21-120: Differential and Integral Calculus
Lecture #27 Outline

Read: Section 5.2 of the textbook

Objectives and Concepts:

- The definite integral is the limit of the rectangle approximation to the area under the curve as the number of rectangles goes to infinity.
- Properties of the definite integral give us ways to manipulate definite integral expressions to find other definite integrals or to estimate the value of the definite integral.

Suggested Textbook Exercises:

- 5.2: 61-83 odd, 89-109 odd.

The Definite Integral

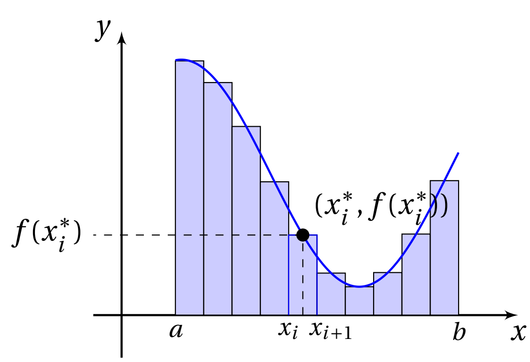
In the previous lecture, we saw that the area A of the region S that lies under the graph of a continuous function f is the sum of the areas of the approximating rectangles (i.e., the Riemann sum): $A \approx \sum_{i=1}^{n} f(x_i)\Delta x$. If we take limit as the number of rectangles goes to infinity, we will arrive at the true value of the area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Definition of the Definite Integral: If f is a function defined for $a \le x \le b$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$. We let $x_0 = a$, $x_1, \ldots, x_n = b$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any **sample points** in these subintervals, so x_i^* lies in the subinterval $[x_{i-1}, x_i]$. Then the **definite integral of f from a to b** is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

$$\underbrace{\int_a^b}_{\text{integral sign}} \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{variable of integration}}$$

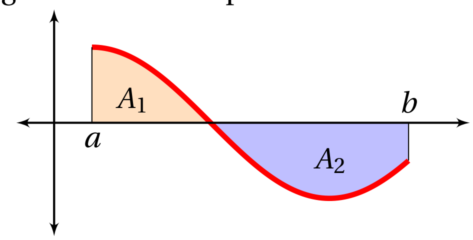


provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is **integrable** on $[a,b]$, and we call a the **lower limit of integration** and b the **upper limit of integration**.

> **Theorem:** If f is continuous on $[a, b]$, or if f has only finitely many jump discontinuities, then f is integrable on $[a, b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

The definite integral finds the **net area** between a and b, which is interpreted as the difference of the areas above and below the x-axis. A definite integral can be interpreted as the difference of areas:

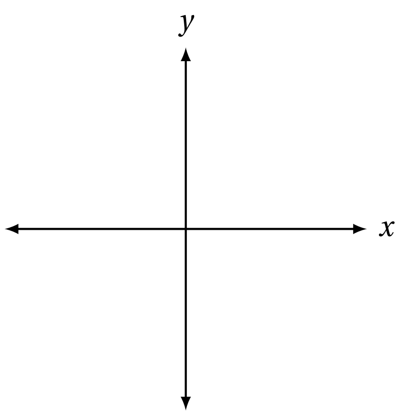
$$\int_a^b f(x)\,dx = A_1 - A_2$$



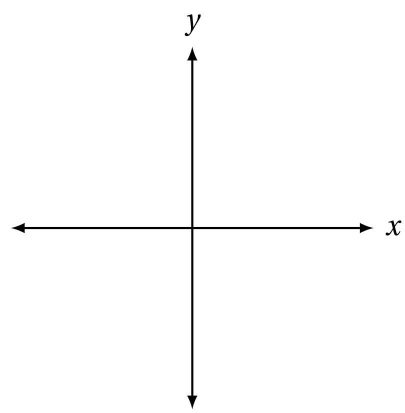
where A_1 is the area of the region above the x-axis (and below the graph of f) and A_2 is the area of the region below the x-axis (and above the graph of f).

Example 1: Use the graphs and your knowledge of areas to evaluate each integral.

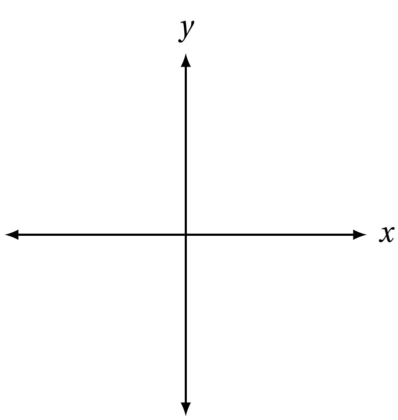
(a) $\int_0^3 2\,dx$



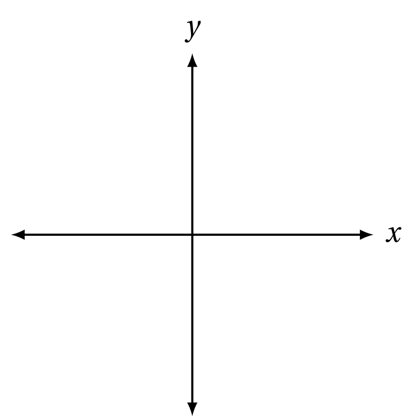
(c) $\int_{-4}^0 \sqrt{16 - x^2}\,dx$



(b) $\int_{-3}^2 (x + 1)\,dx$



(d) $\int_0^b 4x\,dx$



Properties of the Definite Integral: Let f and g be continuous functions.

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b c\,dx = c(b-a)$ for any constant c
3. $\displaystyle\int_a^b \big[f(x)+g(x)\big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\displaystyle\int_a^b \big[f(x)-g(x)\big]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is any constant
6. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
7. $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

Comparison Properties of the Integral:

8. If $m \le f(x) \le M$ over $[a,b]$, then
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$
9. If $f(x) \le g(x)$ over $[a,b]$, then
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

Example 2: Explain why
$$\int_0^{\sqrt{\pi}} \sin(x^2)\,dx \le \sqrt{\pi}$$

Example 3: Write the expression below as an integral over the interval $[-1, 0]$:

$$\lim_{n\to\infty} \sum_{i=1}^{n} 2x_i^3 \Delta x =$$

Example 4: Given the following information, evaluate each integral below.

$$\int_1^9 f(x)\,dx = -1 \qquad \int_7^9 f(x)\,dx = 5 \qquad \int_7^9 h(x)\,dx = 4$$

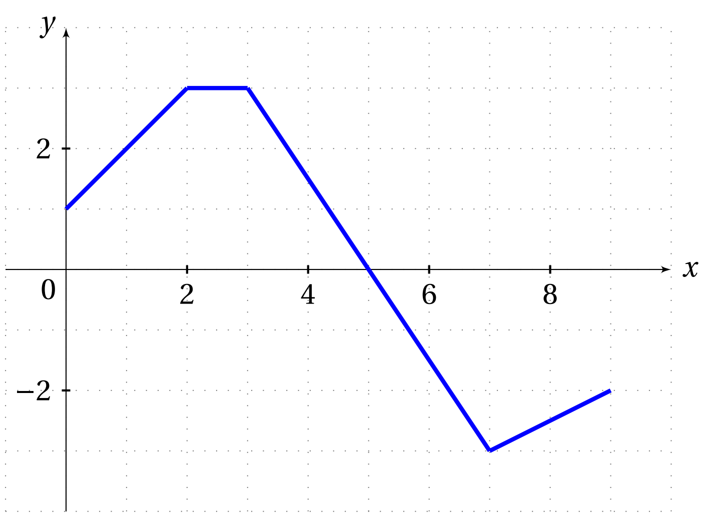
(a) $\displaystyle\int_1^9 2f(x)\,dx$

(b) $\displaystyle\int_7^9 \big(f(x) + 3h(x)\big)\,dx$

(c) $\displaystyle\int_1^9 \big(3 + f(x)\big)\,dx$

(d) $\displaystyle\int_1^7 f(x)\,dx$

Example 5: Given below is the graph of a function $f(x)$. Find the following definite integrals.



(a) $\displaystyle\int_0^2 f(x)\,dx$

(b) $\displaystyle\int_0^5 f(x)\,dx$

(c) $\displaystyle\int_5^7 f(x)\,dx$

(d) $\displaystyle\int_0^9 f(x)\,dx$