THE DOWNSIDE OF PUBLIC INFORMATION IN CONTRACTING*

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Abstract

We propose a model of delegated investment with a public signal that suggests that (i) contracts do not have to refer to the public signal in order to overcome incentive problems; (ii) contracts include references to the public signal not to address incentive problems, but rather to help agents compete; and, in contrast to the contracting literature, (iii) decreasing the precision of the public signal leads to Pareto improvements. We apply this framework to a problem of delegated portfolio choice in which contracts make references to credit ratings. Our model suggests that wider rating categories make everyone better off.

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1 Introduction

Expert delegated asset managers invest on behalf of inexpert clients. They offer contracts to their client which often make reference to credit ratings.[1] But why do they propose compensation schemes that depend on public information, such as credit ratings, even though clients employ them for their private information? The contracting literature suggests that contracting on a public signal can mitigate the incentive problem between a principal and his agent (CITE). Do references to credit ratings mitigate delegated asset managers' incentive to shift risk?

We propose a model of delegated investment with a public signal that suggests that (i) contracts do not have to refer to the public signal in order to overcome the incentive problem; (ii) contracts include references to the public signal not to address the incentive problem, but rather to help agents compete; and, in contrast to the contracting literature, (CITE) (iii) decreasing the precision of the public signal leads to Pareto improvements.

A clear regulatory prescription follows from this last result: broaden ratings categories, i.e. coarsen the contractible public information partition. Our suggestion is consistent with regulators' assertions that institutions should quit responding robotically to ratings. For example, in 2010 the Financial Stability Board told the G20 Finance Ministers that

> Investment managers and institutional investors must not mechanistically rely on CRA ratings for assessing the creditworthiness of assets. This principle applies across the full range of investment managers and of institutional investors, including money market funds, pension funds, collective investment schemes (such as mutual funds and investment companies), insurance companies and securities firms... [Investment managers should limit] the proportion of a portfolio that is CRA ratings-reliant.

We build a model with two key frictions: first, agents have private information and, second, the principal and the agents differ in their attitudes toward risk. The agents' private information creates the motive for delegation and the difference in risk

[1] According to the Bank for International Settlements (2003), "it is common, for example, for fixed income investment mandates to restrict the manager's investment choices to investment grade credits"; that is to say that they restrict their portfolios to securities rated BBB- or higher by Standard & Poor's or Baa3 or higher by Moody's.

attitudes creates the misalignment of incentives. Both the principal and the agents are risk averse, but we make no assumption as to who is more risk averse. Further, the difference between the risk aversion coefficients of the principal and the agent can be arbitrarily large. However, we require that the utility functions of the principal and the agents are in the same class of hyperbolic absolute risk-aversion—i.e. that the absolute risk tolerance of the agent is an affine transformation of the absolute risk tolerance of the principal.

The timing of the model is as follows: first, identical agents offer contracts competitively. Each agent's contract can depend on the final wealth, the agent's action and the realization of the public signal, but not on the agent's private information. The agents offer the contracts before the realization of the public signal and before they learn their private information. Second, the public signal realizes and the principal decides which agent to employ to invest his wealth on his behalf. Third, the agent learns his private information and takes an action. The agent's private information pertains to the conditional distribution of final wealth given each of his possible actions. Finally wealth realizes and the principal and agent divide it according to the initial contract.

The first result is that the contract that depends on final wealth alone both solves the incentive problem and implements efficient risk-sharing. The reason is that the contract that implements efficient risk-sharing makes the principal and agent equally sensitive to the final payoff; since the only incentive problem comes from the difference in risk aversion, this optimal sharing rule aligns the agent's incentives with the principal's. Therefore the principal can delegate the decision to the agent knowing that the agent will act in their joint interest given the contract is the efficient sharing rule. Put differently, the first best action is incentive compatible, thus there is no need to introduce the public signal into the contract. Note that this intuition is robust only if the principal's and agent's preferences belong to the same HARA class.[2]

The second result is that the equilibrium contract does indeed depend on the public signal even though it does not mitigate the incentive the problem. To see why this is the case, suppose an equilibrium in which all agents offer contracts that do not depend on the public signal and observe that an agent has a profitable deviation. Because agents are competitive, in any equilibrium in which contracts do not depend on the public signal, agents must break even in expectation across all realizations of the

[2]The intuition is true locally. The HARA assumption is necessary for it to extend globally. See Pratt (2000) for an exploration of this issue but not in an optimal contracting context.

public signal. Thus, for realizations of the public signal for which the surplus is high, the employed agent receives more than his reservation utility. But now a competing agent can undercut him in this high surplus state by offering a contract contingent on the public signal. Extending this argument implies that agents must break even not only in expectation, but also for every realization of the public signal. They achieve this by writing the public signal into their contracts.

The third main result is that decreasing the precision of the public signal is Pareto improving. Since by the last result above, agents receive the same payoff (their reservation utility) for each realization of the public signal, they do not bear any risk over the realization of the public signal. Therefore, the principal bears all the risk associated with the public signal. That is to say that the agent's competition prevents them from providing insurance to the principal. But, decreasing the precision of the public signal attenuates the negative welfare effects that result from the failure of insurance. To see the advantage of a less precise public signal more clearly, consider the extreme case of a fully uninformative public signal. This is equivalent to the case of contracting without a public signal. In this case, by the first result above, the optimal contract implements both efficient risk-sharing and solves the incentive problem. Therefore, the only effect of decreasing the precision of the public signal is to improve the insurance that the agent provides to the principal: decreasing the precision of the public signal makes everyone better-off.

Our model provides some useful insight into the role of credit ratings in the delegated asset management industry. One of the most important functions of ratings is their role in institutional asset management contracts. We apply our framework to a specific model of delegated portfolio choice, interpreting the public signal as the credit rating of a risky security. We make the model concrete by considering a two asset world with a riskless bond and a risky security. The agent's private information is his knowledge of the risky security's return distribution and his action is the allocation of the principal's wealth to the risky security. For this part of the paper we restrict attention to the case in which both the principal and agent have quadratic utility, although they still differ in their aversion to risk. In this setting we can solve not only for the optimal contract but also for the equilibrium action/portfolio weight in closed-form. This allows us to establish the main results via explicit calculation. In particular, to show that decreasing the precision of the credit rating improves welfare, we write down the players' indirect utilities explicitly and compare them across different ratings partitions. Our application is more than an illustration of our theoretical analysis. It comes with a strong policy prescription: broaden ratings categories to improve risk-sharing.

This example also allows us to demonstrate that at least two predictions of our model are consistent with stylized facts. First, the equilibrium contract is affine in wealth, as are most real-wold asset managers' contracts. Second, the equilibrium contract is higher powered when ratings are good, which we interpret as an economic boom. The prediction is consistent with empirical evidence on fund flows: that capital flows from money market funds to equity funds as economic conditions improve.

- Literature

Lemma **??** is closely related to Wilson's 1984 result on the "revelation of information for joint production", where he proves that when the efficient sharing rule is affine, truthful revelation is a Nash equilibrium. We import the methodology for connecting risk-sharing with implementation into the principal-agent setting, emphasizing the explicit (direct) implementation and, further, that the optimal sharing rule is the investor's optimal contract by the equivalence of the principal-agent problem and social planner's problem above. Note that Wilson's proof exploits that when the efficient sharing rule is affine its derivative is constant and cancels out of his problem's first-order condition; we instead use that in our case the optimal allocation is independent of the welfare weight.

2 Model

The model constitutes an extensive game of incomplete information in which agents first compete in contracts in the hope of being employed by a single investor and then invest his capital on his behalf.

Players

There is a single principal with a unit wealth and von Neumann–Morgenstern utility u_{P} and at least two competitive agents with von Neumann–Morgenstern utility u_{A} and outside option $\bar{u}$. The principal and the agents differ in their risk aversion. We make no assumption as to whether the principal or the agent is more risk averse, but, for the proof of our main result, we require that both utility functions are in the same class of hyperbolic absolute risk-aversion. Specifically, their absolute risk tolerances are affine with the same slope,

$$-\frac{u'_{\mathrm{P}}(w)}{u''_{\mathrm{P}}(w)} = a_{\mathrm{P}} + bw \tag{1}$$

and

$$-\frac{u'_{\mathrm{A}}(w)}{u''_{\mathrm{A}}(w)} = a_{\mathrm{A}} + bw \tag{2}$$

for $a_i > -bw$ for all w and for $i \in \{\mathrm{P}, \mathrm{A}\}$.[3] Note that this assumption imposes no restriction on the magnitude of the difference between the principal's and agent's risk aversions. When we consider the application to delegated asset management (Section 4) we assume that players have quadratic utility; quadratic utility satisfies conditions (1) and (2) with $b = -1$.

Agents have private information, captured by their type σ. A public signal ρ conveys information about σ. In the application to delegated asset management, σ represents agents' expert knowledge about the risk of the market securities and ρ represents the securities' credit ratings.

Actions and Contracts

The principal wishes to delegate investment to an agent because he is better informed; however, he anticipates a misalignment of investment incentives since his risk aversion differs from the agents'.

Contracts attempt to align incentives to mitigate the downside of delegated asset management. Each agent a offers contract Φ_a which may depend on the final wealth w, the public signal ρ, and his action x. The agent chooses x after he has entered contract. The action choice affects only the distribution of the final wealth $\tilde{w}(x)$. We assume that $\tilde{w}$ is a concave function of x for every state of the world. In our portfolio management application in Section 4, we interpret x as the proportion of wealth invested in an asset. Note that the agent's type σ does not enter the contract because it is not verifiable; however, ρ may enter the contract as a proxy.

[3] For example, when $b = 0$ conditions (1) and (2) imply that the principal and the agents have exponential utility with constant coefficients of absolute risk aversion a_{P}^{-1} and a_{A}^{-1}.

Timing

After agents announce their contracts, the principal observes ρ and employs an agent who chooses x after learning σ. Then, wealth realizes and players divide it according to the initial contract. Formally, the timing is as follows:

1. Agents simultaneously offer contracts Φ_a.
2. σ and ρ realize.
3. The principal observes ρ and the profile of contracts $\{\Phi_a\}_a$ and hires an agent a^*.
4. Agent a^* chooses x.
5. Final wealth realizes and it is distributed such that agent a^* is awarded $\Phi_{a^*}(w)$ and the investor keeps $w - \Phi_{a^*}(w)$.

Note that key to our timing is that players learn ρ after agents offer contracts but before the principal employes an agent. In Section 4.2, we demonstrate that our results are robust to the inclusion of a second public signal that realizes after the agent has been employed. Nevertheless, the timing is sensitive to the agent's offering contracts before they learn σ. Our timing shuts down any signaling incentives.

Note on Notation

We frequently omit the arguments of variables. The contract Φ always depends on wealth w, the agent's action x, and the public signal ρ, as well as the offering agent a, but we frequently write just $\Phi(w)$. The agent chooses the action given his type σ, but we usually write just x for $x(\sigma)$. Later we will introduce a social planner's problem, in which the welfare function places weight μ_ρ on the agent given the realization ρ of the public signal. We sometimes suppress this dependence and write μ for μ_ρ. Finally, the social planner's sharing rule φ depends on final wealth directly and on the public signal indirectly via the welfare weight. While we sometimes write formally $\varphi_{\mu_\rho}(w)$, we frequently abbreviate to $\varphi_\mu(w)$ or even just $\varphi(w)$.

3 General Case: Results

Competition Is Rating-by-Rating

We first show that agents must break even for every realization of the public signal. This will allow us to transform our game into a family of principal-agent problems, one for each realization of the public signal. That is to say that for every realization of the public signal the agent must offer the contract that maximizes the principal's utility and assures him at least his reservation payoff.

Lemma 3.1. *The employed agent a^* breaks even for each realization ρ of the public signal, or*

$$\mathbb{E}\left[u_{\mathrm{A}}\Big(\Phi_{a^*}(\tilde{w})\Big)\,\Big|\,\tilde{\rho}=\rho\right]=\bar{u}$$

for all ρ.

Proof. The proof is in Appendix A.1.1. □

That agents receive their reservation utility in equilibrium is unsurprising because they are competitive. The takeaway from Lemma 3.1 above is that agents receive their reservation utility for *every realization of the public signal*. There cannot be an equilibrium in which agents break even in expectation over all possible realizations (unless they break even for every realization). In fact, if that is the case, then an agent who receives less than his reservation utility for some realization of the public signal must receive in excess of his reservation utility for another realization. But since the agent is getting strictly in excess of his reservation utility for this realization, another agent can undercut him by offering a contract that grants him more than his reservation utility and allocates more of the surplus to the principal.

The proof is by contraction. It is standard except for one subtlety. We first suppose that an agent receives strictly in excess of his reservation utility for some realization of the signal. This agent must therefore be employed given this realization. But then another agent, otherwise unemployed and receiving his reservation utility, would undercut the employed agent for this realization of the signal. Therefore, à la Bertrand competition, the agents must break-even given this realization. The only subtlety of the proof is that agents' contracts affect their incentives and hence their actions. Thus, when a deviant agent offers the principal a contract, the principal must take the effect of this contract on the agent's action into account. Our proof circumvents this issue by constructing a deviation that preserves the incentives of the originally employed agent while allocating more surplus to the principal. Specifically, if the supposed equilibrium contract is Φ the deviation $\Phi_\varepsilon(w) := u_{\mathrm{A}}^{-1}\big(u_{\mathrm{A}}(\Phi(w)-\varepsilon)\big)$ preserves the employed agent's incentives.

The argument in the proof of Lemma 3.1 also implies that the contract must maximize the principal's utility for every realization of the signal ρ as is summarized in Corollary 3.1 below. The reason is that if the employed agent does not maximize the principal's utility, then another agent can deviate to a contract more favorable to the principal that also leaves him a small surplus above $\bar{u}$.

Corollary 3.1. *If Φ_{a^*} is the contract of the employed agent a^* given rating $\hat{\rho}$ and there is another contract $\hat{\Phi}$ such that*

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\hat{\Phi}\big(\tilde{w}\big)\Big)\;\big|\;\tilde{\rho}=\hat{\rho}\right] > \mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\Phi_{a^*}\big(\tilde{w}\big)\Big)\;\big|\;\tilde{\rho}=\hat{\rho}\right],$$

then it must be that

$$\mathbb{E}\left[u_{\mathrm{A}}\Big(\hat{\Phi}\big(\tilde{w}\big)\Big)\;\big|\;\tilde{\rho}=\hat{\rho}\right] < \bar{u}.$$

Principal-Agent Formulation

Lemma 3.1 and Corollary 3.1 taken together say that the principal chooses the contract that maximizes his expected utility subject to the constraint that the agent receives his reservation utility for every realization of the signal ρ. That is to say that the equilibrium contract solves the principal-agent problem for every ρ. The twist on a standard principal-agent problem is that the agent's participation constraint depends on the public signal.

Proposition 3.1. *For each realization ρ of the public signal, the contract of the employed agent $a^*(\rho)$ solves the following principal-agent problem:*

$$\begin{cases} \textit{Maximize}\;\; \mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(x)-\Phi\big(\tilde{w}(x),x,\rho\big)\Big)\;\big|\;\tilde{\rho}=\rho\right] \\[2ex] \textit{subject to}\;\; \mathbb{E}\left[u_{\mathrm{A}}\Big(\Phi\big(\tilde{w}(x),x,\rho\big)\Big)\;\big|\;\tilde{\rho}=\rho\right]=\bar{u}\;\;\textit{and} \\[2ex] x\in\arg\max\left\{\mathbb{E}\left[u_{\mathrm{A}}\big(\Phi\big(\tilde{w}(\xi),\xi,\rho\big)\;\big|\;\tilde{\sigma}=\sigma\right]\;\;;\;\xi\in\mathbb{R}\right\} \end{cases} \tag{3}$$

over contracts Φ.

Equilibrium Contract as the Solution of a Social Planner's Problem

For each realization of the public signal ρ, we transform the principal-agent problem into a social planner's problem. The social planner will maximize social welfare subject to the agent's incentive compatibility constraint. Call the agent's welfare weight μ_ρ for a given ρ. This will coincide with the Lagrange multiplier on the agent's participation constraint in the principal-agent problem for a given ρ. This approach allows us eliminate the agent's participation constraints temporarily to focus on incentive compatibility.

Now use the method of Lagrange multipliers to eliminate the participation constraint and say that the problem is to maximize

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(x)-\Phi\big(\tilde{w}(x),x,\rho\big)\Big)+\mu_\rho\Big[u_{\mathrm{A}}\Big(\Phi\big(\tilde{w}(x),x,\rho\big)-\bar{u}\Big]\,\middle|\,\tilde{\rho}=\rho\right]$$

subject to

$$x\in\arg\max\left\{\mathbb{E}\left[u_{\mathrm{A}}\Big(\Phi\big(\tilde{w}(\xi),\xi,\rho\big)\,\middle|\,\tilde{\sigma}=\sigma\right]\ ;\ \xi\in\mathbb{R}\right\}$$

over contract Φ.

For any Lagrange multiplier μ_ρ we solve the following social planner's problem for Φ and x:

$$\left\{\begin{array}{l}\text{Maximize }\ \mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(x)-\Phi\big(\tilde{w}(x),x,\rho\big)\Big)+\mu_\rho u_{\mathrm{A}}\Big(\Phi\big(\tilde{w}(x),x,\rho\big)\,\middle|\,\tilde{\rho}=\rho\right]\\[2ex] \text{subject to }\ x\in\arg\max\left\{\mathbb{E}\left[u_{\mathrm{A}}\Big(\Phi\big(\tilde{w}(\xi),\xi,\rho\big)\,\middle|\,\tilde{\sigma}=\sigma\right]\ ;\ \xi\in\mathbb{R}\right\}.\end{array}\right. \tag{4}$$

Below we solve the problem for a generic Lagrange multiplier and only later we use the agent's binding participation constraint to solve for μ_ρ for each ρ. Transforming the game into a social planner's problem reveals that the task is to trade off efficient risk sharing with implementing efficient investment.

The Efficient Sharing Rule Implements Efficient Investment

We now find the contract that solves the social planner's problem. We do this by characterizing the first-best contract and action—i.e. those that the social planner would choose if he had perfect information. We then show that given the first-best contract the first-best action is incentive compatible, so the solution to the social planner's problem coincides with the first-best outcome. Thus, in fact, there is no tension between risk-sharing and efficient investment in equilibrium.

Proposition 3.2. *If the contract is the efficient sharing rule, then the incentive compatible action is the social optimum.*

Namely, if φ maximizes

$$u_{\mathrm{P}}(w-\varphi)+\mu_\rho u_{\mathrm{A}}(\varphi)$$

then

$$x \in \arg\max \left\{\mathbb{E}\left[u_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(\xi)\big)\Big) \,\Big|\, \tilde{\sigma}=\sigma\right]\right\}$$

implies

$$x \in \arg\max \left\{\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(\xi)-\varphi\big(\tilde{w}(\xi)\big)\Big)+\mu_\rho u_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(\xi)\big)\Big) \,\Big|\, \tilde{\sigma}=\sigma\right]\right\}.$$

Proof. The proof is in Appendix A.1.2. □

The main takeaway of Proposition 3.2 is that for any ρ the efficient contract implements the efficient action.

In the proof we first find the efficient φ. We then demonstrate that, given this φ, the agent would choose the social optimum. That is to say that the action that the agent chooses coincides with the action a social planner would choose if he had the agent's private information.

To understand the connection between incentive alignment and risk sharing, recall that a sharing rule φ is efficient if it maximizes $u_{\mathrm{P}}(w-\varphi)+\mu_\rho u_{\mathrm{A}}(\varphi)$ for each realization of w or

$$u'_{\mathrm{P}}(w-\varphi(w)) = \mu_\rho u'_{\mathrm{A}}(\varphi(w)). \tag{5}$$

On the other hand, the sharing rule φ aligns the incentives of the principal and the agent globally if one's utility function is affine transformation of the other's given the sharing rule φ, or

$$u_{\mathrm{P}}(w-\varphi(w)) = \alpha u_{\mathrm{A}}(\varphi(w))+\beta$$

for some $\alpha>0$ and $\beta\in\mathbb{R}$. Differentiating this condition with respect to w gives

$$u'_{\mathrm{P}}(w-\varphi(w)) = \frac{\alpha\varphi'(w)}{1-\varphi'(w)}u'_{\mathrm{A}}(\varphi(w)).$$

This last condition coincides with the condition above of efficient risk sharing (condition (5)) exactly when $\mu_\rho = \dfrac{\alpha\varphi'(w)}{1-\varphi'(w)}$, which is possible if and only if φ' is a constant or φ is affine. The only remaining step in the argument is to show that the efficient sharing rule is affine for the preferences we consider, which we show in Lemma A.1 in Appendix A.1.2.

Coarser Public Signals Are Pareto-Improving

Proposition 3.2 shows that the optimal contract eliminates the incentive problem for every σ and the risk-sharing problem for every ρ. The problem remains to share risk over realizations of the public signal. The next result states that less precise public signals Pareto dominate more precise public signals. The reason is that the public signal does not mitigate the incentive problem but only hinders risk-sharing.

From now on, since the optimal contract solves the incentive problem, we omit incentive constraints and focus directly on the social planner's problem (with complete information) as per Proposition 3.2.

Proposition 3.3. *Coarser public signals Pareto-dominate finer ones: for any signal $\tilde{\rho}_c$ and $\tilde{\rho}_f$ such that $\sigma(\tilde{\rho}_c) \subset \sigma(\tilde{\rho}_f)$, the ex ante equilibrium utility of all players is weakly higher given $\tilde{\rho}_c$ than $\tilde{\rho}_f$.*

Proof. Below we omit the dependence of φ on x because by Proposition 3.2 the efficient x is chosen for every σ independently of ρ. Below call $\varphi_{\mu_{\rho_f}}$ and $\varphi_{\mu_{\rho_c}}$ the efficient sharing rules associated with fine and coarse public signals respectively.

First, the agent's participation constraint given $\tilde{\rho}_f$ is

$$\mathbb{E}\left[u_{\mathrm{A}}\left(\varphi_{\mu_{\rho_f}}(\tilde{w})\right)\,\middle|\,\tilde{\rho}_f=\rho\right]=\bar{u}.$$

Now, since $\sigma(\tilde{\rho}_c) \subset \sigma(\tilde{\rho}_f)$, use the law of iterated expectations and the condition above to observe that

$$\mathbb{E}\left[u_{\mathrm{A}}\left(\varphi_{\mu_{\rho_f}}(\tilde{w})\right)\,\middle|\,\tilde{\rho}_c\right]=\mathbb{E}\left[\mathbb{E}\left[u_{\mathrm{A}}\left(\varphi_{\mu_{\rho_f}}(\tilde{w})\right)\,\middle|\,\tilde{\rho}_f\right]\,\middle|\,\tilde{\rho}_c\right]=\mathbb{E}\left[\bar{u}\,\middle|\,\tilde{\rho}_c\right]=\bar{u}.$$

This says that $\varphi_{\mu_{\rho_f}}$ satisfies the participation constraint given ρ_c. Since $\varphi_{\mu_{\rho_c}}$ solves the principal-agent problem given ρ_c—viz. it maximizes the principal's utility given the agent's participation constraint—

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\varphi_{\mu_{\rho_c}}(\tilde{w})\Big)\,\Big|\,\tilde{\rho}_c\right] \geq \mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\varphi_{\mu_{\rho_f}}(\tilde{w})\Big)\,\Big|\,\tilde{\rho}_c\right].$$

Now we use the inequality above and we apply the law of iterated expectations again to prove that the principal is better off given the coarser ratings, namely

$$\begin{aligned}\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\varphi_{\mu_{\rho_c}}(\tilde{w})\Big)\right] &= \mathbb{E}\left[\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\varphi_{\mu_{\rho_c}}(\tilde{w})\Big)\,\Big|\,\tilde{\rho}_c\right]\right] \\ &\geq \mathbb{E}\left[\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\varphi_{\mu_{\rho_f}}(\tilde{w})\Big)\,\Big|\,\tilde{\rho}_c\right]\right] = \mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\varphi_{\mu_{\rho_f}}(\tilde{w})\Big)\right].\end{aligned}$$

Since agents always break-even and the principal is better off with coarser public signals $\tilde{\rho}_c$ Pareto dominates $\tilde{\rho}_f$. □

The main step of the proof is to show that a contract that is feasible given a fine signal structure is also feasible given a coarse signal structure. This follows directly from the law of iterated expectations. Since coarsening the signal structure expands the set of feasible contracts, it can only increase the principal's objective (recall that the incentive constraints are not binding by Proposition 3.2). Since the agent always breaks even, increasing the principal's profits constitutes a Pareto improvement.

The intuition behind this result comes from Lemma 3.1. Because competition makes agents break even state-by-state, there is one participation constraint for each realization of the public signal. Thus, with a finer signal structure there are more realizations of the public signal and, thus, more constraints on the principal's objective. Because we know from Proposition 3.2 that the efficient action is always taken, these constraints restrict only risk-sharing between the principal and the agent. A finer signal structure shuts down risk-sharing and reduces welfare.

4 An Example: Portfolio Choice with Quadratic Utility

Setup

To fix ideas we consider the specific case of portfolio choice with quadratic utility. This example allows us to solve the model explicitly and thus it exposes the forces behind the more general proofs above.

The portfolio choice model has a risk-free bond with gross return R_f and a risky asset with random gross return $\tilde{R}$. The agent's type σ will be the standard deviation of $\tilde{R}$ and ρ will be an imperfect public signal about this risk parameter. Call ρ the credit rating of the risky security. The agent's action x represents the proportion of wealth invested in the risky security; therefore,

$$\tilde{w}(x) = R_f + x\big(\tilde{R} - R_f\big).$$

We assume that all players have quadratic utility,

$$u_i(w) = -\frac{1}{2}\big(a_i - w\big)^2$$

for $i \in \{\mathrm{A}, \mathrm{P}\}$. The investor differs from the agents in his risk aversion. Note that the coefficient of absolute risk tolerance is $a_i - w$, so these utility functions are in the same class of hyperbolic absolute risk aversion as defined in equations (1) and (2).

We make some restrictions on the distribution of $\tilde{R}$ to simplify the belief updating. We assume that the mean return $\bar{R}$ of the risky asset is known and independent of the agent's type σ. In fact, since with quadratic utility players' expected utility depends on only the mean and variance of the distribution, all relevant asymmetric information is about the variance σ^2. Note that this assumption implies that the credit rating is informative only about the asset's risk and not about its expected return,

$$\mathbb{E}\left[\tilde{R} \,\middle|\, \tilde{\rho} = \rho\right] = \mathbb{E}\big[\tilde{R}\big].$$

With quadratic utility, players' marginal utility is decreasing when their wealth is large. We restrict parameters to ensure that players' wealth is not so large. In particular, it must be that the wealth of the principal and that of the agent are not too large, or, respectively,

$$w - \Phi(w) < a_{\mathrm{P}}$$

and

$$\Phi(w) < a_{\mathrm{A}}.$$

A sufficient condition for this is

$$\operatorname{supp} \tilde{w} \subset [0, a_{\mathrm{P}} + a_{\mathrm{A}}). \tag{6}$$

To guarantee this, make the technical assumption that

$$\big(\bar{R} - R_f\big)\big(R - \bar{R}\big) \leq \sigma^2 \tag{7}$$

for all pairs (σ, R).[4]

4.1 Results

Competition Is Rating-by-Rating

Lemma 3.1 implies that agents must break even for each realization of the credit rating. Recall that the reason is that competition in contracts is Bertrand-like in the sense that the employed agent will receive his reservation utility conditional on any realization of the credit rating $\tilde{\rho}$; further agents act so as to maximize the investor's expected utility conditional on every ρ subject to their participation constraints.

The proof of Lemma 3.1 is in Appendix A.1.1, but re-iterating the main argument with these specific utility functions can clarify the proof. Recall that the proof is by contradiction. Supposing an equilibrium in which an agent receives in excess of his reservation utility for some realization of the public signal, a deviating agent can undercut him. However, we must be careful to take into account the effect of the new contract on the agent's incentives. We construct a deviation that does not distort incentives. With the current utility specification we can write it explicitly. In particular, if the initial contract given a rating $\hat{\rho}$ is $\hat{\Phi}$, the contract for $\varepsilon > 0$ is

$$\hat{\Phi}_\varepsilon(w) := u_{\mathrm{A}}^{-1}\Big(u_{\mathrm{A}}\Big(\hat{\Phi}\big(\tilde{w}\big) - \varepsilon\Big)\Big) = a_{\mathrm{A}} - \sqrt{\big(a_{\mathrm{A}} - \hat{\Phi}(w)\big)^2 - 2\varepsilon}$$

gives the agent identical incentives to Φ and allocates more surplus to the principal.

Principal-Agent Formulation and Social Planner's Problem

Lemma 3.1 asserts that agents compete rating-by-rating, maximizing investor welfare subject to their participation constraints, that is to say that for every realization ρ of the credit ratings the contract of the employed agent and the corresponding portfolio weight solve the principal-agent problem of Proposition 3.1. Using the method of

[4]Condition (7), sufficient for condition (6), comes from solving the game assuming that the agent's participation constraint binds, then writing a sufficient condition for it to bind in light of the equilibrium.

Lagrange multipliers we can transform the principal-agent problem into the social planner's problem summarized by the system (4). Now, unlike in the general case, we can compute simple expressions not only for the optimal contract but also for the agent's action x and the Lagrange multiplier/welfare weight μ_ρ.

The Efficient Sharing Rule Implements Efficient Investment

First find the optimal sharing rule using the first order condition in equation (5),

$$u'_{\mathrm{P}}\big(w - \varphi_\mu(w)\big) = \mu\, u'_{\mathrm{A}}\big(\varphi_\mu(w)\big),$$

or, for quadratic utility,

$$w - \varphi_\mu(w) - a_{\mathrm{P}} = \mu\big(\varphi_\mu(w) - a_{\mathrm{A}}\big)$$

for all w. Thus the unconstrained efficient sharing rule is

$$\varphi_\mu(w) = a_{\mathrm{A}} + \frac{w - a_{\mathrm{P}} - a_{\mathrm{A}}}{1+\mu}. \tag{8}$$

Observe that the standard deviation σ does not enter the expression, and thus that the social planner need not know the true variance to implement optimal risk sharing.

Given the optimal sharing rule, we now calculate the first-best investment in the risky security x^* in the sense that x^* is the investment that the social planner would make if he knew the standard deviation σ. The first-best will be useful in finding the solution to the second-best problem in which the social planner knows only ρ and the agent chooses x, which in turn constitutes the equilibrium of the model. The reason that it is useful to compute the first-best outcome is that we proceed to show that it is an attainable outcome of the second-best problem. Thus we solve the game by showing that the social optimum is attainable.

The social planner finds x^* by computing the maximum of

$$\begin{aligned} &\mathbb{E}\left[u_{\mathrm{P}}\left(R_f + x\big(\tilde{R} - R_f\big) - \varphi_\mu\Big(R_f + x\big(\tilde{R} - R_f\big)\Big)\right) \,\middle|\, \tilde{\sigma} = \sigma\right] \\ &\qquad + \mu\mathbb{E}\left[u_{\mathrm{A}}\left(\varphi_\mu\Big(R_f + x\big(\tilde{R} - R_f\big)\Big)\right) \,\middle|\, \tilde{\sigma} = \sigma\right], \end{aligned} \tag{9}$$

over all x. Mechanical computations collected in Appendix A.2.1 reveal that the optimal investment is

$$x^*(\sigma) = \frac{\big(\bar{R} - R_f\big)\big(a_{\mathrm{P}} + a_{\mathrm{A}} - R_f\big)}{\sigma^2 + \big(\bar{R} - R_f\big)^2}. \tag{10}$$

Note that the optimal investment does not depend on the welfare weight μ, thus the social planner chooses the same x^* for all μ, even as $\mu \to \infty$. But, now, in the limit as $\mu \to \infty$, since the social planner puts all the weight on the agent, his objective coincides with the agent's. Put differently, if the contract is the efficient sharing rule, the agent always takes the socially optimal action. This observation implies Proposition 3.2 in the context of this example.

The Break-even Welfare Weight and Ex Ante Utility

Now we can characterize the employed agent's contract explicitly by finding the Lagrange multiplier μ_ρ for each ρ. For a given contract φ_{μ_ρ} the agent must break even, so we can determine μ_ρ directly from the agent's participation constraint:

$$\mathbb{E}\left[u_{\mathrm{A}}\left(\varphi_{\mu_\rho}\Big(R_f + x(\tilde{\sigma})\big(\tilde{R} - R_f\big)\Big)\right) \,\Big|\, \tilde{\rho} = \rho\right] = \bar{u}. \tag{11}$$

This equation combined with the closed-form expressions for φ_{μ_ρ} and $x^*(\sigma)$ above allow us to compute μ_ρ in closed-form. A string of calculations employing the law of iterated expectations (cf. Appendix A.2.2), says

$$\big(1 + \mu_\rho\big)^2 = \frac{\big(a_{\mathrm{P}} + a_{\mathrm{A}} - R_f\big)^2}{2\,|\bar{u}|}\, \mathbb{E}\left[\frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + \big(\bar{R} - R_f\big)^2} \,\middle|\, \tilde{\rho} = \rho\right]. \tag{12}$$

This formula will be useful to express the ex ante utility of the principal and then to see constructively how changing the coarseness of the ratings partition affects investor welfare. In particular, within the framework of the example, we will be able to provide a less abstract proof of Proposition 3.3.

Before we proceed to the welfare analysis, we highlight one insight that the expression for the Lagrange multiplier offers. The mapping

$$\tilde{\sigma}^2 \mapsto \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + \big(\tilde{R} - R_f\big)^2}$$

under the expectation operator in equation (11) is concave, so that if the distribution of $\tilde{\sigma}^2$ spreads out (for example in the sense of second-order stochastic dominance), then μ_ρ decreases, suggesting that the more distribution risk the agent faces, the less the investor must compensate him despite his risk aversion, as captured by the social planner's lower welfare weight. This observation presents a puzzle: why would the agent, who is risk-averse, prefer a riskier distribution? The puzzle finds its resolution in the observation that higher dispersion of the variance comes with option value, and thus convexity, making him risk-loving over this kind of risk. The reason is that his investment decision comes after the realization of the variance, and thus the riskier decisions come with option value allowing him to adjust his investment decision to market conditions: when σ^2 is very low he will invest a lot in the risky asset, while when it is high he will invest relatively more in the riskless bond.

Now return to the main analysis. To analyze welfare we use the equilibrium welfare weight to find a formula for the investor's equilibrium expected utility given the rating ρ,

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(x) - \varphi\big(\tilde{w}(x)\big)\Big) \;\Big|\; \tilde{\rho} = \rho\right] = \bar{u}\,\mu_\rho^2 \tag{13}$$

(see Appendix A.2.3 for the short calculation). Thus his ex ante expected utility

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(x) - \varphi\big(\tilde{w}(x)\big)\Big)\right] = \bar{u}\,\mathbb{E}\left[\mu_{\tilde{\rho}}^2\right]. \tag{14}$$

Coarser Credit Ratings Are Pareto-Improving

Since competition means that agents always receive their reservation utilities, the main result that coarsening credit ratings makes everyone better-off follows from comparing the ex ante expected utility of the investor across ratings systems, using formula (14) above combined with the connection between convex functions, second-order stochastic dominance, and the law of iterated expectations.

Within the setting of the example we can now provide a constructive proof for Proposition 3.3 above, which says that coarse credit ratings Pareto dominate finer ones.

Our proof has two main steps. First to show that the investor's ex ante expected utility is minus the expectation of a convex function,

$$\bar{u}\,\mathbb{E}\left[\mu_{\tilde{\rho}}^2\right] = -c\,\mathbb{E}\left[f\Big(\mathbb{E}\left[Y \mid \tilde{\rho}\,\right]\Big)\right]$$

for $c > 0$, $f'' > 0$ and a random variable Y; and second to show that the expectation conditional on coarse ratings second-order stochastically dominates the expectation conditional on fine ratings,

$$\mathbb{E}[Y \mid \tilde{\rho}_c] \overset{\text{SOSD}}{\succ} \mathbb{E}[Y \mid \tilde{\rho}_f],$$

whence utility is greater under coarse ratings because minus a convex function is a concave function, and, à la risk aversion, the expectation of a concave function of a stochastically dominated random variable is greater than the expectation of the function of the dominated variable.

Step 1: Rewrite the investor's ex ante expected utility:

$$\begin{aligned}
\bar{u}\,\mathbb{E}\left[\mu_{\tilde{\rho}}^2\right] &= \bar{u}\,\mathbb{E}\left[\left(\sqrt{\frac{(a_{\mathrm{P}} + a_{\mathrm{A}} - R_f)^2}{2|\bar{u}|}\mathbb{E}\left[\frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + (\bar{R} - R_f)^2} \,\middle|\, \tilde{\rho}\right]} - 1\right)^2\right] \\
&= \frac{\bar{u}(a_{\mathrm{P}} + a_{\mathrm{A}} - R_f)^2}{\sqrt{2|\bar{u}|}}\mathbb{E}\left[\left[\sqrt{\mathbb{E}\left[\frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + (\bar{R} - R_f)^2} \,\middle|\, \tilde{\rho}\right]} - 1\right]^2\right] \\
&= -c\,\mathbb{E}\left[f\Big(\mathbb{E}[Y|\tilde{\rho}]\Big)\right]
\end{aligned}$$

where

$$\begin{aligned}
c &:= \sqrt{|\bar{u}|/2}\,(a_{\mathrm{P}} + a_{\mathrm{A}} - R_f)^2, \\
f(z) &:= \left(\sqrt{z} - 1\right)^2,
\end{aligned}$$

and

$$Y := \frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + \left(\bar{R} - R_f\right)^2}.$$

Note that $c > 0$ and $f''(z) = z^{3/2}/2 > 0$.

Step 2: By definition,

$$\mathbb{E}[Y \mid \tilde{\rho}_c] \overset{\text{SOSD}}{\succ} \mathbb{E}[Y \mid \tilde{\rho}_f]$$

if there exists a random variable $\tilde{\varepsilon}$ such that

$$\mathbb{E}[Y \mid \tilde{\rho}_f] = \mathbb{E}[Y \mid \tilde{\rho}_c] + \tilde{\varepsilon}$$

and

$$\mathbb{E}\left[\tilde{\varepsilon}\,\middle|\,\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\right] = 0.$$

For $\tilde{\varepsilon} = \mathbb{E}\left[Y \mid \tilde{\rho}_f\right] - \mathbb{E}\left[Y \mid \tilde{\rho}_c\right]$ from the above, the condition is

$$\mathbb{E}\left[\mathbb{E}\left[Y \mid \tilde{\rho}_f\right] - \mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\,\middle|\,\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\right] = 0$$

or

$$\mathbb{E}\left[\mathbb{E}\left[Y \mid \tilde{\rho}_f\right]\,\middle|\,\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\right] = \mathbb{E}\left[Y \mid \tilde{\rho}_c\right].$$

Given the assumption $\sigma(\tilde{\rho}_c) \subset \sigma(\tilde{\rho}_f)$ and since conditioning destroys information—$\sigma\big(\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\big) \subset \sigma(\tilde{\rho}_c)$—apply the law of iterated expectations firstly to add and then to delete conditioning information to calculate that

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[Y \mid \tilde{\rho}_f\right]\,\middle|\,\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{E}\left[Y \mid \tilde{\rho}_f\right]\,\middle|\,\tilde{\rho}_c\right]\,\middle|\,\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\,\middle|\,\mathbb{E}\left[Y \mid \tilde{\rho}_c\right]\right] \\
&= \mathbb{E}\left[Y\,\middle|\,\tilde{\rho}_c\right],
\end{aligned}$$

as desired.

Asset Manager's Observed Contracts

The agent's equilibrium contract is

$$\varphi_{\mu_\rho}(w) = a_{\mathrm{A}} + \frac{w - a_{\mathrm{P}} - a_{\mathrm{A}}}{1 + \mu_\rho} \tag{15}$$

where μ_ρ is defined in equation (12).

The compensation contract is affine in wealth, as are typical asset management contracts. For the next result (and the next result only), consider a simplified but realistic credit rating rule. Let $\tilde{\rho}$ define a partition of the realization of the variance $\sigma_0^2 < \sigma_1^2 < \cdots$, namely

$$\mathbb{P}\{\tilde{\sigma}^2 \in [\sigma_i^2, \sigma_{i+1}^2) \,|\, \rho_i\} = 1.$$

Proposition 4.1. *For $i < j$, $\mu_{\rho_i} < \mu_{\rho_j}$. Increases in the expected variance decrease the power of the contract, i.e. the slope in the wealth.*

Proof. Since

$$\frac{\sigma^2}{\sigma^2 + \left(\bar{R} - R_f\right)^2}$$

is increasing in σ^2,

$$\mathbb{E}\left[\frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + \left(\bar{R} - R_f\right)^2}\,\middle|\,\rho_i\right] < \mathbb{E}\left[\frac{\sigma_{i+1}^2}{\sigma_{i+1}^2 + \left(\bar{R} - R_f\right)^2}\right] < \mathbb{E}\left[\frac{\tilde{\sigma}^2}{\tilde{\sigma}^2 + \left(\bar{R} - R_f\right)^2}\,\middle|\,\rho_{i+1}\right].$$

Immediately from equation (12), $\mu_{\rho_i} < \mu_{\rho_{i+1}}$ and by induction $\mu_{\rho_i} < \mu_{\rho_j}$ whenever $i < j$. Combined with equation (15), the result implies that lower expected variances correspond to steeper wealth compensation for agents. □

In the model, ratings describe the variance of the market portfolio. Define a "boom" a realization of $\tilde{\rho}$ implying low expected variance. With this interpretation, proposition 4.1 says that employed agents have higher powered contracts in booms than in busts. Since, almost uniformly, equity funds offer higher powered contracts than money market funds, the model predicts that the in-flows to equity funds relative to money market funds will be procyclical. Using a sample of US mutual fund data from 1991 to 2008, Chalmers et al. (2010) finds that investors direct funds away from money market funds towards equity funds when aggregate economic conditions improve, in line with our prediction.

4.2 Extensions

Imperfect Private Information

Suppose that the agent receives an imperfect signal about the variance. Namely, he observes the realization of a random variable $\tilde{s}$ that is not independent of $\tilde{\sigma}$. Then, equation (10) for the portfolio allocation becomes

$$x(\rho, s) = \frac{\left(\bar{R} - R_f\right)\left(a_{\mathrm{P}} + a_{\mathrm{A}} - R_f\right)}{\operatorname{Var}\left[\tilde{R} \mid \rho, s\right] + \left(\bar{R} - R_f\right)^2}.$$

The optimal contract is $\varphi_\mu(R_f + x(\rho, s)(R - R_f))$ (where an equation analogue to (12) determines μ).

Clearly, whenever $\sigma(\tilde{\rho}) \subset \sigma(\tilde{s})$, then $x(\rho, s)$ does not depend on ρ and our main result remains unchanged. If, instead, $\sigma(\tilde{\rho}) \not\subset \sigma(\tilde{s})$ then a trade-off arises: finer credit ratings still shut down risk sharing, but they increase allocational efficiency, i.e. the portfolio weight is closer to first best. The net welfare effect is then ambiguous.

Our model and policy prescriptions therefore apply to markets in which delegated portfolio managers are better informed than credit rating agencies.

Additional Ratings' Changes

In our model, ratings realize once, after agents offer contracts but before the investor employs an agent. In reality, ratings upgrades and downgrades are frequent and investors and agents have long-term relationships. In the model, if ratings change after the investor has employed an agent, the optimal contract above still induces the agent to invest efficiently. The new rating influences the portfolio allocation only insofar as it improves the agent's information (cf. the preceding discussion of imperfect signals). Ratings changes after the investor and agent commit to a relationship never decrease efficiency and can be beneficial if they improve information. Our model therefore suggests that investment mandates matter only because funds are looking to attract new investors or because their current investors may withdraw their funds. The idea finds support in the observation that hedge funds, who raise money infrequently via long-term contracts do not use investment mandates.

5 Conclusion

Motivated by delegated asset managers' frequent references to credit ratings in the contracts they offer their clients, we study a delegation problem with adverse selection in the presence of a public signal. We characterize the optimal contract between a risk-averse principal and a risk-averse agent and show that while it does indeed depend on the public signal, contracting on the public signal does not mitigate the incentive problem. In fact, in contrast to previous literature, we find that decreasing the precision of the public signal is Pareto improving. The reason is that contracting on final wealth implements efficient investment, so contracting on the public signal serves only to inhibit risk-sharing. Agents include the public signal in their contracts only to help them compete.

We apply the model to a classical delegated portfolio management setting in which delegated asset managers' make a portfolio choice decision on behalf of their clients.

In this setting, we interpret the public signal as a credit rating. Our main policy prescription is that credit rating agencies should provide information in forms prohibitive to their inclusion in rigid contracts. Our recommendation is consistent with the popular suggestion that markets should eliminate the mechanistic reliance on ratings. Our model also suggests investment mandates may contribute to the cyclicality of mutual fund flows, providing further motivation for their abolition.

A Appendices

A.1 General Case

A.1.1 Proof of Lemma 3.1

Suppose, in anticipation of a contradiction, an equilibrium in which the employed agent offers contract $\hat{\Phi}$ given signal $\hat{\rho}$ and receives strictly in excess of his reservation utility,

$$\mathbb{E}\left[u_{\mathrm{A}}\Big(\hat{\Phi}(\tilde{w})\Big)\,\Big|\,\tilde{\rho}=\hat{\rho}\right]>\bar{u}. \tag{16}$$

We now show that another agent $\hat{\mathrm{A}}$ has a profitable deviation. In order for a contract $\hat{\Phi}_\varepsilon$ to be a profitable deviation for $\hat{\mathrm{A}}$ it must (i) make the principal employ him given $\hat{\rho}$ and (ii) give him expected utility greater than his reservation utility $\bar{u}$ given $\hat{\rho}$. The subtlety in this proof is that $\hat{\mathrm{A}}$'s contract determines not only the allocation of surplus, but also his action x. To circumvent the effect of changing actions on payoffs, we construct $\hat{\Phi}_\varepsilon$ to induce the agent to choose the same action that he would have chosen under $\hat{\Phi}$, but still to change the division of surplus. To summarize, $\hat{\Phi}_\varepsilon$ is a profitable deviation if given $\hat{\rho}$ (i) it gives the principal higher utility than does $\hat{\Phi}$,

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\hat{\Phi}_\varepsilon(\tilde{w})\Big)\mid\tilde{\rho}=\hat{\rho}\right]>\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}-\hat{\Phi}(\tilde{w})\Big)\mid\tilde{\rho}=\hat{\rho}\right],$$

(ii) it gives the agent utility in excess of $\bar{u}$,

$$\mathbb{E}\left[u_{\mathrm{A}}\Big(\hat{\Phi}_\varepsilon(\tilde{w})\Big)\,\Big|\,\tilde{\rho}=\hat{\rho}\right]>\bar{u},$$

and (iii) the set of incentive compatible actions under $\hat{\Phi}$ and $\hat{\Phi}_\varepsilon$ coincide,

$$\arg\max_x\left\{\mathbb{E}\left[u_{\mathrm{A}}\Big(\hat{\Phi}_\varepsilon(\tilde{w})\Big)\,\Big|\,\tilde{\sigma}=\sigma\right]\right\}=\arg\max_x\left\{\mathbb{E}\left[u_{\mathrm{A}}\Big(\hat{\Phi}(\tilde{w})\Big)\,\Big|\,\tilde{\sigma}=\sigma\right]\right\}.$$

One example of contract that satisfies these three conditions is

$$\hat{\Phi}_\varepsilon(\tilde{w}):=u_{\mathrm{A}}^{-1}\Big(u_{\mathrm{A}}\Big(\hat{\Phi}(\tilde{w})-\varepsilon\Big)\Big) \tag{17}$$

given $\hat{\rho}$, so that

$$u_{\mathrm{A}}\big(\hat{\Phi}_\varepsilon\big)=u_{\mathrm{A}}\big(\hat{\Phi}\big)-\varepsilon. \tag{18}$$

Since $u'_{\mathrm{P}} > 0$, a sufficient condition for $\hat{\Phi}_\varepsilon$ to satisfy condition (i) is that

$$\tilde{w} - \hat{\Phi}_\varepsilon(\tilde{w}) > \tilde{w} - \hat{\Phi}(\tilde{w}),$$

or, substituting from equation (17),

$$\hat{\Phi}(\tilde{w}) > u_{\mathrm{A}}^{-1}\Big(u_{\mathrm{A}}\Big(\hat{\Phi}(\tilde{w}) - \varepsilon\Big)\Big),$$

which is satisfied for $\varepsilon > 0$ by the inverse function theorem since $u'_{\mathrm{A}} > 0$.

Condition (ii) holds for $\epsilon > 0$ and sufficiently small. This follows from equation (18) and inequality (16) with the continuity of u_{A}.

Finally, condition (iii) is immediate from equation (18) since affine transformations of utility do not affect choices.

Thus the investor will employ agent $\hat{\mathrm{A}}$ who will receive, given $\hat{\rho}$, utility greater than the utility that he would have received in the supposed equilibrium (in the supposed equilibrium he was unemployed and he was obtaining $\bar{u}$). Thus $\hat{\Phi}_\varepsilon$ is a profitable deviation for $\hat{\mathrm{A}}$ and Φ cannot be the contract of an agent employed at equilibrium given $\hat{\rho}$.

We have shown that the agent's expected utility given any ρ cannot exceed $\bar{u}$. To conclude the proof, note that his utility can never be strictly less than $\bar{u}$ because then his expected utility would be less than his reservation utility.

A.1.2 Proof of Proposition 3.2

The proof relies on the following lemma.

Lemma A.1. *The efficient sharing rule φ is affine in wealth w.*

Proof. Assumptions (1) and (2) imply that

$$u_{\mathrm{P}}(w) = \frac{1}{b-1}\left(\frac{w}{b} + \frac{a_{\mathrm{P}}}{b^2}\right)^{\frac{b-1}{b}}$$

and

$$u_{\mathrm{A}}(w) = \frac{1}{b-1}\left(\frac{w}{b} + \frac{a_{\mathrm{A}}}{b^2}\right)^{\frac{b-1}{b}}.$$

The contract that implements efficient risk sharing is that which maximizes social surplus (for appropriate welfare weight μ) for every realization of wealth. Now compute the efficient sharing rule directly via the first the first order approach:

$$\frac{\partial}{\partial \varphi}\Big(u_{\mathrm{P}}(w-\varphi)+\mu_\rho u_{\mathrm{A}}(\varphi)\Big)=0$$

so

$$\Big(\frac{w-\varphi}{b}+\frac{a_{\mathrm{P}}}{b^2}\Big)^{-\frac{1}{b}}=\mu_\rho\Big(\frac{\varphi}{b}+\frac{a_{\mathrm{A}}}{b^2}\Big)^{-\frac{1}{b}}.$$

This implies

$$\varphi(w)=\frac{a_{\mathrm{P}}-\mu_\rho^{-b}a_{\mathrm{A}}+bw}{b\left(1+\mu_\rho^{-b}\right)},$$

which is affine in w. □

The lemma implies that φ' is constant, which allows us to pass it under the expectation operator, a key step in the proof.

We begin the proof of Proposition 3.2 with the agent's incentive problem given the contract φ and show through a series of manipulations that his incentives are aligned with those of the social planner. We rely on the fact that $u'_{\mathrm{P}}(w-\varphi)=\mu_\rho u'_{\mathrm{A}}(\varphi)$, from the definition of efficient risk sharing.

Incentive compatibility implies the first-order condition

$$\frac{\partial}{\partial x}\mathbb{E}\left[u_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(x)\big)\Big)\,\Big|\,\tilde{\sigma}=\sigma\right]=0$$

or

$$\mathbb{E}\left[u'_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(x)\big)\Big)\varphi'\big(\tilde{w}(x)\big)\tilde{w}'(x)\,\Big|\,\tilde{\sigma}=\sigma\right]=0.$$

By lemma A.1 φ' is a constant, thus we can remove it from the equation above to get

$$\mathbb{E}\left[u'_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(x)\big)\Big)\tilde{w}'(x)\,\Big|\,\tilde{\sigma}=\sigma\right]=0.$$

And, since $u'_{\mathrm{P}}(w-\varphi)-\mu_\rho u'_{\mathrm{A}}(\varphi)=0$, the equation above re-writes as

$$\mathbb{E}\left[u'_{\mathrm{P}}\Big(\tilde{w}(x)-\varphi\big(\tilde{w}(x)\big)\Big)\tilde{w}'(x)\,\Big|\,\tilde{\sigma}=\sigma\right]=0. \tag{19}$$

Now, in order to back out the social planner's objective from (19) observe that it suffices to add

$$\mathbb{E}\left[\varphi'\big(\tilde{w}(x)\big)\tilde{w}'(x)\left[u'_{\mathrm{P}}\Big(\tilde{w}(x)-\varphi\big(\tilde{w}(x)\big)\Big)-\mu_\rho u'_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(x)\big)\Big)\right]\,\Big|\,\tilde{\sigma}=\sigma\right], \tag{20}$$

which is zero since $u'_{\mathrm{P}}(w-\varphi)-\mu_\rho u'_{\mathrm{A}}(\varphi)=0$. Now adding expression (20) to equation (19) gives

$$\mathbb{E}\left[\Big(\tilde{w}'(x)-\varphi'\big(\tilde{w}(x)\big)\tilde{w}'(x)\Big)u'_{\mathrm{P}}\Big(\tilde{w}(x)-\varphi\big(\tilde{w}(x)\big)\Big)\,\Big|\,\tilde{\sigma}=\sigma\right]+ \\ +\mu_\rho\mathbb{E}\left[\varphi'\big(\tilde{w}(x)\big)\tilde{w}'(x)u'_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(x)\big)\Big)\,\Big|\,\tilde{\sigma}=\sigma\right]=0$$

or

$$\frac{\partial}{\partial x}\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}(x)-\varphi\big(\tilde{w}(x)\big)\Big)+\mu_\rho u_{\mathrm{A}}\Big(\varphi\big(\tilde{w}(x)\big)\Big)\,\Big|\,\tilde{\sigma}=\sigma\right]=0.$$

This is the first-order condition of the social planner's problem if he knows σ. Since u_{P} and u_{A} are concave and $\tilde{w}$ is concave in x, the first order condition implies a global maximum, viz. the incentive compatible x is a social optimum.

A.2 Application: Portfolio Choice

A.2.1 Computation of Optimal Investment

The problem stated in line (9) is to find the optimal investment x^* given the optimal sharing rule stated in equation (8), namely

$$\varphi_\mu(w)=A+Bw,$$

where

$$A=-\frac{\mu a_{\mathrm{A}}+a_{\mathrm{P}}}{1+\mu}\quad\text{and}\quad B=\frac{1}{1+\mu}. \tag{21}$$

That is, x^* must maximize the expectation

$$-\frac{1}{2}\,\mathbb{E}\Bigg[\left(R_f+x(\tilde{R}-R_f)-A-B\left(R_f+x(\tilde{R}-R_f)\right)-a_{\mathrm{P}}\right)^2 \\ +\mu\left(\left(A+B\left(R_f+x(\tilde{R}-R_f)\right)-a_{\mathrm{A}}\right)^2\right)\,\Big|\,\tilde{\sigma}=\sigma\Bigg]$$

over all x. Thus the first-order condition says that for optimum x^*

$$\mathbb{E}\Bigg[(1-B)(\tilde{R}-R_f)\left(R_f + x^*(\tilde{R}-R_f) - A - B\left(R_f + x^*(\tilde{R}-R_f)\right) - a_{\mathrm{P}}\right)$$
$$+\mu\, B(\tilde{R}-R_f)\left(A + B\left(R_f + x^*(\tilde{R}-R_f)\right) - a_{\mathrm{A}}\right)\,\Big|\,\tilde{\sigma}=\sigma\Bigg] = 0,$$

thus

$$x^* = \frac{\left(\bar{R}-R_f\right)}{\mathbb{E}\left[(\tilde{R}-R_f)^2 \,\middle|\, \tilde{\sigma}=\sigma\right]}\left(\frac{(1-B)(A+a_{\mathrm{P}}) - \mu\, B(A-a_{\mathrm{A}})}{(1-B)^2 + b^2\mu} - R_f\right).$$

Substituting in for A and B from equation (21) in the numerator gives

$$(1-B)(A+a_{\mathrm{P}}) - \mu B(A-a_{\mathrm{A}}) = \frac{\mu\,(a_{\mathrm{A}}+a_{\mathrm{P}})}{1+\mu}$$

and substituting in for A and B from equation (21) in the denominator gives

$$(1-B)^2 + B^2\mu = \frac{\mu}{1+\mu}.$$

Therefore

$$\begin{aligned} x &= \frac{\left(\bar{R}-R_f\right)\left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\right)}{\mathbb{E}\left[\left(\tilde{R}-R_f\right)^2 \,\middle|\, \tilde{\sigma}=\sigma\right]} \\ &= \frac{\left(\bar{R}-R_f\right)\left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\right)}{\sigma^2 + \left(\bar{R}-R_f\right)^2}. \end{aligned}$$

A.2.2 Computation of the Social Planner's Weight

Immediately from plugging in the expressions for u_{A}, φ_{μ_ρ}, and x^* into equation (11), observe that

$$
\begin{aligned}
2|\bar{u}|\left(1+\mu_\rho\right)^2 &= \mathbb{E}\left[\left(R_f + \frac{\left(\bar{R}-R_f\right)\left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2}\left(\tilde{R}-R_f\right) - a_{\mathrm{P}} - a_{\mathrm{A}}\right)^2 \,\middle|\, \tilde{\rho}=\rho\right] \\
&= \left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\right)^2 \mathbb{E}\left[\left(\frac{\left(\bar{R}-R_f\right)\left(\tilde{R}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2} - 1\right)^2 \,\middle|\, \tilde{\rho}=\rho\right] \\
&= \left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\right)^2 \left\{1 - 2\mathbb{E}\left[\frac{\left(\bar{R}-R_f\right)\left(\tilde{R}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2} \,\middle|\, \tilde{\rho}=\rho\right] + \right. \\
&\qquad \left. + \,\mathbb{E}\left[\left(\frac{\left(\bar{R}-R_f\right)\left(\tilde{R}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2}\right)^2 \,\middle|\, \tilde{\rho}=\rho\right]\right\}.
\end{aligned}
\tag{22}
$$

Applying the law of iterated expectations gives

$$
\begin{aligned}
&1 - \frac{2|\bar{\mu}|\left(1+\mu_\rho\right)^2}{\left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\right)^2} \\
&= 2\mathbb{E}\left[\mathbb{E}\left[\frac{\left(\bar{R}-R_f\right)\left(\tilde{R}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2} \,\middle|\, \tilde{\sigma}\right] \,\middle|\, \tilde{\rho}=\rho\right] - \mathbb{E}\left[\mathbb{E}\left[\left(\frac{\left(\bar{R}-R_f\right)\left(\tilde{R}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2}\right)^2 \,\middle|\, \tilde{\sigma}\right] \,\middle|\, \tilde{\rho}=\rho\right] \\
&= 2\mathbb{E}\left[\frac{\left(\bar{R}-R_f\right)\mathbb{E}\left[\left(\tilde{R}-R_f\right) \,\middle|\, \tilde{\sigma}\right]}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2} \,\middle|\, \tilde{\rho}=\rho\right] + \mathbb{E}\left[\frac{\left(\bar{R}-R_f\right)^2\mathbb{E}\left[\left(\tilde{R}-R_f\right)^2 \,\middle|\, \tilde{\sigma}\right]}{\left(\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2\right)^2} \,\middle|\, \tilde{\rho}=\rho\right]
\end{aligned}
$$

and since

$$
\mathbb{E}\left[\left(\tilde{R}-R_f\right)^2 \,\middle|\, \tilde{\sigma}\right] = \tilde{\sigma}^2 + \left(\bar{R}-R_f\right)^2
$$

we have

$$
\begin{aligned}
&1-\frac{2|\bar{\mu}|\big(1+\mu_\rho\big)^2}{\big(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\big)^2}\\
&=\big(\bar{R}-R_f\big)^2\left\{\mathbb{E}\left[\frac{2}{\tilde{\sigma}^2+\big(\bar{R}-R_f\big)^2}\,\middle|\,\tilde{\rho}=\rho\right]-\mathbb{E}\left[\frac{1}{\tilde{\sigma}^2+\big(\bar{R}-R_f\big)^2}\,\middle|\,\tilde{\rho}=\rho\right]\right\}\\
&=\mathbb{E}\left[\frac{\big(\bar{R}-R_f\big)^2}{\tilde{\sigma}^2+\big(\bar{R}-R_f\big)^2}\,\middle|\,\tilde{\rho}=\rho\right].
\end{aligned}
$$

Finally, solve for $(1+\mu_\rho)^2$ and cross multiply to recover equation (12).

A.2.3 Computation of Expected Utility Given ρ

Plug in to equation (13) and compute, maintaining at first the shorthand

$$
\tilde{w}:=\tilde{w}\big(x^*(\tilde{\sigma})\big)=R_f+x^*(\tilde{\sigma})\big(R-R_f\big),
$$

that is:

$$
\begin{aligned}
&\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}\big(x^*(\tilde{\sigma})\big)-\varphi_{\mu_\rho}\big(\tilde{w}\big(x^*(\tilde{\sigma})\big)\big)\Big)\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{1}{2}\mathbb{E}\left[\Big(a_{\mathrm{P}}-\tilde{w}+\varphi_{\mu_\rho}\big(\tilde{w}\big)\Big)^2\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{1}{2}\mathbb{E}\left[a_{\mathrm{P}}-\tilde{w}+a_{\mathrm{A}}+\frac{\tilde{w}-a_{\mathrm{P}}-a_{\mathrm{A}}}{1+\mu_\rho}\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{1}{2}\mathbb{E}\left[a_{\mathrm{P}}-\tilde{w}+a_{\mathrm{A}}+\frac{\tilde{w}-a_{\mathrm{P}}-a_{\mathrm{A}}}{1+\mu_\rho}\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{1}{2}\left(\frac{\mu_\rho}{1+\mu_\rho}\right)^2\mathbb{E}\left[\big(a_{\mathrm{P}}+a_{\mathrm{A}}-\tilde{w}\big)^2\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{1}{2}\left(\frac{\mu_\rho}{1+\mu_\rho}\right)^2\mathbb{E}\left[\Big(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f-x^*(\tilde{\sigma})\big(\tilde{R}-R_f\big)\Big)^2\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{1}{2}\left(\frac{\mu_\rho}{1+\mu_\rho}\right)^2\mathbb{E}\left[\left(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f-\big(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\big)\frac{\big(\bar{R}-R_f\big)\big(\tilde{R}-R_f\big)}{\tilde{\sigma}^2+\big(\bar{R}-R_f\big)^2}\right)^2\,\middle|\,\tilde{\rho}=\rho\right]\\
&=-\frac{\big(a_{\mathrm{P}}+a_{\mathrm{A}}-R_f\big)^2}{2}\left(\frac{\mu_\rho}{1+\mu_\rho}\right)^2\mathbb{E}\left[\left(1-\frac{\big(\bar{R}-R_f\big)\big(\tilde{R}-R_f\big)}{\tilde{\sigma}^2+\big(\bar{R}-R_f\big)^2}\right)^2\,\middle|\,\tilde{\rho}=\rho\right].
\end{aligned}
$$

Now, from equation (22) above,

$$\mathbb{E}\left[\left(1-\frac{\left(\bar{R}-R_f\right)\left(\tilde{R}-R_f\right)}{\tilde{\sigma}^2+\left(\bar{R}-R_f\right)^2}\right)^2 \,\middle|\, \tilde{\rho}=\rho\right]=2|\bar{u}|\left(\frac{1+\mu_\rho}{a_{\mathrm{P}}+a_{\mathrm{A}}-R_f}\right)^2,$$

so, finally,

$$\mathbb{E}\left[u_{\mathrm{P}}\Big(\tilde{w}\big(x^*(\tilde{\sigma})\big)-\varphi\big(\tilde{w}\big(x^*(\tilde{\sigma})\big),\tilde{\sigma},\rho\big)\Big) \,\Big|\, \tilde{\rho}=\rho\right]=\bar{u}\,\mu_\rho^2.$$

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