21-120: Differential and Integral Calculus
Recitation #8 Outline: 09/19/24

1. Find all vertical and horizontal asymptotes (if any) of the following functions and justify all your work:

   (a) $f(x) = \dfrac{x^2+7x+6}{x+1}$

   (b) $f(x) = \dfrac{x+1}{x^2+7x+6}$

   (c) $f(x) = x - \dfrac{1}{x^2}$

   (d) $f(x) = \dfrac{x\sin x}{x^2-1}$

   Solution:

   (a) Note that $f(x) = x+6$ for $x \neq -1$. The graph of this function is just a line with a hole in it. This function doesn't have any horizontal or vertical asymptotes.

   (b) Note that $f(x) = \dfrac{1}{x+6}$ for $x \neq -1$. Since $x = -6$ is the only value of x at which the limit of f is infinite, $x = -6$ is the only vertical asymptote. For horizontal asymptotes, note that $\lim\limits_{x\to\infty} f(x) = 0$ and $\lim\limits_{x\to-\infty} f(x) = 0$, so $y = 0$ is the only horizontal asymptote.

   (c) Note that $f(x) = \dfrac{x^3-1}{x^2}$. The point $x = 0$ is the only point at which the limit of f is infinite (in fact, $\lim\limits_{x\to 0} f(x) = -\infty$), so $x = 0$ is the vertical asymptote. One can also show (by dividing both numerator and denominator by x^2) that $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$, and since these two limits are infinite, there are no horizontal asymptotes.

   (d) The only points at which the limit of f is infinite are $x = 1$ and $x = -1$, so these are the vertical asymptotes. (In fact, only one-sided limits at $x = 1$ and $x = -1$ exist, but this still tells us that $x = 1$ and $x = -1$ are vertical asymptotes.) To find the horizontal asymptotes, we need to determine the limits as x approaches positive and negative infinity:

   $$\lim_{x\to\infty} \frac{x\sin x}{x^2-1} = \lim_{x\to\infty} \frac{x\sin x \cdot \frac{1}{x^2}}{(x^2-1)\cdot\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{\sin x}{x}}{1-\frac{1}{x^2}} = \frac{\lim_{x\to\infty} \frac{\sin x}{x}}{\lim_{x\to\infty}\left(1-\frac{1}{x^2}\right)} = \frac{0}{1} = 0$$

   Here, we used the fact that $\lim_{x\to\infty} \frac{\sin x}{x} = 0$, which follows from the Squeeze Theorem (note that $-\frac{1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$ for $x > 0$). Thus, $y = 0$ is a horizontal asymptote. You can verify that this is the only horizontal asymptote by also showing that $\lim\limits_{x\to-\infty} \dfrac{x\sin x}{x^2-1} = 0$.

2. Is the following function differentiable at $x = 1$? Prove your answer using the limit definition of the derivative.

   $$f(x) = \begin{cases} -x^2+2 & \text{if } x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

   Solution:

   The answer is no because

   $$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

does not exist. It does not exist because the right and left limits are not equal. Indeed,

$$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \frac{(1+h)-(-1^2+2)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

but

$$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} \frac{-(1+h)^2+2-(-1^2+2)}{h} = \lim_{h\to 0^-} \frac{-1-2h-h^2+2-1}{h} = -2.$$

3. Using the Squeeze Theorem, prove that

$$\lim_{x\to\infty} \frac{2-\cos x}{x+3} = 0.$$

Solution:

To prove that

$$\lim_{x\to\infty} \frac{2-\cos x}{x+3} = 0$$

using the Squeeze Theorem, we start by analyzing the function $\frac{2-\cos x}{x+3}$.

First, note that $\cos x$ oscillates between -1 and 1. Therefore:

$$1 \le 2-\cos x \le 3$$

We can use these bounds to establish inequalities for $\frac{2-\cos x}{x+3}$ for sufficiently large x:

$$\frac{1}{x+3} \le \frac{2-\cos x}{x+3} \le \frac{3}{x+3}$$

(Note that we can divide everything by $x+3$ without flipping the inequality symbols becase we are interested in very large xs since we are looking at the limit as $x\to\infty$.) Now, consider the limits of the bounding functions as x approaches infinity:

$$\lim_{x\to\infty} \frac{1}{x+3} = 0$$
$$\lim_{x\to\infty} \frac{3}{x+3} = 0$$

Since both bounding functions approach 0 as x approaches infinity, by the Squeeze Theorem, it follows that:

$$\lim_{x\to\infty} \frac{2-\cos x}{x+3} = 0$$

4. Using the Intermediate Value Theorem, prove that the equation

$$x^3 \cos x = 4$$

has at least one solution in $(-\infty, +\infty)$.[1]

[1] Note that you can solve this problem without introducing the function $x^3\cos x - 4$.

Solution:

Let $f(x) = x^3 \cos x$. Note that $f(0) = 0$. Now we need to find some number x_0 such that $f(x_0) > 4$. It will then follow from the Intermediate Value Theorem that the equation $f(x) = 4$ has a solution in $(0, x_0)$. It's easy to guess what x_0 should be (there are many choices for x_0). For example, we can take $x_0 = 2\pi$, so that $f(2\pi) = (2\pi)^3 > 4$. Given the explanation just above, we are done.

5. Let

$$f(x) = \begin{cases} 3x & \text{if } x > 1 \\ x^3 & \text{if } x < 1 \end{cases}$$

Is it possible to find a value a such that $f(1) = a$ which makes the modified f continuous at all points?

Solution:

To determine if it is possible to find a value a such that $f(1) = a$ makes the modified f continuous at all points, including at $x = 1$, we need to check the continuity of f at $x = 1$. Specifically, we need to verify whether we can make f continuous by appropriately defining $f(1)$ as a.

Let's analyze the function f and the behavior around $x = 1$.

Calculate the left limit as $x \to 1^-$:

As x approaches 1 from the left, the function $f(x) = x^3$. Therefore, the left limit is:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^3 = 1^3 = 1$$

Calculate the right limit as $x \to 1^+$:

As x approaches 1 from the right, the function $f(x) = 3x$. Therefore, the right limit is:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 3x = 3 \cdot 1 = 3$$

Check the continuity condition at $x = 1$:

For f to be continuous at $x = 1$, the following condition must be met:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

From the calculations:

$$\lim_{x \to 1^-} f(x) = 1$$
$$\lim_{x \to 1^+} f(x) = 3$$

For the function to be continuous at $x = 1$, these two limits must be equal, and this should also be equal to $f(1)$.

However, $1 \neq 3$, so the left limit and right limit are not equal. This implies that there is no value a such that $f(1) = a$ can make the function continuous at $x = 1$.

6. At which points (if any) are the functions below discontinuous? Classify each discontinuity.

(a) $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 3 - x & \text{if } x > 1 \end{cases}$

(b) $f(x) = \dfrac{x+1}{x^2 + 7x + 6}$

(c) $f(x) = \begin{cases} \dfrac{x^2 + 7x + 6}{x+1} & \text{if } x \neq -1 \\ 5 & \text{otherwise} \end{cases}$

Solution:

(a) For the function

$$f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

we need to check the continuity at $x = 1$, where the definition of the function changes.

Left limit at $x = 1$:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1^2 = 1.$$

Right limit at $x = 1$:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3 - x) = 3 - 1 = 2.$$

Function value at $x = 1$:

$$f(1) = 1^2 = 1.$$

Since the left limit (1) and the right limit (2) are not equal, $f(x)$ is discontinuous at $x = 1$. This is a jump discontinuity.

(b) Note that

$$f(x) = \frac{x+1}{(x+1)(x+6)} = \frac{1}{x+6}, \quad \text{for } x \neq -1.$$

We have two discontinuities: at $x = -1$ (removable discontinuity) and $x = -6$ (infinite discontinuity).

(c) Note that for $x \neq -1$,

$$\frac{x^2 + 7x + 6}{x+1} = \frac{(x+1)(x+6)}{x+1} = x + 6.$$

Thus, the function simplifies to:

$$f(x) = \begin{cases} x + 6 & \text{if } x \neq -1 \\ 5 & \text{otherwise} \end{cases}$$

Left limit at $x = -1$:

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (x + 6) = -1 + 6 = 5.$$

Right limit at $x = -1$:

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (x + 6) = -1 + 6 = 5.$$

Function value at $x = -1$:

$$f(-1) = 5.$$

Since the left limit, right limit, and function value at $x = -1$ are all equal, $f(x)$ is continuous at $x = -1$. There is no discontinuity here.

7. For which values of a is the function

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ a + x & \text{if } x \geq 0 \end{cases}$$

continuous on $(-\infty, +\infty)$?

Solution:

To determine for which values of a the function

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ a + x & \text{if } x \geq 0 \end{cases}$$

is continuous on $(-\infty, +\infty)$, we need to ensure that $f(x)$ is continuous at $x = 0$, where the definition of the function changes.

A function is continuous at a point if the left limit, the right limit, and the function value at that point are all equal.

Left limit at $x = 0$:

When $x < 0$, $f(x) = e^x$. So, the left limit is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1.$$

Right limit at $x = 0$:

When $x \geq 0$, $f(x) = a + x$. So, the right limit is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (a + x) = a + 0 = a.$$

Function value at $x = 0$:

Since $x = 0$ falls into the case $x \geq 0$, we have

$$f(0) = a + 0 = a.$$

For $f(x)$ to be continuous at $x = 0$, the left limit, right limit, and the function value at $x = 0$ must be equal. Therefore, we need

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0).$$

Substituting the limits and function value, we get

$$1 = a = a.$$

Thus, a must be equal to 1 for $f(x)$ to be continuous on $(-\infty, +\infty)$.

So, the function $f(x)$ is continuous on $(-\infty, +\infty)$ if and only if $a = 1$.

8. Give an example of a function f such that

$$\lim_{x \to \infty} f(x) = -\infty$$

and prove that your answer works using the ϵ-δ definition. (The proof should be very easy.)

Solution: There are many examples to this, here is one (not the simplest one). Let $f(x) = -x^2$. To show that $\lim_{x\to\infty} f(x) = -\infty$, we need to show that for any $M > 0$ there exists $\Delta > 0$ such that for all x satisfying $x > \Delta$, it holds that $f(x) < -M$. Fix $M > 0$ and let $\delta = \sqrt{M}$. We need to show that for any x satisfying $x > \Delta$, we have $-x^2 < -M$. How do we show this? Note that for all $x > \Delta$, we have $x^2 > \Delta^2$, and hence $-x^2 < -\Delta^2$, and the last inequality can be equivalently written as $-x^2 < -M$ by our choice of Δ.

9. Let

$$f(x) = \frac{1}{\sqrt{1+x}}$$

(a) Using the *definition* of the derivative only, find a formula for $f'(x)$.

(b) Find an equation of the tangent line to f at $x = 1$.

Solution:

(a) **Finding the derivative using the definition:**

To find the derivative of $f(x) = \frac{1}{\sqrt{1+x}}$ using the definition of the derivative, we use:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Here, $f(x) = \frac{1}{\sqrt{1+x}}$, so:

$$f(x+h) = \frac{1}{\sqrt{1+(x+h)}} = \frac{1}{\sqrt{1+x+h}}.$$

The difference quotient is:

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{\sqrt{1+x+h}} - \frac{1}{\sqrt{1+x}}}{h}.$$

To simplify this, we find a common denominator:

$$\frac{f(x+h) - f(x)}{h} = \frac{\sqrt{1+x} - \sqrt{1+x+h}}{h\sqrt{1+x}\sqrt{1+x+h}}.$$

To handle the numerator, multiply and divide by the conjugate:

$$\frac{\sqrt{1+x} - \sqrt{1+x+h}}{h\sqrt{1+x}\sqrt{1+x+h}} \cdot \frac{\sqrt{1+x} + \sqrt{1+x+h}}{\sqrt{1+x} + \sqrt{1+x+h}} = \frac{(\sqrt{1+x} - \sqrt{1+x+h})(\sqrt{1+x} + \sqrt{1+x+h})}{h\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x} + \sqrt{1+x+h})}.$$

Simplify the numerator:

$$(\sqrt{1+x})^2 - (\sqrt{1+x+h})^2 = (1+x) - (1+x+h) = -h.$$

Thus, if $h \neq 0$, we get:

$$\frac{f(x+h) - f(x)}{h} = \frac{-h}{h\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x} + \sqrt{1+x+h})} = \frac{-1}{\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x} + \sqrt{1+x+h})}.$$

Taking the limit as $h \to 0$ gives the derivative:

$$f'(x) = \lim_{h\to 0} \frac{-1}{\sqrt{1+x}\sqrt{1+x+h}(\sqrt{1+x}+\sqrt{1+x+h})} = \frac{-1}{\sqrt{1+x}\cdot\sqrt{1+x}\cdot 2\sqrt{1+x}} = -\frac{1}{2(1+x)^{3/2}}.$$

Thus, the formula for the derivative is:

$$f'(x) = -\frac{1}{2(1+x)^{3/2}}.$$

(b) **Finding the equation of the tangent line at $x = 1$:**
First, evaluate $f(1)$:

$$f(1) = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}.$$

Find $f'(1)$:

$$f'(1) = -\frac{1}{2(1+1)^{3/2}} = -\frac{1}{2\cdot 2^{3/2}} = -\frac{1}{4\sqrt{2}}.$$

The equation of the tangent line at $x = 1$ is given by:

$$y - f(1) = f'(1)(x-1).$$

Substituting $f(1)$ and $f'(1)$:

$$y - \frac{1}{\sqrt{2}} = -\frac{1}{4\sqrt{2}}(x-1).$$

10. Using differentiation rules, find the derivative of the function

$$f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right).$$

Solution:

To find the derivative of the function

$$f(x) = 3x\left(18x^4 + \frac{13}{x+1}\right),$$

first expand the function:

$$f(x) = 3x\cdot 18x^4 + 3x\cdot\frac{13}{x+1}.$$

Simplify the expression:

$$f(x) = 54x^5 + \frac{39x}{x+1}.$$

Now, differentiate each term separately.

1. For the first term $54x^5$:

$$\frac{d}{dx}(54x^5) = 54\cdot 5x^4 = 270x^4.$$

2. For the second term $\frac{39x}{x+1}$, use the quotient rule where $u = 39x$ and $v = x+1$:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}.$$

Here, $u = 39x$, $u' = 39$, $v = x+1$, and $v' = 1$. Thus:

$$\frac{d}{dx}\left(\frac{39x}{x+1}\right) = \frac{39\cdot(x+1) - 39x\cdot 1}{(x+1)^2} = \frac{39x+39-39x}{(x+1)^2} = \frac{39}{(x+1)^2}.$$

Combining these results, the derivative of $f(x)$ is:

$$f'(x) = 270x^4 + \frac{39}{(x+1)^2}.$$

11. Evaluate the following limits (if they exist):

(a) $\lim\limits_{x\to 4} \frac{1}{\sqrt{x}-2}$

(b) $\lim\limits_{x\to -\infty} \frac{4x}{\sqrt{x^2-1}}$

(c) $\lim\limits_{x\to 2^+} \left(3\sqrt{x-2} + 5\cdot\frac{x^2+x-6}{x-2}\right)$

Solution:

(a) As x approaches 4 from the right, $\sqrt{x}-2$ approaches 0 from the right, so

$$\lim_{x\to 4^+} \frac{1}{\sqrt{x}-2} = +\infty.$$

However, as x approaches 4 from the left, $\sqrt{x}-2$ approaches 0 from the left, so

$$\lim_{x\to 4^-} \frac{1}{\sqrt{x}-2} = -\infty.$$

Thus,

$$\lim_{x\to 4} \frac{1}{\sqrt{x}-2}$$

does not exist.

(b) We have:

$$\lim_{x\to -\infty} \frac{4x}{\sqrt{x^2-1}} = \lim_{x\to -\infty} \frac{4x}{\sqrt{x^2}\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to -\infty} \frac{4x}{|x|\sqrt{1-\frac{1}{x^2}}}$$

Since $|x| = -x$ when x is negative, the rightmost limit above simplifies to

$$\lim_{x\to -\infty} \frac{4x}{-x\cdot\sqrt{1-\frac{1}{x^2}}}.$$

We can now divide both the numerator and the denominator by x, so the limit simplifies to

$$\lim_{x\to -\infty} \frac{4}{-\sqrt{1-\frac{1}{x^2}}}.$$

Since $\frac{1}{x^2}$ approaches 0 as x approaches $-\infty$, the limit above is -4.

(c) First, simplify $\dfrac{x^2+x-6}{x-2}$:

$$x^2+x-6=(x-2)(x+3)$$

So, for $x \neq 2$,

$$\frac{x^2+x-6}{x-2}=x+3$$

Therefore, the limit becomes:

$$\lim_{x\to 2^+}\left(3\sqrt{x-2}+5(x+3)\right)$$

As $x \to 2^+$, $\sqrt{x-2} \to 0$, so:

$$\lim_{x\to 2^+}\left(3\sqrt{x-2}+5(x+3)\right)=0+5(2+3)=25$$