21-120: Differential and Integral Calculus
Lecture #28 Outline

Read: Section 5.3 of the textbook

Objectives and Concepts:

- The Mean Value Theorem for Integrals states that a continuous function on a closed interval takes on its average value at some point in that interval.
- The Fundamental Theorem of Calculus (FTC) is a principle that states that differentiation and integration are inverse operations.
- Part I of the FTC states that the derivative of a definite integral is the integrand.
- Part II of the FTC states that the definite integral of f from a to b is any antiderivative of f evaluated at b minus the antiderivative evaluated at a.

Suggested Textbook Exercises:

- 5.3: 149-163 odd, 171-197 odd.

The Mean Value Theorem for Integrals

We first start with the definition of the average value of a function over an interval:

Definition: Let $f(x)$ be continuous on the interval $[a,b]$. Then the **average value of f on** $[a,b]$, denoted f_{ave}, is given by

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

We can examine some average value calculations soon, but the main reason for discussing this quantity is to describe the Mean Value Theorem for Integrals, which states that a continuous function on a closed interval will take on its average value at some point in the integral.

The Mean Value Theorem for Integrals: If f is continuous on $[a,b]$, then there is at least one point $c \in [a,b]$ such that

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx, \quad \text{or} \quad \int_a^b f(x)\,dx = f(c)(b-a).$$

The proof of the MVT for Integrals is straightforward - first, note that if f is continuous on $[a,b]$, then the Extreme Value Theorem guarantees that f achieves a minimum value m and a maximum value M somewhere in that interval. Thus $m \le f(x) \le M$. From the properties of definite integrals, we know that this means

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a) \quad \Longrightarrow \quad m \le \frac{1}{b-a}\int_a^b f(x)\,dx \le M.$$

Now, since the Intermediate Value Theorem guarantees that f takes on all values between m and M (as f is continuous), there must be at least one value c such that $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$, proving the theorem.

The Fundamental Theorem of Calculus, Part I

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. It gives the precise inverse relationship between the derivative and the integral. As we will see, differentiation and integration are inverse operations, which means that the two (seemingly unrelated) motivating problems of finding the instantaneous rate of change of a function and finding the area between a curve and the x-axis are in fact intricately related. There are two main parts to the Fundamental Theorem - the first part we will describe tells us that the derivative of an integral yields the integrand.

The Fundamental Theorem of Calculus, (Part I): If f is continuous on $[a,b]$, then the function F defined by

$$F(x) = \int_a^x f(t)dt \qquad\qquad a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b), and $F'(x) = f(x)$ over $[a,b]$.

If f is a positive function, then $F(x) = \displaystyle\int_a^x f(t)\,dt$ can be interpreted as the area under the graph of f from a to x, where x can be any number between a and b. Think of $F(x)$ as the "area so far" function.

We pause briefly before presenting the proof to mention that having the variable upper limit of integration x means that the definite integral above actually yields a function of x, and not just a number (as in the case when the limits of integration are constants). Also, the t variable in the integrand is often referred to as a "dummy variable" - it merely serves as a placeholder for the independent variable in the function.

Proof of FTC1: Using the definition of the derivative, we have

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\
&= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt.
\end{aligned}$$

The last two lines above are due to the properties of definite integrals. Now the last integral above is just the average value of the function f over the interval $[x, x+h]$. Then the MVT for Integrals implies that there is some c_h (the choice of c depends on the value of h) in that interval such that

$$f(c_h) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Since we are taking the limit as $h \to 0$, we are effectively forcing $c_h \to x$. Because f is continuous, we can use the Direct Substitution Property, which will imply that

$$F'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c_h) = \lim_{c_h \to x} f(c_h) = f(x),$$

proving the theorem. ■

Using Leibniz notation for derivatives, we can write FTC1 as $\frac{d}{dx}\int_a^x f(t)dt = f(x)$. This gives us a way to compute some special derivatives.

Example 1: Find the derivative of the function $f(x) = \int_1^x \left(t^3 - 4t^2 + 2t\right)dt$

Note: Sometimes we have to use the Chain Rule in conjunction with FTC1:

Example 2: Find the derivative of the function $f(x) = \int_1^{\sin(x)} 3t^2\,dt$

Sometimes we must also use some of the properties of integrals.

Example 3: Find the derivative of the function $G(x) = \int_x^{x^3} \cos\sqrt{t}\,dt$

Example 4: Find the derivative of the function $y = \int_{1-3x}^{1} \frac{u^3}{1+u^2}\,du$

The Fundamental Theorem of Calculus, Part II

We now present the second part of the Fundamental Theorem of Calculus, which gives us an easy way to evaluate definite integrals using their antiderivatives.

The Fundamental Theorem of Calculus, (Part II): If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

This means that, to evaluate the definite integral of f from a to b, we just need to evaluate any antiderivative of f at b and subtract the value of the antiderivative at a. Note that the constant $+C$ that is attached to the most general antiderivative does not impact the value of the definite integral:

$$\int_a^b f'(x)\,dx = (f(b) + C) - (f(a) + C) = f(b) - f(a).$$

Also, we use a vertical bar with the limits of integration on the top and bottom right to denote the result of the definite integral:

$$\int_a^b f'(x)\,dx = \big(f(x)\big)\Big|_a^b = f(b) - f(a).$$

Proof sketch: Given a large enough number n, we dived $[a,b]$ into equally-spaced subintervals $[x_0, x_1]$, $[x_1, x_2], \dots, [x_{n-1}, x_n]$, each with width Δx so that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x.$$

Since n is large, Δx is very small, thus we also have that

$$f(x_{i-1}) \approx \frac{F(x_{i-1} + \Delta x) - F(x_{i-1})}{\Delta x} = \frac{F(x_i) - F(x_{i-1})}{\Delta x}.$$

We thus have

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{N} \Big(\frac{F(x_i) - F(x_{i-1})}{\Delta x}\Big)\Delta x = \sum_{i=1}^{N} \Big(F(x_i) - F(x_{i-1})\Big).$$

To conclude we note that

$$\begin{aligned}\sum_{i=1}^{N} \Big(F(x_i) - F(x_{i-1})\Big) &= \Big(F(x_n) - F(x_{n-1})\Big) + \Big(F(x_{n-1}) - F(x_{n-2})\Big) + \ldots + \Big(F(x_2) - F(x_1)\Big) + \Big(F(x_1) - F(x_0)\Big) \\ &= F(x_n) - F(x_0) \\ &= F(b) - F(a),\end{aligned}$$

thus in total we have found that

$$\int_a^b f(x)\,dx \approx F(b) - F(a).$$

Turning the $\approx$ into an $=$ requires the use of the Mean Value Theorem. The proof can be found in the textbook. ■

Alternative proof: Let f be a function continuous on $[a,b]$ and let F be any of its antiderivatives. Set

$$G(x) = \int_a^x f(t)\,dt.$$

By the first part of the Fundamental Theorem of Calculus, G is also an antiderivative of f. This being so, we have $F'(x) = G'(x)$ for all x in (a,b), and by a corollary from the Mean Value Theorem (see Lecture 18), $F - G$ must be constant on (a,b); i.e., $F(x) = G(x) + C$ for some constant C, for all x in (a,b).

Note that equation $F(x) = G(x) + C$ also holds when $x = a$ and $x = b$. This is proved by taking the limits of both sides of this equation as $x \to a^+$, and, separately, as $x \to b^-$, and using the Direct Substitution Property to evaluate them. (Note that the Direct Substitution Property can be applied since F and G are continuous on $[a,b]$, so that $\lim\limits_{x\to a^+} F(x) = F(a)$, $\lim\limits_{x\to a^+} G(x) = G(a)$, and similarly $\lim\limits_{x\to b^-} F(x) = F(b)$, $\lim\limits_{x\to b^-} G(x) = G(b)$.)

Thus, we have $F(x) = G(x) + C$ for all x in $[a,b]$. Therefore we can use this formula with $x = a$ and $x = b$. So we have

$$F(b) - F(a) = [G(b) + C] - [G(a) + C] = G(b) - G(a) = G(b) - 0 = G(b) = \int_a^b f(t)\,dt,$$

where we have used that

$$G(a) = \int_a^a f(t)\,dt = 0.$$

Thus, we have shown that $\int_a^b f(t)\,dt = F(b) - F(a)$, which finishes the proof. ■

Example 5: Evaluate $\displaystyle\int_{\pi/6}^{\pi/3} \sec^2 x\,dx$.

Example 6: Evaluate $\displaystyle\int_{-2}^{4} (5x^2 - 4x)\,dx$.