21-120: Differential and Integral Calculus
Recitation #22 Outline: 11/19/24

1. Evaluate each indefinite integral.

(a) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$ (b) $\displaystyle\int \sin x \sin(\cos x)\,dx$ (c) $\displaystyle\int \frac{x^2}{x^3+1}\,dx$

Solution:

(a) Let $u = \ln x$ so that $du = \frac{1}{x}\,dx$. Then the integral becomes:

$$\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{u^3}{3} + C = \frac{(\ln x)^3}{3} + C.$$

(b) Let $u = \cos x$ so that $du = -\sin x\,dx$. Then the integral becomes:

$$\int \sin x \sin(\cos x)\,dx = -\int \sin u\,du = \cos u + C = \cos(\cos x) + C.$$

(c) Let $u = x^3 + 1$ so that $du = 3x^2\,dx$. Rewrite the integral:

$$\int \frac{x^2}{x^3+1}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+1| + C.$$

2. Evaluate each definite integral.

(a) $\displaystyle\int_0^1 \frac{e^x+1}{e^x+x}\,dx$ (b) $\displaystyle\int_0^3 \frac{dx}{5x+1}$ (c) $\displaystyle\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx$

Solution:

(a) Let $u = e^x + x$, so that $du = (e^x+1)\,dx$. The limits of integration change as follows: when $x = 0$, $u = e^0 + 0 = 1$; and when $x = 1$, $u = e^1 + 1 = e + 1$. Therefore:

$$\int_0^1 \frac{e^x+1}{e^x+x}\,dx = \int_1^{e+1} \frac{du}{u} = \ln|u|\Big|_1^{e+1} = \ln(e+1) - \ln(1) = \ln(e+1).$$

(b) Let $u = 5x+1$, so that $du = 5\,dx$, or equivalently $dx = \frac{1}{5}du$. The limits of integration change as follows: when $x = 0$, $u = 5(0)+1 = 1$; and when $x = 3$, $u = 5(3)+1 = 16$. Therefore:

$$\int_0^3 \frac{dx}{5x+1} = \int_1^{16} \frac{1}{5}\cdot\frac{du}{u} = \frac{1}{5}\int_1^{16}\frac{du}{u} = \frac{1}{5}\ln|u|\Big|_1^{16} = \frac{1}{5}(\ln(16) - \ln(1)) = \frac{1}{5}\ln(16).$$

(c) Let $u = \sqrt{2x+1}$, so that $du = \frac{2}{2\sqrt{2x+1}}\,dx = \frac{1}{\sqrt{2x+1}}\,dx$. Thus, $dx = u\,du$. The limits of integration change as follows: when $x = 0$, $u = \sqrt{2(0)+1} = 1$; and when $x = 4$, $u = \sqrt{2(4)+1} = 3$. Substitute into the integral:

$$\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx = \int_1^3 \frac{\frac{u^2-1}{2}}{u}\cdot u\,du.$$

Simplify:

$$\frac{\frac{u^2-1}{2}}{u} \cdot u = \frac{u^2-1}{2}.$$

Thus, we now have:

$$\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx = \int_1^3 \frac{u^2-1}{2}\,du = \frac{1}{2}\int_1^3 (u^2-1)\,du.$$

Now, split the integral:

$$\frac{1}{2}\int_1^3 (u^2-1)\,du = \frac{1}{2}\left(\int_1^3 u^2\,du - \int_1^3 1\,du\right).$$

Evaluate each term:

$$\int u^2\,du = \frac{u^3}{3}, \quad \int 1\,du = u.$$

Apply the limits:

$$\int_1^3 u^2\,du = \frac{1}{3}\left[u^3\right]_1^3 = \frac{1}{3}(27-1) = \frac{26}{3},$$

$$\int_1^3 1\,du = [u]_1^3 = 3-1 = 2.$$

Now combine the results:

$$\frac{1}{2}\left(\frac{26}{3}-2\right) = \frac{1}{2}\left(\frac{26}{3}-\frac{6}{3}\right) = \frac{1}{2}\cdot\frac{20}{3} = \frac{10}{3}.$$

Thus:

$$\int_0^4 \frac{x}{\sqrt{2x+1}}\,dx = \frac{10}{3}.$$

3. Let f be an everywhere-continuous function.

   (a) Find $\int_0^2 f(2x)\,dx$ knowing that $\int_0^4 f(x)\,dx = 10$.

   (b) Find $\int_0^3 x f(x^2)\,dx$ knowing that $\int_0^9 f(x)\,dx = 4$.

   Solution:

   (a) Use substitution to simplify the integral. Let $u = 2x$, so that $du = 2dx$, or equivalently, $dx = \frac{du}{2}$.
   The limits change as follows: when $x = 0$, $u = 0$; and when $x = 2$, $u = 4$.
   The integral becomes:

   $$\int_0^2 f(2x)\,dx = \int_0^4 f(u)\cdot\frac{du}{2} = \frac{1}{2}\int_0^4 f(u)\,du.$$

   We are given that $\int_0^4 f(x)\,dx = 10$, so:

   $$\frac{1}{2}\int_0^4 f(u)\,du = \frac{1}{2}\cdot 10 = 5.$$

Therefore:

$$\int_0^2 f(2x)\,dx = 5.$$

(b) Again, use substitution. Let $u = x^2$, so that $du = 2x\,dx$, or equivalently, $x\,dx = \frac{du}{2}$.
The limits change as follows: when $x = 0$, $u = 0$; and when $x = 3$, $u = 9$.
The integral becomes:

$$\int_0^3 x f(x^2)\,dx = \int_0^9 f(u)\cdot\frac{du}{2} = \frac{1}{2}\int_0^9 f(u)\,du.$$

We are given that $\int_0^9 f(x)\,dx = 4$, so:

$$\frac{1}{2}\int_0^9 f(u)\,du = \frac{1}{2}\cdot 4 = 2.$$

Therefore:

$$\int_0^3 x f(x^2)\,dx = 2.$$

4. Let f be an everywhere-continuous function. Show that

$$\int_0^a x^3 f(x^2)\,dx = \frac{1}{2}\int_0^{a^2} x f(x)\,dx$$

for any $a > 0$.

Solution:

Let $u = x^2$, so that $du = 2x\,dx$, or equivalently, $x\,dx = \frac{du}{2}$.

We can rewrite the integrand as:

$$x^3 f(x^2) = x\cdot x^2 f(x^2).$$

Now, substituting $x^2 = u$, $f(x^2) = f(u)$, and $x\,dx = \frac{du}{2}$, and realizing that the new limits of integration are from 0 to a^2, we get:

$$\int_0^a x^3 f(x^2)\,dx = \int_0^{a^2} \frac{u}{2} f(u)\,du.$$

Thus, we have:

$$\int_0^a x^3 f(x^2)\,dx = \frac{1}{2}\int_0^{a^2} u f(u)\,du.$$

Finally, since u in the integral on the right is just a dummy variable, we can replace it by x, getting:

$$\int_0^a x^3 f(x^2)\,dx = \frac{1}{2}\int_0^{a^2} x f(x)\,dx.$$