21-120: Differential and Integral Calculus
Lecture #5 Outline

Read: Section 4.6 of the textbook

Objectives and Concepts:

- A function f can have a (finite) limit at infinity (or negative infinity) if the function values approach a single value L for $x \to \infty$ ($x \to -\infty$), or the function can have an infinite limit if the function grows or decreases without bound at infinity (or negative infinity).
- If a function has a (finite) limit at infinity, it has a horizontal asymptote there.
- Horizontal asymptotes of a rational function are easy to identify by examining the leading terms in the numerator and denominator.

Suggested Textbook Exercises:

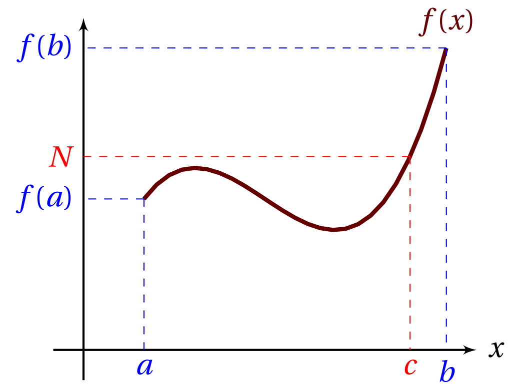
- 4.6: 251-287 odd.

The Intermediate Value Theorem

Intermediate Value Theorem: Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in $(a.b)$ such that $f(c) = N$.

What the IVT means is that a continuous function cannot jump around and "miss" a value between $f(a)$ and $f(b)$, because it is continuous!

If you choose any N between $f(a)$ and $f(b)$, there is at least one c between a and b with $f(c) = N$.



The Intermediate Value Theorem can be used to show that a particular equation has a real solution.

Example 8: Show that the equation $\cos x = x$ has a solution in the interval $(0,1)$.

Solution: Note that the equation $\cos x = x$ is equivalent to the equation $\cos x - x = 0$. Let $f(x) = \cos x - x$. We want to show that there is an x, $0 < x < 1$, such that $f(x) = 0$. Since f is continuous on $[0,1]$, $f(0) = 1 - 0 = 1 > 0$, and $f(1) = \cos 1 - 1 < 0$ (why?), we have that there must be a c in $(0,1)$ such that $f(c) = 0$.

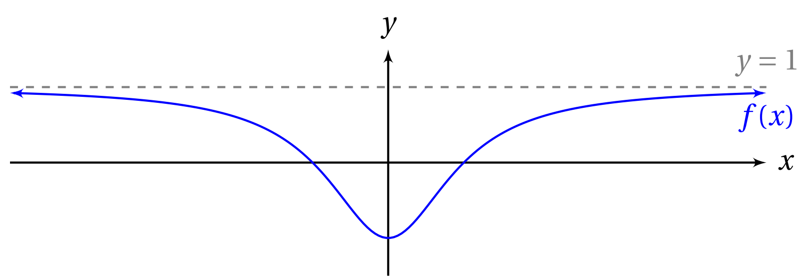
Example 9: Show that the equation $\ln x = 3 - 2x$ has at least one real solution.

Limits at Infinity and Horizontal Asymptotes

Consider the function $f(x) = \dfrac{x^2-1}{x^2+1}$. What happens to this function as x becomes very large or very small? In other words, what happens as $x \to \infty$ or $x \to -\infty$? Now, it is true that as $x \to \infty$, $1/x \to 0$, so it also makes sense that $1/x^2 \to 0$ as $x \to \infty$. Then we have

$$f(x) = \frac{x^2-1}{x^2+1} = \frac{x^2-1}{x^2+1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}.$$

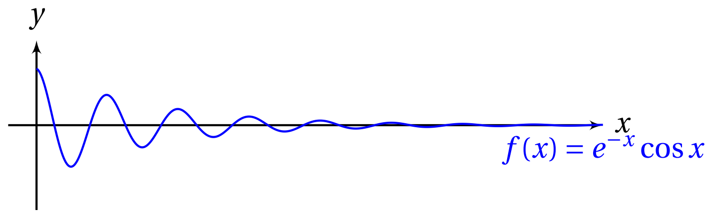
So it seems that, as $x \to \infty$, $f(x) \to \dfrac{1-0}{1+0}$. We can actually see this on the graph:



Symbolically, we express the "long term" behavior of this function by writing $\displaystyle\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$.

Definition: Let f be a function defined over some interval (a,∞). Then $\displaystyle\lim_{x\to\infty} f(x) = L$ means the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large. Likewise, if f is a function defined over some interval $(-\infty, a)$, then $\displaystyle\lim_{x\to-\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative. In either case, the line $y = L$ is called a **horizontal asymptote**.

WARNING: A common misconception is that a horizontal asymptote is a line that the function approaches, but never crosses. This isn't always the case, as you can see with $f(x) = e^{-x}\cos x$.



Theorem: If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^r} = 0$$

To use the above theorem to help us evaluate limits at infinity, we find the *largest power* of x in the **denominator** and divide every term by that power of x.

Example 1: Evaluate $\displaystyle\lim_{x\to-\infty} \frac{4x^2 - 2x + 1}{6x^3 + 1}$

Example 2: Evaluate $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 3x + 1}{x - 4x^{3/2}}$

Note: We need to be careful when considering limits at infinity of functions with radicals. Consider the limit

$$\lim_{x\to\infty} \frac{3x+4}{\sqrt{2x^2-5}}.$$

As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. You cannot "plug in $+\infty$" and then reduce ∞/∞ in a meaningful way. The quotient ∞/∞ is called an **indeterminate form** and we need special tools to understand it.

To evaluate the limit at infinity, as before we first divide both the numerator and the denominator by the highest power of $|x|$ that appears in the denominator. Then we can use the Limit Laws. Keep in mind that:

- If $x > 0$, then $\sqrt{x^2} = x$. So for limits as $x \to \infty$, use $\sqrt{x^2} = x$.
- If $x < 0$, then $\sqrt{x^2} = -x$. So for limits as $x \to -\infty$, use $\sqrt{x^2} = -x$.

Example 3: Evaluate $\displaystyle\lim_{x\to-\infty} \frac{3x+4}{\sqrt{2x^2-5}}$

Example 4: Evaluate $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+2}}{3x+6}$ and $\lim\limits_{x\to-\infty} \dfrac{\sqrt{x^2+2}}{3x+6}$

Example 6: Evaluate $\lim\limits_{x\to-\infty} \dfrac{1-2e^x}{1+e^x}$ and $\lim\limits_{x\to\infty} \dfrac{1-2e^x}{1+e^x}$.

Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function where $p(x)$ and $q(x)$ are polynomials defined by

$$p(x) = a_n x^n + \cdots + a_1 x + a_0 \qquad \text{and} \qquad q(x) = b_m x^m + \cdots + b_1 x + b_0.$$

Then the **vertical asymptotes** of $f(x)$ occur where $q(x) = 0$ while $p(x) \neq 0$. The **horizontal asymptotes** of $f(x)$ depend on the degree n of $p(x)$ and degree m of $q(x)$, as follows:

1. If $n < m$ (the denominator dominates), then $y = 0$ is a horizontal asymptote.
2. If $n = m$, then $y = \dfrac{a_n}{b_m}$ (the ratio of the leading coefficients) is a horizontal asymptote.
3. If $n > m$ (the numerator dominates), then no horizontal asymptote exists.

- **To find all vertical asymptotes, find all infinite limits.**
- **To find all horizontal asymptotes, find all limits at infinity.**

Example 7: Find the vertical and horizontal asymptotes of

$$y = \frac{2x^2 - x - 1}{x^2 + 6x - 16}.$$