21-120: Differential and Integral Calculus
Lecture #10 Outline

Read: Section 3.5 of the textbook

Objectives and Concepts:

- We can use the fact that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ to evaluate other limits.
- The derivative of $\sin x$ is $\cos x$, the derivative of $\cos x$ is $-\sin x$, and these can be used to find the derivatives of the other trig functions.

Suggested Textbook Exercises:

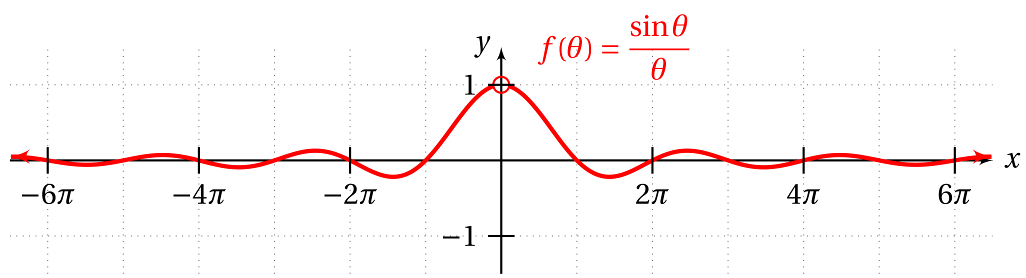
- 3.5: 175-183 odd, 191-213 odd.

Derivatives of Trigonometric Functions

In order to derive the derivatives of trig functions, we first need to examine two limits:

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta}, \qquad \lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta}.$$

Here is a graph of the function $y = \frac{\sin(\theta)}{\theta}$

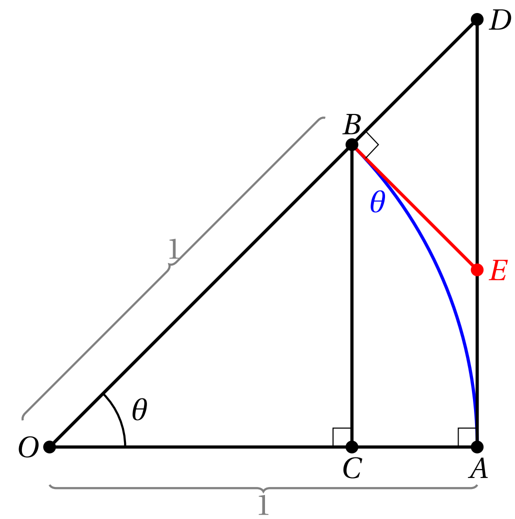


Notice this function is undefined when $\theta = 0$. Now, it is clear from the graph that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

You can also prove this limit by using some geometry and the Squeeze Theorem. Note that in the drawing below, since the hypotenuse of the triangle BCO is 1, we have that $\sin\theta$ is simply the length of the segment $\overline{BC}$. Also, since the radius of the blue arc is 1, the length of the arc from A to B is θ. It's obvious that the length of $\overline{BC}$ is less than θ, so we have that $\sin\theta < \theta$, which implies

$$\frac{\sin\theta}{\theta} < 1.$$



Now, it is clear that θ is less in length than the two segments $\overline{AE}$ and $\overline{EB}$, and we also see that $\overline{EB}$ is shorter than $\overline{ED}$. So, with absolute values representing the lengths of segments, we have

$$\theta < |\overline{AE}| + |\overline{EB}| < |\overline{AE}| + |\overline{ED}| = |\overline{AD}|.$$

Also note that since $|\overline{OA}| = 1$, we have that $\tan\theta = |\overline{AD}|$. Thus, the above inequality means

$$\theta < \tan\theta = \frac{\sin\theta}{\cos\theta},$$

or,

$$\cos\theta < \frac{\sin\theta}{\theta}.$$

Thus we have

$$\cos\theta < \frac{\sin\theta}{\theta} < 1,$$

and the Squeeze Theorem implies, since $\lim_{\theta\to 0}\cos\theta = 1$, that

Theorem: $\displaystyle\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1.$

Now, we also have

Theorem: $\displaystyle\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0.$

Proof:

$$\begin{aligned}
\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} &= \lim_{\theta\to 0}\left(\frac{\cos(\theta)-1}{\theta}\cdot\frac{\cos(\theta)+1}{\cos(\theta)+1}\right) && \text{(multiply by the conjugate of the numerator)}\\
&= \lim_{\theta\to 0}\frac{\cos^2(\theta)-1}{\theta(\cos(\theta)+1)}\\
&= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos(\theta)+1)} && \text{(because } \sin^2\theta+\cos^2\theta = 1\text{)}\\
&= \lim_{\theta\to 0}\left(\frac{-\sin\theta}{\cos(\theta)+1}\cdot\frac{\sin\theta}{\theta}\right)\\
&= \left(\lim_{\theta\to 0}\frac{-\sin\theta}{\cos(\theta)+1}\right)\cdot\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right) && \text{(because each limit exists)}\\
&= \frac{0}{2}\cdot 1 = 0.
\end{aligned}$$

We can use these theorems to compute some limits that we could previously not find.

Example 1: Compute the following limits (if possible). You may only use the facts that

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1 \qquad \text{and} \qquad \lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta}=0$$

(a) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{\sin x}$

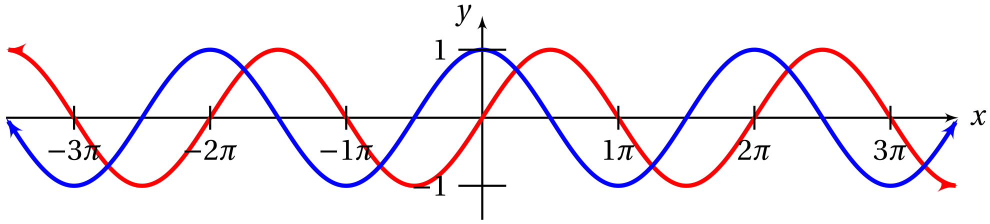
(b) $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{3x}$

(c) $\displaystyle\lim_{y\to 0}\frac{2y}{\tan y}$

(d) $\displaystyle\lim_{z\to 0}\frac{\sin(\sin z)}{\sin z}$

(e) $\displaystyle\lim_{v\to 0}\frac{\sin(5v)}{2v+\sin(4v)}$

If we examine the graph of the function $f(x) = \sin(x)$ (red) and use the interpretation of $f'(x)$ (blue) as the slope of the tangent line to the sine curve in order to sketch the graph of f', then what happens?



Theorem: $\dfrac{d}{dx}(\sin x) = \cos x$

Proof: Recall the *Sum Identity for Sine*:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

We will use the limit definition of the derivative. Let $f(x) = \sin(x)$. Then

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} && \text{(definition)}\\
&= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}\\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} && \text{(sum identity)}\\
&= \lim_{h\to 0}\left[\frac{\sin x\cos h - \sin x}{h} + \frac{\cos x\sin h}{h}\right]\\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)\right] && \text{(factoring)}\\
&= \lim_{h\to 0}\sin x\left(\frac{\cos h - 1}{h}\right) + \lim_{h\to 0}\cos x\left(\frac{\sin h}{h}\right) && \text{(limit law)}\\
&= (\sin x)\left(\lim_{h\to 0}\frac{\cos h - 1}{h}\right) + (\cos x)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) && \text{(because } x \text{ is independent of } h\text{)}\\
&= (\sin x)\cdot 0 + (\cos x)\cdot 1 = \cos x.
\end{aligned}$$

Using the same methods as the previous proof, we could also prove the following:

Theorem: $\dfrac{d}{dx}(\cos x) = -\sin x$

Together with the Quotient Rule, the derivatives of the sine and cosine function allow us to find the derivative of the remaining trigonometric functions. For reference, here they are:

Derivatives of Trigonometric Functions:

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 2: Use the quotient rule to find the derivative of $f(x) = \tan x$.

Example 3: Find the derivative of $y = 5\cot x + x^3 \sec x$.

Example 4: Find an equation of the tangent line to the curve $y = \sec x - 2\cos x$ when $x = \frac{\pi}{3}$.

Example 5: Find the derivative of $f(x) = \dfrac{\sin x \cos x + 4x^2}{\tan x}$.