Recitation 20 Solutions

1. (a) We first split $[2,5]$ into n intervals by letting $\Delta x = \frac{5-2}{n} = \frac{3}{n}$, $x_0 = 2$

$x_i = x_0 + i\Delta x = 2 + i\frac{3}{n}$ so that $x_n = 5$. Then choosing the right-endpoint approximation

$$\int_2^5 f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n (4-2x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left(4-2\left(2+i\tfrac{3}{n}\right)\right)\tfrac{3}{n}$$

We then note that $\sum_{i=1}^n (\cancel{4}-2(\cancel{2}+i\frac{3}{n}))\frac{3}{n} = \sum_{i=1}^n (-2i\frac{3}{n})\frac{3}{n} = -\frac{18}{n^2}\sum_{i=1}^n i = -\frac{18}{n^2}\frac{n(n+1)}{2} = -9\frac{n(n+1)}{n^2}$,

Thus

$$\int_2^5 f(x)\,dx = \lim_{n\to\infty} -9\frac{n(n+1)}{n^2} = -9.$$

The average value of f on $[2,5]$ is thus equal to $\frac{-9}{5-2} = -3$.

(b) We first split $[0,2]$ into n intervals by letting $\Delta x = \frac{2-0}{n} = \frac{2}{n}$, $x_0 = 0$

$x_i = x_0 + i\Delta x = i\frac{2}{n}$ so that $x_n = 2$. Then choosing the right-endpoint approximation

$$\int_0^2 g(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n g(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n (2x_i - x_i^2)\Delta x = \lim_{n\to\infty} \sum_{i=1}^n \left(2\cdot i\tfrac{2}{n} - \left(i\tfrac{2}{n}\right)^2\right)\tfrac{2}{n}$$

We then note $\sum_{i=1}^n \left(2\cdot i\frac{2}{n} - \left(i\frac{2}{n}\right)^2\right)\frac{2}{n} = \sum_{i=1}^n \left(i\frac{4}{n} - i^2\frac{4}{n^2}\right)\frac{2}{n} = \frac{8}{n^2}\sum_{i=1}^n i - \frac{8}{n^3}\sum_{i=1}^n i^2 = 4\frac{n(n+1)}{n^2} - \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3}$

Thus

$$\int_0^2 g(x)\,dx = \lim_{n\to\infty} 4\frac{n(n+1)}{n^2} - \frac{4}{3}\frac{n(n+1)(2n+1)}{n^3} = 4 - \frac{8}{3} = \frac{4}{3}$$

The average value of g on $[0,2]$ is thus equal to $\frac{\frac{4}{3}}{2-0} = \frac{2}{3}$.

2. The Fundamental Theorem of Calculus shows that differentiation and integration are inverse processes in that

$$f(x) \xrightarrow{\int} \int_a^x f(t)\,dt \xrightarrow{\frac{d}{dx}} \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$f(x) - f(a) \xrightarrow{\frac{d}{dx}} f'(x) \xrightarrow{\int} \int_a^x f'(t)\,dt = f(x) - f(a)$$

Thus integrating and then differentiating gives the same function back or differentiating and then integrating gives the same function back.

3. (a) The FTC1 implies that

$$g'(x) = \frac{d}{dx}\int_0^x (2+\sin(t))\,dt = 2+\sin(x).$$

(b) The FTC2 says that

$$g(x) = F(x) - F(0) \quad \text{where } F(x) \text{ is an antiderivative of } 2+\sin(x).$$

We have $\underbrace{\int 2+\sin(x)\,dx}_{\text{indefinite integral}} = 2x - \cos(x) + C$. Thus

$$g(x) = 2x - \cos(x) + C - (2\cdot 0 - \cos(0) + C)$$
$$= 2x - \cos(x) + 1$$

We therefore find that $g'(x) = 2+\sin(x)$ as in (a).

4. (a) First we compute that

$$F(x) = \int\left(\frac{4}{5}x^3 - \frac{3}{4}x^2 + \frac{2}{5}x\right)dx = \frac{1}{5}x^4 - \frac{1}{4}x^3 + \frac{1}{5}x^2 + C$$

thus the FTC2 implies that

$$\int_0^2 \left(\frac{4}{5}t^3 - \frac{3}{4}t^2 + \frac{2}{5}t\right)dt = F(2) - F(0)$$

$$= \frac{1}{5}2^4 - \frac{1}{4}2^3 + \frac{1}{5}2^2 + C - (0 - 0 + 0 + C)$$

$$= \frac{16}{5} - \frac{8}{4} + \frac{4}{5} = 2$$

(b) Again we compute the antiderivative

$$F(\theta) = \int \sin(\theta)\,d\theta = -\cos(\theta) + C$$

thus $\displaystyle\int_{\pi/6}^{\pi} \sin(\theta)\,d\theta = F(\pi) - F\left(\frac{\pi}{6}\right)$

$$= -\cos(\pi) - \left(-\cos\left(\frac{\pi}{6}\right)\right) = 1 + \frac{\sqrt{3}}{2}$$

(c) Again we compute the antiderivative

$$F(x) = \int 2^x \, dx = \frac{2^x}{\ln(2)} + C$$

thus $$\int_{-3}^{e} 2^x \, dx = F(e) - F(-3)$$

$$= \frac{2^e - 2^{-3}}{\ln(2)}$$

5. (a) FTC1 implies that

$$\frac{d}{dx} F(x) = \frac{d}{dx} \int_3^x e^{-t^2} \, dt = e^{-x^2}$$

(b) Let $H(x) = \int_0^x \ln(2+u) \, du$, so that $H'(x) = \ln(2+x)$.

Then $G(x) = H(\sin(x))$ hence

$$G'(x) = H'(\sin(x)) \frac{d}{dx} \sin(x)$$

$$= \ln(2+\sin(x)) \cos(x).$$

(c) Letting $L(x) = \int_0^x (u^2 - u) \, du$ so that $L'(x) = x^2 - x$

$$K(x) = L(1-2x) - L(x^2)$$

$$K'(x) = L'(1-2x) \cdot (-2) - L'(x^2) \cdot 2x$$

$$= \left((1-2x)^2 - (1-2x)\right) \cdot (-2) - (x^4 - x^2) \cdot 2x$$