21-120: Differential and Integral Calculus
Recitation #7 Outline: 09/03/24

1. Find the derivative of the given functions.

   (a) $q(x) = \dfrac{5x^2}{4x+3}$, (b) $s(t) = \dfrac{\sqrt[3]{t}}{t-5}$, (c) $p(x) = 2x^5(4x^2+x)$

   Solution:

   - Let $u(x) = 5x^2$ and $v(x) = 4x+3$. By the Quotient Rule:

   $$q'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2} = \frac{10x(4x+3) - 4(5x^2)}{(4x+3)^2} = \frac{20x^2+30x}{(4x+3)^2}.$$

   -Let $u(t) = t^{\frac{1}{3}}$ and $v(t) = t-5$. By the Quotient rule:

   $$s'(t) = \frac{u'(t)v(t) - u(t)v'(t)}{v^2(t)} = \frac{\frac{1}{3}t^{-2/3}(t-5) - (t^{\frac{1}{3}})(1)}{(t-5)^2} = \frac{-2t-5}{3t^{\frac{2}{3}}(t-5)^2}.$$

   -Let $u(x) = 2x^5$ and $v(x) = 4x^2+x$. By the Product rule,

   $$p'(x) = u'(x)v(x) + u(x)v'(x) = 10x^4(4x^2+x) + (8x+1)(2x^5) = 56x^6 + 12x^5.$$

   (One can also expand and compute the derivative.)

2. Find the derivative of

   $$f(x) = 10\sqrt[5]{x^3} - \sqrt{x^7} + 6\sqrt[3]{x^8} - 3.$$

   and the derivative of

   $$g(y) = \frac{y^5 - 5y^3 + 2y}{y^3}.$$

   Solution:

   - Note that $f(x) = 10x^{\frac{3}{5}} - x^{\frac{7}{2}} + 6x^{\frac{8}{3}} - 3$. By the Sum and Power Rules,

   $$f'(x) = 6x^{-\frac{2}{5}} - \frac{7}{2}x^{\frac{5}{2}} + 16x^{\frac{5}{3}}.$$

   -We can simplify the value of $g(y)$.Note that $g(y) = y^2 - 5 + 2y^{-2}$. Thus, by the Sum and the Power rules:

   $$g'(y) = 2y - 4y^{-3}.$$

3. Find the equation of the line passing through the point $P(3,3)$ (meaning that $x = 3$ and $y = 3$) and tangent to the graph of $f(x) = \frac{6}{x-1}$.

   Solution The equation of the tangent line at $x = 3$ is:

   $$y = f'(3)(x-3) + f(3) = f'(3)(x-3) + 3.$$

We have to compute $f'(3)$ so we need to calculate the derivative of f. Let $u(x) = 6$ and $v(x) = x-1$. Then

$$f'(x) = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)} = \frac{-6}{(x-1)^2}.$$

So

$$f'(3) = \frac{-6}{4} = \frac{-3}{2}.$$

Thus,

$$y = \frac{-3}{2}(x-3) + 3 = \frac{-3}{2}x + \frac{15}{2}.$$

4. For $p(x) = f(x)g(x)$,use the Product rule to find $p'(2)$ if $f(2) = 3, f'(2) = -4, g(2) = 1$ and $g'(2) = 6$.

   Solution

   Since $p(x) = f(x)g(x)$, $p'(x) = f'(x)g(x) + g'(x)f(x)$ and hence

$$p'(2) = f'(2)g(2) + f(2)g'(2) = (-4)\cdot 1 + 3\cdot 6 = 14.$$

5. For

$$k(x) = 3h(x) + x^2 g(x),$$

   find $k'(x)$.

   Solution:

$$\begin{aligned}
k'(x) &= (3h(x) + x^2 g(x))' \\
&= (3h(x))' + (x^2 g(x)) && \text{(Apply the Sum Rule)} \\
&= 3(h(x))' + ((x^2)'g(x) + (g(x))'x^2) && \text{(Apply the Constant Multiple and Product rules)} \\
&= 3h'(x) + 2xg(x) + g'(x)x^2
\end{aligned}$$

6. For $k(x) = f(x)g(x)h(x)$, express $k'(x)$ in terms of $f(x), g(x), h(x)$ and their derivatives.

   Solution:

   One can think of k(x)=(f(x)g(x)) $\cdot h(x)$. Thus

$$\begin{aligned}
k'(x) &= (f(x)g(x))' \cdot h(x) + (f(x)g(x)) \cdot (h(x))' && \text{(Apply the Product Rule)} \\
&= (f'(x)g(x) + g'(x)f(x))h(x) + h'(x)f(x)g(x) && \text{(Apply the Product Rule)} \\
&= f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).
\end{aligned}$$