Reduced Wigner coefficients for unitary representations of the Lie superalgebra $gl(m|n)$

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Abstract

In this paper fundamental Wigner coefficients are determined algebraically by considering the eigenvalues of certain generalized Casimir invariants. Here this method is applied in the context of both type 1 and type 2 unitary representations of the Lie superalgebra $gl(m|n)$.

1 Introduction

The application of the theory of Lie Superalgebras to problems in mathematical physics relies on the construction of an explicit set of basis states of an irreducible representation and also the explicit determination of Wigner (Clebsch-Gordan) coefficients. Prior to the development of Lie superalgerbra theory, the foundational papers by Gel'fand and Tsetlin [4, 5] gave a novel construction of basis vectors for the irreducible representations of the unitary and orthogonal groups. Further work by Baird and Biedenharn [6] gave a proof of Gel'fand and Tsetlin's results while also giving , for the first time, both the fundamental and reduced Wigner coefficients for the Lie group U(n). In the same paper, the factorization of a matrix element into a Wigner coefficient and a reduced matrix element was also examined.

The utility of the characteristic identities (polynomial identities satisfied by generators of the Lie algebra) towards these goals became apparent during the 1970s and 80s and resulted in the algebraic determination of reduced matrix elements, [9] raising and lowering generators [10] and matrix elements, [8, 11]. In the papers [2, 3] the matrix elements of unitary representations of $gl(m|n)$ were given explicitly using similar techniques. The resulting closed-form expressions were obtained by utilizing the factorization of a matrix element into a Wigner coefficient and a reduced matrix element. The vector Wigner coefficients thus obtained allow us in this paper to obtain all fundamental $gl(m|n)$ Wigner coefficients (WCs) in the Gelfand-Tsetlin (GT) basis for both type 1 and type 2 unitary representations. Although the resulting expressions for these Reduced Wigner coefficients have appeared in [7], in this presentation we obtain our results from first principles using algebraic methods that can be directly generalised to other Lie algebras as well as the quantum case.

2 Characteristic identities and associated invariants

We utilise the notation used in the series [1–3]. The generators of the Lie superalgebra $gl(m|n)$ are denoted E_{ij} where $1 \leq p, q \leq m + n$. The values $1 \leq i, j \leq m$ are called *even* while the values $m < i, j \leq m + n$ are called *odd*.

The graded index notation will be used where the Latin indices $i, j, ...$ are used for even quantities and the Greek indices $\mu, \nu, ...$ for odd quantities. The grading operator ( ) will then give

$$(i) = 0,\ (\mu) = 1$$

and then set of generators is then given by

$$E_{ij}, E_{i\mu}, E_{\mu i}, E_{\mu\nu}.$$

We reserve the indices p, q to be ungraded and to range fully from 1 to $m+n$.

The adjoint tensor operator $\bar{\mathcal{A}}$ constructed in [1] plays an important role in what follows and is defined as the $m+n$ square matrix with entries

$$\bar{\mathcal{A}}_{pq} = -(-1)^{(p)(q)} E_{qp}.$$

When $\bar{\mathcal{A}}$ acts on an irreducible $gl(m|n)$ module $V(\Lambda)$ of highest weight Λ it will satisfy the characteristic identity

$$\prod_{i=1}^{m}(\bar{\mathcal{A}} - \bar{\alpha}_i) \prod_{\mu=1}^{n}(\bar{\mathcal{A}} - \bar{\alpha}_\mu) = 0 \tag{1}$$

where the adjoint roots $\bar{\alpha}_i, \bar{\alpha}_\mu$ are given in terms of the highest weight labels

$$\Lambda = (\Lambda_{i=1}, ..., \Lambda_{i=m}|\Lambda_{\mu=1}, ...\Lambda_{\mu=n})$$

as

$$\bar{\alpha}_i = i - 1 - \Lambda_i, \quad \bar{\alpha}_\mu = \Lambda_\mu + m + 1 - \mu.$$

Immediately from the characteristic identity, we see that for each integer r where $1 \leq r \leq m+n$ there exists a projection operator

$$\bar{P}[r] : V(\varepsilon_1) \otimes V(\Lambda) \longrightarrow V(\Lambda + \epsilon_r)$$

given by

$$\bar{P}[r] = \prod_{k \neq r}^{m+n} \left(\frac{\bar{\mathcal{A}} - \bar{\alpha}_k}{\bar{\alpha}_r - \bar{\alpha}_k} \right).$$

Similarly we have the vector matrix $\mathcal{A}$ with entries

$$\mathcal{A}_{pq} = -(-1)^{(p)} E_{pq}.$$

that satisfy the polynomial identities

$$\prod_{i=1}^{m}(\mathcal{A} - \alpha_i) \prod_{\mu=1}^{n}(\mathcal{A} - \alpha_\mu) = 0 \tag{2}$$

where

$$\alpha_i = \Lambda_i + m - n - i, \quad \alpha_\mu = \mu - \Lambda_\mu - n.$$

The associated projection operator

$$P[r] : V^*(\varepsilon_1) \otimes V(\Lambda) \longrightarrow V(\Lambda - \epsilon_r)$$

is then given by

$$P[r] = \prod_{k \neq r}^{m+n} \left(\frac{\mathcal{A} - \alpha_k}{\alpha_r - \alpha_k} \right).$$

We will now show that the eigenvalues of the invariants

$$\bar{c}_r = \bar{P}[r]_{m+n}^{\ m+n}$$

and

$$c_r = P[r]^{m+n}_{\ m+n}$$

are essentially squares of reduced Wigner coefficients.

We note that there is an invariant, nondegenerate sesquilinear form on the tensor product space $V(\Lambda) \otimes V(\mu)$ induced by the natural forms on $V(\Lambda)$ and $V(\mu)$ [1]. This induced form is necessarily unique by Schur's lemma. The restriction of this form to the isotypic component of a given typical submodule is nondegenerate, which allows the definition of Wigner coefficients with respect to the form. Note also, that in the case when $V(\Lambda)$ and $V(\mu)$ both are type 1 unitary or type 2 unitary modules then the induced form gives rise to an inner product. However, when we have a type 1 unitary module tensored with a type 2 unitary module, we may still define Wigner coefficients even though the form induced by the tensor product space is no longer positive definite.

Let e_i denote the Gelfand-Tsetlin (GT) basis states for the vector module $V(\varepsilon_1)$ and e^Λ_β denote the (GT) basis states for the irreducible module $V(\Lambda)$ of highest weight Λ. Then the matrix elements of $\bar{P}[r]$ can be given in the form

$$\left\langle e^\Lambda_\beta | \bar{P}[r]^j_i | e^\Lambda_\alpha \right\rangle = \sum_\gamma \left\langle e^\Lambda_\beta \otimes e_i | e^{\Lambda+\varepsilon_r}_\gamma \right\rangle \left\langle e^{\Lambda+\varepsilon_r}_\gamma | e_j \otimes e^\Lambda_\alpha \right\rangle, \tag{3}$$

where

$$\left\langle e^{\Lambda+\varepsilon_r}_\gamma | e_j \otimes e^\Lambda_\alpha \right\rangle,$$

are the vector (fundamental) Wigner coefficients. Similarly by denoting $\bar{e}_i$ to be the Gelfand-Tsetlin (GT) basis states of the dual vector module we have

$$\left\langle e^\Lambda_\beta | P[r]^j_i | e^\Lambda_\alpha \right\rangle = \sum_\gamma \left\langle e^\Lambda_\beta \otimes \bar{e}_i | e^{\Lambda-\varepsilon_r}_\gamma \right\rangle \left\langle e^{\Lambda-\varepsilon_r}_\gamma | \bar{e}_j \otimes e^\Lambda_\alpha \right\rangle, \tag{4}$$

where

$$\left\langle e^{\Lambda-\varepsilon_r}_\gamma | \bar{e}_j \otimes e^\Lambda_\alpha \right\rangle,$$

are the dual fundamental Wigner coefficients. The above quantities may be expressed in terms of the $gl(m|n)$ highest weight labels of Λ and the highest weight labels of the $gl(m|n-1)$ subalgebra.

From Schur's lemma we observe that the fundamental WCs factorize as follows

$$\left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda+\varepsilon_{0_r} \\ [\Lambda_0'] \end{array} \right| e_i \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \left.\right\rangle = \left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda+\varepsilon_{0_r} \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left.\right\rangle \left\langle \begin{array}{c} \lambda+\varepsilon_{0_r} \\ [\Lambda_0'] \end{array} \right| e_i \otimes \begin{array}{c} \lambda \\ [\Lambda_0] \end{array} \left.\right\rangle, \tag{5}$$

$$\left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda-\varepsilon_{0_r} \\ [\Lambda_0'] \end{array} \right| \bar{e}_i \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \left.\right\rangle = \left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda-\varepsilon_{0_r} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left.\right\rangle \left\langle \begin{array}{c} \lambda-\varepsilon_{0_r} \\ [\Lambda_0'] \end{array} \right| \bar{e}_i \otimes \begin{array}{c} \lambda \\ [\Lambda_0] \end{array} \left.\right\rangle, \tag{6}$$

$$1 < i < m+n,$$

where $1 < k \leq m+n$, $1 < r \leq m+n-1$, Λ denotes the highest weight of $gl(m|n)$, λ denotes the highest weight of $gl(m|n-1)$ and $[\Lambda_0]$ denotes the GT pattern of the $gl(m|n-2)$ subalgebra. In addition, the first term on the rhs of equations (5) and (6) is a reduced Wigner coeficient (RWC) which is independent of the highest weight labels of $[\Lambda_0]$ and $[\Lambda_0']$.

Setting $i = m+n$ gives

$$\left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda \\ [\Lambda_0'] \end{array} \right| e_{m+n} \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \left.\right\rangle = \delta_{[\Lambda_0'][\Lambda]} \left\langle \begin{array}{c} \Lambda+\varepsilon_k \\ \lambda \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left.\right\rangle, \tag{7}$$

and

$$\left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda \\ [\Lambda_0'] \end{array} \right| \bar{e}_{m+n} \otimes \begin{array}{c} \Lambda \\ \lambda \\ [\Lambda_0] \end{array} \left.\right\rangle = \delta_{[\Lambda_0'][\Lambda]} \left\langle \begin{array}{c} \Lambda-\varepsilon_k \\ \lambda \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left.\right\rangle, \tag{8}$$

where the RWC on the rhs is independent of $[\Lambda_0]$ and $[\Lambda_0']$ as expected. The WCs in equations (7) and (8) are given by the eigenvalues of the invariants

$$\bar{c}_r = \bar{P}[r]_{m+n}^{m+n}, \quad c_r = P[r]^{m+n}_{\ \ m+n}$$

since from equations (3) and (4) we have

$$\bar{c}_r = \left| \left\langle \begin{array}{c} \Lambda+\varepsilon_r \\ \lambda \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left.\right\rangle \right|^2,$$

$$c_r = \left| \left\langle \begin{array}{c} \Lambda-\varepsilon_r \\ \lambda \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \dot{0} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \left.\right\rangle \right|^2, \tag{9}$$

where $\bar{e}_r$ denote the Gelfand-Tsetlin (GT) basis states of the adjoint vector module.

We therefore see that the matrix elements of the projections $\bar{P}[r]_{m+n}^{m+n}$ and $P[r]^{m+n}_{\ \ m+n}$ determine the squares of the fundamental Wigner coefficients and depend only on the top two rows of the corresponding Gelfand-Tsetlin basis states.

3 Reduced Wigner coefficients

The above derivations may be repeated for the $gl(m|n-1)$ case so that we have tensor operators given by the $m+n-1$ square matrices

$$\bar{\mathcal{A}}_{0pq} = -(-1)^{(p)(q)}E_{qp}, \qquad \mathcal{A}_{0pq} = -(-1)^{(p)}E_{pq}.$$

that satisfy the usual polynomial identities

$$\prod_{i=1}^{m}(\bar{\mathcal{A}}_0 - \bar{\alpha}_{0i})\prod_{\mu=1}^{n+1}(\bar{\mathcal{A}}_0 - \bar{\alpha}_{0\mu}) = 0$$
$$\prod_{i=1}^{m}(\mathcal{A}_0 - \alpha_{0i})\prod_{\mu=1}^{n+1}(\mathcal{A}_0 - \alpha_{0\mu}) = 0$$

with roots given by

$$\begin{aligned}
\bar{\alpha}_{0i} &= i - 1 - \lambda_i, & 1 \le i \le m,\\
\bar{\alpha}_{0\mu} &= \lambda_\mu + m + 1 - \mu, & 1 \le \mu \le n-1,\\
\alpha_{0i} &= \lambda_i + m - n + 1 - i, & 1 \le i \le m,\\
\alpha_{0\mu} &= \mu - \lambda_\mu - n + 1, & 1 \le \mu \le n-1.
\end{aligned}$$

Similarly, the $gl(m|n+1)$ projection operators are given by

$$\bar{P}_0[r] = \prod_{k\neq r}^{m+n+1}\left(\frac{\bar{\mathcal{A}}_0 - \bar{\alpha}_{0k}}{\bar{\alpha}_{0r} - \bar{\alpha}_{0k}}\right), \quad P_0[r] = \prod_{k\neq r}^{m+n+1}\left(\frac{\mathcal{A}_0 - \alpha_{0k}}{\alpha_{0r} - \alpha_{0k}}\right), \tag{10}$$

The betweenness conditions imply [1], for $1 \le i \le m$, that we have only two cases

$$\alpha_i = \begin{cases} \alpha_{0i}, & \Lambda_i = 1 + \lambda_i \\ \alpha_{0i} - 1, & \Lambda_i = \lambda_i \end{cases}$$

We therefore define the following index sets

$$\begin{aligned}
I_0 &= \{1 \le i \le m \mid \alpha_{0i} = \alpha_i\},\\
\bar{I}_0 &= \{1 \le i \le m \mid \alpha_{0i} = 1 + \alpha_i\},\\
I_1 &= \{1 \le \mu \le n-1\},\\
I &= I_0 \cup I_1,\\
I' &= \bar{I}_0 \cup I_1,\\
\tilde{I} &= I \cup \{m+n\},\\
\tilde{I}' &= I' \cup \{m+n\}.
\end{aligned} \tag{11}$$

By considering the characteristic identities satisfied by the invariants $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}_0$ we may obtain the invariant $\bar{c}_r$ as a rational polynomial in terms of the roots $\bar{\alpha}_i, \bar{\alpha}_\mu$ and $\bar{\alpha}_{0i}, \bar{\alpha}_{0\mu}$. Specifically, we have [1]

$$\bar{c}_r = \prod_{k\in\tilde{I}',k\neq r}(\bar{\alpha}_r - \bar{\alpha}_k)^{-1}\prod_{k\in I'}\left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}\right), \quad r \in \tilde{I}', \tag{12}$$

and

$$c_r = \prod_{k\in\tilde{I},k\neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)}\right), \quad r \in \tilde{I} \tag{13}$$

Note that equations (12) and (13) are positive as expected and serve to determine the Wigner coefficients of equation (9).

The definitions of the projections $\bar{Q}[r]$ and $\bar{P}[r]$ allow the calculation of the invariant $\bar{\rho}_{ru}$ in the following expressions [2]

$$(\bar{P}_0[u]\bar{P}[r]\bar{P}_0[u])_p{}^q = \bar{\rho}_{ru}\bar{P}_0[u]_p{}^q, \tag{14}$$
$$(P_0[u]P[r]P_0[u])_p{}^q = \rho_{ru}P_0[u]_p{}^q \tag{15}$$

where $\bar{\rho}_{ru}$ and ρ_{ru} are $gl(m|n-1)$ invariant operators given by

$$\begin{aligned}
&\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} + 1)^{-1}(\bar{\alpha}_r - \bar{\alpha}_{0u})^{-1}\bar{c}_r\bar{\delta}_u, \quad (u) = 1,\\
&\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} + 1)^{-1}(\bar{\alpha}_r - \bar{\alpha}_{0u})^{-1}\bar{c}_r\bar{\delta}_u \quad (u) = 0, u \neq r,\\
&\bar{\rho}_{uu} = \bar{c}_u\bar{\delta}_u, \quad (u) = 0,\\
&\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} - 1)^{-1}(\bar{\alpha}_r - \bar{\alpha}_{0u})^{-1}\bar{c}_r\bar{\delta}_u \quad gl(m) \text{ case},
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&\rho_{ru} = (\alpha_r - \alpha_{0u} + 1)^{-1}(\alpha_r - \alpha_{0u})^{-1}c_r\delta_u, \quad (u) = 1,\\
&\rho_{ru} = (\alpha_r - \alpha_{0u} + 1)^{-1}(\alpha_r - \alpha_{0u})^{-1}c_r\delta_u \quad (u) = 0, u \neq r,\\
&\rho_{uu} = c_u\delta_u, \quad (u) = 0,\\
&\rho_{ru} = (\alpha_r - \alpha_{0u} - 1)^{-1}(\alpha_r - \alpha_{0u})^{-1}c_r\delta_u \quad gl(m) \text{ case},
\end{aligned} \tag{17}$$

whose eigenvalues determine the square of $gl(m|n) : gl(m|n-1)$ reduced vector Wigner coefficients via

$$\bar{\rho}_{ru} = \left|\left\langle \begin{array}{c} \Lambda + \varepsilon_r \\ \lambda + \varepsilon_{0_u} \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle\right|^2, \tag{18}$$
$$\rho_{ru} = \left|\left\langle \begin{array}{c} \Lambda - \varepsilon_r \\ \lambda - \varepsilon_{0_u} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle\right|^2,$$

and where

$$\bar{\delta}_u = (-1)^{|I'|} \prod_{k\in I',k\neq u} \left(\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(s)}\right)^{-1} \prod_{k\in\tilde{I}'} (\bar{\alpha}_k - \bar{\alpha}_{0u}), u \in I, \tag{19}$$
$$\delta_u = (-1)^{|I|} \prod_{k\in I,k\neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}\right)^{-1} \prod_{k\in\tilde{I}} (\alpha_k - \alpha_{0u}), \quad u \in I', \tag{20}$$

are the reduced matrix elements. Note that $\bar{\rho}_{ru}$ in equation (16) is non-vanishing only when $r \in \tilde{I}'$ and $u \in I$ while ρ_{ru} in equation (17) is non-vanishing only when $r \in \tilde{I}$ and $u \in I'$.

Substituting the expressions for c_r and δ_u into equation (17) initially gives

$$\rho_{ru} = (\alpha_r - \alpha_{0u} + 1)^{-1}(\alpha_r - \alpha_{0u})^{-1} \prod_{k\in\tilde{I},k\neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)}\left(\right)\right.$$
$$\times (-1)^{|I|} \prod_{k\in I,k\neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}\right)^{-1} \prod_{k\in\tilde{I}} (\alpha_k - \alpha_{0u}) \,, \quad r\in\tilde{I}, u\in I'$$

Now $(\alpha_r - \alpha_{0u})^{-1}$ will cancel the corresponding term from δ_u while $(\alpha_r - \alpha_{0u} + 1)^{-1}$ will only cancel a term in c_r when u is odd. We therefore have

$$\rho_{ru} = (\alpha_r - \alpha_{0u} + 1)^{-1} \prod_{k\in\tilde{I},k\neq r} (\alpha_r - \alpha_k)^{-1} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)}\right)$$
$$\times (-1)^{|I|} \prod_{k\in I,k\neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}\right)^{-1} \prod_{k\in\tilde{I},k\neq r} (\alpha_k - \alpha_{0u}) \,, \quad r\in\tilde{I}, u\in I'$$

for u even and

$$\rho_{ru} = (-1)^{|I|} \prod_{k\in\tilde{I},k\neq r} \left(\frac{\alpha_k - \alpha_{0u}}{\alpha_r - \alpha_k}\right) \prod_{k\in I,k\neq u} \left(\frac{\alpha_r - \alpha_{0k} - (-1)^{(k)}}{\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}}\right), \quad r\in\tilde{I}, u\in I_1$$

for u odd.

In the same manner, substituting the expressions for $\bar{c}_r$ and $\bar{\delta}_u$ into equation (16) gives

$$\bar{\rho}_{ru} = (\bar{\alpha}_r - \bar{\alpha}_{0u} + 1)^{-1} \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k)^{-1} \prod_{k\in I'} \left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}\right)$$
$$\times (-1)^{|I'|} \prod_{k\in I',k\neq u} \left(\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}\right)^{-1} \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_k - \bar{\alpha}_{0u}) \,, \quad r\in\tilde{I}', u\in I$$

for u even and

$$\bar{\rho}_{ru} = (-1)^{|I'|} \prod_{k\in\tilde{I}',k\neq r} \left(\frac{\bar{\alpha}_k - \bar{\alpha}_{0u}}{\bar{\alpha}_r - \bar{\alpha}_k}\right) \prod_{k\in I',k\neq u} \left(\frac{\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}}{\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}}\right), \quad r\in\tilde{I}', u\in I_1$$

for u odd. Note that the expressions for ρ_{ru} and $\bar{\rho}_{ru}$ always evaluate to positive values.

4 Summary

The expressions for the RWCs of equations (5) and (6) together with their phases [2,3] are then given by

$$\left\langle \begin{array}{c} \Lambda+\varepsilon_r \\ \lambda+\varepsilon_{0_u} \end{array} \middle| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)} S(r-u)$$
$$\times \left[\frac{(-1)^{|I'|} \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_k - \bar{\alpha}_{0u}) \prod_{k\in I'} \left(\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}\right)}{(\bar{\alpha}_r - \bar{\alpha}_{0u} + 1) \prod_{k\in\tilde{I}',k\neq r} (\bar{\alpha}_r - \bar{\alpha}_k) \prod_{k\in I',k\neq u} \left(\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}\right)} \right]^{1/2}, \quad r\in\tilde{I}', u\in I \tag{21}$$

$$\left\langle \begin{array}{c} \Lambda+\varepsilon_r \\ \lambda+\varepsilon_{0_u} \end{array} \right| \begin{array}{c} \varepsilon_1 \\ \varepsilon_{0_1} \end{array} ; \left. \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)} S(r-u)$$

$$\times \left[(-1)^{|I'|} \prod_{k\in \tilde{I}', k\neq r} \left(\frac{\bar{\alpha}_k - \bar{\alpha}_{0u}}{\bar{\alpha}_r - \bar{\alpha}_k} \right) \prod_{k\in I', k\neq u} \left(\frac{\bar{\alpha}_r - \bar{\alpha}_{0k} - (-1)^{(k)}}{\bar{\alpha}_{0u} - \bar{\alpha}_{0k} - (-1)^{(k)}} \right) \right]^{1/2}, \quad r \in \tilde{I}', u \in I_1 \tag{22}$$

$$\left\langle \begin{array}{c} \Lambda-\varepsilon_r \\ \lambda-\varepsilon_{0_u} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \left. \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)+(r)+(u)} S(r-u)$$

$$\times \left[\frac{(-1)^{|I|} \prod_{k\in I} \left(\alpha_r - \alpha_{0k} - (-1)^{(k)}\right) \prod_{k\in \tilde{I}, k\neq r} (\alpha_k - \alpha_{0u})}{(\alpha_r - \alpha_u + 1) \prod_{k\in \tilde{I}, k\neq r} (\alpha_r - \alpha_k) \prod_{k\in I, k\neq u} \left(\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}\right)} \right]^{1/2}, \quad r \in \tilde{I}, u \in I' \tag{23}$$

$$\left\langle \begin{array}{c} \Lambda-\varepsilon_r \\ \lambda-\varepsilon_{0_u} \end{array} \right| \begin{array}{c} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_{0_1} \end{array} ; \left. \begin{array}{c} \Lambda \\ \lambda \end{array} \right\rangle = (-1)^{(r)(u)+(r)+(u)} S(r-u)$$

$$\times \left[(-1)^{|I|} \prod_{k\in \tilde{I}, k\neq r} \left(\frac{\alpha_k - \alpha_{0u}}{\alpha_r - \alpha_k} \right) \prod_{k\in I, k\neq u} \left(\frac{\alpha_r - \alpha_{0k} - (-1)^{(k)}}{\alpha_{0u} - \alpha_{0k} - (-1)^{(k)}} \right) \right]^{1/2}, \quad r \in \tilde{I}, u \in I_1 \tag{24}$$

where the positive square root is always taken, odd indices are considered greater than even indices,

$$S(x) = sgn(x), \quad S(0) = 1.$$

and where we recall

$$\begin{aligned}
\bar{\alpha}_{0i} &= i - 1 - \Lambda_i, & 1 \le i \le m-1,\\
\bar{\alpha}_{0\mu} &= \Lambda_\mu + m + 1 - \mu, & 1 \le \mu \le n-1,\\
\alpha_{0i} &= \Lambda_i + m - n + 1 - i, & 1 \le i \le m-1,\\
\alpha_{0\mu} &= \mu - \Lambda_\mu - n + 1, & 1 \le \mu \le n-1,\\
\bar{\alpha}_i &= i - 1 - \tilde{\Lambda}_i, & 1 \le i \le m,\\
\bar{\alpha}_\mu &= \tilde{\Lambda}_\mu + m + 1 - \mu, & 1 \le \mu \le n,\\
\alpha_i &= \tilde{\Lambda}_i + m - n - i, & 1 \le i \le m,\\
\alpha_\mu &= \mu - \tilde{\Lambda}_\mu - n, & 1 \le \mu \le n.
\end{aligned}$$

To guarantee the existence of an orthonormal basis it is essential to utilize equations (21) and (22) when Λ is the highest weight of a type 1 (covariant) unitary module and equations (23) and (24) when Λ is the highest weight of a type 2 (contravariant) unitary module. It is important and interesting to note that these final equations may still be used to give Wigner coefficients in the mixed tensor case since we have a unique invariant nondegenerate sesquilinear form under which the construction of Wigner coefficients is possible.

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