2) = 1, \text{supp}(F_2) = [0,2] \right\} \tag{19}$$

where $\triangle$ denotes the set of all probability distribution functions on $[0,\infty)$.

Since $\text{U}[0,2] \in \text{BR}_2(\text{U}[0,2])$, $F_1^* = F_2^* = \text{U}[0,2]$ is an equilibrium.

4 Notes

4.1 The Exact Law of Large Numbers

Myerson is careful not to apply the law of large numbers to integrals, but to consider the limit of a finite population. In fact, for independent Bernoulli random variables $\mathbf{1}_v$, the random function

$$v \mapsto \mathbf{1}_v \tag{20}$$

is almost surely not Lesbegue measurable. The result in a modified measure space is known as the exact law of large numbers. See Duffie and Sun 2004. (The absence of measurability did not stop economists from using the result freely long before it was understood.)

4.2 Tie-Breaking

Sometimes tie-breaking rules are very important, and in continuous games the choice of the rule can determine whether and equilibrium exists. Here the society's tie-breaking rule given a tie in the numbers of votes is important, but the voter's tie-breaking rule given a tie in his proposals is not.