Let r_1 and r_2 be uniformly distributed numbers. If we consider a small rectangle of side lengths $dr_1\ dr_2$, around point $r_1'\ r_2'$

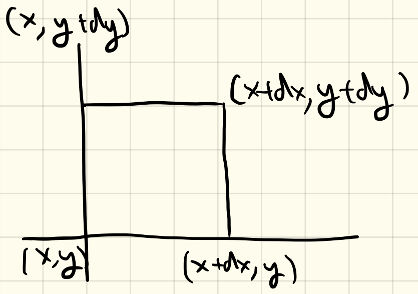
then the expected number of points in the rectangle is $N \cdot P(r_1', r_2')\, dr_1 dr_2$

where P is the probability density, and N is the total number of points

Now, if the points are individually transformed by $y_1 = (-2\ln r_1)^{1/2}\cos(2\pi r_2)$ and $y_2 = (-2\ln r_1)^{1/2}\sin(2\pi r_2)$, then the new probability density will be $P'(y_1, y_2)$

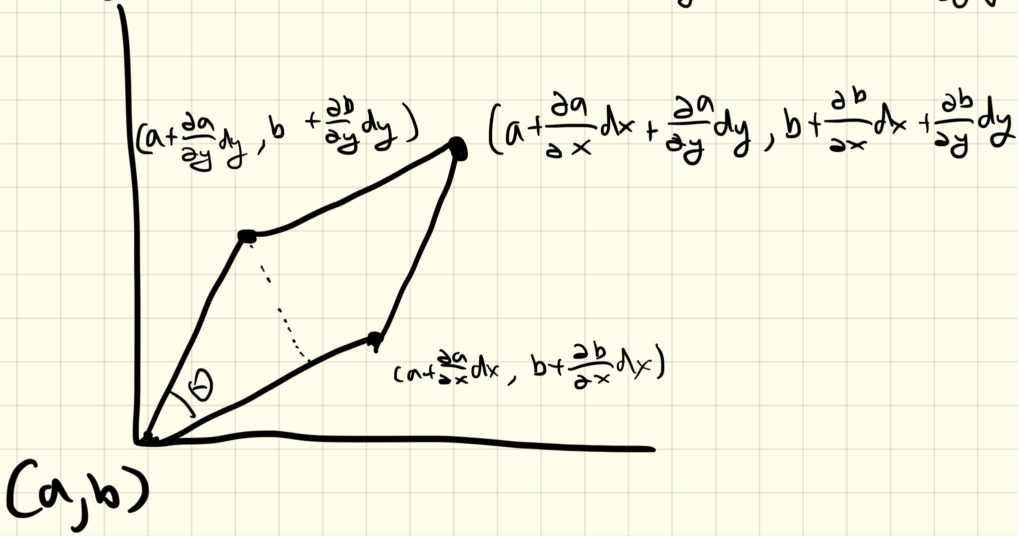
Then, we can get the same number of expected points by taking $N \cdot P'(y_1, y_2)\, dr_1 dr_2 \cdot S$ where S is the area scaling factor

To find S, consider a rectangle with vertices



The rectangle is linearly transformed into a parallelogram using the approximation

$$\left(a(x+dx,y+dy),\ b(x+dx,y+dy)\right) \approx \left(a(x,y)+\frac{\partial a}{\partial x}dx+\frac{\partial a}{\partial y}dy,\ b(x,y)+\frac{\partial b}{\partial x}dx+\frac{\partial b}{\partial y}dy\right)$$



To find the area of this new parallelogram, we imagine a parallelogram made of two vectors $\vec{u}=(a,b)$ and $\vec{v}=(c,d)$. The angle between them is

$$\arccos\left(\frac{ac+bd}{\sqrt{(a^2+b^2)(c^2+d^2)}}\right)$$ using the fact that $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$

Since the area of a parallelogram is $|\vec{u}||\vec{v}|\sin\theta$, we need to find $\sin\theta$. $\sin\theta=\sqrt{1-\cos^2\theta}=$

$$\sqrt{\frac{(a^2+b^2)(c^2+d^2)-(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}}=\frac{|ad-bc|}{\sqrt{(a^2+b^2)(c^2+d^2)}}$$

Applying this formula to our parallelogram made of two vectors $\vec{u}=\left(\frac{\partial a}{\partial x}dx,\ \frac{\partial b}{\partial x}dx\right)$ and $\vec{v}=\left(\frac{\partial a}{\partial y}dy,\ \frac{\partial b}{\partial y}dy\right)$, we get that the area is

$$A=\left|\frac{\partial a}{\partial x}\frac{\partial b}{\partial y}-\frac{\partial b}{\partial x}\frac{\partial a}{\partial y}\right|dx\,dy=\left|\frac{\partial(a,b)}{\partial(x,y)}\right|dx\,dy$$

Going back to the original problem, we now know that the area scaling factor S is the jacobian determinant $\left|\frac{\partial(h_1, h_2)}{\partial(r_1, r_2)}\right|$.

Since $N \cdot P(r_1, r_2) dr_1 dr_2 = N \cdot P'(h_1, h_2) dr_1 dr_2 \left|\frac{\partial(h_1, h_2)}{\partial(r_1, r_2)}\right|$,

we can conclude that $P'(h_1, h_2) = \underbrace{P(r_1, r_2)}_{\text{uniform, so} = 1} \left|\frac{\partial(h_1, h_2)}{\partial(r_1, r_2)}\right|^{-1}$

So, the P' is just the inverse jacobian determinant $J^{-1} = \left| \frac{\partial(r_1, r_2)}{\partial(\zeta_1, \zeta_2)} \right|^{-1}$.

All that remains is to find it explicitly

$$J = \left(\frac{-\cos(2\pi r_2)}{\sqrt{2} r_1 \sqrt{-\ln(r_1)}} \cdot 2\sqrt{2}\pi \sqrt{-\ln(r_1)} \cos(2\pi r_2) \right)$$

$$- \left(-2\sqrt{2}\pi \sqrt{-\ln(r_1)} \sin(2\pi r_2) \cdot \frac{-\sin(2\pi r_2)}{\sqrt{2} r_1 \sqrt{-\ln(r_1)}} \right)$$

$$= \left(\frac{-2\pi \cos^2(2\pi r_2)}{r_1} - \frac{2\pi \sin^2(2\pi r_2)}{r_1} \right)$$

$$= \frac{-2\pi}{r_1}$$

And its inverse is $J^{-1} = \frac{-r_1}{2\pi}$

From the original transformation we get $r_1 = \exp\left[-\frac{\zeta_1^2 + \zeta_2^2}{2} \right]$

And plugging r_1 in we get $J^{-1} = \frac{1}{2\pi} \exp\left[-\frac{\zeta_1^2 + \zeta_2^2}{2} \right]$

Since ζ_1 and ζ_2 are unrelated variables, $P'(\zeta_1, \zeta_2) = P'(\zeta_1) P'(\zeta_2)$

Since they're identically transformed,

$$P'(\zeta_1) = P'(\zeta_2) = \sqrt{J^{-1}} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\zeta^2}{2} \right)$$