21-120: Differential and Integral Calculus
Lecture #33 Outline

Read: Section 3.1 of the OpenStax Calculus 2 Textbook

Objectives and Concepts:

- Integration by Parts is a technique derived from the Product Rule for derivatives that allows us to rewrite an integral of the product of two functions in a possibly more simplified way that is easier to integrate.
- The appropriate choice of functions u and dv in integration by parts depends on how easy they are to differentiate and integrate, respectively. We can use the LIATE approach to help us choose which function would be the u and the dv.

Suggested Textbook Exercises:

- 3.1: 9-35 odd, 39-45 odd, 53-57 odd

Integration By Parts

So far we have explored how to reverse-engineer most of our standard differentiation rules, but this still leaves us with many integrands that we do not yet know how to antidifferentiate. For example, when we see products of distinct functions in the integrand, such as the problems

$$\int xe^{3x}\,dx, \qquad \text{or} \qquad \int (x^2+1)\cos x\,dx,$$

we would like to find a way to evaluate these integrals, or at least reduce these integrals to easier problems.

If we examine the product rule for derivatives,

$$\frac{d}{dx}\big(f(x)g(x)\big) = f(x)g'(x) + g(x)f'(x),$$

and we take the indefinite integral of both sides of the equation, we will arrive at the relationship

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx.$$

Letting $u = f(x)$ and $v = g(x)$, we can write the above formula in the form

$$uv = \int u\,dv + \int v\,du \quad \Longrightarrow \quad \int u\,dv = uv - \int v\,du.$$

Integration by Parts: Let$u = f(x)$ and $v = g(x)$ be functions with continuous derivatives. Then, the integration-by-parts formula for the integral involving these two functions is:

$$\int u\,dv = uv - \int v\,du.$$

Note that in the above formula, we can replace the problem of integrating u times dv with the problem of integrating v times du. There are two things at play here:

1. Ideally, the new integral would be easier, so we should seek to choose u such that du is simpler, or at least more complementary to v.
2. In order to write the new integral, we would need to find v from our choice of dv, so we need to choose dv to be something that is fairly easy to find the antiderivative of.

This means that we need to be careful about which functions we choose to assign to the variables u and v. Let's look at an example.

Example 1: Evaluate $\displaystyle\int xe^{3x}\,dx$.

Solution: Here the two distinct functions in the integrand are x and e^{3x}. Using our rationale above, it seems to be a good idea to choose u to be x, since du will then just be $1\,dx$. This choice of u would mean that we must assign $dv = e^{3x}\,dx$. That's not too bad, as then we would just have $v = \dfrac{1}{3}e^{3x}$. So it seems as these choices might work, so let's try it out:

$$\begin{aligned}
\int xe^{3x}\,dx &= \int u\,dv && \left(u = x, \quad dv = e^{3x}\,dx\right)\\
&= uv - \int v\,du \\
&= \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}\,dx && \left(du = dx, \quad v = \frac{1}{3}e^{3x}\right)\\
&= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C.
\end{aligned}$$

Example 2: Evaluate $\displaystyle\int (x^2+1)\cos x\,dx$.

Solution: Here, if we think along the same lines as the previous example, we see that a good choice for u would be the polynomial x^2+1, and $dv = \cos x$ is also a good choice as it has an easy antiderivative.

$$\begin{aligned}
\int (x^2+1)\cos x\,dx &= \int u\,dv && \left(u = (x^2+1), \quad dv = \cos x\,dx\right)\\
&= uv - \int v\,du \\
&= (x^2+1)\sin x - \int 2x\sin x\,dx && (du = 2xdx, \quad v = \sin x)\\
&= (x^2+1)\sin x - 2\int u\,dv && (u = x, \quad dv = \sin x\,dx)\\
&= (x^2+1)\sin x - 2\left(uv - \int v\,du\right)\\
&= (x^2+1)\sin x - 2\left(-x\cos x + \int \cos x\right) && (du = dx, \quad v = -\cos x)\\
&= (x^2+1)\sin x + 2x\cos x - 2\sin x + C.
\end{aligned}$$

When we use Integration by Parts, the most challenging task is usually identifying the appropriate choices for u and dv. As we have seen in the previous examples, u should be a function whose derivative is easy to integrate, and dv should be a function that is easy to find the antiderivative of. Using these ideas, we can construct a hierarchy of functions to look for. Usually the best choice for u is found as we progress through the acronym "LIATE":

- **L - logarithmic:** Functions whose outermost operation is a logarithm usually convert to algebraic functions when you differentiate them.
- **I - inverse trigonometric:** Inverse trig functions usually have algebraic derivatives.
- **A - algebraic:** Algebraic functions include polynomials, rational functions, and functions with roots of polynomials. These may simplify when you differentiate them.
- **T - trigonometric:** Trig functions have trig function derivatives, so choosing u to be a trig function may not be of much benefit.
- **E - exponential:** Exponential functions have exponential function derivatives so the problem is usually not simplified any by choosing u to be an exponential function.

Note that the choice of dv would go in reverse - you should choose exponential functions first, and logarithmic functions last. As we saw above in Example 2, you often need to apply Integration by Parts more than once. Also, sometimes you will only seemingly have a single function as the integrand - one that is easy to differentiate but hard to integrate. In these cases, it may be useful to consider simply setting $dv = dx$, which would imply that $v = x$.

Example 3: Evaluate each integral.

(a) $\displaystyle\int x \ln x \, dx$

(b) $\displaystyle\int t \sec^2 2t \, dt$

(c) $\int x^2 \arctan x \, dx$

(d) $\int \ln x \, dx$

(e) $\int \arcsin t \, dt$