21-120: Differential and Integral Calculus
Recitation #10 Outline: 09/26/24

1. For each of the following, find $(f^{-1})'(a)$:

   (a) $f(x) = x^2 + 3x + 2,\ x \geq -\dfrac{3}{2},\ a = 2$

   (b) $f(x) = x - \dfrac{2}{x},\ x < 0,\ a = 1$

   (c) $f(x) = x + \sin x,\ a = 0$

   (d) $f(x) = x + \sqrt{x},\ a = 2$

   Solution:

   (a) We need to use the formula
   $$\left(f^{-1}\right)'(a) = \frac{1}{f'(f^{-1}(a))} \quad \text{with} \quad a = 2$$
   In order to use this formula, we need to know what $f^{-1}(2)$ is. That is, we need to find some x such that $f(x) = 2$ (and such that $x \geq -\frac{3}{2}$, since this restriction is given in the statement of the problem). It's easy to guess that such an x is 0, i.e., $f^{-1}(2) = 0$. Alternatively, you can solve the equation $f(x) = 2$ (and choose the non-negative root).
   Now that we know $f^{-1}(2)$, we can find $f'(x)$ and plug in $f^{-1}(2)$ for x. We have $f'(x) = 2x+3$, and so $f'(f^{-1}(2)) = f'(0) = 3$. Therefore,
   $$(f^{-1})'(2) = \frac{1}{3}.$$

   (b) Again, we first find $f^{-1}(1)$, which is the negative number x such that $f(x) = 1$ (it should be negative because of the restriction $x < 0$). Such an x is easy to guess, it is $x = -1$. So we have $f^{-1}(1) = -1$.
   Now we need to find $f'(x)$ at $x = -1$. We have $f'(x) = 1 + \frac{2}{x^2}$, and hence $f'(-1) = 3$. Thus, $(f^{-1})'(1) = \frac{1}{3}$.

   (c) This is similar to the previous parts. In this case, $f^{-1}(0) = 0$ (we guessed it). Then, $f'(x) = 1 + \cos(x)$ and hence $f'(0) = 2$. Thus, $(f^{-1})'(0) = \frac{1}{2}$.

   (d) In this case, $f^{-1}(2) = 1$, $f'(x) = 1 + \frac{1}{2\sqrt{x}}$, and hence $f'(1) = \frac{3}{2}$. Thus, $(f^{-1})'(2) = \frac{2}{3}$.

2. In the problem above, why do some parts have restrictions on x while others don't? What would go wrong if we removed the requirement $x \geq -\dfrac{3}{2}$ in part (a)?

   Solution: The restrictions on x are given in cases where the functions are not one-to-one on their natural domain. If a function is not one-to-one, then the inverse does not exist, and it doesn't make sense to talk about the derivative of the inverse function in such cases. For example, the function f given by $f(x) = x^2 + 3x + 2$ is not one-to-one on its natural domain (which consists of all real numbers), and so f^{-1} does not exist. But if we add the restriction $x \geq -\frac{3}{2}$, then f on this restricted domain will become one-to-one, and it will make sense to talk about its inverse f^{-1}. The functions in (c) and (d) are one-to-one on their natural domains, so no restrictions are necessary for them.

3. For each function f below, find the equation of the tangent line to the graph of f^{-1} at the specified point P, *without* directly using the Inverse Function Theorem. That is, first write an equation for the tangent line for f at the appropriate point, and then convert the equation into an equation of the tangent line for f^{-1} at the point P.

(a) $f(x) = (x^3+1)^4,\ P(16,1)$

(b) $\sqrt{x-4},\ P(2,8)$

Solution:

(a) We are asked to find the tangent line to f^{-1} at $P(16,1)$. To do this, we can first find the tangent line to f at $Q(1,16)$ and then swap the roles of x and y.
First, we compute $f'(x)$:
$$f(x) = (x^3+1)^4.$$
Using the chain rule:
$$f'(x) = 4(x^3+1)^3 \cdot 3x^2 = 12x^2(x^3+1)^3.$$
Next, we evaluate $f'(1)$:
$$f'(1) = 12(1)^2(1^3+1)^3 = 12\cdot 1\cdot 2^3 = 12\cdot 8 = 96.$$
The tangent line to f at the point $Q(1,16)$ is therefore:
$$y - 16 = 96(x-1).$$
To get the tangent line to f^{-1} at $P(16,1)$, we swap the roles of x and y:
$$x - 16 = 96(y-1).$$
This simplifies to:
$$y = \frac{1}{96}x + \frac{5}{6}.$$
Thus, the equation of the tangent line to f^{-1} at $P(16,1)$ is:
$$y = \frac{1}{96}x + \frac{5}{6}.$$

(b) We are asked to find the tangent line to f^{-1} at $P(2,8)$. To do this, we can first find the tangent line to f at $Q(8,2)$ and then swap the roles of x and y.
We have
$$f(x) = \sqrt{x-4}$$
and differentiating,
$$f'(x) = \frac{1}{2\sqrt{x-4}}$$
At $x = 8$, we have
$$f'(8) = \frac{1}{2\sqrt{8-4}} = \frac{1}{4}.$$
Thus, the tangent line to f at $Q(8,2)$ is
$$y = 2 + \frac{1}{4}(x-8).$$
To get the tangent line to f^{-1} at $P(2,8)$, we swap the roles of x and y, giving
$$x = 2 + \frac{1}{4}(y-8).$$
This simplifies to
$$y = 4x,$$
which is the equation of the tangent line to f^{-1} at $P(2,8)$.

4. For each function f below, find the equation of the tangent line to the graph of f^{-1} at the specified point P, using the Inverse Function Theorem. Check that your answers agree with the answers to the previous problem.

   (a) $f(x) = (x^3+1)^4,\ P(16,1)$ (b) $\sqrt{x-4},\ P(2,8)$

   Solution:

   (a) We are asked to find the equation of the tangent line to f^{-1} at $P(16,1)$. According to the Inverse Function Theorem, the slope of the tangent line to f^{-1} at $P(16,1)$ is given by $\frac{1}{f'(f^{-1}(16))}$.
   First, we find $f'(x)$. We have
   $$f(x) = (x^3+1)^4.$$
   Using the chain rule,
   $$f'(x) = 4(x^3+1)^3 \cdot 3x^2 = 12x^2(x^3+1)^3.$$
   Now we need to find $f^{-1}(16)$. Since $f(1) = (1^3+1)^4 = 16$, we have $f^{-1}(16) = 1$.
   Next, we evaluate $f'(1)$:
   $$f'(1) = 12(1^2)(1^3+1)^3 = 12 \cdot 2^3 = 96.$$
   Therefore, the slope of the tangent line to f^{-1} at $P(16,1)$ is
   $$\frac{1}{f'(1)} = \frac{1}{96}.$$
   Using the point-slope form of the equation of a line, the tangent line is
   $$y - 1 = \frac{1}{96}(x-16).$$
   This answer is equivalent to what we got in the previous problem.

   (b) We are asked to find the equation of the tangent line to f^{-1} at $P(2,8)$ for the function $f(x) = \sqrt{x-4}$. By the Inverse Function Theorem, the slope of the tangent line is $\frac{1}{f'(f^{-1}(2))}$.
   First, we find $f'(x)$. We have
   $$f(x) = \sqrt{x-4},$$
   and differentiating,
   $$f'(x) = \frac{1}{2\sqrt{x-4}}.$$
   Next, we need to find $f^{-1}(2)$. Since $f(8) = \sqrt{8-4} = 2$, we have $f^{-1}(2) = 8$.
   Now, evaluate $f'(8)$:
   $$f'(8) = \frac{1}{2\sqrt{8-4}} = \frac{1}{4}.$$
   Therefore, the slope of the tangent line to f^{-1} at $P(2,8)$ is
   $$\frac{1}{f'(8)} = 4.$$
   Using the point-slope form of the equation of a line, the tangent line is
   $$y - 8 = 4(x-2).$$
   Note that the above equation is the same as $y = 4x$, which we got in the previous problem.

5. Find the derivatives of the following functions:

(a) $y = \arccos(\sqrt{x})$

(b) $y = \sec^{-1}(-x)$

(c) $y = \sqrt{\csc^{-1}(x)}$

(d) $y = x\csc^{-1}(x)$

Solution:

(a) For $y = \arccos(\sqrt{x})$:
$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = -\frac{1}{2\sqrt{x(1-x)}}$$

(b) For $y = \sec^{-1}(-x)$:
$$\frac{dy}{dx} = \frac{1}{(-x)\sqrt{(-x)^2-1}} \cdot (-1) = \frac{1}{x\sqrt{x^2-1}}$$

(c) For $y = \sqrt{\csc^{-1}(x)}$:
$$\frac{dy}{dx} = \frac{1}{2\sqrt{\csc^{-1}(x)}} \cdot \frac{d}{dx}(\csc^{-1}(x)) = \frac{1}{2\sqrt{\csc^{-1}(x)}} \cdot \frac{-1}{|x|\sqrt{x^2-1}}$$

(d) For $y = x\csc^{-1}(x)$:
$$\frac{dy}{dx} = \csc^{-1}(x) + x \cdot \frac{d}{dx}(\csc^{-1}(x)) = \csc^{-1}(x) + x \cdot \frac{-1}{x\sqrt{x^2-1}}$$

Simplifying gives:
$$\frac{dy}{dx} = \csc^{-1}(x) - \frac{1}{\sqrt{x^2-1}}$$

6. For each of the following, use the given values to find $(f^{-1})'(a)$: functions:

(a) $f(\pi) = 0,\ f'(\pi) = -1,\ a = 0$

(b) $f(6) = 2,\ f'(6) = 1/3,\ a = 2$

(c) $f(1) = 0,\ f'(1) = -2,\ a = 0$

(d) $f(\sqrt{3}) = 1/2,\ f'(\sqrt{3}) = 2/3,\ a = 1/2$

Solution:

(a) For $f(\pi) = 0$, $f'(\pi) = -1$, $a = 0$:
$$f^{-1}(0) = \pi \implies (f^{-1})'(0) = \frac{1}{f'(\pi)} = \frac{1}{-1} = -1.$$

(b) For $f(6) = 2$, $f'(6) = \frac{1}{3}$, $a = 2$:
$$f^{-1}(2) = 6 \implies (f^{-1})'(2) = \frac{1}{f'(6)} = \frac{1}{\frac{1}{3}} = 3.$$

(c) For $f(1) = 0$, $f'(1) = -2$, $a = 0$:
$$f^{-1}(0) = 1 \implies (f^{-1})'(0) = \frac{1}{f'(1)} = \frac{1}{-2} = -\frac{1}{2}.$$

(d) For $f(\sqrt{3}) = \frac{1}{2}$, $f'(\sqrt{3}) = \frac{2}{3}$, $a = \frac{1}{2}$:

$$f^{-1}\left(\frac{1}{2}\right) = \sqrt{3} \implies (f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'(\sqrt{3})} = \frac{1}{\frac{2}{3}} = \frac{3}{2}.$$

7. Suppose $f(t) = t^3 + 4t + 2$. Find the slope of the tangent line to the graph of $g(x) = xf^{-1}(x)$ at the point $x = 7$.

Solution:

We need to find $g'(x)$ at $x = 7$. To differentiate $g(x)$, we apply the product rule:

$$g'(x) = f^{-1}(x) + x \cdot (f^{-1})'(x).$$

So

$$g'(7) = f^{-1}(7) + 7 \cdot (f^{-1})'(7).$$

Let's first find $f^{-1}(7)$. This amounts to solving the equation $f(x) = 7$:

$$x^3 + 4x + 2 = 7 \implies x^3 + 4x - 5 = 0.$$

By trial (together with the rational root theorem), we find that $x = 1$ is a root:

$$1^3 + 4(1) - 5 = 0.$$

Thus, $f(1) = 7$, so $f^{-1}(7) = 1$.

Now we need to find $(f^{-1})'(7)$.

$$(f^{-1})'(7) = \frac{1}{f'\left(f^{-1}(7)\right)} = \frac{1}{f'(1)} = \frac{1}{3(1)+4} = \frac{1}{7}$$

Therefore,

$$g'(7) = 1 + 7 \cdot \left(\frac{1}{7}\right) = 1 + 1 = 2.$$