21-120: Differential and Integral Calculus
Lecture #9 Outline

Read: Section 3.3 of the textbook

Objectives and Concepts:

- The derivative of a constant function is 0.
- The derivative of x^n is nx^{n-1} for any real number $n \neq 0$.
- The derivative of a constant multiple of a function $cf(x)$ is $cf'(x)$.
- The derivative of a sum or difference of functions is the sum or difference of the derivatives.
- The derivative of the product $f(x)g(x)$ is $f(x)g'(x) + g(x)f'(x)$.
- The derivative of the quotient $f(x)/g(x)$ is $\dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.

Suggested Textbook Exercises:

- 3.3: 107-117 odd, 123-133 odd, 137-143 odd.

Differentiation Rules

We begin our discussion of differentiation rules by examining the four basic rules:

Theorem: Let c and n be (real) constants, and let f and g be differentiable functions. Then the following rules hold:

- **The Constant Rule:** $\dfrac{d}{dx}(k) = 0.$
- **The Constant Multiple Rule:** $\dfrac{d}{dx}\big(cf(x)\big) = c\dfrac{d}{dx}f(x).$
- **The Power Rule:** $\dfrac{d}{dx}\big(x^n\big) = nx^{n-1}.$
- **The Sum and Difference Rule:** $\dfrac{d}{dx}\big(f(x) \pm g(x)\big) = \dfrac{d}{dx}f(x) \pm \dfrac{d}{dx}g(x)$

All of these rules can be proved fairly easily by using the limit definition of the derivative (with the exception of the Power Rule, which requires a result known as the Binomial Theorem). As an example, let's prove the Sum Rule.

Proof: Let f and g be differentiable functions and let $S(x) = f(x) + g(x)$. Then

You can combine these rules when taking derivatives.

Example 1: Differentiate each function.

(a) $y = \sqrt{x} - 3x$

(b) $g(x) = \dfrac{3}{x^{3/4}} - 6x^{-2} + 10$

Example 2: Find all points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

The Product and Quotient Rules

The Product Rule: If f and g are both differentiable functions, then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right].$$

$$(fg)' = fg' + gf'$$

The product rule can also be proved from the limit definition of the derivative. The trick is to "add the right form of 0 in the right place."

Proof: Let $F(x) = f(x)g(x)$ where f and g are differentiable functions. Then

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} =$$

Example 3: Differentiate $g(z) = \left(2z^2 - 5z + 1\right)\left(z^{-4/3} + \sqrt{z}\right)$

The Quotient Rule: If f and g are both differentiable functions, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Example 4: Find the derivative of each of the given functions.

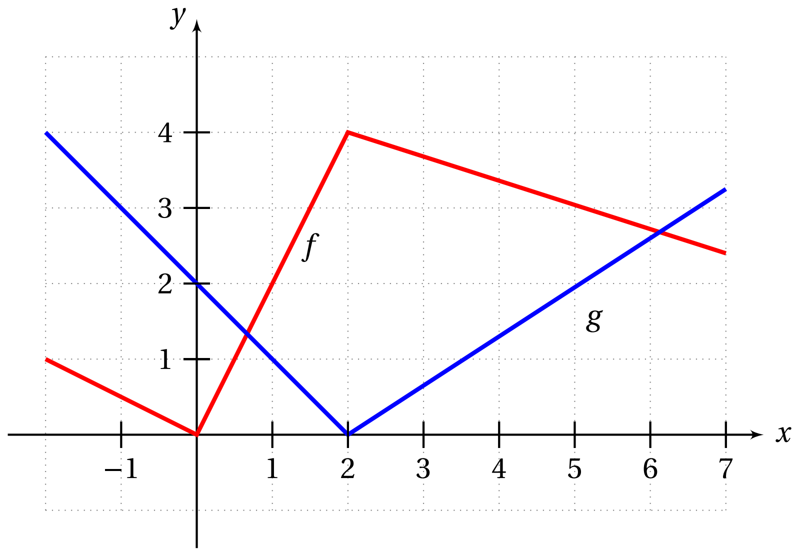
(a) $y = \dfrac{2x}{7 - 4x^3 + x}$

(b) $h(x) = \dfrac{2x^2 - 3x}{5\sqrt[3]{x} + 1}$

Note: Don't automatically use the Quotient Rule *every* time you see a quotient. Sometimes there's an easier way to do it!

Example 5: Find the derivative of $F(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$

Example: The graphs of two functions $f(x)$ and $g(x)$ are shown below.



Let $p(x) = f(x)g(x)$, $q(x) = \dfrac{f(x)}{g(x)}$, and $r(x) = f(x)(g(x) - 1)$.

(a) Where is $f'(x)$ undefined?

(b) Where is $g'(x)$ undefined?

(c) Compute $p'(1)$.

(d) Compute $q'(5)$.

(e) Compute $r'(-1)$.

Example 6: Suppose f and g are differentiable functions about which we know very little. In fact, all we know about these functions is in the following table of data:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
-2	3	1	-5	8
-1	-9	7	4	1
0	5	9	9	-3
1	3	-3	2	6
2	-5	3	8	?

This isn't a lot of information. For example, we can't compute $f'(3)$ with any accuracy. But we are still able to figure some things out using the rules of differentiation.

(a) Let $h(x) = \left(\sqrt[3]{x}\right)^4$. Find $h'(0)$.

(b) Let $p(x) = -4f(x)g(x)$. Find $p'(1)$.

(c) Let $k(x) = \dfrac{3x^2+1}{g(x)}$. Find $k'(-2)$.

(d) Let $r(x) = x^3 g(x)$. If $r'(2) = -48$, find $g'(2)$.

(e) Let $m(x) = \dfrac{1}{f(x)}$. Find $m'(1)$.