21-120: Differential and Integral Calculus
Recitation #4

1. (a) Using the Squeeze Theorem, evaluate the following limit:

$$\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$$

Solution: We see that $-1 \le \cos\left(\frac{2}{x}\right) \le 1$ multiplying both sides with the positive number x^4 one gets

$$-x^4 \le x^4 \cos(\frac{2}{x}) \le x^4$$

Since $\lim_{x\to 0}(-x^4) = 0$ and $\lim_{x\to 0} x^4 = 0$, by the Squeeze Theorem one concludes that

$$\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0$$

(b) Using the Squeeze Theorem, evaluate the following limit:

$$\lim_{x\to 1}(x-1)^2 \cos\left(\frac{1}{x-1}\right)$$

Solution: Since $-1 \le \cos\left(\frac{1}{x-1}\right) \le 1$, multiplying both sides by the positive number $(x-1)^2$ one gets that:

$$-(x-1)^2 \le (x-1)^2 \cos(\frac{1}{x-1}) \le (x-1)^2.$$

Given that $\lim_{x\to 1}(-(x-1)^2) = \lim_{x\to 1}(x-1)^2 = 0$, by the Squeeze Theorem we get that

$$\lim_{x\to 1}(x-1)^2 \cos\left(\frac{1}{x-1}\right) = 0.$$

(c) Let f be a function. If $4x-9 \le f(x) \le x^2-4x+7$ for $x \ge 0$, using the Squeeze Theorem, evaluate the following limit

$$\lim_{x\to 4} f(x)$$

Solution:

Since $\lim_{x\to 4}(4x-9) = 7$ and $\lim_{x\to 4}(x^2-4x+7) = 7$, by the Squeeze Theorem we have $\lim_{x\to 4} f(x) = 7$.

2. We consider the function f defined as follows:

$$f(x) = \begin{cases} 6x+8 & \text{if } x \le -1 \\ -3x+7 & \text{if } -1 < x < 2 \\ x-1 & \text{if } x \ge 2. \end{cases}$$

Is the function f continuous at -1 ? Is the function f continuous at 2?

Solution: *Continuity at -1?* First, note that $f(-1) = 6\cdot(-1)+8 = 2$ and that $\lim_{x\to(-1)^+} f(x) = (-3)\cdot(-1)+7 = 10$. Thus, f is not continuous at -1. *Continuity at 2?* First, note that $f(2) = 2-1 = 1$ and $\lim_{x\to 2^-} f(x) = -3\cdot 2+7 = 1$ and that $\lim_{x\to 2^+} f(x) = 2-1 = 1$. Thus, f is continuous at 2.

3. Let a and b be two real numbers. We consider the function

$$f(x) = \begin{cases} ax^2 + bx + 1 & \text{if } x < 2 \\ x^2 + ax + b & \text{if} x \geq 2. \end{cases}$$

Give a condition on the real numbers a and b for the function f to be continuous everywhere.

Solution: The function f is continuous on $(-\infty, 2)$ and on $(2, +\infty)$. It remains to determine under which condition it is continuous at 2. Note that $f(2) = 4 + 2a + b$. Note that $\lim_{x\to 2^-} f(x) = 4a + 2b + 1$ and $\lim_{x\to 2^+} f(x) = 4 + 2a + b$. To ensure that the function f is continuous at $x = 2$, it suffices that $4a + 2b + 1 = 4 + 2a + b$,or equivalently, $2a + b - 3 = 0$.

4. For all real numbers x, let f be the following function

$$f(x) = x^5 - 2x - 4.$$

Calculate $f(1)$ and $f(2)$. Explain why the equation $x^5 = 2x + 4$ has at least one solution in the interval $[1, 2]$.

Solution: Note that $f(1) = -5 < 0$ and $f(2) = 24 > 0$. Since the function f is continuous on $[1, 2]$ by the Intermediate Value Theorem there exists a real number $c \in [1, 2]$ such that $f(c) = 0$. Thus, we have $c^5 - 2c - 4 = 0$ or equivalently $c^5 = 2c + 4$.

5. Let f be a function continuous on $(0, 1)$ such that, for every real number x in this interval, $0 \leq f(x) \leq 1$. Show that there exists a real number $x \in [0, 1]$ such that $f(x) = x$.

Hint: Consider for all $0 \leq x \leq 1$, the function $g(x) = f(x) - x$.

Solution: If $f(0) = 0$ or $f(1) = 1$, then the result is immediate.

Assume, therefore, that $f(0) \neq 0$ and $f(1) \neq 1$.

For any real number $x \in [0, 1]$, define $g(x) = f(x) - x$. The function g is continuous because it is the difference of two continuous functions.

We have: $g(0) = f(0) - 0 = f(0) > 0$ (since $f(0) > 0$) and
$g(1) = f(1) - 1 < 0$ (because $f(1) < 1$).

Therefore, by the Intermediate Value Theorem, there exists a real number $x \in [0, 1]$ such that $g(x) = 0$ i.e., $f(x) - x = 0$, or equivalently, $f(x) = x$.