21-120: Differential and Integral Calculus
Lecture #26 Outline

Read: Section 5.1 of the textbook

Objectives and Concepts:

- Sigma (summation) notation can be used to write many long sums in a concise way. Some general formulas are available for basic sums.
- To estimate the area under a curve, we divide the area into rectangles whose heights are easy to compute. The more rectangles we draw, the better our approximation is.
- The Riemann sums is an approximation to the net area between a curve and the horizontal axis.

Suggested Textbook Exercises:

- 5.1: 1-27 odd, 43, 45.

Sigma Notation

In mathematics we use the capital Greek letter sigma, $\sum$, to represent the summation of objects. In many cases, the objects being summed can be generalized to a pattern that might depend on an **index** (common choices of indices are i, j, and k). For example, the sum of the integers 3 through 8 can be written compactly as

$$\sum_{i=3}^{8} i = 3+4+5+6+7+8,$$

and the sum of the reciprocals of the cubes of the first 100 integers can be written as

$$\sum_{k=1}^{100} \frac{1}{k^3} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{99^3} + \frac{1}{100^3}.$$

There are some common sums that show up often in mathematics: the sum of the first n integers, the sum of the first n squares, and the sum of the first n cubes.

$$\sum_{i=1}^{n} i = 1+2+\cdots+(n-1)+n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+(n-1)^2+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3+2^3+\cdots+(n-1)^3+n^3 = \frac{n^2(n+1)^2}{4}$$

Sigma notation satisfies many properties, for example, if c is a constant,

$$\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i, \qquad \sum_{i=1}^{n} (c a_i) = c \sum_{i=1}^{n} a_i.$$

Another useful property is related to indexing: if $m < n$ are integers, then

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i.$$

Example 1: Find $\sum_{i=1}^{7} \left(2i^2 - 3i + 4\right)$

Example 2: Find $\sum_{k=5}^{10} k^3$.

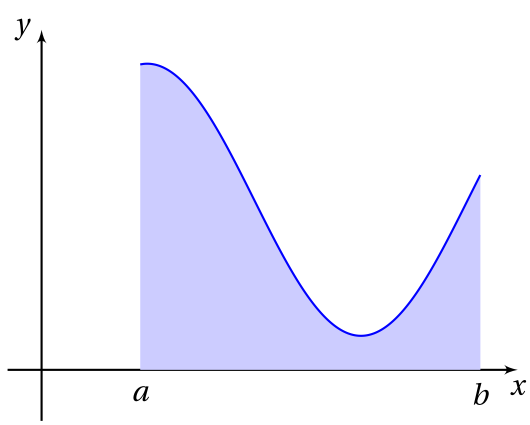
Approximating Areas and Riemann Sums

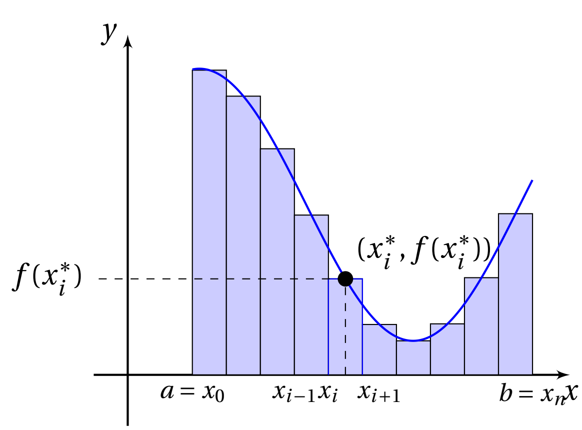
Our goal is to solve the **area problem**: given a curve $y = f(x)$ on a closed interval $[a, b]$, find the (net) area between the curve and the x-axis from $x = a$ to $x = b$. **Note:** The area itself is **signed** - area that appears above the x-axis is considered to be positive area, while the area below the x-axis is considered to be negative.

We begin with a simple approach: what if we were to approximate the area between the curve and the x-axis by dividing the area into rectangles, calculating the area of each of the rectangles, and then summing up all of those areas? We could do this by first creating a **partition** of the closed interval $[a, b]$ by splitting it into several subintervals $[x_i, x_{i+1}]$:

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$

On each subinterval, we would then draw the rectangle that has a width of $x_{i+1} - x_i$ and a height of $f(x_i^*)$, where x_i^* is any value satisfying $x_i \le x_i^* \le x_{i+1}$.





If we let $\Delta x_i = x_{i+1} - x_i$ represent the width of the ith rectangle, then we can use sigma notation to write the total area approximation using n rectangles:

$$\begin{pmatrix}\text{approximation}\\ \text{of area}\end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix}\text{height of}\\ i\text{th rectangle}\end{pmatrix}\begin{pmatrix}\text{width of}\\ i\text{th rectangle}\end{pmatrix} = \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

The sum of these areas of rectangles is known as a **Riemann sum**. When the rectangles all have equal width, we often replace Δx_i with just Δx.

Definition: Let $f(x)$ be a continuous function on $[a,b]$, and let $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ be a Riemann sum for $f(x)$. The **area** A **between the curve** $y = f(x)$ **and the** x**-axis on** $[a,b]$ is given by the limit of the Riemann sum as the number of rectangles goes to infinity:

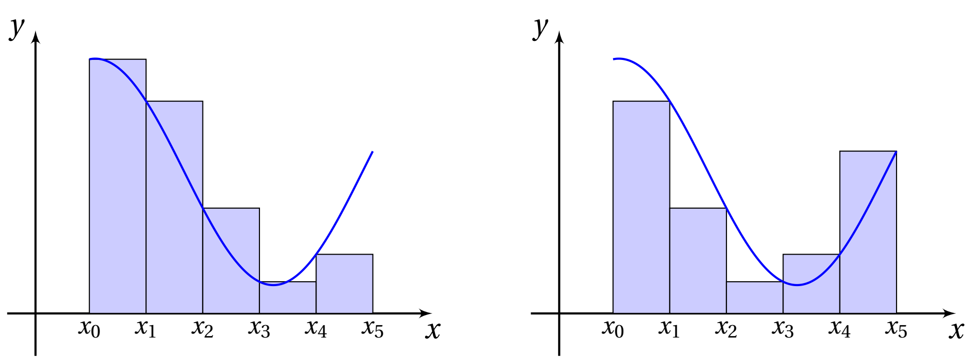
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i.$$

Left and Right Hand Sums

It is fairly common to approximate the area between a curve and the horizontal axis by dividing the interval $[a,b]$ into rectangles of equal width, and to use a choice (consistent across all of the rectangles) of x-value in each subinterval for finding the height of the rectangle. Often, the left endpoint x_{i-1}, or the right endpoint x_i are used to find the height.

Left and Right Endpoint Approximation: Let f be a continuous function over the interval $[a,b]$ and let $n \ge 1$ be some integer. Divide the interval $[a,b]$ into n equal-length subintervals $[x_{i-1}, x_i]$ $(i = 1, \ldots, n)$ with width $\Delta x = (b-a)/n$. Then we can define the approximate areas L_n and R_n as follows:

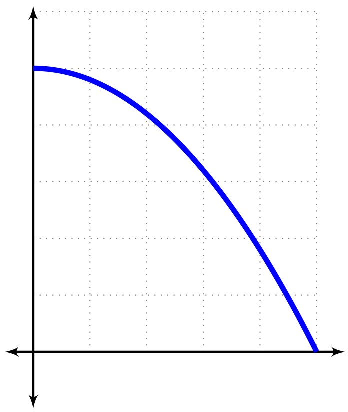
- $L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x$ is the **left endpoint approximation** to $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$.
- $R_n = \sum_{i=1}^{n} f(x_i)\Delta x$ is the **right endpoint approximation** to $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$.

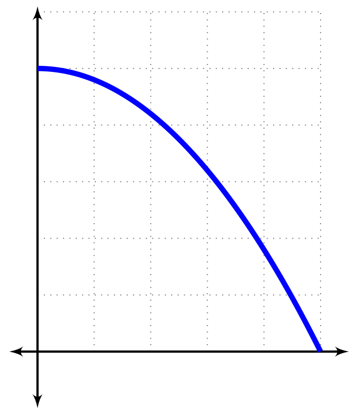


Left Endpoint Approximation   Right Endpoint Approximation

Example 3: Estimate the area under the graph of $f(x) = 25 - x^2$ from $x = 0$ to $x = 5$ using 5 rectangles and (a) right endpoints; (b) left endpoints. Values for this function are found in the table below.

x	0	1	2	3	4	5
$f(x)$	25	24	21	16	9	0





Example 4: Let $f(x) = \sqrt{x}$. Estimate the area under the graph of $f(x)$ from $x = 0$ to $x = 4$ using four approximating rectangles and right endpoints. Is this an underestimate or an overestimate?

The Distance Problem: Consider the distance problem in which we are asked to find the distance traveled by an object during a certain time period if the velocity of the object is known at all times.

Example 5: Speedometer readings for a motorcycle at 12-second intervals are given in the table.

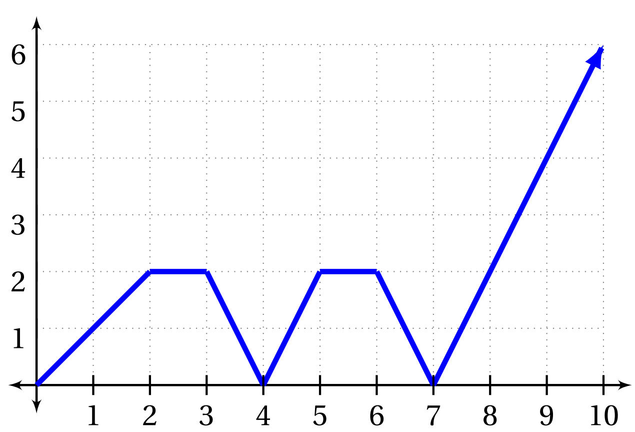
t (s)	0	12	24	36	48	60
v (ft/s)	30	28	25	22	24	27

(a) Estimate the distance traveled by the motorcycle during this time period using velocities at the beginning of the time intervals.

(b) Give another estimate using the velocities at the end of the time periods.

The example above is similar to the Area Problem because if we sketch a graph of the velocity function and draw rectangles whose heights are the initial velocities for each time interval, then the area of each rectangle can be interpreted as **distance traveled during the time period**. The sum of the areas of the rectangles is interpreted as an estimate for the **total distance traveled**.

Example 6: Here is a graph of a particle's velocity (in m/s) at time t:



Find the distance traveled for each specified interval:

- $0 < t < 2$:
- $2 < t < 3$:
- $3 < t < 5$:
- $5 < t < 7$:
- $7 < t < 10$:

What is the total distance the particle traveled from $t = 0$ to $t = 10$?