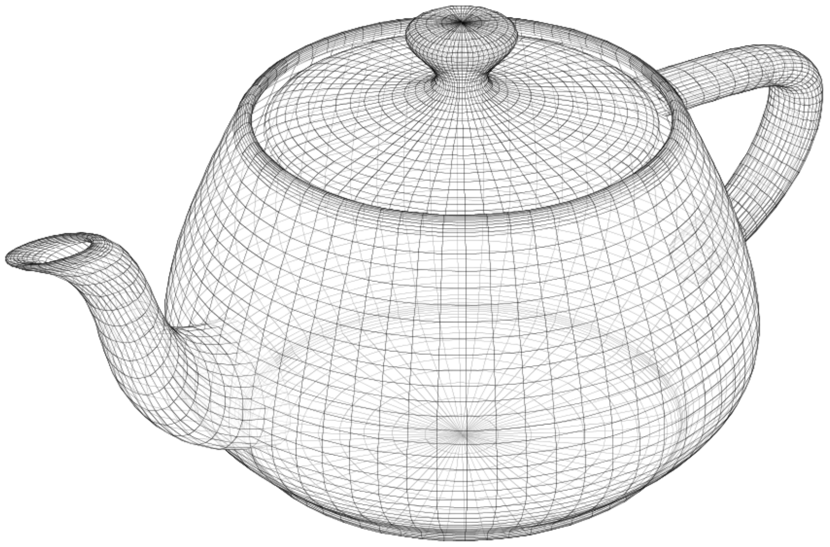

Euler's Method

Interactive Computer Graphics

Professor Eric Shaffer

ILLINOIS

Flow Fields and Differential Equations

A vector field can be thought of as a field of velocities

Each vector describes velocity at a point…velocity is first derivative of position

Flow of a vector field tells us where a particle ends up

- after a certain time
- given a particular starting point

Requires the solution of an ordinary differential equation (ODE)

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \vec{v}(\mathbf{x}(u))du$$

Analytical solution generally does not exist…need to use a numerical method

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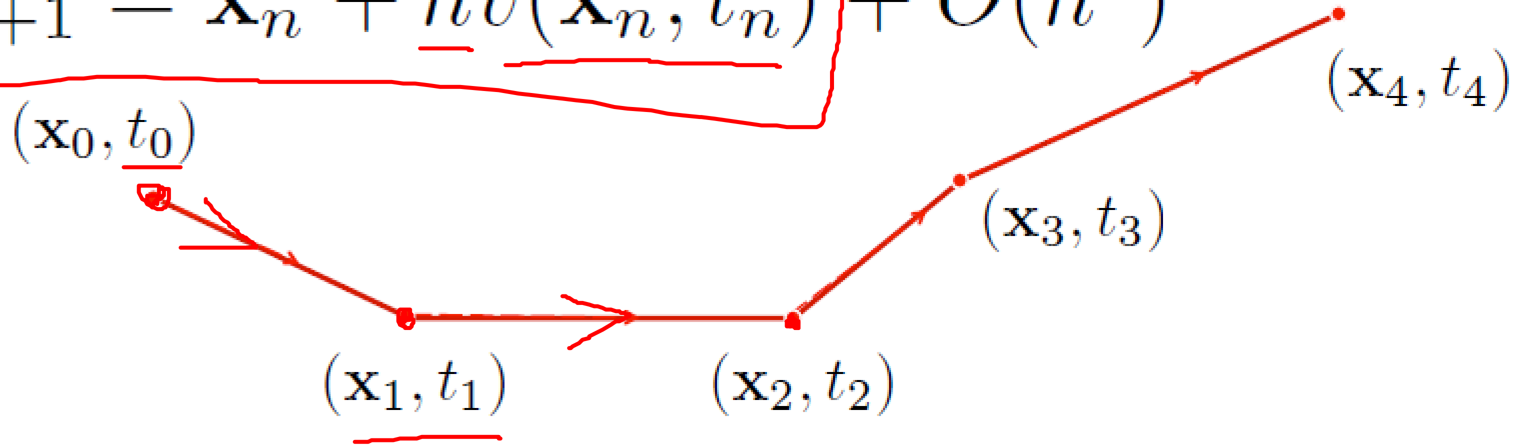
Euler's Method

$$\boxed{\mathbf{x}_{n+1} = \mathbf{x}_n + h\vec{v}(\mathbf{x}_n, t_n)} + O(h^2)$$



- Simple
- Fast
- Inaccurate
- Unstable

But beloved in computer graphics where things just need to look good…in visualization they should probably be accurate as well.

Understanding Error

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h\vec{v}(\mathbf{x}_n, t_n) + O(h^2)$$

Two sources of error

1. Rounding error
    - due to the finite precision of floating-point arithmetic
2. Truncation error (or discretization error)
    - e.g. approximating an infinite process with a finite number of steps

- For ODEs truncation error is usually dominant
- The two types error are not independent
- Reducing the step-size will typically reduce truncation error but may increase rounding error

Truncation Error

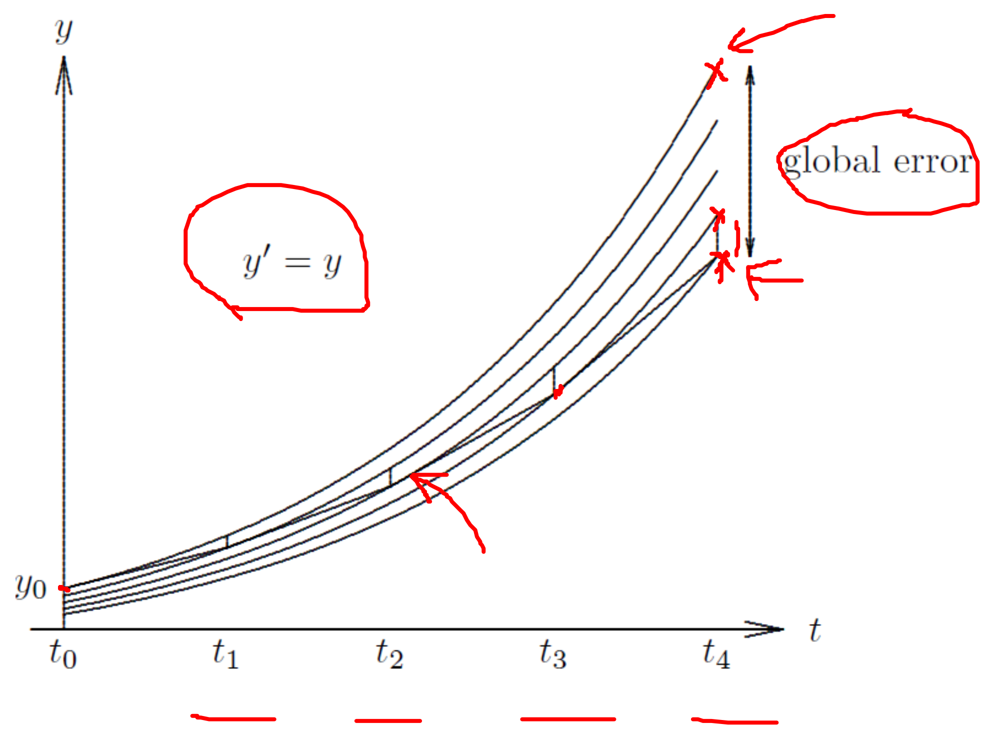
At kth step

- Global error is the cumulative overall error

$$\boldsymbol{e}_k = \boldsymbol{y}_k - \boldsymbol{y}(t_k)$$

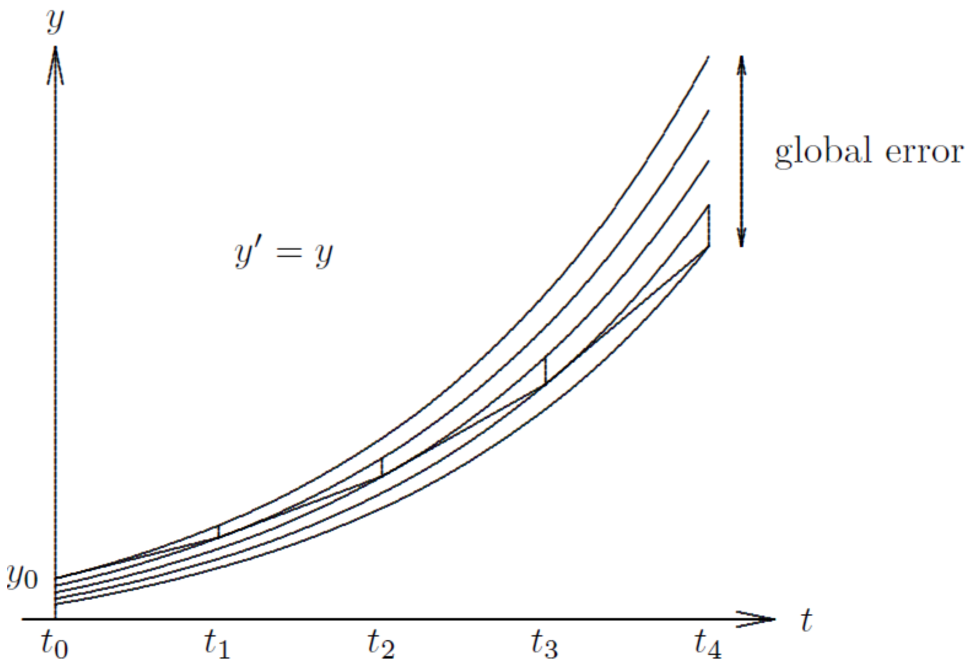
- Local error is the error in one step

$$\boldsymbol{\ell}_k = \boldsymbol{y}_k - \boldsymbol{u}_{k-1}(t_k)$$



Truncation Error for Euler's method

- Global error $O(h)$
- Local error is $O(h^2)$



This is called *first order accurate* $O(h^2)$

*p*th order accurate when $\ell_k = \mathcal{O}(h_k^{p+1})$

1D Example: Euler's Method

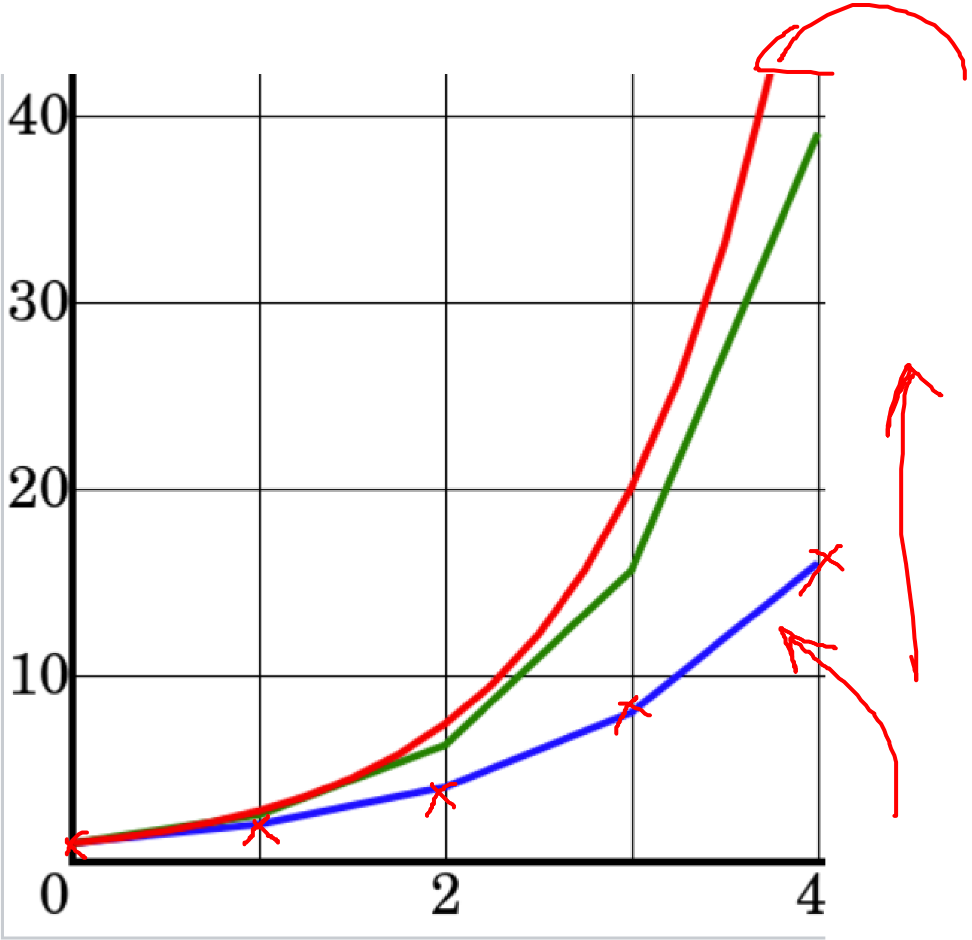


Illustration of numerical integration for the equation $y' = y, y(0) = 1$.

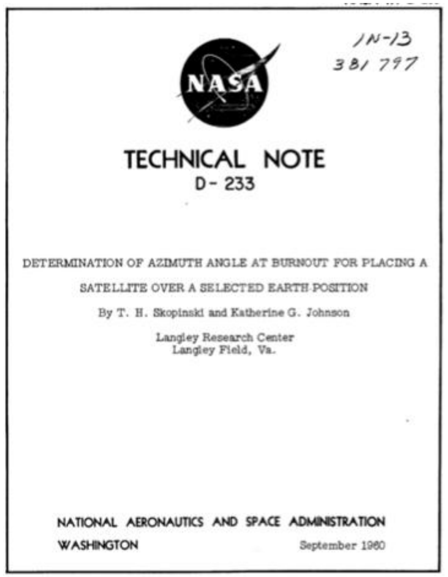
$$y_{n+1} = y_n + hf(t_n, y_n)$$

n	y_n	t_n	$f(t_n, y_n)$	h	Δy	y_{n+1}
0	1	0	1	1	1	2
1	2	1	2	1	2	4
2	4	2	4	1	4	8
3	8	3	8	1	8	16

Euler’s Method in History

Euler's Method scene in Hidden Figures (2016)
https://youtu.be/v-pbGAts_Fg

Used to calculate trajectories by NASA for Project Mercury…apparently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TECHNICAL NOTE
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DETERMINATION OF AZIMUTH ANGLE AT BURNOUT FOR PLACING A SATELLITE OVER A SELECTED EARTH POSITION

By T. H. Skopinski and Katherine G. Johnson

Langley Research Center
Langley Field, Va.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON
September 1960

ILLINOIS