21-120: Differential and Integral Calculus
Recitation #15 Outline: 10/22/24

1. Given that

$$\lim_{x\to a} f(x) = 0 \qquad \lim_{x\to a} g(x) = 0 \qquad \lim_{x\to a} h(x) = 1 \qquad \lim_{x\to a} p(x) = \infty \qquad \lim_{x\to a} q(x) = \infty$$

which of the following limits are indeterminate forms? For those that are not, evaluate the limit where possible.

(a) $\lim_{x\to a} \frac{f(x)}{g(x)}$

(b) $\lim_{x\to a} \frac{f(x)}{p(x)}$

(c) $\lim_{x\to a} \frac{p(x)}{q(x)}$

(d) $\lim_{x\to a} \frac{p(x)}{f(x)}$

(e) $\lim_{x\to a} \frac{h(x)}{p(x)}$

(f) $\lim_{x\to a} f(x)p(x)$

(g) $\lim_{x\to a} p(x)q(x)$

(h) $\lim_{x\to a} h(x)p(x)$

Solution:

(a) $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{0}{0}$. This is an indeterminate form.

(b) $\frac{\lim_{x\to a} f(x)}{\lim_{x\to a} p(x)} = \frac{0}{\infty}$. This is <u>not</u> an indeterminate form and $\lim_{x\to a} \frac{f(x)}{p(x)} = 0$.

(c) $\frac{\lim_{x\to a} p(x)}{\lim_{x\to a} q(x)} = \frac{\infty}{\infty}$. This is an indeterminate form.

(d) $\frac{\lim_{x\to a} p(x)}{\lim_{x\to a} f(x)} = \frac{\infty}{0}$. This is <u>not</u> an indeterminate form and we cannot evaluate $\lim_{x\to a} \frac{p(x)}{f(x)}$ without more information.

(e) $\frac{\lim_{x\to a} h(x)}{\lim_{x\to a} p(x)} = \frac{1}{\infty}$. This is <u>not</u> an indeterminate form and $\lim_{x\to a} \frac{h(x)}{p(x)} = 0$.

(f) $\lim_{x\to a} f(x) \lim_{x\to a} p(x) = 0\cdot\infty$. This is an indeterminate form.

(g) $\lim_{x\to a} p(x) \lim_{x\to a} q(x) = \infty\cdot\infty$. This is <u>not</u> an indeterminate form and $\lim_{x\to a} p(x)q(x) = \infty$.

(h) $\lim_{x\to a} h(x) \lim_{x\to a} p(x) = 1\cdot\infty$. This is <u>not</u> an indeterminate form and $\lim_{x\to a} h(x)p(x) = \infty$.

2. Find the limit using l'Hospital's rule.

(a) $\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}}$

(b) $\lim_{t\to 0} \frac{8^t - 5^t}{t}$

(c) $\lim_{x\to-\infty} x\ln\left(1-\frac{1}{x}\right)$

(d) $\lim_{x\to 0} \frac{x-\sin(x)}{x-\tan(x)}$

(e) $\lim_{x\to\infty} \sqrt{x}e^{-x/2}$

(f) $\lim_{x\to 0} \frac{x3^x}{3^x-1}$

Solution:

(a) This limit has indeterminate form $\frac{\infty}{\infty}$. Applying l'Hospital's Rule we see that

$$\lim_{x\to\infty} \frac{\ln(x)}{\sqrt{x}} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}\sqrt{x}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x} = \lim_{x\to\infty} \frac{2}{\sqrt{x}} = 0.$$

(b) This limit has indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule we see that

$$\lim_{t\to 0}\frac{8^t-5^t}{t}\overset{\frac{0}{0}}{=}\lim_{t\to 0}\frac{\frac{d}{dt}(8^t-5^t)}{\frac{d}{dt}t}=\lim_{t\to 0}\frac{\ln(8)8^t-\ln(5)5^t}{1}=\ln(8)8^0-\ln(5)5^0=\ln(8)-\ln(5).$$

(c) This limit has indeterminate form $\infty\cdot 0$. We thus first rewrite the function so that the limit has indeterminate form $\frac{0}{0}$,

$$x\ln\left(1-\frac{1}{x}\right)=\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}.$$

Applying l'Hospital's Rule we see that

$$\lim_{x\to-\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}\overset{\frac{0}{0}}{=}\lim_{x\to-\infty}\frac{\frac{d}{dx}\ln\left(1-\frac{1}{x}\right)}{\frac{d}{dx}\frac{1}{x}}=\lim_{x\to-\infty}\frac{\frac{1}{x^2-x}}{-\frac{1}{x^2}}=\lim_{x\to-\infty}-\frac{x^2}{x^2-x}=-1.$$

(d) This limit has indeterminate form $\frac{0}{0}$. As the derivative of the numerator and denominator are somewhat complicated we first compute and factor them

$$\frac{d}{dx}(x-\sin(x))=1-\cos(x),$$

$$\frac{d}{dx}(x-\tan(x))=1-\sec^2(x)=\sec^2(x)(\cos^2(x)-1)=\sec^2(x)(\cos(x)-1)(\cos(x)+1).$$

Applying l'Hospital's Rule we see that

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin(x)}{x-\tan(x)}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\frac{d}{dx}(x-\sin(x))}{\frac{d}{dx}(x-\tan(x))}&=\lim_{x\to 0}\frac{1-\cos(x)}{\sec^2(x)(\cos(x)-1)(\cos(x)+1)}\\&=\lim_{x\to 0}-\frac{\cos^2(x)}{(\cos(x)+1)}\\&=-\frac{1)}{(1+1)}\\&=-\frac{1}{2}.\end{aligned}$$

(e) This limit has indeterminate form $\infty\cdot 0$. We thus first rewrite the function so the limit has indeterminate form $\frac{\infty}{\infty}$,

$$\sqrt{x}e^{-x/2}=\frac{\sqrt{x}}{e^{x/2}}.$$

Applying l'Hospital's Rule we see that

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to\infty}\frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}e^{x/2}}=\lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2}e^{x/2}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}e^{x/2}}=0.$$

What would have happened if we instead tried to rewrite the function so the limit has indeterminate form $\frac{0}{0}$?

(f) This limit has indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule we see that

$$\lim_{x\to 0}\frac{x3^x}{3^x-1}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\frac{d}{dx}x3^x}{\frac{d}{dx}(3^x-1)}=\lim_{x\to 0}\frac{3^x+x\ln(3)3^x}{\ln(3)3^x}=\lim_{x\to 0}\frac{1+x\ln(3)}{\ln(3)}=\frac{1}{\ln(3)}.$$

3. If f' is continuous, $f(2)=0$, and $f'(2)=7$, evaluate

$$\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}$$

Solution: Since f is differentiable, f is continuous and we find that

$$\lim_{x\to 0}f(2+3x)+f(2+5x)=f(2)+f(2)=0,$$

thus this limit has the indeterminate form of $\dfrac{0}{0}$. Applying l'Hospitals Rule we thus find that

$$\begin{aligned}\lim_{x\to 0}\frac{f(2+3x)+f(2+5x)}{x}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\frac{d}{dx}(f(2+3x)+f(2+5x))}{\frac{d}{dx}x}&=\lim_{x\to 0}\frac{3f'(2+3x)+5f'(2+5x)}{1}\\&=\lim_{x\to 0}3f'(2+3x)+5f'(2+5x)\end{aligned}$$

where we've used the Chain Rule in the second equality. Since f' is continuous

$$\lim_{x\to 0}3f'(2+3x)+5f'(2+5x)=3f'(2)+5f'(2)=8f'(2)=8\cdot 7=56.$$

4. If an object with mass m is dropped from rest, one model for its speed v after t seconds, taking air resistance into account, is

$$v=\frac{mg}{c}(1-e^{-ct/m})$$

where g is the acceleration due to gravity and c is a positive constant that governs the strength of the air resistance.

(a) Calculate $\lim_{t\to\infty}v$. What is the meaning of this limit?

(b) For fixed t, use l'Hospital's Rule to calculate $\lim_{c\to 0^+}v$. What can you conclude about the velocity of a falling object in a vacuum?

Solution:

(a) If $c>0$ then

$$\lim_{t\to\infty}v=\lim_{t\to\infty}\frac{mg}{c}(1-e^{-ct/m})=\frac{mg}{c}(1-\lim_{t\to\infty}e^{-ct/m})=\frac{mg}{c}(1-0)=\frac{mg}{c}.$$

This shows that as $t\to\infty$ the velocity approaches some finite positive constant. This is called the **terminal velocity**, or the maximum speed an object can fall at given there is air resistance c.

(b) Fixing t and applying l'Hospitals Rule

$$\lim_{c\to 0^+}v=\lim_{c\to 0^+}\frac{mg(1-e^{-ct/m})}{c}\overset{\frac{0}{0}}{=}\lim_{c\to 0^+}\frac{\frac{d}{dc}mg(1-e^{-ct/m})}{\frac{d}{dc}c}=\lim_{c\to 0^+}\frac{mg(\frac{t}{m}e^{-ct/m})}{1}=gt.$$

Thus the velocity of v increases linearly at rate g. This says that when there is no air resistance, an object accelerates ($a=\frac{d}{dt}v=g$) at a constant rate of g.