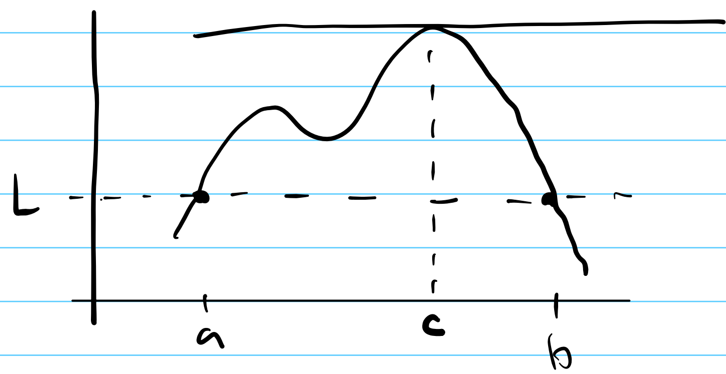
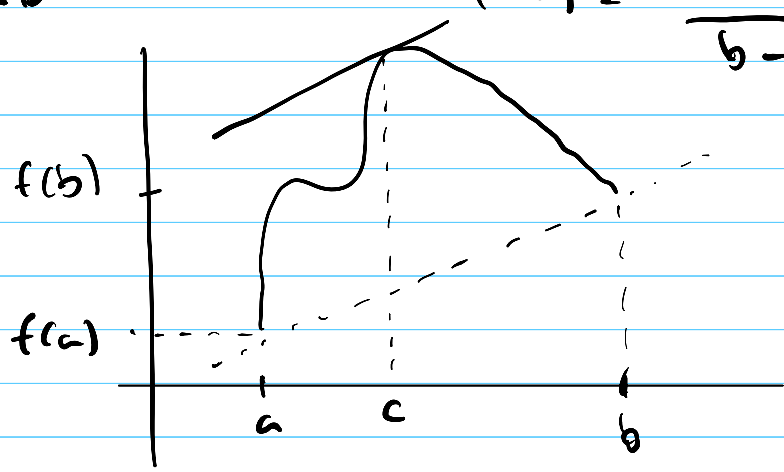
Recitation #14

1. First let's remember what Rolle's Theorem and the MVT state.

$\boxed{\text{Rolle's Theorem}}$ Let f be a continuous function on $[a,b]$ and differentiable on (a,b) so that $f(a)=f(b)$. Then there exists $a<c<b$ so that $f'(c)=0$.



$\boxed{\text{Mean Value Theorem}}$ Let f be a continuous function on $[a,b]$ and differentiable on (a,b). Then there exists $a<c<b$ so that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

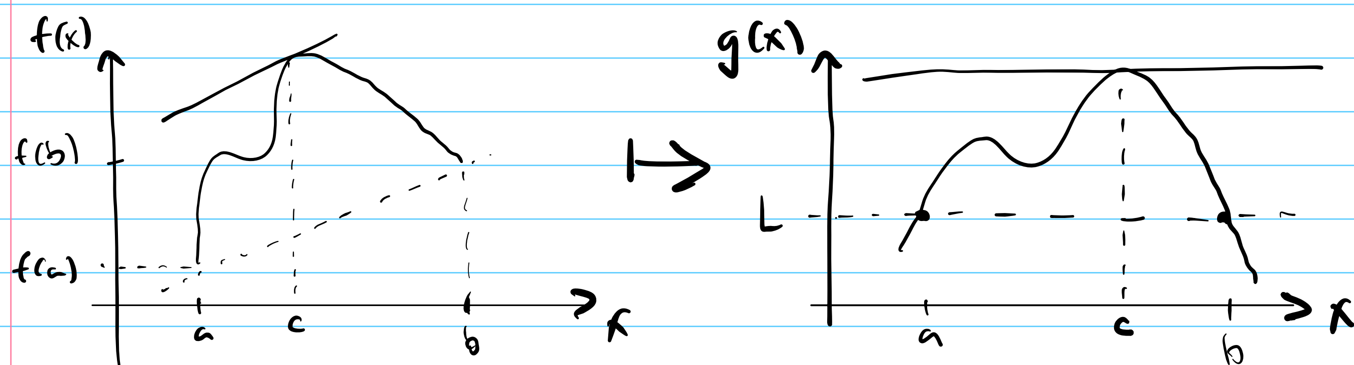


Our goal is thus to show that the MVT is true using Rolle's Theorem. Given a function f satisfying the hypotheses of the MVT we will do this by constructing a new function g that satisfies the hypotheses of Rolle's Theorem.

If f is continuous on $[a,b]$ and differentiable on (a,b) we define a new function

$$g(x) := f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a) \right]$$

Essentially what we've done is taken



That is, $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b) since $f(x)$ is and

$$g(a) = f(a) - \left[\frac{f(b)-f(a)}{b-a}(a-a) + f(a) \right] = 0$$

$$g(b) = f(b) - \left[\frac{f(b)-f(a)}{b-a}(b-a) + f(a) \right] = 0$$

Thus g satisfies the conditions of Rolle's Theorem. Rolle's theorem then implies that there exists $c \in (a,b)$ so that

$$g'(c) = 0$$

But

$$g'(x) = \frac{d}{dx}\left(f(x) - \left[\frac{f(b)-f(a)}{b-a}(x-a) + f(a)\right]\right)$$

$$= f'(x) - \left(\frac{f(b)-f(a)}{b-a}\right)$$

Hence

$$0 = g'(c) = f'(c) - \left(\frac{f(b)-f(a)}{b-a}\right)$$

that is

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

We've just shown the conclusion of the MVT!

2. (a) $\cos(x)$ is continuous and differentiable on $\mathbb{R}$, thus $\cos(x)$ is continuous on $[\frac{\pi}{2}, \frac{5\pi}{2}]$ and differentiable on $(\frac{\pi}{2}, \frac{5\pi}{2})$.

$$\frac{\cos(\frac{5\pi}{2}) - \cos(\frac{\pi}{2})}{\frac{5\pi}{2} - \frac{\pi}{2}} = \frac{0-0}{2\pi} = 0$$

The MVT implies that there exists c in $(\frac{\pi}{2}, \frac{5\pi}{2})$ so that $f'(c) = 0$.

$\frac{d}{dx}\cos(x) = -\sin(x) = 0$ when $x = 2n + k\pi$

Thus $f'(c) = 0$ for $\frac{\pi}{2} < c < \frac{5\pi}{2}$ when $c = \pi$ or $c = 2\pi$.

(b) $\frac{x}{x+1}$ is continuous and differentiable when $x \neq -1$, thus it is continuous on $[0,3]$ and differentiable on $(0,3)$.

$$\frac{f(3)-f(0)}{3-0} = \frac{\frac{3}{4}-0}{3} = \frac{1}{4}$$

Thus the MVT says there exist c in $(0,3)$ so that $f'(c) = \frac{1}{4}$.

$$\frac{d}{dx}\frac{x}{x+1} = \frac{1}{(x+1)^2}$$

Thus $f'(c) = \frac{1}{4}$ when

$\frac{1}{(c+1)^2} = \frac{1}{4}$ hence $(c+1)^2 = 4$

hence $c+1 = \pm 2$

hence $c = 1$ or -3

Thus in $[0,3]$ $f'(c) = \frac{1}{4}$ when $c=1$.

3. $f(3) - f(0) = 2 - |2\cdot 3 - 1| - (2 - |2\cdot 0 - 1|)$

$= 2 - |6-1| - (2 - |-1|)$

$= 2 - 5 - 2 + 1$

$= -4$

$$f'(x) = 0 - 2\frac{d}{dx}|x - \tfrac{1}{2}| = \begin{cases} 2 & x < \frac{1}{2} \\ -2 & x > \frac{1}{2} \end{cases}$$

$-2 \cdot 3 \neq -4, \quad 2 \cdot 3 \neq -4$

thus $\quad f'(c) \cdot (3-0) \neq f(3) - f(0)$

for all $\quad c \neq \frac{1}{2}$.

This does not contradict the MVT since $f(x)$ is not differentiable at $x = \frac{1}{2}$, hence is not differentiable on $(0,3)$.

4. If $2x + \cos(x) = 0$ had two or more roots then there would exist $a < b$ so that $2a + \cos(a) = 2b + \cos(b) = 0$. Since $f(x) = 2x + \cos(x)$ satisfies the conditions of Rolle's Theorem on $[a,b]$, it must then be the case that there exists $c \in (a,b)$ so that $f'(c) = 0$. However

$$f'(x) = 2 - \sin(x) \neq 0 \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Thus this is impossible, thus $2x + \cos(x) = 0$ has at most 1 real root.

5. By the MVT there exists c in $(1,4)$ so that

$$f'(c) = \frac{f(4) - f(1)}{4-1} \qquad \text{i.e.} \qquad f(4) = f(1) + f'(c)(4-1)$$

$$= 10 + f'(c) \cdot 3$$

Since $f'(x) \geq 2$ for $1 \leq x \leq 4$ $\quad f'(c) \geq 2 \quad$ thus

$$f(4) \geq 10 + 2 \cdot 3 = 16$$

i.e. it must be the case that $f(4) \geq 16$.

6. By the MVT for all $a < b$ there exists $a < c < b$ so that

$f'(c) = \frac{f(b) - f(a)}{b - a}$ i.e. $f(b) - f(a) = f'(c)(b-a)$

i.e. $|f(b) - f(a)| = |f'(c)|\,|b-a|$

Thus if $|f'(x)| \leq M$ then $|f(b) - f(a)| \leq M\,|b-a|$ i.e. $f(x)$ must be Lipschitz.