Recitation 9 Solutions

1. (a) Since $f(x)$ is differentiable at $g(a)$

$$\lim_{y \to g(a)} \frac{f(y) - f(g(a))}{y - g(a)} = f'(g(a)).$$

That is to say as y becomes arbitrarily close (but not equal) to $g(a)$, the difference ratio becomes arbitrarily close to $f'(g(a))$.

On the other hand, by assumption, $g(x) \neq g(a)$ when $x \neq a$ and

$$\lim_{x \to a} g(x) = g(a)$$

since $g(x)$ is continuous at a. That is, as x becomes arbitrarily close (but not equal to) a, $g(x)$ becomes arbitrarily close (but not equal to) $g(a)$.

Together these facts imply that as x becomes arbitrarily close to a, $g(x)$ becomes arbitrarily close to $g(a)$, thus

$$\frac{f(g(x)) - f(g(a))}{g(x) - g(a)}$$

becomes arbitrarily close to $f'(g(a))$ i.e.

$$\lim_{x \to a} \frac{f(g(x)) - f(g(a))}{g(x) - g(a)} = f'(g(a))$$

(b) Since $g(x)$ is differentiable at a, it is continuous at a. We thus find that

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} \frac{f(g(x)) - f(g(a))}{x - a}$$

$$= \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{x - a} \right) \times \left(\frac{g(x) - g(a)}{g(x) - g(a)} \right)$$

$$= \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \right) \times \left(\frac{g(x) - g(a)}{x - a} \right)$$

$$(*) \quad = \lim_{x \to a} \left(\frac{f(g(x)) - f(g(a))}{g(x) - g(a)} \right) \times \lim_{x \to a} \left(\frac{g(x) - g(a)}{x - a} \right)$$

$$= f'(g(a))\, g'(a)$$

where in line $(*)$ we've used (a) and the product rule for limits.

2. (a) $f'(x) = 4(5x^6 + 2x^3)^{4-1} \cdot (5 \cdot 6x^{6-1} + 2 \cdot 3x^{3-1})$

$$= 4(5x^6 + 2x^3)^3 \cdot (30x^5 + 6x^2)$$

(b) $y' = \frac{1}{2} \frac{1}{\left(\sqrt{\frac{x}{x+1}}\right)} \cdot \frac{d}{dx} \left(\frac{x}{x+1} \right)$

$$= \frac{1}{2} \sqrt{\frac{x+1}{x}} \cdot \left(\frac{\cancel{x} + 1 - \cancel{x}}{(x+1)^2} \right)$$

$$= \frac{1}{2} \sqrt{\frac{x+1}{x}} \left(\frac{1}{(x+1)^2} \right)$$

(c) $k'(h) = -\sin\left(\sqrt{h^2 - 2}\right) \cdot \frac{d}{dx}\left(\sqrt{h^2 - 2}\right)$

$$= -\sin\left(\sqrt{h^2 - 2}\right) \cdot \frac{1}{2} \frac{1}{\sqrt{h^2 - 2}} \cdot \frac{d}{dx}(h^2 - 2)$$

$$= -\sin\left(\sqrt{h^2 - 2}\right) \cdot \frac{1}{\cancel{2}} \frac{1}{\sqrt{h^2 - 2}} \cdot \cancel{2}h$$

$$= -\frac{h}{\sqrt{h^2 - 2}} \sin\left(\sqrt{h^2 - 2}\right)$$

3. (a) $y' = \cos(3x) \cdot 3$

$y'' = -\sin(3x) \cdot 3 \cdot 3$

$= -9\sin(3x)$

(b) $y' = \frac{1}{2} \frac{1}{\sqrt{1-\sec(x)}} \frac{d}{dx}(1-\sec(x))$

$= -\frac{1}{2} \frac{\sec(x)\tan(x)}{\sqrt{1-\sec(x)}}$

Simplify $= -\frac{1}{2} \frac{\sin(x)}{\cos^2(x)\sqrt{1-\frac{1}{\cos(x)}}}$

$= -\frac{1}{2} \frac{\sin(x)}{\cos^{3/2}(x)\sqrt{\cos(x)-1}}$

$y'' = -\frac{1}{2}\left(\frac{\cos(x)\left(\cos^{3/2}(x)\sqrt{\cos(x)-1}\right) - \sin(x)\left[\frac{d}{dx}\left(\cos^{3/2}(x)\sqrt{\cos(x)-1}\right)\right]}{\left(\cos^{3/2}(x)\sqrt{\cos(x)-1}\right)^2}\right)$

$= -\frac{1}{2}\left(\frac{\cos(x)}{\cos^{3/2}(x)\sqrt{\cos(x)-1}}\right.$

$\left. - \frac{\sin(x)\left(\frac{3}{2}\cos^{\frac{1}{2}}(x)(-\sin(x))\sqrt{\cos(x)-1} + \cos^{3/2}(x)\cdot\frac{1}{2}\left(\frac{-\sin(x)}{\sqrt{\cos(x)-1}}\right)\right)}{\cos^3(x)(\cos(x)-1)}\right)$

4. (a) $y' = \frac{1}{2} \frac{1}{\sqrt{1+x^3}} \cdot \frac{d}{dx}(1+x^3)$

$= \frac{3}{2} \frac{x^2}{\sqrt{1+x^3}}$

at $x = 2$ $\quad y' = \frac{3}{2} \frac{2^2}{\sqrt{1+2^3}} = \frac{3}{2} \frac{4}{\sqrt{9}} = 2$

Thus the tangent line at $(2,3)$ is

$$y = 3 + 2(x-2) = 2x - 1$$

(b) $y' = \cos(\sin(x))\cos(x)$

at $x = \pi$ $\quad y' = \cos(\sin(\pi))\cos(\pi)$

$$= \cos(0)\cdot(-1)$$

$$= -1$$

Thus the tangent line at $(\pi, 0)$ is

$$y = 0 - 1(x - \pi) = \pi - x$$

5. (a) $F'(x) = g'(f(x))\, f'(x)$

$$F'(2) = g'(f(2))\, f'(2)$$

$$= g'(1)\cdot 1$$

$$= 2\cdot 1 = 2$$

(b) $G'(x) = g'(h(f(x)))\, h'(f(x))\, f'(x)$

$$G'(1) = g'(h(f(1)))\, h'(f(1))\, f'(1)$$

$$= g'(h(3))\, h'(3)\cdot 3$$

$$= g'(2)\cdot 4\cdot 3$$

$$= 3\cdot 4\cdot 3 = 36$$

(c) $H'(x) = g'(g(g(g(x))))\, g'(g(g(x)))\, g'(g(x))\, g'(x)$

$$H'(2) = g'(g(g(g(2))))\, g'(g(g(2)))\, g'(g(2))\, g'(2)$$

$$= g'(g(g(1)))\, g'(g(1))\, g'(1) \cdot 3$$

$$= g'(g(2))\, g'(2) \cdot 2 \cdot 3$$

$$= g'(1) \cdot 3 \cdot 2 \cdot 3$$

$$= 2 \cdot 3 \cdot 2 \cdot 3$$

$$= 4 \cdot 9$$

$$= 36$$