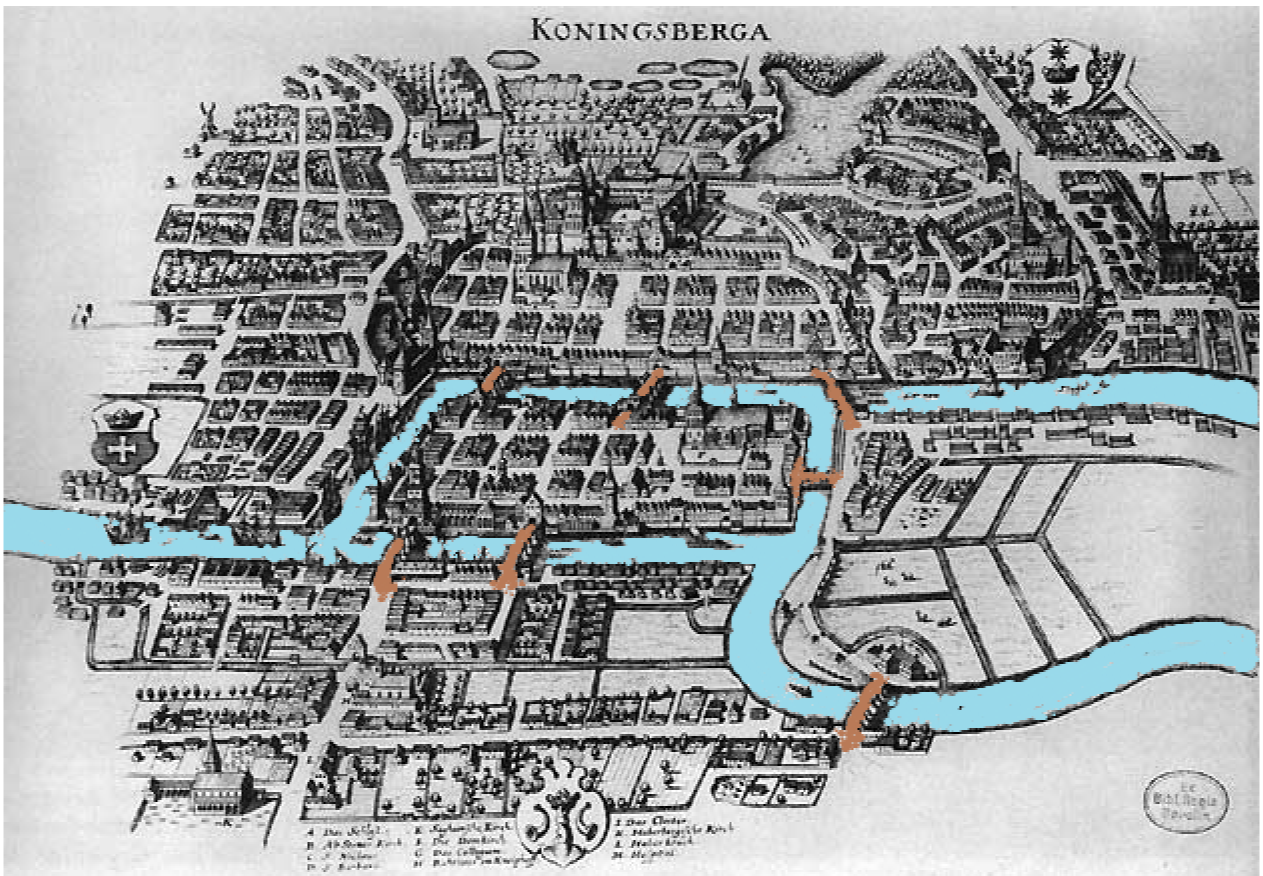
Lecture Notes
for
Math 1030



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October 7, 2021 at 6:27pm

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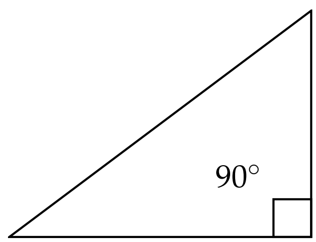
List of Theorems

Chapter 1

Trigonometry

1.1 The Pythagorean Theorem

Trigonometry is the study of triangles, especially **right triangles**. A **right triangle** is a triangle that has a 90° angle which is often denoted by drawing a little square in the corner where two sides meet at a perpendicular or 90° angle.



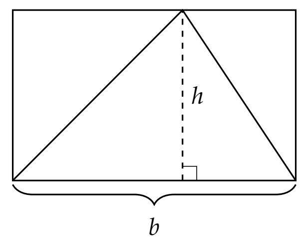
We begin our study of Trigonometry by proving three important theorems. A **theorem** is a fact. But a theorem is more than just a fact, it is a true statement about mathematical objects. Mathematical objects can include a huge variety of things like shapes, numbers, equations, inequalities, graphs, sets, plus many many more. Usually, a theorem applies to a whole class of mathematical objects like all triangles or all graphs.

Theorems require a **proof**, which is just a convincing explanation of why the statement or theorem is true. It is common to denote the end of a proof with a small square, that looks like this: $\square$ or $\blacksquare$. We will present several theorems in this book, but we won't supply a proof for each one because some of them will require techniques or knowledge beyond the scope of our study. Hopefully, the few we do present will give you an idea of what a proving a theorem entails.

Theorem (Area of a Triangle).
The area of a triangle with base, b, and height, h is:

$$A = \frac{1}{2}bh \tag{1.1}$$

***Proof*.**

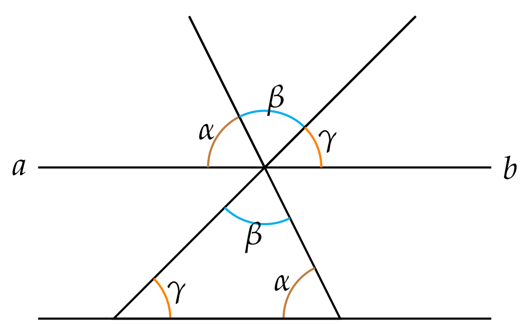


The height of the triangle divides the largest rectangle into two sub-rectangles. Since the triangle covers exactly half of each sub-rectangle, the triangle occupies exactly half of the area of the outer rectangle which is bh. □

It is a common convention to use Greek letters to denote angles, especially the letters α "alpha", β "beta", γ "gamma", θ "theta", but sometimes Roman letters such as x, y, z, t or capital letters such as A, B, C are used as well.

> **Theorem (Sum of Interior Angles of a Triangle).**
> The interior angles of a triangle sum to $180°$.

***Proof*.**



Draw a line segment between points a and b parallel to the base of the triangle. Extending the edges of the triangle past the parallel line generates angles which correspond with angles α and γ. Similarly, the two angles marked β are vertical angles and thus equal. Finally, the three angles at the top of the figure sum to a straight line and thus sum to $180°$. □

The next theorem is the most important one in the study of Trigonometry.

> **Theorem (The Pythagorean Theorem).**
> If a, b, c are lengths of the legs of a *right* triangle where c is the *hypotenuse*, then
>
> $$a^2 + b^2 = c^2 \tag{1.2}$$

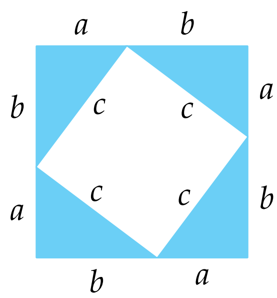


Figure 1.1: Four exact copies of a right triangle arranged to make two squares.

***Proof*.** First, note that the outer (partly blue shaded) quadrilateral is a square because each side has the same length, $a + b$, and because each corner has an angle of $90°$. Second, note that the inner tilted quadrilateral is also a square because each side has the same length, namely c, and each of its corner angles must be $90°$. This is due to the fact that the two non-right angles in the given right triangle must sum to $90°$ because the sum of all angles in *any* triangle is $180°$. Finally, we can derive equation (1.2) by equating the area of the whole outer square to the area of the sum of its pieces.

$$\begin{aligned} \text{Area of outer square} &= \text{Sum of the area of the pieces} \\ (a+b)(a+b) &= c^2 + 4\left(\tfrac{1}{2}ab\right) \\ a^2 + \cancel{2ab} + b^2 &= c^2 + \cancel{2ab} \\ a^2 + b^2 &= c^2 \end{aligned}$$

□

Exercise 1

Laptop computer screens are measured along the diagonal, i.e. from one corner of the screen to the opposite corner. If a 14 in laptop screen is 8 in tall, how wide is the screen?

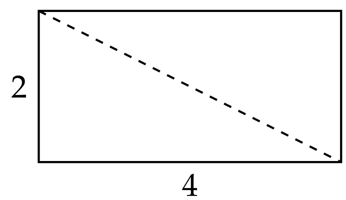
Exercise 2

To safely use a ladder, the base of the ladder should form a $75°$ angle with level ground. A simple way to ensure this is to use the “4 to 1” rule, which says that for every 4 ft of height gained you move the base 1 ft away from the vertical.

What is the minimum length ladder needed to safely reach a roof that is 16 ft high?

Exercise 3

A phone line would normally be installed along two consecutive edges of a rectangular 2 mi by 4 mi field. If the phone line costs $2500 mi^{-1}$ to install, how much will be saved if the phone line is installed diagonally across the field instead of along two of the edges?



1.1.1 Converse of the Pythagorean Theorem

A **conditional statement** is a statement of the form "If P then Q" where P and Q are both sub-statements. P is called the **hypothesis** and Q is called the **conclusion**. For example

> **Conditional:** If Socrates is a man, then Socrates is mortal. (true)

Notice that the above statement is true, that is, the conclusion is a consequence of the hypothesis being true. We can also form three related conditional statements (see Table 1.1). The **converse** of the statement above, is not true:

> **Converse:** If Socrates is mortal, then Socrates is a man. (false)

There are lots of things that are mortal that are not men, for example Socrates could refer to a woman or a horse. Thus a conditional statement and its converse need not both be true. But sometimes they are.

Statement	English Form	Arrow Notation
Conditional	If P then Q.	$P \Rightarrow Q$
Converse	If Q then P.	$Q \Rightarrow P$
Inverse	If not P then not Q.	$\neg P \Rightarrow \neg Q$
Contrapositive	If not Q then not P.	$\neg Q \Rightarrow \neg P$

Table 1.1: Four types of conditional statements

The Pythagorean theorem is a **conditional statement** of the form "If P then Q", where (after removing details) we essentially have:

Pythagorean Theorem: If right triangle, then $a^2 + b^2 = c^2$.

The converse of the Pythagorean theorem is of the form "If Q, then P", that is,

Converse of Pythagorean Theorem: If $a^2 + b^2 = c^2$, then right triangle.

It turns out that the converse of the Pythagorean theorem is also true.

Theorem (Converse of the Pythagorean Theorem).
If a, b, c are sides of a triangle with c the longest side and $a^2 + b^2 = c^2$, then the triangle is a *right* triangle.

Proof. See Appendix A. □

Exercise 4
Use the converse of the Pythagorean theorem to prove that a triangle with sides (5,12,13) is a right triangle.

When both a conditional and its converse are always true we call it a **biconditional**. This means that P implies Q but also Q implies P, in arrow notation: $P \Leftrightarrow Q$. In other words, statements P and Q are different but they both mean the same thing. One side of the Pythagorean theorem is about lengths and the other is about angles. And what it means is that (in the context of triangles), knowing something about lengths of the legs tells you something about the interior angles and vice versa. You can think of it as a *bridge* between the concept of length and the concept of angle. This bridge leads to two new concepts: Pythagorean triples and a way to classify triangles.

1.1.2 Pythagorean Triples

Definition.
A **Pythagorean triple** is any triple of whole numbers, (a, b, c), which satisfy $a^2 + b^2 = c^2$ and thus are the legs of a right triangle.

Example 1
Some examples of Pythagorean triples are:

$$(3, 4, 5), (6, 8, 10), (5, 12, 13), (7, 24, 25), (9, 40, 41)$$

These are useful to remember mostly because Math teachers like to use them in homework problems and tests, and it can save you time if you notice that you are dealing with one of these special triangles.

Exercise 5
Can you take any Pythagorean triple and generate a new triple just by multiplying every number by a whole number, like 2 or 3? Can you prove it?

1.1.3 Classification of Triangles

The converse of the Pythagorean theorem allows us to use the sides of a triangle to determine something about one of the angles. If we combine this fact with the fact that all of the interior angles of a triangle must sum to 180°, then we can classify triangles into three categories.

Definition.
obtuse angle: an angle between 90° and 180°.

acute angle: an angle between 0° and 90°.

obtuse triangle: a triangle which has one obtuse angle.

acute triangle: a triangle where all angles are acute.

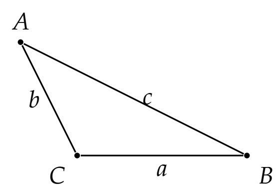
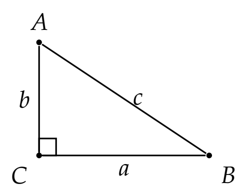
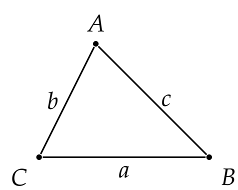
Condition	Example	Category
$c^2 > a^2 + b^2$		obtuse
$c^2 = a^2 + b^2$		right
$c^2 < a^2 + b^2$		acute

Table 1.2: Classification of triangles by sides

Exercise 6
Classify the triangles with sides $(5, 6, 7)$ and $(5, 6, 8)$.

1.1.4 The Distance Formula

We can use the Pythagorean theorem to find the distance between any two points in the Cartesian plane. The basic idea is to construct a right triangle with hypotenuse corresponding to a segment between the two points and one leg parallel to the x-axis and another leg parallel to the y-axis.

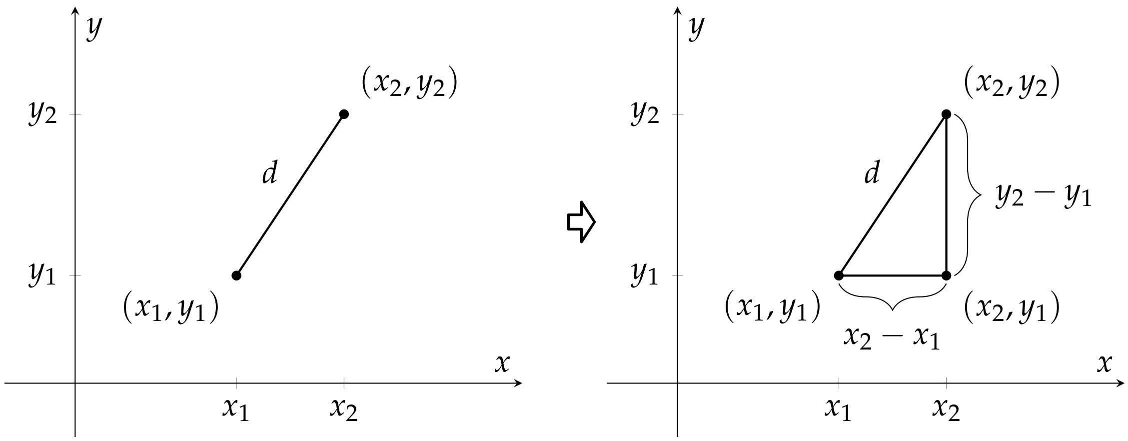


Figure 1.2: The Pythagorean theorem can be used to determine distance bewteen points.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{1.3}$$

Exercise 7
Find the distance between the points $(-1, 5)$ and $(2, -2)$.

Formula (1.3) can be hard to remember. I find it easier to just remember the idea of equating the distance to the hypotenuse and then using a sketch and the Pythagorean theorem to compute the distance.

1.2 Trigonometric Functions

The Pythagorean theorem allows us to determine the length of an unknown side of a right triangle when we know two of the other sides. But in some circumstances, we may only know an interior angle and a side. In this section we will define six functions that will allow us to determine sides when we only know a side and an angle.

In order to define these functions we use three terms to identify the edges or legs of the triangle: **hypoteneuse**, **opposite**, and **adjacent**. First, the term **hypotenuse** always refers to the longest edge of the triangle. Next, we have the side **opposite** the angle of interest, say θ. Finally, we have the side **adjacent** to the angle of interest. Figure 1.3 demonstrates how the adjacent and opposite sides switch depending on where θ is.

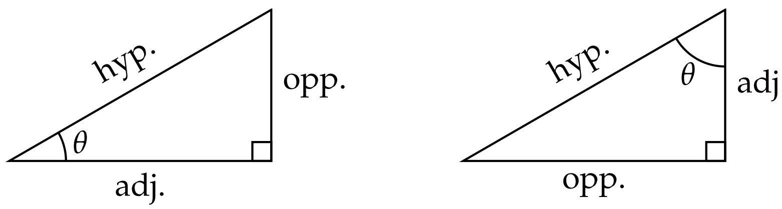


Figure 1.3: Opposite and adjacent sides depend on where θ is located.

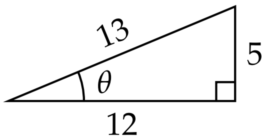
1.2.1 The Sine, Cosine, and Tangent Functions

Definition.
Given a *right* triangle with angle θ, we define three functions in terms of ratios of the sides of the triangle.

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \tan(\theta) = \frac{\text{opp}}{\text{adj}} \tag{1.4}$$

A mnemonic for remembering all three definitions is SOH-CAH-TOA.

Example 2
Use the triangle below to compute: $\sin(\theta), \cos(\theta)$ and $\tan(\theta)$.



Solution:

$$\sin(\theta) = \frac{5}{13} \qquad \cos(\theta) = \frac{12}{13} \qquad \tan(\theta) = \frac{5}{12}$$

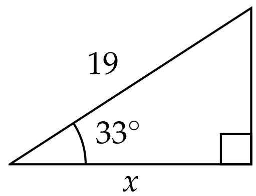
△

The three trigonometric functions we just defined are **side finders**. Later we will introduce functions that are **angle finders**.

Make sure your calculator is in *degrees* mode and not radians mode.

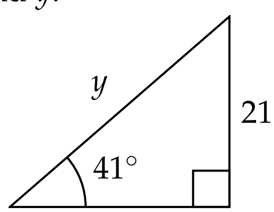
Exercise 8

Find x.



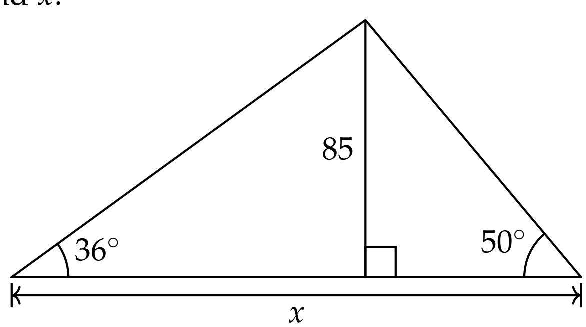
Exercise 9

Find y.



Exercise 10

Find x.



1.2.2 Inverse Trigonometric Functions

Not all functions have an inverse, but the three trigonometric functions we just defined do! Recall that a function is a mapping between two sets which we call the domain and codomain. The **domain** of a function is the set of acceptable inputs, while a function's **codomain** is its output set.

An inverse function maps in the reverse direction, so it swaps the domain and codomain sets. Thus while the three trigonometric functions eat angles and output ratios, the three *inverse* trigonometric functions eat ratios and output angles. You should think of them as **angle finders**.

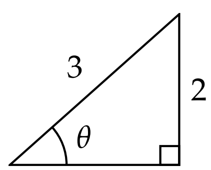
Since we have defined the three trigonometric functions via *right* triangles only, the domains and codomains are somewhat restricted. If you take a full course on Trigonometry, you will learn how these functions can be extended to larger domains and codomains.

Function : Domain $\mapsto$ Codomain	Inverse : Domain $\mapsto$ Codomain
$\sin(x) : [0°, 90°] \mapsto [0, 1]$	$\sin^{-1}(x) : [0, 1] \mapsto [0°, 90°]$
$\cos(x) : [0°, 90°] \mapsto [0, 1]$	$\cos^{-1}(x) : [0, 1] \mapsto [0°, 90°]$
$\tan(x) : [0°, 90°) \mapsto [0, \infty)$	$\tan^{-1}(x) : [0, \infty) \mapsto [0°, 90°)$

Table 1.3: The three Trigonometric functions map angles to ratios, while their inverses map ratios to angles.

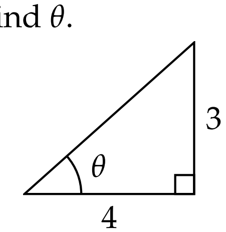
Exercise 11

Find θ.



Exercise 12

Find θ.



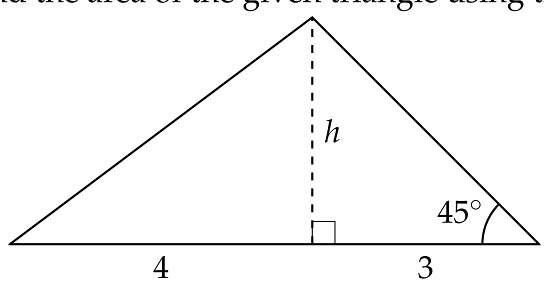
Exercise 13

A new 500 ft long zip line is going to be installed at Lagoon. The zip line will have a vertical distance of 200 ft. At what angle will the zip line meet the ground? What is the horizontal distance of the zip line?

1.2.3 Finding the Area of a Triangle

Exercise 14

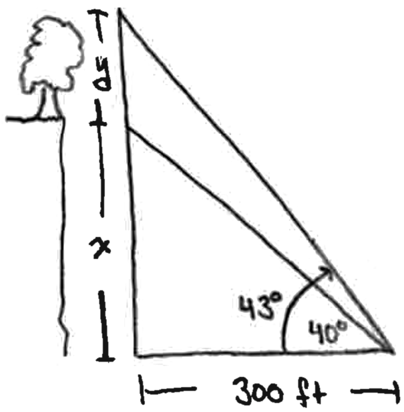
Find the area of the given triangle using trigonometric functions.



1.2.4 Angles of Elevation and Depression

Exercise 15

A person stands 300 ft from the base of a cliff. The angle of elevation to the bottom of a tree atop the cliff is 40°. The angle of elevation to the top of the tree is 43°. How tall is the tree?



Exercise 16

A statue is located 300 ft from a building. From a window in the building, a person determines that the angle of elevation to the top of the statue is 34° and the angle of depression to the bottom of the statue is 22°. How tall is the statue?

Beware the functions we define in this section only apply to *right* triangles.

Chapter 2

Problem Solving

2.1 Exploiting Fractions for Fun and Profit

2.1.1 Absolute and Relative Change

In this section we will see how to use the two arithemtic operations of subtraction and division to measure how something changes over time.

Definition.
Absolute change is the difference:

$$\text{absolute change} = \text{new} - \text{old}$$

Absolute change can be useful for comparing things that are similar. For example you could compare how many inches two fourteen year old boys grew in one year. But suppose the things you wish to compare are related but on very different size scales.

Example 3 (Stock Investments Part I)

Suppose you invest $50,000 in Acme corporation stock and after one year the stock is now worth $52,000. This is an absolute change of $2,000. Now suppose you also invest $500 in Hooli corporation and after one year the stock is worth $550. Which investment was better?

Solution:

$$\text{absolute change in Acme} = \$52,000 - \$50,000 = \$2,000$$
$$\text{absolute change in Hooli} = \$550 - \$500 = \$50$$

In terms of absolute change the yield from the Acme investment is larger, so it is better right? Well what if we had switched our investment strategy and swapped the initial stock purchase for each company? To answer this question we need to see how big the absolute change in value was relative to the initial outlay of money. We need relative change. △

Definition.
Relative change is the ratio of absolute change to the original (old) value.

$$\text{relative change} = \frac{\text{new} - \text{old}}{\text{old}}$$

Example 4 (Stock Investments Part II)
What is the relative change in value of each stock purchase?

Solution:

$$\text{relative change in Acme} = \frac{\$52,000 - \$50,000}{\$50,000} = \frac{\$2,000}{\$50,000} = 0.04$$
$$\text{relative change in Hooli} = \frac{\$550 - \$500}{\$500} = \frac{\$50}{\$500} = 0.10$$

Hooli's relative change was higher than Acme's relative change, and thus the Hooli stock was the better investment. △

Note.
Notice that absolute change has units attached to the value. In the above examples the units were dollars ($). But when you compute relative change, the numerator and denominator will have the same units and thus the units will cancel and you are left with a pure (unitless) number.

2.1.2 Percentages

Definition.
A **percent** or **percentage** is a fraction, where the denominator is 100, but for ease of writing we drop the denominator and replace it with the percent symbol %. For example,

$$\frac{3}{4} = 0.75 = \frac{75}{100} = 75\%$$

Percentages can be negative and even larger than 100, for example

$$250\% = \frac{250}{100} = 2.5$$

Exercise 17
Express the following values as percentages.

1. $\dfrac{1}{5}$

2. 0.004

3. 3.11

It is often convenient to express relative change as a percent

Example 5 (Stock Investments Part III)
What was the percent change in value of each stock investment?

Company	Decimal	Fraction	Percent
Acme	.04	$\frac{4}{100}$	4%
Hooli	.10	$\frac{10}{100}$	10%

Exercise 18
Your truck was worth \$28,000 in 2019 and is now worth \$23,500 in 2021. What is the percent change in value of your truck?

Exercise 19
Your \$290,000 home increased in value by 5%. How much is it worth now?

Exercise 20
A store has a clearance rack where every item is marked down 20% off the original price. They have a sale advertising an additional 20% off everything in the store. What percentage of the original price do you end up paying?

2.1.3 Rates

Definition.
A rate is a ratio of two quantities. Rates are also known as fractions.

Exercise 21

If your car can travel 284 mi per fillup and your gas tank holds 9.5 gal, then at what rate does your car consume gasoline, or more commonly, what is its mpg?

2.1.4 Proportionality

Definition.
A proportion is a ratio of two quantities. Thus it is a fraction.

Exercise 22

A map's legend indicates that 0.5 in equals 4 mi. How many miles apart in real life are two points on the map that are separated by 3.5 in?

Exercise 23

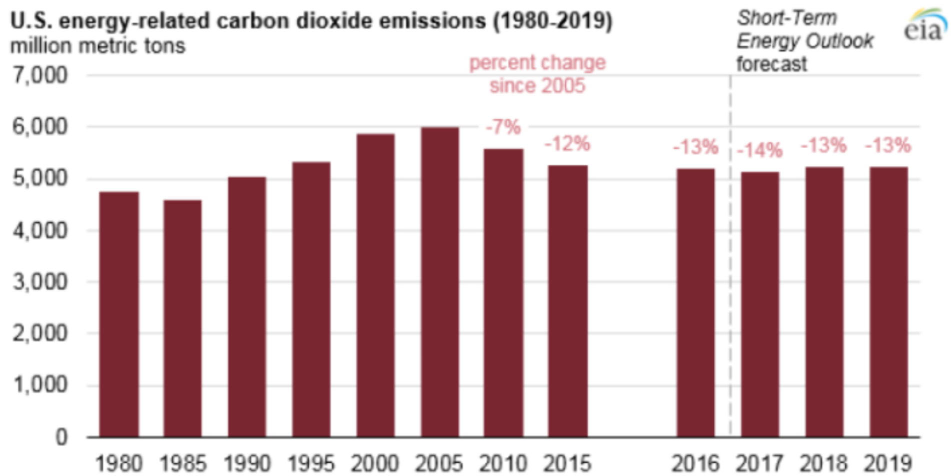
A cookie recipe calls for 2 cups of chocolate chips per batch of 24 cookies. If you wish to make 216 cookies?

2.2 Exploiting Units for Fun and Profit

Today in Energy

February 8, 2018

U.S. energy-related CO2 emissions expected to rise slightly in 2018, remain flat in 2019



Source: U.S. Energy Information Administration, *Monthly Energy Review*, *Short-Term Energy Outlook*

EIA estimates that U.S. energy-related carbon dioxide (CO2) emissions declined by 861 million metric tons (14%) from 2005 to 2017. In the latest *Short-Term Energy Outlook*, EIA projects that CO2 emissions will rise 1.8%, from 5,143 million metric tons in 2017 to 5,237 million metric tons in 2018, then remain virtually unchanged in 2019. In 2019, energy-related CO2 emissions will be about 13% lower than 2005 levels.

1. According to EIA how much carbon did the American energy sector emit in 2017? How much is it predicted to emit in 2018 and 2019?

2. Use the absolute change from 2005 to 2017 given in the first sentence to calculated U.S. energy-related emissions in 2005.

3. Use the result from part 2 to calculate the relative change from 2005 to 2017. Does your value agree with the given percentage?

4. A Type D school bus weighs about 15 metric tons. How many school buses worth of CO_2 did the American energy industry emit in 2005?

5. How many school buses worth of CO_2 is the American energy industry predicted to emit in 2019?

6. According to the American School Bus Council about 480,000 buses are used each day in the United States. Compare the weight of all those buses to the weight of the CO_2 the U.S. energy sector is predicted to emit in 2019.

2.2.1 Unit Conversions

Length	12 in = 1 ft
	3 ft = 1 yd
	5280 ft = 1 mi
Weight	1 lb = 16 oz
	1 ton = 2000 lb
Capacity	1 fl oz = $\frac{1}{8}$ cup
	1 cup = 8 fl oz
	1 pint = 16 fl oz (2 cups)
	1 quart = 32 fl oz (4 cups)
	1 gallon = 128 fl oz (4 quarts)

Table 2.1: Common US unit conversions

Exercise 25

Convert 5 cubic yards to cubic feet.

Exercise 26

During a massive Canadian ice storm, 30 thousand acres of sugarbush crop were damaged, where each acre of sugarbush grows about 80 trees. You can assume that each tree yields $\frac{1}{2}$ gallon of maple syrup a year, and Canada typically produces 8 million gallons of maple syrup per year.

1. What percentage of the maple syrup supply in Canada was destroyed that year?

2. If maple syrup is worth \$39.10 per gallon and the average farmer tends to 45 acres of sugarbush, how much income did the average farmer lose that year?

2.2.2 Working In the Yard

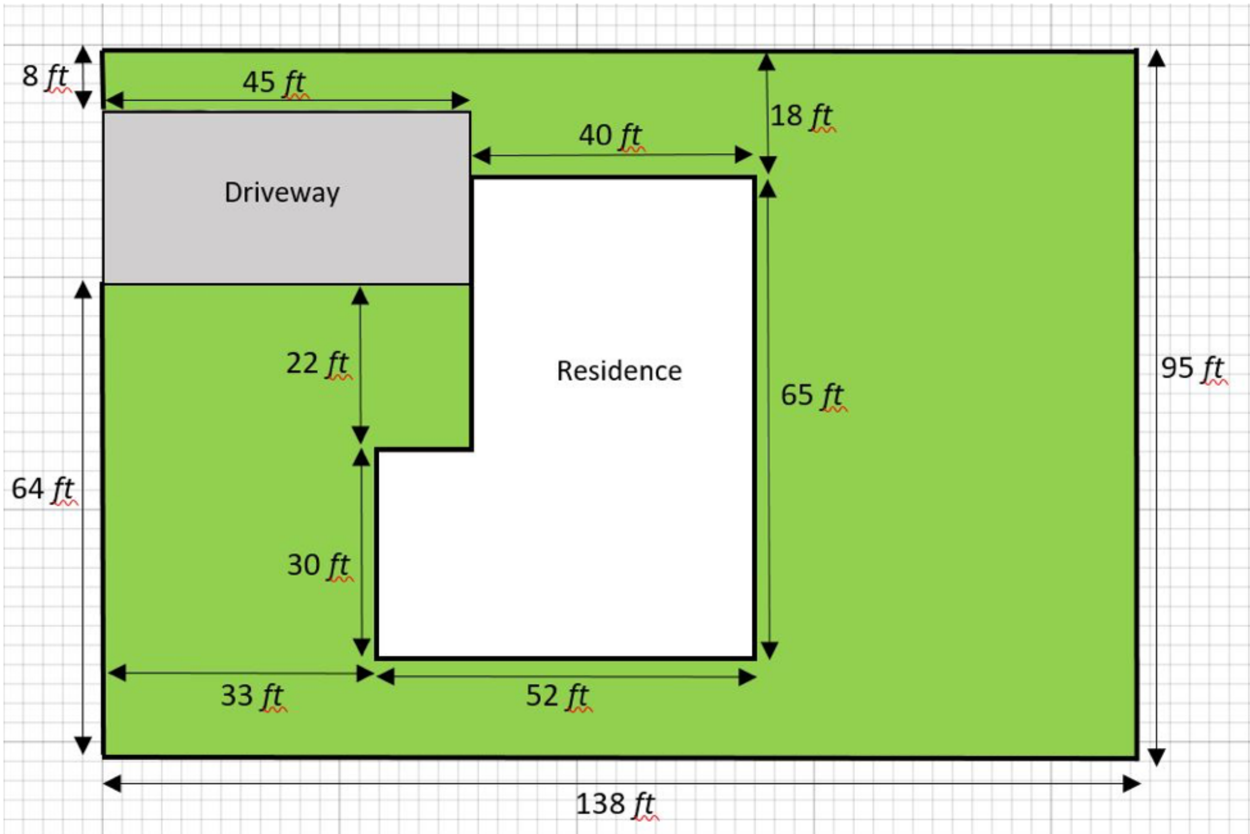


Figure 2.1: Schematic of a plot of land.

Exercise 27

Your neighbor's lawn isn't looking too good. He has decided to remove all the old sod (grass), bring in a new 4 inch layer of topsoil, install new in-ground sprinklers, and reseed the lawn. He seems to think that he'll be able to save money by hauling loads of topsoil from the store himself in his pickup truck, rather than paying for delivery, but you're not sure he's right. You're going to help him.

1. Using Figure 2.1, find the area of the yard via the following steps:

   (a) Find the area of the entire lot.

   (b) Find the area of the residence.

   (c) Find the area of the driveway.

   (d) Find the area of the yard.

2. We are supposed to put a 4 inch layer of topsoil over the area of the yard.

   (a) Find the required topsoil in cubic feet.

   (b) Convert the required volume of topsoil to cubic yards.

   (c) Find the cost of the topsoil. ($18 per cubic yard sold in $\frac{1}{4}$ cubic yard increments)

3. Suppose we pay $30 per truckload on top of the soil cost. Each truckload can deliver up to 18 cubic yards.

   (a) How many truckloads are required?

   (b) How much will the delivery cost?

4. Suppose we use the pickup truck. The bed is 80 inches long, 69 inches wide, and 20 inches tall.

   (a) Calculate the volume of the pickup in cubic inches.

   (b) Convert the volume of the pickup to cubic yards.

   (c) How many trips to the store will we need to make?

   (d) The store is 9 miles away and it takes 20 minutes to drive there. The truck gets 17 miles to the gallon and gasoline costs $3.79 per gallon. How much will we pay to get the soil ourselves?

   (e) How much time will we spend driving?

Chapter 3

Voting Theory

In many decision making situations, it is necessary to gather the group consensus. While the basic idea of voting is fairly universal, many methods for determining a winner have been created.

In deciding upon a winner, there is always one main goal: to reflect the preferences of the people in the most fair way possible. However, as we will see, there is no single criterion for determining whether a voting method is fair. In fact, there are several different criteria which can be used to determine whether a voting method is fair or not.

In this chapter we will look at several voting methods and analyze each method's strengths and weaknesses.

3.1 Plurality Voting Method

In the United States and in many other countries throughout the world, most elections are decided by which ever candidate receives the most votes. Notice that it is *not* necessary to receive a *majority* (more than half) of the votes, just more votes than any other candidate.

> **Definition.**
> The **plurality** voting method chooses the winner based on the candidate which receives more votes than any other candidate. This method is also known as **first-past-the-post** in an analogy to horse racing.

Exercise 28

If Alex, Betty and Chuck are running for the office of club president, then what is the minimum number of votes that one could get to win the election under the plurality method if there are

1. 73 votes cast?

2. 74 votes cast?

When there are only two candidates or choices, plurality voting works extremely well, but as soon as there are three or more candidates or choices, problems can arise.

The first problem is that under plurality voting, the winner may not receive a majority of the votes. This seems unfair to many voters. The second problem is that plurality voting tends to discourage the development of third parties and reward the two major parties. This is because plurality voting gives only the winner in each district a seat, a party that consistently comes in second or third in many or most districts will not gain any seats in the legislature, even if it receives a substantial minority of the vote.

Definition.
Tactical voting, or **insincere voting** occurs in elections with more than two candidates, when a voter supports another candidate more strongly than their sincere preference in order to prevent an undesirable outcome.

Because of tactical or insincere voting, voters sometimes feel like they must choose between the lesser of two evils. This can lead to a general lack of enthusiasm for voting.

Lack of enthusiasm is not the only problem that plurality voting tends to create. Plurality voting sometimes leads to polarization of the electorate into supporting just two parties.

Consider an election in which 100,000 moderate voters and 80,000 radical voters are to vote for candidates for a single seat or office. If two moderate parties ran candidates and one radical candidate ran (and every voter voted), the radical candidate would tend to win unless one of the moderate candidates gathered fewer than 20,000 votes. Appreciating this risk, moderate voters would be inclined to vote for the moderate candidate they deemed likely to gain more votes, with the goal of defeating the radical candidate. To win, then, either the two moderate parties must merge, or one moderate party must fail, as the voters gravitate to the two strongest parties. If this scenario happens enough times it can push voters to only support the two strongest parties and third parties are often (but not always) shut out of the electoral process.

Plurality voting is a rather blunt tool for determining the will of the people when there are several candidates. This is because each voter only gives one piece of information at the polls, their favorite candidate, and due to insincere voting, that information may not even be accurate. What if we gathered more information from each voter about their preferences? Perhaps the extra information could be used to generate fairer outcomes. This is the general idea behind **ranked choice voting** methods. Many of the voting methods that have been created use this idea to try to make elections fairer.

3.2 Instant Runoff Voting

What if we required the eventual winner to receive a *majority* in order to win? This may not be possible if there are more than two candidates. A way around this problem is to do **runoff voting** where the election is held in stages. After each vote, if a single candidate has a majority of the votes then that candidate wins, but if not, then the candidate with the least votes is eliminated and everyone votes again.

Runoff voting is time consuming and inefficient. One way around the inefficiency of runoff voting is to modify the ballot to allow voters to rank the candidates by order of preference.

Definition.
A **preference ballot** is a ballot in which the voter ranks the choices in order of preference. Once the election has been held we tally the votes and create a tabular summary of voter preferences. We call this tabular summary of the ballots a **preference schedule**.

Exercise 29

Below is an example of a preference schedule for candidates A,B,C and D. Notice that there were a total of $8 + 17 + 4 + 11 = 40$ votes cast. This means that for a candidate to get a *majority* of the votes they must get at least 21 votes because 21 is the smallest whole number that is more than half of 40.

Num. Ballots	8	17	4	11
1st	A	D	B	C
2nd	B	A	A	B
3rd	C	C	D	A
4th	D	B	C	D

Table 3.1: Example preference schedule

1. Who wins under the plurality method?

2. Who is the winner under instant-runoff voting?

Exercise 30

Suppose there is an election with four candidates. Two are from the two dominant political parties: Democrat and Republican. The other two are from third parties: Libertarian and Socialist. A preference schedule is given below.

Num. Ballots	8	18	20	4
1st	L	R	D	S
2nd	R	L	S	D
3rd	D	S	L	R
4th	S	D	R	L

Answer the following questions using the data in the preference schedule above.

1. How many people voted?

2. How many votes are required for a majority?

3. What is the smallest possible number of votes a candidate could win with under the plurality method of choosing?

4. Who wins under instant runoff voting?

3.3 Borda Count

Borda Count is another voting method that uses a preference ballot. It is named after Jean-Charles de Borda, who developed the system in 1770.

Definition.
In the **Borda Count** method, points are assigned to candidates based on their ranking; 1 point for last choice, 2 points for second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Exercise 31
Suppose an election resulted in the following preference schedule. Which of the three candidates: A, B, or C would win under the Borda count method of choosing?

Num. Ballots	13	4	14
1st	B	A	C
2nd	A	B	A
3rd	C	C	B

Exercise 32
Which candidate would win under instant runoff voting?

3.4 Copeland's Method (Pairwise Comparison)

Sometimes a candidate can be preferred by voters when compared to every other candidate, but still not win the election. This seems unfair to most people.

Definition.
If one candidate is preferable to all other candidates in head-to-head match ups, then that candidate is called the **Condorcet candidate**.

Note.
Not every election has a Condorcet candidate.

Exercise 33
The preference schedule from exercise 31 is reproduced below.

Num. Ballots	13	4	14
1st	B	A	C
2nd	A	B	A
3rd	C	C	B

1. Determine if this election had a Condorcet candidate.

2. In exercise 32 we determined the winner of this election under instant-runoff-voting. Is that winner and the Condorcet candidate the same person? Do you think this is fair?

Definition.
In **Copeland's method** we award points to each candidate based upon how that candidate does in pairwise comparisons with every other candidate. The candidate with the most points wins.

1 point for pairwise win
$\frac{1}{2}$ point for pairwise tie
0 points for pairwise loss

Exercise 34
Use Copeland's method to determine the winner of the following preference schedule.

Num. Ballots	13	15	9	11
1st	C	A	D	B
2nd	D	B	B	C
3rd	A	C	A	D
4th	B	D	C	A

3.5 Fairness Criteria

Definition.
A **fairness criterion** is a conditional statement (if—then) about a selection or voting method that seems like it should be satisfied in order for the method to be fair.

Majority Criterion
If there is a candidate that has a majority (more than half) of first-place votes, then that candidate should win.

Condorcet Criterion
If there is a candidate that wins every head-to-head comparison, then that candidate should win.

Monotonicity Criterion
If a candidate wins, and only changes that favor the winner are made to the preference ballots, then that candidate should still win.

Independence of Irrelevant Alternatives (IIA) Criterion
If a non-winning candidate is removed from the election, then that should not change the winner.

	Fairness Criteria			
Voting Method	Majority	Condorcet	Monotonicity	IIA
Plurality	✓	✗	✓	✗
Instant Runoff	✓	✗	✗	✗
Borda Count	✗	✗	✓	✗
Copeland	✓	✓	✓	✗

Table 3.2: Voting Methods and Fairness Criteria
✓ = always satisfied
✗ = may violate

Example 6 (IRV Can Violate the Monotonicity Criterion)
Consider the preference schedule below.

Num. Ballots	37	22	12	29
1st	Adams	Brown	Brown	Carter
2nd	Brown	Carter	Adams	Adams
3rd	Carter	Adams	Carter	Brown

Under instant runoff voting, Carter would be eliminated in the first round, and Adams would be the winner with 66 votes to 34 votes for Brown.

Now suppose that the results were announced, but election officials accidentally destroyed the ballots before they could be certified, and the votes had to be recast. Wanting to "jump on the bandwagon", 10 of the voters who originally voted in the order Brown, Adams, Carter change their vote to favor the presumed winner, changing those votes to Adams, Brown, Carter.

Num. Ballots	47	22	2	29
1st	Adams	Brown	Brown	Carter
2nd	Brown	Carter	Adams	Adams
3rd	Carter	Adams	Carter	Brown

In this re-vote, Brown will be eliminated in the first round, having the fewest first-place votes. After transferring votes, we find that Carter will win this election with 51 votes to Adams' 49 votes! Even though the only vote changes made favored Adams, the change ended up costing Adams the election. This doesn't seem fair. If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning.

Example 7 (Copeland Can Violate the IIA Criterion)

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The preference schedule is shown below:

Num. Ballots	5	5	6	4
1st	D	A	C	B
2nd	A	C	B	D
3rd	C	B	D	A
4th	B	D	A	C

Computation of Copeland point totals is as follows:

Pair	Tallies	A	B	C	D
A vs. B	10 vs. 10	0.5	0.5		
A vs. C	14 vs. 6	1			
A vs. D	5 vs. 15				1
B vs. C	4 vs. 16			1	
B vs. D	15 vs. 5		1		
C vs. D	11 vs. 9			1	
Totals		1.5	1.5	2	1

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:

Num. Ballots	5	5	6	4
1st	A	A	C	B
2nd	C	C	B	A
3rd	B	B	A	C

Re-computation of Copeland point totals is as follows:

Pair	Tallies	A	B	C
A vs. B	10 vs. 10	0.5	0.5	
A vs. C	14 vs. 6	1		
B vs. C	4 vs. 16			1
Totals		1.5	0.5	1

Suddenly Anna is the winner!

3.6 Arrow's Impossibility Theorem

Economist Kenneth Arrow was able to prove in 1949 that when an election has three or more candidates, no ranked choice voting method can simultaneously satisfy all fairness criteria. He won the 1972 Nobel Prize in Economics for this work.

Theorem (Arrow's Impossibility Theorem).
No voting method can satisfy all fairness criteria.

3.6.1 Approval Voting

Definition.
Approval voting is where you mark all candidates you approve of without stating preference, i.e. without ranking the candidates.

Exercise 35
Determine the winner under approval voting.

	Number of Ballots					
Choice	10	21	17	13	18	16
A				✓	✓	✓
B	✓	✓	✓			
C		✓		✓		✓
D	✓	✓			✓	

Num. Ballots	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		✓	✓			✓		✓	✓	✓
Scream	✓		✓	✓		✓	✓		✓	
The Matrix	✓	✓	✓	✓	✓		✓			✓

Table 3.3: Using approval voting to choose a movie to watch.

Exercise 36

Use Table 3.3 and approval voting to determine the movie to watch.

This is an easier method of voting because it does not force the voter to rank choices. The negatives to approval voting are twofold. First, it can easily violate the *majority* fairness criterion. Second, approval voting is susceptible to strategic insincere voting, in whicha voter does not voter their true preference to try to increase the chances of their choice winning. For example, suppose Bob and Alice would much rather watch "Scream". Even though they may be okay with watching "The Matrix", if they remove it from their approval list, its chances of winning decrease and thus their chance to watch "Scream" increases.

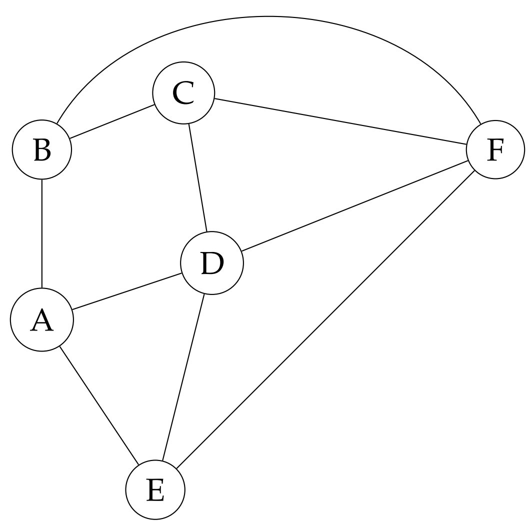
Chapter 4

Graph Theory

4.1 Vocabulary

Definition.
A **vertex** is a dot or circle, often labeled with a letter or short name. An **edge** is a line or arc which connects two **vertices**. A **loop** is a special kind of edge that connects a vertex to itself.

Definition.
A **graph** is a pair of sets, a set of **vertices**, and a set of **edges**.



$$G = (\{A, B, C, D, E, F\}, \{AB, AE, AD, BC, BF, CD, CF, DF, DE, EF\})$$

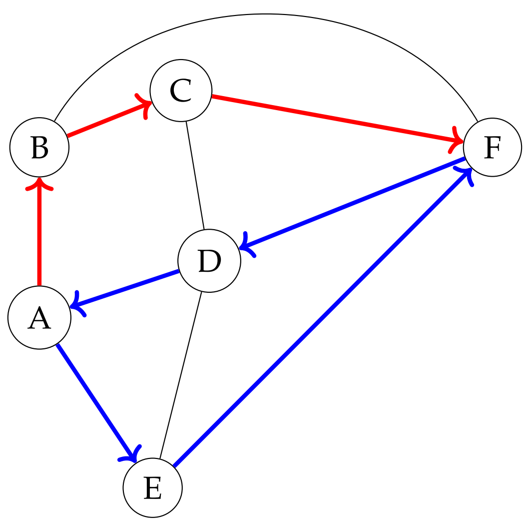
Figure 4.1: Example graph

Definition.
The **degree** of a vertex is the number of edges meeting at that vertex.

Example 8
For the graph in Figure 4.1 and Figure 4.2 below, the vertices have the following degrees:

$$\deg(A) = 3 \qquad \deg(D) = 4$$
$$\deg(B) = 3 \qquad \deg(E) = 3$$
$$\deg(C) = 3 \qquad \deg(F) = 4$$

Definition.
A **path** is a sequence of edges.



$P = AB, BC, CF$

Figure 4.2: Example path in red and example circuit in blue

Note.
For brevity we sometimes denote a path as a sequence of vertices. That is, we will often denote the path in Figure 4.2 as $P = A, B, C, F$

Definition.
A **circuit** is a path that begins and ends at the same vertex.

Definition.
A graph is **connected** if there is a path from any vertex to any other vertex.

Definition.
A **weighted graph** is a graph where each edge has a numerical value associated with it. The weights often correspond with distances, travel time, or travel cost.

Definition.
A **graph algorithm** is like a function, it takes a graph as input and maybe a starting point and outputs either a new graph, or a path or a circuit. An algorithm is essentially a sequence of steps or operations to perform on a graph to obtain a desired result.

Some graph algorithms are **efficient** meaning that they are guaranteed to finish in a short amount of time. Normal execution times could be anywhere from a few milliseconds to a few hours. Some graph algorithms are **optimal** meaning they are guaranteed to find the best possible solution.

4.2 Dijkstra's Algorithm (Shortest Path)

One common usage for graphs is pathfinding. In this case the vertices represent locations, e.g. cities, and the edges represent roads, rail lines, or perhaps connecting flights. Most often these graphs will be weighted where the weights represent distance, travel time, or travel cost.

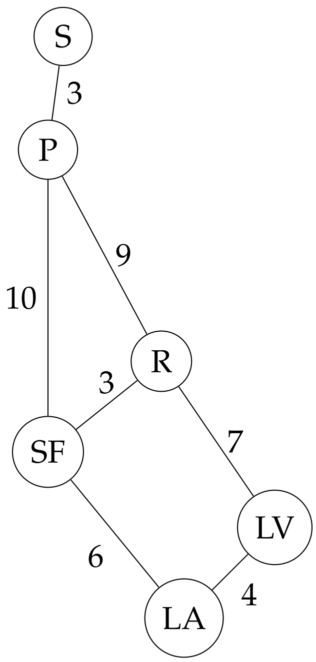
Algorithm (Dijkstra's Algorithm).

1. Mark the ending vertex with a distance of zero. Designate this vertex as current.
2. Find all vertices leading to the current vertex. Calculate their distances to the end. Since we already know the distance the current vertex is from the end, this will just require adding the most recent edge. Don't record this distance if it is longer than a previously recorded distance.
3. Mark the current vertex as visited. We will never look at this vertex again.
4. Mark the vertex with the smallest distance as current, and repeat from step 2.

Note.
Dijkstra's algorithm is both *efficient* and *optimal*. It is so efficient (fast) that it is used by GPS navigation systems to find optimal routes between two points in real time, i.e. while you're driving.

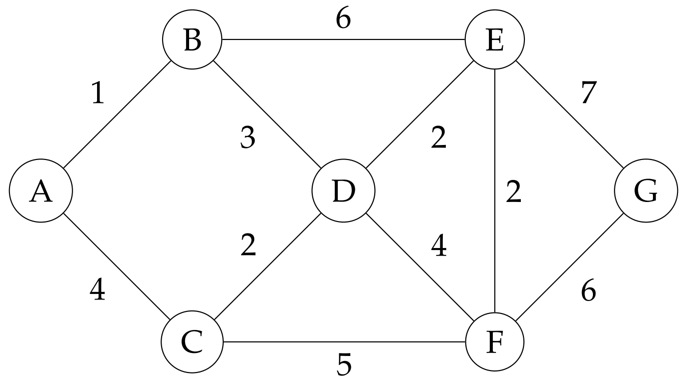
Exercise 37

Use Dijkstra's algorithm to determine the shortest drive time from Seattle to Las Vegas. What is the total drive time? (The graph below is weighted by drive time in hours.)



Exercise 38

Use Dijkstra's algorithm to determine the cheapest path from A to G.



4.3 Hamiltonian Circuits and the Traveling Salesman Problem

Definition.
A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

4.3.1 Brute Force

Algorithm (Brute Force).

1. Create a list of all possible circuits and their costs.
2. Choose the circuit with the least cost.

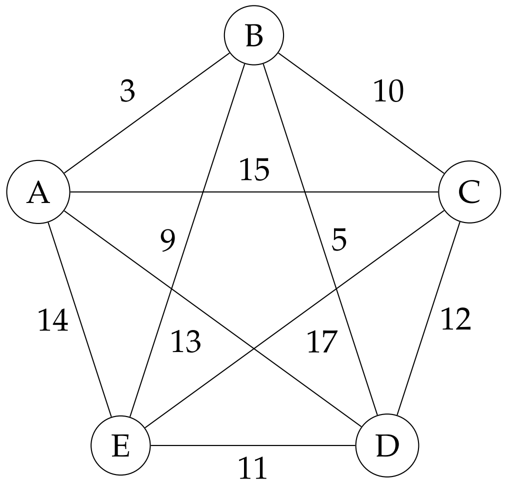
4.3.2 Nearest Neighbor Algorithm (NNA)

Algorithm (Nearest Neighbor Algorithm).

1. Select a starting point.
2. Move to the nearest unvisited vertex (via the edge with smallest weight).
3. Repeat until the circuit is complete.

Exercise 39

Use the graph below to find a Hamiltonian path via NNA beginning at vertex B.



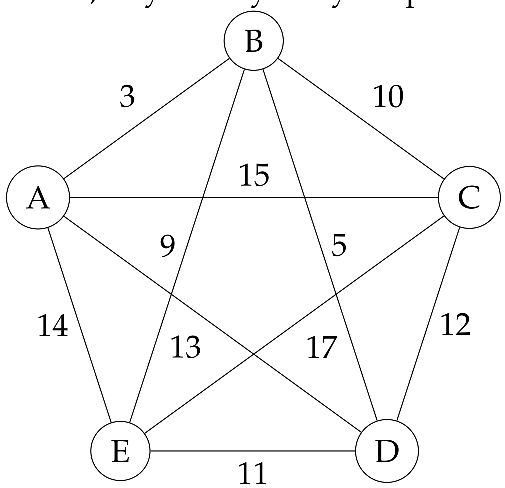
4.3.3 Repeated Nearest Neighbor Algorithm (RNNA)

Algorithm (Repeated Nearest Neighbor Algorithm).

1. Do the Nearest Neighbor Algorithm starting at each vertex.
2. Choose the circuit produced with minimal total weight.

Exercise 40

Use the graph to find a Hamiltonian path via RNNA. This is the same graph as used in Exercise 39, so you may use your previous result where we did NNA starting at vertex B.



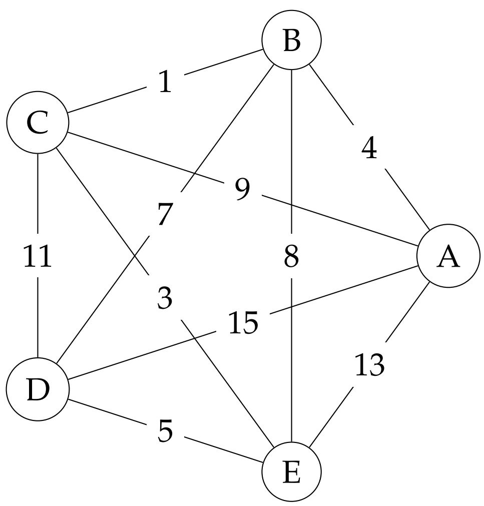
4.3.4 Sorted Edges Algorithm (SEA)

Algorithm (Sorted Edges Algorithm).

1. Select the cheapest unused edge in the graph.
2. Repeat step 1, adding the cheapest unused edge to the circuit, unless:
   (a) adding the edge would create a circuit that doesn't contain all vertices, or
   (b) adding the edge would give a vertex degree 3.
3. Repeat until a circuit containing all vertices is formed.

Exercise 41

Use the graph to find a solution to the traveling salesman problem via the Sorted Edges Algorithm.



4.4 Eulerian Circuits and the Chinese Postman Problem

The picture on the cover of these lecture notes is of the city of K onigsberg which is now called Kaliningrad. At the time that the mathematician Leonhard Euler (pronounced like "oiler") lived there there was a famous question that puzzled the residents. Is it possible to start in one location and take a walk such that you cross each of the seven bridges near the center of town just once and end up back where you started?

4.4.1 Fleury's Algorithm

Algorithm (Chinese Postman — Fleury).

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit, and delete it from the graph.
4. Continue until you're done.

4.5 Spanning Trees and Kruskal's Algorithm

Definition.

- A **tree** is an undirected graph (edges are not arrows) in which any two vertices are connected by exactly one path.
- A **spanning tree** is an undirected subgraph of a graph G that is a tree which includes all the vertices of G.
- A **minimum cost spanning tree** is the spanning tree with the smallest total edge weight.

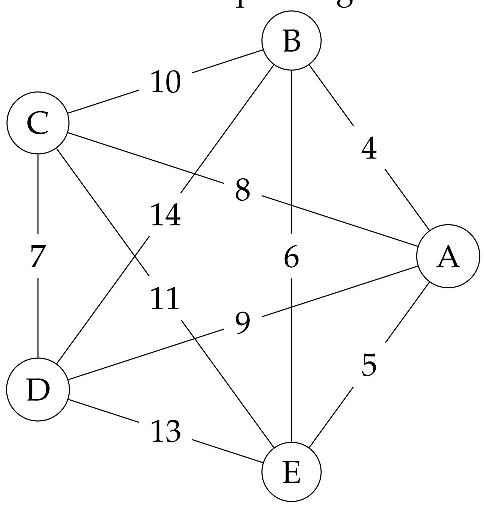
Algorithm (Kruskal's Algorithm).

1. Select the cheapest unused edge in the graph.
2. Repeat step 1, unless
   - adding the edge would create a circuit, or
   - until a spanning tree is formed.

Note.
Similar to Dijkstra's algorithm, Kruskal's algorithm is both *optimal* (it finds the minimum cost spanning tree) and *efficient* (it terminates in a reasonable amount of time).

Exercise 42

A company requires reliable internet and phone connectivity between their five offices (named A, B, C, D and E for simplicity) in New York City. They decide to lease dedicated lines from the phone company. The phone company will charge for each link made. The costs in thousands of dollars per year are shown in the graph. Use Kruskal's algorithm to find a minimal cost spanning tree for the five offices.



Chapter 5

Growth Models

Definition.
A **recursive** function is a function defined in terms of its previous value(s).

Note.
Recursive functions are *easy to define*, but *tedious to evaluate* because you often have to compute several previous values. For this reason, they are often used in computer programs, but not when doing computation by hand.

Example 9 (Fibonacci Sequence)
The famous Fibonacci function, F, is defined *recursively* as follows:

$$F(x) = F(x-2) + F(x-1).$$

Defining $F(0) = 0$ and $F(1) = 1$ yields the following sequence of outputs:

$$\begin{aligned}
&F(0) = 0 \\
&F(1) = 1 \\
&F(2) = F(0) + F(1) = 0 + 1 = 1 \\
&F(3) = F(1) + F(2) = 1 + 1 = 2 \\
&F(4) = F(2) + F(3) = 1 + 2 = 3 \\
&F(5) = F(3) + F(4) = 2 + 3 = 5 \\
&F(6) = F(4) + F(5) = 3 + 5 = 8 \\
&F(7) = F(5) + F(6) = 5 + 8 = 13 \\
&\qquad\vdots
\end{aligned}$$

Each number in the sequence, is the sum of the previous two numbers: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34 \ldots$

5.1 Linear Growth

5.1.1 Recursive Linear Growth Model

Let $P =$ population, $x =$ time in years, then we predict next years population, $P(x+1)$, will be this year's population, $P(x)$, plus a fixed amount, d.

$$\boxed{P(x+1) = P(x) + d} \tag{5.1}$$

Thus **linear growth** (or decay) is characterized by the *difference* in population from two *consecutive* years equaling the constant: d.

$$\boxed{P(x+1) - P(x) = d}$$

5.1.2 Explicit Linear Growth Model

Our goal in this section is to show that the recursive definition of linear growth is equivalent to familiar equational model of lines: $y = mx + b$ where m is the slope and b gives the y-intercept of the graph, see Figure 5.1.

Let $P_0 \equiv P(0)$, and recall the recursive definition: $P(x+1) = P(x) + d$:

$$\begin{aligned}
P(1) &= P(0) + d = P_0 + d & P(1) &= P_0 + d \\
P(2) &= P(1) + d = (P_0 + d) + d & P(2) &= P_0 + 2d \\
P(3) &= P(2) + d = (P_0 + 2d) + d & P(3) &= P_0 + 3d \\
P(4) &= P(3) + d = (P_0 + 3d) + d & P(4) &= P_0 + 4d \\
&\vdots \\
P(x) &= P_0 + xd
\end{aligned}$$

Or equivalently,

$$\boxed{P(x) = dx + P_0}$$

where d is the slope or growth rate, and P_0 is the y–intercept, or initial population.

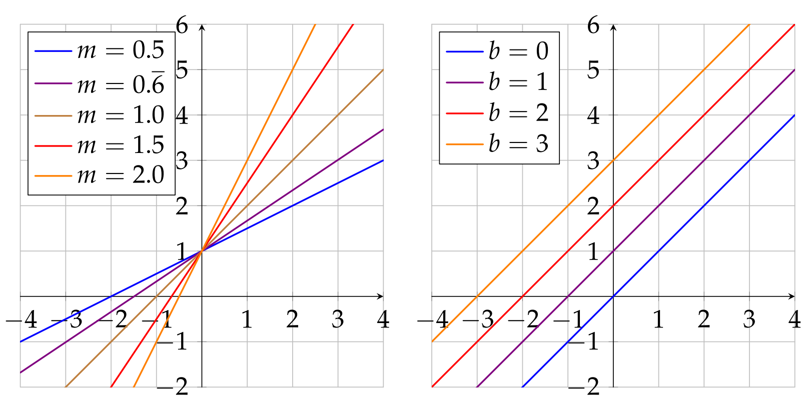


Figure 5.1: Graphs of exponential growth, $y = mx + b$

5.2 Exponential Growth

5.2.1 Recursive Exponential Growth Model

Again, let P = population, x = time measured in years, then we predict the population next year, $P(x+1)$, to be the population this year, $P(x)$, plus a fixed percentage, r, of this year's population:

$$P(x+1) = P(x) + rP(x) \qquad \Rightarrow \qquad \boxed{P(x+1) = P(x)(1+r)} \tag{5.2}$$

Thus **exponential growth** (or decay) is characterized by the *ratio* of populations from *consecutive* years, equaling the constant: $1+r$.

$$\boxed{\frac{P(x+1)}{P(x)} = 1+r}$$

5.2.2 Explicit Exponential Growth Model

Let $P_0 \equiv P(0)$, and recall the recursive definition: $P(x+1) = P(x)(1+r)$

$$\begin{aligned}
P(1) &= P_0(1+r) & P(1) &= P_0(1+r)^1\\
P(2) &= P(1)(1+r) = P_0(1+r)(1+r) & P(2) &= P_0(1+r)^2\\
P(3) &= P(2)(1+r) = P_0(1+r)^2(1+r) & P(3) &= P_0(1+r)^3\\
P(4) &= P(3)(1+r) = P_0(1+r)^3(1+r) & P(4) &= P_0(1+r)^4\\
&\vdots
\end{aligned}$$

$$\boxed{P(x) = P_0(1+r)^x}$$

where r is the growth rate, and P_0 is the y–intercept, or initial population.

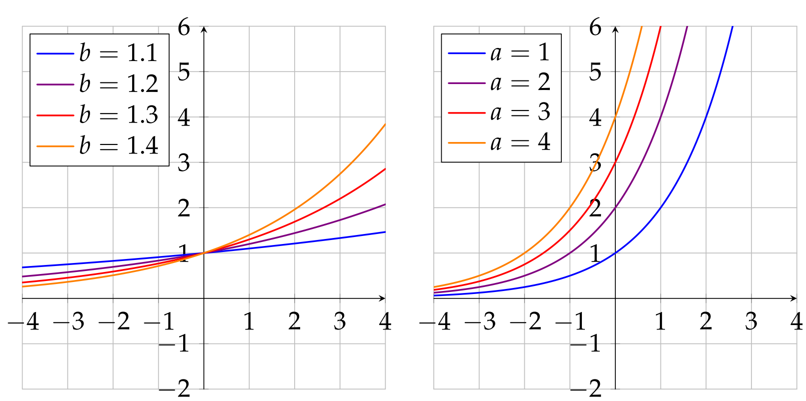


Figure 5.2: Graphs of exponential growth, $y = ab^x$

5.2.3 Summary of Growth Models

	Linear	Exponential
Recursive Form	$P(x+1) = P(x) + d$	$P(x+1) = P(x)(1+r)$
Constant	$P(x+1) - P(x) = d$	$\dfrac{P(x+1)}{P(x)} = 1 + r$
Explicit Form	$P(x) = dx + P_0$	$P(x) = P_0(1+r)^x$
Alternate Form	$y = mx + b$	$y = ab^x$

Exercise 43

Suppose your company is building a solar plant. The plant had 112 solar panels in 2007 and had 884 in 2018. Assuming the growth is *linear*, find an explicit formula for the growth of solar panels.

Exercise 44

Suppose you bought a house for $193,000 and it is worth $299,000 in 2018.

1. Find the rate of growth and common ratio of you bought the house in 1998.

2. Find the rate of growth and common ratio of you bought the house in 2008.

5.3 Logarithms

A **logarithm** (with base b) is the inverse of exponentiation (with base b), symbolically:

$$y = b^x \quad \Leftrightarrow \quad x = \log_b(y).$$

Example 10

The following two equations (statements) are equivalent:

$$8 = 2^3 \quad \Leftrightarrow \quad 3 = \log_2(8).$$

What this means is that a logarithm of a value is an exponent, or we understand logarithms in terms of their equivalent exponential. Thus the expression, $\log_2(32)$, can be turned into a question by writing it as an equation with an unknown, x:

$$\log_2(32) \qquad \longrightarrow \qquad \log_2(32) = x.$$

This equation is equivalent to asking, "Which exponent of 2 equals 32?".

$$\log_2(32) = x \quad \Leftrightarrow \quad 32 = 2^x$$

and of course the answer is 5, thus $\log_2(32) = 5$.

Another way of stating the above equivalence is to think in terms of function composition. Recall that when a function, $f(x)$, and its inverse, $f^{-1}(x)$ are composed via function composition, i.e. $\circ$, then the composition is the *identity* function which just maps the input to the output. Symbolically, this means:

$$\begin{aligned} \log_b(x) \circ b^x &= \mathrm{id}(x) & \qquad \log_b(b^x) &= x \\ b^x \circ \log_b(x) &= \mathrm{id}(x) & \qquad b^{\log_b(x)} &= x. \end{aligned}$$

5.3.1 Properties of Logarithms

There are a few useful properties of the logarithm function we should review:

$$\log(a \cdot b) = \log(a) + \log(b) \tag{5.3}$$

$$\log\left(\frac{a}{b}\right) = \log(a) - \log(b) \tag{5.4}$$

$$\log\left(a^n\right) = n\log(a) \tag{5.5}$$

The easiest way to remember equations (5.3),(5.4) and (5.5) is to remember the following phrases:

> "The log of a product is the sum of the logs."
> "The log of a quotient is the difference of the logs."
> "The log of an exponential is the exponent times the log"

We will use the last property frequently when solving problems involving exponential growth and decay.

Exercise 45

Rewrite the expression $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.

One final useful logarithm formula is the **change of base** formula which allows us to compute logarithms in any given base, b, via either *common logarithms*, i.e. base 10, or *natural logarithms* (base $e \approx 2.718$).

$$\boxed{\log_b(a) = \frac{\log(a)}{\log(b)} = \frac{\ln(a)}{\ln(b)}} \tag{5.6}$$

Example 11

Suppose you need to know what power of 5 equals 100, i.e. $5^x = 100$. The correct value for x must be between 2 and three because:

$$5^2 = 25 < 100 < 125 = 5^3$$

We can solve this problem exactly by translating the exponential equation into the equivalent logarithmic equation,

$$5^x = 100 \quad \Leftrightarrow \quad x = \log_5(100)$$

and then using the change of base formula:

$$\log_5(100) = \frac{\log(100)}{\log(5)} = \frac{\ln(100)}{\ln(5)} \approx 2.8614 \quad \Leftrightarrow \quad 5^{2.8614} \approx 100.$$

Exercise 46

Suppose a Pacific island was discovered by Polynesians around 400 C.E. by 60 people. Assume the the population grows at a constant rate of 1% per year.

1. Find an explicit formula to model this population.

2. How long will it take for this population to reach 300,000 people? What year is this?

3. How long will it take for this population to reach 3 million people? What year?

Exercise 47

Carbon-14 is a radioactive isotope of carbon with a half-life of about 5730 years. What is the yearly decay rate as a percentage?

Exercise 48

If you start with a 5 micro gram sample of pure carbon-14, how much of it will still be carbon-14 after 100 years?

5.4 Logistic Growth

Exponential growth has a serious problem because it predicts "runaway" growth after a certain amount of time. For example, if we model world population via the exponential growth model, it predicts that the earth will eventually reach populations of 100 billion, a trillion and even much higher levels. Now, no one knows exactly how many humans could actually live on Earth and with new technologies that number might substantially increase, but unbounded growth cannot last forever. At some point, people would be standing on top of other people, not to mention valuable resources such as food and water would eventually run out.

The fundamental assumption of the exponential growth model is that the growth rate, r, is constant. However, what social scientists have learned about populations is that they do not grow forever in unbounded fashion. Eventually, a population's growth rate will tend to decrease as competition for valuable resources such as food and water increases. The **logistic growth** model takes this variable growth rate into account by assuming that the growth rate decreases linearly with population size. That is,

$$r_L = r\left(1 - \frac{P}{K}\right)$$

$$\begin{aligned} r &= \text{ base growth rate} \\ P &= \text{ population} \\ K &= \text{ carrying capacity} \end{aligned}$$

If we replace r in the recursive form of the exponential growth model with r_L, then we get:

$$P(x+1) = P(x) + r\left(1 - \frac{P}{K}\right)P(x). \tag{5.7}$$

The above equation is the recursive **logistic growth** model

Exercise 49

Suppose some tiny tropical fish are introduced into a large aquarium and the population grows at 7% annually. A few years later, the caretaker notices that the population is not growing as fast as it did before.

In what follows, assume the tank can sustain a maximum population of 600 fish.

1. If there are currently 400 fish, use the logistic model to predict next year's population.

2. What is the current rate of growth?

3. What do you expect to happen to the growth rate as the population grows?

Exercise 50

Suppose the world's human population has a base rate of growth of roughly 2.8% and the earth has a carrying capacity of around 12 billion humans.

1. Estimate the rate of growth in 1975 when the world population was 4 billion.

2. Estimate the rate of growth in 1987 when the world population was 5 billion.

3. Estimate the rate of growth in 2011 when the world population reached 7 billion.

Chapter 6

Finance

6.1 Simple Interest

Definition.
Simple interest refers to a one-time interest payment. Simple interest is used for short-term loans.

$$
\begin{aligned}
I &= rP_0 \\
A &= P_0 + I \\
A &= P_0 + rP_0 \\
&\boxed{A = P_0(1+r)}
\end{aligned} \tag{6.1}
$$

where,

$$
\begin{aligned}
I &= \text{ interest} \\
A &= \text{ account balance (amount in the account)} \\
P_0 &= \text{ principal (starting amount)} \\
r &= \text{ interest rate (in decimal)}
\end{aligned}
$$

Exercise 51
A friend agrees to loan you \$500 to help you buy a new laptop. You agree to pay them back in 30 days with 3% interest. How much will you owe your friend in 30 days?

6.2 Simple Interest Over Time

Definition.
Simple interest over time refers to a fixed interest payment that accrues once each year. Simple interest is used for bonds. A **bond** is a loan that you make to the government. All levels of government issue bonds; municipal, state and federal.

When you purchase a bond, the government agrees to pay you a fixed interest payment each year until the bond *matures* whereupon they return the initial amount they borrowed from you. A bond matures after a fixed number of years, often 5, 10, 20 or 30 years.

$$
\begin{aligned}
I &= rP_0 n \\
A_n &= P_0 + I \\
A_n &= P_0 + rP_0 n \\
\Aboxed{A_n &= P_0(1 + rn)} \qquad (6.2)
\end{aligned}
$$

where,

$$
\begin{aligned}
I &= \text{ interest} \\
A_n &= \text{ account balance (amount in the account)} \\
P_0 &= \text{ principal (starting amount)} \\
r &= \text{ interest rate (in decimal)} \\
n &= \text{ time (in years)}
\end{aligned}
$$

Note.
Simple interest over time corresponds with *linear growth*.

$$
\begin{aligned}
f(x) &= mx + b, \\
A_n &= (rP_0)n + P_0,
\end{aligned}
$$

where rP_0 is the slope (or fixed amount of yearly increase), and P_0 is the y-intercept.

Exercise 52

Suppose you buy a municipal bond from your local city government to help fund construction projects at the local zoo. The bond costs \$2000, pays 2.5% interest annually and matures in 10 years. How much interest will you earn?

6.3 Compound Interest

Definition.
Compound interest refers to any situation where prior interest payments also accrue interest.

$$\boxed{A_n = P_0\left(1 + \frac{r}{k}\right)^{nk}} \tag{6.3}$$

where,

$$\begin{aligned} A_n &= \text{ account balance after } n \text{ years.} \\ P_0 &= \text{ principal (starting amount)} \\ r &= \text{ annual interest rate (in decimal)} \\ k &= \text{ the number of compounding periods per year} \end{aligned}$$

Note.
If $k = 1$, i.e. there is only one compounding per year then the formula becomes

$$\boxed{A_n = P_0(1 + r)^n,}$$

which is the *exponential population model*.

Exercise 53

Suppose you deposit \$1000 into a bank account which earns 3% interest compounded monthly. How much money will you have after one year?

Answer: \$1,030.42

Exercise 54

A certificate of deposit (CD) is a savings instrument that many banks offer. It usually gives a higher interest rate, but you cannot access your investment for a specified length of time. Suppose you deposit \$3000 in a CD paying 6% interest, compounded monthly. How much will you have in the account after 20 years?

Answer: \$9,930.61

Exercise 55

Again suppose you purchase a CD that pays 6% interest compounded monthly. How long will it take for your initial deposit (the principal, P_0) to double?

Hint: You don't need to know the principal, just use the symbol P_0.

6.4 Saving Annuities

Definition.
A **savings annuity** is an account into which regular payments are made for a fixed number of years. When the annuity matures you receive the payments and the interest that each payment accrued.

$$\boxed{A_n = \frac{d\left[\left(1+\frac{r}{k}\right)^{nk} - 1\right]}{\left(\frac{r}{k}\right)}} \tag{6.4}$$

where,

$$\begin{aligned} A_n &= \text{account balance after } n \text{ years.} \\ d &= \text{regular deposit} \\ r &= \text{annual interest rate (in decimal)} \\ k &= \text{the number of compounding periods per year} \end{aligned}$$

Exercise 56

At the age of 30, Susan, opens a savings annuity account that pays 6% interest compounded monthly. She agrees to make a deposit of \$200 every month for 35 years. How much will the annuity be worth after 35 years when she is 65?

Answer: \$284,000.06

When retirement planning, the object is to determine how much you need to save each month now in order to have a guaranteed fixed income during your retirement years. This requires solving the savings annuity formula for d:

$$\begin{aligned}
A_n &= \frac{d\left[\left(1+\frac{r}{k}\right)^{nk}-1\right]}{\left(\frac{r}{k}\right)} \\
A_n\left(\frac{r}{k}\right) &= d\left[\left(1+\frac{r}{k}\right)^{nk}-1\right] \\
\frac{A_n\left(\frac{r}{k}\right)}{\left[\left(1+\frac{r}{k}\right)^{nk}-1\right]} &= d \\
\boxed{d = \frac{A_n\left(\frac{r}{k}\right)}{\left[\left(1+\frac{r}{k}\right)^{nk}-1\right]}} \qquad (6.5)
\end{aligned}$$

Exercise 57

Suppose you know that you are going to need \$400,000 when you retire. If you don't yet have any retirement savings, but you have 35 years until you will retire, how much do you need to save each month into an annuity that pays 7% interest compounded monthly to amass a \$400,000 nest egg?

Answer: \$222.09

6.5 Payout Annuities

Definition.
A **payout annuity** is an account from which regular payments are made to you for a fixed number of years. After the fixed number of years, the annuity is worth nothing. Payout annuities are often used for retirement, payment of lottery winnings and payment of legal settlements.

$$\boxed{P_0 = \frac{d\left[1-\left(1+\frac{r}{k}\right)^{-nk}\right]}{\left(\frac{r}{k}\right)}} \tag{6.6}$$

where,

$$\begin{aligned} P_0 &= \text{Principal or starting balance} \\ d &= \text{ regular payment} \\ r &= \text{ annual interest rate (in decimal)} \\ k &= \text{ the number of compounding periods per year} \end{aligned}$$

Exercise 58
In retirement, you determine that you will need \$2,500 per month for a total of 30 years. If the annuity earns 6% interest, how much principal will you need to fund this retirement annuity? And how much will the annuity pay out over its lifetime?

Answers: \$416,979.04, \$900,000.00

6.5.1 Retirement Planning

Exercise 59

Suppose you are currently 25 years old, but you plan to begin saving for retirement at age 30 and you plan to retire at age 65. You determine that you would like to have \$4000 per month for 30 years.

1. How much principal will you need at age 65 to fund a payout annuity if the annuity yields 6% interest?

2. How much will you need to save each month beginning at age 30 if your savings annuity yields 8% interest?

3. How much will you need to save each month if you begin saving for retirement at age 25 instead of 30, so you will have $65 - 25 = 40$ total years of saving?

4. How much money did you save for retirement? (Use calculation from question 3.)

5. And how much money in total did you receive from that investment?

Answers: (1) \$667,166.46, (2) \$290.85, (3) \$191.11, (4) \$91,732.80, (5) \$1,440,000.00

6.6 Loans

In this section, you will learn about conventional loans (also called amortized loans or installment loans). Examples include auto loans and home mortgages. These techniques do not apply to payday loans, add-on loans, or other loan types where the interest is calculated up front.

One great thing about loans is that they use exactly the same formula as a payout annuity. To see why, imagine that you had $10,000 invested at a bank, and started taking out payments while earning interest as part of a payout annuity, and after 5 years your balance was zero. Flip that around, and imagine that you are acting as the bank, and a car lender is acting as you. The car lender invests $10,000 in you. Since you're acting as the bank, you pay interest. The car lender takes payments until the balance is zero.

$$\boxed{P_0 = \frac{d\left[1 - \left(1 + \frac{r}{k}\right)^{-nk}\right]}{\left(\frac{r}{k}\right)}}$$

Exercise 60

You can afford to pay $280 per month for a car payment. If you qualify for an auto loan at 3% interest for a term of 60 months (5 years), how expensive of a car can you afford?

Answer: $15,582.66

Often when taking out a loan, you know the principal of the loan, but you need to figure out what that amounts to in monthly payments. To do this we can solve the payout annuity formula (6.6) for d, the payment amount:

$$\begin{aligned} P_0 &= \frac{d\left[1-\left(1+\frac{r}{k}\right)^{-nk}\right]}{\left(\frac{r}{k}\right)} \\ P_0\left(\frac{r}{k}\right) &= d\left[1-\left(1+\frac{r}{k}\right)^{-nk}\right] \\ \frac{P_0\left(\frac{r}{k}\right)}{\left[1-\left(1+\frac{r}{k}\right)^{-nk}\right]} &= d \end{aligned}$$

$$\boxed{d = \frac{P_0\left(\frac{r}{k}\right)}{\left[1-\left(1+\frac{r}{k}\right)^{-nk}\right]}}$$

Exercise 61

You wish to take out a \$300,000 mortgage (home loan). If you qualify for a fixed interest rate of 4% and you choose a 30 year term. How much will your monthly payments be?

Answer: \$1,432.25

6.7 Loan Payoff

To determine the remaining loan balance, we can think “how much loan will these loan payments be able to pay off in the remaining time on the loan?”

Note.
Many people mistakenly believe that the payoff amount is simply their monthly payment times the number of remaining payments. This will always be more than the actual payoff amount.

Exercise 62

Suppose you purchased your home with a 30 year mortgage for $250,000 at a 3.5% interest rate. Your monthly morgage payment is $1122.61.

1. If you have made payments for 5 years (60 payments), what is the payoff amount? In other words how much money would you have to give the lender today to settle the mortgage debt?

2. How much have you payed to the bank (mortgage lender) over five years?

3. How much of the money you payed went towards interest?

Answers: (1) $224,242.68, (2) $67,356.60, (3) $41,599.28

Exercise 63

Suppose your monthly car payment is $366.

1. If your loan has a 5 year term and you have made payments for 3 years (36 payments) what is your loan payoff amount?

2. How much in total will you pay if you continue making payments for two years?

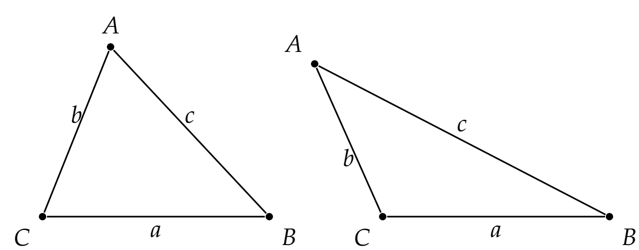
Appendix A

Converse of the Pythagorean Theorem

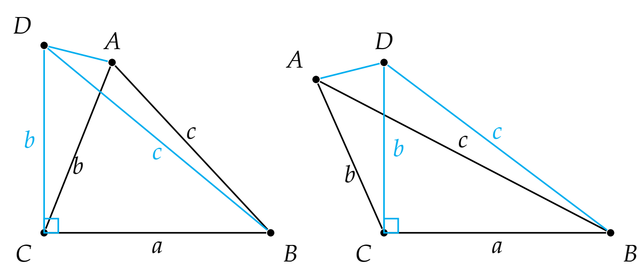
This proof comes from Jim Wilson at the University of Georgia.

The proof uses a technique called **proof by contradiction** which is a little tricky, thus it has been relegated to obscurity in this appendix. The idea of proof by contradiction is to assume that the statement you are trying to prove is false, and then to show that this leads to a logical contradiction, thus the statement must be true. The proof also uses isosceles triangles. An **isosceles** triangle has two sides that are the same length and two angles that are equal.

Proof. Suppose the triangle is *not* a right triangle. Label the vertices A, B and C as pictured. There are two possibilities for the measure of angle C: less than 90° (left picture) or greater than 90° (right picture).



Construct a perpendicular line segment CD as pictured below.



By the Pythagorean theorem, $BD^2 = a^2 + b^2 = c^2$, and so $BD = c$. Thus we have isosceles triangles ACD and ABD. It follows that we have congruent angles $CDA = CAD$ and $BDA = DAB$. But this contradicts the apparent inequalities (see picture) $BDA < CDA = CAD < DAB$ (left picture) or $DAB < CAD = CDA < BDA$ (right picture). □