21-120: Differential and Integral Calculus
Lecture #17 Outline

Read: Section 4.3 of the textbook

Objectives and Concepts:

- Continuous functions on closed intervals reach an absolute maximum and minimum on the interval.
- Local maxima and minima may occur at places where the derivative of the function is zero or the derivative is undefined. A value c such that $f(c)$ is zero or undefined is a critical number of f.
- To find the absolute max and min of a function on a closed interval, check the function values at the endpoints of the interval and at any critical numbers of f.

Suggested Textbook Exercises:

- 4.3: 91-97 odd, 101-133 odd.

Differentials

Let $y = f(x)$ be a differentiable function.

- The differential dx is an independent variable.
- The differential dy is $dy = f'(x)dx$. It is a dependent variable because it depends on the values of x and dx.
- If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy, which represents an **estimated** change in y, is determined.
- If $dx = \Delta x$, the corresponding (**actual**) change in y is $\Delta y = f(x + \Delta x) - f(x)$

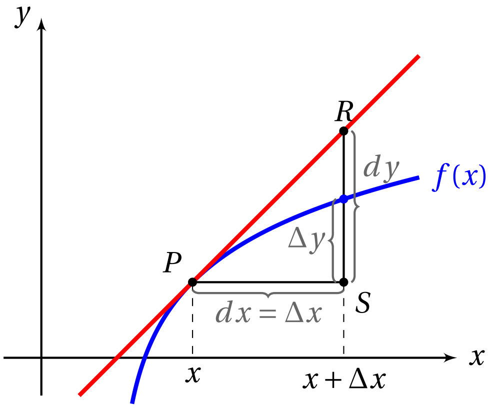
To compute differentials, we simply differentiate both sides of $y = f(x)$:

Example 1: Find dy if $y = \cos(x^4 - 3x^2 + 5)$

Sometimes we have multiple independent variables that contribute to the dependent variable. In these cases we will end up with several factors that affect the differential dy:

Example 2: Find dV if $V = \frac{1}{3}\pi r^2 h$

What's the difference between dy and Δy?



- dy represents the amount that the tangent line rises or falls:

$$dy = f'(x)dx$$

- Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by amount $dx = \Delta x$:

$$\Delta y = f(x+\Delta x) - f(x)$$

If $dx = \Delta x$, the corresponding change in y is

$$\Delta y = f(x+\Delta x) - f(x).$$

For small values of dx, the change in y is approximately the same as the change in the tangent line dy:

$$dy \approx \Delta y, \qquad \text{when } dx \text{ is small.}$$

This is useful because dy may be easier to calculate than Δy. In this case, dy may be thought of as the **error in calculating the value for** y, provided the error of dx is made in estimating x. The ratio $\dfrac{dy}{y}$ is called the **relative error** and is usually given as a percent.

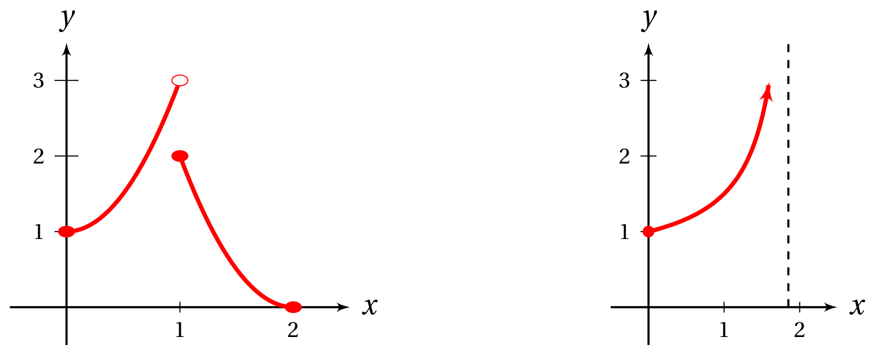
Example 3: The sides of a square field are measured and found to be 50 meters, with a possible error of 0.02 m in the measurement. We calculate this area to be 2500 square meters. Estimate the maximum error and relative error in this calculation.

Maximum and Minimum Values

One of the most important applications of differentiation (and calculus in general) is optimization - finding where **extreme values** (i.e., maximum and minimum values) of a function occur. Before we begin, we first give a formal mathematical definition for global extreme values.

Definition: Let $f(x)$ be a function defined over a domain D. The function f has an **absolute maximum** (or **global** maximum) at c provided $f(c) \geq f(x)$ for all x in D. Similarly, the function f has an **absolute minimum** (or **global** minimum) at c provided $f(c) \leq f(x)$ for all x in D.

Example 4: Determine from the graph whether each function has any absolute extrema on $[0,2]$.



Note that neither function achieved a maximum value over the interval! The following theorem, the proof of which requires advanced mathematical understanding of properties of real numbers, indicates when we can guarantee that a function has an absolute max and min.

The Extreme Value Theorem: If f is continuous at every point in the closed interval $[a,b]$, then f assumes both an absolute maximum value M and an absolute minimum value m somewhere on $[a,b]$.

We can also define local extreme values, which can occur in multiple locations along the graph of a curve.

Definition: A function f has a **local maximum** at c if $f(c) \geq f(x)$ for all x in some open interval containing c. Similarly, a function f has a **local minimum** at c if $f(c) \leq f(x)$ for all x in some open interval containing c.

Fermat's Theorem: If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Warning! The converse of Fermat's Theorem is false - it is not necessarily the case that if $f'(c) = 0$, then f has a local max or min at c.

Consider $g(x) = x^3$. Notice that $g'(x) = 3x^2$ and so $g'(0) = 0$. However, $x = 0$ is neither a maximum nor minimum for this function. Just because the derivative is zero does *NOT* mean the function has an extrema there.

Warning! A function may have a local max or min at places where the derivative is not defined.

Consider $h(x) = |x|$. Notice that $h(x)$ has an absolute minimum at $x = 0$, but this cannot be found by setting $h'(x) = 0$ and solving because $h'(0)$ does not exist.

Definition: A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist. The location $\big(c, f(c)\big)$ is called a **critical point** of $f(x)$.

Theorem: Let f be a function defined on a closed interval $[a,b]$ containing the point c. If $f(c)$ is an extreme value, then c must be an *endpoint of the interval* or a *critical number*; that is,

- c is an endpoint of $[a,b]$; or
- c is an interior point such that $f'(c) = 0$; or
- c is an interior point such that $f'(c)$ is undefined.

This Theorem gives us a way to find absolute extrema. To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

1. Find the critical numbers of $f(x)$ and use them to find the critical points of $f(x)$.
2. Find the values of f at the endpoints.
3. Compare the critical points and the endpoint values. The largest value of f is the absolute maximum value; the smallest value of f is the absolute minimum value.

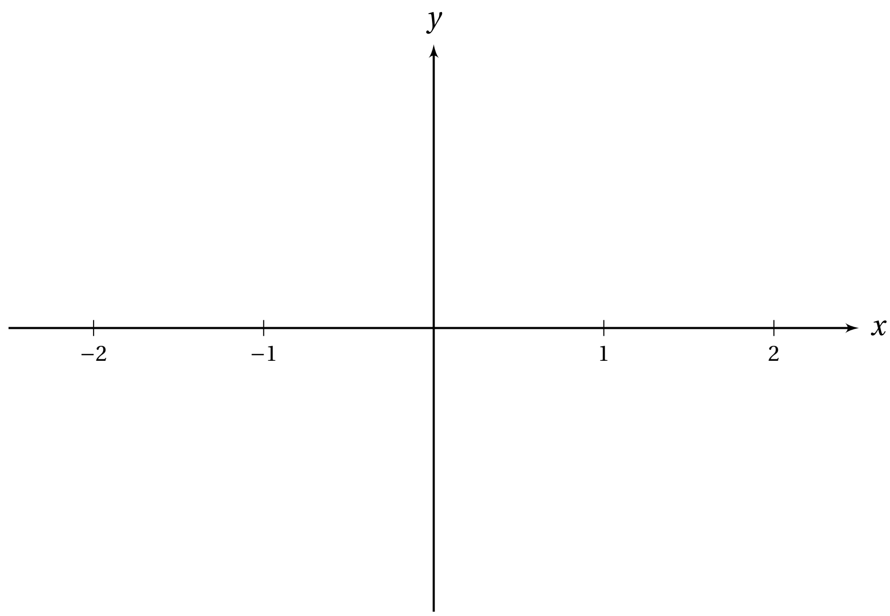
Example 5: Find the absolute max and absolute min value of each function on the given interval.

(a) $y = e^x + e^{-(x+1)}$ over $[-1,1]$

(b) $f(x) = \sin^2(x) - \cos(x)$ over $\left[-\frac{\pi}{4}, \pi\right]$

(c) $f(x) = x(x+1)^{2/3}$ over $[-1, 1]$

Example 6: Sketch a function f on the closed interval $[-2, 2]$ that is defined at every point, but for which the absolute extrema do not exist.



Example 7: Sketch a graph of a function that has two local minima, one local maximum, and no absolute maximum.