**21-120: Differential and Integral Calculus**
**Lecture #20 Outline**

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**Read:** Section 4.8 of the textbook

**Objectives and Concepts:**

- An indeterminate form is an expression (usually in the context of limits) that cannot be evaluated. Examples of indeterminate forms include $0/0$, $\pm\infty/\pm\infty$, $\infty-\infty$, $0\cdot\infty$, $0^0$, $\infty^0$, and $1^\infty$. An indeterminate form can be any value, including $\pm\infty$.
- When evaluating the limit of $f/g$ and arriving at $0/0$ or $\pm\infty/\pm\infty$, one can use L'Hospital's Rule, which says that if the limit of $f/g$ is one of these two indeterminate forms, then $\lim_{x\to a}(f/g) = \lim_{x\to a}(f'/g')$.
- When arriving at a non-quotient indeterminate form, the expression can be manipulated to represent $0/0$ or $\pm\infty/\pm\infty$, and then (and only then) can L'Hospital's Rule be applied.

**Suggested Textbook Exercises:**

- 4.8: 357-395 odd.

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## Indeterminate Forms and L'Hospital's Rule

**Definition:** An **indeterminate form** is an expression that cannot be evaluated, usually obtained in the context of limits. Indeterminate forms could represent any numerical value, as well as $\pm\infty$.

The following are all examples of indeterminate forms:

$$0/0, \quad \pm\infty/\pm\infty, \quad \infty\cdot 0, \quad \infty-\infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

The following are all **not** examples of indeterminate forms (some of these are *undefined* forms):

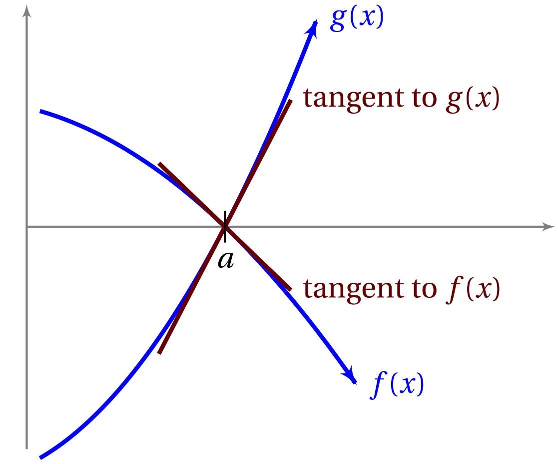
$$1/0, \quad 0/\pm\infty, \quad 1/\pm\infty, \quad \pm\infty/0, \quad \pm\infty/1, \quad 0-0, \quad \infty+\infty, \quad \infty\cdot\infty, \quad 0^1, \quad 1^0, \quad 0^\infty, \quad 0^{-\infty}$$

When dealing with a quotient indeterminate form (one of $0/0$ or $\pm\infty/\pm\infty$), we can use L'Hospital's Rule to gain more information:

**L'Hospital's Rule:** Let $a$ represent any real number or $\pm\infty$. Suppose $f$ and $g$ are differentiable functions over an open interval containing $a$ (except possibly at $a$ itself). If $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ produces an indeterminate form of type $0/0$ or $\pm\infty/\pm\infty$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

This result also holds if the limit is infinite or if the limit is one-sided.




There is a geometric motivation behind L'Hospital's Rule. For now consider the case where $f(x)/g(x) \to 0/0$ as $x \to a$. Then, as can be seen in the graph below, the ratio $f(x)/g(x)$ can be very closely approximated using the linearizations (tangent lines) of $f$ and $g$ for $x$ values near $a$:

$$\frac{f(x)}{g(x)} \approx \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)} = \frac{f'(x)(x-a)}{g'(x)(x-a)} = \frac{f'(x)}{g'(x)}.$$

Above we used the fact that $f(a) = g(a) = 0$.

Formally, in the case where both $f'(x)$ and $g'(x)$ are continuous, we employ the fact that continuity implies the Direct Substitution Property for limits:

$$\lim_{x\to a} f'(x) = f'(a), \qquad \lim_{x\to a} g'(x) = g'(a).$$

This means we can compute the limit of $f$ over $g$ as

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}} = \frac{\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a} \dfrac{g(x)-g(a)}{x-a}} = \frac{f'(x}{g'(x)} = \frac{\lim\limits_{x\to a} f'(x}{\lim\limits_{x\to a} g'(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

More advanced proof approaches relax the assumption that both $f'(x)$ and $g'(x)$ are continuous, and also prove the rule for indeterminate forms of the type $\pm\infty / \pm\infty$.

**Note:** You should always check to see if you actually have one of the valid indeterminate forms, as applying L'Hospital's Rule incorrectly usually produces wrong answers.

**Note:** In some instances, you may need to apply L'Hospital's Rule more than once!

**Example 1:** Find each limit.

(a) $\displaystyle \lim_{x\to\infty} \frac{\ln(2x)}{x}$

(b) $\lim_{x \to 0} \frac{\sin(x) - x}{x^3}$

(c) $\lim_{x \to 0} \frac{x \sin x}{1 - \cos x}$

(d) $\lim_{x \to 0} \frac{x^2}{\ln(\sec x)}$

(e) $\displaystyle\lim_{x\to 0} \frac{x2^x}{3^x - 1}$

(f) $\displaystyle\lim_{x\to\infty} \frac{\sqrt{9x-1}}{\sqrt{x-1}}$

**The Indeterminate Form $0\cdot\infty$**

If the limit of the expression $f \cdot g$ yields the indeterminate form $0\cdot\infty$, rewrite the expression as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$ and then apply L'Hospital's Rule.

**Example 2:** Find each limit.

(a) $\displaystyle\lim_{x\to 0^+} x\ln(x)$

(b) $\lim_{x \to (\pi/2)^-} \left(x - \frac{\pi}{2}\right) \tan(x)$

(c) $\lim_{x \to (\pi/4))} (1 - \tan x) \cot(x)$

(d) $\lim_{x \to -\infty} x^3 e^{3x}$