21-120: Differential and Integral Calculus
Lecture #19 Outline

Read: Section 4.5 of the textbook

Objectives and Concepts:

- The sign of the first derivative of a function $f(x)$ indicates where the graph of f is increasing and decreasing, and can also indicate when a local maximum or minimum value is achieved.
- The second derivative of a function $f(x)$ indicates where the graph of f is concave up or concave down, and the sign of the second derivative can also be used to classify extreme values.

Suggested Textbook Exercises:

- 4.5: 195-229 odd, 241-245 odd.

Derivatives and the Shape of a Graph

Back when we first described the derivative of a function f defined on an interval I, we indicated that the derivative can tell us when a function is increasing or decreasing, namely when the first derivative is positive or negative:

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test: Suppose c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

(b) If f' changes from negative to positive at c, then f has a local minimum at c.

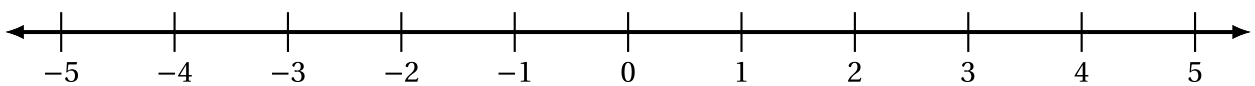
(c) If f' does not change sign at c, then f has no local maximum or minimum at c.

How to Find Local Extrema Using 1st Derivative Test:

1. Find all critical points of f and divide the interval I into smaller intervals using the critical points as endpoints.
2. Analyze the sign of f' in each of the subintervals. If f' is continuous over a given subinterval (which is typically the case), then the sign of f' in that subinterval does not change and, therefore, can be determined by choosing an arbitrary test point x in that subinterval and by evaluating the sign of f' at that test point. Use the sign analysis to determine whether f is increasing or decreasing over that interval.
3. Use the First Derivative Test and the results of step 2 to determine whether f has a local maximum, a local minimum, or neither at each of the critical points.

Example 1: Find where f is increasing and where it is decreasing. Find the local extrema points (if any).

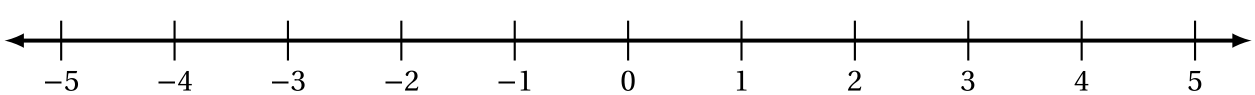
$$f(x) = 2x^3 - 3x^2 - 12x + 7$$



- Increasing: ______________________
- Decreasing: ______________________
- Local max: ______________________
- Local min: ______________________

Example 2: Find where $f(x)$ is increasing and decreasing. Find the local extrema points.

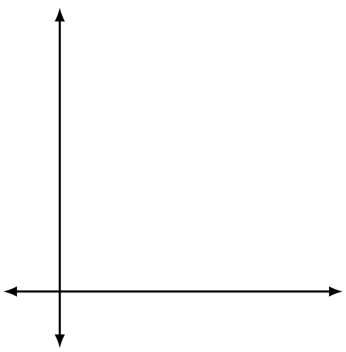
$$f(x) = x^2 e^x$$

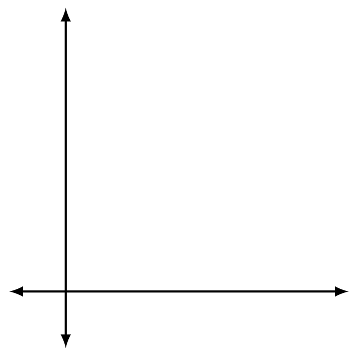


- Increasing: ______________________
- Decreasing: ______________________
- Local max: ______________________
- Local min: ______________________

Definition: If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I. A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes concavity at P.

How does concavity relate to the rate of change of the first derivative?





The Concavity Test:

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

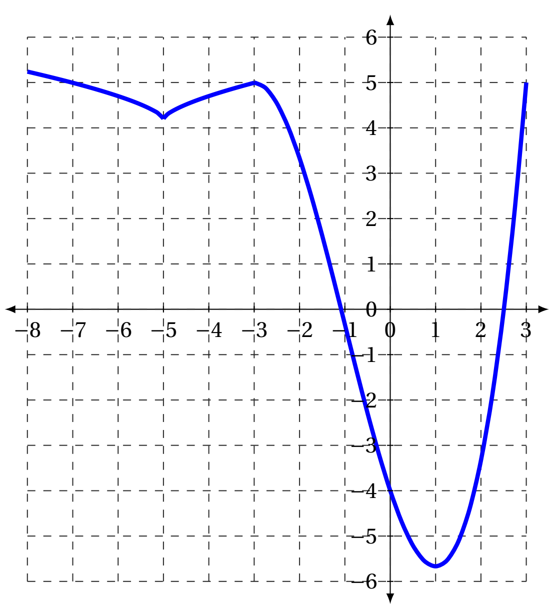
(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

The Second Derivative Test: Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example 3: Use the graph of f to find the following:



(a) Find the intervals where f is concave up.

(b) Find the intervals where f is concave down.

(c) Find the function's inflection points.

(d) Identify the function's local extreme values.

Example 4: Determine the intervals where $f(x)$ is increasing, decreasing, concave down, concave up, and its extrema and inflection points.

$$f(x) = \frac{x}{(x-1)^2}$$

- Increasing: ____________________________
- Decreasing: ____________________________
- Concave up: ____________________________
- Concave down: ____________________________
- Local max: ____________________________
- Local min: ____________________________
- Inflection pts: ____________________________

Example 5: Find all local extrema, intervals of increase/decrease, concavity, inflection points, etc.

$$f(x) = x - 2\sin(x) \quad \text{over } [0, 2\pi]$$

- Increasing: ____________________
- Decreasing: ____________________
- Concave up: ____________________
- Concave down: ____________________
- Local max: ____________________
- Local min: ____________________
- Inflection pts: ____________________