Three-point correlation functions of critical system

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1. INTRODUCTION

At the critical point, the two-point function and three-point correlation function of operators look like[1],

$$\langle\sigma(x)\sigma(y)\rangle = \frac{c_1}{|x-y|^{2\Delta_\sigma}}, \quad \text{with} \quad |x-y| := \sqrt{\sum_i (x^i-y^i)^2}.$$
$$\langle\sigma(x_1)\sigma(x_2)\epsilon(x_3)\rangle = \frac{c_2}{|x_1-x_2|^{2\Delta_\sigma-\Delta_\epsilon}|x_1-x_3|^{\Delta_\sigma}|x_2-x_3|^{\Delta_\sigma}}. \tag{1.1}$$

Here, c_1 and c_2 are arbitrary constants and Δ_σ and Δ_ϵ are the so-called scaling dimensions, which is related to the critical exponents of the critical system. The correlation functions above are defined as the connected correlations, such as

$$\langle ABC\rangle = \langle ABC\rangle_0 - \langle A\rangle_0\langle BC\rangle_0 - \langle B\rangle_0\langle AC\rangle_0 - \langle C\rangle_0\langle AB\rangle_0 + 2\langle A\rangle_0\langle B\rangle_0\langle C\rangle_0. \tag{1.2}$$

For a quantum spin chain of length L with a periodic boundary condition, the formula get modified to be

$$\langle\sigma(x_1)\sigma(x_2)\rangle = c_1\left|\frac{1}{2L\sin(\frac{\pi|x_1-x_2|}{L})}\right|^{2\Delta_\sigma}$$
$$\langle\sigma(x_1)\sigma(x_2)\epsilon(x_3)\rangle = c_2\frac{1}{\left|2L\sin(\frac{\pi|x_1-x_2|}{L})\right|^{2\Delta_\sigma-\Delta_\epsilon}\left|2L\sin(\frac{\pi|x_2-x_3|}{L})\right|^{\Delta_\epsilon}\left|2L\sin(\frac{\pi|x_1-x_3|}{L})\right|^{\Delta_\epsilon}} \tag{1.3}$$

Now the bracket denotes the vacuum expectation values. These are predictions from conformal symmetry; our goal is to verify these formulas.

2. TRANSVERSE FIELD ISING MODEL AND DMRG

We can check these predictions using the DMRG simulation of the transverse field Ising model. The transverse field Ising model has the Hamiltonian

$$\hat{H} = -J\sum_i Z_i Z_{i+1} + h\sum_i X_i. \tag{2.1}$$

We define it on a one-dimensional spin chain, with a periodic boundary condition. When $h > J$, the system is in the disordered phase. When $h < J$, the system is in the spontaneous symmetry-breaking phase. The critical point is at $h = J$, which is described by the two-dimensional Ising CFT. We use DMRG to calculate the ground state wave function at the critical point, and then measure the correlation function. To compare with the lattice result, we need to identify the lattice operators with CFT operators

$$\begin{aligned}\sigma_i &\sim Z_i,\\ \epsilon_i &\sim X_i.\end{aligned} \tag{2.2}$$

We consider a spin chain with length $L = 40$. When calculating the three-point function, we set $x_1 = 1$, $x_2 = 11$, and vary x_3. The results are summarized in Fig. 1.

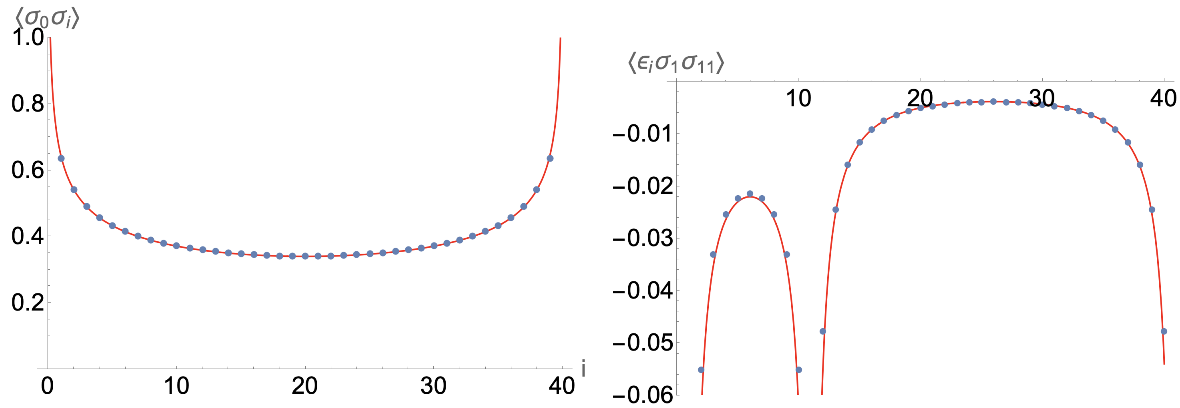

FIG. 1. The two-point $\langle\sigma(x_1)\sigma(x_2)\rangle$ and three-point functions $\langle\epsilon(x_1)\sigma(x_2)\sigma(x_3)\rangle$. The dots are measured at the critical point of the transverse field Ising model, and the red curve is the result predicted by the conformal symmetry. We have used $\Delta_\sigma = \frac{1}{8}$ and $\Delta_\epsilon = 1$.

3. FENDLEY-SENGUPTA-SACHDEV MODEL

There exists a model that is closer to the Rydberg atom experiments, called the Fendley-Sengupta-Sachdev model. The Hamiltonian is

$$\hat{H} = -J\sum_i (d_i^\dagger + d_i) + U\sum_i n_i + V\sum_i n_{i-1}n_{i+1}. \tag{3.1}$$

There can not be more than one boson occupying the same site, that is

$$n_i = 0 \quad \text{or} \quad 1. \tag{3.2}$$

In addition to that, the hardcore bosons cannot occupy neighboring sites; this is often called the Rydberg blockade condition, that is

$$n_i n_{i+1} = 0. \tag{3.3}$$

The phase diagram of this model has been studied using DMRG, see for example[2]. The coupling constants are related to experimental parameters. We identify the ground state of the atom as the state not occupied by the boson $|0\rangle$, and the Rydberg state as the state occupied by the boson $|1\rangle = d^\dagger|0\rangle$. One can change the coupling J by changing the Rabi frequency of the external laser. The frequency of the external laser is adjusted so that the detuning away from resonance of the

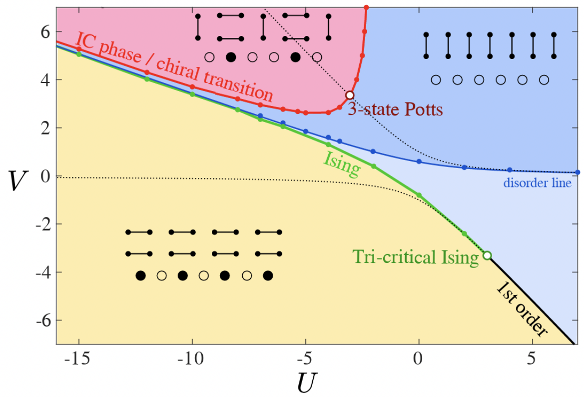

FIG. 2. The phase diagram of the Fendley-Sengupta-Sachdev model. The plot is reproduced from[2].

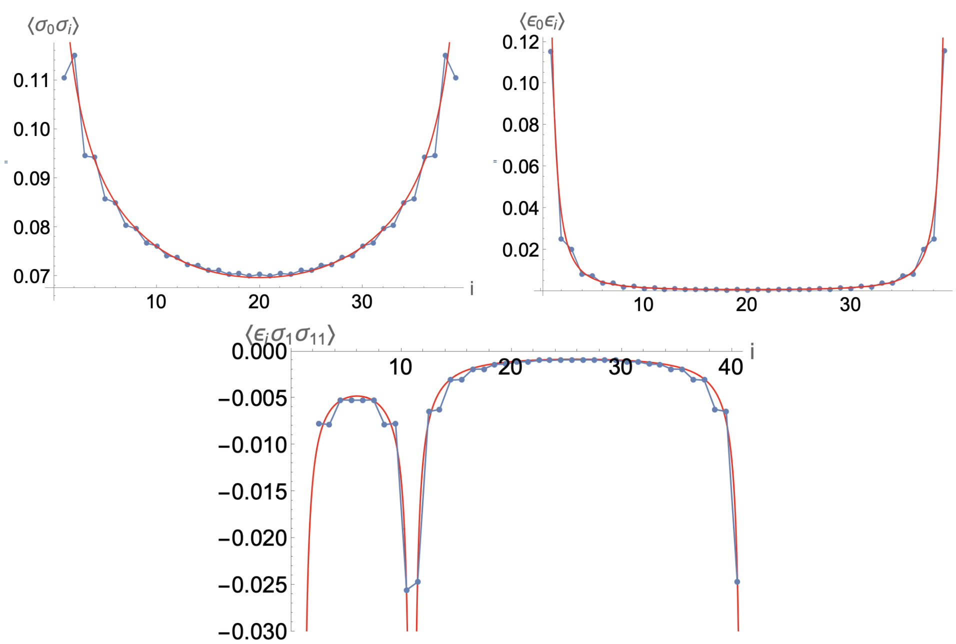

FIG. 3. The two-point and three-point functions of the Fendley-Sengupta-Sachdev model.

$|0\rangle$ to $|1\rangle$ transition is U. (See Sachedev's new book "Quantum Phases of Matter"). We can also replace the last term by a long-range repulsive interaction to represent the van der Waals interaction when both atoms are in their Rydberg states.

We can again study this model using DMRG. We can identify

$$d^\dagger + d \quad \to \quad X. \tag{3.4}$$

Here X is the Pauli Matrix. To impose the Rydberg blockade condition, we add a strong repulsive interaction

$$\hat{H}_{rp} = U_{rp} \sum_i n_i n_{i+1}. \tag{3.5}$$

We can also identify the lattice operators with CFT operators according to,

$$\begin{aligned} \sigma_i \sim & \quad (-1)^i n_i \\ \epsilon_{i+1/2} \sim & \quad (1-n_i)(1-n_{i+1}). \end{aligned} \tag{3.6}$$

The ϵ operator is defined on the bonds. We study a lattice with $L = 40$ sites. We set the coupling constants to be

$$J = 1, \quad U = 0.128, \quad V = -1, \quad \text{and} \quad U_{rp} = 20. \tag{3.7}$$

The two-point $\langle \sigma_0 \sigma_1 \rangle$ and three-point functions $\langle \epsilon_i \sigma_1 \sigma_{11} \rangle$ can be found in Fig. 3.

4. A MORE REALISTIC MODEL

Finally, we can consider a more realistic model with van der Waals $1/r^6$ interaction between the excited atoms. We will follow the same convention as in[3],

$$\hat{H} = \frac{\Omega}{2} \sum_i (d_i^\dagger + d_i) - \Delta \sum_i n_i + C_6 \sum_{ij} \frac{1}{(d_{ij})^6} n_i n_j. \tag{4.1}$$

Here d_{ij} is the distance between the i-th and j-th sites. Since the sites are arranged to form a circle, we get

$$d_{ij} = a \frac{\sin(\frac{|i-j|}{L}\pi)}{\sin(\frac{1}{L}\pi)}, \tag{4.2}$$

where a is the lattice constant. We will set $a = 1$ for later discussion. The sign of the first term does not change the physics. We can flip the sign by the redefinition of state

$$d^\dagger |0\rangle \to -d^\dagger |0\rangle. \tag{4.3}$$

Roughly speaking, in experiment, one can access the parameter space

$$\Omega/C_6 \in [0, 5], \quad \Delta/J \in [-5, 5]. \tag{4.4}$$

The phase diagram near the Ising transition is reported in Fig. 4. The red circle corresponds to the point studied in[3]. Where they studied the model at $R_b/a = \frac{(C_6/\Omega)^{1/6}}{a} = 1.4$ and $\Delta/\Omega = 0.97(5)$. For future reference, we list here the critical points we studied:
$(\Omega/C_6, \Delta/C_6)$ ={{0.04,0.058},{0.066666666,0.0785},{0.1,0.105},{0.1333333334,0.1353333333}, {0.18,0.186}, {0.25,0.26875},{0.332,0.4},{0.4,0.55},{0.45,0.7},{0.47,0.8},{0.484,0.9},{0.488,1.}, {0.488,1.05},{0.474,1.2},{0.456,1.3},{0.4,1.495},{0.3,1.69},{0.2,1.83},{0.1,1.929},{0.04,1.977}}

We can study the correlation function at any point on the critical line. We take

$$\Omega/C_6 = 0.25, \quad \Delta/C_6 = 0.26875. \tag{4.5}$$

[SR: below needs to be modified because interchanging spins is not a symmetry of the Hamiltonian]

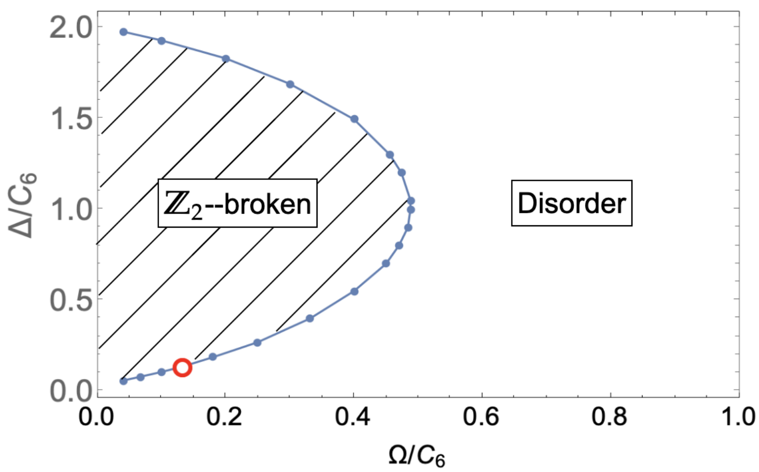

FIG. 4. The phase diagram Fendley-Sengupta-Sachdev model with $1/r^6$ interaction.

We can also modify the map between the lattice operators and CFT operators,

$$\begin{aligned} &\sigma_{i+1/2} \sim (n_i - n_{i+1}), \\ &\epsilon_{i+1/2} \sim n_i + n_{i+1} - (1-n_i)(1-n_{i+1}). \end{aligned} \tag{4.6}$$

Due to the symmetry-breaking pattern, we treat two neighbouring sites as a single unit cell. The above operators are defined when i is an even integer. The results are summarized in Fig. 5, for two point and three point functions.

[1] A. Polyakov, Sov. Phys. JETP **28**, 533 (1969).
[2] N. Chepiga and F. Mila, SciPost Phys. **6**, 033 (2019).
[3] F. Fang *et al.*, (2024), arXiv:2402.15376 [quant-ph].

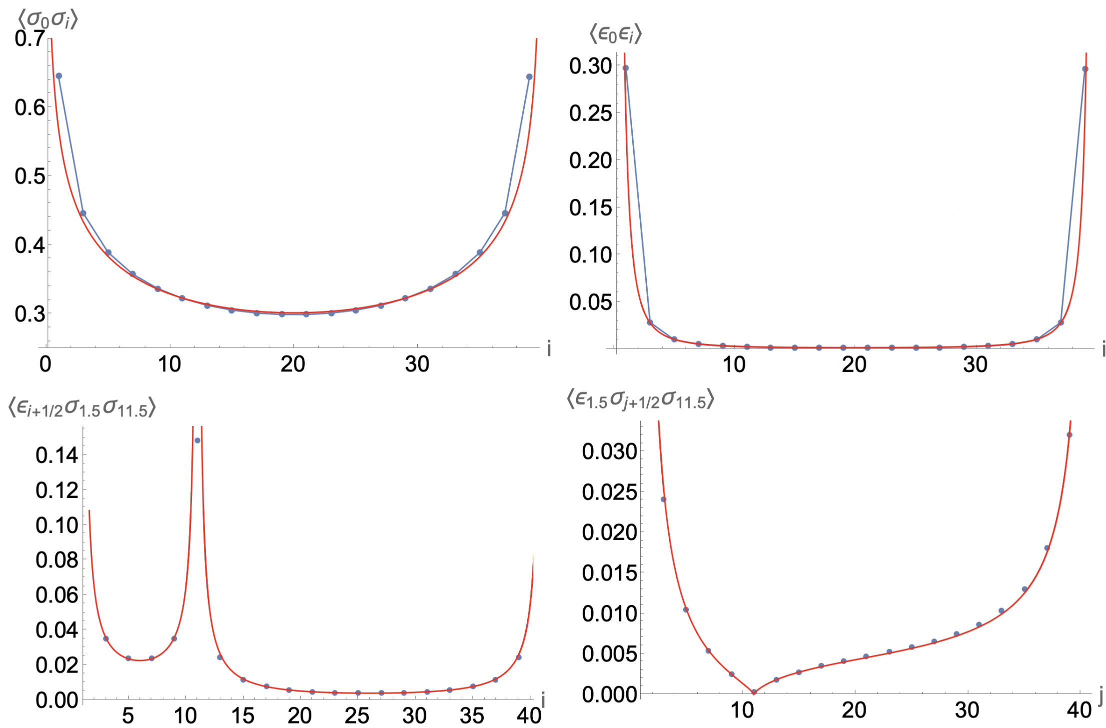

FIG. 5. The two-point and three-point functions of the Fendley-Sengupta-Sachdev model with $1/r^6$ interaction.