21-120: Differential and Integral Calculus
Recitation #11 Outline: 10/01/24

1. (a) Recall that in Lecture 14 we defined e to be the real number which satisfies the following property:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Using this property, show that for any $a > 0$,

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a.$$

(b) Using the limit definition of the derivative and the fact above, prove that for any $a > 0$, we have $\dfrac{d}{dx}\left(a^x\right) = a^x \ln a$.

Solution:

(a) We defined e as the real number that satisfies:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Now, we show that for any $a > 0$:

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a.$$

Since $a^h = e^{h\ln a}$, the limit becomes:

$$\lim_{h\to 0} \frac{e^{h\ln a} - 1}{h}.$$

Let $k = h\ln a$. As $h \to 0$, $k \to 0$. Thus, the limit becomes:

$$\lim_{k\to 0} \frac{e^k - 1}{\frac{k}{\ln a}} = \ln a \cdot \lim_{k\to 0} \frac{e^k - 1}{k}.$$

Using the definition of e, we have:

$$\lim_{k\to 0} \frac{e^k - 1}{k} = 1.$$

Therefore:

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a.$$

(b) Using the limit definition of the derivative, we compute:

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}.$$

This can be rewritten as:

$$\frac{d}{dx}(a^x) = \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}.$$

From part (1), we know:

$$\lim_{h\to 0} \frac{a^h - 1}{h} = \ln a.$$

Therefore:

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

2. Find the derivatives of the functions below. For at least one function of your choice among the given ones, find its derivative in two ways: without using logarithmic differentiation and by using logarithmic differentiation.[1]

   (a) $f(x) = 2^x \cdot \log_7(6x^4+3)^5$;

   (b) $f(x) = 3^{\sin(3x)}$;

   (c) $f(x) = x^\pi \cdot \pi^x$;

   (d) $f(x) = e^{x^3 \ln x}$.

   Solution:

   (a) The function is $f(x) = 2^x \cdot \log_7\left((6x^4+3)^5\right)$. Using the product rule, we have:

   $$f'(x) = \frac{d}{dx}(2^x) \cdot \log_7\left((6x^4+3)^5\right) + 2^x \cdot \frac{d}{dx}\left(\log_7\left((6x^4+3)^5\right)\right).$$

   The derivative of 2^x is $2^x \cdot \ln 2$. For the second term, note that

   $$\log_7\left((6x^4+3)^5\right) = 5\log_7(6x^4+3).$$

   Using this and the chain rule, we get:

   $$\frac{d}{dx}\log_7\left((6x^4+3)^5\right) = \frac{5}{\ln 7} \cdot \frac{1}{6x^4+3} \cdot 24x^3.$$

   Therefore, the derivative is:

   $$f'(x) = 2^x \cdot \ln 2 \cdot \log_7\left((6x^4+3)^5\right) + 2^x \cdot \frac{120x^3}{(6x^4+3)\ln 7}.$$

   (b) The function is $f(x) = 3^{\sin(3x)}$. Using the chain rule, we have:

   $$f'(x) = 3^{\sin(3x)} \ln 3 \cdot \frac{d}{dx}(\sin(3x)).$$

   The derivative of $\sin(3x)$ is $3\cos(3x)$. Therefore:

   $$f'(x) = 3^{\sin(3x)} \ln 3 \cdot 3\cos(3x) = 3^{\sin(3x)} \cdot 3\ln 3 \cdot \cos(3x).$$

   Alternatively, we could have used logarithmic differentiation as follows. First, we apply logarithmic differentiation by taking the natural logarithm of both sides:

   $$\ln(f(x)) = \ln\left(3^{\sin(3x)}\right).$$

[1] In principle, you don't have to use logarithmic differentiation for any of the functions here, but note that in some cases, using it might be a better choice, as it can lead to a simpler solution.

Using the logarithmic property $\ln(a^b) = b\ln(a)$, we get:

$$\ln(f(x)) = \sin(3x)\ln(3).$$

Now, differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left(\sin(3x)\ln(3)\right).$$

The derivative of the left-hand side is:

$$\frac{1}{f(x)}f'(x).$$

For the right-hand side, we differentiate $\sin(3x)\ln(3)$, which gives:

$$\ln(3)\cdot 3\cos(3x).$$

Thus, we have:

$$\frac{1}{f(x)}f'(x) = 3\ln(3)\cdot\cos(3x).$$

Finally, multiply both sides by $f(x) = 3^{\sin(3x)}$ to obtain:

$$f'(x) = 3^{\sin(3x)}\cdot 3\ln(3)\cdot\cos(3x).$$

(c) The function is $f(x) = x^\pi \cdot \pi^x$. Using the product rule, we have:

$$f'(x) = \frac{d}{dx}(x^\pi)\cdot\pi^x + x^\pi\cdot\frac{d}{dx}(\pi^x).$$

The derivative of x^π is $\pi x^{\pi-1}$, and the derivative of π^x is $\pi^x\ln\pi$. Therefore:

$$f'(x) = \pi x^{\pi-1}\cdot\pi^x + x^\pi\cdot\pi^x\ln\pi = \pi^x\left(\pi x^{\pi-1} + x^\pi\ln\pi\right).$$

Alternatively, we could have used logarithmic differentiation. First, take the natural logarithm of both sides:

$$\ln(f(x)) = \ln(x^\pi\cdot\pi^x).$$

Using the properties of logarithms, this can be expressed as:

$$\ln(f(x)) = \ln(x^\pi) + \ln(\pi^x).$$

Applying the logarithmic property $\ln(a^b) = b\ln(a)$, we have:

$$\ln(f(x)) = \pi\ln(x) + x\ln(\pi).$$

Now, differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left(\pi\ln(x) + x\ln(\pi)\right).$$

Using the derivative of the logarithm and the product rule, we obtain:

$$\frac{1}{f(x)}f'(x) = \frac{\pi}{x} + \ln(\pi).$$

Thus, we have:

$$f'(x) = f(x)\left(\frac{\pi}{x} + \ln(\pi)\right).$$

Substituting back for $f(x)$:

$$f'(x) = x^{\pi} \cdot \pi^{x}\left(\frac{\pi}{x} + \ln(\pi)\right).$$

This can be rewritten as:

$$f'(x) = \pi^{x}\left(\pi x^{\pi-1} + x^{\pi}\ln(\pi)\right).$$

(d) The function is $f(x) = e^{x^3 \ln x}$. Using the chain rule, we have:

$$f'(x) = e^{x^3 \ln x} \cdot \frac{d}{dx}(x^3 \ln x).$$

Using the product rule for $x^3 \ln x$, we get:

$$\frac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^2.$$

Therefore, the derivative is:

$$f'(x) = e^{x^3 \ln x} \cdot \left(3x^2 \ln x + x^2\right).$$

Alternatively, we could have used logarithmic differentiation. First, we take the natural logarithm of both sides:

$$\ln(f(x)) = \ln\left(e^{x^3 \ln x}\right).$$

Using the property $\ln(e^u) = u$, this simplifies to:

$$\ln(f(x)) = x^3 \ln x.$$

Now, differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left(x^3 \ln x\right).$$

The derivative of the left-hand side is:

$$\frac{1}{f(x)} f'(x).$$

For the right-hand side, we use the product rule on $x^3 \ln x$:

$$\frac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^2.$$

Thus, we have:

$$\frac{1}{f(x)} f'(x) = 3x^2 \ln x + x^2.$$

Finally, multiply both sides by $f(x) = e^{x^3 \ln x}$ to get:

$$f'(x) = e^{x^3 \ln x} \cdot \left(3x^2 \ln x + x^2\right).$$

3. Find the derivatives of the following functions:[2]

[2] Note that since a variable is raised to a variable power in all these functions, the ordinary rules of differentiation do not apply! The rule $(a^t)' = a^t \ln(a)$ is only valid when a is a constant. If a is an expression containing variables, we cannot use this rule and must instead apply logarithmic differentiation. Similarly, the rule $(t^n)' = nt^{n-1}$ is only valid when n is a constant. If n is an expression containing variables, we cannot use this rule.

(a) $f(x) = (3x)^{2x}$;

(b) $f(x) = x^{\log_2 x^3}$;

(c) $f(x) = (3x^2+4)^{\cos x}$;

(d) $f(x) = (\sin(2x))^{4x}$.

Solution:

(a) The function is $f(x) = (3x)^{2x}$. We apply logarithmic differentiation by taking the natural logarithm of both sides:

$$\ln(f(x)) = \ln\left((3x)^{2x}\right).$$

Using the property $\ln(a^b) = b\ln(a)$, this simplifies to:

$$\ln(f(x)) = 2x\ln(3x).$$

Now, differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(2x\ln(3x)).$$

Using the product and chain rule on $2x\ln(3x)$, we get:

$$\frac{d}{dx}(2x\ln(3x)) = 2\ln(3x) + 2x\cdot\frac{1}{3x}\cdot 3 = 2\ln(3x) + 2.$$

Thus, we have:

$$\frac{1}{f(x)}f'(x) = 2\ln(3x) + 2.$$

Finally, multiply both sides by $f(x) = (3x)^{2x}$ to get:

$$f'(x) = (3x)^{2x}\cdot(2\ln(3x)+2).$$

(b) The function is $f(x) = x^{\log_2 x^3}$. First, take the natural logarithm of both sides:

$$\ln(f(x)) = \ln\left(x^{\log_2 x^3}\right).$$

Using the property $\ln(a^b) = b\ln(a)$, this simplifies to:

$$\ln(f(x)) = \log_2 x^3\cdot\ln(x).$$

Now, rewrite $\log_2 x^3$ as $\frac{3\ln x}{\ln 2}$:

$$\ln(f(x)) = \frac{3\ln x}{\ln 2}\cdot\ln(x) = \frac{3(\ln x)^2}{\ln 2}.$$

Differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left(\frac{3(\ln x)^2}{\ln 2}\right).$$

Using the chain rule, we get:

$$\frac{d}{dx}\left(\frac{3(\ln x)^2}{\ln 2}\right) = \frac{6\ln x}{x\ln 2}.$$

Thus, we have:

$$\frac{1}{f(x)}f'(x) = \frac{6\ln x}{x\ln 2}.$$

Finally, multiply both sides by $f(x) = x^{\log_2 x^3}$ to get:

$$f'(x) = x^{\log_2 x^3}\cdot\frac{6\ln x}{x\ln 2}.$$

(c) The function is $f(x) = (3x^2+4)^{\cos x}$. Taking the natural logarithm of both sides:

$$\ln(f(x)) = \ln\left((3x^2+4)^{\cos x}\right).$$

This simplifies to:

$$\ln(f(x)) = \cos x \ln(3x^2+4).$$

Now, differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left(\cos x \ln(3x^2+4)\right).$$

Using the product rule, we get:

$$-\sin x \ln(3x^2+4) + \cos x \cdot \frac{6x}{3x^2+4}.$$

Thus, we have:

$$\frac{1}{f(x)} f'(x) = -\sin x \ln(3x^2+4) + \frac{6x\cos x}{3x^2+4}.$$

Finally, multiply both sides by $f(x) = (3x^2+4)^{\cos x}$ to get:

$$f'(x) = (3x^2+4)^{\cos x} \cdot \left(-\sin x \ln(3x^2+4) + \frac{6x\cos x}{3x^2+4}\right).$$

(d) The function is $f(x) = (\sin(2x))^{4x}$. Taking the natural logarithm of both sides:

$$\ln(f(x)) = \ln\left((\sin(2x))^{4x}\right).$$

This simplifies to:

$$\ln(f(x)) = 4x\ln(\sin(2x)).$$

Now, differentiate both sides with respect to x:

$$\frac{d}{dx}\ln(f(x)) = \frac{d}{dx}\left(4x\ln(\sin(2x))\right).$$

By the product and chain rule, the right-hand side of the above simplifies to:

$$4\ln(\sin(2x)) + 4x \cdot \frac{2\cos(2x)}{\sin(2x)}.$$

Thus, we have:

$$\frac{1}{f(x)} f'(x) = 4\ln(\sin(2x)) + \frac{8x\cos(2x)}{\sin(2x)}.$$

Finally, multiply both sides by $f(x) = (\sin(2x))^{4x}$ to get:

$$f'(x) = (\sin(2x))^{4x} \cdot \left(4\ln(\sin(2x)) + \frac{8x\cos(2x)}{\sin(2x)}\right).$$

4. Consider the curve given by the following equation:

$$x^3 - x\ln(y) + y^3 = 2x + 5.$$

(a) Find the slope of the tangent line to this curve at the point $(x, y) = (2, 1)$.

(b) Write an equation of the tangent line to this curve at the point $(x, y) = (2, 1)$.

Solution:

(a) To find the slope of the tangent line at the point $(x, y) = (2, 1)$, we first differentiate the given equation implicitly with respect to x.

Differentiating both sides:

$$\frac{d}{dx}(x^3) - \frac{d}{dx}(x\ln(y)) + \frac{d}{dx}(y^3) = \frac{d}{dx}(2x+5).$$

This simplifies to:

$$3x^2 - \left(\ln(y) + x \cdot \frac{1}{y} \cdot \frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 2.$$

Now, rearranging to express $\frac{dy}{dx}$:

$$3x^2 - \ln(y) - 2 = \left(x \cdot \frac{1}{y}\right)\frac{dy}{dx} - 3y^2\frac{dy}{dx}.$$

Grouping the terms involving $\frac{dy}{dx}$:

$$3x^2 - \ln(y) - 2 = \left(-3y^2 + \frac{x}{y}\right)\frac{dy}{dx}.$$

Thus, we have:

$$\frac{dy}{dx} = \frac{3x^2 - \ln(y) - 2}{-3y^2 + \frac{x}{y}}.$$

Now, substituting $x = 2$ and $y = 1$:

$$\left.\frac{dy}{dx}\right|_{(x,y)=(2,1)} = \frac{3(2^2) - \ln(1) - 2}{-3(1^2) + \frac{2}{1}} = -10.$$

Therefore, the slope of the tangent line at the point $(2, 1)$ is -10.

(b) Now, we will write the equation of the tangent line at the point $(2, 1)$. Using the point-slope form of a line, we have:

$$y - y_1 = m(x - x_1),$$

where $m = -10$ and $(x_1, y_1) = (2, 1)$. Substituting these values gives:

$$y - 1 = 10(x - 2).$$