21-120: Differential and Integral Calculus
Lecture #15 Outline

Read: Section 4.1 of the textbook

Objectives and Concepts:

- When continuous quantities are related via a mathematical model, their rates of change with respect to time (or some other independent variable) are also related, provided the quantities are differentiable.
- Many relationships can be established with standard length, area, and volume formulas from basic geometry.

Suggested Textbook Exercises:

- 4.1: 1-29 odd.

Related Rates of Change

As we begin to explore applications of the derivative, we start by trying to understand how related quantities change as an independent variable (in many cases time) changes. A basic geometric problem can be described by the following scenario: if one leg of a right triangle is increasing at a certain rate, and the other leg is decreasing at a different rate, what is the rate of change of the length of the hypotenuse?

To formulate (and hopefully solve) this problem, we first need to draw a simple sketch of what is going on. If you draw a right triangle and label the lengths of the legs a and b and the length of the hypotenuse c (in the space to the right), it is easy to see that these quantities are related by the equation

$$a^2 + b^2 = c^2.$$

Now, if a is changing with respect to some unit of time t, we would describe the rate of change of a as $\frac{da}{dt}$, and we would do the same for b.

Then, if we take the Pythagorean equation and differentiate it implicitly with respect to t, we arrive at the equation

$$(2c)\frac{dc}{dt} = (2a)\frac{da}{dt} + (2b)\frac{db}{dt}.$$

Solving this equation for the rate of change of c we get

$$\frac{dc}{dt} = \frac{a\frac{da}{dt} + b\frac{db}{dt}}{c}.$$

Thus, if we know the quantities a and b (which will give us c) and how fast a and b are changing, we can find the rate of change of the length of the hypotenuse.

This general approach to solving related rates problems follows the process described above. A basic framework is given below.

Steps for Related Rates Problems:

1. **Illustrate:** Draw and label a picture of the scenario.
2. **Translate:** Express what you want to find and what you are given from your picture. Sometimes it is useful to write out the question using words.
3. **Relate:** Find the equations.
4. **Differentiate:** Take the derivative of the equation with respect to time t.
5. **Evaluate:** Plug in the values of the given variables.
6. **Check:** Make sure you answered the question.

Some things to remember when working on related rates problems:

- If a variable y depends on time t, then its derivative $\dfrac{dy}{dt}$ is called a **time rate of change**.
- If x and y are related via an equation, then $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$ are **related rates**.
- In related rate problems, variables are usually functions of time t.
- When differentiating these variables with respect to t, the Chain Rule must be applied.

Note: You will need to recall some basic formulas from geometry, such as

- The area and circumference of a circle: $A = \pi r^2, \quad C = 2\pi r$.
- The volume and surface area of a sphere: $V = \dfrac{4}{3}\pi r^3, \quad A = 4\pi r^2$
- The volume and surface area of a right circular cylinder: $V = \pi r^2 h, \quad A = 2\pi r^2 + 2\pi r h$.
- The volume of a right circular cone: $V = \frac{1}{3}\pi r^2 h$.
- The area of a triangle and the Pythagorean Theorem: $A = \dfrac{1}{2}bh, \quad a^2 + b^2 = c^2$.

The first example we will work on is a classic variation of relating rates of change of the lengths of the sides of a right triangle.

Example 1: A 10-ft ladder rests along a vertical wall. The bottom of the ladder slides away from the wall at a rate of 1 ft/s. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Step 1: Draw a picture. Label it.

Step 2: Translate your picture. What do you know? What do you need to know? Write your question.

Step 3: Find the equation.

Step 4: Differentiate. Use the Chain Rule.

Step 5: Evaluate. Plug in what you know.

Step 6: What's the final answer?

Example 2: A water tank has the shape of an inverted circular cone. The base has a radius of 2 meters and the height is 5 meters. Water is being pumped into the tank at a rate of 2 cubic meters/minute. Find the rate at which the water level is rising when the water is 3 m deep.

Step 1: Draw a picture. Label it.

Step 2: Translate your picture. What do you know? What do you need to know? Write your question.

Step 3: Find the equation.

Step 4: Differentiate. Use the Chain Rule.

Step 5: Evaluate. Plug in what you know.

Step 6: What's the final answer?

Example 3: Two cars start moving from the same point. One travels north at 55 mph and the other travels east at 35 mph. At what rate is the distance between the cars increasing three hours later?

Example 4: Oil spilled from a ruptured tanker spreads in a circular pattern whose radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 feet?