21-120: Differential and Integral Calculus
Lecture #25 Outline

Read: Section 4.10 of the textbook

Objectives and Concepts:

- An antiderivative of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. If F is any antiderivative of f, the most general antiderivative of a function f is $F(x)+C$ where C is an arbitrary constant.
- An initial-value problem is the problem of finding a function from its derivative and a condition that the function must satisfy.
- We use the notation $\int f(x)\,dx$ to represent the process of finding the antiderivative of $f(x)$ as $F(x)+C$. This is known as the indefinite integral of $f(x)$ with respect to x.

Suggested Textbook Exercises:

- 4.10: 465 - 507 odd, 521-524 all

Antiderivatives

As we will learn over the next few weeks, the process of finding functions from their derivatives is perhaps the most important, and likely the most difficult process to undertake. We begin our study of reverse-engineering derivative calculations by defining an antiderivative.

Definition: A function $F(x)$ is an **antiderivative** of a function $f(x)$ if $F'(x) = f(x)$ for all x in the domain of f.

Example 1: If $f(x) = 2x$, then $F(x) = x^2 + 9$ is an antiderivative of $f(x)$ since $F'(x) = f(x)$. Does f have any other antiderivatives?

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is $F(x)+C$, where C represents an arbitrary constant.

Example 2: It is true that $F_1(x) = \cos^2 x + C$ is a general antiderivative of $f(x) = -2\sin x\cos x$. Does f have any other general antiderivatives?

Here is a table of some basic antiderivatives. In the table, a is an arbitrary nonzero constant, and C is an arbitrary constant.

Function	**Antiderivative**	**Function**	**Antiderivative**
x^n where $n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$	a	$ax + C$
$\sin(ax)$	$-\dfrac{1}{a}\cos(ax) + C$	$\cos(ax)$	$\dfrac{1}{a}\sin(ax) + C$
$\sec^2(ax)$	$\dfrac{1}{a}\tan(ax) + C$	$\csc^2(ax)$	$-\dfrac{1}{a}\cot(ax) + C$
$\sec(ax)\tan(ax)$	$\dfrac{1}{a}\sec(ax) + C$	$\csc(ax)\cot(ax)$	$-\dfrac{1}{a}\csc(ax) + C$
e^{ax}	$\dfrac{1}{a}e^{ax} + C$	$\dfrac{1}{x}$	$\ln\lvert x\rvert + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$	$\dfrac{1}{1+x^2}$	$\arctan(x) + C$

Example 3: Find the most general antiderivative for each function.

(a) $f(x) = x^{-2} + \pi - 5e^{-3x} - \dfrac{7}{\sqrt{1-x^2}}$

(b) $f(x) = \dfrac{2}{3}x^{1/3} + \dfrac{1}{x^{1/3}} - \dfrac{1}{x}$

(c) $f(x) = 5\cos(2x) - 3\csc(x)\cot(x) + \sec^2(4x)$

Definition: Any equation in which the unknown is a function and which involves derivatives (or differentials) of this unknown function is called a **differential equation**:

$$\frac{dy}{dx} = f(x)$$

To **solve a differential equation** means to find the unknown function. The problem of finding a function when we know its derivative and its value y_0 at some particular value x_0 is called an **initial value problem**.

Example 4: Find an antiderivative F of f that satisfies the condition $F(1) = 2$ where $f(x) = 3x^4 + x^3 - x$.

Example 5: Solve the initial value problems.

(a) $f'(x) = 1 + \dfrac{1}{x^2}$ where $x > 0$ and $f(1) = 6$

(b) $f''(x) = 3e^x + 5\sin(x)$ where $f(0) = 1$ and $f'(0) = 2$

Notation: We write $\int f(x)\,dx$ for an antiderivative of f. It is called an **indefinite integral**.

$$\int f(x)\,dx = F(x) + C \quad \text{means} \quad F'(x) = f(x).$$

Since the indefinite integral of f represents the most general antiderivative of f, we must include the "$+C$" when we evaluate indefinite integrals. Below is a table of some common indefinite integrals, with a and k being nonzero constants:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \text{ for } n \neq -1 \qquad \int k\,dx = kx + C$$

$$\int x^{-1}\,dx = \ln|x| + C \qquad \int \sqrt{x}\,dx = \frac{2x^{3/2}}{3} + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln(a)} + C$$

$$\int \sin(x)\,dx = -\cos(x) + C \qquad \int \cos(x)\,dx = \sin(x) + C$$

$$\int \sec^2(x)\,dx = \tan(x) + C \qquad \int \csc^2(x)\,dx = -\cot(x) + C$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + C \qquad \int \csc(x)\cot(x)\,dx = -\csc(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \qquad \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

Example 6: Find the following indefinite integrals.

(a) $\displaystyle\int \left(x^{1/3} - \frac{1}{x} + \sec^2(x)\right) dx$

(b) $\displaystyle\int \left(4x^3 + \csc(x)\cot(x) - 3^x\right) dx$