21-120: Differential and Integral Calculus
Recitation #13 Outline: 10/08/24

1. Calculate the absolute extrema of the function $f(x) = x^3 - 3x$ in the interval $[-2,2]$.

   Solution:

   Notice that this function is always differentiable in the interval. Let's calculate the derivative and set it to zero. We obtain that $f'(x) = 3x^2 - 3 = 0$, which implies that $x^2 = 1$, leading to $x = -1$ or $x = 1$. Let's calculate $f(1) = -2$ and $f(-1) = 2$. We will also evaluate the function at the endpoints of the interval $[-2,2]$, that is, at -2 and 2. We find that $f(-2) = -2$ and $f(2) = 2$. By the Extreme Value Theorem, $\max_{x\in[-2,2]} f(x) = 2$ and $\min_{x\in[-2,2]} f(x) = -2$.

2. Find the absolute extrema of the function $f(x) = |x^2 - 9|$ in the interval $[-4,4]$.

   Solution:

   Notice that the function $f(x) = x^2 - 9$ if $x^2 - 9 \geq 0$, that is, if $x \geq 3$ or $x \leq -3$, and the function is equal to $9 - x^2$ if $x^2 - 9 \leq 0$, that is, if $-3 \leq x \leq 3$. Now, by taking the derivative in the first case and in the second case and setting it to zero, we obtain that $x = 0$. Let's evaluate at $x = 0$, yielding $f(0) = 9$. We must calculate where the function is not differentiable, which occurs at $x = -3$ and $x = 3$, giving us $f(3) = f(-3) = 0$. Finally, we evaluate at the endpoints (-4 and 4), yielding $f(4) = f(-4) = 7$. Thus, by the Extreme Value Theorem, $\max_{x\in[-4,4]} f(x) = 9$ and $\min_{x\in[-4,4]} f(x) = 0$.

3. Calculate the absolute maxima and minima of the function $f(x) = \dfrac{\ln(x)}{x}$ in the interval $[1,100]$.

   Solution:

   Note that f is differentiable in the entire interval. By the Quotient rule, the derivative of f equals to $f'(x) = \dfrac{1-\ln(x)}{x^2}$. Thus, if $f'(x) = 0$ then $x = e$ and in this case $f(e) = \dfrac{\ln(e)}{e} = \dfrac{1}{e}$. Moreover, let us compute the values of the function in 1 and 100 : $f(1) = \dfrac{\ln(1)}{1} = 0$ and $f(100) = \dfrac{2\ln(10)}{100}$. Thus, by the Extreme Value Theorem, $\max_{x\in[1,100]} f(x) = \dfrac{1}{e}$ and $\min_{x\in[1,100]} f(x) = 0$.

4. Find the absolute maximum and absolute minimum values of $f(x) = \dfrac{x^2-4}{x^2+4}$ on the interval $[-4,4]$.

   Solution:

   First, find the critical points by finding where the derivative equals zero. The derivative is given by:

   $$f'(x) = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{(2x^3+8x) - (2x^3-8x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}.$$

   Therefore, $f'(x) = 0$ when $16x = 0$, meaning that $x = 0$ is the only critical point. Plugging in the endpoints and the critical point gives:

   $$f(-4) = \frac{(-4)^2-4}{(-4)^2+4} = \frac{12}{20} = \frac{3}{5},$$

   $$f(0) = \frac{0^2-4}{0^2+4} = \frac{-4}{4} = -1,$$

$$f(4) = \frac{4^2 - 4}{4^2 + 4} = \frac{12}{20} = \frac{3}{5}.$$

Therefore, by the Extreme Value Theorem; f achieves its absolute minimum of -1 at $x = 0$ and its absolute maximum of $\frac{3}{5}$ at both $x = -4$ and $x = 4$.

5. Suppose the side length of a cube is measured to be 5 cm with an accuracy of 0.1 cm. Use differentials to estimate the error in the computed volume of the cube.

Solution:

The measurement of the side length is accurate to within ± 0.1 cm. Therefore,

$$-0.1 \leq dx \leq 0.1.$$

The volume of a cube is given by $V = x^3$, which leads to

$$dV = 3x^2\,dx.$$

Using the measured side length of 5 cm, we can estimate the error as:

$$-3(5)^2(0.1) \leq dV \leq 3(5)^2(0.1).$$

Thus,

$$-7.5 \leq dV \leq 7.5.$$