21-120: Differential and Integral Calculus
Lecture #6 Outline

Read: Section 2.5 of the textbook

Objectives and Concepts:

- A more formal and rigorous mathematical definition of $\lim_{x\to a} f(x) = L$ specifies exactly how close input values must be to a in order to guarantee that output values are within a given tolerance of L.
- Graphically, the formal definition amounts to finding an open interval around a so that all of the graph of f is within a small interval of L.

Suggested Textbook Exercises:

- 2.5: 176-185.

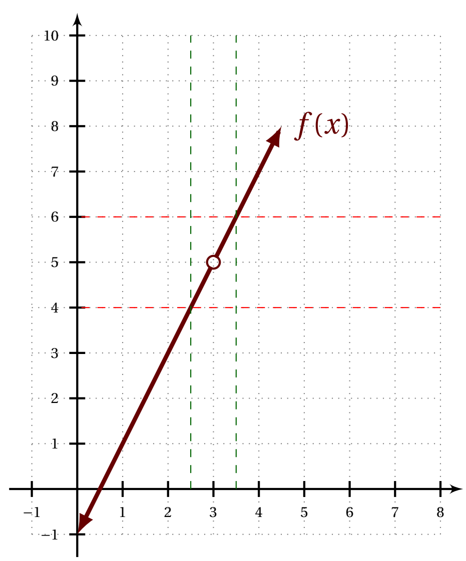
The Precise Definition of a Limit

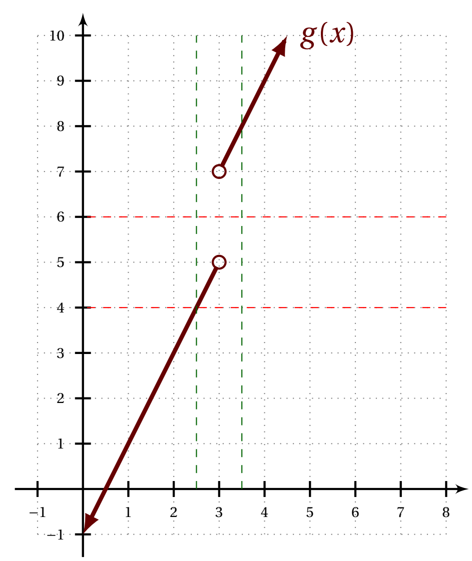
Recall that we previously defined the limit of a function as x approaches a by stating that the limit exists and is equal to L if we can make the outputs of $f(x)$ arbitrarily close to L by choosing inputs x that are sufficiently close to a. This definition was somewhat hand-wavy because we did not really define what "sufficiently close" means. Before defining this formally, let's consider a motivational example.

Consider the functions f and g graphed below. In case you're wondering what formulas they are given by, $f(x) = 2x-1$ for $x \neq 3$, and

$$g(x) = \begin{cases} 2x-1 & \text{if } x < 3, \\ 2x+1 & \text{if } x > 3. \end{cases}$$

Both f and g are undefined at $x = 3$. As you can see, f has a limit at $x = 3$, whereas g does not. How do we quantify this distinction? Here is a standard way of doing so.





First, consider f. The fact that it has a limit at $x = 3$ is justified by the following observation: if we take a horizontal strip around $y = 5$ — for example, a strip of width 2 centered at $y = 5$ (in the figure, it is bounded by the red dashed lines) — then we can always find a vertical strip around $x = 3$ (in the figiure, it is bounded by the green dashed lines) such that, no matter which x-value we choose from the vertical strip, the corresponding $f(x)$-value will lie inside the pre-chosen horizontal strip.

For instance, if you take $x = 3.1$ (which lies in the vertical strip), the corresponding $f(x)$-value is $f(3.1) = 5.2$, which falls within the horizontal strip (since the strip contains y-values between 4 and 6). Similarly, for any other x-value within the vertical strip, the corresponding $f(x)$-value will also fall inside the horizontal strip. (Also note that the choice of the vertical strip shown in the figure is not unique; any thinner vertical strip will also have the desired property. However, if you take a wider vertical strip, the desired property may no longer hold. Think about this! For example, consider the vertical strip that extends from $x = 2$ to $x = 4$ and find an x-value within this strip such that the corresponding $f(x)$-value lies outside the horizontal red strip.)

Now consider g. Unlike f, the function g does not have a limit at $x = 3$. This is reflected in the fact that, if we try to apply the same reasoning as above, it will fail. Indeed, let us again take a horizontal strip of width 2 centered at $y = 5$. In the case of f, we were able to choose a vertical strip around $x = 3$ such that all x-values from that vertical strip yielded $f(x)$-values within the horizontal strip. For g, however, this is impossible. We now explain why this is the case.

An attempt to choose a vertical strip is shown with the green dashed lines in the figure. Notice that this does not work because, if we take $x = 3.1$ (which lies within the vertical strip), the corresponding value of g, $g(3.1) = 7.2$, does not lie within the horizontal strip (which only contains y-values between 4 and 6).

We might try shrinking the vertical strip, but this will not help: there will always be points slightly to the right of $x = 3$ inside *any* vertical strip with the property that the corresponding $g(x)$-values are outside the red horizontal strip.[1] For example, if your right boundary of the vertical strip is 3.0001 and the left boundary is 2.9999 (that is, if the vertical strip is extremely slim – its width is 0.0002!), then $x = 3.0000001$ will still be within the vertical strip, but $g(3.0000001)$ will *not* fall inside the horizontal strip because $f(3.0000001) = 7.0000002$, which is not between 4 and 6.

This discussion yields the following informal definition of the limit, to be made precise momentarily.

Informal Definition: Let $f(x)$ be a function defined on an open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, written as

$$\lim_{x \to a} f(x) = L,$$

if, for any horizontal strip around $y = L$, there exists a vertical strip around $x = a$ such that all x-values within this vertical strip yield $f(x)$-values that lie within the horizontal strip (and do not go beyond it).

[1] A fine detail: the vertical strip we are talking about must include some points to the right of $x = 3$ and some points to the left of $x = 3$. And moreorver, it should be symmetric about $x = 3$. This should become clearer when we discuss the role of δ.

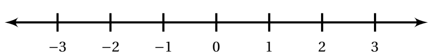
Note that above we checked that a suitable vertical strip exists for the function f if the width of the horizontal strip is 2. To prove that the limit of f at $x = 3$ indeed equals 5, we need to do more than just this. We must find a vertical strip with the desired properties stated above for *any* horizontal strip. Its like a game: someone gives you a strip around $y = L$, and you need to find a vertical strip around $x = a$ with the properties stated above. If you can always win this game, no matter what horizontal strip your opponent gives you, then the function is continuous at $x = a$.

Our next step is to make the discussion above more precise. But before doing so, we need to quickly review distances in order to formally discuss the widths of strips.

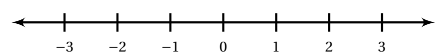
A Quick Review of Distances

Given a number x on the number line, the value $|x|$ represents the distance between x and 0 on the number line. The value $|x-3|$ represents the distance between x and 3, the value $|x+5|$ represents the distance between x and -5, and so on.

If $|x| = 3$, then $x = 3$ or $x = -3$.

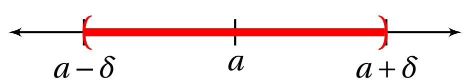


If $|x| < 3$, then $-3 < x < 3$.

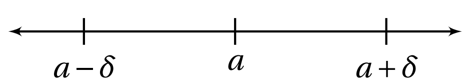


If $|x-3| < 2$, then:

In general, $|x-a| < \delta$ (for $\delta > 0$) represents all numbers within δ units of a. Graphically,



What about $0 < |x-a| < \delta$?



The formal definition of a limit

We can now state the formal definition of the limit.

> **Definition:** Let $f(x)$ be a function defined on an open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, written as
>
> $$\lim_{x\to a} f(x) = L,$$
>
> if, for every $\varepsilon > 0$ there is a number $\delta > 0$ such that
>
> $$\text{if} \quad 0 < |x-a| < \delta, \quad \text{then} \quad \left|f(x) - L\right| < \varepsilon.$$

Note that $|f(x) - L| < \varepsilon$ is equivalent to $L - \varepsilon < f(x) < L + \varepsilon$, and $0 < |x - a| < \delta$ is equivalent to $a - \delta < x < a + \delta$ together with the extra condition that $x \neq a$. So here is an equivalent version of the above definition:

> **Definition:** Let $f(x)$ be a function defined on an open interval that contains the number a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, written as
>
> $$\lim_{x\to a} f(x) = L,$$
>
> if, for every $\varepsilon > 0$ there is a number $\delta > 0$ such that
>
> $$\text{if} \quad a-\delta < x < a+\delta \text{ and } x \neq a, \quad \text{then} \quad L-\varepsilon < f(x) < L+\varepsilon.$$

You should think of ε as half the width of the horizontal strip discussed above and δ as half the width of the vertical strip. The statement

$$\text{if } a-\delta < x < a+\delta \text{ and } x \neq a, \text{ then } L-\varepsilon < f(x) < L+\varepsilon$$

says the following:

For any x-value within the vertical strip (except the x-value in the middle of the strip) of length 2δ, the corresponding $f(x)$-value will lie strictly within the horizontal strip of length 2ε without going beyond its boundaries.

Let's go back to our functions f and g from the beginning. Let's focus on f. Matching the letters in the definition and in our discussion of f, we see that $L = 5$ and $a = 3$. What about ε and δ? Remember that we took a horizontal strip of width 2, and this corresponds to ε being 1. The lower boundary of our horizontal strip was $L - \varepsilon = 5 - 1 = 4$, and the upper boundary was $L + \varepsilon = 5 + 1 = 6$. As for δ, we took $\delta = 0.5$, so the left boundary of the vertical strip was $3 - 0.5 = 2.5$ and the right boundary of the vertical strip was $3 + 0.5 = 3.5$. See the picture on the next page.

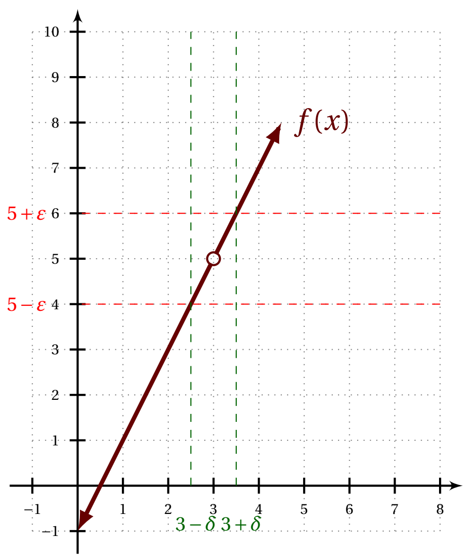
With our choice of $\varepsilon = 1$ and $\delta = 0.5$, it is clear from the picture that if $3-\delta < x < 3+\delta$ (and $x \neq 3$), then $5-\varepsilon < f(x) < 5+\varepsilon$. (Equivalently, if $0 < |x-3| < \delta$, then $|f(x)-5| < \varepsilon$.) This statement is simply saying that all x-values from the green vertical strip yield $f(x)$-values that lie inside the red horizontal strip (and do not go beyond it). We can also check this algebraically. Indeed, suppose $0 < |x-3| < \delta$. Then by the definition of f,

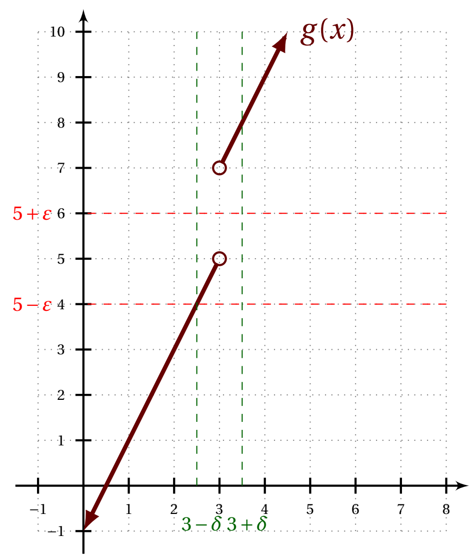
$$|f(x)-5| = \underline{\hspace{8cm}} < 2\cdot\delta.$$

But we said that $\delta = 0.5$, so it follows that

$$|f(x)-5| < 2\cdot 0.5 = 1,$$

which we wanted to prove.





But note that we haven't fully proved that $\lim_{x\to 3} f(x) = 5$ yet, because the formal definition says that for *any* ε we need to produce a δ such that something holds. But we have only done so for $\varepsilon = 1$.

Example 1: Use the ε-δ definition of the limit to *prove* the statement

$$\lim_{x\to 3} f(x) = 5,$$

where f is the function from above, i.e., $f(x) = 2x-1$ for $x \neq 3$ and $f(x)$ is undefined at $x = 3$.

First Step (Scratchwork): Do some algebra to try to find a relationship between δ and ε. Our goal is to find a number δ so that

$$\text{if } 0 < |x-a| < \delta \qquad \text{then} \qquad \big|f(x)-L\big| < \varepsilon.$$

In this case, since $a =$ ________ and $L =$ ________, our goal is to find a δ so that

We start "backwards": plug in for $f(x)$:

Simplify:

This suggests that taking $\delta =$ ________ would be a good idea...

Second Step: Doing the proof.

Claim: Let $f(x) = 2x - 1$ for $x \neq 3$ (and $f(x)$ is undefined at $x = 3$). Then

$$\lim_{x \to 3} f(x) = 5.$$

Proof:
Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{2}$. If $0 < |x - 3| < \delta$, then

$$\begin{aligned} \left| f(x) - 5 \right| &= |(2x - 1) - 5| \\ &= |2x - 6| \\ &= |2(x - 3)| \\ &= 2 \cdot |x - 3| \\ &< 2 \cdot \delta \\ &= 2 \cdot \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

Thus, if $0 < |x - 3| < \delta$, then $\left| f(x) - L \right| < \varepsilon$. Therefore, by the precise definition of a limit,

$$\lim_{x \to 3} f(x) = 5$$

The above case is pretty simple because the vertical strip was symmetric about $x = a$ (in that case we had $a = 3$). However, sometimes the x-interval induced by $|f(x) - L| < \varepsilon$ is not "centered" around a. In these cases, we have to choose the **smaller** of two distances to serve as our δ. First, we try to understand how the logic works by answering some true-false questions about inequalities.

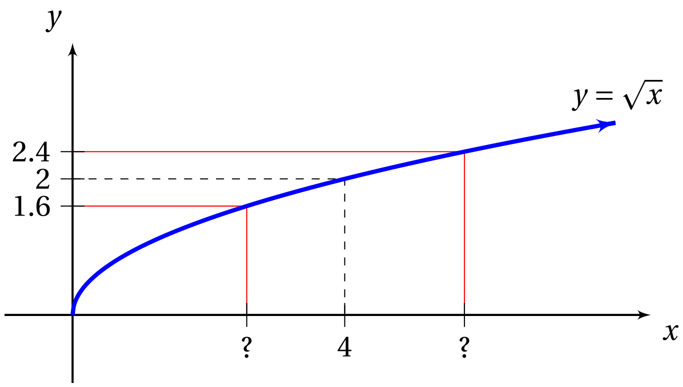
Example 2: Determine if the following statements are true or false. It may be helpful to analyze the statements using algebra or number lines.

(a) If $1 < x < 3$, then $1 < 2x - 1 < 4$.

(b) If $4 < x < 6$, then $|3x - 15| < 5$.

Example 3: Use the given graph below to find a number δ such that

$$\text{if } 0 < |x-a| < \delta \qquad \text{then} \qquad \left|f(x) - L\right| < 0.4.$$



$a =$ __________

$L =$ __________

$\varepsilon =$ __________

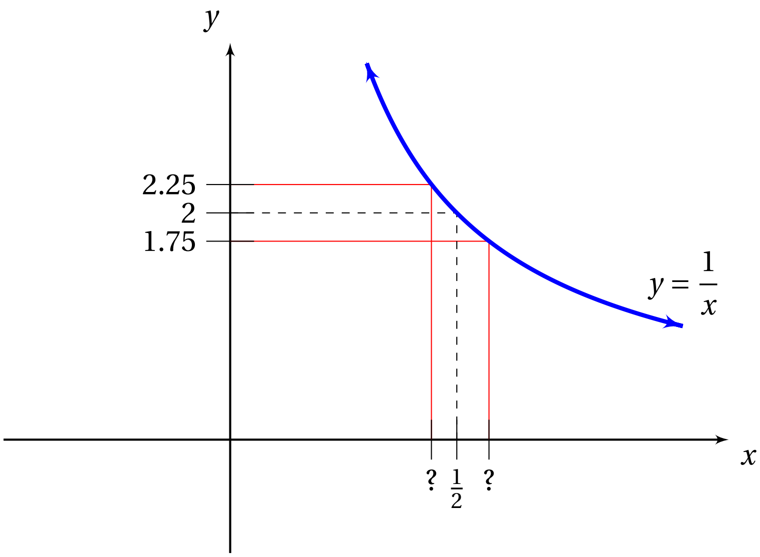
$f(x) =$ __________

$\delta =$ __________

The limit the graph is meant to display is:

Example 4: Use the given graph below to find a number δ such that

$$\text{if } 0 < |x - a| < \delta \qquad \text{then} \qquad \left|f(x) - L\right| < 0.25.$$



$a =$ __________

$L =$ __________

$\varepsilon =$ __________

$f(x) =$ __________

$\delta =$ __________

The limit the graph is meant to display is:

Example 5: Consider the following limit: $\lim_{x\to 2} x^2 = 4$. Given $\varepsilon = 1$, use the graph to find a number $\delta > 0$ such that for all x,

$$\text{if } 0 < |x - a| < \delta \qquad \text{then} \qquad \left|f(x) - L\right| < \varepsilon.$$