21-120: Differential and Integral Calculus
Lecture #2 Outline

Read: Section 2.2 of the textbook

Objectives and Concepts:

- The limit of a function f at a number a represents the behavior of the function for input values x near a. This limit may or may not exist.
- Left and right-hand limits are defined by examining values of x that approach a from either the left or the right side, respectively.
- We define the limit to be a number L if the values of f approach L as x approaches a from both sides of a.
- We define the limit to be $\pm\infty$ if the values of f grow (either positively or negatively) without bound as x approaches a from one or both sides.
- A function that has a (one-sided or two-sided) infinite limit at a has the vertical asymptote $x = a$.

Suggested Textbook Exercises:

- 2.2: 46-80 all.

The Limit of a Function

As we were trying to describe the instantaneous rate of change of a curve $y = f(x)$, we found that the slope of the tangent line could be described by some sort of limiting process. This is the limit of a function that we formally define and present notation for below.

Definition: We write
$$\lim_{x\to a} f(x) = L$$
and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a (on either side of a) but not equal to a.

We use the arrow symbol "→" to represent the word "approach." The symbols "$x \to a$" indicates that x is getting closer and closer to a (but x is not equal to a). If x is approaching a from the **right** side (that is, through numbers that are larger than a), then we write
$$x \to a^+.$$
Similarly, if x is approaching a from the **left** side (that is, through numbers that are smaller than a), then we write
$$x \to a^-.$$
If the limits from both directions are equal to the number L, we say $f(x) \to L$ as $x \to a$.

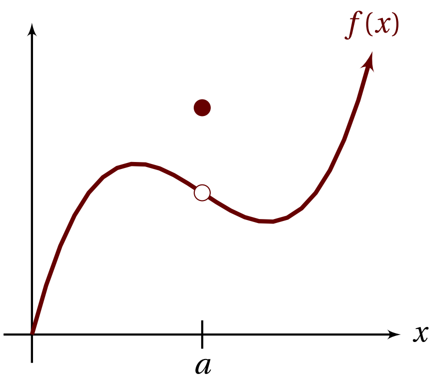
Theorem: A function $f(x)$ has a limit as x approaches a if and only if it has a left-side limit, it has a ride-side limit, and these one-sided limits agree.

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L \text{ and } \lim_{x\to a^+} f(x) = L.$$

$\lim_{x\to a} f(x) = L$ **if** f **approaches the same value** L **from both sides of** a**.**

Limits of a function at a point can be observed by examining at the graph of the function, if it is available.

Example: Consider the function $f(x)$ graphed below. Are the following statements true or false? Why?



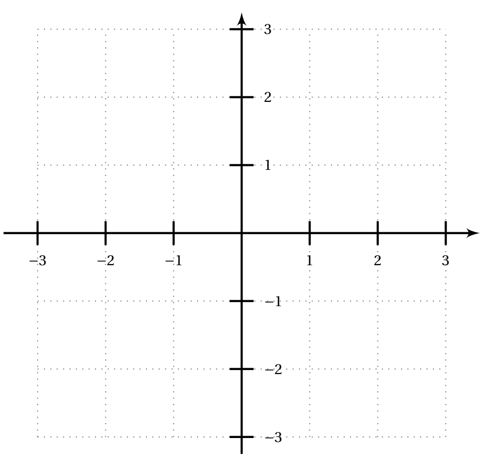
_____ $x = a$ is in the domain of $f(x)$.

_____ $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

_____ $\lim_{x\to a} f(x)$ exists.

_____ $\lim_{x\to a} f(x) = f(a)$.

Example: Graph the function $f(x) = \dfrac{|2-2x|}{x-1}$ and find the indicated limits.



$\lim_{x\to -1} f(x) =$ ____________

$\lim_{x\to 1^-} f(x) =$ ____________

$\lim_{x\to 1^+} f(x) =$ ____________

$\lim_{x\to 1} f(x) =$ ____________

Infinite Limits and Vertical Asymptotes

In many cases, a function's values may increase without bound as x approaches a from the left, the right, or from both sides. Whenever we find that a function has an infinite limit at a value a, the function will have a **vertical asymptote** there.

Definition: Let $f(x)$ be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the value of $f(x)$ increases without bound as x approaches a. Similarly,

$$\lim_{x \to a} f(x) = -\infty$$

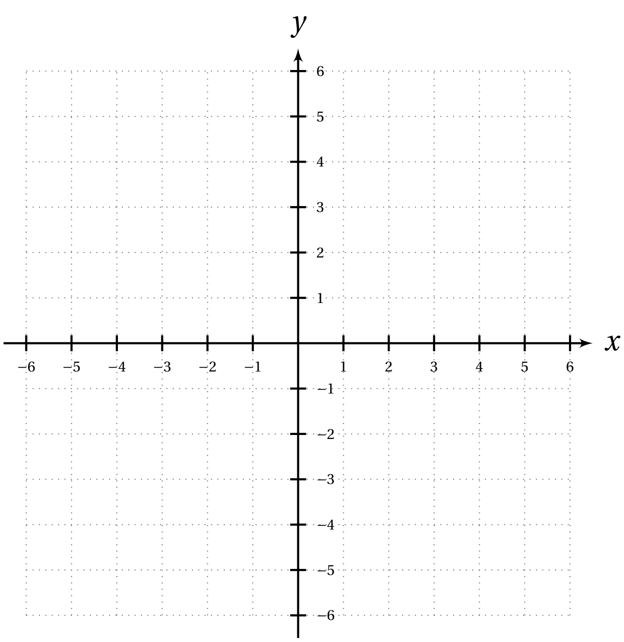
means that the value of $f(x)$ decreases without bound as x approaches a.

Definition: The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

Example: Sketch the graph of a function that satisfies all of the given conditions.



$f(1) = 1$ and $f(4) = -1$

$$\lim_{x \to 1} f(x) = 3$$

$\lim_{x \to 4^-} f(x) = 3$ and $\lim_{x \to 4^+} f(x) = -3$

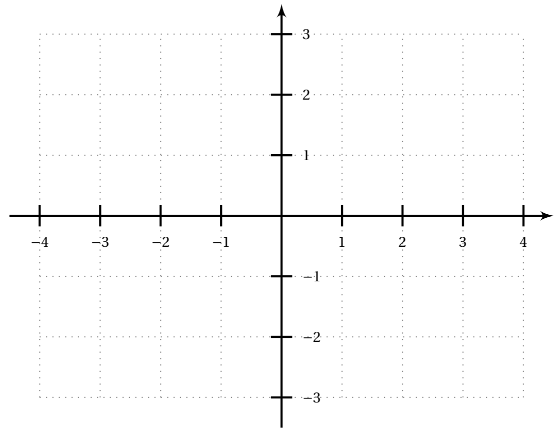
$\lim_{x \to -4^+} f(x) = \infty$ and $\lim_{x \to -4^-} f(x) = -\infty$

Example: Determine the infinite limit.

$$\lim_{x \to 2^+} \ln(x-2)$$

Example: Graph the piecewise function given below. Determine the values of a for which $\lim_{x\to a} f(x)$ does not exist.

$$f(x) = \begin{cases} -2, & x < -1 \\ x+1, & -1 \le x < 0 \\ 1, & x = 0 \\ 1-x & 0 < x \le 2 \\ 0, & x > 2 \end{cases}$$



Example: Evaluate the limits:

a) $\lim_{x\to 1^+} \frac{x}{x-1}$

b) $\lim_{x\to 1^-} \frac{x}{x-1}$

c) $\lim_{x\to 1} \frac{x}{x-1}$

d) $\lim_{x\to -1^-} \frac{x}{x-1}$

Example: Evaluate the following limits:

a) $\lim_{x\to (\pi/2)^+} \tan x$

b) $\lim_{x\to (\pi/2)^-} \tan x$

c) $\lim_{x\to 0^-} \cot x$

d) $\lim_{x\to 0} \cot x$

Example: Evaluate the limits:

a) $\lim_{x\to 1^+} \frac{x^2+1}{x^2-1}$

b) $\lim_{x\to 1^-} \frac{x^2+1}{x^2-1}$

c) $\lim_{x\to 1} \frac{x^2+1}{x^2-1}$

d) $\lim_{x\to -1^-} \frac{x^2+1}{x^2-1}$