21-120: Differential and Integral Calculus
Lecture #32 Outline

Review of Integration

Here are several indefinite integrals/antiderivatives that you should be able to solve easily by now. Note that these are simply differentiation rules in reverse with the exception of a few details. Here u is any function of x such that du contains the chain rule derivative of that function of x.

1. $\displaystyle\int k\,du = ku + C$
2. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\displaystyle\int \frac{1}{u}\,du = \ln|u| + C$
4. $\displaystyle\int e^u\,du = e^u + C$
5. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C$
6. $\displaystyle\int \sin u\,du = -\cos u + C$
7. $\displaystyle\int \cos u\,du = \sin u + C$
8. $\displaystyle\int \sec^2 u\,du = \tan u + C$
9. $\displaystyle\int \sec u \tan u\,du = \sec u + C$
10. $\displaystyle\int \csc^2 u\,du = -\cot u + C$
11. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$
12. $\displaystyle\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\left(\frac{u}{a}\right) + C$
13. $\displaystyle\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
14. $\displaystyle\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

There are a few more that we can use u-substitution to derive and add to the list above. These are given below:

15. $\displaystyle\int \tan u\,du = \ln|\sec u| + C$
16. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$
17. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
18. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

For example, we can easily derive #15 above by writing $\tan x$ in terms of sines and cosines:

$$\begin{aligned}
\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx \qquad \text{(let } u = \cos x, \text{then } du = -\sin x\,dx\text{)} \\
&= -\int \frac{1}{u}\,du \\
&= -\ln|u| + C = \ln\left|\frac{1}{u}\right| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C.
\end{aligned}$$

The antiderivative of $\sec x$ is somewhat more complicated, as it requires us to algebraically manipulate the integrand prior to making an appropriate substitution:

$$\begin{aligned}
\int \sec x\,dx &= \int \sec x\left(\frac{\sec x+\tan x}{\sec x+\tan x}\right)dx \\
&= \int \frac{\sec^2 x+\sec x\tan x}{\tan x+\sec x}\,dx \qquad \left(u=\tan x+\sec x,\quad du=(\sec^2 x+\sec x\tan x)\,dx\right) \\
&= \int \frac{1}{u}\,du \\
&= \ln|u|+C=\ln|\sec x+\tan x|+C.
\end{aligned}$$

Using these 18 antiderivative/integration formulas should give you a solid foundation when trying to solve any integration problem. Of course, you may need to make an appropriate substitution (or two) in order to get your integrand into one of those forms. Additionally, you might also need to perform some algebraic manipulation of the integrand to get it into a usable form. Some techniques that might be necessary are completing the square, multiplying a fraction by 1, long division of polynomials, common denominators, separation of fractions, using trig identities, and more. Also, don't forget about the properties of integrals we discussed before the Fundamental Theorem of Calculus. Let's look at some examples:

Example 1: Evaluate each integral.

(a) $\displaystyle\int \frac{dx}{\sqrt{6x-x^2}}$

(b) $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

(c) $\displaystyle\int \frac{dx}{1-\cos x}$

(d) $\displaystyle\int \frac{2x^2+4x}{5x+1}\,dx$

(e) $\displaystyle\int \frac{4x-1}{\sqrt{1-x^2}}\,dx$

(f) $\displaystyle\int \frac{dx}{(1+\sqrt{x})^3}$

(g) $\displaystyle\int_{-\pi/3}^{\pi/3} x\cos x\,dx$