Recitation 3 Solutions

1. (a) The two functions are not equal because one of them is defined for all x, whereas the other is defined only for $x \neq 2$

(b) It holds by the theorem from page 1 of lecture notes #3.

(Note that the function

$$f(x) = \frac{x^2 + x - 6}{x-2}$$

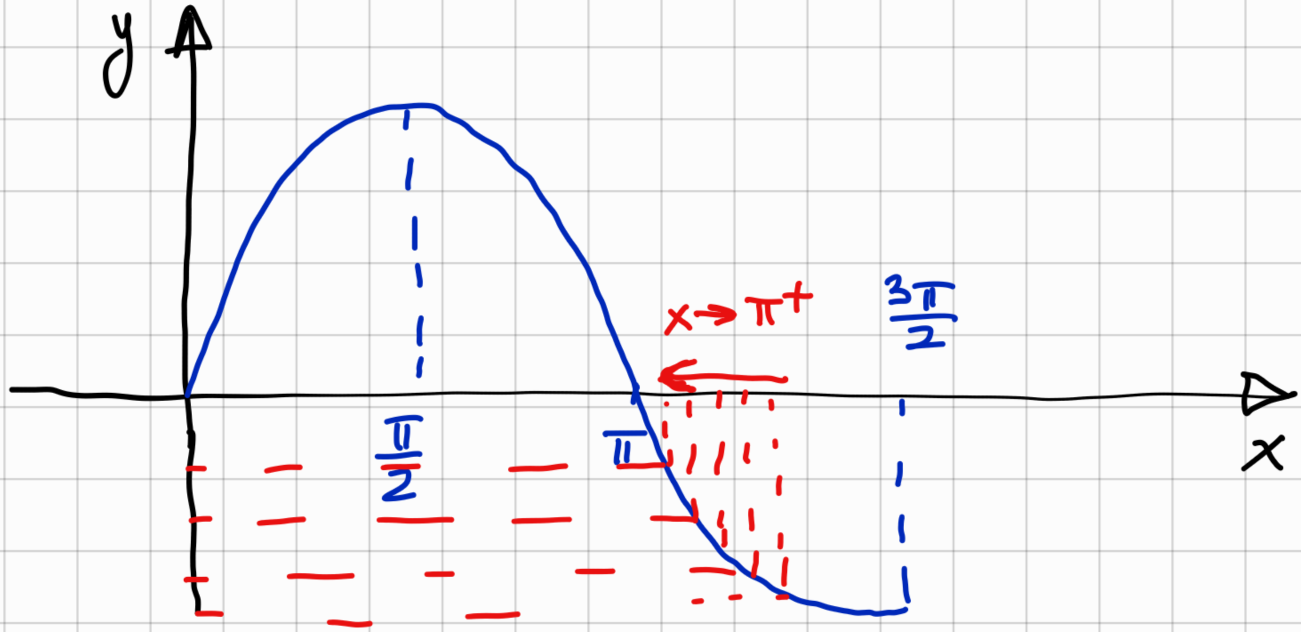
differs from the function

$$g(x) = x + 3$$

only at $x = 2$, so lims at $x=2$ are the same by that theorem)

2. a) We have $\csc x = \dfrac{1}{\sin x}$

So $\lim\limits_{x \to \pi^+} \csc x = \lim\limits_{x \to \pi^+} \dfrac{1}{\sin x}$



As $x \to \pi^+$, the values of $\sin x$ approach 0 and also stay negative. So for x-values close to π and to the right of π, we have:

$$\csc x = \frac{1}{\text{very small negative number}}$$

So $\lim_{x \to \pi^+} \csc x = -\infty$

b) Note that by an argument similar to a),

$$\lim_{x \to \pi^-} \csc x = +\infty$$

so $\lim_{x \to \pi} \csc x$ DNE

c) This limit doesn't exist.

Indeed, suppose it does exist. Then for any sequence of x-values that approaches 0 from the right, $\sin \frac{1}{x}$ should approach some fixed quantity

But here is a sequence of x-values for which this fails: $\frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \dots$

x	$\sin\frac{1}{x}$
$\frac{2}{\pi}$	1
$\frac{2}{3\pi}$	-1
$\frac{2}{5\pi}$	1
$\frac{2}{7\pi}$	-1
...	...

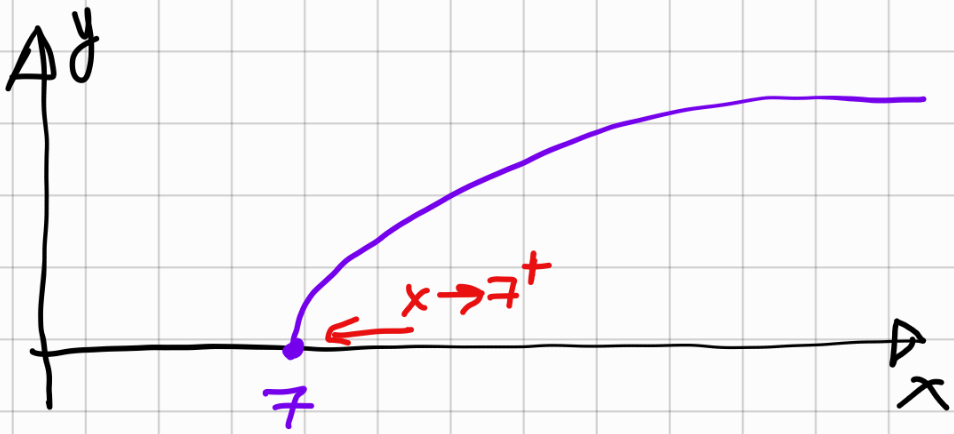
For this sequence of x-values that approach 0 from the right, $\sin\frac{1}{x}$ doesn't approach any one number, it oscillates between -1 & 1.

d) As $x \to 7$ from the left, $x-7$ approaches 0 while also staying negative.

So for x close to 7 and to the left of 7, we have;

$$\frac{1}{(x-7)(x-1)} = \frac{1}{\left(\begin{array}{c}\text{very small}\\ \text{negative \#}\end{array}\right)\cdot\left(\begin{array}{c}\text{some}\\ \text{positive \#}\end{array}\right)}$$

So $\lim_{x \to 7^-} \dfrac{1}{(x-7)(x-1)} = -\infty$

e) 

You can apply the limit law for $\sqrt{\ }$:

$$\lim_{x \to 7^+} \sqrt{x-7} = \sqrt{7-7} = 0$$

Note that the limit laws apply for right limits at $x=a$ only if the function is defined for x-values close to $x=a$ to the right of $x=a$.

Our function $\sqrt{x-7}$ is defined for values close to $x=7$ to the right of $x=7$

(e.g. $x = 7.001$), and that's why it works.

f) This limit DNE because $\sqrt{x-7}$ is not defined for values close to $x=7$ to the left of $x=7$ (and hence $\lim_{x \to 7^-} \sqrt{x-7}$ DNE, let alone $\lim_{x \to 7} \sqrt{x-7}$).

3. a) $\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{f(x)}{x^2} \cdot x^2 \right)$

$= \lim_{x \to 0} \frac{f(x)}{x^2} \cdot \lim_{x \to 0} x^2$

$= 3 \cdot 0 = 0$

b) $\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \left(\frac{f(x)}{x^2} \cdot x \right)$

$$= \lim_{x \to 0} \frac{f(x)}{x^2} \cdot \lim_{x \to 0} x$$

$$= 3 \cdot 0 = 0$$

4. a) $\lim_{x \to -1} (f(x) \cdot g(x))$

$$= \lim_{x \to -1} f(x) \cdot \lim_{x \to -1} g(x)$$

$$= 1 \cdot 2 = 2$$

b) $\lim_{x \to -1} (x^3 \cdot g(x))$

$$= \lim_{x \to -1} x^3 \cdot \lim_{x \to -1} g(x)$$

$$= (-1)^3 \cdot 2 = -2$$

c) $\lim_{x \to -1} (f(x) \cdot g(0))$

$$= \lim_{x \to -1} f(x) \cdot \lim_{x \to -1} g(0)$$

$$= 1 \cdot g(0) = g(0) = 1$$

d) $\lim_{x \to -1} (f(x) \cdot g(2))$ DNE

since $g(2)$ is undefined

e) $\lim_{x \to 2} \frac{x^2+x}{g(x)} = \frac{\lim_{x \to 2}(x^2+x)}{\lim_{x \to 2} g(x)}$

$= \frac{2^2+2}{2} = 3$

f) $\lim_{x \to 0} (f(x)+g(x))$ DNE

because : $\lim_{x \to 0^+} (f(x)+g(x))$

$= \lim_{x \to 0^+} f(x) + \lim_{x \to 0^+} g(x)$

$= 2+1 = 3$, whereas

$\lim_{x \to 0^-} (f(x)+g(x))$

$$= \lim_{x \to 0^-} f(x) + \lim_{x \to 0^-} g(x)$$

$$= 2 + 3 = 5$$

g) $\lim_{x \to 0} (f(x) + g(0))$

$$= \lim_{x \to 0} f(x) + \lim_{x \to 0} g(0)$$

$$= 2 + 2 = 4$$

h) $\lim_{x \to 0} (f(0) + g(x))$ DNE

because: $\lim_{x \to 0^+} (f(0) + g(x))$

$$= \lim_{x \to 0^+} f(0) + \lim_{x \to 0^+} g(x)$$

$= 2 + 1 = 3$, whereas

$\lim_{x \to 0^-} (f(0) + g(x))$

$$= \lim_{x\to 0^-} f(0) + \lim_{x\to 0^-} g(x)$$

$$= 2 + 3 = 5$$

i) $$\lim_{x\to 0} (10560 \cdot f(x))$$

$$= \lim_{x\to 0} 10560 \cdot \lim_{x\to 0} f(x)$$

$$= 10560 \cdot 2 = 21120$$

5. a) $$\lim_{t\to 3} \left(\sqrt{t} + \frac{t}{2t-5}\right) = \sqrt{3} + 3$$

b) $$\lim_{t\to 1} 2 = 2$$

c) $$\lim_{x\to -4} \frac{\sqrt{x^2+9} - 5}{x+4}$$

$$= \lim_{x\to -4} \frac{(\sqrt{x^2+9} - 5)(\sqrt{x^2+9} + 5)}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x\to -4} \frac{x^2 + 9 - 25}{(x+4)(\sqrt{x^2+9} + 5)}$$

$$= \lim_{x \to -4} \frac{(x-4)(x+4)}{(x+4)(\sqrt{x^2+9}+5)}$$

$$= \lim_{x \to -4} \frac{x-4}{\sqrt{x^2+9}+5} = \frac{-8}{5+5} = -\frac{4}{5}$$

d) $$\lim_{x \to 16} \frac{4-\sqrt{x}}{16x-x^2}$$

$$= \lim_{x \to 16} \frac{4-\sqrt{x}}{x(4-\sqrt{x})(4+\sqrt{x})}$$

$$= \lim_{x \to 16} \frac{1}{x(4+\sqrt{x})} = \frac{1}{16(4+4)} = \frac{1}{128}$$

e) $$\lim_{h \to 0} \frac{(x+h)^2-x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h}$$

$$= \lim_{h \to 0} (2x+h) = 2x$$

f) $\lim_{h \to 0} \left(\frac{1}{h} - \frac{1}{h^2+h} \right)$

$= \lim_{h \to 0} \frac{(h+1)-1}{h(h+1)}$

$= \lim_{h \to 0} \frac{1}{h+1} = 1$