Maximum radii of rolling spheres

Two lines $g : \vec{x}$ and $h : \vec{x}$ are given:

- $g : \vec{x} = \vec{p} + \lambda \vec{v}$
- $h : \vec{x} = \vec{q} + \mu \vec{u}$

Suppose a sphere S with radius $r > 0$ rolls on g tangentially.
Additionaly r should not be chosen such that h is touching S.

Because the radius of S is unchanged during its' movement, the center M is always parallel to g with distance r.
There is a line $m : \vec{x}$, that contains all M, and thus is parallel to g.

- $M \in m$
- $d(M; g) = r$
- $m \parallel g$.

There may exist a point M, where both g and h are touching S at one point. For such M, the distance of M to g is equal to that of M to h. Both these distances will be equal to r as previously stated.

$$d(M; g) = d(M; h) = r$$

Construction of $m : \vec{x}$

Because m has a distance r to g, a vector $\vec{n}$ with length r perpendicular to g can be constructed which will represent m. There is a group of vectors that can represent that vector n.

- $\vec{n} \perp g$
- $|\vec{n}| = r$
  Because it may be important where the sphere rolls on g, let the group of vectors be $\vec{n}_\theta$, where $\vec{n}_0$ ($\theta = 0$) then is the vector that is "straight above" g. This results to the following:
- $\{\vec{n}_0 \times \vec{v}\} \perp \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
  We know that the vector $\vec{n}_0 \times \vec{v}$ is perpendicular to the $x_1 x_2$ plane by description. This allows us to reconstruct the structure to find a vector $\vec{n}_0$:
- $\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v} \right] \perp \vec{n}_0$
  The bracket creates a vector that is perpendicular to the line $g : \vec{x}$. When we combine this with the line again, we create a factor of $\vec{n}_0$:

$$\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v} \right] \times \vec{v} = \omega^{-1} \cdot \vec{n}_0$$

$$\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right] \times \vec{v} = \omega^{-1} \cdot \vec{n}_0$$

$$\left[\begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \right] \times \vec{v} = \omega^{-1} \cdot \vec{n}_0$$

$$\begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \omega^{-1} \cdot \vec{n}_0$$

$$\begin{pmatrix} v_1 v_3 \\ v_2 v_3 \\ -v_2^2 - v_1^2 \end{pmatrix} = \omega^{-1} \cdot \vec{n}_0$$

The vector on the left side is always pointing towards below the line g, hence I will only flip the vector with the factor of -1 to make the vector face the upper side of g. This makes it possible to normalize the vector without any directional issues (to follow the definition of $\vec{n}_0$):

$$\begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} = \omega^{-1} \cdot \vec{n}_0$$

As of now the left side is a factor of the vector we need. The length of the vector will be r.

$$
\begin{aligned}
n_0 &:= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\left| \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_1^2 + v_2^2 \end{pmatrix} \right|} \\
&= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\sqrt{(-v_1 v_3)^2 + (-v_2 v_3)^2 + (v_1^2 + v_2^2)^2}} \\
&= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2 v_3^2 + v_2^2 v_3^2 + v_1^4 + 2 v_1^2 v_2^2 + v_2^4}}
\end{aligned}
$$

For simplification, I will represent this factor as ω.
- $\omega = \frac{1}{\sqrt{v_1^2 v_3^2 + v_2^2 v_3^2 + v_1^4 + 2 v_1^2 v_2^2 + v_2^4}}$

This vector allows us to now define the line $m : \vec{x}$:

$$m : \vec{x} = \vec{p} + \vec{n}_0 + \lambda \vec{v}$$

> Note that the 0 is a parameter for rotation, not a unit vector.

Construction of $m_\theta : \vec{x}$

We will now rotate the line m around g with distance r.

> For rotation we can use $\vec{i} \cos(\theta) + \vec{j} \sin(\theta)$, where $\vec{i} \perp \vec{j}$.

From above we know that for this equation:

- $\vec{i} = \vec{n}_0$

- $\phi^{-1}\vec{j} = \phi^{-1}\vec{n}_{\frac{\pi}{2}} = \phi^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v}$

Again we have a factor for a vector $(\vec{j})$. Vector $\vec{i}$ already has length 1 by definition.

$$n_{\frac{\pi}{2}} := \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \cdot \frac{r}{\left| \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \right|}$$

$$:= \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \cdot \frac{r}{\sqrt{(-v_2)^2 + (v_1)^2 + (0)^2}}$$

$$:= \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2 + v_2^2}}$$

Again we separate the factor with a variable.

- $\phi = \frac{1}{\sqrt{v_1^2 + v_2^2}}$

For the next, $\hat{z}$ is the unit vector of the third axis x_3 (z)

$$\vec{n}_\theta = \vec{i}\cos(\theta) + \vec{j}\sin(\theta)$$
$$= \vec{n}_0 \cos(\theta) + \vec{n}_{\frac{\pi}{2}}\sin(\theta)$$
$$= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2 v_3^2 + v_2^2 v_3^2 + v_1^4 + 2v_1^2 v_2^2 + v_2^4}} \cdot \cos(\theta) + \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2 + v_2^2}} \cdot \sin(\theta)$$
$$= \begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot r \cdot \omega \cdot \cos(\theta) + \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \cdot r \cdot \phi \cdot \sin(\theta)$$
$$= r \cdot \left[\begin{pmatrix} -v_1 v_3 \\ -v_2 v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \omega \cdot \cos(\theta) + \begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \cdot \phi \cdot \sin(\theta) \right]$$
$$= r \cdot \left[-\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v} \right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v} \right) \cdot \phi \cdot \sin(\theta) \right]$$
$$= r \cdot \left[-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta) \right]$$

This makes it able to represent $m_\theta : \vec{x}$ as follows:

$$m_\theta : \vec{x} = \vec{p} + r \cdot \left[-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta) \right] + \lambda \vec{v}$$

Calculating for r

> For two lines, their distance is defined as follows:

- $d(g_1; g_2) = \frac{\left| \vec{n} \circ \left(\overrightarrow{OR} - \overrightarrow{OP} \right) \right|}{|\vec{n}|}$

We know that the distance of the two lines is exactly r. $\vec{n}$ here is the cross product from both the directions. $\overrightarrow{OR}$ will here be the independent vector from $m_\theta : \vec{x}$. $\overrightarrow{OP} = \vec{q}$.

$$d(m_\theta; h) = \frac{\left| \vec{n} \circ \left(\overrightarrow{OR} - \overrightarrow{OP} \right) \right|}{|\vec{n}|}$$

$$r = \frac{\left| [\vec{v} \times \vec{u}] \circ (\vec{p} + r \cdot [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] - \vec{q}) \right|}{|\vec{v} \times \vec{u}|}$$

$$r = \frac{\left| [\vec{v} \times \vec{u}] \circ \vec{p} + r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] - [\vec{v} \times \vec{u}] \circ \vec{q} \right|}{|\vec{v} \times \vec{u}|} \qquad | \cdot |\vec{v} \times \vec{u}|$$

$$|\vec{v} \times \vec{u}| \cdot r = \left| [\vec{v} \times \vec{u}] \circ \vec{p} + r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] - [\vec{v} \times \vec{u}] \circ \vec{q} \right|$$

Die linke Seite kann per definition nicht negativ werden. Zunächst wird die rechte Seite als positiver Wert behandelt, danach als negativ.

Positiv:

$$|\vec{v} \times \vec{u}| \cdot r = \left| [\vec{v} \times \vec{u}] \circ \vec{p} + r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] - [\vec{v} \times \vec{u}] \circ \vec{q} \right|$$

$$|\vec{v} \times \vec{u}| \cdot r = [\vec{v} \times \vec{u}] \circ \vec{p} + r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] - [\vec{v} \times \vec{u}] \circ \vec{q}$$

$$|\vec{v} \times \vec{u}| \cdot r - r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] = [\vec{v} \times \vec{u}] \circ \vec{p} - [\vec{v} \times \vec{u}] \circ \vec{q}$$

$$\left(|\vec{v} \times \vec{u}| - [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] \right) \cdot r = [\vec{v} \times \vec{u}] \circ \vec{p} - [\vec{v} \times \vec{u}] \circ \vec{q}$$

$$r = \frac{[\vec{v} \times \vec{u}] \circ \vec{p} - [\vec{v} \times \vec{u}] \circ \vec{q}}{|\vec{v} \times \vec{u}| - [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)]}$$

$$r = \frac{[\vec{v} \times \vec{u}] \circ (\vec{p} - \vec{q})}{|\vec{v} \times \vec{u}| - [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)]}$$

Negativ:

$$|\vec{v} \times \vec{u}| \cdot r = \left| [\vec{v} \times \vec{u}] \circ \vec{p} + r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] - [\vec{v} \times \vec{u}] \circ \vec{q} \right|$$

$$|\vec{v} \times \vec{u}| \cdot r = -[\vec{v} \times \vec{u}] \circ \vec{p} - r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] + [\vec{v} \times \vec{u}] \circ \vec{q}$$

$$|\vec{v} \times \vec{u}| \cdot r + r \cdot [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] = -[\vec{v} \times \vec{u}] \circ \vec{p} + [\vec{v} \times \vec{u}] \circ \vec{q}$$

$$\left(|\vec{v} \times \vec{u}| + [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)] \right) \cdot r = -[\vec{v} \times \vec{u}] \circ \vec{p} + [\vec{v} \times \vec{u}] \circ \vec{q}$$

$$r = \frac{-[\vec{v} \times \vec{u}] \circ \vec{p} + [\vec{v} \times \vec{u}] \circ \vec{q}}{|\vec{v} \times \vec{u}| + [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)]}$$

$$r = \frac{[\vec{v} \times \vec{u}] \circ (-\vec{p} + \vec{q})}{|\vec{v} \times \vec{u}| + [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)]}$$

$$r = \frac{-[\vec{v} \times \vec{u}] \circ (\vec{p} - \vec{q})}{|\vec{v} \times \vec{u}| + [\vec{v} \times \vec{u}] \circ [-(\hat{z} \times \vec{v}) \times \vec{v} \cdot \omega \cdot \cos(\theta) + (\hat{z} \times \vec{v}) \cdot \phi \cdot \sin(\theta)]}$$