21-120: Differential and Integral Calculus
Lecture #12 Outline

Read: Section 3.7 of the textbook

Objectives and Concepts:

- The Inverse Function Theorem gives the relationship between the derivatives of a differentiable function $f(x)$ and its inverse function $f^{-1}(x)$ when it is defined.
- The derivatives of inverse trigonometric functions can be found by applying the Inverse Function Theorem.

Suggested Textbook Exercises:

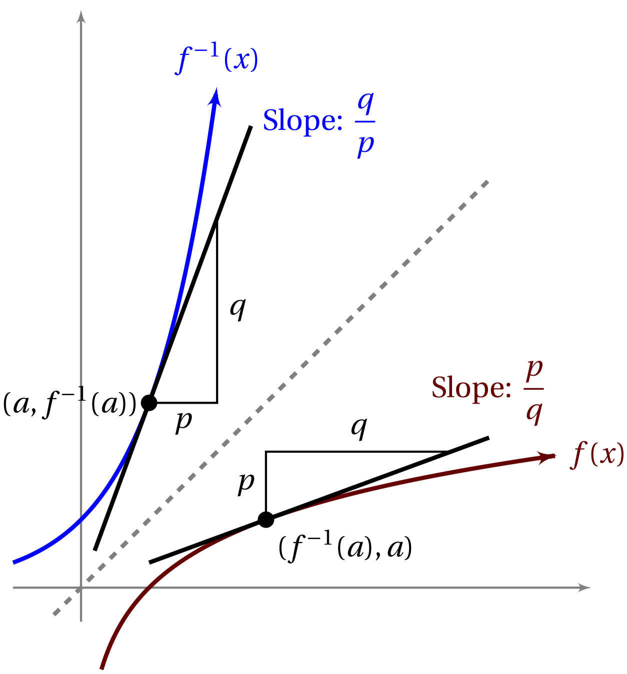
- 3.7: 261-293 odd.

The Derivatives of Inverse Functions

Recall that when $y = f(x)$ is a one-to-one function (i.e., its graph passes the Horizontal Line Test), we can define the inverse function $f^{-1}(x)$, and the graph of $y = f^{-1}(x)$ is obtained by reflecting the graph $y = f(x)$ across the line $y = x$.

Now, if f is differentiable at a, then the inverse function f^{-1} will also be differentiable at $f(a)$. As the roles of x and y switch when considering the graph of the inverse function, this means the lines tangent to f and f^{-1} (at the points $(a, f(a))$ and $(f(a), a)$, respectively) will have reciprocal slopes. This establishes the relationship

$$\left(f^{-1}\right)'(a) = \frac{1}{f'\left(f^{-1}(a)\right)}.$$



We can also compute this directly using the Chain Rule. Since $f(f^{-1}(x)) = x$, we have, differentiating both sides,

$$1 = f'\left(f^{-1}(x)\right) \cdot \left(f^{-1}\right)'(x),$$

and the theorem below is obtained when solving for $\left(f^{-1}\right)'(x)$.

The Inverse Function Theorem: Let f be a function that is both invertible and differentiable. Let $y = f^{-1}(x)$ be the inverse of $f(x)$. For all x satisfying $f'\left(f^{-1}(x)\right) \neq 0$, we have that

$$\frac{dy}{dx} = \frac{d}{dx}\left(f^{-1}(x)\right) = \left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

We can use the Inverse Function Theorem as a way to find tangent lines of inverse functions at given values, even without finding the description of the inverse function itself!

Example 1: Let $f(x) = x^3 + x^{1/3} + 1$ for $x > 0$. Find the equation of the line tangent to $f^{-1}(x)$ when $x = 3$.

Example 2: For the function $f(x) = \dfrac{4}{1+x^2}$ (for $x \geq 0$), find the equation of the line tangent to $f^{-1}(x)$ at the point $(2, 1)$.

Recall the inverse trigonometric functions from Precalculus. (It is important to pay close attention to the ranges of the inverse trig functions - these ranges are the restricted domains of the standard trigonometric functions.)

Inverse Trigonometric Functions:

$$\begin{array}{lcl}
\sin^{-1} x = y & \iff & \sin y = x \quad \text{and} \quad y \in [-\pi/2, \pi/2] \\
\cos^{-1} x = y & \iff & \cos y = x \quad \text{and} \quad y \in [0, \pi] \\
\tan^{-1} x = y & \iff & \tan y = x \quad \text{and} \quad y \in (-\pi/2, \pi/2) \\
\csc^{-1} x = y & \iff & \csc y = x \quad \text{and} \quad y \in (0, \pi/2] \cup (\pi, 3\pi/2] \\
\sec^{-1} x = y & \iff & \sec y = x \quad \text{and} \quad y \in [0, \pi/2) \cup [\pi, 3\pi/2) \\
\cot^{-1} x = y & \iff & \cot y = x \quad \text{and} \quad y \in (0, \pi)
\end{array}$$

We can also use the Inverse Function Theorem to find the derivatives of these inverse trigonometric functions.

Example 3: Use the Inverse Function Theorem to find $f'(x)$ when $f(x) = \arcsin x$. Recall that $\arcsin x$ is defined on the interval $[-1, 1]$ and has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The derivatives of the other inverse trigonometric functions can be found in the same way.

Derivatives of the Inverse Trigonometric Functions:

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$$

Example 4: Find the derivative of each function:

(a) $f(x) = x^3 \sin^{-1}(\sqrt{x})$

(b) $y = \sqrt{\arctan(3x)}$

(c) $g(x) = \arccos\left(\frac{x^2+1}{4}\right)$

(d) $y = \cot^{-1}\sqrt{9-x^2}$