Maximum radii of rolling spheres

Two lines $g : \vec{x}$ and $h : \vec{x}$ are given:

- $g : \vec{x} = \vec{p} + \lambda\vec{v}$
- $h : \vec{x} = \vec{q} + \mu\vec{u}$

Let a sphere S of radius $r > 0$ roll tangentially along g. Additionally, choose r such that the sphere S does not touch the line h.

Because the radius of S is unchanged during its' movement, the center M is always parallel to g with distance r.
There is a line $m : \vec{x}$, that contains all M, and thus is parallel to g.

- $M \in m$
- $d\,(M; g) = r$
- $m \parallel g$.

There may exist a point M, where both g and h are touching S at one point. For such M, the distance of M to g is equal to that of M to h. Both these distances will be equal to r as previously stated.

$$d\,(M; g) = d\,(M; h) = r$$

Construction of $m : \vec{x}$

Because m has a distance r to g, a vector $\vec{n}$ with length r perpendicular to g can be constructed which will represent m. There is a group of vectors that can represent that vector n.

- $\vec{n} \perp g$
- $|\vec{n}| = r$
  Because it may be important where the sphere rolls on g, let the group of vectors be $\vec{n}_\theta$, where $\vec{n}_0$ ($\theta = 0$) then is the vector that is "straight above" g. This results to the following:
- $\{\vec{n}_0 \times \vec{v}\} \perp \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
  We know that the vector $\vec{n}_0 \times \vec{v}$ is perpendicular to the x_1x_2 plane by description. This allows us to reconstruct the structure to find a vector $\vec{n}_0$:
- $\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v}\right] \perp \vec{n}_0$
  The bracket creates a vector that is perpendicular to the line $g : \vec{x}$. When we combine this with the line again, we create a factor of $\vec{n}_0$:

$$\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \vec{v}\right] \times \vec{v} = \omega^{-1} \cdot \vec{n}_0$$
$$\left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right] \times \vec{v} = \omega^{-1} \cdot \vec{n}_0$$
$$\left[\begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix}\right] \times \vec{v} = \omega^{-1} \cdot \vec{n}_0$$
$$\begin{pmatrix} -v_2 \\ v_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \omega^{-1} \cdot \vec{n}_0$$
$$\begin{pmatrix} v_1v_3 \\ v_2v_3 \\ -v_2^2 - v_1^2 \end{pmatrix} = \omega^{-1} \cdot \vec{n}_0$$

  The vector on the left side is always pointing towards below the line g, hence I will only flip the vector with the factor of -1 to make the vector face the upper side of g. This makes it possible to normalize the vector without any directional issues (to follow the definition of $\vec{n}_0$):

$$\begin{pmatrix} -v_1v_3 \\ -v_2v_3 \\ v_2^2 + v_1^2 \end{pmatrix} = \omega^{-1} \cdot \vec{n}_0$$

  As of now the left side is a factor of the vector we need. The length of the vector will be r.

$$\begin{aligned} n_0 &:= \begin{pmatrix} -v_1v_3 \\ -v_2v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\left|\begin{pmatrix} -v_1v_3 \\ -v_2v_3 \\ v_1^2 + v_2^2 \end{pmatrix}\right|} \\ &= \begin{pmatrix} -v_1v_3 \\ -v_2v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\sqrt{(-v_1v_3)^2 + (-v_2v_3)^2 + (v_1^2 + v_2^2)^2}} \\ &= \begin{pmatrix} -v_1v_3 \\ -v_2v_3 \\ v_2^2 + v_1^2 \end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2v_3^2 + v_2^2v_3^2 + v_1^4 + 2v_1^2v_2^2 + v_2^4}} \end{aligned}$$

  For simplification, I will represent this factor as ω.
- $\omega = \frac{1}{\sqrt{v_1^2v_3^2 + v_2^2v_3^2 + v_1^4 + 2v_1^2v_2^2 + v_2^4}}$

This vector allows us to now define the line $m : \vec{x}$:

$$m : \vec{x} = \vec{p} + \vec{n}_0 + \lambda\vec{v}$$

> Note that the 0 is a parameter for rotation, not a unit vector.

Construction of $m_\theta : \vec{x}$

We will now rotate the line m around g with distance r.

> For rotation we can use $\vec{i}\cos(\theta) + \vec{j}\sin(\theta)$, where $\vec{i} \perp \vec{j}$.

From above we know that for this equation:

- $\vec{i} = \vec{n}_0$
- $\phi^{-1}\vec{j} = \phi^{-1}\vec{n}_{\frac{\pi}{2}} = \phi^{-1}\begin{pmatrix}0\\0\\1\end{pmatrix} \times \vec{v}$

  Again we have a factor for a vector ($\vec{j}$). Vector $\vec{i}$ already has length 1 by definition.

$$
\begin{aligned}
n_{\frac{\pi}{2}} &:= \begin{pmatrix}-v_2\\v_1\\0\end{pmatrix} \cdot \frac{r}{\left|\begin{pmatrix}-v_2\\v_1\\0\end{pmatrix}\right|} \\
&:= \begin{pmatrix}-v_2\\v_1\\0\end{pmatrix} \cdot \frac{r}{\sqrt{(-v_2)^2 + (v_1)^2 + (0)^2}} \\
&:= \begin{pmatrix}-v_2\\v_1\\0\end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2 + v_2^2}}
\end{aligned}
$$

  Again we separate the factor with a variable.

- $\phi = \frac{1}{\sqrt{v_1^2+v_2^2}}$

  For the next, $\hat{z}$ is the unit vector of the third axis x_3 (z)

$$
\begin{aligned}
\vec{n}_\theta &= \vec{i}\cos(\theta) + \vec{j}\sin(\theta) \\
&= \vec{n}_0\cos(\theta) + \vec{n}_{\frac{\pi}{2}}\sin(\theta) \\
&= \begin{pmatrix}-v_1v_3\\-v_2v_3\\v_2^2+v_1^2\end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2v_3^2 + v_2^2v_3^2 + v_1^4 + 2v_1^2v_2^2 + v_2^4}} \cdot \cos(\theta) + \begin{pmatrix}-v_2\\v_1\\0\end{pmatrix} \cdot \frac{r}{\sqrt{v_1^2+v_2^2}} \cdot \sin(\theta) \\
&= \begin{pmatrix}-v_1v_3\\-v_2v_3\\v_2^2+v_1^2\end{pmatrix} \cdot r \cdot \omega \cdot \cos(\theta) + \begin{pmatrix}-v_2\\v_1\\0\end{pmatrix} \cdot r \cdot \phi \cdot \sin(\theta) \\
&= r \cdot \left[\begin{pmatrix}-v_1v_3\\-v_2v_3\\v_2^2+v_1^2\end{pmatrix} \cdot \omega \cdot \cos(\theta) + \begin{pmatrix}-v_2\\v_1\\0\end{pmatrix} \cdot \phi \cdot \sin(\theta)\right] \\
&= r \cdot \left[-\left(\begin{pmatrix}0\\0\\1\end{pmatrix} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\begin{pmatrix}0\\0\\1\end{pmatrix} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] \\
&= r \cdot \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]
\end{aligned}
$$

This makes it able to represent $m_\theta : \vec{x}$ as follows:

$$m_\theta : \vec{x} = \vec{p} + r \cdot \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] + \lambda\vec{v}$$

Calculating for r

> For two lines, their distance is defined as follows:

- $d(g_1; g_2) = \frac{\left|\vec{n} \circ \left(\overrightarrow{OR} - \overrightarrow{OP}\right)\right|}{|\vec{n}|}$

We know that the distance of the two lines is exactly r. $\vec{n}$ here is the cross product from both the directions. $\overrightarrow{OR}$ will here be the independent vector from $m_\theta : \vec{x}$. $\overrightarrow{OP} = \vec{q}$.

$$
\begin{aligned}
d(m_\theta; h) &= \frac{\left|\vec{n} \circ \left(\overrightarrow{OR} - \overrightarrow{OP}\right)\right|}{|\vec{n}|} \\
r &= \frac{\left|\left[\vec{v} \times \vec{u}\right] \circ \left(\vec{p} + r \cdot \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] - \vec{q}\right)\right|}{|\vec{v} \times \vec{u}|} \\
r &= \frac{\left|\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] - \left[\vec{v} \times \vec{u}\right] \circ \vec{q}\right|}{|\vec{v} \times \vec{u}|} \qquad | \cdot |\vec{v} \times \vec{u}| \\
|\vec{v} \times \vec{u}| \cdot r &= \left|\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] - \left[\vec{v} \times \vec{u}\right] \circ \vec{q}\right|
\end{aligned}
$$

The left side can not be negative by definition. The absolute value on the right side will be split into positive and negative.
Positive:

$$
\begin{aligned}
|\vec{v} \times \vec{u}| \cdot r &= \left|\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] - \left[\vec{v} \times \vec{u}\right] \circ \vec{q}\right| \\
|\vec{v} \times \vec{u}| \cdot r &= \left[\vec{v} \times \vec{u}\right] \circ \vec{p} + r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] - \left[\vec{v} \times \vec{u}\right] \circ \vec{q} \\
|\vec{v} \times \vec{u}| \cdot r - r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] &= \left[\vec{v} \times \vec{u}\right] \circ \vec{p} - \left[\vec{v} \times \vec{u}\right] \circ \vec{q} \\
\left(|\vec{v} \times \vec{u}| - \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]\right) \cdot r &= \left[\vec{v} \times \vec{u}\right] \circ \vec{p} - \left[\vec{v} \times \vec{u}\right] \circ \vec{q} \\
r &= \frac{\left[\vec{v} \times \vec{u}\right] \circ \vec{p} - \left[\vec{v} \times \vec{u}\right] \circ \vec{q}}{|\vec{v} \times \vec{u}| - \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]} \\
r_1 &= \frac{\left[\vec{v} \times \vec{u}\right] \circ \left(\vec{p} - \vec{q}\right)}{|\vec{v} \times \vec{u}| - \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]}
\end{aligned}
$$

Negative:

$$
\begin{aligned}
|\vec{v} \times \vec{u}| \cdot r &= \left|\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] - \left[\vec{v} \times \vec{u}\right] \circ \vec{q}\right| \\
|\vec{v} \times \vec{u}| \cdot r &= -\left[\vec{v} \times \vec{u}\right] \circ \vec{p} - r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] + \left[\vec{v} \times \vec{u}\right] \circ \vec{q} \\
|\vec{v} \times \vec{u}| \cdot r + r \cdot \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] &= -\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + \left[\vec{v} \times \vec{u}\right] \circ \vec{q} \\
\left(|\vec{v} \times \vec{u}| + \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]\right) \cdot r &= -\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + \left[\vec{v} \times \vec{u}\right] \circ \vec{q} \\
r &= \frac{-\left[\vec{v} \times \vec{u}\right] \circ \vec{p} + \left[\vec{v} \times \vec{u}\right] \circ \vec{q}}{|\vec{v} \times \vec{u}| + \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]} \\
r &= \frac{\left[\vec{v} \times \vec{u}\right] \circ \left(-\vec{p} + \vec{q}\right)}{|\vec{v} \times \vec{u}| + \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]} \\
r_2 &= \frac{-\left[\vec{v} \times \vec{u}\right] \circ \left(\vec{p} - \vec{q}\right)}{|\vec{v} \times \vec{u}| + \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]}
\end{aligned}
$$

> The range for possible values in the denominator of r_1:

We note the following:

$$|\vec{v} \times \vec{u}| - \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]$$

By definition of the term colored green, the length of this vector is 1. With the scalar product being $\vec{a} \circ \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\alpha)$, we can alter our term:

$$\left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right] = \left|\vec{v} \times \vec{u}\right| \cdot \cos(\alpha)$$

We can see that with the whole denominator being $\left|\vec{v} \times \vec{u}\right| - \left|\vec{v} \times \vec{u}\right| \cdot \cos(\alpha)$
it will have values in $\left[0; \left|\vec{v} \times \vec{u}\right|\right]$.
This makes it always be zero positive.

$$\begin{aligned} r_1 &= \frac{\left[\vec{v} \times \vec{u}\right] \circ \left(\vec{p} - \vec{q}\right)}{\left|\vec{v} \times \vec{u}\right| - \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \omega \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \phi \cdot \sin(\theta)\right]} \\ &= \frac{\left[\vec{v} \times \vec{u}\right] \circ \left(\vec{p} - \vec{q}\right)}{\left|\vec{v} \times \vec{u}\right| - \left[\vec{v} \times \vec{u}\right] \circ \left[-\left(\hat{z} \times \vec{v}\right) \times \vec{v} \cdot \frac{1}{\left|\left(\hat{z} \times \vec{v}\right) \times \vec{v}\right|} \cdot \cos(\theta) + \left(\hat{z} \times \vec{v}\right) \cdot \frac{1}{\left|\hat{z} \times \vec{v}\right|} \cdot \sin(\theta)\right]} \end{aligned}$$