21-120: Differential and Integral Calculus
Lecture #36 Outline

Summary of The Course Content

Overall this semester we focused on three main topics: limits, differentiation, and integration.

- **Limits** are mathematical expressions that allow us to understand a function's behavior at or near various points, including points where the function may not be defined. Precisely, the limit exists (and equals a finite value) when the function's output values can be made arbitrarily close to a given value by taking input values that are either sufficiently close to a given input value or are sufficiently large (or sufficiently large and negative). Similar precise definitions exist when the limit is said to be infinite, otherwise, the limit does not exist. While the behavior of a function in the limit can often be understood by examining the graph of a function, the graph may not always be available, so we must resort to mathematical methods (i.e., algebraic or calculus-based methods) to compute many limits. Limits are of paramount importance in the study of calculus, and the broader field of mathematics referred to as analysis.

- **Differentiation** is the process of finding the instantaneous rate of change of a function, either at a single point in the domain of the function, or at an arbitrary point in the domain. **Differentiation is just the limit of a difference quotient**, and rules for finding derivatives are found by applying the definition of the derivative to various functions. The derivative has a wide array of useful applications, including linearization, finding local and global extreme values, how related quantities experience change at different rates, influencing the shape and nature of graphs of functions, and solving applied optimization problems.

- **Integration** is motivated by the classical problem of finding the area of an oddly-shaped region - **definite integrals are just limits of Riemann sums**. Both parts of the Fundamental Theorem of Calculus reveal the deep connections between the inverse processes of integration and differentiation, tying two distinct classical problems and seemingly disjoint processes together into one of the most significant human intellectual achievements of the last 500 years. Integration (antidifferentiation) is a moderately more challenging process than differentiation, which is why techniques like u-substitution and integration by parts require so much practice to master.

The Final Exam

The 21-120 Final Exam is cumulative and will cover all content presented in the course. That being said, it is unlikely that you will see an equal distribution of problems from Exams 1, 2, and 3 show up on the final exam, simply because the first several weeks of the semester were spent laying the foundations of what constituted the focus of the course (limits, derivatives, integrals). Here are some things to expect on the final, and some recommendations for preparing for it.

What to Expect on the Final Exam:

- The final exam is 3 hours long. However, since it only constitutes 30% of your final course grade, we usually write the final exam so it would ideally take a maximum of 1.75-2.0 hours for you to complete (i.e., no longer than twice the the length of a regular midterm exam). Since you will have all three hours, this means you should have plenty of time to work at your own pace and check your solutions multiple times.

- The format will be similar to the collective format of the midterm exams.
  - There will be several limit problems, several derivative problems, and several integral problems. These will include all techniques we have employed to calculate limits, derivatives, and integrals, including algebraic techniques, direct substitution, L'Hospital's Rule, the limit definition of the derivative, implicit differentiation, the Inverse Function Theorem, logarithmic differentiation, integration by substitution and by parts.
  - There will be several application-type problems, which could include IVT, related rates, linearization, max/min, MVT (derivatives), shapes of graphs, optimization, position/velocity/displacement/distance, and any other types of applications we have discussed this semester.
  - There will also be several conceptual questions, possibly including short and long answer questions, true/false, matching/multiple choice, and so on (for example, the types of short and long answer questions that you have seen on midterm exams this semester).
  - If you need a specific nontrivial (i.e. non-circle, rectangle, triangle, or rectangular box) formula from geometry to set up a particular problem, it will be given.
  - Topics that will **NOT** be on the exam include: the epsilon-delta definition of the limit, Riemann sums, or financial-related optimization problems.

How to Prepare for the Final Exam: There will be no new study guide for the final exam, as more than enough material already exists for you to review and rework in an effort to prepare for the exam.

- The first suggestion is to review your performance on Midterm Exams 1-3, paying close attention to the mistakes you made. Address those issues first - check the exam solutions, and make sure you fully understand why points were deducted for the errors that you made.
- The Midterm Exams (and their solutions) that were given in the other Lectures of 21-120 will be made available in your Canvas course by the end of the day Wednesday 12/4. If you would like to see more examples of the types of problems that have appeared on exams throughout the semester, please consult those exams.
- Review the Exam 1-3 Study Guides closely, being sure to review both concepts and practice problems.
- As the only topic that was not covered on the Exam 1-3 study guides is integration by parts, be sure to work some problems from the OpenStax Calculus 2 textbook if you want more practice.
- After the exam study guides, reviewing lecture notes are an excellent next step, as you can review definitions, theorems, and concepts alongside the examples we worked in class.
- After lecture notes, recitation problems and their solutions are a good resource.
- Homework problems and their solutions can also be used for preparing for the exam.
- Finally, every set of lecture notes comes with suggested textbook problems that all have answers in the back of the textbook.

What To Expect in Future Calculus Courses

Calculus II-type Courses:

- **21-122: Integration and Approximation:** This is a standard science and engineering Calculus II course, which is usually considered to be significantly more challenging than 21-120. You will spend several weeks just on further techniques of integration (parts, trig integrals, trig substitution, partial fraction decomposition, improper integrals, numerical integration). Subsequently, you will work with infinite sequences and series (i.e., sums of an infinite number of terms), developing a number of tests for convergence or divergence of a given series. Then you will discuss Taylor, Maclaurin, and Power series, which give an alternate representation of some functions under certain conditions. Finally, the semester closes with a collection of other topics, including some differential equations, parametric curves, polar coordinates, and an introduction to vectors as a preparation for multivariable calculus.

- **21-124: Calculus II for Biologists and Chemists:** This is intended as a second calculus course for biology and chemistry majors. It uses a variety of computational techniques based around the use of MATLAB or a similar system. Topics to be covered include: Integration: techniques and numerical integration. Ordinary differential equations: techniques for solving ODEs and numerical methods. Modeling with ODEs (e.g., infection, population models). Linear algebra: matrices, complex numbers, eigenvalues, eigenvectors. Systems of ordinary differential equations (if time allows: stability of differential systems). Probability: discrete and continuum probability, conditional probability and independence, limit theorems, important distributions, probabilistic models.

Multivariable Calculus Courses:

- **21-256 Multivariate Analysis:** (Most Tepper and DC students) Matrix algebra: vectors, matrices, systems of equations, dot product, cross product, lines and planes. Optimization: partial derivatives, the chain rule, gradient, unconstrained optimization, constrained optimization (Lagrange multipliers and the Kuhn-Tucker Theorem). Improper integrals. Multiple integration: iterated integrals, probability applications, triple integrals, change of variables. (Prerequisite: 21-120)

- **21-254 Linear Algebra and Vector Calculus for Engineers:** (Most CIT students) This course will introduce the fundamentals of vector calculus and linear algebra. The topics include vector and matrix operations, determinants, linear systems, matrix eigenvalue problems, vector differential calculus including gradient, divergence, curl, and vector integral calculus including line, surface, and volume integral theorems. Lecture and assignments will emphasize the applications of these topics to engineering problems. (Prerequisite: 21-122)

- **21-259 Calculus in Three Dimensions:** (Most Math & Physics students) Vectors, lines, planes, quadratic surfaces, polar, cylindrical and spherical coordinates, partial derivatives, directional derivatives, gradient, divergence, curl, chain rule, maximum-minimum problems, multiple integrals, parametric surfaces and curves, line integrals, surface integrals, Green-Gauss theorems. (Prerequisite: 21-122)

- **21-266 Vector Calculus for Computer Scientists:** (Most SCS students) This course is an introduction to vector calculus making use of techniques from linear algebra. Topics covered include scalar-valued and vector-valued functions, conic sections and quadric surfaces, new coordinate systems, partial derivatives, tangent planes, the Jacobian matrix, the chain rule, gradient, divergence, curl, the Hessian matrix, linear and quadratic approximation, local and global extrema, Lagrange multipliers, multiple integration, parametrised curves, line integrals, conservative vector fields, parametrised surfaces, surface integrals, Green's theorem, Stokes's theorem and Gauss's theorem. (Prerequisites: 21-122, 21-241)