Recitation 19 Solutions

1. (a) Using that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$$\int 4x^5 + 2x^3 + 7x^2 dx = 4\frac{x^6}{6} + 2\frac{x^4}{4} - 7\frac{x^3}{3} + C \boxed{= \frac{2}{3}x^6 + \frac{1}{2}x^4 - \frac{2}{3}x^3 + C}$$

(b) Using that $\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + C$ and $\int \frac{1}{x^2+1}dx = \arctan(x) + C$

$$\int \cos\left(\frac{x}{2}\right) - \frac{3}{x^2+1}dx = \frac{1}{\left(\frac{1}{2}\right)}\sin\left(\frac{x}{2}\right) - 3\arctan(x) + C$$

$$\boxed{= 2\sin\left(\frac{x}{2}\right) - 3\arctan(x) + C}$$

(c) Using that $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$$\int 3^e - \frac{1}{x^{4/3}}dx = \int 3^e - x^{-4/3}dx = 3^e x - \frac{x^{-4/3+1}}{\left(-\frac{4}{3}+1\right)} + C$$

$$\boxed{= 3^e x + 3x^{-1/3} + C}$$

2. (a) $$\sum_{i=4}^{8}\left((i+2)^2 - 1\right) = \sum_{i=4}^{8}(i+2)^2 - \sum_{i=4}^{8}1$$

$$= \sum_{j=6}^{10} j^2 - 5 \qquad (\text{letting } j = i+2)$$

$$= \sum_{j=1}^{10} j^2 - \sum_{j=1}^{5} j^2 - 5$$

$$= \frac{10\cdot 11\cdot 21}{6} - \frac{5\cdot 6\cdot 11}{6} - 5 \qquad \left(\text{using } \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\right)$$

$$\boxed{= 325}$$

(b) $$\sum_{k=3}^{12}(k^3 + k^2) = \sum_{k=3}^{12}k^3 + \sum_{k=3}^{12}k^2$$

$$= \sum_{k=1}^{12}k^3 - \sum_{k=1}^{2}k^3 + \sum_{k=0}^{12}k^2 - \sum_{k=1}^{2}k^3$$

$$= \frac{12^2 13^2}{4} - \frac{2^2 3^2}{4} + \frac{12 \cdot 13 \cdot 25}{6} - \frac{2 \cdot 3 \cdot 5}{6}$$

$$\boxed{= 6720}$$

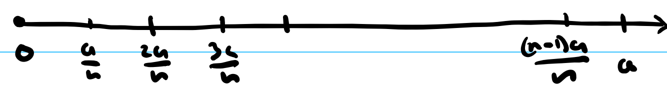
3. (a) $\displaystyle\sum_{i=1}^{n} \frac{i a^2}{n^2} = \frac{a^2}{n^2} \sum_{i=1}^{n} i = \frac{a^2}{n^2} \frac{n(n+1)}{2} = \frac{a^2}{2}\left(1+\frac{1}{n}\right)$

thus $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i a^2}{n^2} = \lim_{n\to\infty} \frac{a^2}{2}\left(1+\frac{1}{n}\right) \boxed{= \frac{a^2}{2}}$

(b) $\displaystyle\sum_{i=1}^{n} \frac{i^2 a^3}{n^3} = \frac{a^3}{n^3} \sum_{i=1}^{n} i^2 = \frac{a^3}{n^3} \frac{n(n+1)(2n+1)}{6} = \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$

thus $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2 a^3}{n^3} = \lim_{n\to\infty} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{a^3}{6} \cdot 1 \cdot 2 \boxed{= \frac{a^3}{3}.}$

Area interpretation: Consider the partition of $[0,a]$ into n intervals of length $\frac{a}{n}$ given by $x_0 = 0$, $x_i = \frac{ai}{n}$ so that $x_n = a$.



Then the right endpoint approximation to the area between $f(x) = x$ and the x-axis over $[0,a]$ is

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{ai}{n}\right)\frac{a}{n} = \sum_{i=1}^{n} \left(\frac{ai}{n}\right)\frac{a}{n} = \sum_{i=1}^{n} \frac{i a^2}{n^2}.$$

Thus the area under $f(x) = x$ on $[0,a]$ is equal to

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i a^2}{n^2} = \frac{a^2}{2}$$

$\frac{a^2}{2}$

Similarly, the right endpoint approximation to the area between $g(x) = x^2$ and the x-axis over $[0,a]$ is

$$\sum_{i=1}^{n} g(x_i)\,\Delta x = \sum_{i=1}^{n} g\left(\frac{ai}{n}\right)\frac{a}{n} = \sum_{i=1}^{n} \left(\frac{ai}{n}\right)^2\frac{a}{n} = \sum_{i=1}^{n} \frac{i^2 a^3}{n^3}$$

Thus the area under $g(x) = x^2$ on $[0,a]$ is equal to

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2 a^3}{n^3} = \frac{a^3}{3}.$$

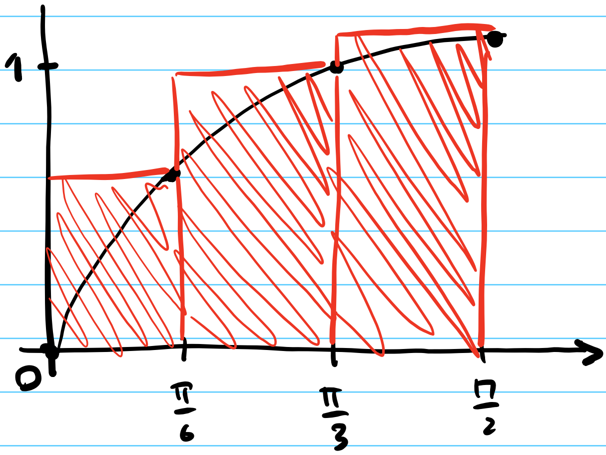
$\frac{a^3}{3}$

4. (a) First we split $[0, \frac{\pi}{2}]$ into three equally sized intervals $x_0 = 0$, $x_1 = \frac{\pi}{6}$, $x_2 = \frac{\pi}{3}$, $x_4 = \frac{\pi}{2}$ so that $\Delta x = \frac{\pi}{6}$. The corresponding right endpoint approximation is

$$\sin\left(\frac{\pi}{6}\right)\cdot\frac{\pi}{6} + \sin\left(\frac{\pi}{3}\right)\cdot\frac{\pi}{6} + \sin\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{6} = \left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1\right)\frac{\pi}{6}$$

$$= \left(\frac{3+\sqrt{3}}{2}\right)\frac{\pi}{6}$$

$$\approx 1.2388$$



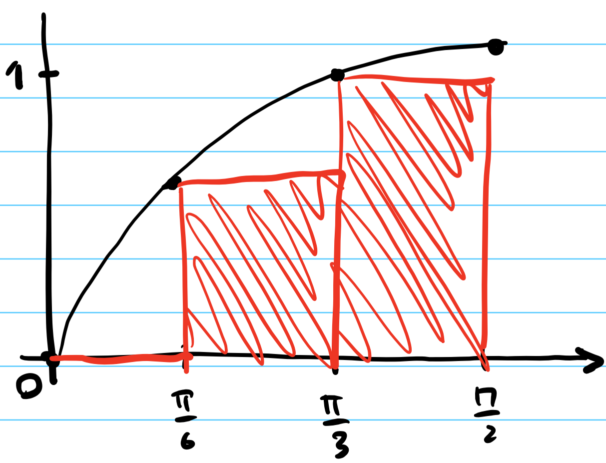
This is clearly an overestimate of the actual area under the curve.

(b) Using the left endpoint approximation we find

$$\sin(0)\frac{\pi}{6} + \sin\left(\frac{\pi}{6}\right)\frac{\pi}{6} + \sin\left(\frac{\pi}{3}\right)\frac{\pi}{6} = \left(0 + \frac{1}{2} + \frac{\sqrt{3}}{2}\right)\frac{\pi}{6}$$

$$= \left(\frac{1+\sqrt{3}}{2}\right)\frac{\pi}{6}$$

$$\approx 0.7152$$



This is clearly an underestimate of the actual area under the curve.

Indeed we will soon see that the exact area is equal to 1.

5. Since the rate decreased as time passed a lower estimate is found using right endpoints and an upper estimate is found using left endpoints.

Lower estimate:

$2(7.6)+2(6.8)+2(6.2)+2(5.7)+2(5.3) = 63.2$ L

Upper estimate:

$2(8.7)+2(7.6)+2(6.8)+2(6.2)+2(5.7) = 70$ L.