21-120: Differential and Integral Calculus
Lecture #14 Outline

Read: Section 3.9 of the textbook

Objectives and Concepts:

- The derivative of the general exponential function $y = a^{g(x)}$ is $y' = a^{g(x)}(\ln a)(g'(x))$.
- The derivative of $y = \log_a(g(x))$ is found to be $y' = g'(x)/(g(x)\ln a)$ via implicit differentiation.
- When taking the derivative of $y = \ln(g(x))$ where $g(x)$ has many factors, it is often easier to expand $\ln(g(x))$ using the laws of logarithms before finding y'.
- Logarithmic differentiation is a process by which the derivative of $y = f(x)$ is found by first taking ln of both sides of the equation, simplifying $\ln(f(x))$ and then using implicit differentiation.
- Logarithmic differentiation must be used to find the derivative of $y' = (f(x))^{g(x)}$.

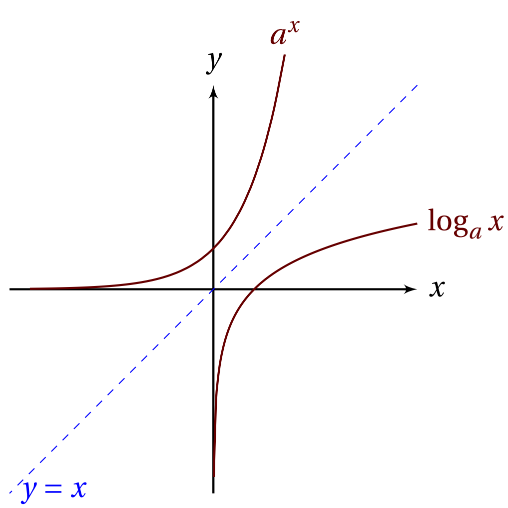
Suggested Textbook Exercises:

- 3.9: 331-353 odd.

The Derivatives of Logarithmic and Exponential Functions

Definition: Let $a > 0$, $a \neq 1$. Then the exponential function $f(x) = a^x$ has the inverse function $f^{-1}(x) = \log_a x$, the **logarithmic function with base** a.

$$\log_a x = y \quad \Longleftrightarrow \quad a^y = x$$



Function	Domain	Range
a^x	$(-\infty, \infty)$	$(0, \infty)$
$\log_a x$	$(0, \infty)$	$(-\infty, \infty)$

For **any** real number x, we have

$$\log_a(a^x) = x.$$

For a real number $x > 0$, we have

$$a^{\log_a x} = x.$$

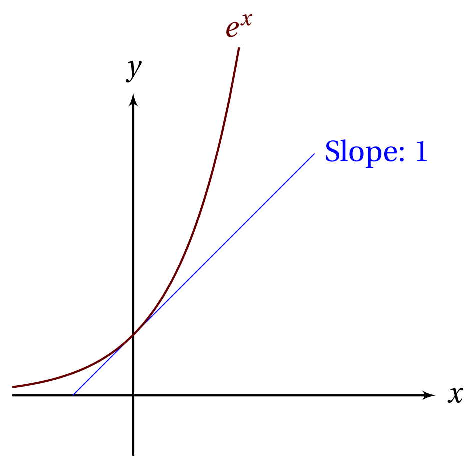
Laws of Logarithms: If x and y are positive numbers, then

1. $\log_a(xy) = \log_a x + \log_a y$ $\qquad (a^x a^y = a^{x+y})$
2. $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$ $\qquad \left(\frac{a^x}{a^y} = a^{x-y}\right)$
3. $\log_a(x^r) = r\log_a x$ (where r is any real number) $\qquad \left((a^x)^r = a^{xr}\right)$

The **natural exponential function** $f(x) = e^x$ is the exponential function whose base is the irrational number e.

Definition: The number e is the constant $e \approx 2.71828$ such that

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1$$



Geometrically, this means that of all possible exponential functions $y = a^x$, the function $f(x) = e^x$ is the one whose tangent line at $(0, 1)$ has slope $f'(0)$ that is exactly 1.

The Natural Exponential Function Rule: $\dfrac{d}{dx}\left(e^x\right) = e^x$.

Proof: Let $f(x) = e^x$. According to the limit definition of the derivative,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} =$$

Definition: The inverse function of the natural exponential function e^x is the **natural logarithm** function $\ln x$.

$$\ln x = y \quad \Longleftrightarrow \quad e^y = x$$

$$\ln(e^x) = x, \quad x \in \mathbb{R} \qquad\qquad e^{\ln x} = x, \quad x > 0$$

Change of Base Formula: $\log_a x = \dfrac{\ln x}{\ln a} \quad (a > 0, a \neq 1)$

The derivative of $f(x) = a^x$ can be found in a similar way as the Natural Exponential Rule, but by using the fact that $\lim_{h\to 0} \dfrac{a^h - 1}{h} = \ln a$.

Theorem: If $f(x) = a^x$ is an exponential function with base $a > 0$, then

$$f'(x) = a^x \ln(a).$$

Finally, when you apply the Chain Rule to an exponential function with base a, we derive the following theorem:

The General Exponential Rule: If $g(x) = a^{f(x)}$ where $f(x)$ is a differentiable function, then

$$g'(x) = f'(x)a^{f(x)}(\ln a).$$

Example 1: Find the derivative of $f(x) = e^{x \sec x}$.

Example 2: Find the derivative of $f(x) = 2^{3^{x^2}}$.

To find the derivative of $y = \ln x$, we write $e^y = x$ and use implicit differentiation:

$$e^y = x \quad \Longrightarrow \quad e^y y' = 1 \quad \Longrightarrow \quad y' = \frac{1}{e^y} = \frac{1}{x}.$$

The derivative of $y = \log_a x$ is found similarly. We first write $a^y = x$ and note that since $a^y = e^{y \ln a}$,

$$e^{y \ln a} = x \quad \Longrightarrow \quad a^y (\ln a) y' = 1 \quad \Longrightarrow \quad y' = \frac{1}{(\ln a)e^y} = \frac{1}{x \ln a}.$$

Derivative of Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}\big(\ln(f(x))\big) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\big(\log_a x\big) = \frac{1}{x\ln a} \qquad \frac{d}{dx}\big(\log_a(f(x))\big) = \frac{f'(x)}{(\ln a)f(x)}$$

Example 3: Find the derivative of each of the following.

(a) $y = 3x^4\ln(x^2) - \dfrac{x^3}{9}$

(b) $y = \log_{10}(x^2 + 2x)$

Note that we can use the laws of logarithms to simplify complicated functions prior to taking their derivatives:

Example 4: Find the derivative.

$$y = \ln\left(\frac{(x^2+1)^4}{\sqrt{4-3x^2}}\right)$$

The derivative of positive functions can be found by taking the natural logarithm of both sides before differentiating. This is a process called **logarithmic differentiation:**

Steps in Logarithmic Differentiation:

1. Take the natural logarithm of both sides of an equation $y = f(x)$. Use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

Example 5: Use logarithmic differentiation to find the derivative.

(a) $y = \sqrt{(x^2+1)(x-1)}$

(b) $y = \dfrac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}$

While logarithmic differentiation can be useful when dealing with complicated-looking functions, it is a necessity when dealing with functions of the form $y = (f(x))^{g(x)}$.

Example 6: Use logarithmic differentiation to find y' where $y = (\sin x)^{5x}$.

Example 7: Use logarithmic differentiation to find y' where $y = \left(x^2+1\right)^{\tan x}$.