PHYS516 ASSIGNMENT 2 — MONTE CARLO BASICS
Due January 27 (Fri), 2023

Submit to Blackboard by 11:59 pm. Please create a single file (*e.g.*, in PDF format) that has all materials (source code, plots and explanation) and your name in it.

Part I—Programming: Testing the Central Theorem of Monte Carlo Estimate

In this assignment, you will numerically test the dependence of the Monte Carlo (MC) error,

$$\mathrm{Std}\{\overline{f(x)}\} = \frac{\mathrm{Std}\{f(x)\}}{\sqrt{M}},$$

on the sample size M in the sample mean integral of π, mean.c (see §1 in the lecture note on "Monte Carlo Basics"):

$$\frac{1}{M}\sum_{n=1}^{M}\frac{4}{1+r_n^2} = \overline{\frac{4}{1+r_n^2}} \approx \pi \quad (r_n \in [0,1]).$$

(Assignment)

1. (**Monte Carlo estimate**) Plot your MC estimate of π *along with an error bar* using the unbiased estimate of its standard deviation,

$$\sqrt{\frac{\overline{f^2} - \left(\bar{f}\right)^2}{M-1}}$$

(where $f(r_n) = 4/(r_n^2+1)$) as a function of $\log_{10}M$ for M = 10, 10^2, ..., 10^6. ***Submit the source code and the plot***.

2. (**Monte Carlo error**) We next perform a numerical experiment to directly measure the standard deviation of the MC estimate. To do so, for each of the above M values, estimate π for N_{seed} times using N_{seed} different random-number seeds (use N_{seed} = 100). Calculate the standard deviation σ_M of these N_{seed} estimates, $\pi_1, \pi_2, \ldots, \pi_{N_{\mathrm{seed}}}$:

$$\sigma_M = \sqrt{\frac{1}{N_{\mathrm{seed}}}\sum_{i=1}^{N_{\mathrm{seed}}}\pi_i^2 - \left[\frac{1}{N_{\mathrm{seed}}}\sum_{i=1}^{N_{\mathrm{seed}}}\pi_i\right]^2}.$$

Plot the measured values of $\log_{10}\sigma_M$ as a function of $\log_{10}M$ for M = 10, 10^2, ..., 10^6, along with its unbiased estimate from question 1 above (are they similar?). If the MC error decreases as $\sigma = C/\sqrt{M}$ (C is the standard deviation of the same quantity in the underlying population), then

$$\log_{10}\sigma_M = \log_{10}C - \frac{1}{2}\log_{10}M = \log_{10}C + p\log_{10}M$$

so that you can fit your data to a line with slope, or power $p = -0.5$. Use the least square fit (see the lecture note on "Least square fit of a line") to obtain the power p in your plot of σ_M measurement. ***Submit the source code, the plot, and the estimated value of the power***.

Part II—Derivation: Nonuniform Random Number Generation by Transformation

Submit the answer to the following question. Include all the algebra and proof steps, and explain what they mean *in your own words* (as you practiced in assignment 1).

(Assignment) Prove that the Box-Muller algorithm below generates a normally distributed random number, following the lecture slides on "Monte Carlo Basics".

Box-Muller algorithm: Generates a normally distributed random number ζ with unit variance,

$$\rho(\zeta) = \sqrt{\frac{1}{2\pi}}\exp\left(-\frac{\zeta^2}{2}\right).$$ [1]

(a) Generate uniform random numbers r_1 and r_2 in the range (0, 1)

(b) Calculate $\zeta_1 = (-2\ln r_1)^{1/2}\cos(2\pi r_2)$ and $\zeta_2 = (-2\ln r_1)^{1/2}\sin(2\pi r_2)$

Then both ζ_1 and ζ_2 are the desired normally distributed random number, and either can be used as ζ.

[1] For the normalization of the normal distribution function, see Appendix A on p. 21 of the lecture note on "Monte Carlo basics". For its variance, see the Problem in p. 23 of the same lecture note.