21-120: Differential and Integral Calculus
Recitation #21 Outline: 11/14/24

1. Evaluate each definite integral.

(a) $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz$

(b) $\displaystyle\int_0^{1} (x^e + e^x)\,dx$

(c) $\displaystyle\int_0^{1} (5x - 5^x)\,dx$

(d) $\displaystyle\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta$

(e) $\displaystyle\int_0^{\pi} g(t)\,dt$ where $g(x) = \begin{cases} \sin x & \text{if } 0 \le x < \pi/2, \\ \cos x & \text{if } \pi/2 \le x \le \pi \end{cases}$

Solution:

(a)

$$\begin{aligned}
\int_1^{18} \sqrt{\frac{3}{z}}\,dz &= \int_1^{18} \frac{\sqrt{3}}{\sqrt{z}}\,dz \\
&= \sqrt{3}\int_1^{18} \frac{1}{\sqrt{z}}\,dz \\
&= \sqrt{3}\int_1^{18} z^{-\frac{1}{2}}\,dz \\
&= \sqrt{3}\cdot\left(\frac{z^{\frac{1}{2}}}{\frac{1}{2}}\right)\Bigg|_1^{18} \\
&= 2\sqrt{3}\cdot\left(\sqrt{18}-\sqrt{1}\right) \\
&= 2\sqrt{3}\cdot\left(3\sqrt{2}-1\right) \\
&= 6\sqrt{6}-2\sqrt{3}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_0^{1} (x^e + e^x)\,dx &= \int_0^1 x^e\,dx + \int_0^1 e^x\,dx \\
&= \left(\frac{x^{e+1}}{e+1}\right)\Bigg|_0^1 + \left(e^x\right)\Big|_0^1 \\
&= \frac{1^{e+1}}{e+1} - \frac{0^{e+1}}{e+1} + (e^1 - e^0) \\
&= \frac{1}{e+1} + (e-1) \\
&= \frac{e^2}{e+1}.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_0^1 (5x - 5^x)\,dx &= \int_0^1 5x\,dx - \int_0^1 5^x\,dx \\
&= 5\left(\frac{x^2}{2}\right)\Big|_0^1 - \left(\frac{5^x}{\ln 5}\right)\Big|_0^1 \\
&= 5 \cdot \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - \left(\frac{5^1}{\ln 5} - \frac{5^0}{\ln 5}\right) \\
&= \frac{5}{2} - \left(\frac{5}{\ln 5} - \frac{1}{\ln 5}\right) \\
&= \frac{5}{2} - \frac{4}{\ln 5}.
\end{aligned}$$

(d)

$$\begin{aligned}
\int_0^{\pi/4} \frac{1+\cos^2\theta}{\cos^2\theta}\,d\theta &= \int_0^{\pi/4} \left(\frac{1}{\cos^2\theta} + 1\right) d\theta \\
&= \int_0^{\pi/4} \sec^2\theta\,d\theta + \int_0^{\pi/4} 1\,d\theta \\
&= \tan\theta\Big|_0^{\pi/4} + \theta\Big|_0^{\pi/4} \\
&= \tan\left(\frac{\pi}{4}\right) - \tan(0) + \left(\frac{\pi}{4} - 0\right) \\
&= 1 - 0 + \frac{\pi}{4} \\
&= 1 + \frac{\pi}{4}.
\end{aligned}$$

(e)

$$\begin{aligned}
\int_0^{\pi} g(t)\,dt &= \int_0^{\pi/2} \sin t\,dt + \int_{\pi/2}^{\pi} \cos t\,dt \\
&= (-\cos t)\Big|_0^{\pi/2} + (\sin t)\Big|_{\pi/2}^{\pi} \\
&= -\cos\left(\frac{\pi}{2}\right) + \cos(0) + \left(\sin(\pi) - \sin\left(\frac{\pi}{2}\right)\right) \\
&= -0 + 1 + (0-1) \\
&= 1 - 1 \\
&= 0.
\end{aligned}$$

2. Identify the roots of the integrand to remove absolute values, then evaluate the integral.

   (a) $\int_{-2}^{3} |x|\,dx$ (b) $\int_{-4}^{-2} |t^2 - 2t - 3|\,dt$

   Solution:

(a) Note that the function $f(x) = |x|$ has root $x = 0$ and f is positive on $(0, +\infty)$ and negative on $(-\infty, 0)$. Thus, we have

$$\begin{aligned}
\int_{-2}^{3} |x|\,dx &= \int_{-2}^{0} (-x)\,dx + \int_{0}^{3} x\,dx \\
&= \left[-\frac{x^2}{2}\right]\Bigg|_{-2}^{0} + \left[\frac{x^2}{2}\right]\Bigg|_{0}^{3} \\
&= -\frac{0^2}{2} + \frac{(-2)^2}{2} + \frac{3^2}{2} - \frac{0^2}{2} \\
&= 2 + \frac{9}{2} \\
&= \frac{13}{2}.
\end{aligned}$$

(b) Note that the function $g(t) = t^2 - 2t - 3$ has roots $t = -1$ and $t = 3$, and g positive on $(-\infty, -1) \cup (3, +\infty)$ and negative on $(-1, 3)$. Thus, $t^2 - 2t - 3 > 0$ for $x \in [-4, -2]$ and therefore $|t^2 - 2t - 3| = t^2 - 2t - 3$ for $x \in [-4, -2]$. Thus, we have

$$\begin{aligned}
\int_{-4}^{-2} |t^2 - 2t - 3|\,dt &= \int_{-4}^{-2} (t^2 - 2t - 3)\,dt \\
&= \left[\frac{t^3}{3} - t^2 - 3t\right]\Bigg|_{-4}^{-2} \\
&= \left(\frac{(-2)^3}{3} - (-2)^2 - 3(-2)\right) - \left(\frac{(-4)^3}{3} - (-4)^2 - 3(-4)\right) \\
&= \left(\frac{-8}{3} - 4 + 6\right) - \left(\frac{-64}{3} - 16 + 12\right) \\
&= \frac{-2}{3} - \left(\frac{-76}{3}\right) \\
&= \frac{74}{3}.
\end{aligned}$$

3. What is wrong with the equations below (if anything)?

(a) $\displaystyle\int_{-1}^{1} \frac{1}{x}\,dx = \ln|x|\Big|_{x=-1}^{x=1} = 0$ (b) $\displaystyle\int_{0}^{\pi} \sec^2 x\,dx = \tan x\Big|_{x=0}^{x=\pi} = 0$

Solution: The first thing we need to point out is that in this course, we only consider integrals which are finite. That is, by our definition, an integral is the limit of some Riemann sum, and we say that a function is integrable if that limit exists and is finite.

If a function f is continuous on $[a, b]$, or if f has a finite number of jump discontinuities on $[a, b]$, then f is integrable over $[a, b]$, i.e., the corresponding limit exists and is finite. But if a function f has an infinite discontinuity at some point in $[a, b]$, then it will not be integrable because the corresponding limit will not be finite. Indeed, suppose f has an infinite discontinuity at some point $\xi \in [a, b]$. Then after we divide $[a, b]$ into n parts, ξ will lie in one of the resulting subintervals, say in $[x_{i-1}, x_i]$. But then by choosing x_i^* from the definition of a Riemann sum to be close to ξ, we can make $f(x_i^*)$ as large in absolute value as we want, and so one of the terms in the Riemann sum (and as a result, the entire Riemann sum) will approach either infinity or negative infinity, making the function f non-integrable.

The function $\frac{1}{x}$ has an infinite discontinuity in $[-1,1]$, and so does the function $\sec^2 x$ in $[0,\pi]$. As a result, these functions are not integrable in the respective intervals (in the sense that the limits of the corresponding Riemann sums are infinite). The FTC is then not applicable (in fact, we only stated FTC under the assumption that the function must be continuous on the entire closed interval over which integration is taking place, and this is violated here). Also, $\ln|x|$ is not an antiderivative of $\frac{1}{x}$ *on the interval* $[-1,1]$; it is only an antiderivative of $\frac{1}{x}$ on e.g. $(0,1]$ or $[-1,0)$. And there's a similar problem with an antiderivative of $\sec^2 x$ on $[0,\pi]$ since $\sec^2 x$ is undefined at $\pi/2$.