21-120: Differential and Integral Calculus
Lecture #4 Outline

Read: Sections 2.3, 2.4 of the textbook

Objectives and Concepts:

- The Squeeze Theorem can be used to find a limit of a function g that is bounded by two other functions f and h, both of whose limits exist and are equal.
- A function is continuous at a number a when it is defined and equal to its limit at a.
- A function is continuous on an interval when it is continuous at every point in the interval.
- A function can fail to be continuous if is contains a jump discontinuity, an infinite (or essential) discontinuity, or a removable discontinuity.
- Polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions, and logarithmic functions are all continuous at every number in their domain.
- The Intermediate Value Theorem (IVT) guarantees that a continuous function defined on $[a,b]$ will take on all values between $f(a)$ and $f(b)$. The IVT can be used to prove an equation has a solution.

Suggested Textbook Exercises:

- 2.3: 83-125 odd.
- 2.4: 131-167 odd.

The Squeeze Theorem

The Squeeze Theorem:
Suppose $f(x) \leq g(x) \leq h(x)$ when x is near a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} g(x) = L$.

The Squeeze Theorem allows us to calculate the limit of a given function (whose limit may be difficult to calculate) by "squeezing" the graph of the function between the graphs of two other functions with known and equal limits as $x \to a$. To use the Squeeze Theorem to find $\lim_{x\to a} f(x)$, you must find a function that serves as an upper bound on $f(x)$ and another function that serves as a lower bound on $f(x)$, and both of those functions must have the same limit as $x \to a$.

Example 1: Suppose that

$$2 - x^2 \le g(x) \le 2\cos x$$

for all x. Find $\lim_{x\to 0} g(x)$. Sketching the graphs of $f(x) = 2 - x^2$ and $h(x) = 2\cos x$ may be helpful.

Example 2: Use the Squeeze Theorem to evaluate

$$\lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right).$$

Note that we *cannot* use

$$\lim_{x\to 0}\left(x^2 \sin\left(\frac{1}{x}\right)\right) = \left(\lim_{x\to 0} x^2\right)\cdot\left(\lim_{x\to 0}\sin\left(\frac{1}{x}\right)\right)$$

because the limit $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ does not exist.

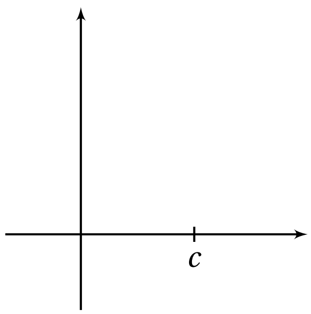
Continuity

Definition: A function $f(x)$ is **continuous at** $x = c$ means

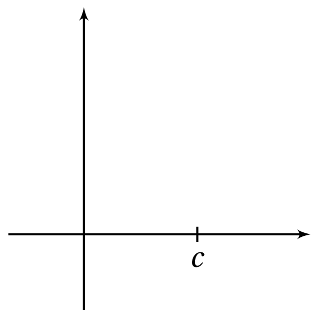
1. $f(c)$ exists;
2. The limit of $f(x)$ as $x \to c$ exists;
3. $\lim_{x \to c} f(x) = f(c)$.

If any of these three statements is not true, we say that $f(x)$ is **discontinuous at** $x = c$. If $f(x)$ is continuous at every number in an interval, then we say f is **continuous on the interval**.

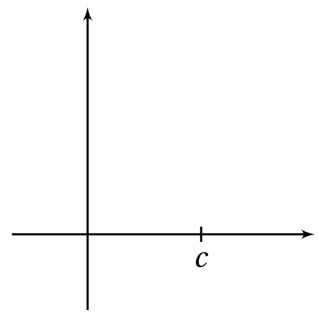
Example 3: On each of the grids below, draw a function $f(x)$ that fails **only** the specified condition.



$f(c)$ exists



$\lim_{x \to c} f(c)$ exists



$\lim_{x \to c} f(x) = f(c)$

Definition: A function $f(x)$ is **continuous from the right at** $x = a$ means

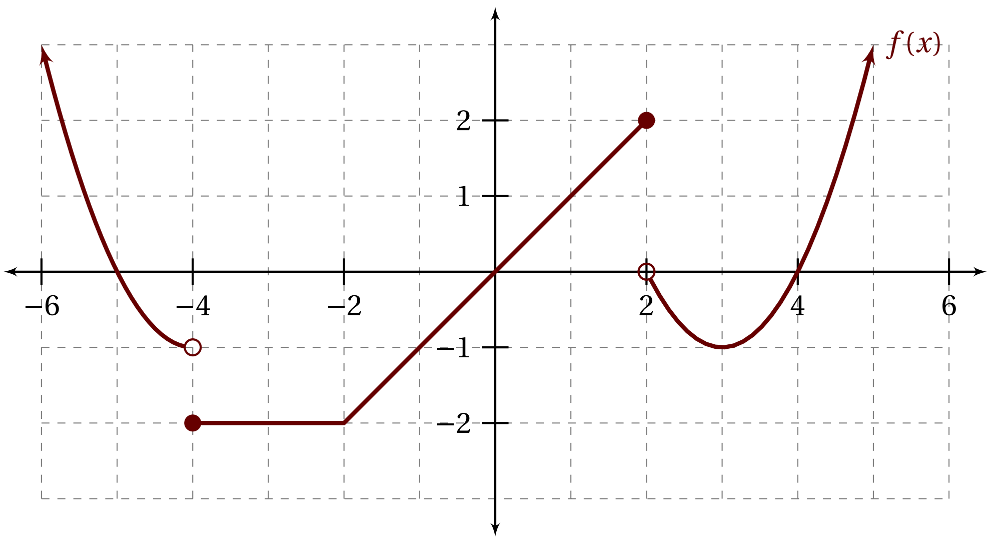
1. $f(a)$ exists;
2. The limit of $f(x)$ as $x \to a^+$ exists;
3. $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is **continuous from the left at** $x = b$ means

1. $f(b)$ exists;
2. The limit of $f(x)$ as $x \to b^-$ exists;
3. $\lim_{x \to b^-} f(x) = f(b)$.

Example 4: The graph of the function $f(x)$ is displayed below. Fill in the table with "Yes" or "No".

c	-4	-2	0	2	3
left continuous at c					
right continuous at c					
continuous at c					



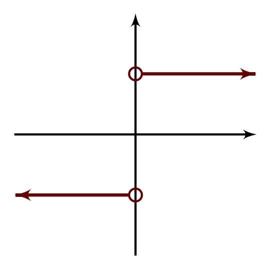
Types of Discontinuities

(a) **Jump Discontinuity:** when the left and right-hand limits are not the same.

$$f(x) = \frac{|x|}{x}$$

is discontinuous at

$x =$ _______

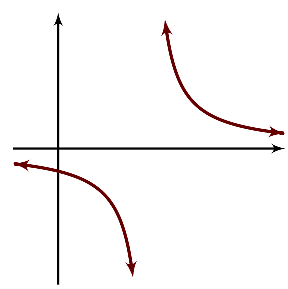


(b) **Infinite (Essential) Discontinuity:** when a limit from one side does not exist or is infinite.

$$f(x) = \frac{1}{x-2}$$

is discontinuous at

$x =$ _______

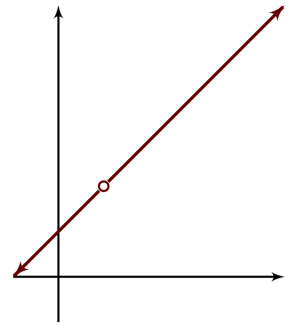


(c) **Removable Discontinuity:** when a limit exists but the function is not equal to its limit there.

$$f(x) = \frac{x^2-1}{x-1}$$

is discontinuous at

$x =$ _______



> **Theorem:** The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trig functions, inverse trig functions, exponential functions, and logarithmic functions.

Example 5: Find all points at which the function $y = \dfrac{x}{\cos(x)}$ is not continuous.

Example 6: State the intervals of continuity of the function.

$$f(x) = \begin{cases} \dfrac{x^2-1}{x+1}, & \text{if } x < -1 \\ -2, & \text{if } -1 \le x < 1 \\ x-2, & \text{if } x \ge 1 \end{cases}$$

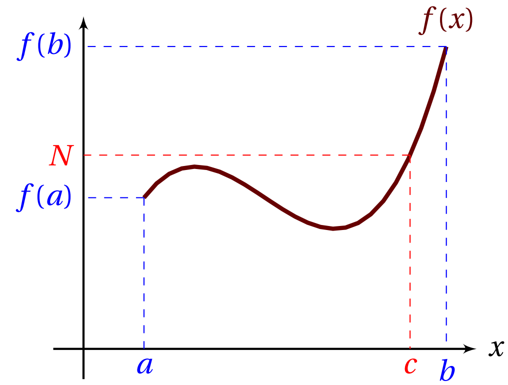
Example 7: Find the values of c and m that make f continuous everywhere.

$$f(x) = \begin{cases} cx^2, & \text{if } x < 1 \\ 4, & \text{if } x = 1 \\ -x^3 + mx, & \text{if } x > 1 \end{cases}$$

Intermediate Value Theorem: Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in $(a.b)$ such that $f(c) = N$.

What the IVT means is that a continuous function cannot jump around and "miss" a value between $f(a)$ and $f(b)$, because it is continuous!

If you choose any N between $f(a)$ and $f(b)$, there is at least one c between a and b with $f(c) = N$.



The Intermediate Value Theorem can be used to show that a particular equation has a real solution.

Example 8: Show that the equation $\cos x = x$ has a solution in the interval $(0, 1)$.

Solution: Note that the equation $\cos x = x$ is equivalent to the equation $\cos x - x = 0$. Let $f(x) = \cos x - x$. We want to show that there is an x, $0 < x < 1$, such that $f(x) = 0$. Since f is continuous on $[0, 1]$, $f(0) = 1 - 0 = 1 > 0$, and $f(1) = \cos 1 - 1 < 0$ (why?), we have that there must be a c in $(0, 1)$ such that $f(c) = 0$.

Example 9: Show that the equation $\ln x = 3 - 2x$ has at least one real solution.