21-120: Differential and Integral Calculus
Recitation #17 Outline: 10/29/24

1. Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1 \qquad \lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not, evaluate the limit where possible.

(a) $\lim_{x\to a}(f(x) - p(x))$
(b) $\lim_{x\to a}(p(x) - q(x))$
(c) $\lim_{x\to a}(p(x) + q(x))$
(d) $\lim_{x\to a}(f(x))^{g(x)}$
(e) $\lim_{x\to a}(f(x))^{p(x)}$
(f) $\lim_{x\to a}(h(x))^{p(x)}$
(g) $\lim_{x\to a}(p(x))^{f(x)}$
(h) $\lim_{x\to a}(p(x))^{q(x)}$
(i) $\lim_{x\to a} \sqrt[q(x)]{p(x)}$

Solution:

(a) This limit yields the form $0-\infty$, which is not an indeterminate form. The limit evaluates to $-\infty$.
(b) This limit yields an indeterminate form of type $\infty-\infty$.
(c) This limit yields the form $\infty+\infty$, which is not an indeterminate form. The limit evaluates to $+\infty$.
(d) This limit yields an indeterminate form of type 0^0.
(e) This limit yields the form 0^∞, which is not an indeterminate form. The limit evaluates to 0.
(f) This limit yields an indeterminate form of type 1^∞.
(g) This limit yields an indeterminate form of type ∞^0.
(h) This limit yields the form ∞^∞, which is not an indeterminate form. The limit evaluates to ∞.
(i) The function inside this limit can be written as $p(x)^{1/q(x)}$, so the limit yields an indeterminate form of type ∞^0.

2. Find the limit using l'Hospital's rule.

(a) $\lim_{x\to 0}(\csc x - \cot x)$
(b) $\lim_{t\to\infty}(x - \ln x)$
(c) $\lim_{x\to 0^+} x^{\sqrt{x}}$
(d) $\lim_{x\to\infty} x^{e^{-x}}$
(e) $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$
(f) $\lim_{x\to 0^+}(1+\sin(3x))^{1/x}$

Solution:

(a) $\lim_{x\to 0}(\csc x - \cot x)$
Rewrite the expressions in terms of sine and cosine:

$$\csc x - \cot x = \frac{1}{\sin x} - \frac{\cos x}{\sin x} = \frac{1-\cos x}{\sin x}.$$

As $x \to 0$, both the numerator and denominator approach zero, giving the indeterminate form $\frac{0}{0}$. Apply l'Hôpital's rule:

$$\lim_{x\to 0}(\csc x - \cot x) \overset{0/0}{=} \lim_{x\to 0}\frac{1-\cos x}{\sin x} = \lim_{x\to 0}\frac{\sin x}{\cos x} = 0.$$

(b) $\lim_{x\to\infty}(x-\ln x)$

As $x\to\infty$, both x and $\ln x$ approach infinity, giving the indeterminate form $\infty-\infty$. We can rewrite the expression as:

$$x-\ln x=\ln(e^x)+\ln(x^{-1})=\ln\left(\frac{e^x}{x}\right).$$

Thus, we have:

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}\ln\left(\frac{e^x}{x}\right).$$

Since the logarithm is continuous, we can interchange the limit and the logarithm:

$$\lim_{x\to\infty}\ln\left(\frac{e^x}{x}\right)=\ln\left(\lim_{x\to\infty}\frac{e^x}{x}\right).$$

Now, we evaluate the limit inside the logarithm:

$$\lim_{x\to\infty}\frac{e^x}{x}\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{e^x}{1}=\infty.$$

Therefore, we have:

$$\lim_{x\to\infty}\ln\left(\frac{e^x}{x}\right)=\ln(\infty)=\infty.$$

Hence, the final result is:

$$\lim_{x\to\infty}(x-\ln x)=\infty.$$

(c) $\lim_{x\to 0^+}x^{\sqrt{x}}$

Let $y=\lim_{x\to 0^+}x^{\sqrt{x}}$. Taking the natural logarithm of both sides, we have:

$$\ln y=\ln\left(\lim_{x\to 0^+}x^{\sqrt{x}}\right)=\lim_{x\to 0^+}\sqrt{x}\ln x.$$

The interchange of limit and logarithm is valid because the logarithm is continuous. As $x\to 0^+$, $\sqrt{x}\ln x$ approaches $0\cdot(-\infty)$, giving the indeterminate form $0\cdot(-\infty)$. Rewrite and apply L'Hôpital's rule:

$$\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{1/\sqrt{x}}\overset{-\infty/\infty}{=}\lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}=\lim_{x\to 0^+}(-2\sqrt{x})=0.$$

Therefore,

$$\ln y=-\infty\implies y=e^0=1.$$

(d) $\lim_{x\to\infty}x^{e^{-x}}$

Let $y=\lim_{x\to\infty}x^{e^{-x}}$. Taking the natural logarithm of both sides, we have:

$$\ln y=\ln\left(\lim_{x\to\infty}x^{e^{-x}}\right)=\lim_{x\to\infty}e^{-x}\ln x.$$

The interchange of limit and logarithm is valid because the logarithm is continuous. As $x\to\infty$, e^{-x} approaches 0 and $\ln x$ approaches ∞, giving the indeterminate form $0\cdot\infty$. We can rewrite this expression to apply L'Hôpital's rule:

$$\lim_{x\to\infty}e^{-x}\ln x=\lim_{x\to\infty}\frac{\ln x}{e^x}\overset{\infty/\infty}{=}\lim_{x\to\infty}\frac{1/x}{e^x}=\lim_{x\to\infty}\frac{1}{xe^x}=0.$$

Thus, we find:

$$\ln y=0\implies y=e^0=1.$$

(e) $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

Let $y = \lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$. Taking the natural logarithm of both sides, we have:

$$\ln y = \ln\left(\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}\right) = \lim_{x\to 1}\tan\left(\frac{\pi x}{2}\right)\ln(2-x).$$

The interchange of limit and logarithm is valid because the logarithm is continuous. As $x \to 1$, $\ln(2-x)$ approaches 0 and $\tan\left(\frac{\pi x}{2}\right)$ approaches either $+\infty$ or $-\infty$ depending on whether x approaches 1 from the left or the right. Let's focus on the limit as $x \to 1^-$ (the case of $x \to 1^+$ is similar and will give the same result). Then we have indeterminate form of type $\infty \cdot 0$. We can rewrite this expression to apply L'Hôpital's rule:

$$\lim_{x\to 1^-}\tan\left(\frac{\pi x}{2}\right)\ln(2-x) = \lim_{x\to 1^-}\frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} \overset{0/0}{=} \lim_{x\to 1^-}\frac{-\frac{1}{2-x}}{-\frac{\pi}{2}\csc^2\left(\frac{\pi x}{2}\right)} = \frac{2}{\pi}.$$

Similarly,

$$\lim_{x\to 1^+}\tan\left(\frac{\pi x}{2}\right)\ln(2-x) = \frac{2}{\pi}.$$

Thus, we find:

$$\ln y = \frac{2}{\pi} \implies y = e^{2/\pi}.$$

(f) $\lim_{x\to 0^+}(1+\sin(3x))^{1/x}$

Let $y = \lim_{x\to 0^+}(1+\sin(3x))^{1/x}$. Taking the natural logarithm of both sides, we have:

$$\ln y = \ln\left(\lim_{x\to 0^+}(1+\sin(3x))^{1/x}\right) = \lim_{x\to 0^+}\frac{1}{x}\ln(1+\sin(3x)).$$

The interchange of limit and logarithm is valid because the logarithm is continuous. As $x \to 0^+$, $\sin(3x)$ approaches 0, giving us the indeterminate form $\frac{0}{0}$. We can rewrite this expression to apply L'Hôpital's rule:

$$\lim_{x\to 0^+}\frac{\ln(1+\sin(3x))}{x} \overset{0/0}{=} \lim_{x\to 0^+}\frac{\frac{3\cos(3x)}{1+\sin(3x)}}{1} = 3.$$

Thus, we find:

$$\ln y = 3 \implies y = e^3.$$

3. Suppose f is a positive function. If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, show that

$$\lim_{x\to a}(f(x))^{g(x)} = 0.$$

This shows that 0^∞ is not an indeterminate form.

Solution:

Let

$$y = \lim_{x\to a}(f(x))^{g(x)}.$$

Taking the natural logarithm of both sides and interchanging the limit with logarithm, as well as using logarithm laws, gives us:

$$\ln y = \lim_{x\to a} g(x)\ln(f(x)) = \lim_{x\to a} g(x)\cdot \lim_{x\to a}\ln(f(x)).$$

(Note that since f is positive for all x, $\ln(f(x))$ is defined for all x.) Since $\lim_{x\to a} f(x) = 0$, we have:

$$\lim_{x\to a}\ln(f(x)) = \ln(\lim_{x\to a} f(x)) = -\infty.$$

(Here we also used that f is a positive function.) Thus, since $\lim_{x\to a} g(x) = \infty$, we have

$$\ln y = \infty\cdot(-\infty) = -\infty.$$

Therefore, we have $y = e^{\ln y} = e^{-\infty} = 0$.