21-120: Differential and Integral Calculus
Recitation #23 Outline: 11/21/24

1. Calculate the following limits:

$$\lim_{x\to 0}\frac{e^x-1-x}{x^2},\quad \lim_{x\to 1}\frac{\ln(x)-1+x}{\arctan(x)-\frac{\pi}{4}},\quad \lim_{x\to+\infty}\ln(1+e^x)\sin\left(\frac{1}{x}\right)$$

Solution:

Let's calculate $\lim_{x\to 0}\frac{\exp(x)-1-x}{x^2}$. The repeated application of L'Hôpital's rule leads to

$$\lim_{x\to 0}\frac{\exp(x)-1-x}{x^2}=\lim_{x\to 0}\frac{\exp(x)-1}{2x}=\lim_{x\to 0}\frac{\exp(x)}{2}=\frac{1}{2}.$$

Let's calculate $\lim_{x\to 1}\frac{\ln(x)-1+x}{\arctan(x)-\frac{\pi}{4}}$. Again, we are dealing with a limit of the form $\frac{0}{0}$. L'Hôpital's rule leads to

$$\lim_{x\to 1}\frac{\ln(x)-1+x}{\arctan(x)-\frac{\pi}{4}}=\lim_{x\to 1}\frac{\frac{1}{x}+1}{\frac{1}{1+x^2}}=4.$$

The last limit is of the form $\frac{\infty}{\infty}$. L'Hôpital's rule leads to studying the limit

$$\lim_{x\to\infty}\frac{\exp(x)}{1+\exp(x)}\Big/\frac{\cos\left(\frac{1}{x}\right)}{x^2\sin^2\left(\frac{1}{x}\right)}.$$

Using limit algebra, it can be seen that this last expression tends to 1.

2. Calculate the following integrals:

$$\int_0^3 |1-x|\,dx$$

and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{x^2\sin(x)}{1+x^6}\,dx$$

Solution

The first integral can be solved as follows:

$$\int_0^3|1-x|dx=\int_0^1(1-x)dx+\int_1^3(x-1)dx=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^3=\frac{5}{2}.$$

For the second integral, it is enough to notice that the integrand is an odd function over a symmetric interval, so the integral is zero.

3. Calculate the following antiderivatives by substitution:

$$\int(\cos(x))^{1234}\sin(x)\,dx$$

$$\int\frac{1}{x\ln(x)}\,dx$$

Solution

For the first antiderivative. Let $u = \cos(x)$ then $du = -\sin(x)dx$. So,

$$\int (\cos(x))^{1234} \sin(x)\,dx = -\int u^{1234} du = -\frac{u^{1235}}{1235} + C = -\frac{(\cos(x))^{1235}}{1235} + C,$$

where C is a constant. For the second antiderivative. Let $u = \ln(x)$. then $du = \frac{1}{x}dx$. So

$$\int \frac{1}{x\ln(x)} dx = \int \frac{1}{u} du = \ln(|\,u\,|) + C = \ln(|\ln(x)|) + C,$$

where C is a constant.

4. Calculate the derivative of the following function:

$$f(x) = \int_0^{x^3} \cos(t^2)\,dt$$

Solution

Let $G(x) = \int_0^x \cos(t^2)dt$. Note that

$$f(x) = G(x^3).$$

So, by the Fundamental Theorem of Calculus and the chain rule:

$$f'(x) = G'(x^3)3x^2 = \cos(x^6)3x^2$$

5. Determine the antiderivative F of the function $f(x) = x^3 - 3x^2 + 7$ on $\mathbb{R}$ such that $F(1) = 2$.

Solution A primitive is $7x - x^3 + \frac{x^4}{4} + C$, where C is a constant. To determine the constant C, it is enough to note that $F(1) = 2$. Therefore,

$$7 - 1 + \frac{1}{4} + C = 2,$$

so $C = -\frac{17}{4}$, and thus

$$F(x) = 7x - x^3 + \frac{x^4}{4} - \frac{17}{4}.$$

6. The height, in meters, of an electrical line of 160 meters can be modeled by the function h defined on $[-80, 80]$ as

$$h(x) = 10\left(e^{x/40} + e^{-x/40}\right).$$

What is the average height of this electrical line?

Solution:

By a formula from the course, the average height of the electrical line is:

$$\frac{1}{160}\int_{-80}^{80} h(x)\,dx = \frac{1}{160}\int_{-80}^{80} 10\left(e^{x/40} + e^{-x/40}\right)dx = \frac{1}{16}\left[40e^{x/40} - 40e^{-x/40}\right]_{-80}^{80}.$$

$$= \frac{5}{2}\left(e^2 - e^{-2} - \left(e^{-2} - e^2\right)\right) = 5\left(e^2 - e^{-2}\right) \approx 36.27\,\text{m}.$$

7. Suppose we want to construct a chocolate box for a friend's anniversary. The open box will be made from a 24-inch by 36-inch cardboard by cutting a square from each corner of the box and folding up the flaps on each side. What is the size of the square that should be cut from each corner in order to obtain a box with the maximum volume?

   Solution

   Let us assume that x is the length of the side of the square to be cut from each corner. Then, the four remaining flaps can be folded to form a box with an open top. Let us assume that V is the volume of the resulting box. A square with side length x in is cut from each corner of the cardboard. The remaining flaps are folded to form a box with an open top.

   We will try to maximize the volume of the box. Therefore, the problem is to maximize V. We are trying to maximize the volume of the box. The volume of a box is given by $V = L \cdot W \cdot H$, where L, W, and H are the length, width, and height, respectively.

   The height of the box is x in, the length is $36-2x$ in, and the width is $24-2x$ in. Therefore, the volume of the box is

$$V(x) = (36-2x)(24-2x)x = 4x^3 - 120x^2 + 864x.$$

   To determine the domain of consideration. Indeed, we need $x > 0$. Furthermore, the side length of the square cannot be greater than or equal to half the length of the shorter side, 24 in, otherwise one of the flaps would be completely cut off. Therefore, we are trying to determine if there is a maximum volume of the box for x in the open interval $(0,12)$. Since V is a continuous function on the closed interval $[0,12]$, we know that V will have an absolute maximum on this closed interval. Hence, we consider V on the closed interval $[0,12]$ and check if the absolute maximum occurs at an interior point.

   Since $V(x)$ is a continuous function on the closed and bounded interval $[0,12]$, V must have an absolute maximum (and an absolute minimum). Since $V(x) = 0$ at the endpoints and $V(x) > 0$ for $0 < x < 12$, the maximum must occur at a critical point. The derivative is

$$V'(x) = 12x^2 - 240x + 864.$$

   To find the critical points, we need to solve the equation

$$12x^2 - 240x + 864 = 0.$$

   If we divide both sides of this equation by 12, the problem simplifies to solving the equation

$$x^2 - 20x + 72 = 0.$$

   Using the quadratic formula, we find that the critical points are

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(1)(72)}}{2} = \frac{20 \pm \sqrt{112}}{2} = \frac{20 \pm 47}{2}.$$

   Since $10 + \sqrt{27}$ is not in the domain of consideration, the only critical point we need to consider is $10 - \sqrt{27}$. Therefore, the volume is maximized when $x = 10 - \sqrt{27}$ in. The maximum volume is

$$V(10 - \sqrt{27}) = 640 + 448\sqrt{7} \approx 1{,}825\,\text{in}^3,$$