21-120: Differential and Integral Calculus
Lecture #16 Outline

Read: Sections 4.1 and 4.2 of the textbook

Objectives and Concepts:

- When f is differentiable at $x = a$, the line tangent to $y = f(x)$ can be used as an approximation to the function $f(x)$ for values of x near a.
- Differentials give ways to quickly estimate how a given change in one or more independent variables impacts the change in the dependent variable.

Suggested Textbook Exercises:

- 4.1: 1-29 odd.
- 4.2: 51-61 odd, 69-85 odd.

More Related Rates Problems

Example 1: A camera is mounted at a point 3000 feet from the base of a rocket launching pad. If the rocket is rising vertically at 880 ft/s when it is 4000 feet above the launching pad, how fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

Example 2: A light is on the top of a 15-ft vertical pole. A 6-ft tall man walks away from the pole base at 5 feet per second. How fast is the tip of his shadow moving when he is 40 feet from the pole?

Linear Approximation and Differentials

We have seen that the line tangent to a curve lies very close to the curve near the point of tangency. In fact, if you zoom in toward a point on the graph of a differentiable function, you can observe that the graph of the curve itself looks more and more like its tangent line. This observation is the basis for a method of finding approximate values of functions. The idea is that it might be easy to calculate the value $f(a)$ of a function, but difficult (or even impossible) to compute values of $f(x)$ nearby but not exactly at $x = a$. So, we settle for the easily computable values of the linear function $y = L(x)$ whose graph is the tangent line of f at the point where $x = a$. This tangent line is described by the equation

$$y = f(a) + f'(a)(x - a).$$

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization of f at a**.

The approximation $f(x) \approx L(x)$ of f by L is the **standard linear approximation** of f at a. The point $x = a$ is called the **center** of the approximation.

We can use linear approximations to find values like $\sqrt{4.02}$ without using a calculator.

Example 3: Let $f(x) = \sqrt{x}$. Find the linear approximation to $f(x)$ when $a = 4$. Use the approximation to estimate $\sqrt{4.02}$.

Example 4: Find the linearization of the function $f(x) = \sin(x)$ at $a = 0$. Use it to approximate $\sin\left(\frac{\pi}{15}\right)$. (*Note:* $\pi/15 \approx 0.20944$).

Common Linear Approximations when $x \approx 0$:

$$\sin(x) \approx x \qquad \cos(x) \approx 1 \qquad (1+x)^k \approx 1 + kx$$

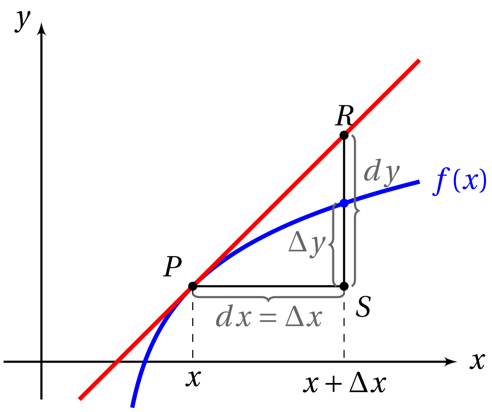
Example 5: Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$. Use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

Differentials

Let $y = f(x)$ be a differentiable function.

- The differential dx is an independent variable.
- The differential dy is $dy = f'(x)dx$. It is a dependent variable because it depends on the values of x and dx.
- If dx is given a specific value and x is taken to be some specific number in the domain of f, then the numerical value of dy is determined.
- If $dx = \Delta x$, the corresponding change in y is $\Delta y = f(x+\Delta x) - f(x)$

What's the difference between dy and Δy?



- dy represents the amount that the tangent line rises or falls:

$$dy = f'(x)dx$$

- Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by amount $dx = \Delta x$:

$$\Delta y = f(x+\Delta x) - f(x)$$

Example 6: Find dy if $y = \cos(x^4 - 3x^2 + 5)$

If $dx = \Delta x$, the corresponding change in y is

$$\Delta y = f(x+\Delta x) - f(x).$$

For small values of dx, the change in y is approximately the same as the change in the tangent line dy:

$$dy \approx \Delta y, \qquad \text{when } dx \text{ is small.}$$

This is useful because dy may be easier to calculate than Δy. In this case, dy may be thought of as the **error in calculating the value for** y, provided the error of dx is made in estimating x. The ratio $\dfrac{dy}{y}$ is called the **relative error** and is usually given as a percent.

Example 7: The sides of a square field are measured and found to be 50 meters, with a possible error of 0.02 m in the measurement. We calculate this area to be 2500 square meters. Estimate the maximum error and relative error in this calculation.