21-120: Differential and Integral Calculus
Recitation #12 Outline: 09/03/24

1. A conical tank is being filled from its vertex at a rate of 9 liters per second. Knowing that the height of the tank is 10 meters and the radius of the top is 5 meters, how fast is the water level rising when it has reached a depth of 6 meters?
Solution: We will express all measurements in meters. If $V(t)$ is the volume of water in the tank at time t measured in seconds, we are given that

$$V'(t) = \frac{9}{10^3} m^3/\text{s}.$$

We know that

$$V(t) = \frac{1}{3}\pi r(t)^2 h(t),$$

where $h(t)$ is the height, measured from the vertex, reached by the water at time t, and $r(t)$ is the radius of the cross-section of the cone at the distance $h(t)$ from the vertex. By similarity of triangles, we deduce that

$$\frac{r}{R} = \frac{h}{H},$$

from which it follows that

$$r = r(t) = \frac{R}{H}h(t) = \frac{1}{2}h(t).$$

Then,

$$V(t) = \frac{1}{12}\pi h(t)^3,$$

and

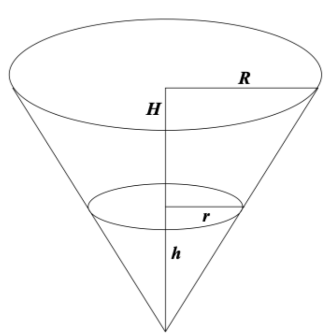
$$V'(t) = \frac{9}{10^3} = \frac{\pi}{4}h(t)^2 h'(t).$$

Now, when $h(t_0) = 6$, we deduce that

$$\frac{9}{10^3} = \frac{\pi}{4}\cdot 36 \cdot h'(t_0),$$

which means

$$h'(t_0) = \frac{1}{10^3\pi}\,\text{m/s} \quad \text{or} \approx 1.146\,\text{m/h}.$$



2. The volume of a cube is increasing at a rate of 70 cm^3 per minute. How fast is the surface area increasing when the length of a side is 12 cm?

   Solution:

   Let $V(t)$ be the volume of the cube, measured in cubic centimeters, at time t, measured in minutes. If $L(t)$ is the length in centimeters of a side at time t, we have $V(t) = L(t)^3$, from which it follows that $L'(t) = \dfrac{V'(t)}{3L(t)^2}$. Given that $V'(t) = 70\text{cm}/\text{min}$, we deduce that when $L(t_0) = 12$, $L'(t_0) = \frac{70}{3(12)^2}$. The surface area of the cube is given by $S(t) = 6L(t)^2$, and we deduce that

$$S'(t_0) = 12L(t_0)L'(t_0) = \frac{70}{3}\,\text{cm}^2/\text{min}.$$

3. A spherical ice ball is melting uniformly over its surface at a rate of 50 cm^3 per minute. How fast is the radius of the ball decreasing when it measures 15 cm?

   Solution:

   The volume of the sphere at time t minutes is given by

$$V(t) = \frac{4}{3}\pi r(t)^3 \text{ cubic centimeters.}$$

   We are told that $V'(t) = -50$. We deduce that

$$-50 = 4\pi r(t)^2 r'(t).$$

   If $r(t_0) = 15$, it follows that

$$r'(t_0) = -\frac{50}{4\pi(15)^2} = -\frac{1}{18\pi}\,\text{cm}/\text{min}.$$

   The derivative is negative, as it should be, since the radius is decreasing.

4. A point P moves along the portion of the parabola $x = y^2$ located in the first quadrant, such that its x-coordinate is increasing at a rate of 5 cm/s. Calculate the rate at which the point P is moving away from the origin when $x = 9$.

   Solution:

   Let $(x(t), y(t))$ be the coordinates, measured in centimeters, of point P at time t measured in seconds. We are given that $y(t) \geq 0$ and that $x(t) = y(t)^2$. The distance from point P to the origin is given by

$$f(t) = \sqrt{x(t)^2 + y(t)^2}$$

   Therefore,

$$f'(t) = \frac{x(t)x'(t) + y(t)y'(t)}{\sqrt{x(t)^2 + y(t)^2}}$$

   What we need is $f'(t_0)$ knowing that $x(t_0) = 9$. In this case, it must be that $y(t_0) = 3$. We also know $x'(t) = 5\,(\text{cm}/\text{s})$. With this, it is easy to deduce the value of $y'(t_0) = \dfrac{x'(t_0)}{2y(t_0)} = \dfrac{5}{6}$. Finally,

$$f'(t_0) = \frac{x(t_0)x'(t_0) + y(t_0)y'(t_0)}{\sqrt{x(t_0)^2 + y(t_0)^2}} = \frac{45 + 3(5/6)}{\sqrt{81+9}} = \frac{19\sqrt{10}}{12}\text{cm}/\text{s}$$