Recitation 26 Solutions

1. (a) Solution 1: $\dfrac{x^2-1}{x^2-3x+2} = \dfrac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-2)} = \dfrac{x+1}{x-2}$

Thus $\displaystyle\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x\to 1} \frac{x+1}{x-2} = \frac{1+1}{1-2} \boxed{= -2}$

Solution 2: $\displaystyle\lim_{x\to 1} x^2-1 = 0 = \lim_{x\to 1} x^2-3x+2$

$$\lim_{x\to 1} \frac{x^2-1}{x^2-3x+2} \overset{\frac{0}{0}}{=} \lim_{x\to 1} \frac{2x}{2x-3} = \frac{2}{2-3} \boxed{= -2}$$

(b) as $x \to 0^+$, $e^{x^2+1} \to e$ and $\sin(x) \to 0^+$

thus $\displaystyle\lim_{x\to 0^+} \frac{e^{x^2+1}}{\sin(x)} = +\infty$

(c) Solution 1: If $x > 0$ $\quad \dfrac{-2x}{\sqrt{2x^2-3}} = \dfrac{-2\cancel{x}}{\cancel{x}\sqrt{2-3x^{-2}}} = \dfrac{-2}{\sqrt{2-3x^{-2}}}$

thus $\displaystyle\lim_{x\to\infty} \frac{-2x}{\sqrt{2x^2-3}} = \lim_{x\to\infty} \frac{-2}{\sqrt{2-3x^{-2}}} = \frac{-2}{\sqrt{2-0}} \boxed{= -\sqrt{2}}$

Solution 2: $\displaystyle\lim_{x\to\infty} -2x = -\infty$, $\displaystyle\lim_{x\to\infty} \sqrt{2x^2-3} = \infty$

$$\lim_{x\to\infty} \frac{-2x}{\sqrt{2x^2-3}} \overset{\frac{-\infty}{\infty}}{=} \lim_{x\to\infty} \left(\frac{-2}{\frac{2x}{\sqrt{2x^2-3}}}\right) = \lim_{x\to\infty} -\frac{\sqrt{2x^2-3}}{x} = \lim_{x\to\infty} -\sqrt{2-3x^{-2}} \boxed{= -\sqrt{2}}$$

2. First, to be continuous we need $\displaystyle\lim_{x\to 0^+} g(x) = \lim_{x\to 0^-} g(x) = g(0)$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x+b = b = g(0)$$

$$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} a^x = 1$$

Thus we need $b = 1$.

To be differentiable, we need $\displaystyle\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^-} \frac{g(x)-g(0)}{x}$

$$\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} = \lim_{x\to 0^+} \frac{x+1-1}{x} = 1$$

$$\lim_{x \to 0^-} \frac{g(x)-g(0)}{x} = \lim_{x \to 0^-} \frac{a^x - 1}{x} = \ln(a)$$

thus we need $\ln(a) = 1$, i.e. $a = e$.

3. Let V be the volume of the balloon and r be the radius.

Then $V = \frac{4\pi}{3} r^3$ thus when $V = 100 \text{ cm}^3$, $r = \left(\frac{3}{4\pi} 100\right)^{1/3} \text{cm}$

Implicitly differentiation $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

thus $\frac{dr}{dt} = \frac{1}{4\pi} r^{-2} \frac{dV}{dt} \boxed{= \frac{1}{4\pi} \left(\frac{3}{4\pi} 100\right)^{-2/3} (5) \text{ cm}^3/\text{s}}$

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4. Let $a, b, c \in (-\infty, +\infty)$. Show that there exists $x \in (0,1)$ such that:

$$4ax^3 + 3bx^2 + 2cx = a + b + c.$$

Solution:

Let $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(x) = ax^4 + bx^3 + cx^2 - (a+b+c)x$$

φ is differentiable, and $\varphi(0) = 0 = \varphi(1)$. It follows that we can apply Rolle's Theorem and thus it exists it exists some $x \in (0,1)$ such that $\varphi'(x) = 0$ which is exactly what we had to show.

5. Determine the following antiderivatives :

(a) $\displaystyle\int \frac{dt}{\sqrt{t} + \sqrt{t^3}}$

(b) $\displaystyle\int \frac{\ln t}{t + t(\ln t)^2}\, dt$

(c) $\displaystyle\int \frac{e^{2t}}{e^t + 1}\, dt$

Solution:

(a)

$$\int \frac{1}{\sqrt{t}+\sqrt{t^3}}\, dt \underset{u=\sqrt{t}}{=} \int \frac{2u\, du}{u + u^3} = \int \frac{2\, du}{1+u^2} = 2\arctan(u) + C = 2\arctan(t) + C$$

(b)

$$\int \frac{\ln t}{t + t(\ln t)^2}\, dt \underset{u=\ln t}{=} \int \frac{ue^u\, du}{e^u + e^u u^2} = \int \frac{u\, du}{1+u^2} = \frac{1}{2}\ln(1+u^2) + C == \frac{1}{2}\ln(1 + (\ln t)^2) + C$$

(c)

$$\int \frac{e^{2t}}{e^t+1}\, dt \underset{u=e^t}{=} \int \frac{u\, du}{u+1} = \int \left(1 - \frac{1}{u+1}\right) du = u - \ln(1+u) + C = e^t - \ln(1+e^t) + C$$

6. Calculate the following integrals:

(a) $\displaystyle\int_0^1 \arctan t\, dt$

(b) $\displaystyle\int_0^{1/2} \arcsin t\, dt$

(c) $\displaystyle\int_0^1 t\arctan t\, dt$

Solution:

By integration by parts with $u = \arctan(t)$, $du = \frac{1}{1+t^2}dt$, $dv = 1$ and $v = t$ one has:

(a)

$$\begin{aligned}\int_0^1 \arctan(t)\,dt &= [t\arctan(t)]_0^1 - \int_0^1 \frac{t}{1+t^2}\,dt\\ &= \frac{\pi}{4} - \frac{1}{2}\left[\ln(1+t^2)\right]_0^1\\ &= \frac{\pi}{4} - \frac{\ln 2}{2}.\end{aligned}$$

(b) By integration by parts with $u = \arcsin(t)$, $du = \frac{1}{\sqrt{1-t^2}}dt$, $dv = 1$ and $v = t$ one has:

$$\begin{aligned}\int_0^{1/2} \arcsin(t)\,dt &= [t\arcsin(t)]_0^{1/2} - \int_0^{1/2} \frac{t}{\sqrt{1-t^2}}\,dt\\ &= \frac{\pi}{12} + \left[\sqrt{1-t^2}\right]_0^{1/2}\\ &= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.\end{aligned}$$

(c) By integration by parts, with $u = \arctan(t)$, $du = \frac{1}{1+t^2}dt$, $dv = t\,dt$ and $v = \frac{t^2}{2}$ one has

$$\begin{aligned}\int_0^1 t\arctan(t)\,dt &= \frac{1}{2}\left[t^2\arctan(t)\right]_0^1 - \frac{1}{2}\int_0^1 \frac{t^2}{1+t^2}\,dt\\ &= \frac{\pi}{8} - \frac{1}{2}[t - \arctan(t)]_0^1 = \frac{\pi}{4} - \frac{1}{2}.\end{aligned}$$

7. Let $f : (-\infty, +\infty) \to (-\infty, +\infty)$ be a continuous function. The function $F(x)$ is defined as follows

$$F(x) = \int_1^x \left(t\int_1^t f(s)\,ds\right)dt.$$

Compute $F'(1)$.

Solution:

By the Fundamental Theorem of Calculus, we have that

$$F'(x) = x\int_1^x f(s)\,ds,$$

from which we can see that

$$F'(1) = 1\cdot\int_1^1 f(s)\,ds = 0.$$