21-120: Differential and Integral Calculus
Recitation #16 Outline: 10/24/24

1. Find the following limits. You may use the fact that $\lim_{t\to 0}\frac{\sin t}{t} = 1$ and $\lim_{t\to 0}\frac{\cos t - 1}{t} = 0$. You may NOT use L'Hôpital's rule.

   (a) $\lim_{x\to 0}(\cot(2x)\cdot\sin x)$; (b) $\lim_{x\to 0}\frac{\sec x - 1}{x}$.

   Solution:

   (a)
$$\begin{aligned}\lim_{x\to 0}(\cot(2x)\cdot\sin x) &= \lim_{x\to 0}\left(\frac{\cos(2x)}{\sin(2x)}\cdot\sin x\right)\\ &= \lim_{x\to 0}\frac{\cos(2x)\cdot\sin x}{\sin(2x)}\\ &= \lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{2x}{\sin(2x)}\cdot\frac{\cos(2x)}{2}\right)\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\lim_{x\to 0}\frac{2x}{\sin(2x)}\cdot\lim_{x\to 0}\frac{\cos(2x)}{2}\\ &= 1\cdot 1\cdot\frac{1}{2}\\ &= \frac{1}{2}.\end{aligned}$$

   (b)
$$\begin{aligned}\lim_{x\to 0}\frac{\sec x - 1}{x} &= \lim_{x\to 0}\frac{\frac{1}{\cos x} - 1}{x}\\ &= \lim_{x\to 0}\frac{1-\cos x}{x\cos x}\\ &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\lim_{x\to 0}\frac{1}{\cos x}\\ &= 0\cdot 1\\ &= 0.\end{aligned}$$

2. Determine if the Mean Value Theorem can be applied to the function $f(x) = x^2 - x^{2/3}$ on the the interval $[-1, 8]$. If so, find all possible values of c that satisfy the conclusion of the Mean Value Theorem.

   Solution:

   First, check if $f(x) = x^2 - x^{2/3}$ satisfies the conditions for applying the Mean Value Theorem (MVT) on the interval $[-1, 8]$. The MVT requires that:

   (a) f be continuous on $[-1, 8]$.

   (b) f be differentiable on $(-1, 8)$.

1. Continuity: The function f is continuous on $[-1,8]$ because both x^2 and $x^{2/3}$ are continuous for all $x \in [-1,8]$.

2. Differentiability: Now, find the derivative of f:

$$f'(x) = 2x - \frac{2}{3}x^{-1/3}.$$

The derivative $f'(x)$ exists for all $x \neq 0$, so $f(x)$ is differentiable on $(-1,8)$ except at $x = 0$.

Since f is not differentiable at $x = 0$, the Mean Value Theorem cannot be applied on the interval $[-1,8]$.

3. Without graphing anything, explain why the equation $2^x - 2x + 1 = 0$ cannot have more than two real solutions.

Solution: Assume, for the sake of contradiction, that the equation $2^x - 2x + 1 = 0$ has three distinct real solutions, say x_1, x_2, x_3, and assume $x_1 < x_2 < x_3$. This means that $f(x_1) = f(x_2) = f(x_3) = 0$, where $f(x) = 2^x - 2x + 1$.

Since f is continuous on both $[x_1, x_2]$ and $[x_2, x_3]$, and differentiable on both (x_1, x_2) and (x_2, x_3), and since $f(x_1) = f(x_2) = f(x_3)$, by Rolle's Theorem, there must exist $c_1 \in (x_1, x_2)$ and $c_2 \in (x_2, x_3)$ such that:

$$f'(c_1) = 0 \quad \text{and} \quad f'(c_2) = 0.$$

That is, the derivative of f must vanish at two distinct points c_1 and c_2 in the intervals (x_1, x_2) and (x_2, x_3), respectively.

Now, compute the derivative of f:

$$f'(x) = 2^x \ln 2 - 2.$$

Now, let's determine whether $f'(x)$ can vanish at two distinct points. We have:

$$f'(x) = 2^x \ln 2 - 2.$$

Setting $f'(x) = 0$, we get:

$$2^x \ln 2 - 2 = 0,$$

which simplifies to:

$$2^x = \frac{2}{\ln 2}.$$

Taking the logarithm base 2 on both sides, we obtain:

$$x = \log_2\left(\frac{2}{\ln 2}\right).$$

Thus, the equation $f'(x) = 0$ has exactly one real solution.

Since $f'(x)$ cannot have two distinct zeros, this contradicts the earlier conclusion that $f'(x)$ must be zero at two distinct points, c_1 and c_2. Therefore, the assumption that the equation $2^x - 2x + 1 = 0$ has (at least) three distinct real solutions must be false.

Thus, the equation cannot have more than two real solutions.

4. Using an appropriate linear approximation, estimate $(1.01)^{-3}$.

Solution:

To estimate $(1.01)^{-3}$ using linear approximation, we can start by defining $f(x) = x^{-3}$. Then we need to estimate $f(1.01)$. We will choose $x = 1$ as the point around which to linearize, since 1.01 is close to 1.

First, we calculate $f(1)$:

$$f(1) = 1^{-3} = 1.$$

Next, we find the derivative $f'(x)$:

$$f'(x) = -3x^{-4}.$$

Now, evaluate the derivative at $x = 1$:

$$f'(1) = -3 \cdot 1^{-4} = -3.$$

Using the linear approximation formula:

$$f(x) \approx f(a) + f'(a)(x-a),$$

where $a = 1$ and $x = 1.01$, we have:

$$f(1.01) \approx f(1) + f'(1)(1.01-1).$$

Substituting the values, we get:

$$f(1.01) \approx 1 + (-3)(0.01) = 1 - 0.03 = 0.97.$$

5. Find $\dfrac{dy}{dx}$ for each of the following:

(a) $\cos(x^2+2y) + xe^{y^2} = 1$;

(b) $y = (2x - e^{8x})^{\sin(2x)}$.

Solution:

(a) To find $\frac{dy}{dx}$ using implicit differentiation for the equation

$$\cos(x^2+2y) + xe^{y^2} = 1,$$

we differentiate both sides with respect to x:

1. Differentiate the left-hand side:

- The derivative of $\cos(x^2+2y)$ using the chain rule is:

$$-\sin(x^2+2y)\left(\frac{d}{dx}(x^2+2y)\right) = -\sin(x^2+2y)\left(2x + 2\frac{dy}{dx}\right).$$

- The derivative of xe^{y^2} using the product rule is:

$$e^{y^2} + x \cdot e^{y^2} \cdot \frac{d}{dx}(y^2) = e^{y^2} + x \cdot e^{y^2} \cdot (2y\frac{dy}{dx}).$$

2. Therefore, the left-hand side becomes:

$$-\sin(x^2+2y)(2x+2\frac{dy}{dx})+e^{y^2}+xe^{y^2}(2y\frac{dy}{dx}).$$

3. The right-hand side, being a constant, differentiates to 0.
Putting it all together, we have:

$$-\sin(x^2+2y)(2x+2\frac{dy}{dx})+e^{y^2}+xe^{y^2}(2y\frac{dy}{dx})=0.$$

4. Rearranging the equation to isolate $\frac{dy}{dx}$:

$$-\sin(x^2+2y)(2x+2\frac{dy}{dx})+e^{y^2}+2xye^{y^2}\frac{dy}{dx}=0.$$

Expanding and grouping terms gives:

$$-\sin(x^2+2y)(2x)+e^{y^2}=\sin(x^2+2y)(2\frac{dy}{dx})-2xye^{y^2}\frac{dy}{dx}.$$

5. Factoring out $\frac{dy}{dx}$:

$$-\sin(x^2+2y)(2x)+e^{y^2}=\left(\sin(x^2+2y)2-2xye^{y^2}\right)\frac{dy}{dx}.$$

6. Solving for $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{-\sin(x^2+2y)(2x)+e^{y^2}}{\sin(x^2+2y)(2)-2xye^{y^2}}.$$

Thus, the derivative $\frac{dy}{dx}$ in terms of x and y is given by:

$$\frac{dy}{dx}=\frac{-2x\sin(x^2+2y)+e^{y^2}}{2\sin(x^2+2y)-2xye^{y^2}}.$$

(b) First, we take the natural logarithm of both sides:

$$\ln y=\sin(2x)\ln(2x-e^{8x}).$$

Next, we differentiate both sides with respect to x:
1. The left-hand side differentiates to:

$$\frac{1}{y}\frac{dy}{dx}.$$

2. The right-hand side requires the product rule and the chain rule:

$$\frac{d}{dx}\left[\sin(2x)\ln(2x-e^{8x})\right]=\cos(2x)\cdot 2\ln(2x-e^{8x})+\sin(2x)\cdot\frac{1}{2x-e^{8x}}\cdot\frac{d}{dx}(2x-e^{8x}).$$

3. The derivative of $2x-e^{8x}$ is:

$$2 - 8e^{8x}.$$

4. Substituting this back into the equation gives:

$$\frac{d}{dx}\left[\sin(2x)\ln(2x - e^{8x})\right] = \cos(2x)\cdot 2\ln(2x - e^{8x}) + \sin(2x)\cdot\frac{1}{2x - e^{8x}}\cdot(2 - 8e^{8x}).$$

Putting it all together, we have:

$$\frac{1}{y}\frac{dy}{dx} = 2\cos(2x)\ln(2x - e^{8x}) + \sin(2x)\cdot\frac{2 - 8e^{8x}}{2x - e^{8x}}.$$

Now, we can isolate $\frac{dy}{dx}$:

$$\frac{dy}{dx} = y\left[2\cos(2x)\ln(2x - e^{8x}) + \sin(2x)\cdot\frac{2 - 8e^{8x}}{2x - e^{8x}}\right].$$

Finally, substituting y back gives us:

$$\frac{dy}{dx} = (2x - e^{8x})^{\sin(2x)}\left[2\cos(2x)\ln(2x - e^{8x}) + \sin(2x)\cdot\frac{2 - 8e^{8x}}{2x - e^{8x}}\right].$$

6. Let $f(x) = \dfrac{e^{-3x}}{x^2+1}$. Find the equation of the tangent line to f^{-1} at $(1,0)$.

Solution: To find the equation of the tangent line to the inverse function f^{-1} at the point $(1,0)$, we first calculate $f'(x)$.

Using the quotient rule, we have:

$$f'(x) = \frac{(e^{-3x})'(x^2+1) - (e^{-3x})(x^2+1)'}{(x^2+1)^2}.$$

Calculating the derivatives:

$$(e^{-3x})' = -3e^{-3x} \quad \text{and} \quad (x^2+1)' = 2x.$$

Substituting these into the quotient rule gives:

$$f'(x) = \frac{-3e^{-3x}(x^2+1) - e^{-3x}(2x)}{(x^2+1)^2}.$$

Next, we evaluate $f'(0)$:

$$f'(0) = \frac{-e^0(3(0^2+1) + 2\cdot 0)}{(0^2+1)^2} = -3.$$

To find the slope of the tangent line to f^{-1} at the point $(1,0)$, we use the Inverse Function Theorem:

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

Since $f^{-1}(1) = 0$:

$$(f^{-1})'(1) = \frac{1}{f'(0)} = \frac{1}{-3} = -\frac{1}{3}.$$

The equation of the tangent line at the point $(1,0)$ with slope $-\frac{1}{3}$ can be written in point-slope form:

$$y - 0 = -\frac{1}{3}(x-1).$$

7. Let $f(x) = x^{1/3}(x-2)$. Without doing any calculations, explain why this function must have both absolute maximum and minimum on $[-1,3]$. Then find the critical numbers as well as the absolute maximum and minimum values of f on the interval $[-1,3]$.

Solution: By the Extreme Value Theorem, since f is continuous on the closed interval $[-1,3]$, it must attain both an absolute maximum and minimum on that interval.

Next, we find the critical numbers of f. To do this, we first calculate the derivative $f'(x)$ using the product rule:

$$f'(x) = \left(x^{1/3}\right)'(x-2) + x^{1/3}(x-2)'.$$

Calculating the derivatives:

$$\left(x^{1/3}\right)' = \frac{1}{3}x^{-2/3} \quad \text{and} \quad (x-2)' = 1.$$

Substituting these back into the expression for f', we have:

$$f'(x) = \frac{1}{3}x^{-2/3}(x-2) + x^{1/3}(1) = \frac{1}{3}\frac{x-2}{x^{2/3}} + x^{1/3}.$$

At this point, it is clear that $x = 0$ is a critical number, since f' is undefined at this point. To find the other critical numbers, we set $f'(x) = 0$:

$$\frac{1}{3}\frac{x-2}{x^{2/3}} + x^{1/3} = 0.$$

Putting everything under a common denominator gives:

$$\frac{x-2+3x}{3x^{2/3}} = 0 \quad \Rightarrow \quad x - 2 + 3x = 0.$$

This gives $x = \frac{1}{2}$.

Thus, the critical numbers are $x = 0$ and $x = \frac{1}{2}$.

Next, we evaluate the values of f at the critical numbers and the endpoints of the interval $[-1,3]$:

1. For the endpoint $x = -1$:

$$f(-1) = (-1)^{1/3}(-1-2) = -1 \cdot (-3) = 3.$$

2. For the critical number $x = 0$:

$$f(0) = (0)^{1/3}(0-2) = 0 \cdot (-2) = 0.$$

3. For the critical number $x = \frac{1}{2}$:

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^{1/3}\left(\frac{1}{2} - 2\right) = \frac{1}{2^{1/3}}\left(-\frac{3}{2}\right) = -\frac{3}{2^{4/3}}.$$

4. For the endpoint $x = 3$:

$$f(3) = 3^{1/3}(3-2) = 3^{1/3}(1) = 3^{1/3}.$$

On $[-1,3]$, the absolute maximum of f occurs at $x = -1$ with $f(-1) = 3$, and the absolute minimum of f occurs at $x = \frac{1}{2}$ with $f\left(\frac{1}{2}\right) = -\frac{3}{2^{4/3}}$.

8. Let $f(x) = x^2 - x - \ln x$.

(a) Find the intervals on which f is increasing and decreasing.

(b) Find the local minimum and maximum values of f (if any).

(c) Find the inflection points of f (if any), and the intervals of concavity.

Solution:

(a) To find the intervals on which f is increasing or decreasing, we first calculate the derivative $f'(x)$:

$$f'(x) = 2x - 1 - \frac{1}{x}.$$

Note that f' is undefined at $x = 0$, but in this case $x = 0$ is not a critical number, since it does not lie in the domain of f. (Any critical number must lie in the domain of the function.) Thus there aren't critical numbers arising from f' being undefined. To find the other kind of critical numbers, we set $f'(x) = 0$:

$$2x - 1 - \frac{1}{x} = 0.$$

Putting everything under a common denominator gives:

$$\frac{2x^2 - x - 1}{x} = 0.$$

A fraction is zero exactly when the numerator is zero and the denominator is non-zero. We set the numerator equal to zero:

$$2x^2 - x - 1 = 0.$$

Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $a = 2, b = -1, c = -1$:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}.$$

Thus, the solutions to $f'(x) = 0$ are:

$$x = 1 \quad \text{and} \quad x = -\frac{1}{2}.$$

However, $x = -\frac{1}{2}$ is not a critical number since it does not lie in the domain of f. The only critical number is $x = 1$.

Since f is undefined for $x \leq 0$, we test the intervals $(0,1)$ and $(1,\infty)$:
- For $x \in (0,1)$, choose $x = 0.5$:

$$f'(0.5) = 2(0.5) - 1 - \frac{1}{0.5} = 1 - 1 - 2 = -2 < 0 \quad \Rightarrow \quad f \text{ is decreasing.}$$

- For $x \in (1,\infty)$, choose $x = 2$:

$$f'(2) = 2(2) - 1 - \frac{1}{2} = 4 - 1 - 0.5 = 2.5 > 0 \quad \Rightarrow \quad f \text{ is increasing.}$$

Therefore, f is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

(b) To find the local minimum and maximum values of f, we evaluate f at the critical point $x = 1$:

$$f(1) = 1^2 - 1 - \ln(1) = 1 - 1 - 0 = 0.$$

Since f changes from decreasing to increasing at $x = 1$, it is a local minimum.

(c) To find the inflection points and the intervals of concavity, we calculate the second derivative $f''(x)$:

$$f''(x) = 2 + \frac{1}{x^2}.$$

Setting $f''(x) = 0$:

$$2 + \frac{1}{x^2} = 0 \quad \Rightarrow \quad \text{No real solutions (since } 2 + \frac{1}{x^2} > 0 \text{ for } x).$$

Therefore, $f''(x) > 0$ for all x, indicating that f is concave up on $(0,\infty)$ and there are no inflection points.