21-120: Differential and Integral Calculus
Lecture #30 Outline

Read: Sections 5.5, 5.6, 5.7 of the textbook

Objectives and Concepts:

- Integration by substitution (or u-substitution) is a technique for integrating functions whose derivatives are also present in the integrand. Substitution can be thought of as the chain rule for derivatives in reverse.
- To use u-substitution, we must substitute the entire integrand and the variable of integration completely.
- When performing u-substitution in a definite integral, you should change the limits of integration to reflect the new variable. This often simplifies the problem and makes the antiderivative easier to evaluate at the limits of integration.

Suggested Textbook Exercises:

- 5.5: 257-287 odd, 293-297 odd, 307-311 odd.
- 5.6: 321-365 odd, 373-381 odd.
- 5.7: 391-405 odd, 411-415 odd, 423-431 odd.

u-Substitution

So far, our anti-derivative formulas don't tell us how to evaluate integrals like

$$\int 2x\sqrt{1+x^2}\,dx$$

For problems like these, in which the integrand looks like it was obtained by using the chain rule for derivatives, we use the following approach:

> ***u*-Substitution:** If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then
> $$\int f\big(g(x)\big)\,g'(x)\,dx = \int f(u)\,du$$

Notation: it is permissible to operate with dx and du after integral signs as if they were differentials or symbols. To use substitution, we must replace **everything** after the integral sign with expressions in the new variable u and the differential du.

In the example above, we see that $f(x) = \sqrt{x}$ and $g(x) = 1 + x^2$. So letting $u = 1 + x^2$, we compute differentials of both sides to find that $du = 2x\,dx$. Then

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C.$$

You should always write your final answer in terms of the variable in the original problem statement when evaluating an indefinite integral.

Example 1: Evaluate each integral.

(a) $\displaystyle\int \sqrt[5]{10-7x}\,dx$

(b) $\displaystyle\int x^3(1-x^4)^5\,dx$

(c) $\displaystyle\int \frac{x}{x^2+1}\,dx$

(d) $\displaystyle\int e^x \sin\left(e^x\right)\, dx$

Sometimes we make substitutions to simply rewrite an integrand in a simpler form. These substitutions aren't always obvious at first - you may need to try more than one substitution before you find something that works.

Example 2: Evaluate each integral.

(a) $\displaystyle\int \frac{x}{\sqrt{x-4}}\, dx$

(b) $\displaystyle\int \frac{y^2}{(y+1)^4}\, dy$

Substitution for Definite Integrals: If g' is continuous on $[a,b]$ and if f is continuous on the range of $u = g(x)$, then

$$\int_a^b f\big(g(x)\big)\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

This means that when performing substitution for a definite integral, we must remember to change the limits of integration.

Example 3: Evaluate each integral.

(a) $\displaystyle\int_1^4 (2x+1)\sqrt{x^2+x}\, dx$

(b) $\displaystyle\int_0^{\pi/4} \sin^3(2x)\cos(2x)\, dx$

(c) $\displaystyle\int_0^1 \frac{x+1}{(x^2+2x+6)^2}\,dx$

(d) $\displaystyle\int_{1/2}^1 \sin\left(\frac{\pi}{2}x-\frac{\pi}{4}\right)\,dx$

(e) $\displaystyle\int_2^3 xe^{-x^2}\,dx$