21-120: Differential and Integral Calculus
Lecture #11 Outline

Read: Section 3.6 of the textbook

Objectives and Concepts:

- The derivative of the composite function $f(g(x)$ is $f'(g(x))\cdot g'(x)$. In Leibniz notation, if $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Suggested Textbook Exercises:

- 3.6: 215-251 odd.

The Chain Rule

So far we know how to compute derivatives of most functions by themselves, for example, if $f(x) = \sqrt{x}$ and $g(x) = 2x^2 - 1$, it's easy to compute $f'(x)$ and $g'(x)$. But what about the derivative of the composite function $f(g(x)) = \sqrt{2x^2-1}$?

> **The Chain Rule:** If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f\circ g$ defined by $F(x) = f\left(g(x)\right)$ is differentiable at x and F' is given by the product
>
> $$F'(x) = f'\left(g(x)\right)\cdot g'(x)$$
>
> In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then
>
> $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$\underbrace{\frac{d}{dx}}_{\text{derivative}}\ \underbrace{f}_{\substack{\text{outer}\\\text{function}}}\ \underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} = \underbrace{f'}_{\substack{\text{derivative}\\\text{of outer}\\\text{function}}}\ \underbrace{(g(x))}_{\substack{\text{evaluated}\\\text{at inner}\\\text{function}}} \cdot \underbrace{g'(x)}_{\substack{\text{derivative}\\\text{of inner}\\\text{function}}}$$

For the function $F(x) = \sqrt{2x^2-1}$, we can find functions f and g such that $F(x) = f(g(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = 2x^2 - 1$, then $F(x) = f(g(x)) = f(2x^2-1) = \sqrt{2x^2-1}$. Now $f'(x) = \frac{1}{2\sqrt{x}}$ and $g'(x) = 4x$, so this means that

$$F'(x) = f'(g(x))\cdot g'(x) = f'(2x^2-1)\cdot(4x) = \frac{1}{2\sqrt{2x^2-1}}\cdot(4x) = \frac{2x}{\sqrt{2x^2-1}}.$$

> **The Power Rule Combined with the Chain Rule::** If n is any real number and $u = g(x)$ is differentiable, then
>
> $$\frac{d}{dx}\left[g(x)\right]^n = n\left[g(x)\right]^{n-1}\cdot g'(x)$$

Example 1: Differentiate the following functions.

(a) $y = \sqrt[3]{x^3 - 1}$

(b) $g(x) = \sec(x^3 - 3x)$

(c) $f(x) = \dfrac{2x\cos(5x)}{\sqrt{x}+1}$

(d) $y = \sin\left(\cos\left(3x+2\right)\right)$

(e) $f(t) = \left(4\cos(2x) - 3\csc(5x^3)\right)^{-3}$

Example 2: Find the equation of the line tangent to the curve $y = 2\tan\left(\frac{\pi x}{4}\right)$ when $x = 1$.

Example 3: Find all locations where the function $f(x) = 2\sqrt{x^2+1}$ has a horizontal tangent line.

Example 4: Find $f''(x)$ if $f(x) = 2\sqrt{x^2+1}$.

Example 5: Find $f^{(7)}(x)$ if $f(x) = \cos(2x+1)$.

Example 6: Suppose that $p(x) = f(x)g(x)$ and $F(x) = f\left(g(x)\right)$. Also,

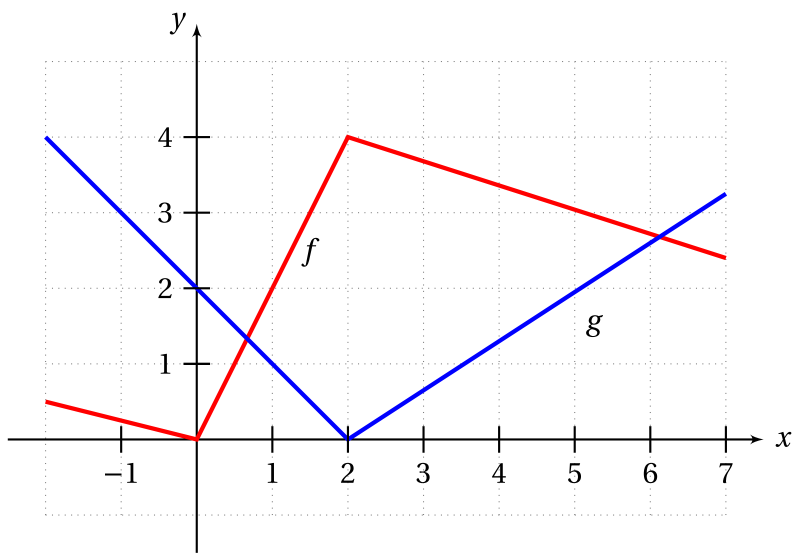
$$f(2) = 3 \qquad g(2) = 5 \qquad g'(2) = 4 \qquad f'(2) = -2 \qquad f'(5) = 11.$$

Find:

(a) $p'(2)$

(b) $F'(2)$

Example 7: The graphs of two functions $f(x)$ and $g(x)$ are shown below.



Let $F(x) = f\left(g(x)\right)$, $G(x) = g\left(f(x)\right)$, and $H(x) = f\left(f(x)\right)$. Find each derivative, if it exists. If it does not exist, explain why.

(a) $F'(1)$

(b) $G'(2)$

(c) $H'(5)$