21-120: Differential and Integral Calculus
Lecture #18 Outline

Read: Section 4.4 of the textbook

Objectives and Concepts:

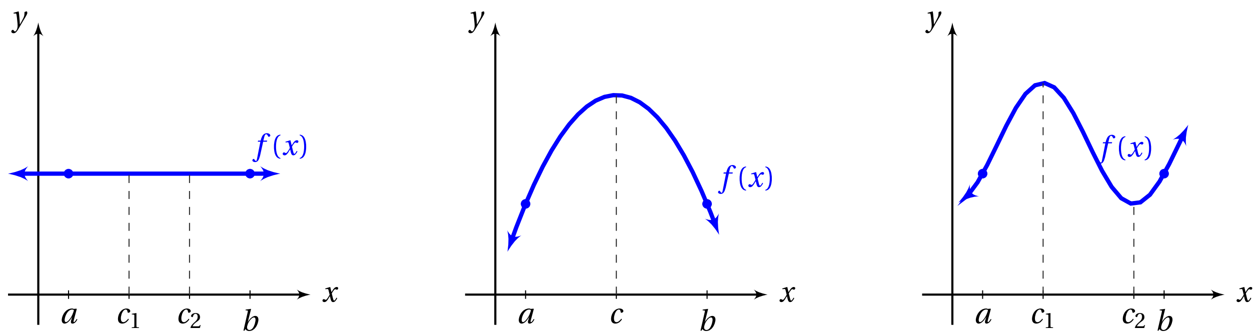
- Rolle's Theorem says that any differentiable function defined on $[a,b]$ with $f(a) = f(b) = 0$ has at least one number c between a and b such that $f'(c) = 0$.
- The Mean Value Theorem says that any differentiable function defined on $[a,b]$ with $f(a) = f(b) = 0$ has at least one number c between a and b such that $f'(c)$ is equal to the average rate of change of f on $[a,b]$.
- The Mean Value Theorem/Rolle's Theorem can be used to show that a function has at most one real root in a certain interval.

Suggested Textbook Exercises:

- 4.4: 149-155 odd, 161-183 odd.

The Mean Value Theorem

We begin our discussion of the Mean Value Theorem by first observing that in each graph below, the continuous and differentiable function $f(x)$ satisfies the property that $f(a) = f(b)$. Note that in each case, f has one or more horizontal tangent lines between a and b.



This observation is a mathematical result known as Rolle's Theorem:

Rolle's Theorem: Let f be a function that satisfies the following three conditions:

1. f is continuous on the interval $[a,b]$;
2. f is differentiable on the interval (a,b);
3. $f(a) = f(b)$

Then there is a number c in (a,b) so that $f'(c) = 0$.

Example 1: Verify the function below satisfies all three conditions of Rolle's Theorem on the given interval. Find all numbers c that satisfy the conclusion of Rolle's Theorem.

$$f(x) = 3x^2 - 12x + 5 \quad \text{over } [1,3]$$

The main use of Rolle's Theorem is proving the Mean Value Theorem, which generalizes Rolle's theorem by considering functions that do not necessarily have equal value at the endpoints.

The Mean Value Theorem: Let f be a function that satisfies the following conditions:

1. f is continuous on the interval $[a,b]$;
2. f is differentiable on the interval (a,b).

Then there is a number c in (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or, equivalently, $f(b) - f(a) = f'(c)(b-a)$.

Below we sketch the main idea behind the Mean Value Theorem:

Example 2: Find the number c guaranteed by the Mean Value Theorem for $f(x) = \arcsin(x)$ over $[-1, 1]$.

Example 3: Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

Some other results to know:

Theorem: If $f'(x) = 0$ for all x in the interval (a, b), then f is constant on (a, b).

Proof: To show that f has a constant value in the interval, we will show that if x_1 and x_2 are two points in the interval, then $f(x_1) = f(x_2)$. Suppose $x_1 < x_2$. According to the Mean Value Theorem, we have that there is a c, $x_1 < c < x_2$, such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

But $f'(c) = 0$, so $f(x_2) - f(x_1) = 0$, or $f(x_1) = f(x_2)$. Since x_1 and x_2 were any two numbers in (a, b), f is constant on all of (a, b).

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); in other words, $f(x) = g(x) + c$.

Example 4: Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?

Example 5: Does there exist an everywhere-differentiable function f such that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?

Example 6: The equation $2x - 1 - \sin(x) = 0$ has exactly one real solution. We will prove this fact. Let $f(x) = 2x - 1 - \sin(x)$.

(a) Compute $f(0)$ and $f(\pi)$.

(b) Is $f(x)$ continuous over the interval $[0, \pi]$?

(c) Is $f(x)$ differentiable over the interval $(0, \pi)$?

By the Intermediate Value Theorem, $f(x)$ has a real root in the interval $[0, \pi]$. How do we know it doesn't have more than one real root?

Suppose $f(x) = 0$ has two real roots a and b in the interval $[0, \pi]$. Then $f(a) = f(b) = 0$. According to **Rolle's Theorem**, there is a number c between a and b so that $f'(c) = 0$.

- Compute $f'(x)$. Is it ever equal to zero?

Since $f'(x) \neq 0$, we have a contradiction. Thus $f(x)$ has only one real root.