21-120: Differential and Integral Calculus
Lecture #3 Outline

Read: Section 2.3 of the textbook

Objectives and Concepts:

- The limit of a polynomial or rational function f at a number a, where a is in the domain of f, is simply $f(a)$.
- When you cannot use direct substitution, you can find $\lim_{x\to a} f(x)$ if there is a function $g(x)$ such that $f(x) = g(x)$ everywhere except $x = a$ and $\lim_{x\to a} g(x)$ exists.
- The Limit Laws can be used to break down limit calculations into smaller problems, provided all of the limits exist.

Suggested Textbook Exercises:

- 2.3: 83-125 odd.

The Limit Laws

The Direct Substitution Property: If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

It turns out the Direct Substitution Property works for other types of functions as well. Functions with this property are called *continuous* at a.

Example: Compute the following limits using the Direct Substitution Property, if possible.

(a) $\lim\limits_{x\to 3}\left(x^3 + 2x^2 - 1\right)$

(b) $\lim\limits_{x\to 3}\dfrac{x^2-1}{x-1}$

Theorem: If $f(x) = g(x)$ when $x \neq a$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x),$$

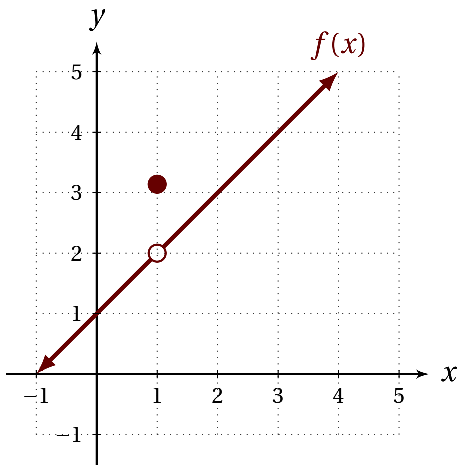
provided the limits exist.

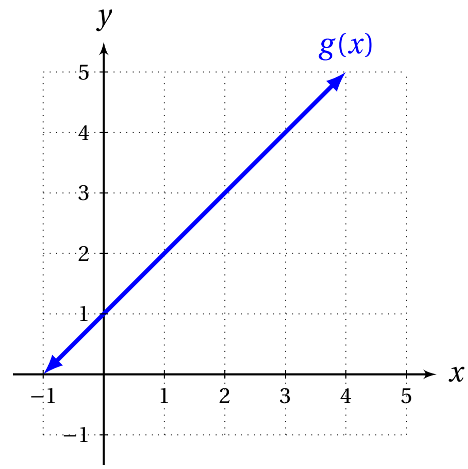
If f and g are the same everywhere except when $x = a$, then they have the same limit at a.

This is actually the way we will find most limits: we will find a function g that is equal to f everywhere except at a, and then find the limit of g. Sometimes the function g is obvious, however most of the time we will need to *algebraically manipulate* the expression that defines f into a form that can be evaluated at a.

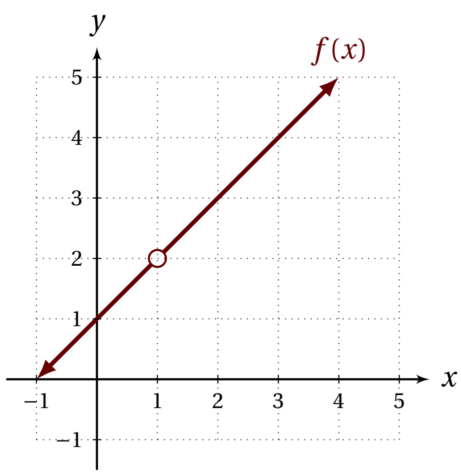
For example, consider the limit $\lim\limits_{x\to 1} f(x)$ where $f(x) = \begin{cases} x+1 & \text{if } x \neq 1, \\ \pi & \text{if } x = 1. \end{cases}$

By looking at the graph of f, the choice of g is clear: $g(x) = x+1$ because it is the same function with the exception of what happens at $x = 1$.





Now consider $\lim\limits_{x\to 1} f(x)$ where $f(x) = \dfrac{x^2-1}{x-1}$.



Note that, algebraically,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1.$$

But is is **very important** to note that the functions $f(x) = (x^2-1)/(x-1)$ and $g(x) = x+1$ are **not the same function**. The domain of f is

$$(-\infty, -1) \cup (-1, \infty)$$

while the domain of g is $(-\infty, \infty)$.

Example: Evaluate $\lim\limits_{x\to 4} \dfrac{x^2-4x}{x^2-3x-4}$, if it exists.

Example: Find each limit, if it exists.

(a) $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$

(b) $\displaystyle\lim_{x\to 10} \left(\frac{1}{x-10} - \frac{20}{x^2-100}\right)$

(c) $\displaystyle\lim_{x\to -6} \frac{2x+12}{|x+6|}$

The Limit Laws:
Suppose that c is a constant, n is positive, and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a}\left[cf(x)\right] = c\,\lim_{x\to a} f(x)$
3. $\lim_{x\to a}\left[f(x)g(x)\right] = \left(\lim_{x\to a} f(x)\right)\cdot\left(\lim_{x\to a} g(x)\right)$
4. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$
5. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
6. $\lim_{x\to a} c = c$
7. $\lim_{x\to a} x = a$
8. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$

Example: Given that $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = -2$, find

(a) $\lim_{x\to a}\left(f(x)+g(x)\right)$

(b) $\lim_{x\to a} f(x)(3g(x)+2)$

(c) $\lim_{x\to a}\dfrac{f(x)}{f(x)+g(x)}$

(d) $\lim_{x\to a}\sqrt[3]{2f(x)-g(x)}$

Example: Compute the limits below, if they exist.

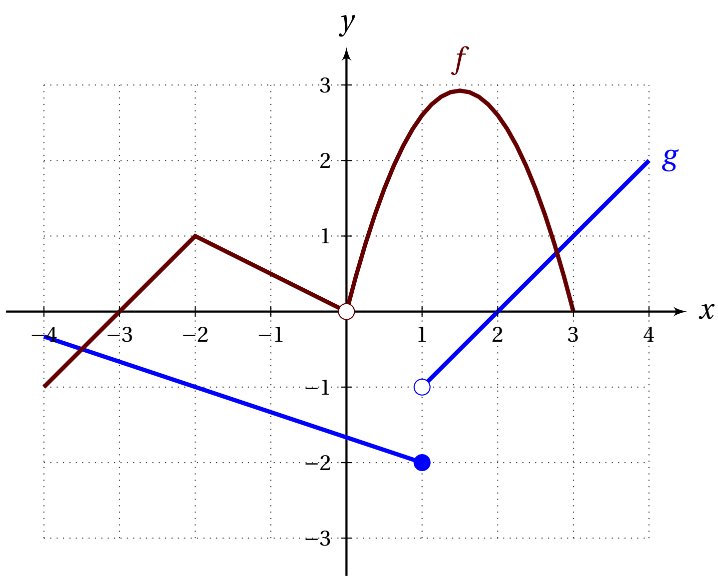
(a) $\lim\limits_{x\to 0}(5x-8)^{1/3}$

(c) $\lim\limits_{x\to 2}\left(x^2-x\right)^4$

(b) $\lim\limits_{h\to 0}\dfrac{3}{\sqrt{16+3h}+4}$

(d) $\lim\limits_{x\to 10000}\pi$

Example 5: Use the graphs of f and g below to evaluate the following limits (if possible).



$\lim\limits_{x\to -2}\left(f(x)+5g(x)\right)$

$\lim\limits_{x\to 1^+}\left[f(x)g(x)\right]$

$\lim\limits_{x\to 1^-}\left[f(x)g(x)\right]$

$\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}$