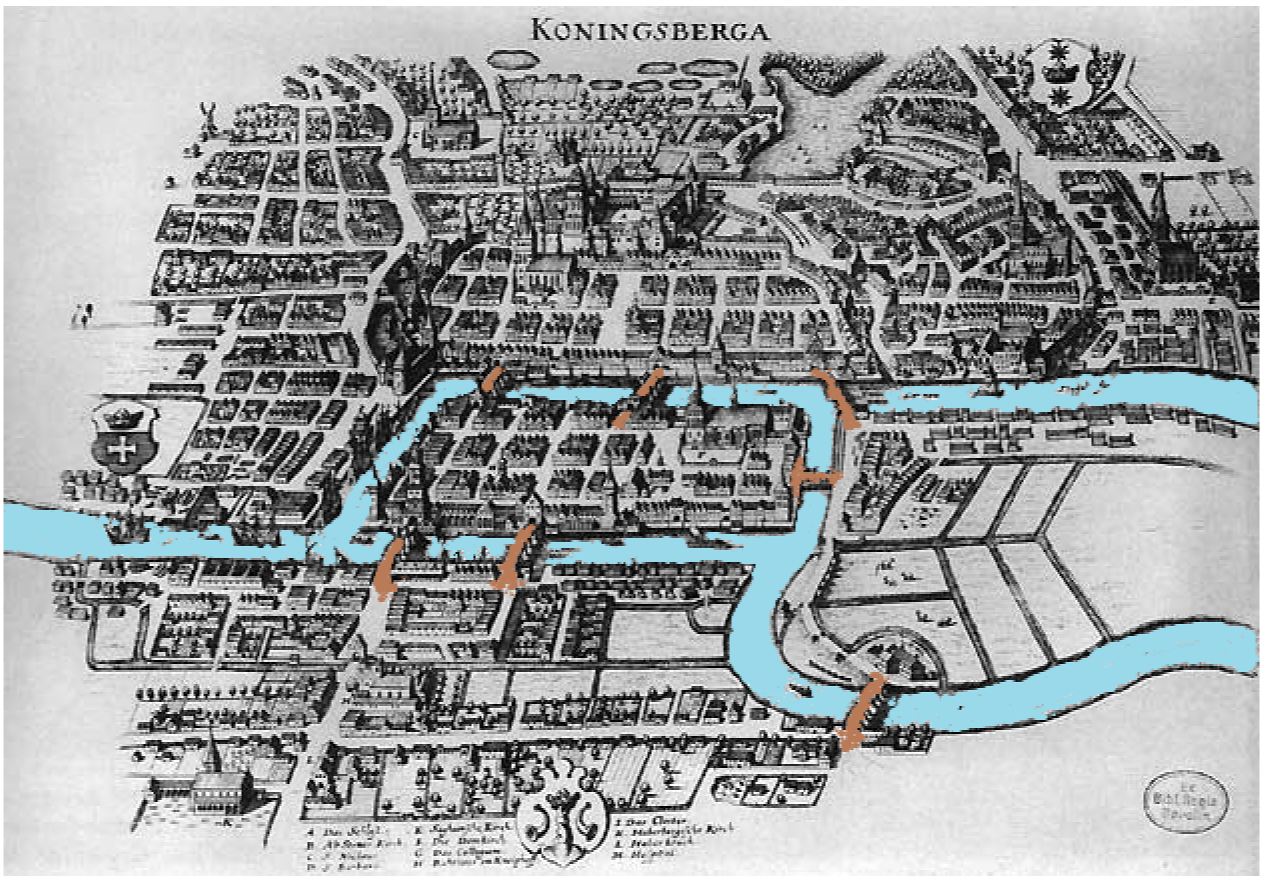
Lecture Notes
for
Math 1030



Jason D. Underdown

Salt Lake Community College

August 26, 2021 at 4:59pm

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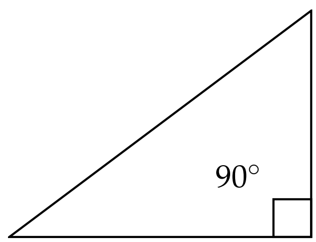
List of Theorems

Chapter 1

Trigonometry

1.1 The Pythagorean Theorem

Trigonometry is the study of triangles, especially **right triangles**. A **right triangle** is a triangle that has a 90° angle which is often denoted by drawing a little square in the corner where two sides meet at a perpendicular or 90° angle.



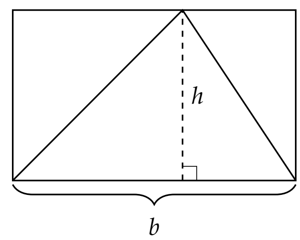
We begin our study of Trigonometry by proving three important theorems. A **theorem** is a fact. But a theorem is more than just a fact, it is a true statement about mathematical objects. Mathematical objects can include a huge variety of things like shapes, numbers, equations, inequalities, graphs, sets, plus many many more. Usually, a theorem applies to a whole class of mathematical objects like all triangles or all graphs.

Theorems require a **proof**, which is just a convincing explanation of why the statement or theorem is true. It is common to denote the end of a proof with a small square, that looks like this: □ or ■. We will present several theorems in this book, but we won't supply a proof for each one because some of them will require techniques or knowledge beyond the scope of our study. Hopefully, the few we do present will give you an idea of what a proving a theorem entails.

Theorem (Area of a Triangle).
The area of a triangle with base, b, and height, h is:

$$A = \frac{1}{2}bh \tag{1.1}$$

***Proof*.**

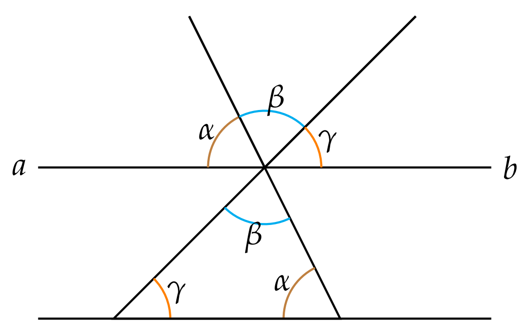


The height of the triangle divides the largest rectangle into two sub-rectangles. Since the triangle covers exactly half of each sub-rectangle, the triangle occupies exactly half of the area of the outer rectangle which is bh. □

It is a common convention to use Greek letters to denote angles, especially the letters α "alpha", β "beta", γ "gamma", θ "theta", but sometimes Roman letters such as x, y, z, t or capital letters such as A, B, C are used as well.

Theorem (Sum of Interior Angles of a Triangle).
The interior angles of a triangle sum to 180°.

***Proof*.**



Draw a line segment between points a and b parallel to the base of the triangle. Extending the edges of the triangle past the parallel line generates angles which correspond with angles α and γ. Similarly, the two angles marked β are vertical angles and thus equal. Finally, the three angles at the top of the figure sum to a straight line and thus sum to 180°. □

The next theorem is the most important one in the study of Trigonometry.

Theorem (The Pythagorean Theorem).
If a, b, c are lengths of the legs of a *right* triangle where c is the *hypotenuse*, then

$$a^2 + b^2 = c^2 \tag{1.2}$$

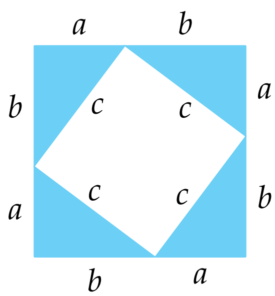


Figure 1.1: Four exact copies of a right triangle arranged to make two squares.

***Proof*.** First, note that the outer (partly blue shaded) quadrilateral is a square because each side has the same length, $a+b$, and because each corner has an angle of $90°$. Second, note that the inner tilted quadrilateral is also a square because each side has the same length, namely c, and each of its corner angles must be $90°$. This is due to the fact that the two non-right angles in the given right triangle must sum to $90°$ because the sum of all angles in *any* triangle is $180°$. Finally, we can derive equation (1.2) by equating the area of the whole outer square to the area of the sum of its pieces.

$$\begin{aligned}\text{Area of outer square} &= \text{Sum of the area of the pieces}\\ (a+b)(a+b) &= c^2 + 4\left(\tfrac{1}{2}ab\right)\\ a^2 + \cancel{2ab} + b^2 &= c^2 + \cancel{2ab}\\ a^2+b^2 &= c^2\end{aligned}$$ □

Exercise 1

Laptop computer screens are measured along the diagonal, i.e. from one corner of the screen to the opposite corner. If a 14 in laptop screen is 8 in tall, how wide is the screen?

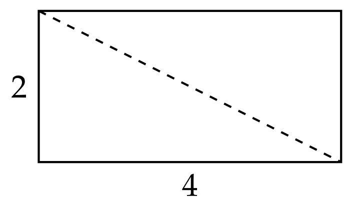
Exercise 2

To safely use a ladder, the base of the ladder should form a $75°$ angle with level ground. A simple way to ensure this is to use the “4 to 1” rule, which says that for every 4 ft of height gained you move the base 1 ft away from the vertical.

What is the minimum length ladder needed to safely reach a roof that is 16 ft high?

Exercise 3

A phone line would normally be installed along two consecutive edges of a rectangular 2 mi by 4 mi field. If the phone line costs \$2500 mi^{-1} to install, how much will be saved if the phone line is installed diagonally across the field instead of along two of the edges?



1.1.1 Converse of the Pythagorean Theorem

A **conditional statement** is a statement of the form "If P then Q" where P and Q are both sub-statements. P is called the **hypothesis** and Q is called the **conclusion**. For example

Conditional: If Socrates is a man, then Socrates is mortal. (true)

Notice that the above statement is true, that is, the conclusion is a consequence of the hypothesis being true. We can also form three related conditional statements (see Table 1.1). The **converse** of the statement above, is not true:

Converse: If Socrates is mortal, then Socrates is a man. (false)

There are lots of things that are mortal that are not men, for example Socrates could refer to a woman or a horse. Thus a conditional statement and its converse need not both be true. But sometimes they are.

Statement	English Form	Arrow Notation
Conditional	If P then Q.	$P \Rightarrow Q$
Converse	If Q then P.	$Q \Rightarrow P$
Inverse	If not P then not Q.	$\neg P \Rightarrow \neg Q$
Contrapositive	If not Q then not P.	$\neg Q \Rightarrow \neg P$

Table 1.1: Four types of conditional statements

The Pythagorean theorem is a **conditional statement** of the form "If P then Q", where (after removing details) we essentially have:

Pythagorean Theorem: If right triangle, then $a^2 + b^2 = c^2$.

The converse of the Pythagorean theorem is of the form "If Q, then P", that is,

Converse of Pythagorean Theorem: If $a^2 + b^2 = c^2$, then right triangle.

It turns out that the converse of the Pythagorean theorem is also true.

Theorem (Converse of the Pythagorean Theorem).
If a, b, c are sides of a triangle with c the longest side and $a^2 + b^2 = c^2$, then the triangle is a *right* triangle.

***Proof*.** See appendix A. □

Exercise 4

Use the converse of the Pythagorean theorem to prove that a triangle with sides (5,12,13) is a right triangle.

When both a conditional and its converse are always true we call it a **biconditional**. This means that P implies Q but also Q implies P, in arrow notation: $P \Leftrightarrow Q$. In other words, statements P and Q are different but they both mean the same thing. One side of the Pythagorean theorem is about lengths and the other is about angles. And what it means is that (in the context of triangles), knowing something about lengths of the legs tells you something about the interior angles and vice versa. You can think of it as a *bridge* between the concept of length and the concept of angle. This bridge leads to two new concepts: Pythagorean triples and a way to classify triangles.

1.1.2 Pythagorean Triples

Definition.
A **Pythagorean triple** is any triple of whole numbers, (a, b, c), which satisfy $a^2 + b^2 = c^2$ and thus are the legs of a right triangle.

Example 1
Some examples of Pythagorean triples are:

$$(3, 4, 5), (6, 8, 10), (5, 12, 13), (7, 24, 25), (9, 40, 41)$$

These are useful to remember mostly because Math teachers like to use them in homework problems and tests, and it can save you time if you notice that you are dealing with one of these special triangles.

Exercise 5
Can you take any Pythagorean triple and generate a new triple just by multiplying every number by a whole number, like 2 or 3? Can you prove it?

1.1.3 Classification of Triangles

The converse of the Pythagorean theorem allows us to use the sides of a triangle to determine something about one of the angles. If we combine this fact with the fact that all of the interior angles of a triangle must sum to 180°, then we can classify triangles into three categories.

> **Definition.**
> **obtuse angle:** an angle between 90° and 180°.
>
> **acute angle:** an angle between 0° and 90°.
>
> **obtuse triangle:** a triangle which has one obtuse angle.
>
> **acute triangle:** a triangle where all angles are acute.

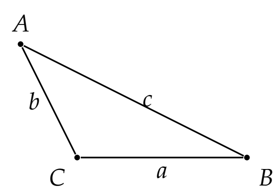
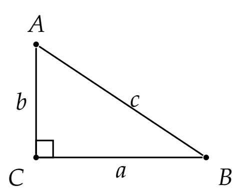
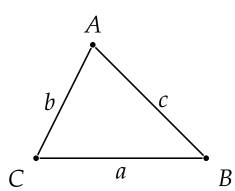
Condition	Example	Category
$c^2 > a^2 + b^2$		obtuse
$c^2 = a^2 + b^2$		right
$c^2 < a^2 + b^2$		acute

Table 1.2: Classification of triangles by sides

Exercise 6
Classify the triangles with sides $(5, 6, 7)$ and $(5, 6, 8)$.

1.1.4 The Distance Formula

We can use the Pythagorean theorem to find the distance between any two points in the Cartesian plane. The basic idea is to construct a right triangle with hypotenuse corresponding to a segment between the two points and one leg parallel to the x-axis and another leg parallel to the y-axis.

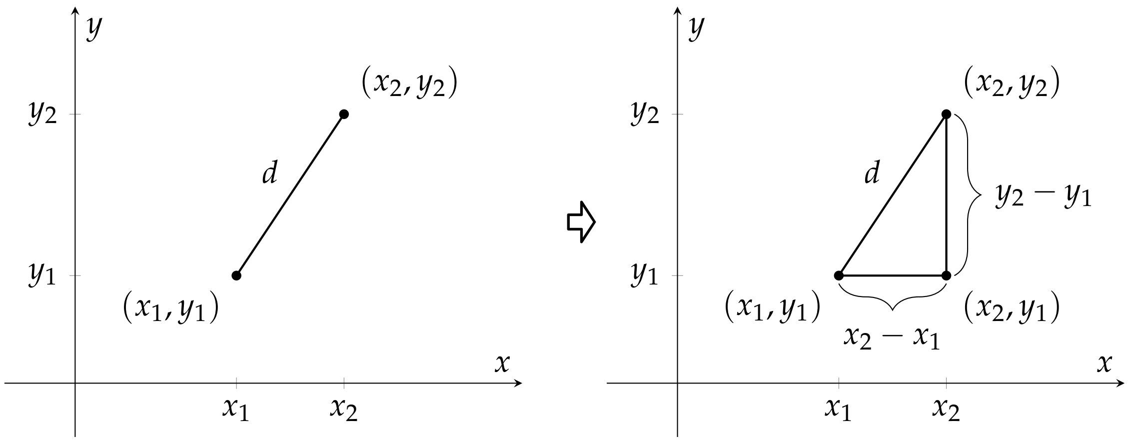


Figure 1.2: The Pythagorean theorem can be used to determine distance bewteen points.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$\boxed{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}} \tag{1.3}$$

Exercise 7
Find the distance between the points $(-1, 5)$ and $(2, -2)$.

Formula (1.3) can be hard to remember. I find it easier to just remember the idea of equating the distance to the hypotenuse and then using a sketch and the Pythagorean theorem to compute the distance.

1.2 Trigonometric Functions

The Pythagorean theorem allows us to determine the length of an unknown side of a right triangle when we know two of the other sides. But in some circumstances, we may only know an interior angle and a side. In this section we will define six functions that will allow us to determine sides when we only know a side and an angle.

In order to define these functions we use three terms to identify the edges or legs of the triangle: **hypoteneuse**, **opposite**, and **adjacent**. First, the term **hypotenuse** always refers to the longest edge of the triangle. Next, we have the side **opposite** the angle of interest, say θ. Finally, we have the side **adjacent** to the angle of interest. Figure 1.3 demonstrates how the adjacent and opposite sides switch depending on where θ is.

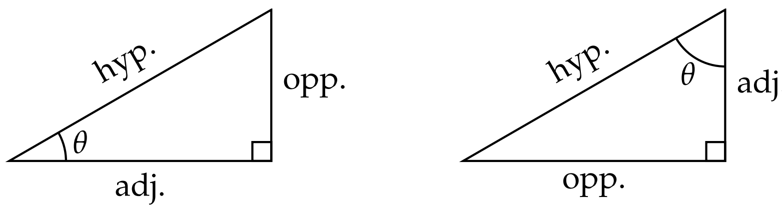


Figure 1.3: Opposite and adjacent sides depend on where θ is located.

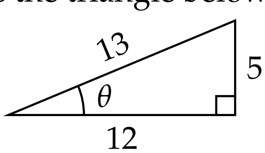
1.2.1 The Sine, Cosine, and Tangent Functions

Definition.
Given a *right* triangle with angle θ, we define three functions in terms of ratios of the sides of the triangle.

$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \tan(\theta) = \frac{\text{opp}}{\text{adj}} \tag{1.4}$$

A mnemonic for remembering all three definitions is SOH-CAH-TOA.

Example 2
Use the triangle below to compute: $\sin(\theta), \cos(\theta)$ and $\tan(\theta)$.



Solution:

$$\sin(\theta) = \frac{5}{13} \qquad \cos(\theta) = \frac{12}{13} \qquad \tan(\theta) = \frac{5}{12}$$

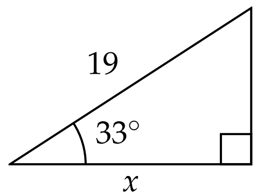
△

The three trigonometric functions we just defined are **side finders**. Later we will introduce functions that are **angle finders**.

Make sure your calculator is in *degrees* mode and not radians mode.

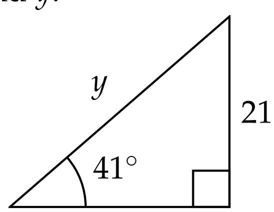
Exercise 8

Find x.



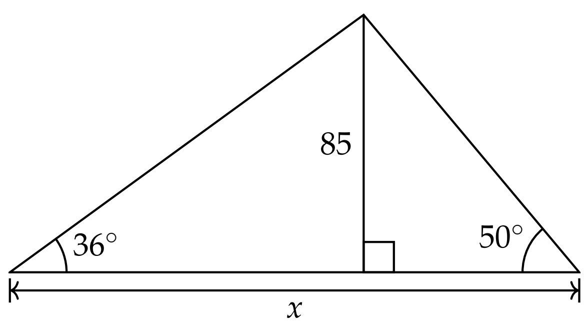
Exercise 9

Find y.



Exercise 10

Find x.



1.2.2 Inverse Trigonometric Functions

Not all functions have an inverse, but the three trigonometric functions we just defined do! Recall that a function is a mapping between two sets which we call the domain and codomain. The **domain** of a function is the set of acceptable inputs, while a function's **codomain** is its output set.

An inverse function maps in the reverse direction, so it swaps the domain and codomain sets. Thus while the three trigonometric functions eat angles and output ratios, the three *inverse* trigonometric functions eat ratios and output angles. You should think of them as **angle finders**.

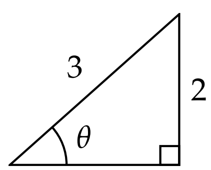
Since we have defined the three trigonometric functions via *right* triangles only, the domains and codomains are somewhat restricted. If you take a full course on Trigonometry, you will learn how these functions can be extended to larger domains and codomains.

Function : Domain $\mapsto$ Codomain	Inverse : Domain $\mapsto$ Codomain
$\sin(x) : [0°, 90°] \mapsto [0, 1]$	$\sin^{-1}(x) : [0, 1] \mapsto [0°, 90°]$
$\cos(x) : [0°, 90°] \mapsto [0, 1]$	$\cos^{-1}(x) : [0, 1] \mapsto [0°, 90°]$
$\tan(x) : [0°, 90°) \mapsto [0, \infty)$	$\tan^{-1}(x) : [0, \infty) \mapsto [0°, 90°)$

Table 1.3: The three Trigonometric functions map angles to ratios, while their inverses map ratios to angles.

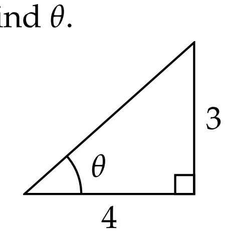
Exercise 11

Find θ.



Exercise 12

Find θ.



Exercise 13

A new 500 ft long zip line is going to be installed at Lagoon. The zip line will have a vertical distance of 200 ft. At what angle will the zip line meet the ground? What is the horizontal distance of the zip line?

1.2.3 Finding the Area of a Triangle

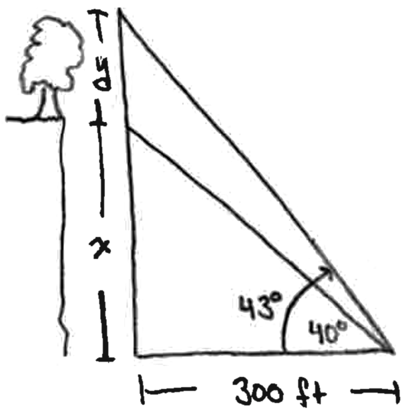
Exercise 14

Find the area of the given triangle using trigonometric functions.

1.2.4 Angles of Elevation and Depression

Exercise 15

A person stands 300 ft from the base of a cliff. The angle of elevation to the bottom of a tree atop the cliff is 40°. The angle of elevation to the top of the tree is 43°. How tall is the tree?



Exercise 16

A statue is located 300 ft from a building. From a window in the building, a person determines that the angle of elevation to the top of the statue is 34° and the angle of depression to the bottom of the statue is 22°. How tall is the statue?

Beware the functions we define in this section only apply to *right* triangles.

Chapter 2

Problem Solving

2.1 Exploiting Fractions for Fun and Profit

2.1.1 Absolute and Relative Change

In this section we will see how to use the two arithemtic operations of subtraction and division to measure how something changes over time.

Definition.
Absolute change is the difference:

$$\text{absolute change} = \text{new} - \text{old}$$

Absolute change can be useful for comparing things that are similar. For example you could compare how many inches two fourteen year old boys grew in one year. But suppose the things you wish to compare are related but on very different size scales.

Example 3 (Stock Investments Part I)

Suppose you invest $50,000 in Acme corporation stock and after one year the stock is now worth $52,000. This is an absolute change of $2,000. Now suppose you also invest $500 in Hooli corporation and after one year the stock is worth $550. Which investment was better?

Solution:

$$\begin{aligned} \text{absolute change in Acme} &= \$52,000 - \$50,000 = \$2,000 \\ \text{absolute change in Hooli} &= \$550 - \$500 = \$50 \end{aligned}$$

In terms of absolute change the yield from the Acme investment is larger, so it is better right? Well what if we had switched our investment strategy and swapped the initial stock purchase for each company? To answer this question we need to see how big the absolute change in value was relative to the initial outlay of money. We need relative change. △

Definition.
Relative change is the ratio of absolute change to the original (old) value.

$$\text{relative change} = \frac{\text{new} - \text{old}}{\text{old}}$$

Example 4 (Stock Investments Part II)
What is the relative change in value of each stock purchase?

Solution:

$$\text{relative change in Acme} = \frac{\$52,000 - \$50,000}{\$50,000} = \frac{\$2,000}{\$50,000} = 0.04$$
$$\text{relative change in Hooli} = \frac{\$550 - \$500}{\$500} = \frac{\$50}{\$500} = 0.10$$

Hooli's relative change was higher than Acme's relative change, and thus the Hooli stock was the better investment. △

Note.
Notice that absolute change has units attached to the value. In the above examples the units were dollars ($). But when you compute relative change, the numerator and denominator will have the same units and thus the units will cancel and you are left with a pure (unitless) number.

2.1.2 Percentages

Definition.
A **percent** or **percentage** is a fraction, where the denominator is 100, but for ease of writing we drop the denominator and replace it with the percent symbol %. For example,

$$\frac{3}{4} = 0.75 = \frac{75}{100} = 75\%$$

Percentages can be negative and even larger than 100, for example

$$250\% = \frac{250}{100} = 2.5$$

Exercise 17
Express the following values as percentages.

1. $\dfrac{1}{5}$

2. 0.004

3. 3.11

It is often convenient to express relative change as a percent

Example 5 (Stock Investments Part III)
What was the percent change in value of each stock investment?

Company	Decimal	Fraction	Percent
Acme	.04	$\frac{4}{100}$	4%
Hooli	.10	$\frac{10}{100}$	10%

Exercise 18

Your truck was worth \$28,000 in 2019 and is now worth \$23,500 in 2021. What is the percent change in value of your truck?

Exercise 19

Your \$290,000 home increased in value by 5%. How much is it worth now?

Exercise 20

A store has a clearance rack where every item is marked down 20% off the original price. They have a sale advertising an additional 20% off everything in the store. What percentage of the original price do you end up paying?

2.1.3 Rates

Definition.
A rate is a ratio of two quantities. Rates are also known as fractions.

Exercise 21

If your car can travel 284 mi per fillup and your gas tank holds 9.5, then at what rate does your car consume gasoline, or more commonly, what is its mpg?

2.1.4 Proportionality

Definition.
A proportion is a ratio of two quantities. Thus it is a fraction.

Exercise 22

A map's legend indicates that equals 4 mi. How many miles apart in real life are two points on the map that are separated by ?

Exercise 23

A cookie recipe calls for 2 cups of chocolate chips per batch of 24 cookies. If you wish to make 216 cookies?

2.2 Exploiting Units for Fun and Profit

2.2.1 Unit Conversions

2.2.2 Geometry: Area and Volume Formulas

2.2.3 Linear, Areal and Volumetric Densities

Chapter 3

Voting Theory

3.1 Plurality Method

Exercise 24

If Alex, Betty and Chuck are running for the office of club president, then what is the minimum number of votes that one could get to win the election under the plurality method if there are

1. 73 votes cast?

2. 74 votes cast?

3.2 Instant Runoff Voting (Ranked Choice Voting)

3.2.1 Preference Schedules

3.3 Borda Count

3.4 Copeland's Method (Pairwise Comparison)

3.5 Fairness Criteria

Definition.
A **fairness criterion** is a conditional statement (if—then) about a selection or voting method that seems like it should be satisfied in order for the method to be fair.

Plurality Criterion
If there is a candidate that gets more votes than any other candidate, then that candidate should win.

Majority Criterion
If there is a candidate that has a majority (more than half) of first-place votes, then that candidate should win.

Condorcet Criterion
If there is a candidate that wins every head-to-head comparison, then that candidate should win.

Monotonicity Criterion
If a candidate wins, and only changes that favor the winner are made to the preference ballots, then that candidate should still win.

Independence of Irrelevant Alternatives (IIA) Criterion
If a candidate wins, and only losing candidates are removed from the preference ballots, then that candidate should still win.

3.6 Arrow's Impossibility Theorem

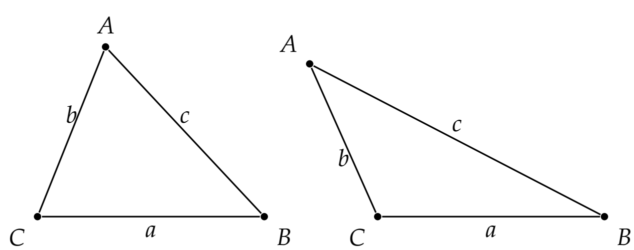
Appendix A

Converse of the Pythagorean Theorem

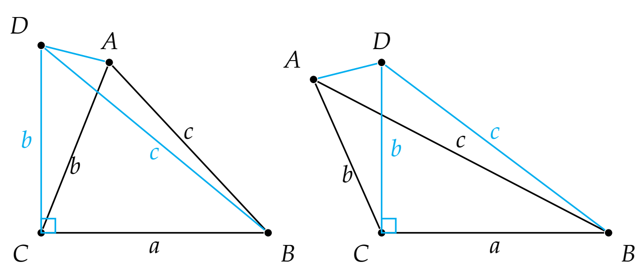
This proof comes from Jim Wilson at the University of Georgia.

The proof uses a technique called **proof by contradiction** which is a little tricky, thus it has been relegated to obscurity in this appendix. The idea of proof by contradiction is to assume that the statement you are trying to prove is false, and then to show that this leads to a logical contradiction, thus the statement must be true. The proof also uses isosceles triangles. An **isosceles** triangle has two sides that are the same length and two angles that are equal.

***Proof*.** Suppose the triangle is *not* a right triangle. Label the vertices A, B and C as pictured. There are two possibilities for the measure of angle C: less than $90°$ (left picture) or greater than $90°$ (right picture).



Construct a perpendicular line segment CD as pictured below.



By the Pythagorean theorem, $BD^2 = a^2 + b^2 = c^2$, and so $BD = c$. Thus we have isosceles triangles ACD and ABD. It follows that we have congruent angles $CDA = CAD$ and $BDA = DAB$. But this contradicts the apparent inequalities (see picture) $BDA < CDA = CAD < DAB$ (left picture) or $DAB < CAD = CDA < BDA$ (right picture). $\square$