Reduced Wigner coefficients for unitary representations of the Lie superalgebra $gl(m|n)$

Jason L. Werry, Phillip S. Isaac and Mark D. Gould

School of Mathematics and Physics, The University of Queensland, St Lucia QLD 4072, Australia.

Abstract

In this paper fundamental Wigner coefficients are determined algebraically by considering the eigenvalues of certain generalized Casimir invariants. Here this method is applied in the context of both type 1 and type 2 unitary representations of the Lie superalgebra $gl(m|n)$.

1 Introduction

The application of the theory of Lie Superalgebras to problems in mathematical physics relies on the construction of an explicit set of basis states of an irreducible representation and also the explicit determination of Wigner (Clebsch-Gordan) coefficients. Prior to the development of Lie superalgerbra theory, the foundational papers by Gel'fand and Tsetlin [4, 5] gave a novel construction of basis vectors for the irreducible representations of the unitary and orthogonal groups. Further work by Baird and Biedenharn [6] gave a proof of Gel'fand and Tsetlin's results while also giving , for the first time, both the fundamental and reduced Wigner coefficients for the Lie group U(n). In the same paper, the factorization of a matrix element into a Wigner coefficient and a reduced matrix element was also examined.

The utility of the characteristic identities (polynomial identities satisfied by generators of the Lie algebra) towards these goals became apparent during the 1970s and 80s and resulted in the algebraic determination of reduced matrix elements, [10] raising and lowering generators [11] and matrix elements, [9, 12]. In the papers [2, 3] the matrix elements of unitary representations of $gl(m|n)$ were given explicitly using similar techniques. The resulting closed-form expressions were obtained by utilizing the factorization of a matrix element into a Wigner coefficient and a reduced matrix element. The vector Wigner coefficients thus obtained allow us in this paper to obtain all fundamental $gl(m|n)$ Wigner coefficients (WCs) in the Gelfand-Tsetlin (GT) basis for both type 1 and type 2 unitary representations. Although the resulting expressions for these Reduced Wigner coefficients have appeared in [7], in this presentation we obtain our results from first principles using algebraic methods that can be directly generalised to other Lie algebras as well as the quantum case.

2 Characteristic identities and associated invariants

We utilise the notation used in the series [1–3]. The generators of the Lie superalgebra $gl(m|n)$ are denoted E_{ij} where $1 \leq p, q \leq m + n$. The values $1 \leq i, j \leq m$ are called *even* while the values $m < i, j \leq m + n$ are called *odd*.

The graded index notation will be used where the Latin indices $i, j, ...$ are used for even quantities and the Greek indices $\mu, \nu, ...$ for odd quantities. The grading operator ( ) will then give

$$(i) = 0, \ (\mu) = 1$$

and then set of generators is then given by

$$E_{ij}, E_{i\mu}, E_{\mu i}, E_{\mu\nu}.$$

We reserve the indices p, q to be ungraded and to range fully from 1 to $m + n$.

A weight Λ may be expanded in terms of the elementary weights ε_p $(1 \leq p \leq m + n)$ so that

$$\Lambda = \sum_{p=1}^{m} \Lambda_p \varepsilon_p$$

where ϵ_p is the $m + n$-tuple with 1 in position p and zeros elsewhere.

The adjoint tensor operator $\bar{\mathcal{A}}$ constructed in [1] plays an important role in what follows and is defined as the $m + n$ square matrix with entries

$$\bar{\mathcal{A}}_{pq} = -(-1)^{(p)(q)} E_{qp}.$$

When $\bar{\mathcal{A}}$ acts on an irreducible $gl(m|n)$ module $V(\Lambda)$ of highest weight Λ it will satisfy the characteristic identity

$$\prod_{i=1}^{m} (\bar{\mathcal{A}} - \bar{\alpha}_i) \prod_{\mu=1}^{n} (\bar{\mathcal{A}} - \bar{\alpha}_\mu) = 0 \tag{1}$$

where the adjoint roots $\bar{\alpha}_i, \bar{\alpha}_\mu$ are given in terms of the highest weight labels

$$\Lambda = (\Lambda_{i=1}, ..., \Lambda_{i=m} | \Lambda_{\mu=1}, ... \Lambda_{\mu=n})$$

as

$$\bar{\alpha}_i = i - 1 - \Lambda_i, \quad \bar{\alpha}_\mu = \Lambda_\mu + m + 1 - \mu.$$

Immediately from the characteristic identity, we see that for each integer r where $1 \leq r \leq m + n$ there exists a projection operator

$$\bar{P}[r] : V(\varepsilon_1) \otimes V(\Lambda) \longrightarrow V(\Lambda + \epsilon_r)$$

given by

$$\bar{P}[r] = \prod_{k \neq r}^{m+n} \left(\frac{\bar{\mathcal{A}} - \bar{\alpha}_k}{\bar{\alpha}_r - \bar{\alpha}_k} \right).$$

Similarly we have the vector matrix $\mathcal{A}$ with entries

$$\mathcal{A}_{pq} = -(-1)^{(p)} E_{pq}.$$

that satisfy the polynomial identities

$$\prod_{i=1}^{m}(\mathcal{A}-\alpha_i)\prod_{\mu=1}^{n}(\mathcal{A}-\alpha_\mu)=0 \tag{2}$$

where

$$\alpha_i=\Lambda_i+m-n-i,\quad \alpha_\mu=\mu-\Lambda_\mu-n.$$

The associated projection operator

$$P[r]:V^*(\varepsilon_1)\otimes V(\Lambda)\longrightarrow V(\Lambda-\epsilon_r)$$

is then given by

$$P[r]=\prod_{k\neq r}^{m+n}\left(\frac{\mathcal{A}-\alpha_k}{\alpha_r-\alpha_k}\right).$$

We will now show that the eigenvalues of the invariants

$$\bar{c}_r=\bar{P}[r]_{m+n}^{\;m+n}$$

and

$$c_r=P[r]^{m+n}_{\;m+n}$$

are essentially squares of reduced Wigner coefficients.

Remark 1: It is important to note that within the $gl(m|n)$ projection operator expressions we may have $\alpha_r=\alpha_k$ for some $r\neq k$. This is related to the occurrence of atypical irreducible representations in the tensor product of $V(\varepsilon_1)\otimes V(\Lambda)$ or $V(\varepsilon_1)^*\otimes V(\Lambda)$. The set of Λ for which this happens, however, is closed in the Zariski topology [8] on H^*. It follows that the roots of the characteristic identity are distinct on an open and hence dense subset of H^*. Hence without loss of generality, we will make the assumption that the roots are distinct. Furthermore, the invariants derived in this paper are given by rational polynomial functions which are continuous in the Zariski topology.

Remark 2: When applying the projection operators we must take care to distinguish between type 1 unitary and type 2 unitary modules. These classes of modules are defined as follows. Consider the positive-definite, invariant, sesquilinear form defined on an irreducible highest weight module $V(\lambda)$ of $gl(m|n)$

$$\langle a_{pq}|w\rangle=(-1)^{\epsilon[(p)+(q)]}\langle v|a_{qp}w\rangle,\epsilon\in\{0,1\}.$$

When $\epsilon=0$ we call $V(\Lambda)$ a type 1 unitary irrep while for $\epsilon=1$ we call $V(\Lambda)$ a type 2 unitary irrep. The induced form on a tensor product space $V(\Lambda)\otimes V(\mu)$ is non-degenerate only when restricted to a direct summand of the tensor product space and only when $V(\Lambda)$ and $V(\mu)$ are of the same type. This induced form is necessarily unique by Schur's lemma. We then define Wigner coefficients with respect to this form.

Due to these considerations we now consider the type 1 unitary and type 2 unitary cases of Wigner coefficients seperately.

Let e_i denote the Gelfand-Tsetlin (GT) basis states for the (type 1 unitary) vector module $V(\varepsilon_1)$ and e^Λ_β denote the (GT) basis states for the irreducible type 1 unitary module $V(\Lambda)$ of highest weight Λ. Then the matrix elements of $\bar{P}[r]$ can be given in the form

$$\left\langle e^\Lambda_\beta | \bar{P}[r]^j_i | e^\Lambda_\alpha \right\rangle = \sum_\gamma \left\langle e^\Lambda_\beta \otimes e_i | e^{\Lambda+\varepsilon_r}_\gamma \right\rangle \left\langle e^{\Lambda+\varepsilon_r}_\gamma | e_j \otimes e^\Lambda_\alpha \right\rangle, \tag{3}$$

where

$$\left\langle e^{\Lambda+\varepsilon_r}_\gamma | e_j \otimes e^\Lambda_\alpha \right\rangle,$$

are the vector (fundamental) Wigner coefficients.

Similarly by denoting $\bar{e}_i$ to be the Gelfand-Tsetlin (GT) basis states of the dual vector module we have

$$\left\langle e^\Lambda_\beta | P[r]^j_i | e^\Lambda_\alpha \right\rangle = \sum_\gamma \left\langle e^\Lambda_\beta \otimes \bar{e}_i | e^{\Lambda-\varepsilon_r}_\gamma \right\rangle \left\langle e^{\Lambda-\varepsilon_r}_\gamma | \bar{e}_j \otimes e^\Lambda_\alpha \right\rangle, \tag{4}$$

where

$$\left\langle e^{\Lambda-\varepsilon_r}_\gamma | \bar{e}_j \otimes e^\Lambda_\alpha \right\rangle,$$

are the dual fundamental Wigner coefficients. The above quantities may be expressed in terms of the $gl(m|n)$ highest weight labels of Λ and the highest weight labels of the $gl(m|n-1)$ subalgebra.

We now extend our notation to let $\bar{\varepsilon}_k$ denote the weights of the $gl(m|n)$ contravariant vector irrep. These weights are given by $\bar{\varepsilon}_k = -\varepsilon_{m+n+1-k} (1 \le k \le m+n)$. We similarly define the $gl(m|n-1)$ fundamental weights $\varepsilon_{0_{m+n+1-r}} = -\bar{\varepsilon}_{0_r} (1 \le r \le m+n-1)$.

From Schur's lemma we observe that the fundamental WCs factorize as follows

$$\left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda+\varepsilon_{0_r} \\ [\Lambda'_0] \end{array} \middle| \, e_i \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \right\rangle = \left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda+\varepsilon_{0_r} \end{array} \middle| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle \left\langle \begin{array}{c} \lambda+\varepsilon_{0_r} \\ [\Lambda'_0] \end{array} \middle| \, e_i \otimes \begin{array}{c} \lambda \\ [\Lambda_0] \end{array} \right\rangle, \tag{5}$$

$$\left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda-\varepsilon_{0_r} \\ [\Lambda'_0] \end{array} \middle| \, \bar{e}_i \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \right\rangle = \left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda-\varepsilon_{0_r} \end{array} \middle| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle \left\langle \begin{array}{c} \lambda-\varepsilon_{0_r} \\ [\Lambda'_0] \end{array} \middle| \, \bar{e}_i \otimes \begin{array}{c} \lambda \\ [\Lambda_0] \end{array} \right\rangle, \tag{6}$$

$$1 < i < m+n,$$

where $1 < k \le m+n$, $1 < r \le m+n-1$, Λ denotes the highest weight of $gl(m|n)$, λ denotes the highest weight of $gl(m|n-1)$ and $[\Lambda_0]$ denotes the GT pattern of the $gl(m|n-2)$ subalgebra. In addition, the first term on the rhs of equations (5) and (6) is a reduced Wigner coeficient (RWC) which is independent of the highest weight labels of $[\Lambda_0]$ and $[\Lambda'_0]$.

Setting $i = m+n$ gives

$$\left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda \\ [\Lambda'_0] \end{array} \middle| \, e_{m+n} \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \right\rangle = \delta_{[\Lambda'_0][\Lambda]} \left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda \end{array} \middle| \begin{array}{c} \varepsilon_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle, \tag{7}$$

and

$$\left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda \\ [\Lambda_0'] \end{array} \right| \bar{e}_{m+n}\otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \left. \vphantom{\begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array}} \right\rangle = \delta_{[\Lambda_0'][\Lambda]} \left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left. \vphantom{\begin{array}{c} \Lambda \\ \lambda \end{array}} \right\rangle, \tag{8}$$

where the RWC on the rhs reduces to the WC in this case which are independent of $[\Lambda_0]$ and $[\Lambda_0']$ as expected. The WCs in equations (7) and (8) are given by the eigenvalues of the invariants

$$\bar{c}_r = \bar{P}[r]_{m+n}^{\;m+n}, \quad c_r = P[r]^{m+n}_{\;\;m+n}$$

since from equations (3) and (4) we have

$$\begin{aligned} \bar{c}_r &= \left| \left\langle \begin{array}{c} \Lambda+\varepsilon_r \\ \lambda \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left. \vphantom{\begin{array}{c} \Lambda \\ \lambda \end{array}} \right\rangle \right|^2, \\ c_r &= \left| \left\langle \begin{array}{c} \Lambda-\varepsilon_r \\ \lambda \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left. \vphantom{\begin{array}{c} \Lambda \\ \lambda \end{array}} \right\rangle \right|^2, \end{aligned} \tag{9}$$

where $\bar{e}_r$ denote the Gelfand-Tsetlin (GT) basis states of the adjoint vector module.

We therefore see that the matrix elements of the projections $\bar{P}[r]_{m+n}^{\;m+n}$ and $P[r]^{m+n}_{\;\;m+n}$ determine the squares of the fundamental Wigner coefficients and depend only on the top two rows of the corresponding Gelfand-Tsetlin basis states.

3 Reduced Wigner coefficients

The above derivations may be repeated for the $gl(m|n-1)$ case so that we have the corresponding $(m+n-1)\times(m+n-1)$ square matrices

$$\bar{\mathcal{A}}_{0pq} = -(-1)^{(p)(q)}E_{qp}, \qquad \mathcal{A}_{0pq} = -(-1)^{(p)}E_{pq}.$$

that satisfy the usual polynomial identities on an irreducible module with highest weight λ:

$$\begin{aligned} \prod_{i=1}^{m}(\bar{\mathcal{A}}_0-\bar{\alpha}_{0i})\prod_{\mu=1}^{n+1}(\bar{\mathcal{A}}_0-\bar{\alpha}_{0\mu}) &= 0 \\ \prod_{i=1}^{m}(\mathcal{A}_0-\alpha_{0i})\prod_{\mu=1}^{n+1}(\mathcal{A}_0-\alpha_{0\mu}) &= 0. \end{aligned}$$

Here the roots are given by

$$\begin{aligned} \bar{\alpha}_{0i} &= i-1-\lambda_i, & 1\le i\le m, \\ \bar{\alpha}_{0\mu} &= \lambda_\mu+m+1-\mu, & 1\le \mu\le n-1, \\ \alpha_{0i} &= \lambda_i+m-n+1-i, & 1\le i\le m, \\ \alpha_{0\mu} &= \mu-\lambda_\mu-n+1, & 1\le \mu\le n-1. \end{aligned}$$

Similarly, the $gl(m|n-1)$ projection operators are given by

$$\bar{P}_0[r] = \prod_{k\neq r}^{m+n+1} \left(\frac{\bar{\mathcal{A}}_0 - \bar{\alpha}_{0k}}{\bar{\alpha}_{0r} - \bar{\alpha}_{0k}} \right), \quad P_0[r] = \prod_{k\neq r}^{m+n+1} \left(\frac{\mathcal{A}_0 - \alpha_{0k}}{\alpha_{0r} - \alpha_{0k}} \right), \tag{10}$$

The betweenness conditions imply [1], for $1 \leq i \leq m$, that we have only two cases

$$\alpha_i = \begin{cases} \alpha_{0i}, & \Lambda_i = 1 + \lambda_i \\ \alpha_{0i} - 1, & \Lambda_i = \lambda_i \quad . \end{cases}$$

We therefore define the following index sets

$$\begin{aligned} I_0 &= \{1 \leq i \leq m \mid \alpha_{0i} = \alpha_i\}, \\ \bar{I}_0 &= \{1 \leq i \leq m \mid \alpha_{0i} = 1 + \alpha_i\}, \\ I_1 &= \{1 \leq \mu \leq n-1\}, \\ I &= I_0 \cup I_1, \\ I' &= \bar{I}_0 \cup I_1, \\ \tilde{I} &= I \cup \{m+n\}, \\ \tilde{I}' &= I' \cup \{m+n\}. \end{aligned} \tag{11}$$

By considering the characteristic identities satisfied by the matrices $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_0$ we may obtain the invariant $\bar{c}_r$ as a rational polynomial in terms of the roots $\bar{\alpha}_i, \bar{\alpha}_\mu$ and $\bar{\alpha}_{0i}, \bar{\alpha}_{0\mu}$. Specifically, we have [1]

$$\bar{c}_r = \prod_{k\in\tilde{I}', k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k)^{-1} \prod_{k\in I'} \left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)} \right), \quad r \in \tilde{I}', \tag{12}$$

and

$$c_r = \prod_{k\in\tilde{I}, k\neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)} \right), \quad r \in \tilde{I}. \tag{13}$$

Note that equations (12) and (13) are positive as expected and serve to determine the Wigner coefficients of equation (9).

The definitions of the projections $\bar{P}_0[r]$ and $\bar{P}[r]$ allow the calculation of the invariant $\bar{\rho}_{ru}$ in the following expressions [2]

$$(\bar{P}_0[u]\bar{P}[r]\bar{P}_0[u])_p{}^q = \bar{\rho}_{ru}\bar{P}_0[u]_p{}^q, \tag{14}$$

$$(P_0[u]P[r]P_0[u])_p{}^q = \rho_{ru}P_0[u]^q{}_p. \tag{15}$$

Explicitly, $\bar{\rho}_{ru}$ and ρ_{ru} are $gl(m|n-1)$ invariant operators given by

$$\begin{aligned} &\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} + 1)^{-1}(\bar{\alpha}_r - \bar{\alpha}_{0u})^{-1}\bar{c}_r\bar{\bar{\delta}}_u, \quad (u) = 1, \\ &\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} + 1)^{-1}(\bar{\alpha}_r - \bar{\alpha}_{0u})^{-1}\bar{c}_r\bar{\bar{\delta}}_u \quad (u) = 0, u \neq r, \\ &\bar{\rho}_{uu} = \bar{c}_u\bar{\bar{\delta}}_u, \quad (u) = 0, \\ &\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} - 1)^{-1}(\bar{\alpha}_r - \bar{\alpha}_{0u})^{-1}\bar{c}_r\bar{\bar{\delta}}_u \quad gl(m) \text{ case}, \end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\rho_{ru} = (\alpha_r - \alpha_{0u} + 1)^{-1}(\alpha_r - \alpha_{0u})^{-1} c_r \delta_u, \quad (u) = 1, \\
&\rho_{ru} = (\alpha_r - \alpha_{0u} + 1)^{-1}(\alpha_r - \alpha_{0u})^{-1} c_r \delta_u \quad (u) = 0, u \neq r, \\
&\rho_{uu} = c_u \delta_u, \quad (u) = 0, \\
&\rho_{ru} = (\alpha_r - \alpha_{0u} - 1)^{-1}(\alpha_r - \alpha_{0u})^{-1} c_r \delta_u \quad gl(m) \text{ case},
\end{aligned} \tag{17}$$

whose eigenvalues determine the square of $gl(m|n) : gl(m|n-1)$ reduced vector Wigner coefficients via

$$\bar{\rho}_{ru} = \left| \left\langle \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda + \varepsilon_{0_u} \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle \right|^2, \tag{18}$$

$$\rho_{ru} = \left| \left\langle \begin{array}{c} \Lambda - \varepsilon_r \\ \lambda - \varepsilon_{0_u} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle \right|^2,$$

and where

$$\bar{\delta}_u = (-1)^{|I'|} \prod_{k \in I', k \neq u} \left(\bar{\alpha_{0u}} - \bar{\alpha_{0k}} - (-1)^{(s)} \right)^{-1} \prod_{k \in \tilde{I}'} \left(\bar{\alpha}_k - \bar{\alpha_{0u}} \right), u \in I, \tag{19}$$

$$\delta_u = (-1)^{|I|} \prod_{k \in I, k \neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)} \right)^{-1} \prod_{k \in \tilde{I}} \left(\alpha_k - \alpha_{0u} \right), \quad u \in I', \tag{20}$$

are the reduced matrix elements. Note that $\bar{\rho}_{ru}$ in equation (16) is non-vanishing only when $r \in \tilde{I}'$ and $u \in I$ while ρ_{ru} in equation (17) is non-vanishing only when $r \in \tilde{I}$ and $u \in I'$.

Substituting the expressions for c_r and δ_u into equation (17) initially gives

$$\begin{aligned}
\rho_{ru} = {} & (\alpha_r - \alpha_{0u} + 1)^{-1}(\alpha_r - \alpha_{0u})^{-1} \prod_{k \in \tilde{I}, k \neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k \in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)} \right) \\
& \times (-1)^{|I|} \prod_{k \in I, k \neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)} \right)^{-1} \prod_{k \in \tilde{I}} \left(\alpha_k - \alpha_{0u} \right), \quad r \in \tilde{I}, u \in I'
\end{aligned}$$

Now $(\alpha_r - \alpha_{0u})^{-1}$ will cancel the corresponding term from δ_u while $(\alpha_r - \alpha_{0u} + 1)^{-1}$ will only cancel a term in c_r when u is odd. We therefore have

$$\begin{aligned}
\rho_{ru} = {} & (\alpha_r - \alpha_{0u} + 1)^{-1} \prod_{k \in \tilde{I}, k \neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k \in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)} \right) \\
& \times (-1)^{|I|} \prod_{k \in I, k \neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)} \right)^{-1} \prod_{k \in \tilde{I}, k \neq r} \left(\alpha_k - \alpha_{0u} \right), \quad r \in \tilde{I}, u \in I'
\end{aligned}$$

for u even and

$$\rho_{ru} = (-1)^{|I|} \prod_{k \in \tilde{I}, k \neq r} \left(\frac{\alpha_k - \alpha_{0u}}{\alpha_r - \alpha_k} \right) \prod_{k \in I, k \neq u} \left(\frac{\alpha_r - \alpha_{0k} - (-1)^{(k)}}{\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}} \right), \quad r \in \tilde{I}, u \in I_1$$

for u odd.

In the same manner, substituting the expressions for $\bar{c}_r$ and $\bar{\delta}_u$ into equation (16) gives

$$\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} + 1)^{-1} \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k)^{-1} \prod_{k\in I'} \left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}\right)$$
$$\times (-1)^{|I'|} \prod_{k\in I',k\neq u} \left(\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}\right)^{-1} \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_k - \bar{\alpha}_{0u}), \quad r \in \tilde{I}', u \in I$$

for u even and

$$\bar{\rho}_{ru} = (-1)^{|I'|} \prod_{k\in\tilde{I}',k\neq r} \left(\frac{\bar{\alpha}_k - \bar{\alpha}_{0u}}{\bar{\alpha}_r - \bar{\alpha}_k}\right) \prod_{k\in I',k\neq u} \left(\frac{\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}}{\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}}\right), \quad r \in \tilde{I}', u \in I_1$$

for u odd. Note that the expressions for ρ_{ru} and $\bar{\rho}_{ru}$ always evaluate to positive values as expected.

4 Summary

The expressions for the WCs of equations (7) and (8) together with the RWCs of equations (5) and (6) including their phases [2, 3] are then given by

$$\left\langle \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda \end{array} \middle| \begin{array}{c} \varepsilon_1 \\ 0 \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = \left[\prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k)^{-1} \prod_{k\in I'} \left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}\right) \right]^{1/2}, \quad r \in \tilde{I}' \tag{21}$$

$$\left\langle \begin{array}{c} \Lambda - \varepsilon_r \\ \lambda \end{array} \middle| \begin{array}{c} \bar{\varepsilon}_1 \\ 0 \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = \left[\prod_{k\in\tilde{I},k\neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)}\right) \right]^{1/2}, \quad r \in \tilde{I}. \tag{22}$$

$$\left\langle \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda + \varepsilon_{0_u} \end{array} \middle| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)} S(r-u)$$
$$\times \left[\frac{(-1)^{|I'|} \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_k - \bar{\alpha}_{0u}) \prod_{k\in I'} \left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}\right)}{(\bar{\alpha}_r - \bar{\alpha}_{0u} + 1) \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k) \prod_{k\in I',k\neq u} (\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)})} \right]^{1/2}, \quad r \in \tilde{I}', u \in I \tag{23}$$

$$\left\langle \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda + \varepsilon_{0_u} \end{array} \middle| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)} S(r-u)$$
$$\times \left[(-1)^{|I'|} \prod_{k\in\tilde{I}',k\neq r} \left(\frac{\bar{\alpha}_k - \bar{\alpha}_{0u}}{\bar{\alpha}_r - \bar{\alpha}_k}\right) \prod_{k\in I',k\neq u} \left(\frac{\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}}{\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}}\right) \right]^{1/2}, \quad r \in \tilde{I}', u \in I_1 \tag{24}$$

$$\left\langle \begin{array}{c} \Lambda - \varepsilon_r \\ \lambda - \varepsilon_{0_u} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)+(r)+(u)} S(r-u)$$

$$\times \left[\frac{(-1)^{|I|} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)}\right) \prod_{k\in \tilde{I}, k\neq r} (\alpha_k - \alpha_{0u})}{(\alpha_r - \alpha_u + 1) \prod_{k\in \tilde{I}, k\neq r} (\alpha_r - \alpha_k) \prod_{k\in I, k\neq u} (\alpha_{0u} - \alpha_{0k} - (-1)^{(k)})} \right]^{1/2}, \quad r \in \tilde{I}, u \in I' \tag{25}$$

$$\left\langle \begin{array}{c} \Lambda - \varepsilon_r \\ \lambda - \varepsilon_{0_u} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)+(r)+(u)} S(r-u)$$

$$\times \left[(-1)^{|I|} \prod_{k\in \tilde{I}, k\neq r} \left(\frac{\alpha_k - \alpha_{0u}}{\alpha_r - \alpha_k} \right) \prod_{k\in I, k\neq u} \left(\frac{\alpha_r - \alpha_{0k} - (-1)^{(k)}}{\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}} \right) \right]^{1/2}, \quad r \in \tilde{I}, u \in I_1 \tag{26}$$

where the positive square root is always taken, odd indices are considered greater than even indices,

$$S(x) = sgn(x), \quad S(0) = 1.$$

and where we recall

$$\begin{aligned}
\bar{\alpha}_{0i} &= i - 1 - \Lambda_i, & 1 \le i \le m-1,\\
\bar{\alpha}_{0\mu} &= \Lambda_\mu + m + 1 - \mu, & 1 \le \mu \le n-1,\\
\alpha_{0i} &= \Lambda_i + m - n + 1 - i, & 1 \le i \le m-1,\\
\alpha_{0\mu} &= \mu - \Lambda_\mu - n + 1, & 1 \le \mu \le n-1,\\
\bar{\alpha}_i &= i - 1 - \tilde{\Lambda}_i, & 1 \le i \le m,\\
\bar{\alpha}_\mu &= \tilde{\Lambda}_\mu + m + 1 - \mu, & 1 \le \mu \le n,\\
\alpha_i &= \tilde{\Lambda}_i + m - n - i, & 1 \le i \le m,\\
\alpha_\mu &= \mu - \tilde{\Lambda}_\mu - n, & 1 \le \mu \le n.
\end{aligned}$$

Note that the $gl(m|n)$ roots α and $\bar{\alpha}$ are related to the $gl(m|n-1)$ roots α_0 and $\bar{\alpha}_0$ by the substitution $n \to n-1$.

Final remarks...

Acknowledgments

This work was supported by the Australian Research Council through Discovery Project DP140101492.

References

[1] M.D. Gould, P.S. Isaac and J.L. Werry, J. Math. Phys. **54** (2013), 013505.

[2] M.D. Gould, P.S. Isaac and J.L. Werry, J. Math. Phys. **55** (2014), 011703.

[3] J.L. Werry, M.D. Gould and P.S. Isaac, J. Math. Phys. **56** (2015), 121703.

[4] I.M. Gelfand and M.L. Tsetlin, Dokl. Akad. Nauk., SSSR **71** (1950) 825 (Russian). English translation in: I.M. Gelfand, "Collected Papers", Vol II, Berlin: Springer-Verlag (1988) 653.

[5] I.M. Gelfand and M.L. Tsetlin, Dokl. Akad. Nauk., SSSR **71** (1950) 1017 (Russian). English translation in: I.M. Gelfand, "Collected Papers", Vol II, Berlin: Springer-Verlag (1988) 657.

[6] G.E. Baird and L.C. Biedenharn, J. Math. Phys. **4** (1963) 1449.

[7] N.I. Stoilova and J. Van der Jeugt, J. Math. Phys. **51** (2010) 093523.

[8] J.E. Humphreys, Introduction to Lie algebras and representation theory, volume 9 of Graduate Texts in Mathematics. Springer, New York, 1972.

[9] M.D. Gould, J. Math. Phys. **22** (1981) 15.

[10] M.D. Gould, J. Austral. Math. Soc. Ser. B **20** (1978) 401.

[11] M.D. Gould, J. Math. Phys. **21** (1980) 444.

[12] M.D. Gould, J. Math. Phys. **22** (1981) 2376.