21-120: Differential and Integral Calculus
Lecture #21 Outline

Read: Section 4.8 of the textbook

Objectives and Concepts:

- An indeterminate form is an expression (usually in the context of limits) that cannot be evaluated. Examples of indeterminate forms include $0/0$, $\pm\infty/\pm\infty$, $\infty-\infty$, $0\cdot\infty$, 0^0, ∞^0, and 1^∞. An indeterminate form can be any value, including $\pm\infty$.
- When evaluating the limit of f/g and arriving at $0/0$ or $\pm\infty/\pm\infty$, one can use L'Hospital's Rule, which says that if the limit of f/g is one of these two indeterminate forms, then $\lim_{x\to a}(f/g) = \lim_{x\to a}(f'/g')$.
- When arriving at a non-quotient indeterminate form, the expression can be manipulated to represent $0/0$ or $\pm\infty/\pm\infty$, and then (and only then) can L'Hospital's Rule be applied.

Suggested Textbook Exercises:

- 4.8: 357-395 odd.

The Indeterminate Form $\infty - \infty$

In this case, we need to somehow convert the difference $f-g$ into a quotient to obtain $0/0$ or $\pm\infty/\pm\infty$. We can usually do this by getting a common denominator, factoring out a common factor, or multiplying by a conjugate expression.

Example 1: Find each limit.

(a) $\displaystyle\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

(b) $\lim_{x\to\infty}\left(xe^{1/x}-x\right)$

(c) $\lim_{x\to 0}\left(\dfrac{1}{3\sin x}-\dfrac{1}{2x}\right)$

(d) $\lim_{x\to\infty}\left(\ln(4x^2+1)-\ln(x^3+2)\right)$

Exponential Indeterminate Forms

When encountering one of $0^0, \infty^0$, or 1^∞ while trying to evaluate $y = \lim_{x\to a}(f(x))^{g(x)}$, we need to convert the exponential expression into a quotient. This is usually accomplished by the following steps:

1. Take the natural log of both sides: $\ln y = \ln\left(\lim_{x\to a}(f(x))^{g(x)}\right)$.
2. Since ln is a continuous function, we have
$$\ln y = \ln\left(\lim_{x\to a}(f(x))^{g(x)}\right) = \lim_{x\to a}\ln\left((f(x))^{g(x)}\right) = \lim_{x\to a}\left(g(x)\ln f(x)\right).$$
3. The expression $g(x)\cdot\ln f(x)$ usually will yield an indeterminate form of type $0\cdot\infty$. Find $\lim_{x\to a}\left(g(x)\ln f(x)\right) = L$ using manipulation into a quotient and subsequently use L'Hospital's Rule.
4. Since $\ln y = L$, we have that the original limit is $y = e^L$.

Example 2: Find each limit.

(a) $\lim_{x\to 0^+}\left(1+\frac{1}{x}\right)^x$

(b) $\lim_{x \to 0^+} (\sin(x))^{\tan(x)}$

(c) $\lim_{x \to \infty} \left(e^x + x\right)^{1/x}$

(d) $\lim\limits_{x\to 0^+} \left(e^x + 2x\right)^{3/x}$

(e) $\lim\limits_{x\to\infty} \left(\dfrac{x^2+1}{x+2}\right)^{1/x}$