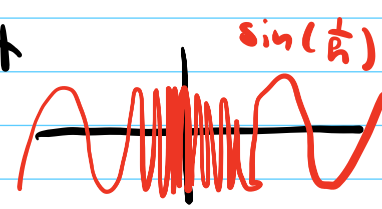
Recitation 6 Solutions

1. (a) $$f'(0) = \lim_{h\to 0} \frac{(h^2+3h-\cancel{2})-(0^2+3\cdot 0-\cancel{2})}{h}$$
$$= \lim_{h\to 0} \frac{h^{\cancel{2}}+3\cancel{h}}{\cancel{h}}$$
$$= \lim_{h\to 0} h+3 \boxed{= 3}$$

(b) $$f'(0) = \lim_{h\to 0} \frac{\cancel{h}\cos h - 0\cdot\cos 0}{\cancel{h}}$$
$$= \lim_{h\to 0} \cos h$$
$$= \cos 0 \boxed{= 1}$$

2. First we note that since $|x\sin(\frac{1}{x})| \le |x||\sin(\frac{1}{x})| \le |x|$
$$-|x| \le x\sin\left(\tfrac{1}{x}\right) \le |x|.$$
Thus since $\lim_{x\to 0} -|x| = 0 = \lim_{x\to 0} |x|$ the Squeeze Theorem implies that $\lim_{x\to 0} f(x) = 0 = f(0)$ hence f is continuous at 0.

To be differentiable at 0 the limit of $\frac{f(h)-f(0)}{h} = \frac{h\sin(\frac{1}{h})-0}{h} = \sin(\frac{1}{h})$ has to exist as $h\to 0$ but



$$\lim_{h\to 0} \sin\left(\tfrac{1}{h}\right) = \text{DNE}$$
since $\sin(\frac{1}{h})$ oscillates infinitely often as $h\to 0$.

4. (a) $f'(2) = \lim_{h \to 0} \dfrac{(2+h)^3 - 3(\cancel{2}+h) + \cancel{1} - (2^3 - \cancel{3 \cdot 2} + \cancel{1})}{h}$

$= \lim_{h \to 0} \dfrac{(4+4h+h^2)(2+h) - 3h - 8}{h}$

$= \lim_{h \to 0} \dfrac{\cancel{8} + 8\cancel{h} + 2h^{\cancel{2}} + 4\cancel{h} + 4h^{\cancel{2}} + h^{\cancel{3}\,2} - 3\cancel{h} - \cancel{8}}{\cancel{h}}$

$= \lim_{h \to 0} 8 + \cancel{2h} + 4 + \cancel{4h} + \cancel{h^2} - 3$

$= 9$

$y = f(2) + f'(2)(x-2) = 3 + 9(x-2) = -15 + 9x$

is the tangent line at $(2,3)$

(b) $f'(1) = \lim_{h \to 0} \left(\dfrac{3(1+h)+1}{(1+h)+3} - \left(\dfrac{3 \cdot 1 + 1}{1+3} \right) \right) \dfrac{1}{h}$

$= \lim_{h \to 0} \left(\dfrac{3(\cancel{1}+h)+\cancel{1} - (\cancel{1}+h)+\cancel{3}}{(1+h)+3} \right) \dfrac{1}{h}$

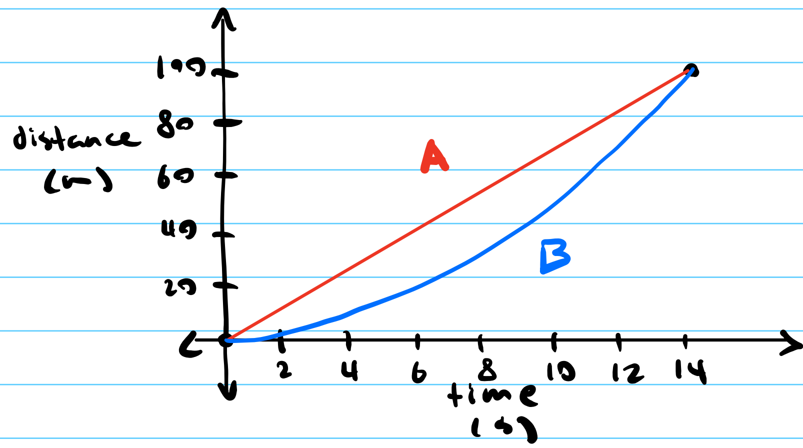
$= \lim_{h \to 0} \dfrac{2\cancel{h}}{\cancel{h}(4+h)}$

$= \dfrac{2}{4} = \dfrac{1}{2}$

$y = f(1) + f'(1)(x-1) = 1 + \frac{1}{2}(x-1) = \frac{1}{2} + \frac{1}{2}x$

is the tangent line at $(1,1)$.

5. (a) $f(0) = g(0) = 0$, $f(14) = \frac{100}{14} \cdot 14 = 100$, $g(14) = \frac{100}{196} 14^2 = 100$.



(b) Runner A ran at a constant speed while runner B's speed increased as the race went on. The two runners' tied.

(c) Velocity of runner A at time $t = f'(t)$

$$f'(t) = \lim_{h \to 0} \left(\frac{\frac{100}{14}(t+h) - \frac{100}{14}t}{h} \right) = \frac{100}{14} \text{ m/s}$$

Velocity of runner B at time $t = g'(t)$

$$g'(t) = \lim_{h \to 0} \frac{100}{14^2} \left(\frac{(t+h)^2 - t^2}{h} \right) = \lim_{h \to 0} \frac{100}{14^2} \left(\frac{\cancel{t^2} + 2\cancel{t}h + h^{\cancel{2}} - \cancel{t^2}}{\cancel{h}} \right)$$

$$= \frac{200}{14^2} t \text{ m/s}$$

$\frac{100}{14} = \frac{200}{14^2} t$ if and only if $t = \frac{14}{2} = 7$.

$\boxed{\text{Note!}}$ the velocity is in m/s since f & g's units are meters and t's units are seconds.

(d) B was running the fastest at time $t=14$ and had velocity

$$\frac{200}{14^2}(14) = \frac{200}{14} = \frac{100}{7} \text{ m/s.}$$

B would then have finished the race in

$100\text{m} / (\frac{100}{7} \text{m/s}) = 7\text{s}$ and would have

beaten runner A by 7s!

6. (a) $f'(5)$ is the instantaneous rate at which the colony is growing at 5 hours. It has units of $\frac{\text{bacteria}}{\text{hour}}$

(b) If there is unlimited space and resources, the more bacteria there are, the faster the rate at which the colony should grow at. We should thus expect that $f'(10) > f'(5)$ (in fact we should expect exponential growth).

If the supply was limited eventually the rate at which the bacteria are reproducing at should go to 0 as the population reaches an equilibrium (potentially this could be that all the bacteria die after all the resources have been used).

(c) Since $\lim_{t\to\infty} e^{-t} = 0$, this should correspond to the situation where there is limited space and resources.