21-120: Differential and Integral Calculus
Lecture #7 Outline

Read: Section 3.1 of the textbook

Objectives and Concepts:

- The slope of the line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is given by the formula
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
and is called the derivative of f at a.
- A function f is differentiable on an interval if the derivative exists at every point in the interval.
- The line tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is given by
$$y = f(a) + f'(a)(x-a).$$

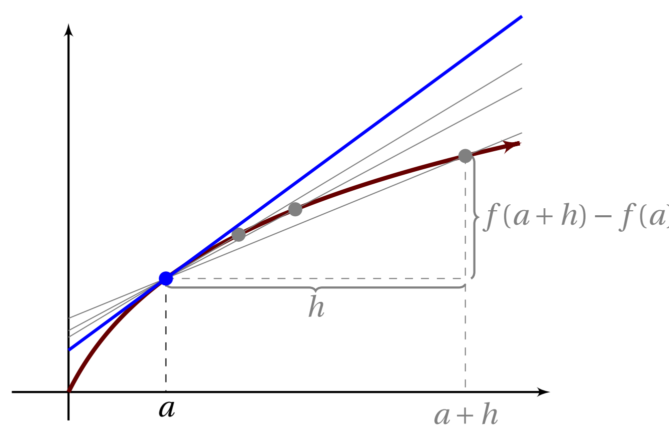
Suggested Textbook Exercises:

- 3.1: 11-29 odd, 39-43 odd.

Defining the Derivative

At the beginning of the semester, we briefly discussed the problem of finding the slope of the line tangent to a curve at a point. What we did was describe a process where we would take the slope of the secant line through the points $(a, f(a))$ and $(a+h, f(a+h))$.

Then, by making h smaller and smaller, we see how the secant line approaches a tangent line. We found that the slope of this tangent line was given by
$$m_{\tan} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$



Now that we have developed the necessary mathematical tool (limits) to deal with understanding expressions such as this, we can go about describing the instantaneous rate of change of a function f at a point $(a, f(a))$.

Definition: The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is the number

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists. If $f'(x)$ exists for each x in an open interval I, then f is said to be **differentiable over the interval** I.

The derivative of f at a is the slope of the line tangent to f at the point $(a, f(a))$.

If $f'(a)$ exists, the **tangent line to the graph of** f **at** a is the line through the point $\big(a, f(a)\big)$ with slope $f'(a)$. The tangent line has equation

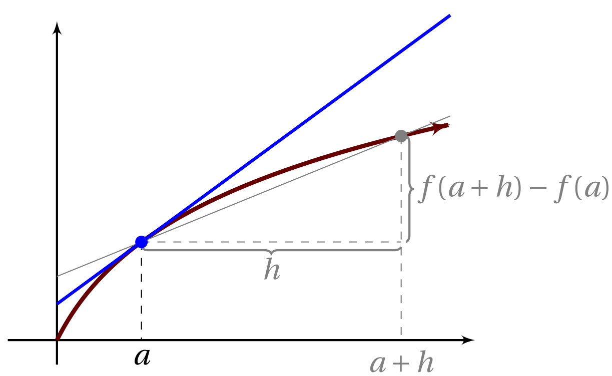
$$y = f(a) + f'(a)(x-a).$$

Example 1: Find $f'(1)$ for $f(x) = 1 + \sqrt{x}$.

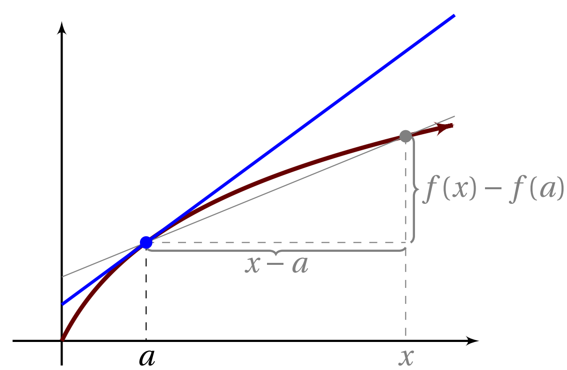
Example 2: For $f(x) = 1 + \sqrt{x}$, find the equation of the line tangent to f at $x = 1$.

Example 3: Find $f'(a)$ for $f(x) = \dfrac{1}{2x+1}$.

Note: An equivalent definition of the derivative is given by $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.



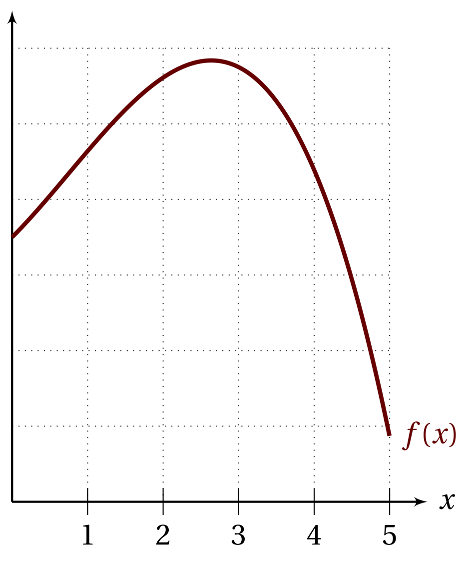
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 3: Use the above definition of $f'(a)$ to find the equation of the line tangent to $f(x) = \dfrac{1-x}{2x}$ at $a = -1$.

Example 4: Use the graph of $f(x)$ to determine which is larger. Explain your answer.



(a) $f(3)$ or $f(4)$?

(b) $f(3) - f(2)$ or $f(2) - f(1)$?

(c) $\dfrac{f(2)-f(1)}{2-1}$ or $\dfrac{f(3)-f(1)}{3-1}$?

(d) $f'(1)$ or $f'(4)$?

(e) $f'(5)$ or $f'(3)$?

Example 5: Find the slope of the line tangent to $f(x) = x^3 - x^2 + 3$ at $x = 2$.

Example 6: Given that $f'(a) = 3a^2 + 2$ for the function $f(x) = x^3 + 2x + 1$, find:

(a) The slope of the tangent line to f at the point corresponding to $x = 2$

(b) The equation of the tangent line to f at the point corresponding to $x = 2$

Example 7: The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$ for some differentiable function f.

(a) What is the meaning of the derivative $f'(8)$? What are its units?

(b) If $f'(8)$ is positive, what does that mean?

(c) Suppose that there is a value of p, call it p^*, such that $f'(p)$ is positive for $p < p^*$ but $f'(p)$ is negative for $p > p^*$. Explain what is happening there.