21-120: Differential and Integral Calculus
Recitation #18 Outline: 10/31/24

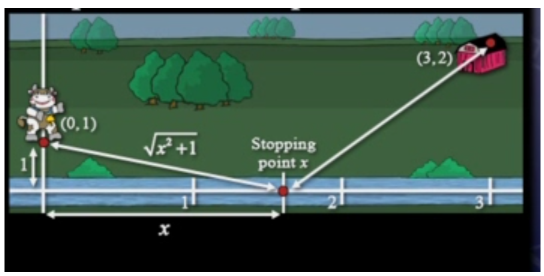
1. Show that among all rectangles with a given perimeter, the one with the largest area is a square.

   Solution:

   Let P_0 be the value of the perimeter of the rectangle, and x and y be the lengths of its sides. We have that $2x+2y=P_0$. The area is given by $xy = x\left(\frac{P_0-2x}{2}\right)$. Therefore, we need to maximize the function $f(x)=\frac{1}{2}P_0x-x^2$, where $0 \le x \le P_0$.

   Differentiating, we get $f'(x)=\frac{1}{2}P_0-2x$, which means the only zero of the derivative is $x_0=\frac{P_0}{4}$. In this case, $y_0=\frac{P_0-2x_0}{2}=x_0$, so the rectangle is, in fact, a square. It is immediate that the value obtained is an absolute maximum of f on the interval $[0,P_0]$.

2. Claudia the cow is 1 mile north of the x-axis river which runs east to west. Her barn is Her barn is 3 miles east and 1mile north of her current position . She wishes to drink from the river and the walk to her barn so as to minimize her total amount of walking. Where on the river should she stop to drink?



   Solution:

   We can position points A and B as in the figure . The length of the path APB is given by

$$f(x)=(x^2+1)^{\frac{1}{2}}+(x^2-6x+13)^{\frac{1}{2}}.$$

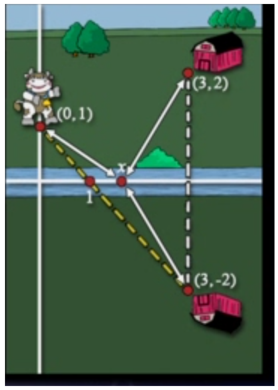
   We need to calculate the absolute minimum of $f(x)$ We have that

$$f'(x)=\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)+\frac{1}{2}(x^2-6x+13)^{\frac{-1}{2}}(2x-6)$$

   By solving $f'(x)=0$, we obtain the solution $\alpha=1$. It is easy to check that α is indeed a minimizer of the function.

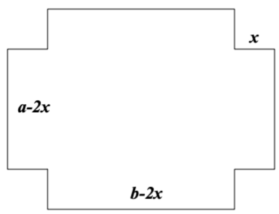
   Can be solved without calculus with a moment's reflection...

   Instead of walking to $(3,2)$ imagine she walked to $(3,-2)$ (the reflection), it is the same distance to walk thorough the drinking point to $(3,2)$ than to the drinking point to $(3,-2)$. Note that the shortest distance is the straight line !



Solution:

3. A box without a lid is to be constructed from a rectangular metal sheet by cutting out equal squares from each corner and folding the edges up. Find the dimensions of the box with the largest volume that can be constructed in this way if the sides of the rectangular sheet measure: a) 10 cm by 10 cm. b) 12 cm by 18 cm.



Solution:

Let a and b be the lengths of the sides of the sheet, and x be the length of the side of the square that will be cut from each corner. The volume of the resulting box is given by $f(x) = (a-2x)(b-2x)x$. Let us define $\gamma = \min\left(\frac{a}{2}, \frac{b}{2}\right)$.

We need to calculate the absolute maximum of the function f in the interval $[0,\gamma]$. Differentiating, we obtain $f'(x) = 12x^2 - 4(a+b)x + ab$. The zeros of the derivative are given by

$$\alpha = \frac{1}{6}\left(a+b-\sqrt{a^2+b^2-ab}\right), \quad \beta = \frac{1}{6}\left(a+b+\sqrt{a^2+b^2-ab}\right).$$

Notice that, since $a^2+b^2-ab > 0$ (this inequality follows from an old friend of ours, namely, $uv \leq \frac{1}{2}(u^2+v^2)$), the roots of f' are real.

Also observe that, since $f(0) = f(\gamma) = 0$, by the Rolle's theorem, at least one of them must be in the interval $(0,\gamma)$. Furthermore, f must achieve an absolute maximum at some point in $[0,\gamma]$, and since it is evident that this point must lie in $(0,\gamma)$, we deduce that this point is either α or β.

The second derivative test allows us to resolve any doubts. We have $f''(x) = -4(a+b-6x)$. Thus,

$$f''(\alpha) = -4(a+b-6\alpha) = -4\sqrt{a^2+b^2-ab}, \quad f''(\beta) = -4(a+b-6\beta) = 4\sqrt{a^2+b^2-ab}.$$

Therefore, $f''(\alpha) < 0$ and $f''(\beta) > 0$. We conclude that the point α is in the interval $(0,\gamma)$, and at this point, the function f reaches its absolute maximum in $[0,\gamma]$.

With some simple calculations, we obtain

$$f(\alpha) = \frac{1}{54}(-2a^3 + 3a^2b + 3ab^2 - 2b^3 + 2(a^2 - ab + b^2)^{3/2}).$$

and one can replace by the values of a, b given in the statement.

4. Calculate the dimensions (radius and height) of a cylindrical can with a capacity of one liter, whose total surface area is minimized.

   Solution:

   Let r be the radius and h the height measured in decimeters. Since the volume is $1\,\mathrm{dcm}^3$, we have $\pi r^2 h = 1$, from which it follows that $h = \frac{1}{\pi r^2}$.

   The total surface area of the can is given by $f(r) = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{2}{r}$. Therefore, we need to calculate the absolute maximum of $f(r)$ when $r > 0$.

   Differentiating, we find $f'(r) = 4\pi r - \frac{2}{r^2} = 2\frac{2\pi r^3 - 1}{r^2}$. We deduce that the derivative has a unique real zero $\alpha = \frac{1}{\sqrt[3]{2\pi}}$.

   Since $f'(r) < 0$ for $0 < r < \alpha$, it follows that f is decreasing in the interval $(0, \alpha]$; and since $f'(r) > 0$ for $\alpha < r$, it follows that f is increasing in the interval $[\alpha, +\infty)$. Consequently, $f(\alpha) \le f(r)$ for all $r > 0$. Thus, the dimensions of the can with minimal lateral surface area are $r = \frac{1}{\sqrt[3]{2\pi}} \approx 0.542\,\mathrm{dcm}$, and $h \approx 1.1\,\mathrm{dcm}$.