21-120: Differential and Integral Calculus
Recitation #25 Outline: 12/03/24

1. Evaluate each indefinite integral.

   (a) $\displaystyle\int e^{2y}\sin(2y)\,dy$ (b) $\displaystyle\int t\tan^2 t\,dt$

Solution:

(a) Let

$$u=\sin(2y),\quad dv=e^{2y}\,dy,$$

so that

$$du=2\cos(2y)\,dy,\quad v=\frac{1}{2}e^{2y}.$$

Applying the integration by parts formula, we get:

$$\int e^{2y}\sin(2y)\,dy=\frac{1}{2}e^{2y}\sin(2y)-\int\frac{1}{2}e^{2y}2\cos(2y)\,dy=\frac{1}{2}e^{2y}\sin(2y)-\int e^{2y}\cos(2y)\,dy.$$

Now, we need to evaluate the remaining integral $\int e^{2y}\cos(2y)\,dy$. Let

$$u=\cos(2y),\quad dv=e^{2y}\,dy,$$

so that

$$du=-2\sin(2y)\,dy,\quad v=\frac{1}{2}e^{2y}.$$

Applying integration by parts again, we get

$$\int e^{2y}\cos(2y)\,dy=\frac{1}{2}e^{2y}\cos(2y)-\int\frac{1}{2}e^{2y}(-2\sin(2y))\,dy=\frac{1}{2}e^{2y}\cos(2y)+\int e^{2y}\sin(2y)\,dy.$$

Thus, we have

$$\begin{aligned}\int e^{2y}\sin(2y)\,dy&=\frac{1}{2}e^{2y}\sin(2y)-\left(\frac{1}{2}e^{2y}\cos(2y)+\int e^{2y}\sin(2y)\,dy\right)\\&=\frac{1}{2}e^{2y}\sin(2y)-\frac{1}{2}e^{2y}\cos(2y)-\int e^{2y}\sin(2y)\,dy.\end{aligned}$$

Now, add $\displaystyle\int e^{2y}\sin(2y)\,dy$ to both sides:

$$\begin{aligned}2\int e^{2y}\sin(2y)\,dy&=\frac{1}{2}e^{2y}\sin(2y)-\frac{1}{2}e^{2y}\cos(2y)-\int e^{2y}\sin(2y)\,dy+\int e^{2y}\sin(2y)\,dy\\&=\frac{1}{2}e^{2y}\sin(2y)-\frac{1}{2}e^{2y}\cos(2y)+\int\left(e^{2y}\sin(2y)-e^{2y}\sin(2y)\right)\,dy\\&=\frac{1}{2}e^{2y}\sin(2y)-\frac{1}{2}e^{2y}\cos(2y)+\int 0\,dy\\&=\frac{1}{2}e^{2y}\sin(2y)-\frac{1}{2}e^{2y}\cos(2y)+C\end{aligned}$$

Dividing both sides by two, we get

$$\int e^{2y}\sin(2y)\,dy = \frac{1}{4}e^{2y}\left(\sin(2y)-\cos(2y)\right)+C.$$

(In the last line, we should have written $C/2$ instead of C, but C represents an arbitrary constant if and only if $C/2$ does. So this abuse of notation is fine. If this bothers you, you can replace C in the last line with C_1, or something similar.)

(b) Let

$$u = t, \quad dv = \tan^2 t\,dt,$$

so that

$$du = dt, \quad v = \tan t - t.^{1}$$

Using the integration by parts, we get

$$\int t\tan^2 t\,dt = t(\tan t - t) - \int (\tan t - t)\,dt.$$

Now, we compute the remaining integral:

$$\int (\tan t - t)\,dt = \int \tan t\,dt - \int t\,dt = -\ln|\cos t| - \frac{t^2}{2}.$$

Thus, we have

$$\int t\tan^2 t\,dt = t\tan t - t^2 - \left(-\ln|\cos t| - \frac{t^2}{2}\right).$$

Simplifying, we obtain

$$\int t\tan^2 t\,dt = t\tan t - \frac{t^2}{2} + \ln|\cos t| + C.$$

2. Evaluate each definite integral.

(a) $\displaystyle\int_0^{1/2} x\cos\pi x\,dx$ (b) $\displaystyle\int_1^{\sqrt{3}} \arctan\left(\frac{1}{\theta}\right)d\theta$

Solution:

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$$\begin{aligned}\int \tan^2 x\,dx &= \int \frac{1-\cos^2 x}{\cos^2 x}\,dx\\ &= \int \frac{1}{\cos^2 x}\,dx - \int 1\,dx\\ &= \int \sec^2 x\,dx - \int 1\,dx\\ &= \tan x - x + C.\end{aligned}$$

(a) Let
$$u = x, \quad dv = \cos(\pi x)\,dx,$$
so that
$$du = dx, \quad v = \frac{\sin(\pi x)}{\pi}.$$
Applying the integration by parts formula, we get:
$$\int_0^{1/2} x\cos(\pi x)\,dx = \left[x \cdot \frac{\sin(\pi x)}{\pi}\right]_0^{1/2} - \int_0^{1/2} \frac{\sin(\pi x)}{\pi}\,dx.$$
The first term is:
$$\left[x \cdot \frac{\sin(\pi x)}{\pi}\right]_0^{1/2} = \frac{1}{2} \cdot \frac{\sin\left(\pi \cdot \frac{1}{2}\right)}{\pi} - 0 = \frac{1}{2\pi} \cdot 1 = \frac{1}{2\pi}.$$
The second term is:
$$\int_0^{1/2} \frac{\sin(\pi x)}{\pi}\,dx = \left[-\frac{\cos(\pi x)}{\pi}\right]_0^{1/2} = -\frac{1}{\pi}\left(\cos\left(\frac{\pi}{2}\right) - \cos(0)\right) = \frac{1}{\pi^2}.$$
Thus, combining everything, we get
$$\int_0^{1/2} x\cos(\pi x)\,dx = \frac{1}{2\pi} - \frac{1}{\pi^2}.$$

(b) Let
$$u = \arctan\left(\frac{1}{\theta}\right), \quad dv = d\theta.$$
Then, we compute du and v:
$$du = \frac{-1}{1+\theta^2}\,d\theta, \quad v = \theta.$$
Applying the integration by parts formula, we get:
$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{\theta}\right)d\theta = \left[\theta \cdot \arctan\left(\frac{1}{\theta}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{\theta}{1+\theta^2}\,d\theta.$$
The first term is:
$$\left[\theta \cdot \arctan\left(\frac{1}{\theta}\right)\right]_1^{\sqrt{3}} = \sqrt{3} \cdot \arctan\left(\frac{1}{\sqrt{3}}\right) - 1 \cdot \arctan(1) = \sqrt{3} \cdot \frac{\pi}{6} - \frac{\pi}{4}.$$
The second term is:
$$\int_1^{\sqrt{3}} \frac{\theta}{1+\theta^2}\,d\theta = \left[\frac{1}{2}\ln(1+\theta^2)\right]_1^{\sqrt{3}} = \frac{1}{2}\ln(4) - \frac{1}{2}\ln(2) = \frac{1}{2}\ln(2).$$
Thus, combining both terms:
$$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{\theta}\right)d\theta = \sqrt{3} \cdot \frac{\pi}{6} - \frac{\pi}{4} + \frac{1}{2}\ln(2).$$

3. (a) Show that

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

where $n \geq 2$ is an integer.

(b) Use the previous part to show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\cos x \sin 2x}{2} + C.$$

(c) Use the previous two parts to evaluate $\int \sin^4 x\,dx$.

Solution:

(a) Let

$$u = \sin^{n-1} x, \quad dv = \sin x\,dx.$$

Then

$$du = (n-1)\sin^{n-2} x \cos x\,dx, \quad v = -\cos x.$$

Applying the integration by parts formula, we get:

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + \int (n-1)\cos^2 x \sin^{n-2} x\,dx.$$

Now, use the identity $\cos^2 x = 1 - \sin^2 x$ to simplify the second term:

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x(1 - \sin^2 x)\,dx.$$

Expanding the second integral:

$$\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx - (n-1)\int \sin^n x\,dx.$$

Finally, move the last term to the left side to isolate the desired integral:

$$\int \sin^n x\,dx + (n-1)\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx.$$

This simplifies to:

$$n\int \sin^n x\,dx = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\,dx.$$

Finally, divide both sides by n to obtain the desired result:

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

(b) To show that

$$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C,$$

we use the result from the previous part, where we derived the following formula for $n = 2$:

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

For $n = 2$, this simplifies to:

$$\begin{aligned}\int \sin^2 x\,dx &= -\frac{1}{2}\cos x \sin x + \frac{1}{2}\int dx \\ &= -\frac{1}{2}\cos x \sin x + \frac{x}{2} + C.\end{aligned}$$

(c) To evaluate

$$\int \sin^4 x\,dx,$$

we use the reduction formula for $\int \sin^n x\,dx$:

$$\int \sin^n x\,dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx.$$

For $n = 4$, this gives:

$$\int \sin^4 x\,dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\int \sin^2 x\,dx.$$

Using the result from the previous part for $\int \sin^2 x\,dx = -\frac{1}{2}\cos x \sin x + \frac{x}{2} + C$, we get the answer:

$$\int \sin^4 x\,dx = -\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}\left(-\frac{1}{2}\cos x \sin x + \frac{x}{2} + C\right).$$