

Word-Level Abstractions for Sequential Design Verification using Algebraic Geometry

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- Contributions
- Motivations
- Previous Work
- Preliminaries
 - Finite fields
 - Polynomial algebra & Algebraic geometry
 - Projection of varieties
- Projection based abstraction
 - Application: **Reachability analysis**
 - Application: **Sequential arithmetic ckt verification**
- UNSAT core based abstraction
 - **UNSAT core extraction using Gröbner basis refutation**
 - Application: Bounded model checking (BMC) with abstraction refinement
- Conclusion & Future work

- Word-level reachability analysis – analog of implicit state enumeration
 - **Q:** Why word-level?

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 - **A:** Data \leftarrow word-level info

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 - Abstraction \rightarrow word-level signature each time-frame
 - Word-level abstraction from bit-level ckts [Pruss, 2015]
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- Implement above algos: C++ & SINGULAR
 - For sequential GF multipliers: overwhelmingly better than contemporary tools

Motivation I: BFS state space traversal

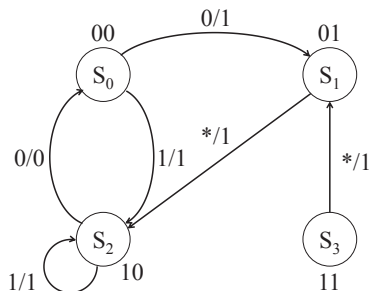


Figure : State Transition Graph

- Initial state: $\{00\}$
- Iteration 1:
 - Start from $\{00\}$
 - One-step transition: $\{01, 10\}$
 - Newly reached: $\{01, 10\}$
- Iteration 2:
 - Start from $\{01, 10\}$
 - One-step transition: $\{00, 10\}$
 - Newly reached: \emptyset
- All reachable states detected.
Final reached states:
 $\{00, 01, 10\}$

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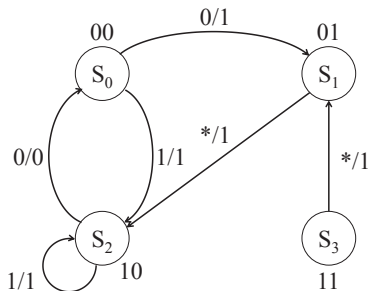


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Final reached states:
 $\{00, 01, 10\}$
- Still need bit-level Boolean variables to represent states

Motivation III: k -BMC with abstraction refinement

ALGORITHM: k -BMC with Abstraction Refinement (L. Zhang'05)

Input: M is the original machine, p is the property to check, k is the number of steps unrolling M

```
1  $k = \text{InitValue}$ ;  
2 if  $k\text{-BMC}(M, p, k)$  is SAT (reachable states after  $k$  steps violates  $p$ ) then  
3   return "Found error trace"  
4 else  
5   Extract UNSAT core  $\mathcal{P}$  of  $k\text{-BMC}$  ;  
6    $M' = \text{ABSTRACT}(M, \mathcal{P})$ ;  
7 end  
8 if  $\text{MODEL-CHECK}(M', p)$  returns PASS then  
9   return "Passing property"  
10 else  
11   Increase bound  $k$ ;  
12   goto Line 2;  
13 end
```

► Go back to Abs refine

- Sequential Equivalence Checking (SEC): bit-blasting, or structural info dependency
 - Usually based on reachability analysis
 - Sequential miter
 - Unroll, then use DDs, SAT or AIGs (Combinational)
 - Induction-based
- Symbolic model checking (counterexample, IC3): SAT/BDDs in nature
- Word-level techniques (term rewriting, uninterpreted function): ?/no encoding
- Algebraic geometry methods
 - Gröber basis in model checking [Avrunin,CAV'96;Vardi,IASTED'07]: Analog of bit-level Boolean functions
 - Abstract word-level polynomial representation for arbitrary combinational ckt [Pruss,TCAD'16]

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 - Abstract word-level polynomial representation for arbitrary combinational ckt [Pruss,TCAD'16]
- No purely word-level sequential verification: data/abstraction/algorithm

- Our proposed state-space model is based on finite field \mathbb{F}_{2^k}
 - \mathbb{B}^k : bit-vector
 - $\mathbb{Z}_{2^k}, \mathbb{R}_q$: approaches not compatible
- Evaluations in $\mathbb{B}^k \Leftrightarrow$ Elements in \mathbb{F}_{2^k}
- Functions $\mathbb{B}^k \rightarrow \mathbb{B}^k \Leftrightarrow \mathcal{F} : \mathbb{F}_{2^k} \rightarrow \mathbb{F}_{2^k}$

Galois field \mathbb{F}_q is a finite field with q elements, $q = p^k$, $p = \text{prime}$

- 0,1 elements, commutative, associate, distributive laws
- Closure property: $+$, $-$, \times , inverse (\div)

Our interest: $\mathbb{F}_q = \mathbb{F}_{2^k}$ ($q = 2^k$)

- \mathbb{F}_{2^k} : k -dimensional extension of $\mathbb{F}_2 = \{0, 1\}$
 - k -bit bit-vector, AND/XOR arithmetic

To construct \mathbb{F}_{2^k}

- $\mathbb{F}_{2^k} \equiv \mathbb{F}_2[x] \pmod{P(x)}$
- $P(x) \in \mathbb{F}_2[x]$, irreducible polynomial of degree k
- Operations performed $\pmod{P(x)}$ and coefficient reduced $\pmod{2}$
 - E.g. $\mathbb{C} = \mathbb{R}[x] \pmod{x^2 + 1}$

Preliminaries: Field construction of \mathbb{F}_8

Consider: $\mathbb{F}_{2^3} = \mathbb{F}_2[x] \pmod{x^3 + x + 1}$

$A \in \mathbb{F}_2[x]$

$A \pmod{x^3 + x + 1} = a_2x^2 + a_1x + a_0$. Let $P(\alpha) = 0$:

- $\langle a_2, a_1, a_0 \rangle = \langle 0, 0, 0 \rangle = 0$
- $\langle a_2, a_1, a_0 \rangle = \langle 0, 0, 1 \rangle = 1$
- $\langle a_2, a_1, a_0 \rangle = \langle 0, 1, 0 \rangle = \alpha$
- $\langle a_2, a_1, a_0 \rangle = \langle 0, 1, 1 \rangle = \alpha + 1$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 0, 0 \rangle = \alpha^2$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 0, 1 \rangle = \alpha^2 + 1$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 1, 0 \rangle = \alpha^2 + \alpha$
- $\langle a_2, a_1, a_0 \rangle = \langle 1, 1, 1 \rangle = \alpha^2 + \alpha + 1$

Theorem (Fermat's Little Theorem over \mathbb{F}_q)

Let $\alpha \in \mathbb{F}_q$, then $\alpha^q = \alpha$. Therefore, $x^q - x$ vanishes on all points in \mathbb{F}_q .

$\{a_2 a_1 a_0\} \in \mathbb{B}^3$	$A \in \mathbb{F}_{2^3}$	\rightarrow	$\{z_2 z_1 z_0\} \in \mathbb{B}^3$	$Z \in \mathbb{F}_{2^3}$
000	0	\rightarrow	000	0
001	1	\rightarrow	001	1
010	α	\rightarrow	111	$\alpha^2 + \alpha + 1$
011	$\alpha + 1$	\rightarrow	111	$\alpha^2 + \alpha + 1$
100	α^2	\rightarrow	101	$\alpha^2 + 1$
101	$\alpha^2 + 1$	\rightarrow	011	$\alpha + 1$
110	$\alpha^2 + \alpha$	\rightarrow	101	$\alpha^2 + 1$
111	$\alpha^2 + \alpha + 1$	\rightarrow	101	$\alpha^2 + 1$

Table : Truth table for mappings in \mathbb{B}^3 and \mathbb{F}_{2^3}

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Table : Truth table for mappings in \mathbb{B}^3 and \mathbb{F}_{2^3}

$$Z = \mathcal{F}(A)$$

$$\begin{aligned}
 &= (\alpha^2 + \alpha + 1)A^7 + (\alpha^2 + 1)A^6 + \alpha A^5 + (\alpha + 1)A^4 \\
 &\quad + (\alpha^2 + \alpha + 1)A^3 + (\alpha^2 + 1)A
 \end{aligned}$$

Let $\mathbb{F}_q = GF(2^k)$, and $\overline{\mathbb{F}_q}$ be its closure

- $\mathbb{F}_q[x_1, \dots, x_n]$: ring of all polynomials with coefficients in \mathbb{F}_q
- Polynomial $f = c_1X_1 + c_2X_2 + \dots + c_tX_t$
 - A monomial ordering is imposed on $f : X_1 > X_2 > \dots > X_t$
 - **Leading term** $lt(f) = c_1X_1$, $tail(f) = c_2X_2 + \dots + c_tX_t$
 - Leading coefficient $lc(f) = c_1$ and leading monomial $lm(f) = X_1$
 - LEX $x > y > z$: $f = -2x^3 + 2x^2yz + 3xy^3$
 - DEGLEX $x > y > z$: $f = 2x^2yz + 3xy^3 - 2x^3$
 - DEGREVLEX $x > y > z$: $f = 3xy^3 + 2x^2yz - 2x^3$
- Leading terms $lt(f)$ play an important role
 - Affect division results!

Preliminaries: Polynomial division

Divide $f = x^3 - 2x^2 + 2x + 8$ by $g = 2x^2 + 3x + 1$

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$$\begin{array}{r} \frac{1}{2}x - \frac{7}{4} \\ \hline 2x^2 + 3x + 1) x^3 - 2x^2 + 2x + 8 \\ - x^3 - \frac{3}{2}x^2 - \frac{1}{2}x \\ \hline - \frac{7}{2}x^2 + \frac{3}{2}x + 8 \\ \frac{7}{2}x^2 + \frac{21}{4}x + \frac{7}{4} \\ \hline \phantom{\frac{7}{2}x^2 + } \frac{27}{4}x + \frac{39}{4} \end{array}$$

Preliminaries: Polynomial division

Divide $f = x^3 - 2x^2 + 2x + 8$ by $g = 2x^2 + 3x + 1$

[illegible]

- The key step in division: $r = f - \frac{lt(f)}{lt(g)} \cdot g$, denoted $f \xrightarrow{g} r$
- Similarly divide f by a set of polynomials $F = \{f_1, \dots, f_s\}$
- Denoted: $f \xrightarrow{f_1, \dots, f_s} r$
 - Remainder r is reduced: no term in r is divisible by $lt(f_i)$

Let $\mathbb{F}_q = GF(2^k)$:

- Given a set of polynomials:
 - $f_1, f_2, \dots, f_s \in \mathbb{F}_q[x_1, \dots, x_n]$
 - Find solutions to $f_1 = f_2 = \dots = f_s = 0$
- **Variety**: Set of ALL solutions to a given system of polynomial equations: $V(f_1, \dots, f_s)$
 - In $\mathbb{R}[x, y]$, $V(x^2 + y^2 - 1) = \{\text{all points on circle} : x^2 + y^2 - 1 = 0\}$
 - In $\mathbb{R}[x]$, $V(x^2 + 1) = \emptyset$
 - In $\mathbb{C}[x]$, $V(x^2 + 1) = \{(\pm i)\}$
- Variety depends on the **ideal** generated by the polynomials.
- Reason about the Variety by analyzing the Ideals

Definition

Ideals of Polynomials: Let $f_1, f_2, \dots, f_s \in \mathbb{F}_q[x_1, \dots, x_n]$. Let

$$J = \langle f_1, f_2, \dots, f_s \rangle = \{f_1 h_1 + f_2 h_2 + \dots + f_s h_s\}, \quad h_i \in \mathbb{F}_q[x_1, \dots, x_n]$$

$J = \langle f_1, f_2, \dots, f_s \rangle$ is an ideal generated by f_1, \dots, f_s and the polynomials are called the generators.

- Different generators can generate the same ideal
- $\langle f_1, \dots, f_s \rangle = \dots = \langle g_1, \dots, g_t \rangle$
- Some generators are a “better” representation of the ideal
- A **Gröbner basis** G is a “canonical” representation of an ideal
 - $I = \langle F \rangle = \langle G \rangle$, and $V(F) = V(G)$
- Map: set of states \rightarrow **variety** of polynomial ideal

Preliminaries: Buchberger's algorithm computes a Gröbner basis

INPUT : $F = \{f_1, \dots, f_s\}$

OUTPUT : $G = \{g_1, \dots, g_t\}$

$G := F$;

REPEAT

$G' := G$

 For each pair $\{f, g\}, f \neq g$ in G' DO

$Spoly(f, g) \xrightarrow{G'}_+ r$

 IF $r \neq 0$ THEN $G := G \cup \{r\}$

UNTIL $G = G'$

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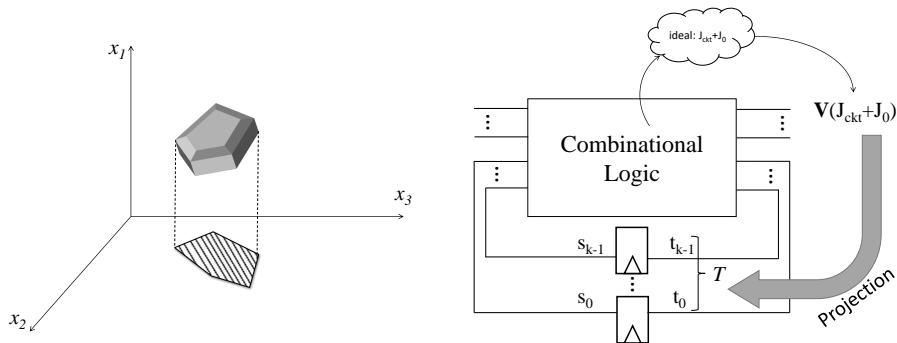
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- $Spoly(f, g) = \frac{L}{lt(f)} \cdot f - \frac{L}{lt(g)} \cdot g$
 $L = \text{LCM}(lm(f), lm(g))$, $lm(f)$: leading monomial of f
- Animation 1...

- Let ideal $I = \langle f_1, f_2, f_3 \rangle$ where
 - $f_1 = x^2 + y + z - 1$
 - $f_2 = x + y^2 + z - 1$
 - $f_3 = x + y + z^2 - 1$
- The Gröbner basis of I with **elimination** (LEX) order ($x > y > z$) is
 - $g_1 = x + y + z^2 - 1$
 - $g_2 = y^2 - y - z^2 + z$
 - $g_3 = 2yz^2 + z^4 - z^2$
 - $g_4 = z^6 - 4z^4 + 4z^3 - z^2$
- Notice that g_2 and g_3 only contain variables y and z
 - Eliminates variable $x \Leftrightarrow \exists_x$ in Boolean formula!
- Similarly, g_4 only contains the variable z and eliminates x and y

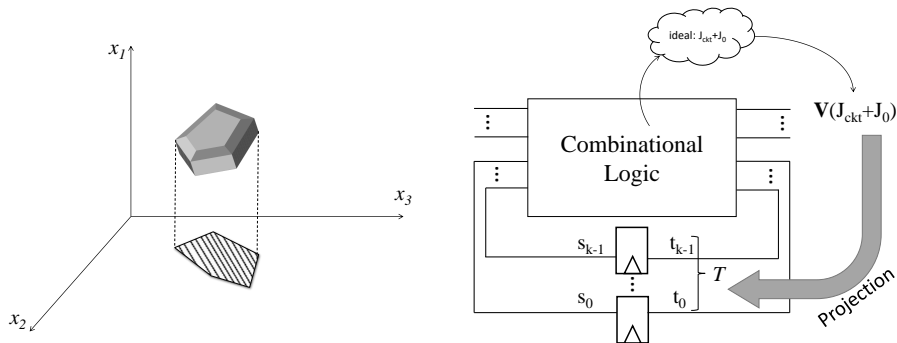
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- Similarly, g_4 only contains the variable z and eliminates x and y
- $\text{GB}(I_{x,y,z}) \cap \mathbb{F}_q[z]$ related to **projection** on variable z !

Elimination by projection



- Projection of variety from $\mathbb{F}[x_1, x_2, x_3]$ to $\mathbb{F}[x_2, x_3]$
- Projection of ckt ideal's variety on next state (NS) variables T

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- Projection of ckt ideal's variety on next state (NS) variables T
- $\text{GB}(J_{ckt} + J_0) \cap \mathbb{F}_2[T] \implies \text{Next state polynomial } f(T)!$

ALGORITHM: Breadth-first Traversal Algorithm

Input: Transition functions Δ , initial state S^0

```
1  $from^0 = reached = S^0$ ;  
2 repeat  
3    $i \leftarrow i + 1$ ;  
4    $to^i \leftarrow \text{Img}(\Delta, from^{i-1})$ ;  
5    $new^i \leftarrow to^i \cap \overline{reached}$ ;  
6    $reached \leftarrow reached \cup new^i$ ;  
7    $from^i \leftarrow new^i$ ;  
8 until  $new^i == 0$ ;  
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- Image function:

$$\text{Img}(\Delta, from) = \exists_s \exists_x [T(s, x, t) \wedge from] = \exists_s \exists_x \bigwedge_{i=1}^n (t_i \oplus \Delta_i) \wedge from$$

- In \mathbb{B}^k , image function $\Leftrightarrow \exists$
- In \mathbb{F}_{2^k} , need to implement **quantifier elimination**

Recall: Breadth-First Traversal Algorithm

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Implement this algorithm by finding analogs in algebraic geometry

Recall: Breadth-First Traversal Algorithm

ALGORITHM: Breadth-first Traversal Algorithm

Input: Transition functions Δ , initial state S^0 ; // polynomial ideal
1 $from^0 = reached = S^0$; // set of states \Leftrightarrow variety of ideal
2 **repeat**
3 $i \leftarrow i + 1$;
4 $to^i \leftarrow \text{lmg}(\Delta, from^{i-1})$;
5 $new^i \leftarrow to^i \cap \overline{reached}$;
6 $reached \leftarrow reached \cup new^i$;
7 $from^i \leftarrow new^i$;
8 **until** $new^i == 0$;
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5    $new^i \leftarrow to^i \cap \overline{reached}$ ;           // ideal quotient & sum
6    $reached \leftarrow reached \cup new^i$ ;           // ideal product
7    $from^i \leftarrow new^i$ ;
8 until  $new^i == 0$ ;
9 return  $reached$ 
```

Implement this algorithm by finding analogs in algebraic geometry

Definition

(Sum/Product of Ideals) If $I = \langle f_1, \dots, f_r \rangle$ and $J = \langle g_1, \dots, g_s \rangle$ are ideals in $\mathbb{F}[x_1, \dots, x_n]$, then the **sum** of I and J is defined as

$$I + J = \langle f_1, \dots, f_r, g_1, \dots, g_s \rangle$$

And the **product** of I and J is defined as

$$I \cdot J = \langle f_i g_j \mid 1 \leq i \leq r, 1 \leq j \leq s \rangle$$

Theorem

If I and J are ideals in $\mathbb{F}[x_1, \dots, x_n]$, then $\mathbf{V}(I + J) = \mathbf{V}(I) \cap \mathbf{V}(J)$ and $\mathbf{V}(I \cdot J) = \mathbf{V}(I) \cup \mathbf{V}(J)$.

Definition

(Quotient of Ideals) If I and J are ideals in $\mathbb{F}[x_1, \dots, x_n]$, then $I : J$ is the set

$$\{f \in \mathbb{F}[x_1, \dots, x_n] \mid f \cdot g \in I, \forall g \in J\}$$

and is called the **ideal quotient** of I by J .

Theorem

Let J_0 be an ideal of vanishing polynomials over $\mathbb{F}_{2^k}[x_1, \dots, x_n]$, then

$$\mathbf{V}(J_0 : J) = \mathbf{V}(J_0) - \mathbf{V}(J) = \overline{\mathbf{V}(J)}$$

- $V(J) \subseteq \mathbb{F}_{2^k}$ in affine space
- Given ideal J , compute J' s.t.
 $V(J') = \overline{V(J)} = \mathbb{F}_{2^k} - V(J) \implies J' = J_0 : J$

Our proposed algorithm of BFS traversal based on algebraic geometry

ALGORITHM: Algebraic Geometry based FSM Traversal

Input: The circuit's characteristic polynomial ideal J_{ckt} , initial state polynomial $\mathbb{F}(S)$, and LEX term order: bit-level variables $x, s, t > \text{PS word } S > \text{NS word } T$

```
1  $from^0 = reached = \mathbb{F}(S)$ ;  
2 repeat  
3    $i \leftarrow i + 1$ ;  
4    $G \leftarrow \text{GB}(\langle J_{ckt}, J_0, from^{i-1} \rangle)$ ;           // This step contains bit-level  
5    $\langle to^i \rangle \leftarrow G \cap \mathbb{F}_{2^k}[T]$ ;           // Only word-level  $S, T$  onwards  
6    $\langle new^i \rangle \leftarrow \langle to^i \rangle + (\langle T^{2^k} - T \rangle : \langle reached \rangle)$ ;  
7    $\langle reached \rangle \leftarrow \langle reached \rangle \cdot \langle new^i \rangle$ ;  
8    $from^i \leftarrow new^i(S \setminus T)$ ;  
9 until  $\langle new^i \rangle == \langle 1 \rangle$ ;  
10 return  $\langle reached \rangle$ 
```

► [Go to example page 2](#)

Full Blown traversal of example circuit using algebraic geometry

- Initial state $from^0 = S(\{00\})$
- Iteration 1:** Compose an elimination ideal J

$$f_1 : t_0 - (xs_0s_1 + xs_0$$
$$+ xs_1 + x + s_0 + s_1 + 1)$$

$$f_2 : t_1 - (xs_0 + x + s_0s_1 + s_0)$$

$$f_3 : S - s_0 - s_1\alpha$$

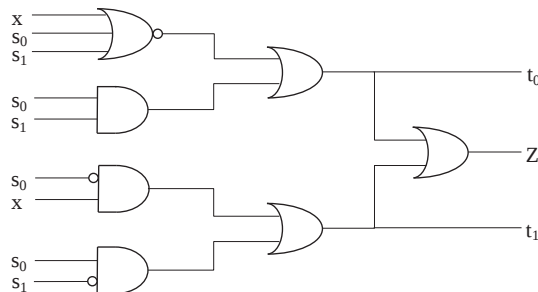
$$f_4 : T - t_0 - t_1\alpha$$

$$f_5 : x^2 - x$$

$$f_6 : s_0^2 - s_0, f_7 : s_1^2 - s_1$$

$$f_8 : t_0^2 - t_0, f_9 : t_1^2 - t_1$$

$$f_{10} : S^4 - S, f_{11} : T^4 - T$$



$$J_{ckt} = \langle f_1, f_2, f_3, f_4 \rangle$$

$$J_0 = \langle f_5, f_6, \dots, f_{11} \rangle$$

Full Blown traversal of example circuit using algebraic geometry

- Newly reached states:

$$\langle T^3 + 1, T^2 + (\alpha + 1)T + \alpha \rangle = \langle T^2 + (\alpha + 1)T + \alpha \rangle (\{01, 10\})$$

- Update current reached states

$$reach = \langle T \cdot T^2 + (\alpha + 1)T + \alpha \rangle = \langle T^3 + (\alpha + 1)T^2 + \alpha T \rangle$$

- Mapping to set of states

$$V(reached) = \{0, 1, \alpha\} \Leftrightarrow \{00, 01, 10\}$$

- Update the present states for next iteration

$$from^1 = \langle S^2 + (\alpha + 1)S + \alpha \rangle$$

- Iteration 2:

- Next state: $to^2 = \langle T^2 + \alpha T \rangle (\{00, 10\})$
- The complement of *reached*:

$$\langle T^4 - T \rangle : \langle T^3 + (\alpha + 1)T^2 + \alpha T \rangle = \langle T + 1 + \alpha \rangle (\{11\})$$

- Newly reached state:

$$\langle T^2 + \alpha T, T + 1 + \alpha \rangle = \langle 1 \rangle$$

- Algorithm terminates
- Return value (final reachable states):

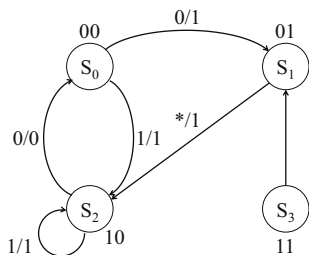
$$reached = \langle T^3 + (\alpha + 1)T^2 + \alpha T \rangle$$

Improve complexity using RATO [Pruss, '15]

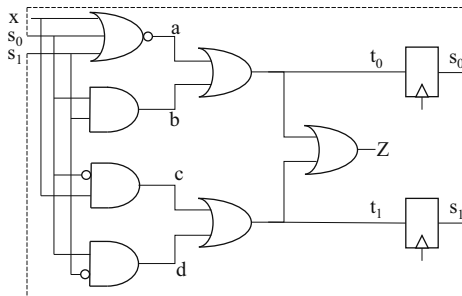
- Directly compute $\text{GB}(J_{ckt} + J_0)$ in \mathbb{F}_q is costly ($q^{O(n)}$)
- GB sensitive to term order \rightarrow Transform to simpler set for GB?

Improve complexity using RATO [Pruss, '15]

- Directly compute $\text{GB}(J_{ckt} + J_0)$ in \mathbb{F}_q is costly ($q^{O(n)}$)
- GB sensitive to term order \rightarrow Transform to simpler set for GB?
- RATO: reverse topological traverse on ckt structure



(a)



(b)

- Definition of RATO:
“bit-level variables ordered reverse topologically ” $> T > S$
- Topology in ckt structure
 - Output of each gate = $lt(f_i)$
- Product criterion: $gcd(lt(f_i), lt(f_j)) = 1 \implies Spoly(f_i, f_j) \xrightarrow{J_{ckt} + J_0} + 0$
- Only one pair of poly with non-relatively-prime leading terms
- *Spoly* division
 - Divide with levelization
 - Only inputs (primary & pseudo) left!

Example of using RATO

- RATO: LEX with $(t_0, t_1) > (a, b, c, d) > (x, s_0, s_1) > T > S$

$$f_1 : a + xs_0s_1 + xs_0 + xs_1 + x + s_0s_1 + s_0 + s_1 + 1$$

$$f_2 : b + s_0s_1 \quad f_3 : c + x + xs_0 \quad f_4 : d + s_0s_1 + s_0$$

$$f_5 : t_0 + ab + a + 1 \quad f_6 : t_1 + cd + c + d \quad f_7 : t_0 + t_1\alpha + T$$

- Spoly reduction gives $T + \mathcal{F}(\text{primary/pseudo inputs})$

$$\begin{aligned} \text{Spoly}(f_5, f_7) &\xrightarrow{J_{ckt} + J_0} T + s_0s_1x + \alpha s_0s_1 \\ &+ (1 + \alpha)s_0x + (1 + \alpha)s_0 + s_1x + s_1 + (1 + \alpha)x + 1 \end{aligned}$$

- Q: How to get rid of bit-level inputs?

- Objective: find $s_i = \mathcal{G}(S)$
- $(s_0 + s_1\alpha + \cdots + s_{k-1}\alpha^{k-1})^{2^n} = s_0^{2^n} + (s_1\alpha)^{2^n} + \cdots + (s_{k-1}\alpha^{k-1})^{2^n}$
- Build system of poly eqns by squaring:

$$\begin{bmatrix} S \\ S^2 \\ S^{2^2} \\ \vdots \\ S^{2^{k-1}} \end{bmatrix} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{k-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(k-1)} \\ 1 & \alpha^4 & \alpha^8 & \cdots & \alpha^{4(k-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{2^{k-1}} & \alpha^{2 \cdot 2^{k-1}} & \cdots & \alpha^{(k-1) \cdot 2^{k-1}} \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{k-1} \end{bmatrix}$$

- Transition function $f_T : T + \mathcal{F}(S, x)$
- Elimination on ideal $\langle f_T, f_S \rangle + J_0$ using $S, x > T$

ALGORITHM: Refined Algebraic Geometry based FSM Traversal

Input: Polynomial ideal J_{ckt} , initial state polynomial $\mathcal{F}(S)$

Output: Final reachable states represented by polynomial $\mathcal{G}(T)$

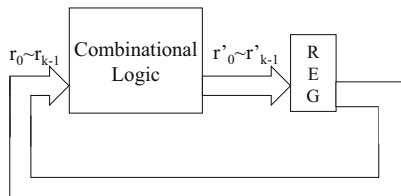
```
1   $from^0 = reached = \mathcal{F}(S);$ 
2   $f_T = \text{Reduce}(\text{Spoly}(f_w, f_g), J_{ckt});$ 
   /* Compute  $\text{Spoly}$  for the critical pair, then reduce it with
      circuit ideal under RATO */
3  Eliminate bit-level variables in  $f_T$ ;
4  repeat
5       $i \leftarrow i + 1;$ 
6       $G \leftarrow \text{GB}(\langle f_T, from^{i-1} \rangle + J_0^{PI});$ 
7       $to^i \leftarrow G \cup \mathbb{F}_{2^k}[T];$ 
8       $\langle new^i \rangle \leftarrow \langle to^i \rangle + (\langle T^{2^k} - T \rangle : \langle reached \rangle);$ 
9       $\langle reached \rangle \leftarrow \langle reached \rangle \cdot \langle new^i \rangle;$ 
10      $from^i \leftarrow new^i(S \setminus T);$ 
11 until  $\langle new^i \rangle == \langle 1 \rangle;$ 
12 return  $\langle reached \rangle$ 
```

Experiment results: word-level traversal

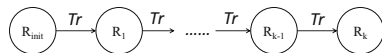
Table : Results of running benchmarks using our tool. Parts I to III denote the time taken by polynomial divisions, bit-level to word-level abstraction and iterative reachability convergence checking part of our approach, respectively.

Benchmark	# States	# iterations	Runtime (sec)			Runtime of VIS (sec)
			I	II	III	
b01	18	5	< 0.01	0.01	0.02	< 0.01
b02	8	5	< 0.01	0.01	< 0.01	< 0.01
b06	13	4	< 0.01	0.07	5.0	< 0.01
s27	6	2	< 0.01	0.01	0.02	< 0.01
s208	16	16	< 0.01	0.32	2.4	< 0.01
s386	13	3	1.0	7.6	8.2	< 0.01
bbara	10	6	0.04	0.01	0.04	< 0.01
beecount	7	3	< 0.01	0.01	0.01	< 0.01
dk14	7	2	45	< 0.01	0.08	< 0.01
donfile	24	3	12316	0.02	1.7	< 0.01

Apply FSM traversal to arithmetic ckt



(a)



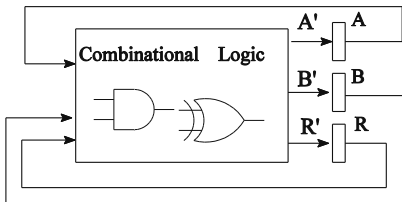
(b)

- Model: restricted Moore finite state machine
 - Some sequential arithmetic circuits will give results after running for k clock cycles
 - The initial operands are preloaded in register files
- State transitions on this model:

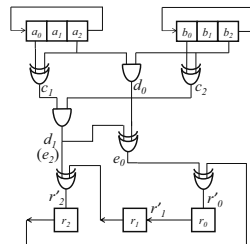
$$R_k = Tr(R_{k-1}) = Tr(Tr(\cdots Tr(R_{init}) \cdots)) = Tr^k(R_{init})$$

- Word-level unrolling

SPEC: $R = A_{init} \cdot B_{init} \pmod{P(\alpha)}$ after k clock cycles



(a)



(b)

- Projection on NS R', A', B'

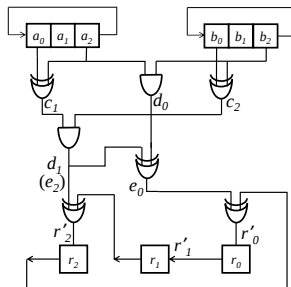
ALGORITHM: Abstraction via implicit unrolling for Sequential GF circuit verification

Input: Circuit polynomial ideal J , vanishing ideal J_0 , initial state ideal

$$R(=0), \mathcal{G}(A_{init}), \mathcal{H}(B_{init})$$

```
1  $from_0(R, A, B) = \langle R, \mathcal{G}(A_{init}), \mathcal{H}(B_{init}) \rangle;$   
2  $i = 0;$   
3 repeat  
4    $i \leftarrow i + 1;$   
5    $G \leftarrow \text{GB}(\langle J + J_0 + from_{i-1}(R, A, B) \rangle)$  with ATO;  
6    $to_i(R', A', B') \leftarrow G \cap \mathbb{F}_{2^k}[R', A', B', R, A, B];$   
7    $from_i \leftarrow to_i(\{R, A, B\} \setminus \{R', A', B'\});$   
8 until  $i == k;$   
9 return  $from_k(R_{final})$ 
```

Experiment on 3-bit RH-SMPO



- The elimination ideal (first iteration):

$$\begin{aligned} J = & d_0 + b_2 \cdot a_2, c_1 + a_0 + a_2, c_2 + b_0 + b_2, d_1 + c_1 \cdot c_2, \\ & e_0 + d_0 + d_1, e_2 + d_1, r'_0 + r_2 + e_0, r'_1 + r_0, r'_2 + r_1 + e_2, \\ & A + a_0\beta + a_1\beta^2 + a_2\beta^4, B + b_0\beta + b_1\beta^2 + b_2\beta^4, \\ & R + r_0\beta + r_1\beta^2 + r_2\beta^4, R' + r'_0\beta + r'_1\beta^2 + r'_2\beta^4; \end{aligned}$$

Experiment on 3-bit RH-SMPO(2)

- $J_0 = \langle x_i^2 - x_i, X^q - X \rangle$
- $from_0 = \{R, A_{init} + a_0\beta + a_1\beta^2 + a_2\beta^4, B_{init} + b_0\beta + b_1\beta^2 + b_2\beta^4\}$

ALGORITHM: Abstraction via implicit unrolling for Sequential GF circuit verification

Input: Circuit polynomial ideal J , vanishing ideal J_0 , initial state ideal

$$R(=0), \mathcal{G}(A_{init}), \mathcal{H}(B_{init})$$

```
1  $from_0(R, A, B) = \langle R, \mathcal{G}(A_{init}), \mathcal{H}(B_{init}) \rangle;$   
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3 repeat  
4    $i \leftarrow i + 1;$   
5    $G \leftarrow \text{GB}(\langle J + J_0 + from_{i-1}(R, A, B) \rangle)$  with ATO;  
6    $to_i(R', A', B') \leftarrow G \cap \mathbb{F}_{2^k}[R', A', B', R, A, B];$   
7    $from_i \leftarrow to_i(\{R, A, B\} \setminus \{R', A', B'\});$   
8 until  $i == k;$   
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```

Experiment on 3-bit RH-SMPO(2)

- $J_0 = \langle x_i^2 - x_i, X^q - X \rangle$
- $from_0 = \{R, A_{init} + a_0\beta + a_1\beta^2 + a_2\beta^4, B_{init} + b_0\beta + b_1\beta^2 + b_2\beta^4\}$
($\beta = \alpha^3$)
- $to_1 : R' + (\alpha^2)A_{init}^4B_{init}^4 + (\alpha^2 + \alpha)A_{init}^4B_{init}^2 + (\alpha^2 + \alpha)A_{init}^4B_{init} + (\alpha^2 + \alpha)A_{init}^2B_{init}^4 + (\alpha^2 + \alpha + 1)A_{init}^2B_{init}^2 + (\alpha^2)A_{init}^2B_{init} + (\alpha^2 + \alpha)A_{init}B_{init}^4 + (\alpha^2)A_{init}B_{init}^2$
- $from_1 =$
 $\{R' + (\alpha^2)A_{init}^4B_{init}^4 + (\alpha^2 + \alpha)A_{init}^4B_{init}^2 + (\alpha^2 + \alpha)A_{init}^4B_{init} + (\alpha^2 + \alpha)A_{init}^2B_{init}^4 + (\alpha^2 + \alpha + 1)A_{init}^2B_{init}^2 + (\alpha^2)A_{init}^2B_{init} + (\alpha^2 + \alpha)A_{init}B_{init}^4 + (\alpha^2)A_{init}B_{init}^2, A_{init} + a_2\alpha^3 + a_0\alpha^6 + a_1\alpha^{12}, B_{init} + b_2\alpha^3 + b_0\alpha^6 + b_1\alpha^{12}\}$
- ...
- After 3 iterations: $to_3 =$
 $\{R' + A_{init}B_{init}, A_{init} + a'_0\alpha^3 + a'_1\alpha^6 + a'_2\alpha^{12}, B_{init} + b'_0\alpha^3 + b'_1\alpha^6 + b'_2\alpha^{12}\}$

ALGORITHM: Abstraction via implicit unrolling for Sequential GF circuit verification

Input: Circuit polynomial ideal J , vanishing ideal J_0 , initial state ideal

$R(=0), \mathcal{G}(A_{init}), \mathcal{H}(B_{init})$

```
1  $from_0(R, A, B) = \langle R, \mathcal{G}(A_{init}), \mathcal{H}(B_{init}) \rangle;$   
2  $i = 0;$   
3 repeat  
4    $i \leftarrow i + 1;$   
5    $f_o \xrightarrow{J+J_0+from_{i-1}(R,A,B)}_+ f_r$  under RATO ;  
6    $to_i(R', A', B') \leftarrow f_r(\{R', A', B'\} \setminus \{r_0, \dots, r_{k-1}, a_0, \dots, a_{k-1}, b_0, \dots, b_{k-1}\});$   
7    $from_i \leftarrow to_i(\{R, A, B\} \setminus \{R', A', B'\});$   
8 until  $i == k;$   
9 return  $from_k(R_{final})$ 
```

Experiment result: sequential GF multiplier verification

- Run-time for verification of bug-free RH-SMPO circuits for SAT, ABC and BDD based methods. $TO = \text{timeout } 14 \text{ hrs}$

	Word size of the operands k -bits			
Solver	11	18	23	33
Lingeling	593	TO	TO	TO
ABC	6.24	TO	TO	TO
BDD	0.1	11.7	1002.4	TO

- Runtime for verification of bug-free Agnew's and RH-SMPO circuits using our approach

Operand size k		36	60	81	100	131	162
RH-SMPO	#Polys	4716	12960	21870	35600	56592	92826
	Runtime	14.3	213.3	1343	4685	26314	124194
Agnew's SMPO	#Polys	2700	7380	13356	20300	34715	52974
	Runtime	10.2	212.0	2684	4686	56568	119441

Abstraction refinement

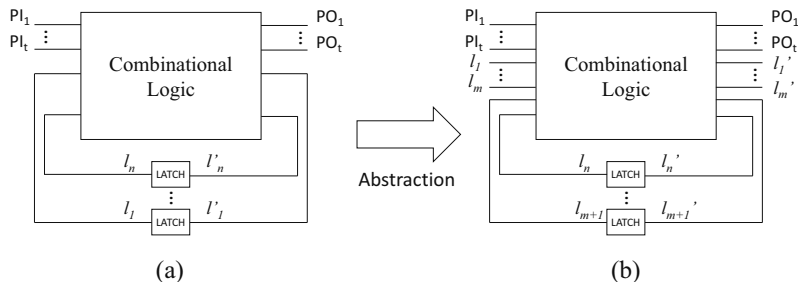


Figure : Abstraction by reducing latches

- Remove “irrelevant” latches, reduce state space
- Provide an over-approximation
- This algorithm requires **UNSAT core extraction**

Theorem (Weak Nullstellensatz)

Let $J = \langle f_1, \dots, f_s \rangle$ be an ideal in the ring $\mathbb{F}[x_1, \dots, x_n]$ and $V_{\overline{\mathbb{F}}}(J)$ be its variety over $\overline{\mathbb{F}}$. Then $V_{\overline{\mathbb{F}}}(J) = \emptyset \iff J = \mathbb{F}[x_1, \dots, x_n] \iff 1 \in J$.

$$V_{\overline{\mathbb{F}}}(J) = \emptyset \iff 1 \in J \iff \text{reduced GB}(J) = \{1\}$$

- Recall Buchberger's algorithm: UNSAT \rightarrow terminates with 1

Theorem (Weak Nullstellensatz)

Let $J = \langle f_1, \dots, f_s \rangle$ be an ideal in the ring $\mathbb{F}[x_1, \dots, x_n]$ and $V_{\mathbb{F}}(J)$ be its variety over \mathbb{F} . Then $V_{\mathbb{F}}(J) = \emptyset \iff J = \mathbb{F}[x_1, \dots, x_n] \iff 1 \in J$.

$$V_{\mathbb{F}}(J) = \emptyset \iff 1 \in J \iff \text{reduced GB}(J) = \{1\}$$

- Recall Buchberger's algorithm: UNSAT \rightarrow terminates with 1
- Information lies in *Spoly*?
- Poly calculus stronger than resolution!

Definition

Assume a polynomial set F is UNSAT. A subset $M \subseteq F$ is an **UNSAT core** if M is also UNSAT. Further, if $\forall f \in M, M \setminus \{f\}$ is SAT, then M is called a **minimal UNSAT core** of F .

Animation 2...

Motivating example

- $f_1 \sim f_9 \in \mathbb{F}_2[a, b, c, d]$
- $F = \{f_1, f_2, \dots, f_9\}$

$$f_1 : abc + ab + ac + bc$$

$$+ a + b + c + 1$$

$$f_2 : b$$

$$f_3 : ac$$

$$f_4 : ac + a$$

$$f_5 : bc + c$$

$$f_6 : abd + ad + bd + d$$

$$f_7 : cd$$

$$f_8 : abd + ab + ad + bd + a + b + d + 1$$

$$f_9 : abd + ab + bd + b$$

- Find a subset $F_c \subset F$ s.t. F_c remains UNSAT

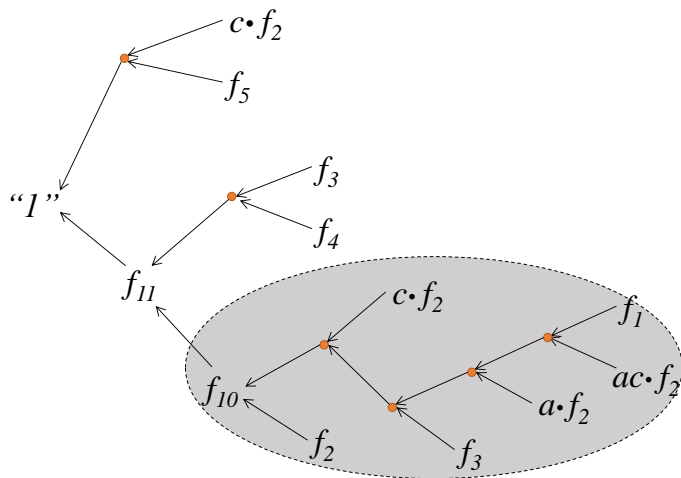
Following Spoly selection strategy

$(f_1, f_2) \rightarrow (f_1, f_3) \rightarrow (f_2, f_3) \rightarrow (f_1, f_4) \rightarrow \dots$, execute Buchberger's algorithm until adding "1" to Gröbner basis

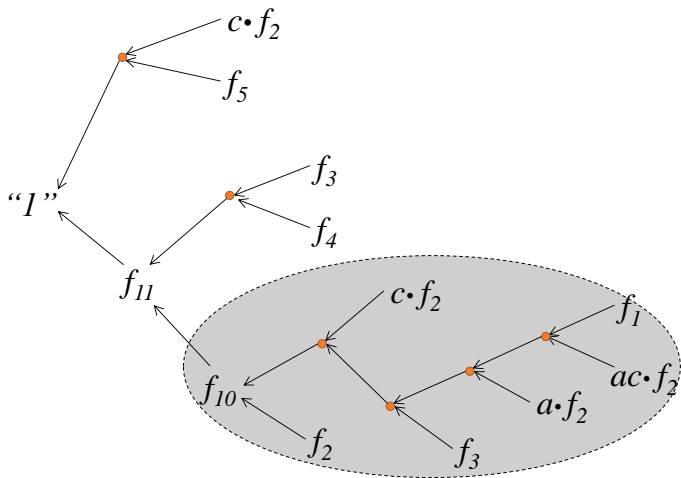
- $\text{Spoly}(f_1, f_2) = f_1 - ac \cdot f_2 \xrightarrow{a \cdot f_2} \xrightarrow{f_3} \xrightarrow{c \cdot f_2} \xrightarrow{f_2} f_{10} = a + c + 1$
- $\text{Spoly}(f_1, f_3) \xrightarrow{F} + 0$
- ...
- $\text{Spoly}(f_3, f_4) \xrightarrow{f_{10}} f_{11} = c + 1$
- $\text{Spoly}(f_2, f_5) \xrightarrow{f_{11}} 1$

$\{f_1, \dots, f_9, f_{10}, f_{11}, 1\}$ is the Gröbner basis generated from Buchberger's algorithm

Motivating Example: Refutation Tree



Motivating Example: Refutation Tree



- $f_{10} = f_1 - acf_2 - af_2 - f_3 - cf_2 - f_2 = f_1 + acf_2 + af_2 + f_3 + cf_2 + f_2$
- $1 = \mathbb{F}(f_1, f_2, f_3, f_4, f_5) \implies 1 \in \langle f_1, f_2, f_3, f_4, f_5 \rangle \implies UNSAT$

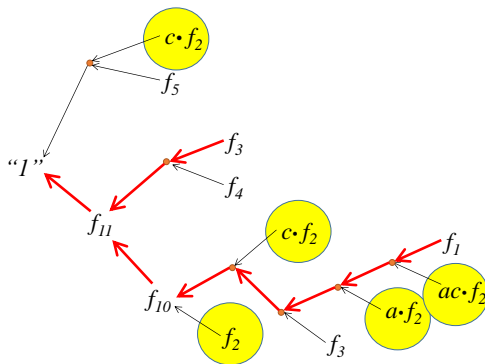
- Expand:

$$\begin{aligned} 1 &= \mathbb{F}(f_1, f_2, f_3, f_4, f_5) \\ &= cf_2 + f_5 + f_{11} \\ &= cf_2 + f_5 + f_3 + f_4 + f_{10} \\ &= (cf_2 + f_5) + \cdots + \mathbf{1} \cdot \mathbf{f}_3 + \cdots + \mathbf{1} \cdot \mathbf{f}_3 + \cdots + (f_1 + acf_2) \end{aligned}$$

- UNSAT core reduced to $\{f_1, f_2, f_4, f_5\}$ which is **minimal**
- GB-core algorithm:
 - Execute Buchberger's algorithm
 - Recording data including Spoly, polynomials for division and remainder
 - Terminate Buchberger's algorithm after recording remainder "1"
 - Build refutation tree and get UNSAT core
 - Analyze recorded data, remove redundant polynomials from the core

Reducing size: iterative refinement

- Spoly pair selection strategy:
 $(f_1, f_2) \rightarrow (f_1, f_3) \rightarrow (f_2, f_3) \rightarrow (f_1, f_4) \rightarrow (f_2, f_4) \rightarrow \dots$
- High likelihood in minimal core \rightarrow Put ahead in Spoly queue \rightarrow Faster approaching “1” in GB-core \rightarrow Smaller core



- **Refutation distance:** shortest path to leaf
- Distance $\downarrow \rightarrow$ LT Degree $\downarrow \rightarrow$ Likelihood in minimal core \uparrow
- **Frequency:** number of times f_i appears in refutation tree

Iterative refinement Example

$$f_1 : x_1x_3 + x_3$$

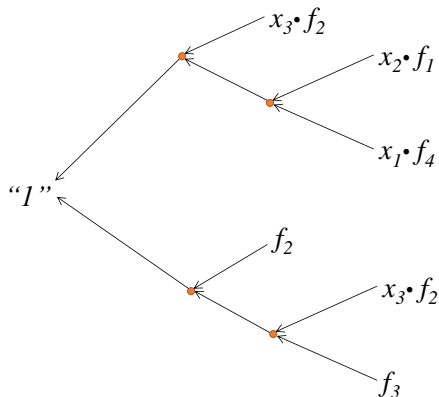
$$f_2 : x_2 + 1$$

$$f_3 : x_2x_3 + x_2$$

$$f_4 : x_2x_3$$

$$f_5 : x_2x_3 + x_2 + x_3 + 1$$

$$f_6 : x_1x_2x_3 + x_1x_3$$



- $F = \{f_1, f_2, \dots, f_6\} \in \mathbb{F}_2[x_1, x_2, x_3]$
- UNSAT core: f_1, f_2, f_3, f_4
- Refutation distance:

f_2	f_1	f_3	f_4
2	3	3	3
- Frequency:

f_1	f_3	f_4
1	1	1
- Reorder: f_2, f_1, f_3, f_4

Iterative refinement: iteration 2

$$f_1 : x_1x_3 + x_3$$

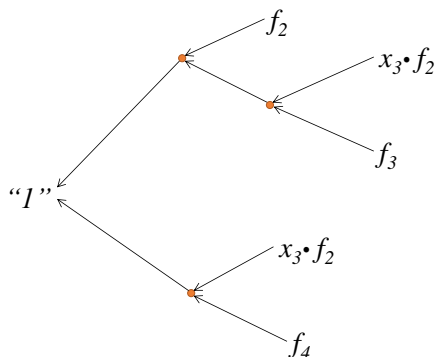
$$f_2 : x_2 + 1$$

$$f_3 : x_2x_3 + x_2$$

$$f_4 : x_2x_3$$

$$f_5 : x_2x_3 + x_2 + x_3 + 1$$

$$f_6 : x_1x_2x_3 + x_1x_3$$



- New order: f_2, f_1, f_3, f_4
- Spoly pairs selection:
 $(f_2, f_1) \rightarrow (f_2, f_3) \rightarrow$
 $(f_1, f_3) \rightarrow (f_2, f_4) \rightarrow$
 $(f_1, f_4) \rightarrow (f_3, f_4)$
- UNSAT core: f_2, f_3, f_4
- Fixpoint reached

- Finding interdependencies: $f_i \in \langle F \setminus \{f_i\} \rangle$?
- Given $F = \{f_1, \dots, f_s\}$, find $f_i = \sum_{j \neq i} h_j f_j$
- Info lost in refutation tree/Buchberger's algorithm?
- Inner loop of Buchberger's algorithm:

$$\text{Spoly}(f, g) \xrightarrow{G'}_{+} r$$

IF $r \neq 0$ THEN $G := G \cup \{r\}$

IF $r = 0$ THEN Discard

- We collect division info when $r = 0$:
 - Record data for Spoly and polynomial division as in GB-core algorithm
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- (c_1, c_2, \dots, c_s) is a **syzygy** on (f_1, f_2, \dots, f_s)

- In a single syzygy, $c_i = 1 \implies f_i = \sum_{j \neq i} h_j f_j$
- In general cases, need to analyze all syzygies recorded
- Collect m syzygies as a system of polynomial equations, or a **Syzygy Matrix**

$$\begin{cases} c_1^1 f_1 + c_2^1 f_2 + \cdots + c_s^1 f_s = 0 \\ c_1^2 f_1 + c_2^2 f_2 + \cdots + c_s^2 f_s = 0 \\ \vdots \\ c_1^m f_1 + c_2^m f_2 + \cdots + c_s^m f_s = 0 \end{cases} \quad \begin{bmatrix} c_1^1 & c_2^1 & \cdots & c_s^1 \\ c_1^2 & c_2^2 & \cdots & c_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ c_1^m & c_2^m & \cdots & c_s^m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_s \end{bmatrix} = 0$$

- Syzygy matrix:

$$\begin{array}{c}
 f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7 \quad f_8 \quad f_9 \quad f_{10} \\
 \text{Spoly}(f_1, f_3) \\
 \text{Spoly}(f_2, f_3) \\
 \text{Spoly}(f_1, f_4) \\
 \text{Spoly}(f_2, f_4) \\
 \text{Spoly}(f_1, f_5)
 \end{array}
 \begin{bmatrix}
 1 & a+c+1 & b+1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & ac & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & c+1 & 1 & b & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & ac+a & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & a+c+1 & 0 & 0 & a & 0 & 0 & 0 & 0 & 1
 \end{bmatrix}$$

- Syzygy matrix:

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
$\text{Spoly}(f_1, f_3)$	1	$a + c + 1$	$b + 1$	0	0	0	0	0	0	1
$\text{Spoly}(f_2, f_3)$	0	ac	b	0	0	0	0	0	0	0
$\text{Spoly}(f_1, f_4)$	1	$c + 1$	1	b	0	0	0	0	0	1
$\text{Spoly}(f_2, f_4)$	0	$ac + a$	0	b	0	0	0	0	0	0
$\text{Spoly}(f_1, f_5)$	1	$a + c + 1$	0	0	a	0	0	0	0	1


- rows: syzygies
- $J = \langle f_1, f_2, \dots, f_9 \rangle$
- $f_{10} \in J$

Syzygy Example: refine syzygy matrix

- Considering

$$f_{10} = f_1 - acf_2 - af_2 - f_3 - cf_2 - f_2 = f_1 + acf_2 + af_2 + f_3 + cf_2 + f_2$$

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}
$\text{Spoly}(f_1, f_3)$	1	$a + c + 1$	$b + 1$	0	0	0	0	0	0	1
$\text{Spoly}(f_2, f_3)$	0	ac	b	0	0	0	0	0	0	0
$\text{Spoly}(f_1, f_4)$	1	$c + 1$	1	b	0	0	0	0	0	1
$\text{Spoly}(f_2, f_4)$	0	$ac + a$	0	b	0	0	0	0	0	0
$\text{Spoly}(f_1, f_5)$	1	$a + c + 1$	0	0	a	0	0	0	0	1
$\text{Spoly}(f_1, f_2)$	[1	$ac+a+c+1$	1	0	0	0	0	0	0	1]



Syzygy example: derive interdependency

$$\begin{array}{c}
 f_1 \quad f_2 \quad f_3 \quad f_4 \quad f_5 \quad f_6 \quad f_7 \quad f_8 \quad f_9 \\
 \text{Spoly}(f_1, f_3) \\
 \text{Spoly}(f_2, f_3) \\
 \text{Spoly}(f_1, f_4) \\
 \text{Spoly}(f_2, f_4) \\
 \text{Spoly}(f_1, f_5)
 \end{array}
 \begin{bmatrix}
 0 & ac & b & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & ac & b & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & ac + a & 0 & b & 0 & 0 & 0 & 0 & 0 \\
 0 & ac + a & 0 & b & 0 & 0 & 0 & 0 & 0 \\
 0 & ac & 1 & 0 & a & 0 & 0 & 0 & 0
 \end{bmatrix}$$

- Column f_3 contains “1”, means:
 $1 \cdot f_3 + acf_2 + af_5 = 0 \Leftrightarrow f_3 = acf_2 + af_5$
- Generalization of this strategy refer to the paper

ALGORITHM: UNSAT core extraction based on Gröbner basis algorithm

Input: A set of UNSAT polynomials F

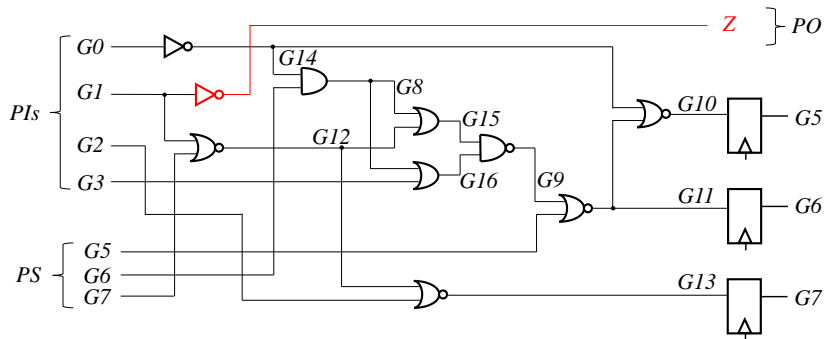
Output: A subset $F' \subset F$ remains UNSAT

```
1  $G \leftarrow F$ ;  
2 repeat  
3    $F' = G$ ;  
4    $G \leftarrow \text{GB-core}(F' \text{ with order } >)$ ;  
5   Update order  $>$ ;  
6 until  $G = F'$ ;  
7  $F' \leftarrow \text{syzygy\_heuristic}(G)$ ;  
8 return  $F'$ 
```

Table of selected benchmarks
I:single GB-core; II: Iterative GB-core; III: syzygy

Benchmark	# Polys	# MUS	Size of Core			#GB-core iterations	Runtime (sec)			Runtime of PicoMUS (sec)
			I	II	III		I	II	III	
5x5 SMPO	240	137	169	137	137	8	1222	1938	1698	<0.1
aim-100	79	22	22	22	22	1	43	0.7	0.2	<0.1
phole4	104	10	16	16	10	1	4.3	0.2	0.5	<0.1
phole5	169	19	30	25	19	3	12	3.2	2.7	<0.1
subset-2	141	19	37	23	21	2	12	1.6	1.1	<0.1
subset-3	118	16	13	12	11	2	8.6	0.2	0.07	<0.1

Application to abstraction refinement



- $PS = \{G7, G6, G5\}$, $NS = \{G13, G11, G10\}$: 8 states
- Property $p = \mathbf{AG}((\neg G13)\mathbf{U}(\neg Z))$
- k -BMC without abstraction refinement: when $k = 3$ prove *PASS*

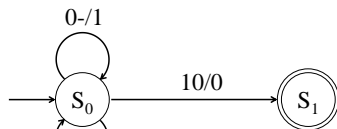
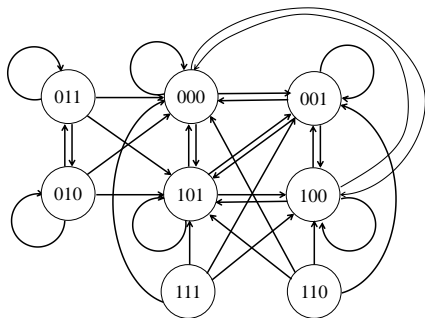
- Circuit ideal when $k = 0$:

$$I = \langle G14 + 1 + G0, G8 + G14 \cdot G6, G15 + G12 + G8 + G12 \cdot G8, \\ G16 + G3 + G8 + G3 \cdot G8, G9 + 1 + G16 \cdot G15, \\ G10 + 1 + G14 + G11 + G14 \cdot G11, G11 + 1 + G5 + G9 + G5 \cdot G9, \\ G12 + 1 + G1 + G7 + G1 \cdot G7, G13 + 1 + G2 + G12 + G2 \cdot G12, \\ Z + 1 + G1, \\ (\text{Initial state } 000) G5, G6, G7 \rangle;$$

- Property: $\neg p = Z \cdot G13 + 1$
- UNSAT core:

$$\text{Core}(I \wedge \neg p) = G12 + 1 + G1 + G7 + G1 \cdot G7, \\ G13 + 1 + G2 + G12 + G2 \cdot G12, \\ Z + 1 + G1, G7;$$

Application to abstraction refinement



$S = \{G13\}$
 $S_0 = \{0\}$
 $S_1 = \{1\}$

- $\{G5/G10, G6/G11\}$: irrelevant
- By removing irrelevant latches, state-space reduced

Conclusion

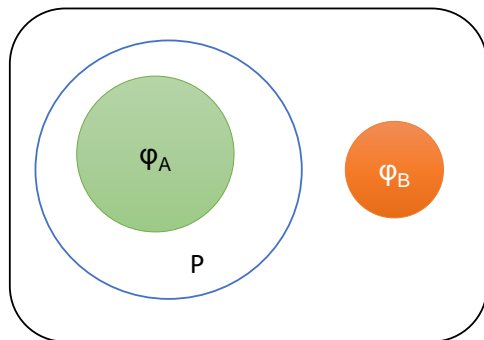
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- Succeed to verify 162-bit, while contemporary fails beyond 23 bit
- UNSAT core extraction for a set of polynomials
- Refine the core using refutation proof & syzygies
- UNSAT core info can be applied to abstraction refinement

- Multivariate polynomial ideals
 - Extend the application of univariate polynomial ideals
- Accelerate GB reduction
 - F_4 algorithm on term-sparse polynomial ideal (parallel computing)
 - ZDDs can represent chain of OR gates logic in linear space complexity (alternative canonical graphic representation)
- Compute Craig's interpolants using algebraic geometry
 - Projection of varieties \implies interpolants

Future work: Craig's interpolants



P (common vars of A, B)

$$A \Rightarrow P \\ B \wedge P: \text{UNSAT}$$

OR

$$\varphi_A \subseteq P \\ \varphi_B \cap P = \emptyset$$

- $A \wedge B = \emptyset$, $A = (\bar{d})(\bar{c})(\bar{a} \vee d)$ and $B = (a \vee b \vee c)(\bar{b})$
- $P = \bar{a} \wedge \bar{c}$ is an interpolant of (A, B)
- Find interpolant from resolution tree [K. Mcmillan '03]
- Algebraic geometry: projection in affine space

- Publications:

- *Formal Verification of Sequential Galois Field Arithmetic Circuits using Algebraic Geometry.* **Xiaojun Sun**, Priyank Kalla, Tim Pruss, Florian Enescu. DATE 2015, Grenoble
- *Finding Unsatisfiable Cores of a Set of Polynomials using the Groebner Basis Algorithm.* **Xiaojun Sun**, Irina Iliaoa, Priyank Kalla, Florian Enescu. CP 2016, Toulouse
- *Word-level Traversal of Finite States Machines using Algebraic Geometry.* **Xiaojun Sun**, Priyank Kalla, Florian Enescu. HLDVT 2016, Santa Cruz
- Journal paper in preparation

- Tools:

My website: ece.utah.edu/~xiaojuns/code.html