21-120: Differential and Integral Calculus
Lecture #8 Outline

Read: Section 3.2 of the textbook

Objectives and Concepts:

- The derivative of a function f is a function whose value is the derivative of f at every point a (i.e., $f'(a)$) in its domain, provided the derivative exists.
- The derivative of a function can be found by evaluating the limit definition of the derivative.
- The graphs of f and its derivative f' are related as f' indicates where f is increasing, decreasing, and constant.
- Second, third, and higher-order derivatives of f can be defined whenever they exist.
- A function fails to be differentiable at a point a when $f'(a)$ does not exist. This can occur at a cusp, a discontinuity, or when the curve has a vertical tangent line.

Suggested Textbook Exercises:

- 3.2: 55-83 odd, 91-97 odd.

The Derivative of a Function

Definition: The derivative of a function f at x, denoted by $f'(x)$, is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided this limit exists.

So the derivative of a function is a function itself - its domain will be a subset of the domain of f. An alternative notation for derivatives was introduced by Leibniz when calculus was first being developed. We saw in the last section that $f'(x)$ is approximated by the average rate of change over smaller and smaller intervals. If $y = f(x)$, this means

$$f'(x) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}.$$

Notation: All of the following represent the derivative of $y = f(x)$.

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}f(x) \qquad D_x f \qquad \dot{y}$$

Now derivatives are limits, so one way they can be calculated is by using the definition above. When we say use the “limit definition of the derivative” we mean finding $f'(x)$ by evaluating the limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

Example 1: Use the limit definition of the derivative to compute $f'(x)$ for $f(x) = \dfrac{2x}{5-x}$.

Example 2: Let $g(x) = x + \sqrt{x}$.

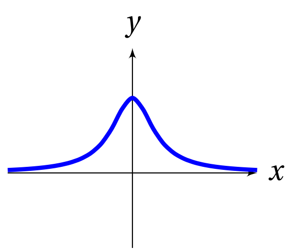
(a) Use the limit definition of the derivative to find the derivative of $g(x)$.

(b) Find an equation of the tangent line to $g(x)$ when $x = 4$.

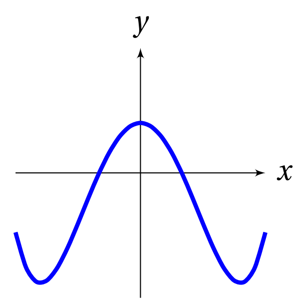
The sign of the derivative $f'(x)$ tells us whether the function f is increasing or decreasing.

- If $f' > 0$ on an interval, then f is <u>increasing</u> over that interval.
- If $f' < 0$ on an interval, then f is <u>decreasing</u> over that interval.
- If $f' = 0$ over an interval, then f is <u>constant</u> over that interval.

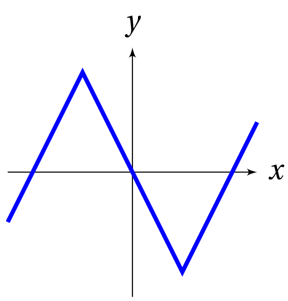
Example 3: Each of the graphs labeled (A)-(D) displays a function $g(x)$, and each of the graphs labeled (I)-(IV) displays a derivative $g'(x)$ and is labeled (I) to (IV). Match each graph with its corresponding derivative graph.



(A)



(B)



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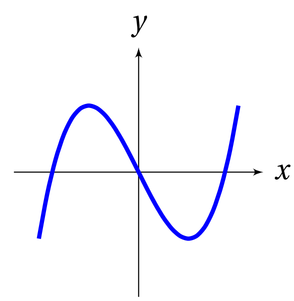

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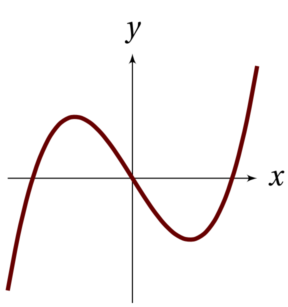
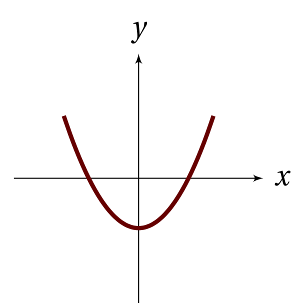
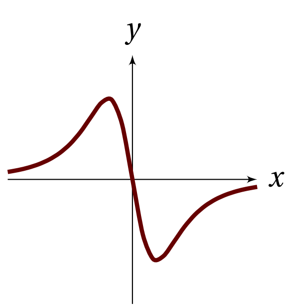


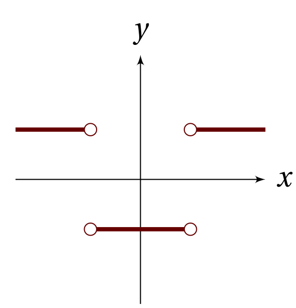
(I)



(II)



(III)



(IV)

Higher-Order Derivatives

Just as $f'(x)$ describes the rate of change of $f(x)$, the **second derivative** $f''(x)$ describes the rate of change of $f'(x)$, provided it exists. All of the following are acceptable notation for the second derivative of $y = f(x)$:

$$f''(x) \qquad f^{(2)}(x) \qquad y'' \qquad \ddot{y} \qquad \frac{d^2}{dx^2}\big(f(x)\big) \qquad \frac{d^2y}{dx^2}$$

The last two notations stem from the idea of applying the differentiation operator to the first derivative:

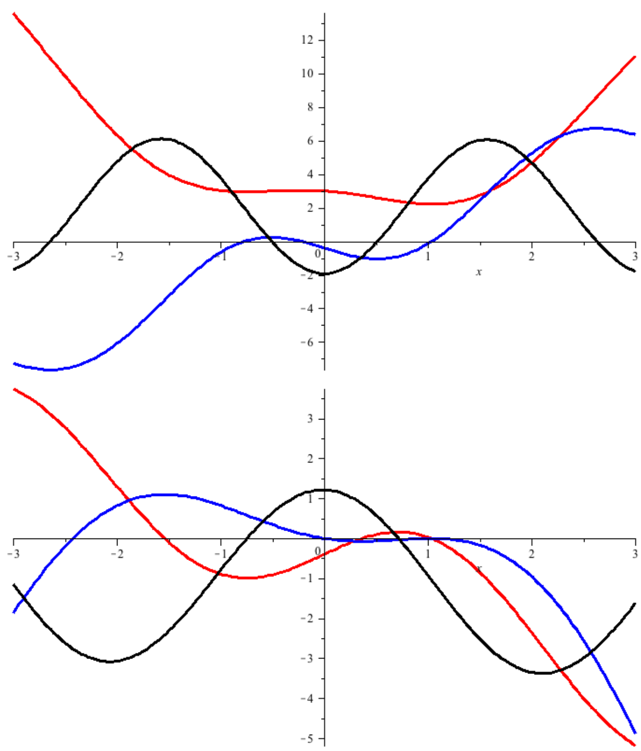
$$\frac{d^2}{dx^2}(f(x)) = \frac{d}{dx}\left(\frac{d}{dx}(f(x))\right), \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The **third derivative** $f'''(x)$ is defined in the same way. The n**th derivative** of f is usually denoted by either $f^{(n)}(x)$ or $\dfrac{d^ny}{dx^n}$.

As we saw from the previous example, we can use our understanding of derivatives as rates of change to help determine how the graphs of f, f', and f'' are related. Here are some facts that you can use in problems such a the example below:

- Since the graph of f' represents the rate of change of the graph of f, f' will be 0 whenever f has a horizontal tangent line. This means that at points where the graph of f changes from increasing to decreasing (or vice versa), the graph of f' will cross the x-axis. Also, when f is increasing, the graph of f' should be above the x-axis, and it should be below the x-axis when f is decreasing.
- Similarly, as the graph of f'' represents the rate of change of f', whenever the graph of f' has a horizontal tangent, the graph of f'' will cross the x-axis. Also, the graph of f'' will be above the x-axis when f' is increasing, and below when f' is decreasing.

Example 4: A function f and its first two derivatives are plotted on each set of axes. Label each graph with f, f', or f''.



How Can A Function Fail To Be Differentiable?

The geometric interpretation of $f'(a)$ as the slope of the tangent line to the graph of $y = f(x)$ when $x = a$ leads to these three situations in which $f'(a)$ is not defined:

1. The graph has a sharp corner or "cusp" at $x = a$, in which case there is no tangent line at $x = a$;
2. The graph has a jump or a hole at $x = a$, in which case there is no tangent line at $x = a$;
3. The graph has a vertical tangent line at $x = a$, in which case there is a tangent line (but its slope is undefined).

Theorem: If $f'(a)$ exists, then f is continuous at a.

Warning: If f is continuous at a, then $f'(a)$ may or may not exist. In other words, continuity does **not** imply differentiability.

Example 5: The graph of $y = f(x)$ is given. Find all x-coordinates where the function is not continuous. Find all x-coordinates where the function is not differentiable.