21-120: Differential and Integral Calculus
Recitation #5

1. Show that the equation $-x^3+x^2-x+2=0$ has at least one solution in the interval $(1,2)$.

   Solution

   For all $x \in \mathbb{R}$, one defines $f(x) := -x^3+x^2-x+2$. The function f is continuous and satisfies
$$f(1) = -1+1-1+2 = 1 > 0,$$
   and
$$f(2) = -8+4-2+2 = -4 < 0.$$
   Thus, by the Intermediate Value Theorem, there exists $c \in (1,2)$ such that $f(c)=0$. Equivalently, the equation $-x^3+x^2-x+2=0$ has at least one solution in the interval $(1,2)$.

2. Show that the equation
$$\cos(x) = \frac{1}{x}$$
   has infinitely many solutions in $(0,+\infty)$.

   Hint: Think about what happens between $2k\pi$ and $2k\pi+\pi$ when $k \geq 1$ is an integer.

   Solution: Let us define for $x>0$, the function $f(x) = \cos(x) - \frac{1}{x}$. Then, f is continuous on $(0,+\infty)$. Let $k \geq 1$ be an integer. Then, we have
$$f(2k\pi) = 1 - \frac{1}{2k\pi} \geq 0,$$
   and
$$f(2k\pi+\pi) = -1 - \frac{1}{2k\pi+\pi} \leq 0.$$
   By the Intermediate Value Theorem, there exists a real number x_k in the interval $[2k\pi, 2k\pi+\pi]$ such that $f(x_k)=0$. Clearly, $x_k < 2(k+1)\pi \leq x_{k+1}$. Then, the real numbers are pairwise disjoint, and the equation has infinitely many solutions.

3. Evaluate the following limits:

   (a) $\displaystyle\lim_{x\to+\infty} \frac{\sqrt{x^2-7}}{3x+5}$

   (b) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^2-7}}{3x+5}$

   (c) $\displaystyle\lim_{x\to+\infty} \left(\sqrt{x^2+6x+1}-x\right)$

   (d) $\displaystyle\lim_{x\to+\infty} \frac{2e^x+1}{e^x-2}$.

   Solution:

(a)-(b) We have that:
$$\sqrt{x^2-7} = \sqrt{x^2\left(1-\frac{7}{x^2}\right)} = |x|\sqrt{1-\frac{7}{x^2}},$$
and
$$3x+5 = 3x(1+\frac{5}{3x}).$$

So

$$\frac{\sqrt{x^2-7}}{3x+5} = \frac{|x|\sqrt{1-\frac{7}{x^2}}}{3x\left(1+\frac{5}{3x}\right)} = \begin{cases} -\frac{\sqrt{1-\frac{7}{x^2}}}{3\left(1+\frac{5}{3x}\right)}, & x<0, \\ \frac{\sqrt{1-\frac{7}{x^2}}}{3\left(1-\frac{5}{3x}\right)}, & x>0. \end{cases}$$

We deduce that

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-7}}{3x+5} = -\frac{1}{3},$$

and

$$\lim_{x\to+\infty} \frac{\sqrt{x^2-7}}{3x+5} = \frac{1}{3}.$$

(c) We have that

$$\sqrt{x^2+6x+1}-x = \frac{\left(\sqrt{x^2+6x+1}-x\right)\left(\sqrt{x^2+6x+1}+x\right)}{\sqrt{x^2+6x+1}+x} = \frac{6x+1}{\sqrt{x^2+6x+1}+x}.$$

Moreover,

$$\sqrt{x^2+6x+1}+x = \sqrt{x^2\left(1+\frac{6}{x}+\frac{1}{x^2}\right)}+x = |x|\sqrt{\left(1+\frac{6}{x}+\frac{1}{x^2}\right)}+x.$$

If $x>0$, we obtain

$$\sqrt{x^2+6x+1}-x = \frac{6x(1+\frac{1}{6x})}{|x|\sqrt{\left(1+\frac{6}{x}+\frac{1}{x^2}\right)}+x} = \frac{6x(1+\frac{1}{6x})}{x\left(\sqrt{1+\frac{6}{x}+\frac{1}{x^2})}+1\right)} = \frac{6\left(1+\frac{1}{6x}\right)}{\sqrt{(1+\frac{6}{x}+\frac{1}{x^2})}+1},$$

so

$$\lim_{x\to+\infty}\left(\sqrt{x^2+6x+1}-x\right) = \lim_{x\to+\infty} \frac{6\left(1+\frac{1}{6x}\right)}{\sqrt{(1+\frac{6}{x}+\frac{1}{x^2})}+1} = 3.$$

(d) Observe that

$$f(x) = \frac{2e^x+1}{e^x-2} = \frac{e^x(2+e^{-x})}{e^x(1-2e^{-x})} = \frac{2+e^{-x}}{1-2e^{-x}}.$$

Thus, using the quotient rule for limits,

$$\lim_{x\to+\infty} f(x) = 2.$$

4. For the function f defined for every $x\in\mathbb{R}$ as follows:

$$f(x) = \frac{3x}{x^2-x-6},$$

determine the equations of all horizontal or vertical asymptotes.

Solutions:

For vertical asymptotes, note that $x^2 - x - 6 = 0$ is equivalent to $(x-3)(x+2) = 0$ and has two solutions $x = 3$ or $x = -2$. There are two vertical asymptotes at $x = 3$ and $x = -2$.

For horizontal asymptotes,

$$f(x) = \frac{3x}{x^2 - x - 6} = \frac{3/x^2}{1 - 1/x - 6/x^2};$$

thus $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{3}{x} = 0$. Similarly, one shows that $\lim_{x\to-\infty} f(x) = 0$. Therefore, $y = 0$ is an horizontal asymptote of the graph of f for both large positive and negative values of x.

5. (a) Show, using the (ϵ, δ) definition that :

$$\lim_{x\to 0} x^2 = 0.$$

Solution:

By definition, "$\lim_{x\to 0} x^2 = 0$" translates into the following:

$$\text{for all } \epsilon > 0, \text{there exists } \delta > 0, \text{for all } x \in \mathbb{R}, |x| \leq \delta \text{ implies } |x^2| \leq \epsilon.$$

We want to show that this statement is true, that is, given a real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that the implication $|x| \leq \delta$ implies $|x^2| \leq \epsilon$ holds for all real numbers x. To do this, it is sufficient to take $\delta = \sqrt{\epsilon}$, and the result holds.

(b) Translate the statement into a mathematical formula (with quantifiers)

$$\lim_{x\to 0} \ln(1 + x) = 0.$$

Solution: By definition of the limit, the sentence can be translated to:

$$\text{for all } \epsilon > 0, \text{there exists } \delta > 0, \text{for all } x \in (-1, +\infty), |x| \leq \delta \text{ implies } |\ln(1 + x)| \leq \epsilon.$$