Sensor Networks and Data Analysis 2 (ELEE08021, AY2022–23) Part I: Signal Analysis Methods

Course Lecture Notes and Tutorial Questions

Dr James R. Hopgood

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Copyright Statement

However, there is some material that has been based on work in a number of previous textbooks, and therefore some sections and paragraphs have strong similarities in structure and wording. These texts have been referenced and include, amongst a number of others, in order of contributions:

- Mulgew B., P. M. Grant, and J. S. Thompson, *Digital Signal Processing: Concepts and Applications*, Palgrave, Macmillan, 2003.

  IDENTIFIERS – *Paperback*, ISBN10: 0333963563, ISBN13: 9780333963562

  See `http://www.homepages.ed.ac.uk/pmg/SIGPRO/`

- Lathi B. P., *Linear Systems and Signals*, Oxford University Press, Inc., 2005.

  IDENTIFIERS – *Hardback*, ISBN10: 0195158334, ISBN13: 9780195158335

Outline of Lecture Contents

Contents

List of Figures

Acronyms

2-D	two-dimensional
3-D	three-dimensional
A2DP	Advanced Audio Distribution Profile
ADC	analogue-to-digital converter
AI	artificial intelligence
AM	amplitude modulated
BSS	blind source separation
CCTV	closed-circuit television
CD	compact disc
CDMA	code division multiple access
CFS	complex Fourier series
CMOS	complementary metal-oxide-semiconductor
COTS	commercial off-the-shelf
CTFT	continuous-time Fourier transform
DAB	digital audio broadcasting
DAC	digital-to-analogue converter
DAW	digital audio workstation
DC	“direct current”
DFT	discrete Fourier transform
DNA	deoxyribonucleic acid
DSP	digital signal processing
DTFT	discrete-time Fourier transform
DVB	digital video broadcasting
DVD	digital versitile disc
DVD-A	digital versitile disc-audio
ECG	electrocardiogram
EEG	electroencephalogram
ESD	energy spectral density

FDM	frequency division multiplexing
FFT	Fast Fourier transform
FLAC	free lossless audio codec
GCF	greatest common factor
GPS	global positioning system
HCI	human-computer interface
IDFT	inverse-DFT
IDTFT	inverse-DTFT
LHS	left hand side
MEG	magnetoencephalography
MEMS	micro-electromechanical systems
MP3	MPEG-1 Audio Layer 3
MPEG	Moving Picture Experts Group
MRI	magnetic resonance imaging
NMRI	nuclear magnetic resonance imaging
ODE	ordinary differential equation
OFDM	orthogonal frequency division multiplexing
RHS	right hand side
RMS	root mean square
ROC	region of convergence
Radar	RAdio Detection And Ranging
SACD	super-audio CD
SAR	synthetic aperture RADAR
SLAM	simultaneous localisation and mapping
UAV	unmanned aerial vehicle
dc	“direct current”
i. t. o.	in terms of

Acronyms

SNADA Sensor Networks and Data Analysis

1
Module Guide, Contents, and Overview

Somewhere in me is a curiosity sensor. I want to know what's over the next hill. You know, people can live longer without food than without information. Without information, you'd go crazy.

Arthur C. Clarke

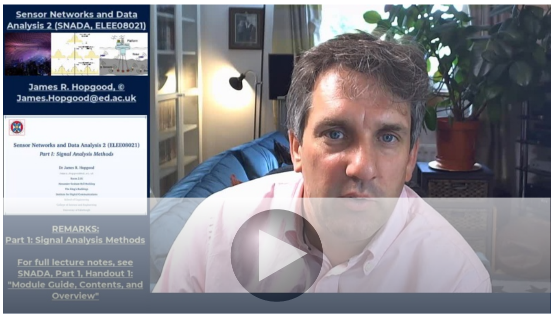
Everything that needs to be said has already been said. But since no one was listening, everything must be said again.

André Gide

This Handout provides a general overview of the course, including a short course descriptor and summary of intended learning outcomes, a preliminary lecture list, and recommended texts for the course. This Handout also provides a list of corrections and changes made since the published version of the lecture notes were released.

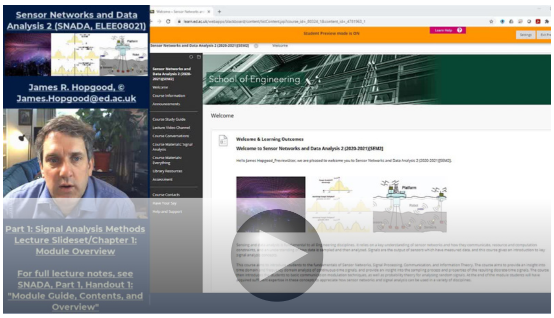
1.1 Welcome

The **Sensor Networks and Data Analysis (SNADA)** module introduces the fundamental conceptual and mathematical tools that are required to analyse and describe data obtained from sensor networks, and the communications technologies required to enable sensor networks in the first place.



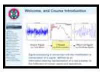
`http://media.ed.ac.uk/media/1_6wt1ez10`

Video Summary: This video introduces the Course Organiser and Lecturer, Dr James Hopgood. The video tells you a little about himself in a professional capacity and his research interests. This video also discusses the Institute of Digital Communications, where Dr Hopgood is a member. For more about Dr Hopgood's research interests, please see `https://www.research.ed.ac.uk/portal/jhopgoo1`.



`http://media.ed.ac.uk/media/1_z4a2hq8b`

Video Summary: This video shows you how to navigate the LEARN virtual learning environment. It shows how to navigate course content and the course guide.



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1.2 Introduction to this course

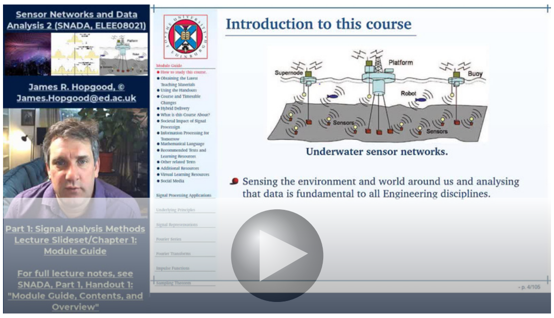
Topic Summary 1 Introduction to SNADA, its applications, and core principles

Topic Objectives:

- Welcome to the course on Sensor Networks and Data Analysis (SNADA).
- Discuss applications of sensor networks from different disciplines.
- Highlight that sensor signals are always subject to noise, interference, and distortion.
- Discuss the nature of output signals of sensor networks.

Topic Activities:

Type	Details	Duration	Progress
Watch video	16 : 17 minute video	$3\times$ video length	
Read Handout	Read page 3 to page 8 and reflect	8 mins/page	



http://media.ed.ac.uk/media/1_evwo1csg

Video Summary: This video introduces students to the Sensor Networks and Data Analysis (SNADA) course. It discusses, at a high level, what a sensor network is, and various applications across the Engineering disciplines, from underwater sensing, to structural health monitoring, and remote sensing. The topic highlights the importance of understanding how signals are created by each sensor modality, the need for reliable network communications, and the need for machine learning at the decision making stage. The topic finishes by giving the key aims of the course, and looking forward to what the rest of the course will offer.

A (wireless) sensor network is a group of spatially dispersed and dedicated sensors for monitoring and recording the physical world around us. The signals collected can either be processed at a central location, or can even be processed in a distributed fashion (in a distributed network). Processing of sensor data is crucial to reduce the impact of noise, interference, and distortion that is inevitable in practical systems.

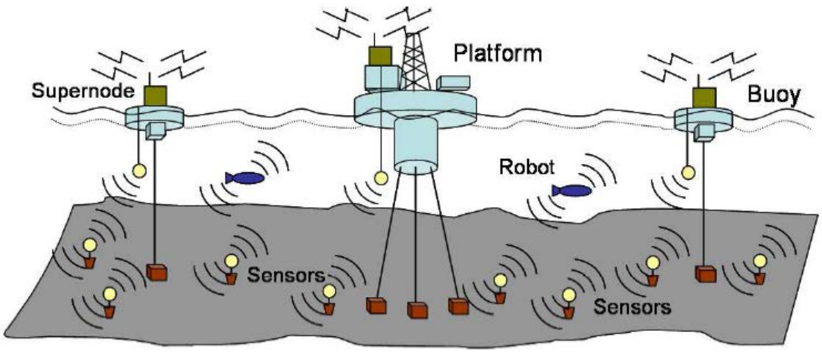


Figure 1.1: Underwater sensor networks.

Sensing the environment and world around us and analysing that data is fundamental to all Engineering disciplines. It relies on a key understanding of sensor networks and how they communicate, resource and computation constraints, and an understanding of how data is sampled and then analysed. Signals are the output of sensors which have measured data, and this course gives an introduction to key signal analysis concepts. The sub-disciplines behind these concepts are Information Theory, Signal Processing, Communications, and Machine Learning, which are at the heart of our modern world, powering today's entertainment and tomorrow's technology.

Applications of sensor networks include:

1. Underwater sensor networks for detecting underwater objects, as shown in Figure 1.1. Improving information processing, optimising sonar performance, and underwater scene analysis is of great importance to industries working in underwater exploration such as mine detection and clearance; the oil and gas companies and their service industries; offshore electrical power industries; civil engineering industries; and wider marine and energy markets.

2. Structural health monitoring is an important Civil Engineering application of sensor networks, such as monitoring bridges (see Figure 1.2).

3. Precision agriculture systems, as shown in Figure 1.7 on page 8. The aim of the adoption of sensor networks in precision agriculture is to measure the different environmental parameters such as humidity, temperature, soil moisture, PH value of soil etc., for enhancing the quantity and quality of crops. Furthermore, the sensor networks are also helping to reduce the consumptions of the natural resources used in farming.

4. Other applications include speech procesing as shown in Figure 1.3, wearable sensors for personal health monitoring as in Figure 1.4, to wide-area surveillance and dynamical sensor arrays as in Figure 1.5. Dynamic and scalable sensor-platforms enable opportunities in remote sensing, disaster monitoring, border surveillance, traffic monitoring and so on.

Note that Sensor Networks and Data Analysis explicitly covers:

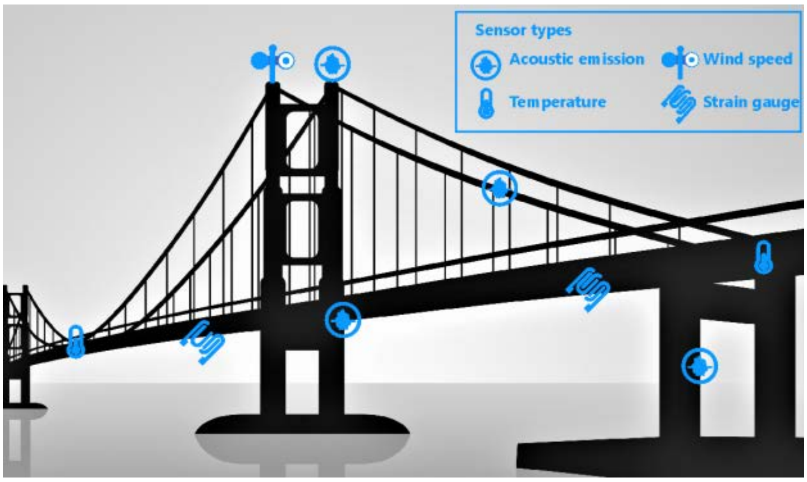


Figure 1.2: Structural health monitoring is an important Civil Engineering application of sensor networks.

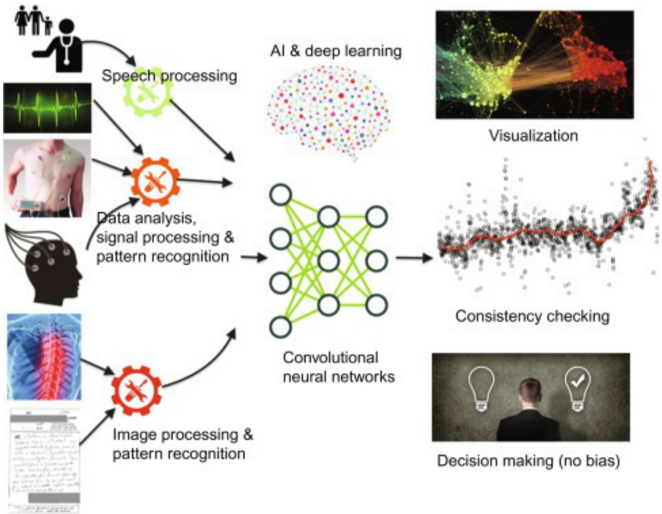


Figure 1.3: Advanced Data Analysis: Smart healthcare and the use of speech processing: can we get more information?

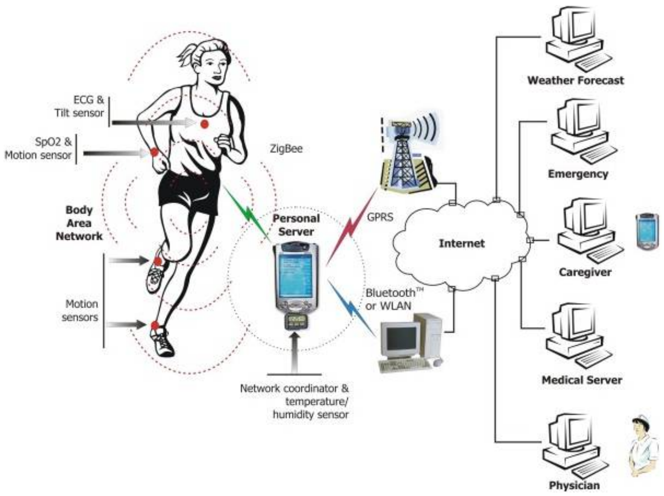


Figure 1.4: Wearable sensors for personal health monitoring.

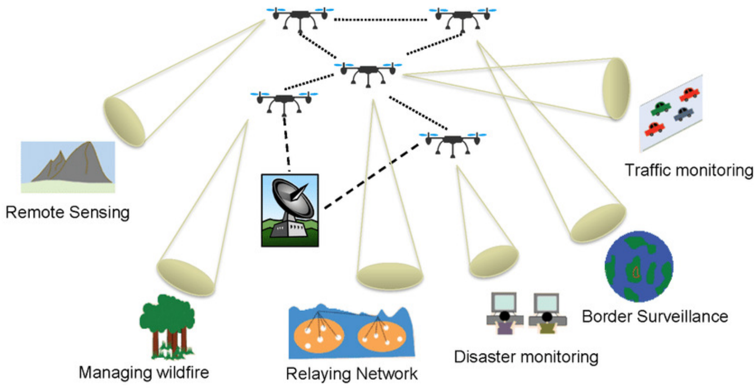


Figure 1.5: Flying ad-hoc networks (FANET) have numerous applications and is an active research area.

- Scenarios in which sensors produce signal data, such as time-series, images, features, or multimodal data.

- The sensors can be commercial off-the-shelf (COTS) devices, but in this course it is assumed the *raw-data* is obtained directly with an understanding of the sensing modality.

- The data analysis aspects in this course consider signal processing and machine learning methods. If you want to know how to use Excel, use databases, or wrangle data, this will need to be addressed in other courses.

- Ultimately, this course uses the general phrase *data analysis* to emphasise the diverse meaning of the term.

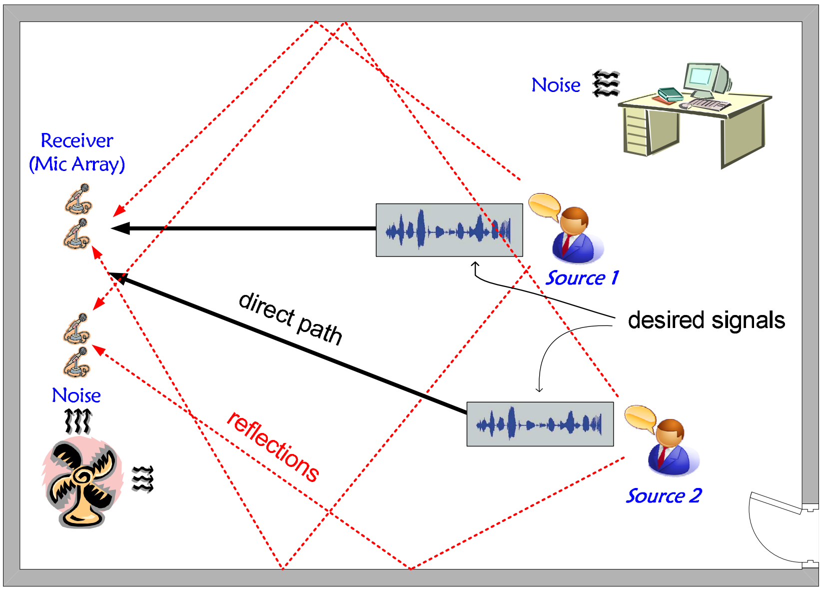


Figure 1.6: Networks don't have to be exotic, heterogeneous and multi-modal, and complicated; they also include wired or wireless homogeneous networks. Acoustic multi-source localisation and blind source separation (BSS) is one such example, and one which uses statistical signal processing and sensor networks.

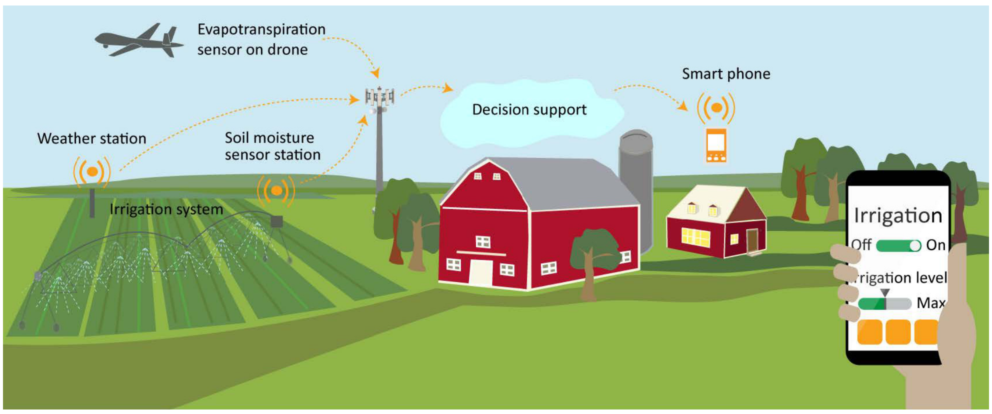


Figure 1.7: A possible configuration of a sensor-network and smartphone-integrated precision agriculture system. By U.S. Government Accountability Office from Washington, DC, United States - Figure 16: Components of a Precision Agriculture System, see `https://commons.wikimedia.org/w/index.php?curid=84564298`

- The aim of sensor networks in precision agriculture is to measure different environmental parameters such as humidity, temperature, soil moisture, PH value of soil etc., for enhancing the quantity and quality of crops.
- Further, the sensor networks are also helping to reduce the consumptions of the natural resources used in farming.

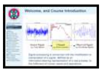
The course aims to provide an insight into time domain and frequency domain analysis of continuous-time signals, and provide an insight into the sampling process and properties of the resulting discrete-time signals. The course then introduces the students to basic communication modulation techniques, as well as probability theory for analysing random signals. At the end of the module students will have acquired sufficient expertise in these concepts to appreciate how sensor networks and signal analysis can be used in a variety of disciplines. It is written at a level which assumes knowledge of undergraduate Engineering mathematics and an awareness of what sensors are, but otherwise should be accessible to most technical students.

In summary, this course will:

- Aim to provide an understanding of the concepts and applications of sensor-networks and the signals they generate.
- Introduce the fundamentals of data and signal-analysis which are the building blocks for signal processing techniques used to analyse sensor data.
- Introduce machine learning techniques which can aid with decision making based on this signal analysis performed above.
- Introduce the fundamentals of communication systems which enable sensor networks to operate.
- At the end of the module, students will have acquired sufficient expertise in these concepts to appreciate how sensor networks and signal analysis can be used in a variety of disciplines.

– End-of-Topic 1: **Introduction to SNADA, its applications, and the core principles involved –**

1.3 How to study this course



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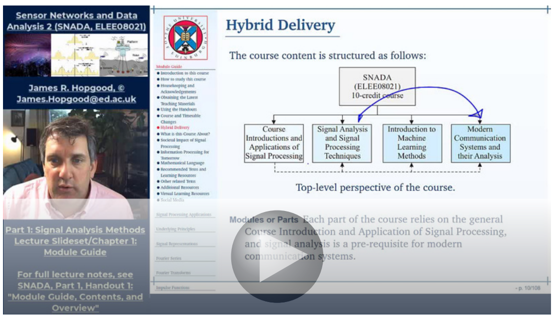
Topic Summary 2 How to Study using the Course Materials

Topic Objectives:

- Welcome to the course on Sensor Networks and Data Analysis (SNADA).
- Make students aware of the handouts and slides available on LEARN, and how to use them.
- Hightlight some simple study skills.
- Raise awareness of Hybrid Delivery methods.

Topic Activities:

Type	Details	Duration	Progress
E-training	Become familiar with LEARN	20 mins	
Watch video	21 : 40 minute video	$\approx 3\times$ video length	
Read Handout	Read page 9 to page 17	$\approx$ 8 mins/page	



http://media.ed.ac.uk/media/1_sugxcd1d

Video Summary: This video welcomes students to the SNADA course, and shows the best way of using the teaching materials to study this course in hybrid teaching mode. The materials available range from handouts and slides on LEARN, through to the course videos, discussion boards and so on. However, the focus of this video is how the course is structured, what the hybrid delivery method is, and how the course is broken down into themes, topics, activities, and learning opportunities.

This course is delivered in three parts with a corresponding teaching team:

Signal Analysis and signal processing techniques, taught by James R. Hopgood for around 5 weeks.

Machine Learning introduction and classification techniques, taught by Elliot

(a) James Hopgood, teaching Signal Analysis and Signal Processing.

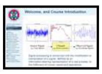
(b) Elliot Crowley, teaching Machine Learning and classification techniques.

(c) Popoola Wasiu, teaching Communications Theory and Information Theory.

Figure 1.8: The teaching team for this course on Sensor Networks and Data Analysis (SNADA).

Crowley for around 2 weeks.

Communication theory and information theory, taught by Popoola Wasiu for around 4 weeks.



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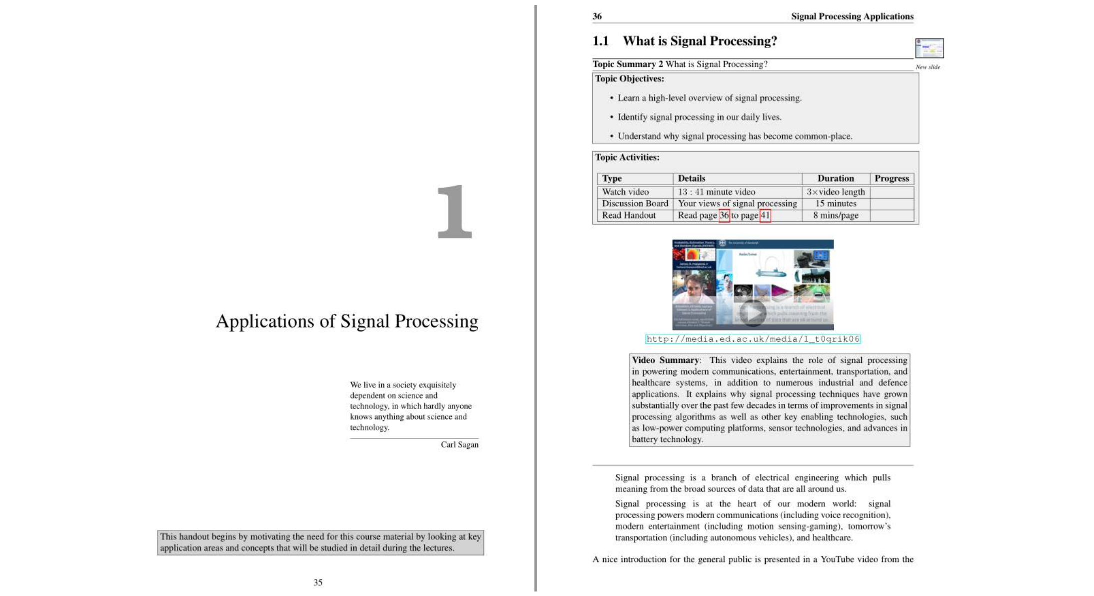
1.3.1 Housekeeping and Acknowledgements

Before we begin, some housekeeping and acknowledgements:

- These lecture notes are intended to cover a wide range of aspects which introduce students to the fundamentals of Sensor Networks, Signal Processing, Communication, and Information Theory.
- These notes have been developed to be a comprehensive set of teaching materials to help you in your studies. As will be discussed in later sections, is should be possible to study this course entirely based on the material provided. Effectively, these notes form the basis of an as yet unpublished course textbook!
- Extended thanks are given to the many undergraduate who, since 2012, have helped proof-read and improve these documents.
- This course is delivered in a hybrid model, whereby students are enabled to transition easily between online and on campus education, and where no differentiation is made between online and on campus students, but rather they are brought together by design. Further details on hybrid delivery is provided in Section 1.3.5.



1

Applications of Signal Processing

We live in a society exquisitely dependent on science and technology, in which hardly anyone knows anything about science and technology.

Carl Sagan

This handout begins by motivating the need for this course material by looking at key application areas and concepts that will be studied in detail during the lectures.

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36 Signal Processing Applications

1.1 What is Signal Processing?

New slide

Topic Summary 2 What is Signal Processing?

Topic Objectives:

- Learn a high-level overview of signal processing.
- Identify signal processing in our daily lives.
- Understand why signal processing has become common-place.

Topic Activities:

Type	Details	Duration	Progress
Watch video	13 : 41 minute video	3×video length	
Discussion Board	Your views of signal processing	15 minutes	
Read Handout	Read page 36 to page 41	8 mins/page	

http://media.ed.ac.uk/media/1_t0qrik06

Video Summary: This video explains the role of signal processing in powering modern communications, entertainment, transportation, and healthcare systems, in addition to numerous industrial and defence applications. It explains why signal processing techniques have grown substantially over the past few decades in terms of improvements in signal processing algorithms as well as other key enabling technologies, such as low-power computing platforms, sensor technologies, and advances in battery technology.

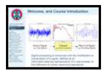
Signal processing is a branch of electrical engineering which pulls meaning from the broad sources of data that are all around us.

Signal processing is at the heart of our modern world: signal processing powers modern communications (including voice recognition), modern entertainment (including motion sensing-gaming), tomorrow's transportation (including autonomous vehicles), and healthcare.

A nice introduction for the general public is presented in a YouTube video from the

Figure 1.9: Example of lecture handouts for signal analysis component of the course.

1.3.2 Obtaining the Latest Version of the Teaching Materials



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KEYPOINT! (Latest Materials). Please note the following:

- The lecture materials are continually being updated throughout the semester, and feedback is welcome. Therefore, any documents published in hardcopy may differ from those available on LEARN.
- In particular, there are likely to be a few typos in any physically published documents, so if there is something that isn't clear, please feel free to contact me so I can correct it (or make it clearer).
- The latest version of the teaching materials can be found online on LEARN.
- An example of what the handouts look like are shown in Figure 1.9, and a screenshot of the LEARN page for SNADA is shown in Figure 1.10.

1.3.3 Using the Handouts

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Before starting this course, it is important to know how to use these notes. We all learn in different ways, and no one teaching style suits us all. Therefore, in this course, I have aimed to provide a variety of different teaching resources that will help you with your learning. It is important that you choose the most appropriate resource for your style.

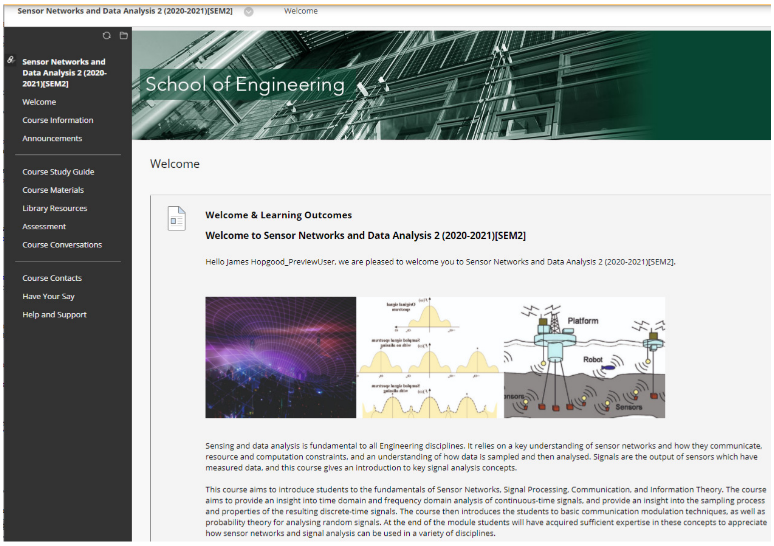


Figure 1.10: The LEARN page for SNADA (at the time of publication).

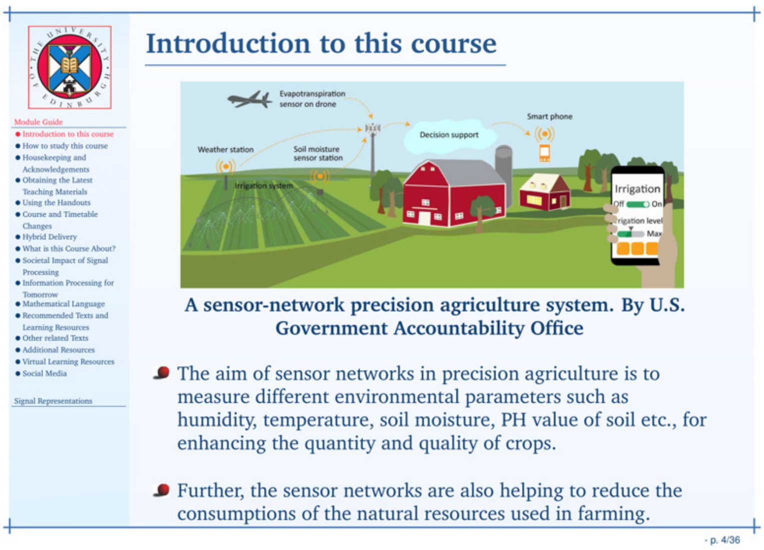


Figure 1.11: A screenshot of the slides used in the lecture videos.

KEYPOINT! (What works for you!). You do not need to use all different versions, just the one that suits you best!

The general idea is as follows:

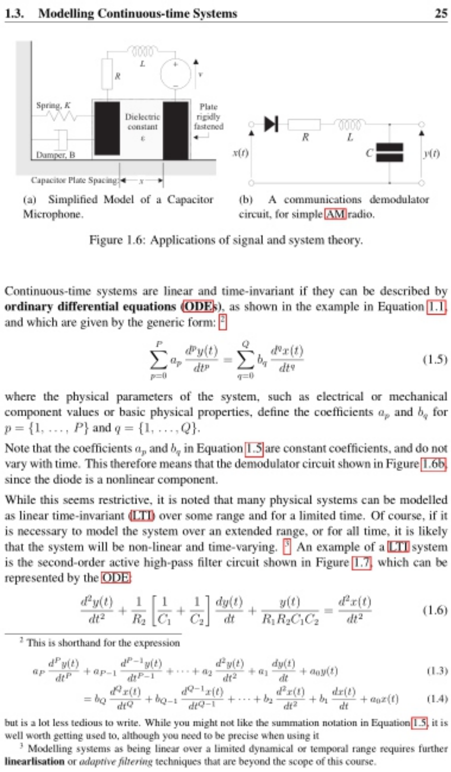
- The basic slides on which the lectures are based, containing the key information needed to follow the course; an example slide is shown in Figure 1.11.
- A full set of notes, which includes extended description, discussion, examples, and so on, which aims to aid your understanding and mastery of the material.
- A reduced version of the notes that contains the most important information for understanding and assessment. These are effectively the basic slides used in lectures.

Specifically, the various set of notes is available for this couse include:

Complete Handouts These handouts include all material seen in the lecture slides, and include **skeleton templates** for making summary notes about the material covered.

Slides and Printed Slides These slides are projected in lectures and are a **subset** of the complete handouts. A printed version is also available, although hard copies are not provided.

Simple Handouts These are effectively the printed slides and **skeleton templates** for making summary notes.



1.3. Modelling Continuous-time Systems 25

(a) Simplified Model of a Capacitor Microphone.

(b) A communications demodulator circuit, for simple AM radio.

Figure 1.6: Applications of signal and system theory.

Continuous-time systems are linear and time-invariant if they can be described by **ordinary differential equations (ODEs)**, as shown in the example in Equation 1.1, and which are given by the generic form: [2]

$$\sum_{p=0}^{P} a_p \frac{d^p y(t)}{dt^p} = \sum_{q=0}^{Q} b_q \frac{d^q x(t)}{dt^q} \tag{1.5}$$

where the physical parameters of the system, such as electrical or mechanical component values or basic physical properties, define the coefficients a_p and b_q for $p = \{1, \dots, P\}$ and $q = \{1, \dots, Q\}$.

Note that the coefficients a_p and b_q in Equation 1.5 are constant coefficients, and do not vary with time. This therefore means that the demodulator circuit shown in Figure 1.6b, since the diode is a nonlinear component.

While this seems restrictive, it is noted that many physical systems can be modelled as linear time-invariant (LTI) over some range and for a limited time. Of course, if it is necessary to model the system over an extended range, or for all time, it is likely that the system will be non-linear and time-varying. [3] An example of a LTI system is the second-order active high-pass filter circuit shown in Figure 1.7, which can be represented by the ODE:

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{R_2}\left[\frac{1}{C_1} + \frac{1}{C_2}\right]\frac{dy(t)}{dt} + \frac{y(t)}{R_1 R_2 C_1 C_2} = \frac{d^2 x(t)}{dt^2} \tag{1.6}$$

[2] This is shorthand for the expression

$$a_P \frac{d^P y(t)}{dt^P} + a_{P-1} \frac{d^{P-1} y(t)}{dt^{P-1}} + \cdots + a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) \tag{1.3}$$

$$= b_Q \frac{d^Q x(t)}{dt^Q} + b_{Q-1} \frac{d^{Q-1} x(t)}{dt^{Q-1}} + \cdots + b_2 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + a_0 x(t) \tag{1.4}$$

but is a lot less tedious to write. While you might not like the summation notation in Equation 1.5, it is well worth getting used to, although you need to be precise when using it

[3] Modelling systems as being linear over a limited dynamical or temporal range requires further **linearlisation** or *adaptive filtering* techniques that are beyond the scope of this course.

If you are using the **full set of notes**, then the slide transition icon can be used to determine when we have moved from section-to-section: to do this, match the section title of the overhead slides by searching for the next *New slide* icon, which is shown in Figure 1.12.

- Other tips for navigating the course materials will be also provided on LEARN.

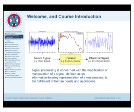


Figure 1.12: New slide icon: used to help keep track of overhead slide changes.

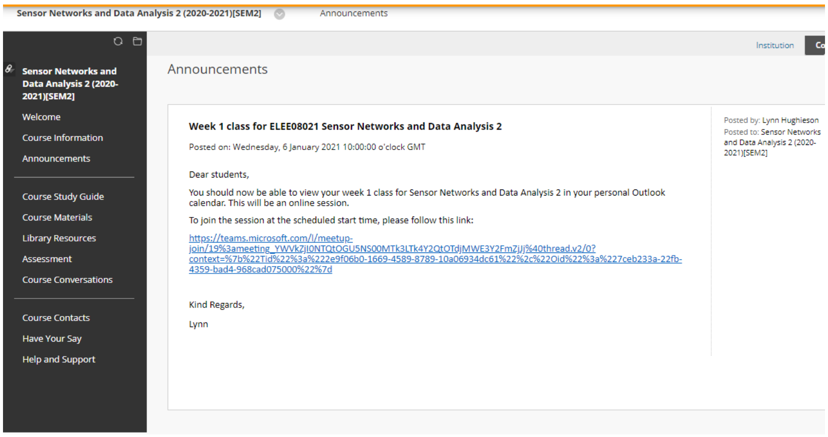
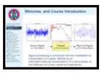


Figure 1.13: Teaching announcements will be made via LEARN, the University virtual learning environment (VLE).

- The aim for this part of the course is to provide teaching materials that suit a wide range of learning approaches. It is not expected that any one approach will work for everyone.
- If something isn't working on the course, please discuss with the teaching team – your feedback is valuable, and often small changes can have significant impacts.



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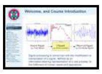
1.3.4 Course and Timetable Changes

Course changes and timetable changes will be notified to the class via LEARN. In particular, any timetable changes will be notified on the official University timetable service. Any major changes may be added to the online version of these notes.



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1.3.5 Hybrid Delivery

This course is delivered using blended teaching methods. As you study this course, you will use a combination of:

Acquisition of knowledge and concepts through watching lecture videos and reading the handouts;

Discussion of ideas and problem solving, through examples classes, and individually or collaboratively through discussion boards;

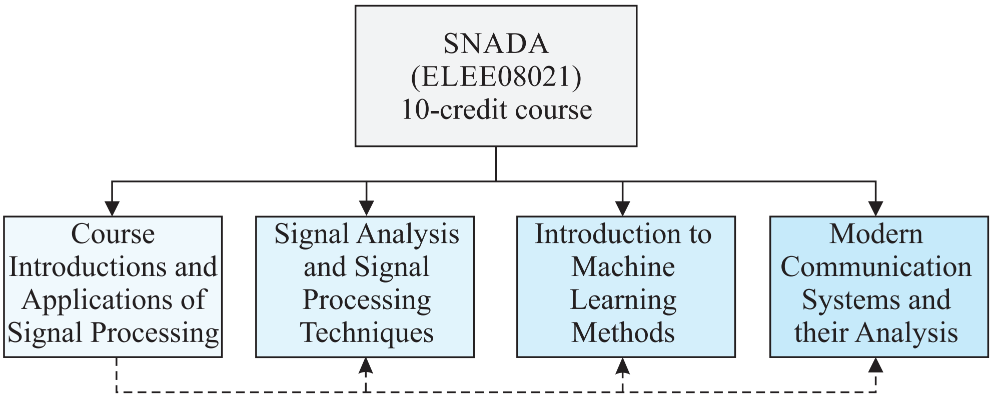


Figure 1.14: Top-level perspective of the course. The overall course is divided into four parts: an introduction and overview; Signal Analysisus and Signal Processing Techniques; an Introduction to Machine Learning Methods; and Modern Communication Systems and their Analysis.

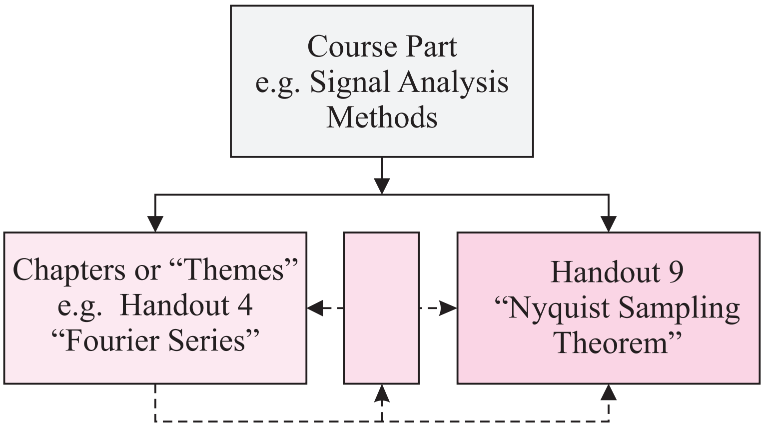


Figure 1.15: Each part or module is divided into a number of chapters or themes.

Investigation of ideas through code-based simulation and other demonstrations;

Practice of problem solving through examples in the notes and tutorial exercises.

Production of your knowledge, skills, and understanding through formative feedback and assessments.

The course content is structured as follows:

Modules or Parts The overall course is divided into three parts, as explained above, and shown in Figure 1.14. Each part of the course relies on the general Course Introduction and Application of Signal Processing, and signal analysis is a pre-requisite for modern communication systems.

Chapters or Themes Each module is divided into a number of chapters, as shown in Figure 1.15. Each chapter is a detailed handout containing a wealth

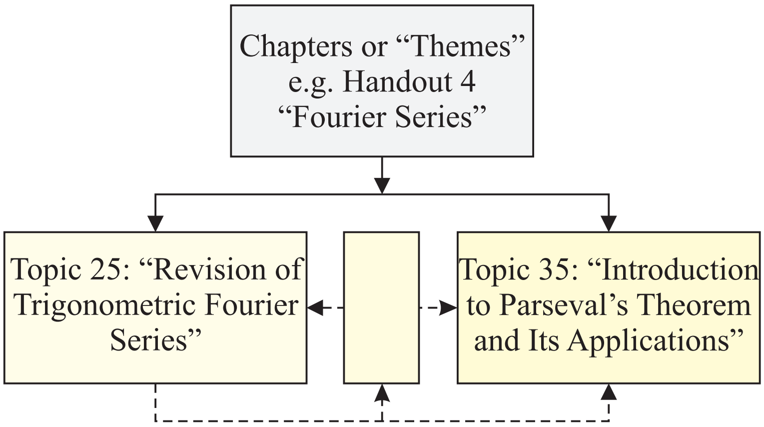


Figure 1.16: Each chapter or *theme* is divided into a number of topics.

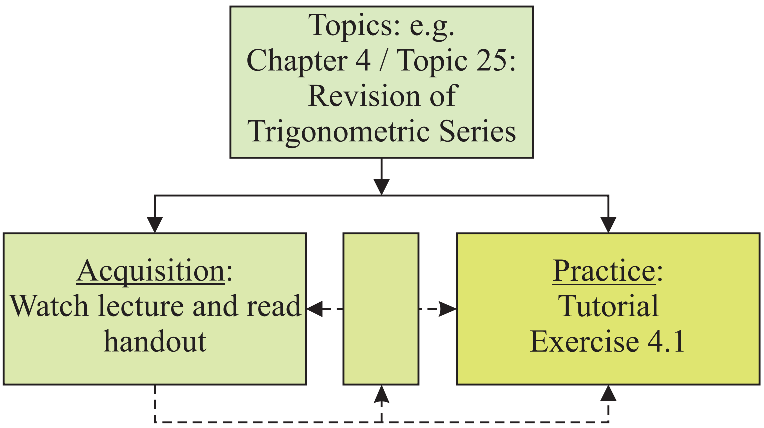
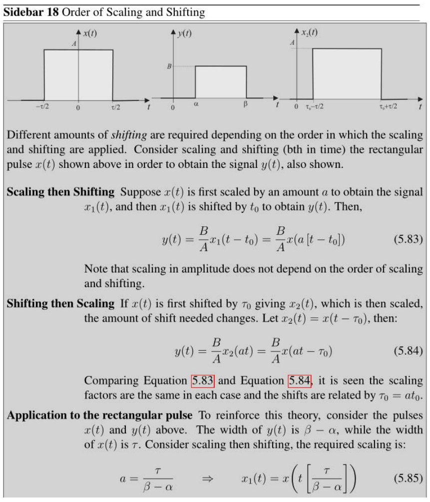


Figure 1.17: Each topic can be learnt through watching lectures, reading the handouts, discussion in examples classes and discussion boards, and practice of example questions.



Sidebar 18 Order of Scaling and Shifting

Different amounts of *shifting* are required depending on the order in which the scaling and shifting are applied. Consider scaling and shifting (bth in time) the rectangular pulse $x(t)$ shown above in order to obtain the signal $y(t)$, also shown.

Scaling then Shifting Suppose $x(t)$ is first scaled by an amount a to obtain the signal $x_1(t)$, and then $x_1(t)$ is shifted by t_0 to obtain $y(t)$. Then,

$$y(t) = \frac{B}{A} x_1(t - t_0) = \frac{B}{A} x(a\,[t - t_0]) \tag{5.83}$$

Note that scaling in amplitude does not depend on the order of scaling and shifting.

Shifting then Scaling If $x(t)$ is first shifted by τ_0 giving $x_2(t)$, which is then scaled, the amount of shift needed changes. Let $x_2(t) = x(t - \tau_0)$, then:

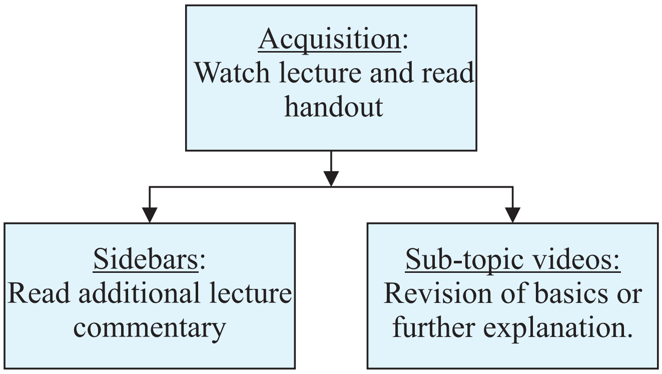
$$y(t) = \frac{B}{A} x_2(at) = \frac{B}{A} x(at - \tau_0) \tag{5.84}$$

Comparing Equation 5.83 and Equation 5.84, it is seen the scaling factors are the same in each case and the shifts are related by $\tau_0 = at_0$.

Application to the rectangular pulse To reinforce this theory, consider the pulses $x(t)$ and $y(t)$ above. The width of $y(t)$ is $\beta - \alpha$, while the width of $x(t)$ is τ. Consider scaling then shifting, the required scaling is:

$$a = \frac{\tau}{\beta - \alpha} \quad \Rightarrow \quad x_1(t) = x\left(t\left[\frac{\tau}{\beta - \alpha}\right]\right) \tag{5.85}$$

(a) Example of a sidebar in the handouts.



(b) Additional videos and commentary available.

Figure 1.18: Subtopics expand on fundamental material to aid understanding, but are not needed to progress with the course.

of information, including examples, further explanations, and tutorial exercises.

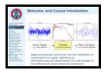
Topics Each chapter is divided into a number of topics, as shown in Figure 1.16. Each topic can be considered as a self-contained study component, although it clearly earlier topics are pre-requisites for latter topics – they are not designed for picking and choosing!

Blended teaching Each topic is divided into a number of learning opportunities, each of which are designed to help you learn and study the course in a structured manner. An example is shown in Figure 1.17.

Further study Some lecture videos and handouts have additional *subtopics* which expand on pre-requisite knowledge or subtle points, for those who want a deeper understanding of the course. An example is shown in Figure 1.18.

– End-of-Topic 2: **How do I study this course?** –

1.4 What is this Course About?



New slide

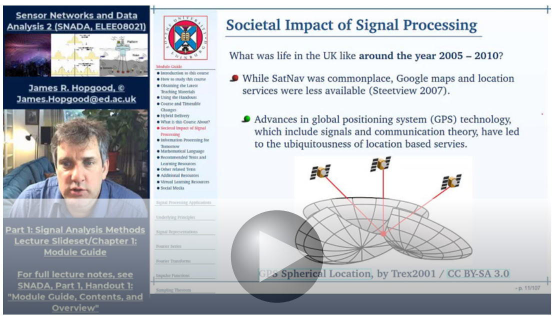
Topic Summary 3 Societal Impact of Signal Processing

Topic Objectives:

- Consider the rôle of sensor networks and information processing in society.
- Examples of advances in applications areas due to improvements in Information Theory.
- Consider future opportunities and priorites for this subject area.

Topic Activities:

Type	Details	Duration	Progress
Watch video	18 : 54 minute video	3× video length	
Read Handout	Read page 18 to page 25	8 mins/page	
Discussion Board	Your views on SNADA	20 minutes	



http://media.ed.ac.uk/media/1_j014ier2

Video Summary: This topic primarily discusses the societal impact of Information Theory, Signal Processing, Communication Theory, and Machine learning, by considering some particular application areas, especially those which rely on the fundamental and important fast Fourier Transform algorithm. The applications chosen are intended to be familiar, but far from being exhaustive. The topic considers how the performance of the solutions in some applications have changed over the past couple of decades. The purpose of this video is to further motivate he context of the course, and to specifically look at processing and communications of signals from sensors. The viewer is encouraged to continue the conversations raised in this video, through the discussion board, or in hybrid seminars.

The primary aim of this lecture module is to introduce the fundamental concepts of **signals**, **systems**, **machine learning,** and **communication theory**, especially in the context of sensor-networks. Sensing and data analysis is fundamental to all Engineering disciplines. It relies on a key understanding of sensor networks and how they communicate, resource and computation constraints, and an understanding of how data is sampled and then analysed. Signals are the output of sensors which have measured data, and this course gives an introduction to key signal analysis concepts.

After an introductory lecture on the different aspects of the course, including **signal processing**, **machine learning**, and **communication systems**, the first half of the course focuses on the fundamentals of **signals** analysis and a brief introduction to **systems**, which are necessary for analysing the output signals of many sensor networks, as well as more general applicability to a variety of Engineering problems as well as engineering solutions to numerous application areas. Applications range from the study of vibrations and control in **Mechanical Engineering**, to the analysis of **Analogue Electronic** filter circuits, to the analysis of modern communication systems, to the design of medical imaging systems.

The middle sixth of the course gives an introduction to machine learning techniques, and the last third of the course covers an introduction to the fundamentals of communications theory.

The course develops a consistent mathematical framework which is the necessary language for working in the fields of **communications**, **automatic control**, **machine learning**, **filter design**, and **signal processing**, as well as other areas of **information processing**.

This course is essentially the first half of an introduction to **signals** and **communication systems** which continues in the third year with the course **ELEE09017: Signal and Communication Systems 3**. A related third year course is **SCEE09002: Control and Instrumentation Engineering 3**, although it is not necessary to take that course to understand the present course. The fourth year offers further related courses in **ELEE10010: Digital Signal Analysis 4** and **ELEE10006: Digital Communications 4**. There may be further options in your fifth year depending on your degree programme.

1.4.1 Societal Impact of Signal Processing

New slide

The nature of **signals, systems, and communications** and the role of **machine learning** will be discussed in the next handout. The purpose of this handout is merely to give a reference to learning outcomes, course structure, lecture lists, recommended text, and any **pre-requisites.** However, in order to start thinking about what this course is actually about from a practical perspective, consider how the fields of signal proessing, communications, and machine learning have influenced applications in the past decades.

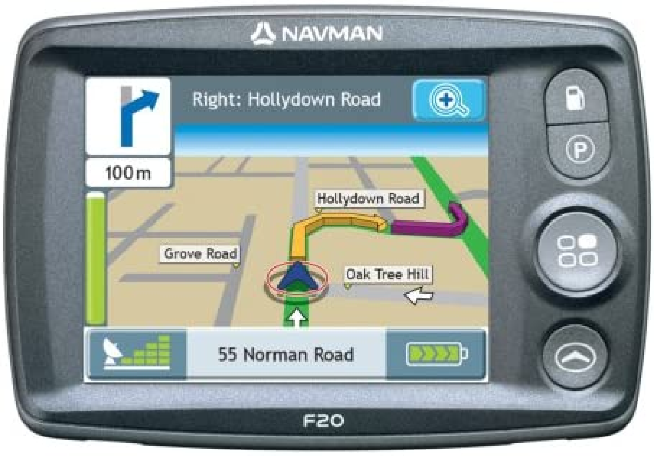


Figure 1.19: Navman F20 (2006), by Raimond Spekking / CC BY-SA 4.0

Example 1.1 (What are Sensor Networks and Information Processing?). First, however, what are your thoughts on the rôle of sensor networks and information processing in society?

1. What specific applications can you think sensor networks would be useful for?
2. What is signal processing and communications?
3. Which applications have *signal processing*, communications, and *machine learning* had an impact on in society?
4. What are the potential ethical issues that need to be considered in these applications?

The following applications are exemplars of how signal processing has had an impact on society. Note that we will be careful here to distinguish between applications that involve a significant **signal processing** component, as opposed to applications that involve purely **artificial intelligence (AI)** or **machine learning**. Unfortunately, due to the hype surrounding machine learning, any boundaries between the two disciplines is becoming more blurred, and in any case most of the following applications utilise a variety of Engineering disciplines – from antenna theory, computer science, material science, and so on.

What was life in the UK like **around the year 2005 – 2010**?

- While SatNav was commonplace, Google maps and location services were less available (Steetview 2007).
  - Advances in global positioning system (GPS) technology, which include signals and communication theory, have led to the ubiquitousness of location based servies see Figure 1.20. Being able to detect weak signals in

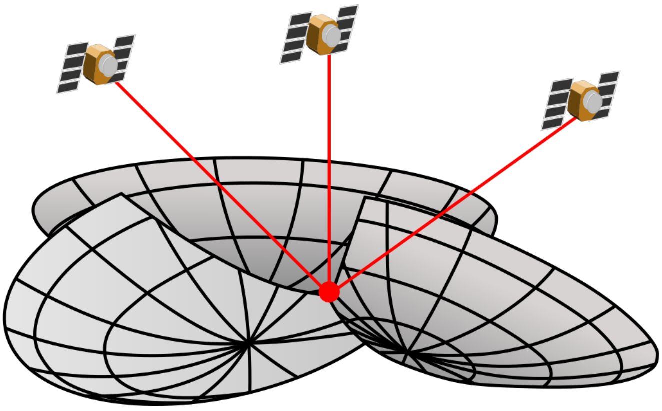


Figure 1.20: GPS Spherical Location, by Trex2001 / CC BY-SA 3.0

Figure 1.21: Close-talk microphones and headsets

substantial amounts of background noise, as well as efficiently solving the equations for localisation of the receiving device are applications of signal processing.

- Speech recognition available with *close-talk* microphones, but often reduced dialogue or very inaccurate.
  - Far field speech recognition using microphone arrays common place (Amazon Echo demonstration), due to improvements in signal processing algorithmic design and real-time processing. While there has been a substantial improvement in cloud-based machine learning technology, ultimately, pre-processing the signals using a micophone array is of considerable benefit.
- Mobile data was possible with 3G networks based on spread-spectrum code

Figure 1.22: The 4th generation Amazon Echo

Figure 1.23: A 3G phone from around 2008.

division multiple access (CDMA) technology. Data-rates relatively slow, and data contracts limited.

- 4G and 5G technology uses orthogonal frequency division multiplexing (OFDM) technology based on the Fast Fourier transform (FFT). The FFT is a practical implementation of the Fourier analysis learnt last year in **Signals and Communication Systems 2**, and is a fundamental signal processing tool. Mobile broadband speeds often offer higher than broadband access via copper-twisted pair (assuming far from cabinet – the last mile problem). Many smartphone apps wouldn't be useable without these improvements in 4G mobile wireless technology. Note that there are also recent advances in 5G technology that will is beyond the scope of this course.

• Video streaming services were relatively limited; internet calls mostly audio

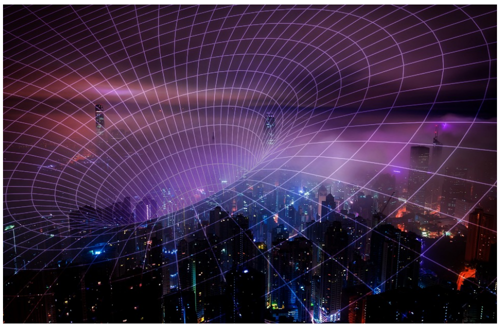


Figure 1.24: 5G Technology

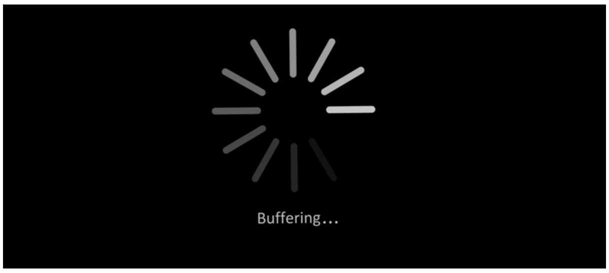


Figure 1.25: Bandwidth limitations and weaker compression technologies prevented full video streaming.

based.

- Improvements in compression technology has led to Zoom, Teams, video conferencing, Netflix and so forth (4K UHD with Dolby Audio).

  Video-conferencing has been an extremely important communication technology during the Covid-19 Pandemic.

Going further back, what was life in the UK like around 20 **to** 15 **years ago**?

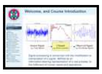
- Home media via VHS tape, with DVD new to the market.
  - Advances in compression and physical media has led to 4k UHD discs with full cimema sound (DTS:X, Dolby Atmos, Auro-3D) (UHD blu-ray far superior to 4k streaming).
- CD technology was now common-place (44.1 kHz at 16 bits).
  - Lossless high-definition audio streaming to any device (including mobile platforms), 192 kHz at 24 bits, multi-channel.
- Broadcast television: via Terresteral, Satelite or Cable in Metroplitan areas – 4 primary channels, 13 or so channels.

Figure 1.26: Ultrasound devices connected to smartphones are here today.

  - Compression technology, Digital-Television services, video-on-demand services.
- Email was available, primarily to colleagues.
  - Instant messaging, big-data handling, internet shopping.

How do you see Signals and Communication Theory improving the world in the next twenty years? *Because you will be building these technologies.*

- All these techniques rely on the FFT – which relies on mathematics, and a clear understanding of the concept of time-frequency signal decompositions.



1.4.2 Information Processing for Tomorrow

New slide

Returning to our earlier question, can you think: How might the rôle of sensor networks, communications, and information processing change society in the future?

- In 2002, civilians users were able to receive a non-degraded GPS signal globally, which soon led to Qualcomm announcing successful tests of assisted GPS for mobile phones in 2004.

  Was the proliferation of location-aware services predictable back then, or was the focus on person-location for emergency services?

  What current in-development Information Engineering solutions will be common-place when you start your graduate career?
- What role will sensor networks play in Smart Transportation in the future?

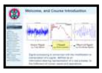
  Why do we still jump on a crowded bus, not knowing that there is an empty bus just a few minutes behind?

- Similarly, What role will signal processing and machine learning play in modern medical diagnostics?

  What role will arrays or networks of pocket-sized ultrasound machines or medical scanners play in society?

– End-of-Topic 3: **Applications of Signal Processing Past, Present, and Future! –**

1.5 Mathematical Language of Signals and Communications



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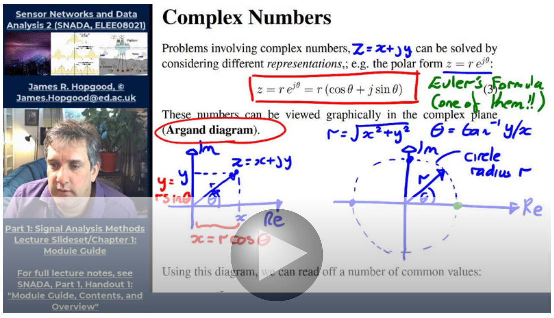
Topic Summary 4 Mathematical Prerequisites for the SNADA course

Topic Objectives:

- Discussion of the need to use mathematics in this Sensor Networks and Data Analysis (SNADA) course.
- General mathematical prerequisites of the course.
- Examples of simplifying some common cases of Polar representations of complex numbers.

Topic Activities:

Type	Details	Duration	Progress
E-training	Become familiar with LEARN	20 mins	
Watch video	20 : 35 minute video	$3\times$ video length	
Study Handout	Read page 26 to page 31	8 mins/page	
Try Example	Work through 1.3 and 1.4	8 mins/page	



http://media.ed.ac.uk/media/1_gs23fzc6

Video Summary: This topic motivates the need for using mathematical analysis in this course on Sensor Networks and Data Analysis, emphasising that relying on pre-existing software solution is limited in developing and verifying solutions to new problems. The Topic then highlights the fairly straightforward mathematical pre-requisites. The Topic continues to then provide some simple examples of manipulating the polar form of complex numbers or complex phasors, which are fundamental to developing the complex Fourier series and complex Fourier transform later in the course.

It is perfectly possible to teach a course in Signals, Systems, and Communications without the use of **Engineering Mathematics** whatsoever. However, the knowledge gained from such a learning process is restricted to rules of thumb, hand-waving arguments, and an extremely vague understanding of the underlying concepts involved. While the notion of teaching a course by avoiding the mathematics sounds pedalogically appealing, it not only underestimates the capabilities that you have as a student, but also essentially sells the subject as something it is not.

As a result, the language used for the study of Signals, Systems, and Communications is mathematics and, as such, is taught in moderate detail. The mathematics is restricted to basic Engineering mathematics, and as such rquires a basic understanding of algebra, calculus, and complex numbers. At this stage, vector calculus, pure mathematics, measure theory, and tensors are **not needed**! As will be seen through the tutorial examples and exam questions, the actual calculations involved when applying the mathematical concepts are mostly restricted to:

1. multiplication of two exponential functions;
$$e^{x+y} = e^x\, e^y \tag{1.1}$$
2. Cartesian $(x + jy)$ and polar $(r\, e^{j\theta})$ manipulation of complex numbers (or **phasors**);
3. calculus integration and differentiation of trigonometric functions (sin and cosines), real and complex exponentials, polynomials, and constants:
$$\frac{d}{dx} e^{ax} = a\, e^{ax} \quad \Rightarrow \int_a^b e^{j\omega\, t}\, d\omega = \left[\frac{e^{j\omega\, t}}{j\, t}\right]_a^b; \tag{1.2}$$
4. roots of quadratic equations $(a\, x^2 + b\, x + c = 0)$;
5. area of simple geometric shapes (triangles mostly).

The way to get confident with mathematics is to try and use it as often as possible, get as much practice solving problems with mathematics as you can, and use it as a tool in your Engineering toolkit.

KEYPOINT! (Simpler than you think). Remember, at the end of the day, the Engineering Mathematics we use is the mathematics you've been studying since high school. The only reason it seems difficult is simply because you're not practicing is as much as you should be!

Example 1.3 (Complex Numbers). Problems involving complex numbers, $z = x + jy$, can be solved by considering different *representations*, and in particular the polar form $z = r\, e^{j\theta}$:
$$z = r\, e^{j\theta} = r\,(\cos\theta + j\sin\theta) \tag{1.10}$$
This is Euler's Equation, or at least, one of them, as there are many others! These numbers can be viewed graphically in the complex plane, which is often called the **Argand diagram**, as shown below. Using this diagram, find simpler expressions for the following common values: $e^{j\frac{\pi}{2}}$; $e^{2\pi j\, m}$, $m \in \mathbb{Z}$; $e^{j\frac{\pi}{6}}$; $e^{j\frac{\pi}{3}}$; $e^{j\frac{\pi}{4}}$; $e^{j\frac{2\pi}{3}}$.

Summary Slide 1 Engineering Mathematics Revision

Complex Numbers

Problems involving complex numbers, , can be solved by considering different *representations*,; e.g. the polar form $z = r\,e^{j\theta}$:

$$z = r\,e^{j\theta} = r\,(\cos\theta + j\sin\theta) \tag{1.3}$$

These numbers can be viewed graphically in the complex plane (**Argand diagram**).

Using this diagram, we can read off a number of common values:

- $z = e^{j\frac{\pi}{2}} = j$ is on the positive imaginary axis;
-
-
-
-
- $e^{j\left(\frac{\pi}{2}+\frac{\pi}{6}\right)} = e^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$, and $e^{j\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{j}{2}$;

Finally, use diagrams to solve values of sinusoids and cosinusoids; e.g. $\cos(\pi m) = (-1)^m$ for integer m.

Summary Slide 2 Engineering Mathematics Revision

Complex Numbers (Continued)

Example 1.2 (Complex Number Manipulations).

Let $j = \sqrt{-1}$. Write down an alternative expression for the value:

$$z = j^j \tag{1.4}$$

SOLUTION. Obviously you could leave this in the form j^j, but that shows no insight. Use $z = r\, e^{j\theta}$ and the unit-circle diagram.

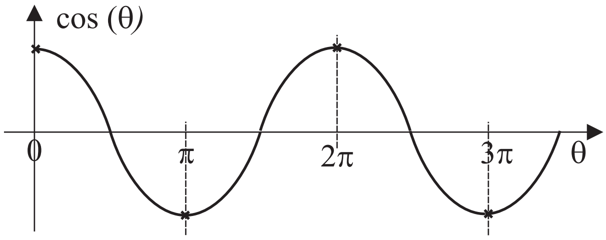
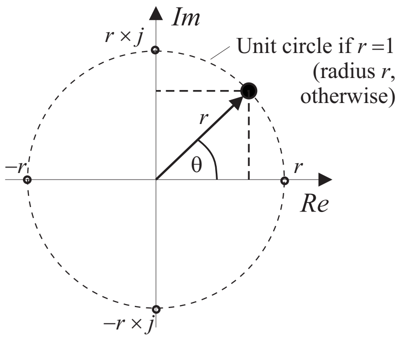


Figure 1.27: Plotting trigonometric functions can often help simplify expressions such as $\cos(\pi m)$.



SOLUTION. Taking each example in turn, consider the position of this complex numbers on the unit circle, which is the circle of radius one.

1. $z = e^{j\frac{\pi}{2}} = j$ is on the positive imaginary axis, where $r = 1$;
2. $z = e^{2\pi jm} = 1 = e^{j0}$ for any integer m;
3. for $z = e^{j\frac{\pi}{6}}$, then $\theta = 30\,\mathrm{deg}$; a neat trick to remember this value is to note the *adjacent* of the triangle in the **unit circle** is longer than the *opposite*; observing $\frac{\sqrt{3}}{2} > \frac{1}{2}$, using basic trigonometic values, $z = e^{j\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{j}{2}$;
4. conversely, it follows that $e^{j\frac{\pi}{3}} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$;
5. moving further around the unit circle, then it follows $e^{j\frac{\pi}{4}} = \frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1 + j)$; similarly, $e^{j\frac{3\pi}{4}} = \frac{\sqrt{2}}{2}(-1 + j)$;
6. $e^{j\left(\frac{\pi}{2}+\frac{\pi}{6}\right)} = e^{j\frac{2\pi}{3}} = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$, and $e^{j\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{j}{2}$;

Finally, use diagrams such as in Figure 1.27 to solve values of sinusoids and cosinusoids; e.g. $\cos(\pi m) = (-1)^m$ for integer m.

Example 1.4 (Multi-choice Question). What form does the complex number $z = j^j$ take?

1. Some hard maths involving elliptic curves and modularity theorems?

2. An easily recognised complex number?
3. A single real number involving e and π?
4. An interesting infinite sequence of real numbers?

- Rather than guessing, can you find the detailed solution to this question?

– End-of-Topic 4: **Overview of the pre-requisite mathematics required in the signal processing component of this course –**

1.6 Course Overview

Topic Summary 5 Course Guide and Text Books

Topic Objectives:

- This Topic is being updated, please ignore.

This section contains the most up to date information regarding the course content, and may differ in minor detail to that which appears on the DPRS at: `http://www.drps.ed.ac.uk/current/dpt/cxelee08021.htm`.

1.6.1 Course Summary (Short Description)

Sensing and data analysis is fundamental to all Engineering disciplines. It relies on a key understanding of sensor networks and how they communicate, resource and computation constraints, and an understanding of how data is sampled and then analysed. Signals are the output of sensors which have measured data, and this course gives an introduction to key signal analysis concepts.

This course aims to:

1. introduce students to the fundamentals of Sensor Networks and Signal Processing, Communications, and Information Theory;
2. provide an insight into time domain and frequency domain analysis of continuous-time signals, and provide an insight into the sampling process and properties of the resulting discrete-time signals;
3. introduce the students to basic communication modulation techniques, as well as probability theory for analysing random signals.

At the end of the module students will have acquired sufficient expertise in these concepts to appreciate how sensor networks and signal analysis can be used in a variety of disciplines.

1.6.2 Summary of Intended (Extended) Learning Outcomes

By the end of the course, a student should be able to:

- distinguish between, and give examples of, deterministic and random, periodic and aperiodic, continuous-time and discrete-time signals;
- determine the fundamental frequency of periodic signals, and determine when the sum of periodic signals might lead to a non-periodic signal;
- evaluate the trigonometric, complex Fourier Series, and Fourier transforms of simple waveforms, provide a physical interpretation for these transforms, plot phase, magnitude, and line spectra, and explain the relationship between the various transforms;
- appreciate the concept of a linear signal decomposition, the concept of orthogonality, the rationale of using complex phasors as a basis function set, and appreciate other orthogonal basis sets such as Walsh functions;
- distinguish between energy and power signals, be able to perform the appropriate calculation for a given signal, and be able to apply Parseval's theorem;
- derive or recall properties of the Fourier transform for simplifying Fourier transform calculations;

- recall the definitions of the Dirac delta impulse function, the Heaviside step function, and their respective properties, including the sifting theorem, and the impulse train;
- understand the structure of a sampled data system, and explain the basic ideas behind sampling a continuous-time signal to obtain a discrete-time signal;
- recall the Nyquist sampling theorem and analyse the effect of sampling on the frequency content of a signal;
- develop the discrete-time Fourier transform (DTFT) and inverse DTFT for the Fourier analysis of discrete-time signals; undertake basic DTFTs of simple signals;
- use the notion of duality to understand sampling in frequency, leading to the notion of temporal-spectral sampling, and therefore appreciate the development of the discrete Fourier transform (DFT) for the analysis of finite-duration discrete-time signals;
- describe various pulse modulation schemes and circuits for their generation and reception, including OOK, FSK, and PSK;
- explain frequency division and time-vision multiplexing, and analyse simple multiplexing communication systems;
- explain how communication signals can be modelled as a random process, and perform simple statistical and probabilistic analysis of simple communication schemes;
- demonstrate an ability of use MATLAB to analyse simple signals and communication systems.

1.7 Course Structure

1.7.1 Teaching Hours

The total contact hours for this course include:

Core teaching hours 22 lectures total, 4 tutorials per student (one tutorial every fortnight for a particular student), 4 examples classes (approximately fortnightly), one 3-hour computing lab.[1]

Teaching Pattern The study pattern and course structure is 2 lectures per week for 11 weeks, with staggered fortnightly tutorials starting from the third week, and four examples class on fortnightly basis, biased towards the end of the semester.

[1]Over the Semester, students will therefore hvae $11 \times 2 + 4 + 4 + 3 = 33$ total hours of teaching, or an average of three hours per week. In some weeks, the loading will naturally be higher, others it will be lower.

Office hours The course lectures will hold an office hour for this course, which is an opportunity to come and discuss any course issues. You are advised to email the course lecturer so both student and lecturer can prepare and plan appropriately to use this hour most efficiently. The lecturers will advertise their office hours via LEARN.

Mock Exam Where possible, a mini mock exam will be held later in the semester in order to provide feedback on your approach and techniques to solving exam questions. Use these to your advantage!

Feedback/forward opportunities Further **feedback** and **feedforward** hours are provided through **office hours**, **tutorials**, and **examples classes**. You are expected to use these opportunities to gain both feedback and feedforward advice and recommendations.

Revision Revision lectures and examples classes will be held after lectures have finished.

Your Study Hours You are expected to use your study hours to practice what you have learnt through the tutorial questions. Do not arrive at tutorials without previously having attempted such questions, as this will simply be a waste of your time.

1.7.2 Preliminary Lecture List (under revision)

1. Course overview, and introduction to signals, systems, communications and the broader topic of signal processing (2 hours).

2. Nature of, and types of signals; definitions of continuous-time, discrete time, periodic, aperiodic, deterministic and random. Introduction to phasors and concept of frequency of single tone, typical signals and signal classification, power and energy (1 hour).

3. Signal decompositions and concept of signal building blocks, Fourier Analysis, including trigonometric and complex Fourier series, Parseval's theorem, physical interpretations, and plotting spectra (2 hours).

4. Development of the Fourier transform and its properties (2 hour).

5. The concept of an impulse, step function, impulse train, sifting theorem, and Fourier transform of periodic signals (1 hour).

6. Introduction to sampled data systems, Nyquist's Sampling Theorem and Discrete-Time Signals (including discrete-time convolution) (2 hours).

7. Development of the DTFT for analysing discrete-time signals, its properties, and application to simple signals (2 hours).

8. Development of the DFT (1 hour).

9. Introduction to communication theory and modulation techniques, including OOK, FSK, and PSK (2 hours)

10. Multiplexing techniques, including Frequency Division Multiplexing and Time Division Multiplexing (2 hours)

11. Basic Information theory and probability (3 hours).

1.8 Related fourth and fifth year modules

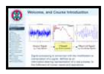
The following third and fourth year modules are closely related to this module. The first three in particular assume a **good working knowledge** of Signals and Communications 2:

- Signals and Communications 3 (*ELEE09027/8)*;
- Control and Instrumentation Engineering 3 (*SCEE09002*);
- Digital Signal Analysis 4 (*ELEE10010*);
- Digital Communications 4 (*ELEE10006*);

The following fifth year modules are closely related to this module.

- Machine Learning in Signal Processing 5 (*ELEE11103*)
- Advanced Wireless Communications 5 (*ELEE11093*)
- Advanced Coding Techniques 5 (*ELEE11092*)

1.9 Recommended Texts and Learning Resources



New slide

KEYPOINT! (About Recommended Books). This section lists some text books that might help in your studies. However, while these books come recommended, the handouts and tutorial sheets, along with material presented in lectures and available on the course website, will all be more than sufficient to study from. The text books listed here are only for additional reading or to verify the material presented in this handouts.

The recommended text for this course, which will also cover next years **Signals and Communications 3** course, as well as some fourth year courses, will be cited throughout the handouts as [Ziemer:2010], with the front cover shown in Figure 1.28:

> Ziemer R. E. and W. H. Tranter, *Principles of Communications: Systems, Modulation, and Noise*, Sixth edition, John Wiley and Sons, 2010.

IDENTIFIERS – *Paperback*, ISBN10: 0470398787, ISBN13: 978-0470398784

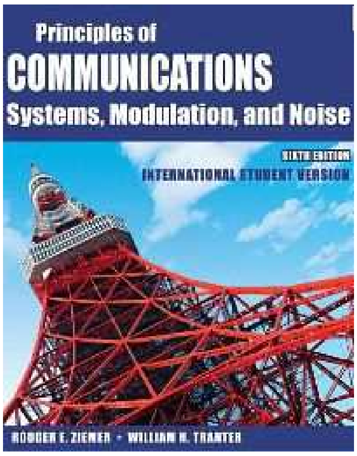
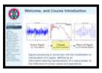


Figure 1.28: The course text book [Ziemer:2010].



1.9.1 Other related Texts

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The **essential book** for the previous incarnation of this course usually held in the third year, *ELEE09007 Electronic Engineering 3: Signals and Systems*, is [Mulgrew:2002], and the full reference is:

> Mulgew B., P. M. Grant, and J. S. Thompson, *Digital Signal Processing*: *Concepts and Applications*, Palgrave, Macmillan, 2003.

IDENTIFIERS – *Paperback*, ISBN10: 0333963563, ISBN13: 9780333963562

See `http://www.homepages.ed.ac.uk/pmg/SIGPRO/`

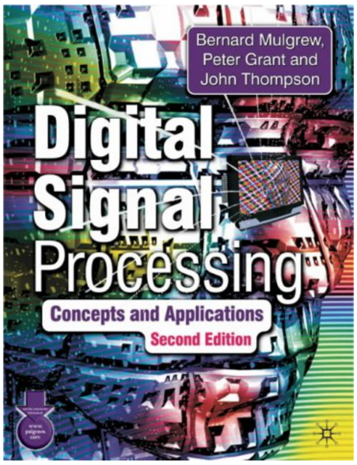
The latest edition was printed in 2003, but any edition will do. This book will be helpful for the first half of this course, and is also the recommended text book for the fourth year module *Digital Signal Analysis*, currently taught by *Dr Dave Laurenson*. You may find a copy in the reserve section of the **Robertson Library**, **Shelfmark: TK5102.5 Mul**. Because this book was used in a previous version of this course, it should be possible to find cheap copies. However, it is recommended that if you buy a book, you should buy [Ziemer:2010].

It can be constructive to get a different perspective on the subject; thus, an alternative presentation of roughly the same material is provided in the following book:

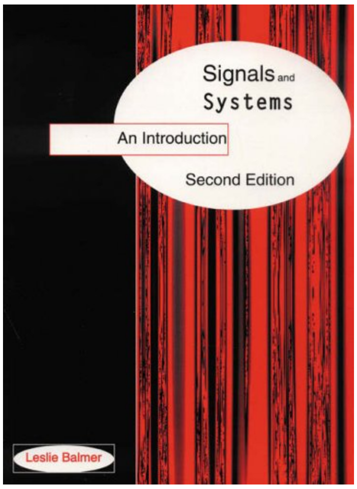
> Balmer L., *Signals and Systems*: *An Introduction*, Second edition, Prentice-Hall, Inc., 1997.

IDENTIFIERS – *Paperback*, ISBN10: 0134954729, ISBN13: 9780134956725

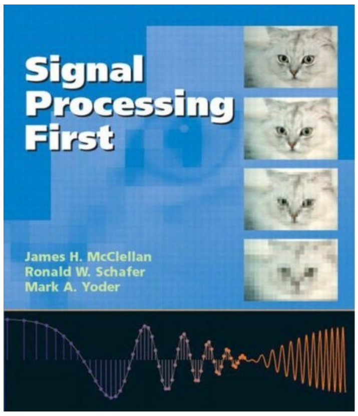
The first page of the preface to the first edition, also included in second edition, is well worth reading, and the appendix on complex numbers may prove useful. There is



(a) From the existing course: [Mulgrew:2002].



(b) An alternative presentation: [Balmer:1997].



(c) For an excellent introduction: [McClennan:2003].

Figure 1.29: Recommended and useful books for the course.

also a first edition of this book, which you may still find in the reserve section of the **Robertson Library**, **Shelfmark: 621.377.6 Bal**.

For an excellent and gentle introduction to signals and systems, with an elegant yet thorough overview of the mathematical framework involved, have a look at the following book, if you can get hold of a copy (but don't go spending money on it):

> McClellan J. H., R. W. Schafer, and M. A. Yoder, *Signal Processing First*, Pearson Education, Inv, 2003.

IDENTIFIERS – *Paperback*, ISBN10: 0131202650, ISBN13: 9780131202658
Hardback, ISBN10: 0130909998, ISBN13: 9780130909992

Finally, a recently published book by **Lathi** has an excellent and extremely clear coverage of Signals and Systems. This book comes highly recommended and may, in future, become the recommended course text book!

> Lathi B. P., *Linear Systems and Signals*, Oxford University Press, Inc., 2005.

IDENTIFIERS – *Hardback*, ISBN10: 0195158334, ISBN13: 9780195158335

1.9.2 Additional Resources

New slide

There are a number of web resources that will be of use. You are strongly encouraged to use these web resources in your studies. However, please be aware that if you use any information from the web in any assessment of your studies, and that material is

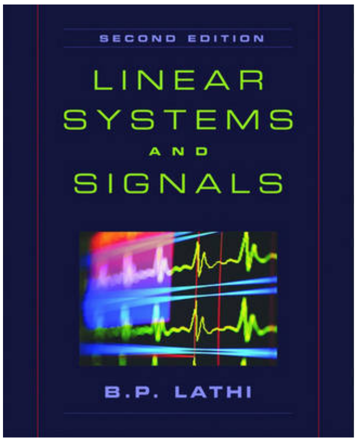


Figure 1.30: A highly recommended text book, that covers a great deal of Signals and Systems in a very clear manner: [Lathi:2005]

(a) The MATLAB logo. MATLAB is a useful utility to experiment with.

(b) Wikipedia, The Free Encyclopedia

Figure 1.31: me useful resources.

factually wrong, you may not receive credit for your efforts. However, naturally, if you quote anything from these lecture notes which is factually wrong, you won't be penalised!

- The extremely comprehensive and interactive mathematics encyclopedia:

  Weisstein E. W., *MathWorld*, From MathWorld - A Wolfram Web Resource, 2008.

  See `http://mathworld.wolfram.com`

- A wide variety of technical lectures can also be found at:

  Connexions, The Connexions Project, 2008.

  See `http://cnx.org`

- The Wikipedia online encyclopedia is very useful, although beware that there is no guarantee that the technical articles are either correct, or comprehensive. However, there are some excellent articles available on the site, so it is worth taking a look.

  *Wikipedia, The Free Encyclopedia*Wikipedia, The Free Encyclopedia, 2001 – present.

  See `http://en.wikipedia.org/`

- The Mathworks website, the creators of MATLAB, contains much useful information:

  MATLAB: The language of technical computing, The MathWorks, Inc., 2008.

  See `http://www.mathworks.com/`

- And, of course, the one website to rule them all:

  Google Search Engine, Google, Inc., 1998 – present.

  See `http://www.google.co.uk`

1.9.3 Virtual Learning Resources

New slide

All course materials can be found on the University LEARN system (Blackboard Software) via MyEd as shown in the screen-shot in Figure 1.32.

Moreover, lecture capture and screen casts, or *video bytes* will be made available. An example of a *video byte* is shown in Figure 1.33.

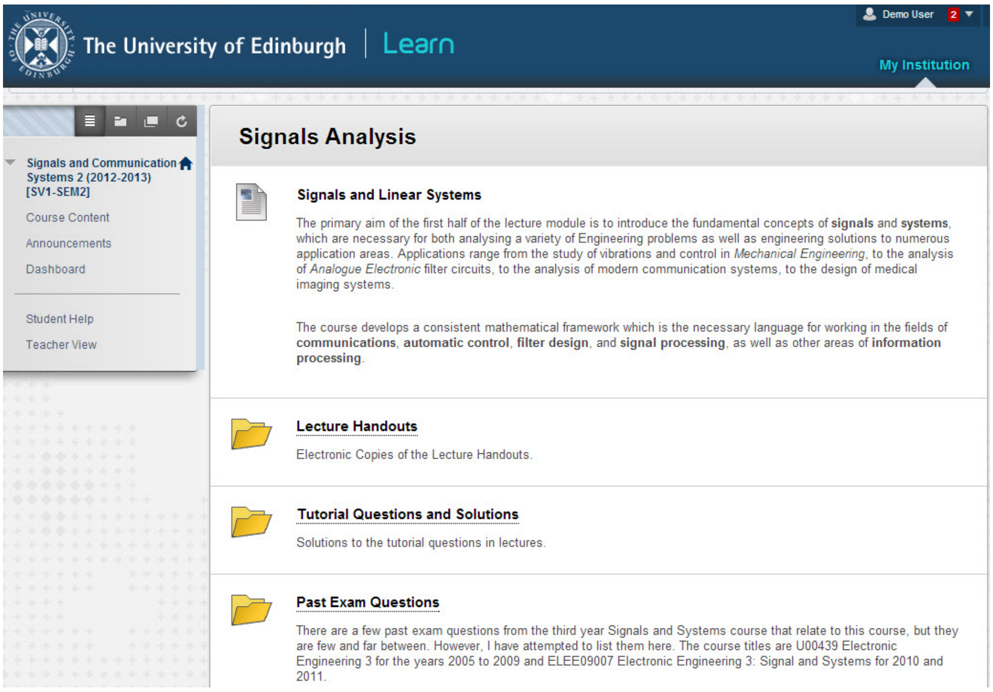
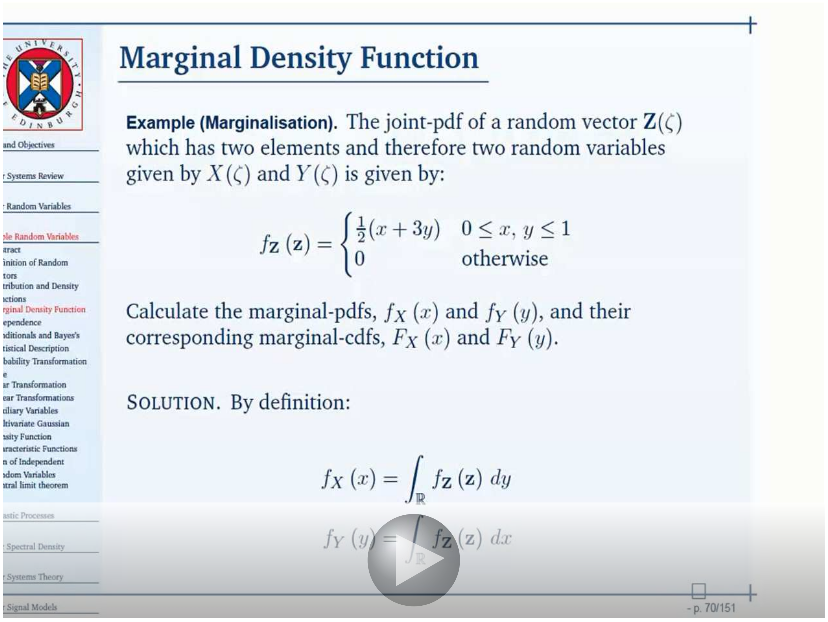


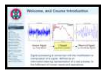
Figure 1.32: Course materials will be on the LEARN system.



`http://youtu.be/iSJvCEJe_N8`

Figure 1.33: An example of an embedded YouTube video clip. These *video bytes* will be used to add further explanation where appropriate.

1.9.4 Social Media



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While all formal announcements regarding the course will be made through the official University channels, such as LEARN and email, there is a Facebook page for this course, as well as a hashtag for Twitter. The rationale for this is that minor updates and non-urgent information can be made rather than sending out large numbers of emails via LEARN; moreover, other material related to the subject area can be posted easily, such as news items and videos. Such items will be posted via Twitter `@HopgoodTeaching` on `http://twitter.com/HopgoodTeaching`; the hashtag `UoE_SCEE08007` will be used for comments relating to this course. The course Facebook page can be found at: `https://www.facebook.com/UoE.SCEE08007`.

Finally, there is a YouTube playlist for lecture captures for this course. The list is hidden from public view, and details will be circulated via LEARN.

– End-of-Topic 5: **further learning resources** –

1.10 Errata, Recent, and Major Changes in Notes

These notes have evolved since a major course revision in 2013, where the entire course was entirely rewritten. The structure of these notes built on existing material based on the textbook:

> Mulgew B., P. M. Grant, and J. S. Thompson, *Digital Signal Processing: Concepts and Applications*, Palgrave, Macmillan, 2003.
>
> **IDENTIFIERS** – *Paperback*, ISBN10: 0333963563, ISBN13: 9780333963562
>
> See http://www.homepages.ed.ac.uk/pmg/SIGPRO/

The notes have undergone minor and major corrections each year, based on feedback from students, and the notes are far from static. In fact, as the course is delivered each year, a number of corrections and changes will have been made since the published version of these lecture notes were printed, as well as future amendments. The corrected versions will be posted on the course web-site and updated continuously. Some changes are not simply corrections, but exist for numerous reasons, including making explanations clearer, providing more detail and so forth.

Previously, a list of these corrections and changes were presented here. However, after experience, it has seemed more appropriate to publish the entire **change-log**, and this is included in reverse chronological order in the final handout (available on LEARN). The changes are listed by year, and then by Chapter, and changes to Tutorial Questions or Solutions are also included.

Very minor updates will be added to Twitter `@HopgoodTeaching` on `http://twitter.com/HopgoodTeaching` rather than sending large numbers of emails out via LEARN. The hashtag `UoE_SCEE08007` will be used for comments relating to this course.

Note that all page numbers and equations references refer to the *published version* of this document. Currently, it is not possible to give the new page numbers and equation references due to the ever-changing re-pagination and equation numbering.

1.11 Tutorial Exercises

Each handout contains a number of tutorial questions of varying standards, covering the various topics raised in the corresponding part of the course. Typically the questions are progressively more difficult, although often the questions might simply cover a different topic. This isn't always the case however.

The topic broadly corresponding to each exercise is highlighted in bold in the parenthesis. Each exercise also has a difficulty rating associated with the question, numbered from 1 (easy) to 5 (difficult), and then the occasional $5*$ question (very difficult) that probably shouldn't be set as a question – but will definitely provide a challenge to some of you. The rating system can approximately be described as follows:

1. A straightforward question that typically requires direct application of a formula or simple method, that does not require any insight into the theory.

2. A question that requires some basic thought about how a formula or method should be applied, but for which the subsequent solution is relatively straightforward.

3. The most common type of exercise, in which the underlying theory requires some insight or in which the solution requires some care in the mathematical solution. However, these exercises are still relatively formulaic, and would be slightly below the standard of an undergraduate exam question.

4. An exercise that requires insight and care in obtaining a solution, and requires understanding rather than following a set procedure to finding the answer. This standard of question will be similar to the standard in the exams, but is might be shorter or longer than a standard exam questions.

5. An exercise that requires both considerable insight, as well as excellent skills in mathematical manipulation. These questions are occasionally long, and require perseverance.

There are currently no tutorial questions associated with this handout, although there is some advice on good practice in the next exercise, which you should practice throughout your course!

Exercise 1.1. KEYPOINT! (Learn good practice). There are some good practices that you can do to help you self-study as well as prepare for assessments such as examinations. These include:

- **Find yourself a nice style of writing out your solutions:**

  Do not write your solutions in the margins of the self-study question papers. This leads to scrappy answers that you will find hard to refer back to. Instead, make some effort to use a separate work book, and neatly write out your full solutions. Use a new sheet of blank paper where possible, as though you were preparing for an exam solution.

  Why? In most **assessment**, most of the marks are about the method, not the final solution. Therefore, it is important to show your working logically (see below).

- **Make sure others can read your work:**

  Your answers need to be legible in the examinations. So write your solutions as though others are going to mark them. Do not write in small font that is illegible except with a magnifying glass.

- **Explain your solutions:**

  Pay attention to your **annotation**, **notation**, and **flow**.

  Notation Make clear what all your symbols mean, don't leave it to chance that the examiner will intepret everything; for example, what does $x(t) \star y(t)$ mean? Convolution? Multiplication, and so forth.

  Annotation Include a few words which indicate what you are doing, and help explain how you go from one step to another. Sometimes, one or two words is sufficient.

  Flow Show a logical progression, or flow, to your working. Don't miss out steps, as this might lead to errors, as well as potentially missed marks in any assessments.

- **Problem Solving Techniques and Strategeis**

  Develop techniques for applying the resouces available, such as drawing a graph or diagram (often used in mathematics and engineering), identify sub-goals, make use of symmetry, identify special cases, and so forth.

Exercise 1.2. KEYPOINT! (Managing Expectations). You need to manage expectations for how you will **LEARN** and how to use the **TEACHING** materials.

- **Make a mistake? Don't worry!**

  Many people learn by trying, making mistakes, but then trying again. The first two steps (trying and failing) are easy; the final step (trying again) is the one that most of us need to work on.

- **Teaching materials will help most of the time:**

  You will receive typed solutions to the self-study questions. However, not everyone learns by simply reading the solution; you need to try the solutions yourself to really understand the thought processes.

  The key issue is to avoid reading someone elses solution and say to yourself "*I can do that*". Many people learn by trying problems without guidance, failing, and learning what they needed to do to make it work.

- **Have faith in the teaching materials:**

  The solutions at the end of the self-study questions are correct, so don't second guess them. If your answers don't match, try again! (Very rarely, there may be minor technical issues such as inequality signs, but please check!).

- **Everyone is different:**

  Some of the questions contain hints; hints are not useful for everyone! Also, some students will find different questions easier than others.

- **Self-study takes time:**

  Remember that a 10-credit course equates to approximately 100-hours of work. Included in this is approximately 33 lectures and tutorials across the semester, as well as revision classes, and the examination itself. This leaves approximately 60-hours of self-study.

  Including the 11-teaching weeks, one revision week, and two-weeks of exam diet, on average, this equates to at least 4-hours of self-directed study per week for each 10-credit course. This time includes reading lecture notes and solving self-study questions.

  The conclusion? The self-study questions will take time, and cannot be completed within the tutorial slot, or within just half an hour or so.

2

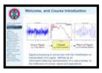
Applications of Signal Processing

We live in a society exquisitely dependent on science and technology, in which hardly anyone knows anything about science and technology.

Carl Sagan

This handout begins by motivating the need for this course material by looking at key application areas and concepts that will be studied in detail during the lectures.

2.1 What is Signal Processing?



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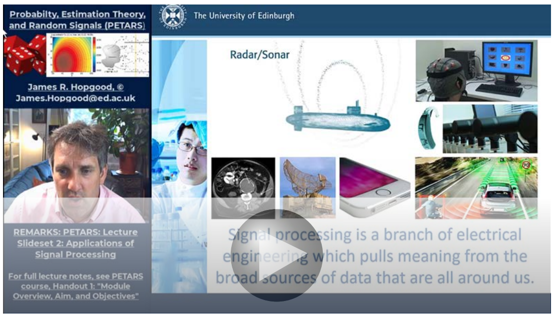
Topic Summary 6 What is Signal Processing?

Topic Objectives:

- Learn a high-level overview of signal processing.
- Identify signal processing in our daily lives.
- Understand why signal processing has become common-place.

Topic Activities:

Type	Details	Duration	Progress
Watch video	13 : 41 minute video	3×video length	
Discussion Board	Your views of signal processing	15 minutes	
Read Handout	Read page 48 to page 53	8 mins/page	



http://media.ed.ac.uk/media/1_t0qrik06

Video Summary: This video explains the role of signal processing in powering modern communications, entertainment, transportation, and healthcare systems, in addition to numerous industrial and defence applications. It explains why signal processing techniques have grown substantially over the past few decades in terms of improvements in signal processing algorithms as well as other key enabling technologies, such as low-power computing platforms, sensor technologies, and advances in battery technology.

Signal processing is a branch of electrical engineering which pulls meaning from the broad sources of data that are all around us.

Signal processing is at the heart of our modern world: signal processing powers modern communications (including voice recognition), modern entertainment (including motion sensing-gaming), tomorrow's transportation (including autonomous vehicles), and healthcare.

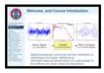
A nice introduction for the general public is presented in a YouTube video from the

`http://youtu.be/R90ciUoxcJU`

Figure 2.1: A video from the IEEE Signal Processing Society explaining *What is Signal Processing?*

IEE Signal Processing Society, as shown in Figure 2.1.

2.1.1 Modern Signal Processing Applications



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The last decade has seen a large number of domestic products which are heavily dependent on sophisticated *signal processing algorithms*. Some of these products are actually worth getting excited about in the sense they are extremely clever, and signal processing isn't restricted to simple removal of basic background noise (either in images or in audio). Some examples include:

- Microsoft Kinect, as shown in Figure 2.3, which includes skeletal tracking, depth estimation, acoustic noise cancellation, and speech identification and recognition; a demonstration of this will be given in lectures;
- Low-cost low-flying unmanned aerial vehicles (UAVs), which includes sophisticated algorithms for self-geolocation using on-board cameras and other sensors, and simultaneous localisation and mapping (SLAM), and on-board sensing of objects and targets; see Figure 2.2
- Video calling such as Skype and Facetime, which requires good audio, image, and video compression for network communication and online streaming;
- Computer-based music analysis, especially for game play, such as Guitar Hero and Rocksmith;

`http://youtu.be/Gj-5RNdUz3I`

Figure 2.2: A research UAV from Ascending Technologies: `http://www.asctec.de/en/uav-uas-drone-products/asctec-firefly`

- Room acoustic calibration (or *correction*) techniques in audio-visual setups (for example, most major audio-visual AV receivers);
- Far-field speech enhancement for voice assistance (Amazon Echo, Google Home);
- Digital image manipulation and processing using desktop software (*Photoshopping images*).

These are domestic applications which have grown over recent years, and of course are in addition to *medical imaging*, *defence*, *meteorological*, and *geophysical* applications, amongst many others as described below. It is important, however, to appreciate *why* digital techniques have grown substantially over the past few decades. Reasons include:

1. the dramatic improvement in computational power available on low-power devices due to the microelectronics revolution and advances in battery power;
2. the almost universal adoption of digital media, both audio and video, over the past two decades;
3. the vast improvements in sensor modalities including micro-electromechanical systems (MEMS) microphones and complementary metal-oxide-semiconductor (CMOS) cameras, as well as other MEMS devices such as accelerometers on mobile devices;

Figure 2.3: Hands-free human-computer interface (HCI).

Figure 2.4: UAVs used for package deliveries.

4. advances in understanding and performance of optimisation algorithms, estimation theory, and signal filters.

Signal processing is the technology that allows the manipulation, efficient storage, and analysis of signals that are recorded using a variety of sensor technologies, on electronic hardware. It is vital to appreciate that many of the electronic products, domestic, civilian, or military, are reliant on the processing of measured signals, from RAdio Detection And Ranging (Radar) (see Figure 2.5), to magnetic resonance imaging (MRI), through to cameras and microphones, or temperature sensors. It is vital to appreciate that most electronic products require some form of signal processing.

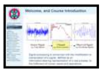
Figure 2.5: Radar of the type used for detection of aircraft. It rotates steadily sweeping the airspace with a narrow beam. Air Force Museum, by Bukvoed / CC BY-SA 3.0.

KEYPOINT! (Discussion Topic). Signal Processing as a subject has strong overlaps with other disciplines, such as machine learning in Computer Science, applied statistics in Mathematics and Econometrics, and remote sensing in the Geosciences. Using the discussion boards, think about and try and answer the questions:

1. What is signal processing and communications?
2. What applications have signal processing, communications, and machine learning had an impact on in society?
3. How do sensors play an important role in signal processing?

– End-of-Topic 6: **What is Signal Processing?** –

2.1.2 The fields of Signal Processing, Automatic Control, and Communications



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Topic Summary 7 Applications of Signal Processing and Communications

Topic Objectives:

- Examples signal processing applications.
- Privacy aware signal processing.
- Example of a signal processing and communication system.

Topic Activities:

Type	Details	Duration	Progress
Watch video	9 : 25 minute video	3×video length	
Read Handout	Read page 54 to page 56	8 mins/page	

http://media.ed.ac.uk/media/1_cwkcy5dq

Video Summary: This video considers in more detail some applications of signal processing, including biomedical, surveillance and homeland security, target tracking and navigation, mobile communications, and speech enhancement and recognition. The video then considers the application of delivering live music to a remote listener wearing a wireless headset. The different signal processing and communication systems involved in this application are discussed. This video provides background information for this course.

Although this course has been written with a bias towards *electronic engineering*, the mathematical tools and techniques introduced are fundamental in many other areas of Engineering. They are not limited to the examples given in this course by any stretch of the imagination. More significantly, this course initially covers continuous-time analogue signals, and then moves onto discrete-time signals. Discrete-time digital signals are the basis of modern digital and statistical signal processing, and is used in a plethora of modern Engineering problems. Modern advances in statistical signal processing, control, and communications include:

Biomedical From medical imaging to analysis and diagnosis, signal processing is now dominant in patient monitoring, preventive health care, and tele-medicine. From analysing electroencephalogram (EEG) scans to MRI (or nuclear magnetic resonance imaging (NMRI)), to classification and analysis of deoxyribonucleic acid (DNA) from micro-arrays, signal processing is required to make sense of the analogue signals to then provide information to clinicians and doctors.

Surveillance and homeland security From fingerprint analysis, voice transcription and communication monitoring, to the analysis of closed-circuit television (CCTV) footage, digital signal processing is applied in many areas of homeland security. It is an especially well-funded area at the moment.

Target tracking and navigation Although radar and sonar principally use analogue signals for *illuminating* an object with either an electromagnetic or acoustic wave, discrete-time signal processing is the primary method for analysing the received data. Typical features for estimation include detecting targets, estimating the position, orientation, and velocity of the object, target tracking and target identification.

Of recent interest is tracking groups of targets, such as a convey of vehicles, or a flock of birds. Attempting to track each individual target is an overly complicated problem, and by considering the group dynamics of a particular scenario, the multi-target tracking problem is substantially simplified.

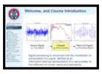
Mobile communications New challenges in mobile communications include next-generation networks; users demand higher data-rates, which in-turn requires higher bandwidth. Typically, higher-bandwidth communication systems have shorter ranges. Rather than have more and more base stations for the mobile network, there is substantial research into mobile ad-hoc networks.

A mobile ad-hoc network is a self-configuring network of mobile routers connected by wireless links, forming an arbitrary topology. The routers are free to move randomly and organize themselves arbitrarily; thus, the network's wireless topology may change rapidly and unpredictably. The challenge is to design a system that can cope with this changing topology, and is a very active area of research in communication theory.

A testament to the change in mobile communications is the availability of cheap mobile broadband modems which provide broadband Internet access which is comparable with fixed-line technologies that were available only a few years ago.

Speech enhancement and recognition Whether for the analysis of a black-box flight recording, for enhancing speech recognition in noisy and reverberant environments, or for the improved acoustic clarity of mobile phone conversations, the enhancement of acoustic signals is still a major aspect of signal processing research.

To consider how signal processing plays a role in modern domestic products, Section 2.1.3 considers how audio is streamed to your phone.



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2.1.3 From Studio to the Ear

As an immediate application of **signal and system** theory, consider the Engineering processes that have occurred in delivering down-loadable music to your phone, either high-definition formats such as free lossless audio codec (FLAC) files (much preferred and strongly encouraged) or lossy-compressed files (if you really really must and don't appreciate sonic quality). A very simplified diagram is shown in Figure 2.6.

A sound is generated in a room, which generates a sound pressure wave which propagates throughout the room until reaching a microphone. This electo-mechanical device converts the sound pressure wave into an **analogue continuous-time signal** which appears as a voltage waveform. This signal is sampled by an analogue-to-digital converter (ADC), which quantises and samples the signal, thereby producing a **discrete-time digital signal** that can be stored in finite-precision memory on a computer or digital recording device. This digital representation can then be processed on a digital audio workstation (DAW) which will compose various audio tracks and add any special-effects. Once the musical track is complete, this can then be delivered via the Internet to an online music server, probably in a compressed format (using perceptual compression). This audio track can then be delivered via a mobile network to a laptop or phone, which can then relay the signal to a set of Bluetooth headphones using the A2DP bluetooth mode. [1] This process involves a number of signal analysis and processing methods, such as sampling the analogue signal to produce a digital signal; it also involves systems, such as the effect of the acoustics on the propagation of sound, or the circuitry within the ADC; it also involves various communication systems, including wired baseband systems, medium-range wireless systems, and short-range personal wireless systems. This course provides an introduction to the understanding and analysis of these systems.

– End-of-Topic 7: **Examples of Signal Processing** –

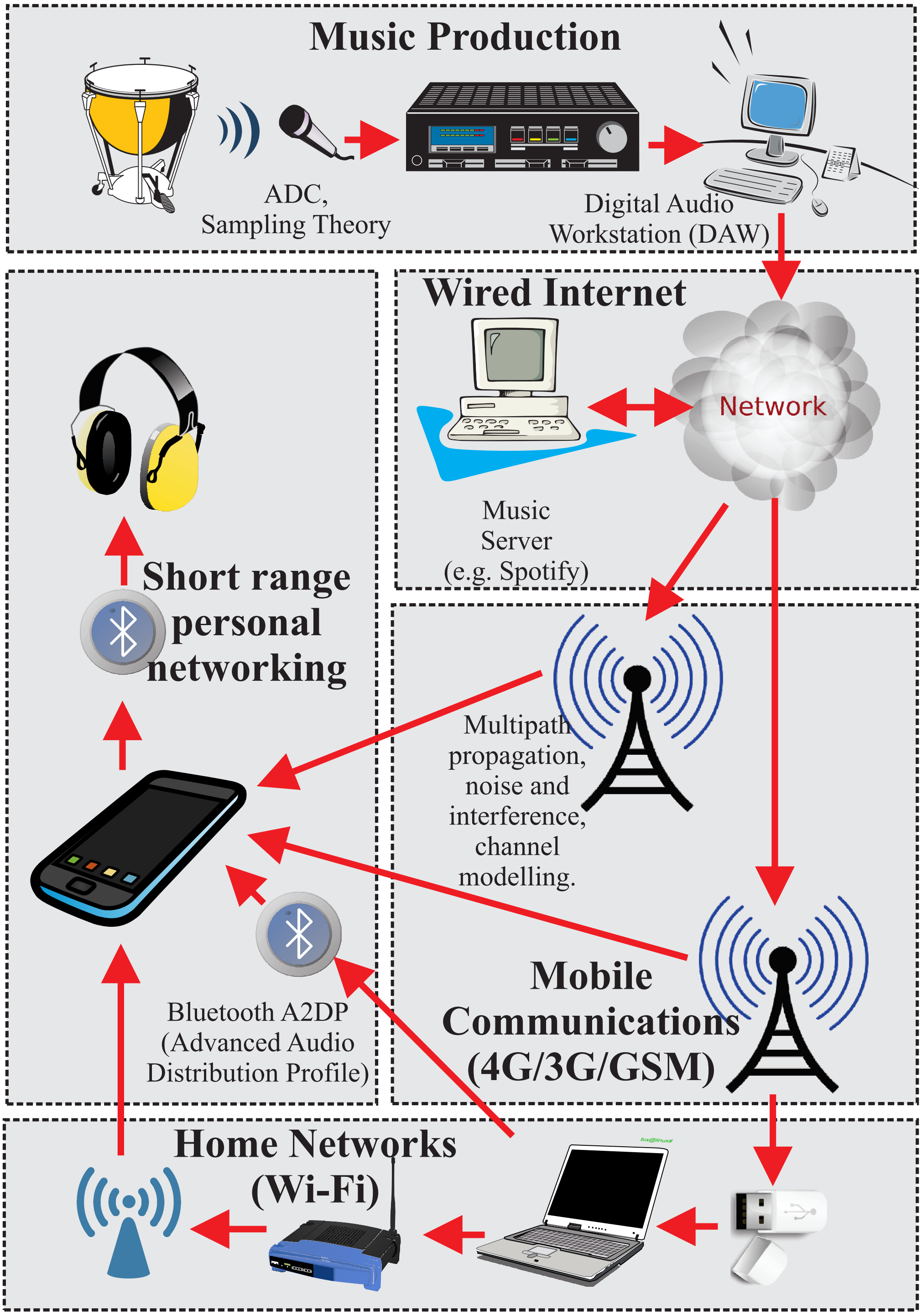
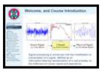


Figure 2.6: From an instrument being played through to listening on Advanced Audio Distribution Profile (A2DP) Bluetooth headphones via a portable media player.



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2.1.4 Case Study: Digital Audio Processing

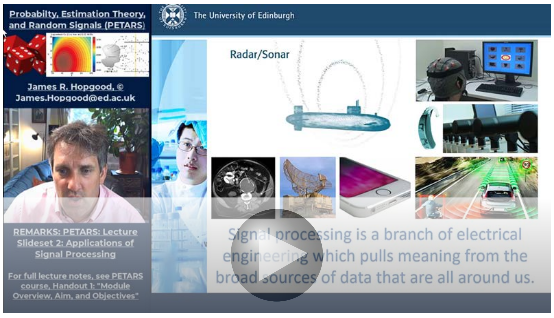
Topic Summary 8 Topic Title TBC

Topic Objectives:

- Objectives TBC.

Topic Activities:

Type	Details	Duration	Progress
Watch video	13 : 41 minute video	3×video length	
Discussion Board	Your views of signal processing	15 minutes	
Read Handout	Read page 58 to page 59	8 mins/page	



http://media.ed.ac.uk/media/1_t0qrik06

Video Summary: To be completed. Video above is a temporary link

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From an electronic Engineering perspective, **signals** and **systems** is the foundation for the revolution in digital audio and video processing. Sophisticated digital electronic devices are common-place in modern everyday life; games consoles, mobile telephones, digital audio recording and playback devices, digital audio broadcasting (DAB), digital video broadcasting (DVB), digital versitile disc (DVD) video, and audio and visual streams using Moving Picture Experts Group (MPEG) compression schemes, are all very familiar to us.

These devices are the direct result of over six decades of research and innovation in the areas of **information theory** and **signal processing**.

It is common knowledge that, for example, a MPEG-1 Audio Layer 3 (MP3) player encodes an audio **signal** as a binary sequence of *ones* and *zeros*. However, such a statement isn't saying very much since, for example, word processing documents are also encoded as *ones* and *zeros*. So what makes an audio file different to an arbitrary electronic document?

To understand thoroughly how MP3 works, more pertinent questions are:

[1]See http://en.wikipedia.org/wiki/Bluetooth_profile#Advanced_Audio_Distribution_Profile_.28A2DP.29

- How is a continuous-time analogue signal turned into a discrete sequence of binary numbers, and what are the properties of this binary sequence?
- How many *ones* and *zeros* are needed to represent the audio signal? If they are stored as bytes, how many bytes are needed to represent each individual audio sample? How many audio samples must be recorded to faithfully reproduce the real-world analogue signal?
- The MP3 standard uses a compression technique based on the characteristics of the human-hearing mechanism; it incorporates a method known as **perceptual masking** which removes (or masks) signal components that are not perceived by the human brain. What tools are used to characterise the properties of human-hearing, and how are these acoustical properties expressed in terms of an algorithm that runs on a **digital signal processing (DSP)**?
- How is an analogue signal recreated from a sequence of *ones* and *zeros*, and how can the deficiencies of our electronic systems be overcome by clever schemes with how the data is encoded in the first place?

The issue of using **signals** and **systems** theory to overcome the deficiencies of electronics is the basis of two recent data-formats that are available for high-quality audio reproduction. The compact disc (CD) player dominated the digital audio market from the mid-1980's until the early 2000's. Although other web-driven formats now dominate, such as MP3 and other proprietary formats, in the 1990's, the music industry initially pushed two new high-end audio formats: SACD and DVD-A. These formats store more data than the traditional CD, despite the fact that CDs already store just enough data to accurately encode the audio stream. By storing much more information than needed, SACD and DVD-A can use several tricks which mean that cheaper and less accurate electronics are needed in the playback device. How exactly do these tricks work? This will be answered later in the course.

The physical-media based SACD and DVD-A are essentially a failed format, primarily because of their high-prices, the lack of interest in multi-channel audio formats at the time, and the fact that there is sufficient download bandwidth to avoid physical-media for music. Nevertheless, stereo HD audio files such as 24/96 formats are increasingly becoming available in a download format such as FLAC and ALAC, amongst others. The insight gained from the SACD and DVD-A are the same as for downloadable HD audio formats, and Sony is in now pushing the hi-res audio format with considerable drive: `http://www.sony.co.uk/electronics/hi-res-audio`.

– End-of-Topic 8: **case studies of signal processing** –

(a) The Blu-Ray Disc Logo

(b) The digital versitile disc-audio (DVD-A) logo.

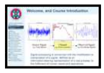
(c) The super-audio CD (SACD) logo.

(d) The free lossless audio codec (FLAC) logo.

(e) Sony and Hi-Res Audio.

Figure 2.7: High-quality audio formats. Note that SACD and DVD-A are essentially a failed format, but HD audio files such as $24/96$ formats are increasingly becoming available in a download format such as FLAC.

2.1.5 Why Study Signals and Communications?



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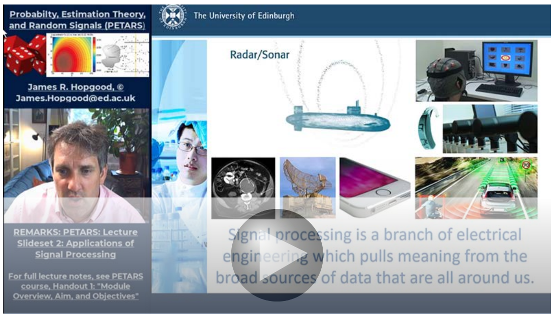
Topic Summary 9 Topic Title TBC

Topic Objectives:

- Objectives TBC.

Topic Activities:

Type	Details	Duration	Progress
Watch video	13 : 41 minute video	3×video length	
Discussion Board	Your views of signal processing	15 minutes	
Read Handout	Read page 61 to page 63	8 mins/page	



`http://media.ed.ac.uk/media/1_t0qrik06`

Video Summary: To be completed. Video above is a temporary link

The need for formal analysis of signals and systems stems from a number of viewpoints which will become apparent as the course progresses. In the meantime, it perhaps is simplest to begin with, as an example, the circuit shown in Figure 2.8. You might have analysed this **linear system** in other courses in your degree; the most likely analysis you will have tried is evaluating the output of the circuit when a sinusoidal signal is applied to the input. We will cover this again in this course, but could you calculate the output of the system if a microphone were connected to the input of the circuit? In such a scenario, the microphone converts a sound pressure wave into an electrical signal as the result of an instrument being played or some arbitrary spoken speech.

KEYPOINT! (Analysing system output to an arbitrary input). Evaluating the output of a linear system to an arbitrary signal is made possible by using signal analysis techniques such as the **Fourier series** and **Fourier transforms**.

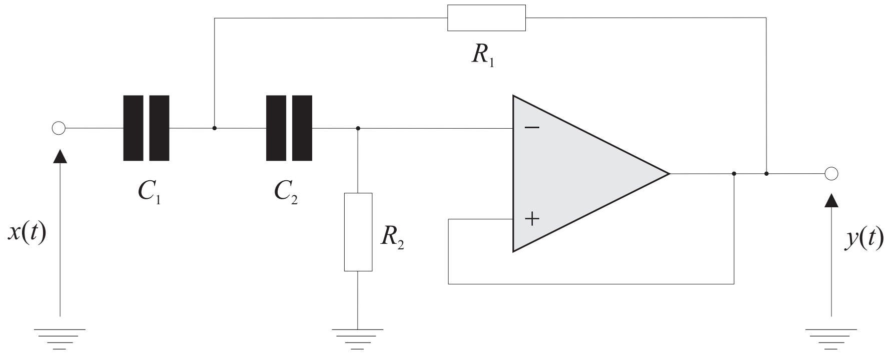
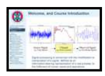


Figure 2.8: Second-order active **high-pass** filter.

Figure 2.9: Person undergoing an magnetoencephalography (MEG). National Institute of Mental Health.

2.2 Fundamental Signal Processing Problems



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Consider three fundamental signal processing problems:

1. Extracting *desired* signals from other signals.
2. Correcting *distortions* in measured signals.
3. Extracting *estimates* of indirect quantities from observed signals.

We shall briefly consider each of these fundamental applications in turn, and then consider what tools we need to solve these problems.

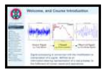
New slide

2.2.1 Extracting Signals from Other Signals

The generic problem of extracting signals from a mixture of other signals covers a wide range of applications, from simple noise reduction or removal, through to signal separation problems. As an example application, consider functional neuroimaging technique for mapping brain activity, called MEG, seen in Figure 2.9. This technique records magnetic fields produced by electrical currents naturally occurring in the brain using very sensitive magnetometers. These signals are extremely small; moreover, due to the number of electrodes present, a number of signals are measured, and there is a variety of interferences from other electromagnetic signals in the human body.

In the examples shown in Figure 2.10a, there are 148 signals of length 1695 samples over 10 seconds, or a sampling frequency of 169.55 Hz. In order to extract the brain activity, it is necessary to remove interference resulting from the heart. This interference overlaps with the desired frequencies in the brain activity, and therefore cannot be removed with a basic filter. This requires a technique called blind source separation (BSS), which requires models for the underlying interfering signals, as well as a model for the system which mixes the signals. The extracted signals are shown in Figure 2.10b, which show the signal resulting from the heart (can you calculate the patient's heart-rate?).

2.2.2 Correcting Distortions in Measured Signals



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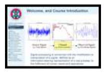
While visible-spectrum camera images are usually very high quality, remote imaging or sensing technologies are significantly less so. Techniques such as synthetic aperture RADAR (SAR) produce noisy images with much distortion. Signal processing techniques can be used to significantly improve the quality of the image, as shown in Figure 2.11.

2.2.3 Indirect Parameter Estimation



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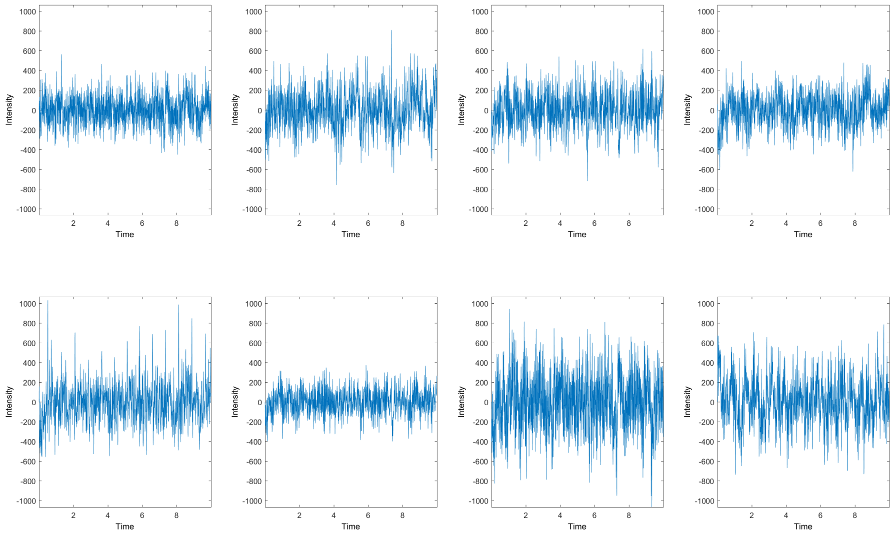
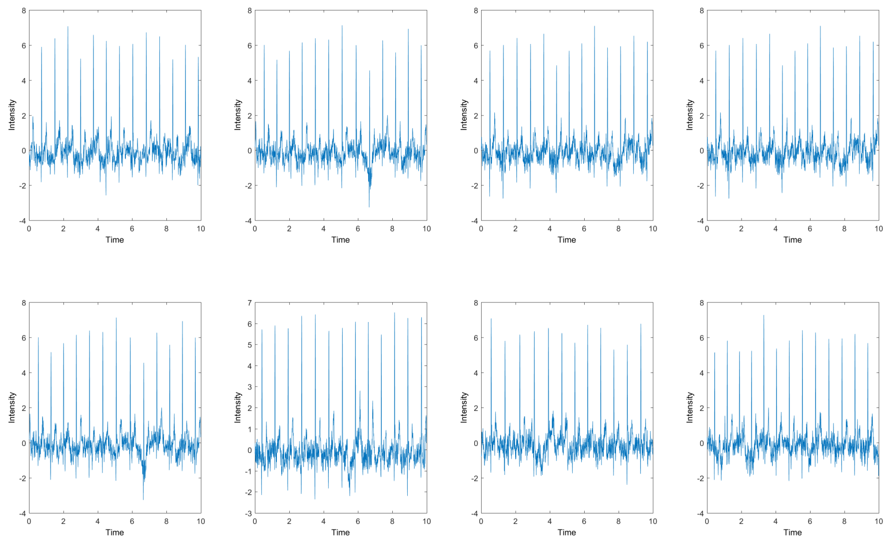
A further application of signal processing is the estimation of a quantity indirectly from measured signals. Figure 2.12 shows a multi-static radar system that uses multiple transmit and receive antenna's to locate an aircraft. The underlying signals are pulse chirps transmitted and received, but the quantity of interest is the actual position of the aircraft.

– End-of-Topic 9: **fundamental signal processing problems –**



(a) Example MEG signals.



(b) Extracted heart interference. Data kindly supplied by Dr Javier Escudero (School of Engineering, University of Edinburgh).

Figure 2.10: Signal processing of MEG signals.

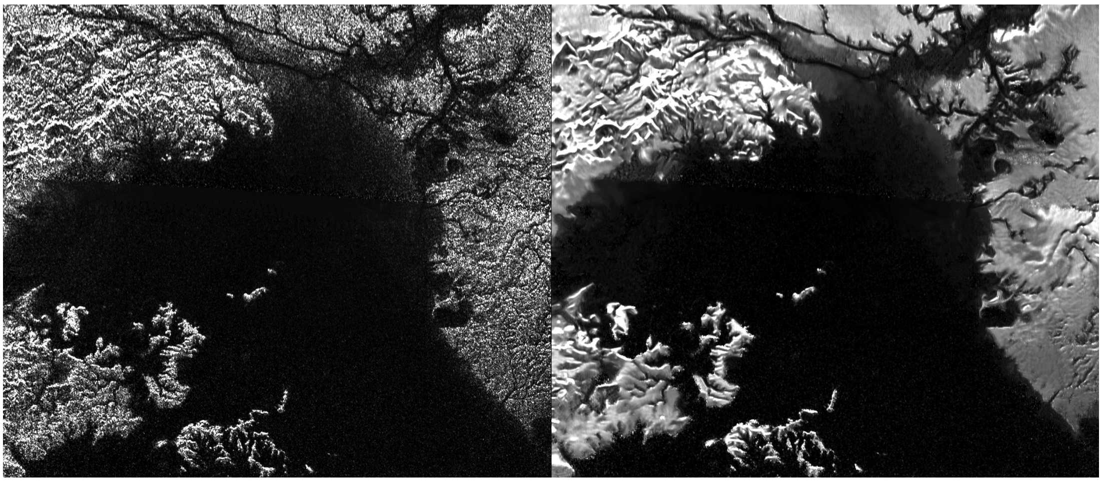


Figure 2.11: SAR and clearer despeckled views of Titan – Ligeia Mare. NASA/JPL-Caltech/ASI. Presented here are side-by-side comparisons of a traditional Cassini SAR view and one made using a new technique for handling electronic noise that results in clearer views of Titan's surface.

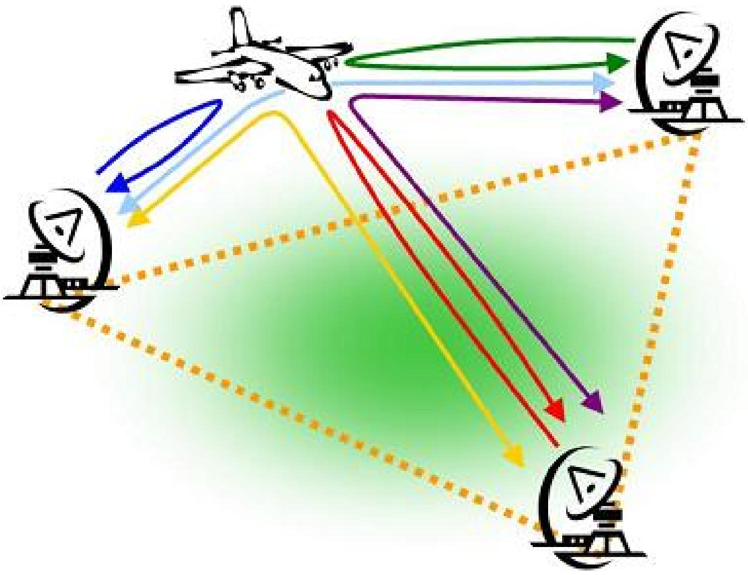
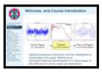


Figure 2.12: A multistatic RADAR Multistatic system, by Srdoughty / CC BY-SA 3.0.

2.2.4 Tools for solving these problems



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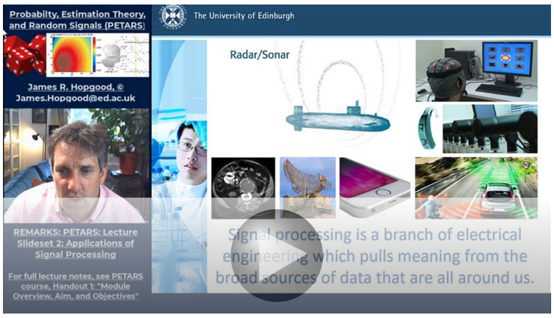
Topic Summary 10 Topic Title TBC

Topic Objectives:

- Objectives TBC.

Topic Activities:

Type	Details	Duration	Progress
Watch video	13 : 41 minute video	3×video length	
Discussion Board	Your views of signal processing	15 minutes	
Read Handout	Read page 66 to page 66	8 mins/page	



`http://media.ed.ac.uk/media/1_t0qrik06`

Video Summary: To be completed. Video above is a temporary link

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In each application scenario considered in this section, it is necessary to:

- Understand the nature and structure of the signal in the real world.
- Understand the nature of how the signal was acquired by our data processing system.
- Understand how the signals are effected by propagation through systems.
- Design systems that can modify or change the signals to our needs.

An example of the different signal processing chains is shown in Figure 2.13, and will be discussed further in lectures (and expanded on here in due course).

– End-of-Topic 10: **The Signal Processing Chain** –

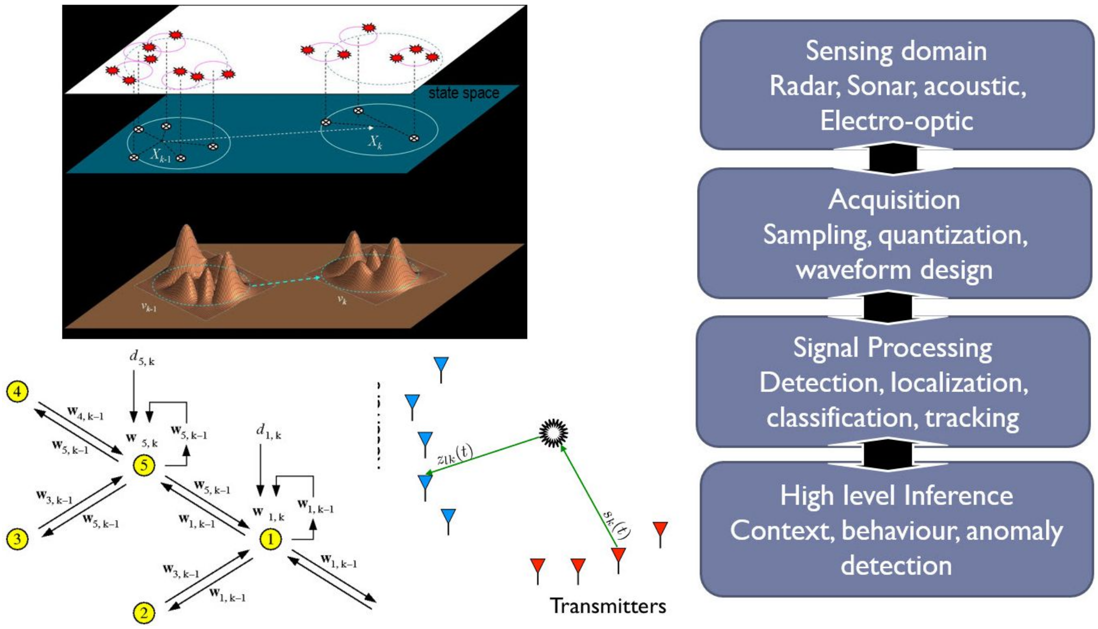


Figure 2.13: The signal processing chain.

2.3 Tutorial Exercises

There are currently no tutorial questions associated with this handout.

3

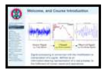
Underlying Principles in Signal Analysis and Sensor Systems

Every line is the perfect length if you don't measure it.

Marty Rubin

This handout begins by motivating the need for this course material by looking at key application areas and concepts that will be studied in detail during the lectures.

3.1 Key Principles



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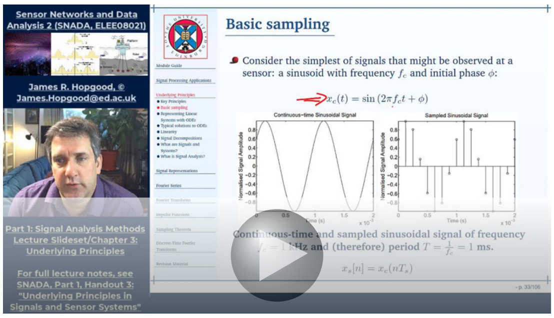
Topic Summary 11 Underlying principles in signal analysis, starting with basic sampling

Topic Objectives:

- Motivates the key principles that will be considered in this chapter.
- Focuses on the first key principle of basic sampling and why it is important.
- Highlights the Nyquist sampling theorem, but discuss why we need to understand it more.

Topic Activities:

Type	Details	Duration	Progress
Watch video	09 : 23 minute video	3× video length	
Read Handout	Read page 70 to page 73 and reflect	8 mins/page	



`http://media.ed.ac.uk/media/1_wfh5nlo6`

Video Summary: This video introduces the key principles and concepts that will be considered in the signal analysis part of the Sensor Networks and Data Analysis (SNADA) course. It discusses the importance of linear system modelling, complex phasors, signal decompositions, and sampling theory. The video then introduces the first key principle, which is on basic sampling techniques. Nyquist's sampling theorem is mentioned, but understanding the origin of this theorem is crucial to understanding its limitations, and how it is possible to beat Nyquist.

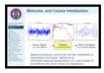
There are a number of key principles that will be used in the study of signals and sensor systems, in order to understand sensor-networks and data analysis techniques. Namely:

- real-world and sensor system modelling using physical principles leading to linear constant-coefficient ordinary differential equations (ODEs);

- the notion that complex exponentials are a fundamental building block to signal and systems analysis;
- the underlying importance of linearity of systems and linear signal decompositions;
- the concept of sampling an analogue continuous-time signal and storing on a finite-precision sampled data machine.

The following sections will take a brief look at these ideas, which will be covered in more depth later in the course.

3.1.1 Basic sampling



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This section gives a very basic introduction to **continuous-time signals** and **digital sampling theory**, and specifically address issues with storing a **continuous-time** real-world **analogue signal** as a **discrete-time** and **quantised** list of numbers in digital memory, which can then be manipulated and **reconstructed** to obtain the original real-world signal.

Consider the simplest of signals that might be observed at a sensor output: a sinusoid with frequency f_c and initial phase ϕ can be expressed as:

$$x_c(t) = \sin(2\pi f_c t + \phi) \tag{3.1}$$

When plotted in continuous-time, the signal $x_c(t)$ is a smooth curve, as shown in Figure 3.1a. In this example, the sinusoid has frequency $f_c = 1$ kHz, and therefore period $T = \frac{1}{f_c} = 1$ ms. [1] A sinusoidal signal can be sampled by measuring the signal $x_c(t)$ at time instances which are multiples of the sampling period $T_s = \frac{1}{f_s}$, where f_s is known as the **sampling frequency**. The time instance is indexed by an integer n, such that the **discrete-time signal** is denoted as:

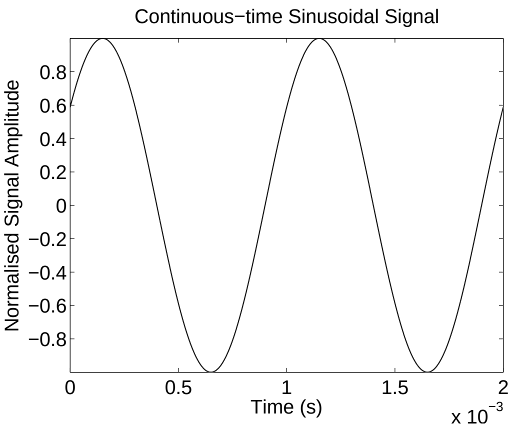
$$x_s[n] = x_c(nT_s) \tag{3.2}$$

An example of the sinusoid sampled is shown in Figure 3.1b. Note that this signal has not been quantised in amplitude in any way yet.

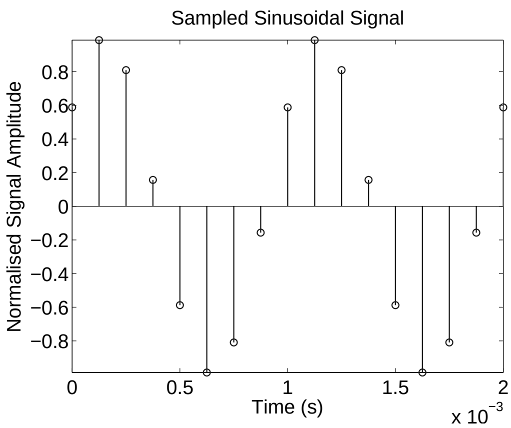
Figure 3.1b shows an idealised sampled signal, where the values *stored* in memory are the values measured specifically at times $t = nT_s$. When these values are sent to the output of a DAC, their values are held for the duration of the **sampling period**. Therefore, the output of the DAC is a **staircase function**, in which the signal is **piecewise-linear constant**.

An example of the input to a ADC and the output of the corresponding DAC is shown in Figure 3.2 and Figure 3.3c, depending on the sampling frequency.

[1] It is, of course, worth noting that the curve in Figure 3.1a is not, in fact, smooth since it has been digitally printed and, therefore, is a sampled image!



(a) Continuous-time sinusoidal signal.



(b) A sampled version of the signal in Figure 3.1a.

Figure 3.1: Continuous-time and sampled sinusoidal signal of frequency $f_c = 1$ kHz and (therefore) period $T = \frac{1}{f_c} = 1$ ms.

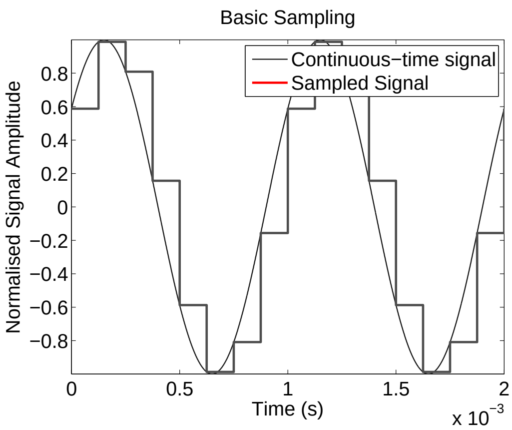
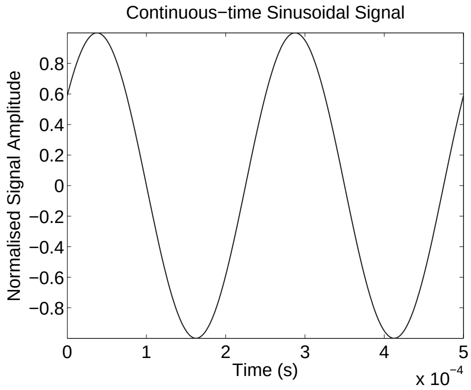
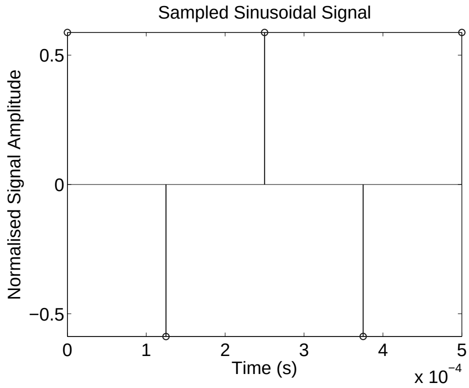


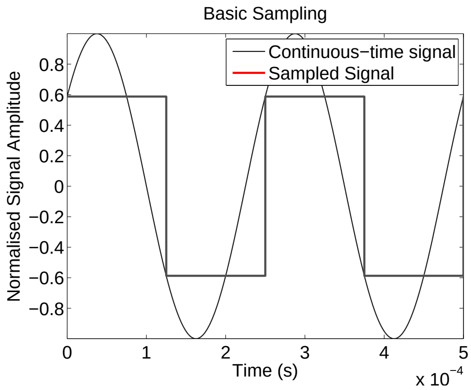
Figure 3.2: A continuous-time signal at the input of an analogue-to-digital converter (ADC), and a piece-wise linear approximation to the signal at the output of a digital-to-analogue converter (DAC).



(a) Continuous-time sinusoidal signal.



(b) A sampled version of Figure 3.3a.



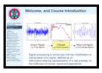
(c) Piece-wise linear approximation to the signal at the output of a DAC.

Figure 3.3: Continuous-time and sampled sinusoidal signal with frequency $f_c = 4$ kHz.

KEYPOINT! (Sampling and Quantisation). Note that sampling in time and quantisation are different! The staircase function at the output of the DAC is due to the signal value being held for a fixed period of time. This is not the same as the staircase function seen when quantising a signal.

– End-of-Topic 11: **Introducing the notion of basic sampling of signals** –

3.1.2 Representing Linear Systems with ODEs



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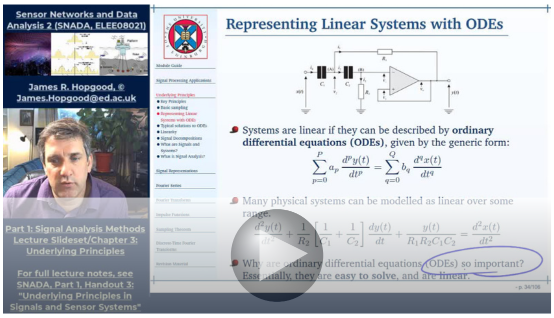
Topic Summary 12 Underlying Concepts in Linear System Theory

Topic Objectives:

- Explains why harmonic signals are so important in signal analysis.
- Explains the importance of system linearity in signal analysis.
- Motivates signal decompositions for analysing complex system responses.

Topic Activities:

Type	**Details**	**Duration**	**Progress**
Watch video	14 : 35 minute video	$3\times$ video length	
Read Handout	Read page 74 to page 78 and reflect	8 mins/page	



`http://media.ed.ac.uk/media/1_e5pirqms`

Video Summary: This topic uses linear system theory to motivate looking at methods for signal decompositions. It starts by observing that linear systems can be described by ordinary differential equations (ODEs), which have harmonic steady-state solutions in response to a harmonic input. This conceptually simple solution enables the frequency response of the system to be easily determined, and therefore the system response to an arbitrary input can be found, if the input signal can be represented in the frequency domain. Fourier analysis enables this, and this Topic provides the narrative journey to explain this.

Signal and system analysis simplifies greatly if the systems are linear; nonlinear systems are significantly more difficult to deal with and are beyond the scope of this course. However, nonlinear will be the topic of future study, and an appreciation of nonlinear systems will be extremely important as an Engineer. [2] Systems are linear

[2]While there are a number of special cases of nonlinear systems which can be analysed *naturally*, most nonlinear systems are analysed by local linear approximations around a point of interest. This can

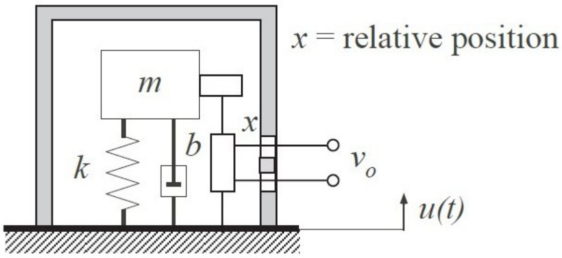


Figure 3.4: A simplified spring-mass representation of an accelerometer.

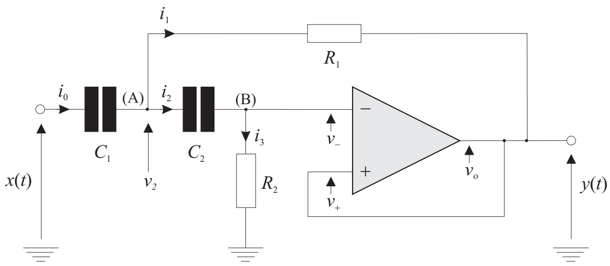


Figure 3.5: Analysing the second-order active **high-pass** filter.

if they can be described by **ordinary differential equations (ODEs)**, given by the generic form: [3]

$$\sum_{p=0}^{P} a_p \frac{d^p y(t)}{dt^p} = \sum_{q=0}^{Q} b_q \frac{d^q x(t)}{dt^q} \tag{3.5}$$

Many physical systems can be modelled as linear over some range. An example is the circuit shown in the figure page 62, which can be represented by the ODE:

$$\frac{d^2 y(t)}{dt^2} + \frac{1}{R_2}\left[\frac{1}{C_1} + \frac{1}{C_2}\right]\frac{dy(t)}{dt} + \frac{y(t)}{R_1 R_2 C_1 C_2} = \frac{d^2 x(t)}{dt^2} \tag{3.6}$$

This can be shown by making the assumptions for a perfect op-amp as listed in Sidebar 1, and then applying circuit analysis as shown in Sidebar 2.

Why are ordinary differential equations (ODEs) so important? Essentially, because they are **easy to solve**, and they are **linear**.

Sidebar 1 Perfect Op-amp Assumptions

Op-amp circuits, such as the one shown in the figure page 62, can be analysed assuming a perfect op-amp, where the following conditions are met:

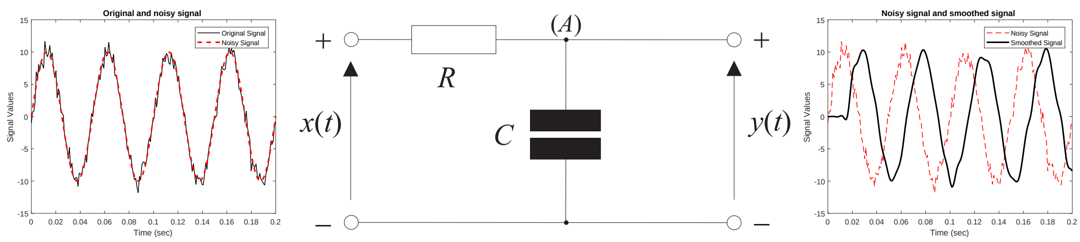
1. the inverting and non-inverting inputs have infinite input impedance, so that no current flows into the inputs;

2. the output has zero impedance, so that the output can deliver as much current as necessary to drive the rest of the circuit;

3. the op-amp has infinite **open-loop** gain, $A \to \infty$, and therefore, since the voltage at the immediate output of the op-amp is given by

$$v_o = A\left(v_- - v_+\right) \tag{3.7}$$

to achieve a finite-valued output voltage, v_o, requires $v_- - v_+ \to 0$, or equivalently the voltage at the inverting and non-inverting inputs must the identical.

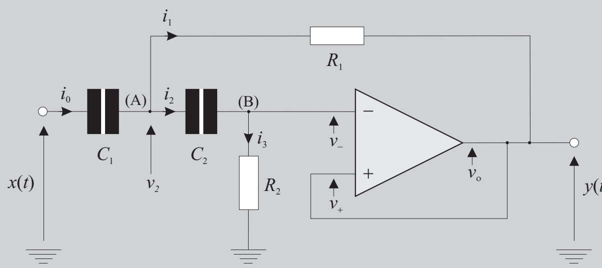


(a) Input waveform (b) Simple low-pass filter. (c) Output signal.

Figure 3.6: Simple low-pass electronic filter using an RC circuit: time domain analysis.

Sidebar 2 Analysing the second-order active **high-pass** filter

The op-amp circuit shown on page 62 and repeated below can be analysed assuming a *perfect op-amp*. The circuit is analysed using the notation shown in the figure below, and note that the *knowns* are C_1, C_2, R_1, R_2, $x(t)$, and $y(t)$.



Using nodal-analysis, summing currents at node (A) gives $i_o - i_1 - i_2 = 0$. Noting the current through a capacitor is proportional to the rate of change of voltage across it:

$$\underbrace{C_1 \frac{d\,(x(t) - v_2)}{dt}}_{i_o} - \underbrace{\frac{v_2 - y(t)}{R_1}}_{i_1} - \underbrace{C_2 \frac{d\,(v_2 - v_-)}{dt}}_{i_2} = 0 \tag{3.8}$$

Similarly, summing currents at node B gives:

$$i_2 - i_3 = 0 \quad \Rightarrow \quad C_2 \frac{d\,(v_2 - v_-)}{dt} = \frac{v_-}{R_2} \tag{3.9}$$

Finally, noting that $v_- = v_+ = y(t)$, then Equation 3.9 gives:

$$C_2 \frac{d\,(v_2 - y(t))}{dt} = \frac{y(t)}{R_2} \quad \text{or} \quad \frac{dv_2}{dt} = \frac{y(t)}{R_2 C_2} + \frac{dy(t)}{dt} \tag{3.10}$$

The expression for the derivative of v_2 in Equation 3.10 can be used in Equation 3.8; however, since Equation 3.8 also contains a term in v_2 directly, it is necessary to differentiate Equation 3.8 before substitution to avoid an expression involving an integral. Thus, differentiating Equation 3.8 gives:

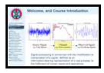
$$C_1 \frac{d^2\,(x(t) - v_2)}{dt^2} - \frac{1}{R_1} \frac{d\,(v_2 - y(t))}{dt} - \frac{1}{R_2} \frac{dy(t)}{dt} = 0 \tag{3.11}$$

Substituting Equation 3.10 gives:

$$C_1 \frac{d^2 x(t)}{dt^2} - \frac{C_1}{R_2 C_2} \frac{dy(t)}{dt} - C_1 \frac{d^2 y(t)}{dt^2} - \frac{y(t)}{R_1 R_2 C_2} - \frac{1}{R_2} \frac{dy(t)}{dt} = 0 \tag{3.12}$$

Dividing by C_1, rearranging, and re-grouping gives the ODE in Equation 3.6. The resulting differential equation may be written in the form of Equation 3.5 with $P = 2$, $a_0 = \frac{1}{R_1 R_2 C_1 C_2}$, $a_1 = \frac{1}{R_2}\left[\frac{1}{C_1} + \frac{1}{C_2}\right]$, $a_2 = 1$, and $Q = 1$, $b_0 = 0$, and $b_1 = 1$.

3.1.3 Typical solutions to ODEs



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The solution to a differential equation such as Equation 3.5 can often be expressed as a decaying exponential, or by a trigonometric function. In particular, if the input is a harmonic function of the form:

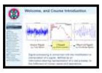
$$x(t) = \alpha \sin(\omega_0 t) + \beta \cos(\omega_0 t) \tag{3.13}$$

then the **steady-state** solution is also sinusoidal: [4]

$$y(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t) \tag{3.14}$$

for some coefficients A and B which are related to the coefficients of the ODE, namely the a_p's and b_q's, as well as the input parameters ω_0, α and β.

Therefore, it is relatively straightforward to evaluate the output of a linear system such as the op-amp circuit on page 62 for a sinusoidal or co-sinusoidal input, what about more complex signals? The first factor in dealing with a more complex signal is to understand why linearity is so important.



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3.1.4 Linearity

Suppose that the system is represented by the ODE in Equation 3.5, and that for a particular input, $x_a(t)$, the solution of the ODE is $y_a(t)$, as indicated in Figure 3.7a. Similarly, for a different input, $x_b(t)$, the output is $y_b(t)$, as indicated in Figure 3.7b

Then, if the input is given by:

$$x(t) = c\, x_a(t) + d\, x_b(t) \tag{3.15}$$

it is straightforward to show that the output is given by:

$$y(t) = c\, y_a(t) + d\, y_b(t) \tag{3.16}$$

– End-of-Topic 12: **Introduction to the basic concepts in linear system theory, motivating signal decomposition theory** –

---

most easily be achieved by approximating systems using Taylor series expansions.

[3] This is shorthand for the expression

$$a_P \frac{d^P y(t)}{dt^P} + a_{P-1} \frac{d^{P-1} y(t)}{dt^{P-1}} + \cdots + a_2 \frac{d^2 y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) \tag{3.3}$$

$$= b_Q \frac{d^Q x(t)}{dt^Q} + b_{Q-1} \frac{d^{Q-1} x(t)}{dt^{Q-1}} + \cdots + b_2 \frac{d^2 x(t)}{dt^2} + b_1 \frac{dx(t)}{dt} + a_0 x(t) \tag{3.4}$$

but is a lot less tedious to write. While you might not like the summation notation in Equation 3.5, it is well worth getting used to, although you need to be precise when using it

[4] The full solution can usually be written as the sum of a transient term, which eventually dies away as $t \to \infty$, and a steady-state term. The transient term is the homogeneous solution to the ODE, while the steady-state is the particular solution dependent on the input.

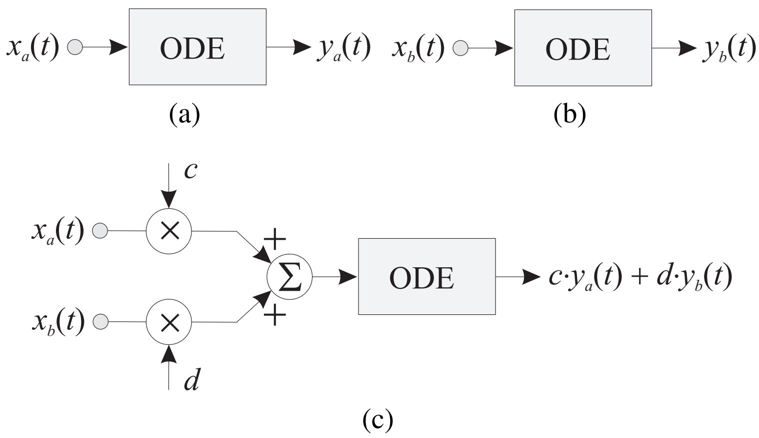


Figure 3.7: Input and output relationships for a linear system.

Sidebar 3 Linearity and ordinary differential equations (ODEs)

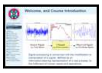
To show that if the input is given by $x(t) = c\,x_a(t) + d\,x_b(t)$, then the output is given by $y(t) = c\,y_a(t) + d\,y_b(t)$, notice that the input-output pairs $(x_a(t)\,,\,y_a(t))$ and $(x_b(t)\,,\,y_b(t))$ satisfy the differential equation, for example:

$$\sum_{p=0}^{P} a_p \frac{d^p y_a(t)}{dt^p} = \sum_{q=0}^{Q} b_q \frac{d^q x_a(t)}{dt^q} \quad \text{and} \quad \sum_{p=0}^{P} a_p \frac{d^p y_b(t)}{dt^p} = \sum_{q=0}^{Q} b_q \frac{d^q x_b(t)}{dt^q} \tag{3.17}$$

If the input is now $x(t) = c\,x_a(t) + d\,x_b(t)$, then note:

$$\sum_{q=0}^{Q} b_q \frac{d^q\,(c\,x_a(t) + d\,x_b(t))}{dt^q} = c\sum_{q=0}^{Q} b_q \frac{d^q x_a(t)}{dt^q} + d\sum_{q=0}^{Q} b_q \frac{d^q x_b(t)}{dt^q} \tag{3.18}$$

$$= c\sum_{p=0}^{P} a_p \frac{d^p y_a(t)}{dt^p} + d\sum_{p=0}^{P} a_p \frac{d^p y_b(t)}{dt^p} = \sum_{p=0}^{P} a_p \frac{d^p\,(c\,y_a(t) + d\,y_b(t))}{dt^p} \tag{3.19}$$



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3.1.5 Signal Decompositions

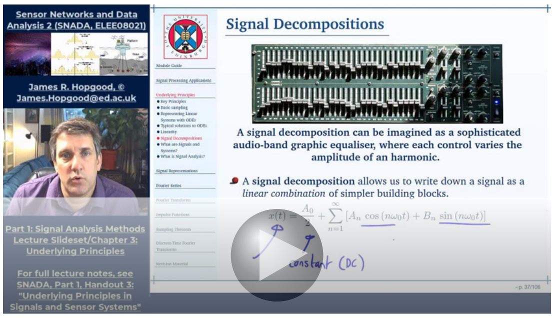
Topic Summary 13 Signal Decompositions as a Building Block in Signal Analysis

Topic Objectives:

- Motivates the Fourier series as the principal signal decomposition to start the journey from.
- Reminder of the form of the Fourier Series, and some examples.
- Reminder of the class of signals on which Fourier series can be applied.

Topic Activities:

Type	**Details**	**Duration**	**Progress**
Watch video	06 : 30 minute video	3× video length	
Read Handout	Read page 80 to page 82 and reflect	8 mins/page	



`http://media.ed.ac.uk/media/1_4pr5opcr`

Video Summary: This topic discusses signal decompositions by recalling the Fourier series decomposition, and explaining how this can be used in finding solutions to ordinary differential equations (ODEs) to an arbitrary input. The Topic highlights a physical, albeit impractical, method for synthesising a Fourier series. It then recalls a couple of classic examples, such as the decomposition of a square wave and sawtooth wave.

Linearity helps when used with another important tool, signal decompositions.

KEYPOINT! (Linear Decompositions). A **signal decomposition** allows us to write down a signal as a *linear combination* of simpler building blocks.

For example, we shall see that some signals can be expressed as a linear combination of sinusoids and co-sinusoids of frequencies which are a multiple of a fundamental:

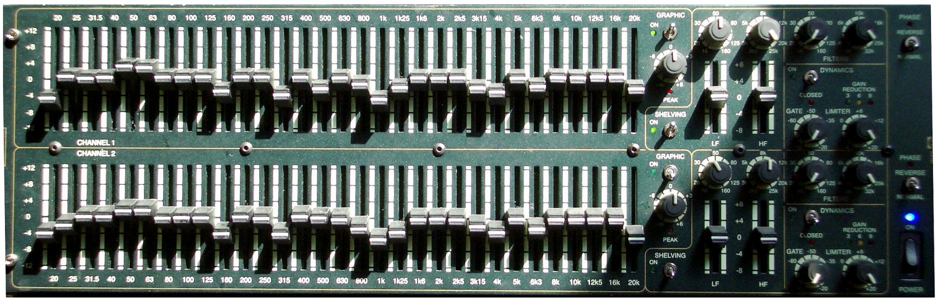


Figure 3.8: A signal decomposition can be imagined as a sophisticated audio-band graphic equaliser, where each control varies the amplitude of an harmonic.

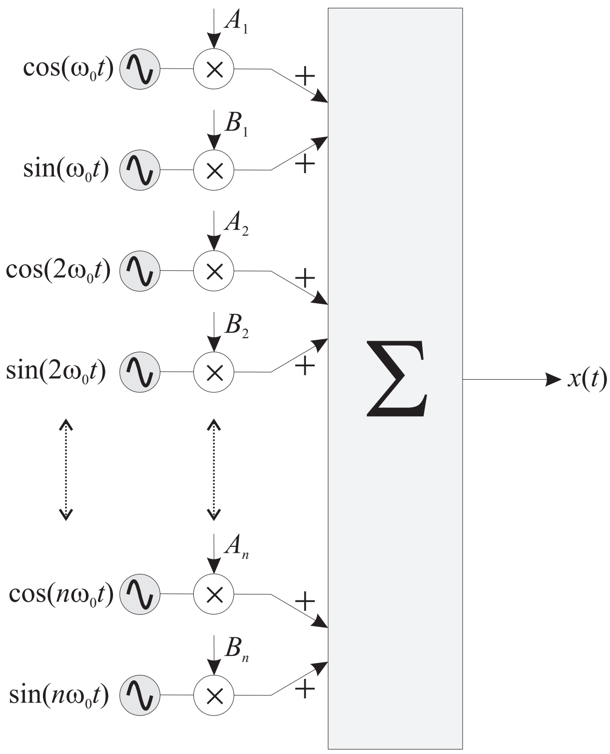


Figure 3.9: Decomposition of a signal as a linear combination of simpler building blocks, in this case sine waves and cosine waves.

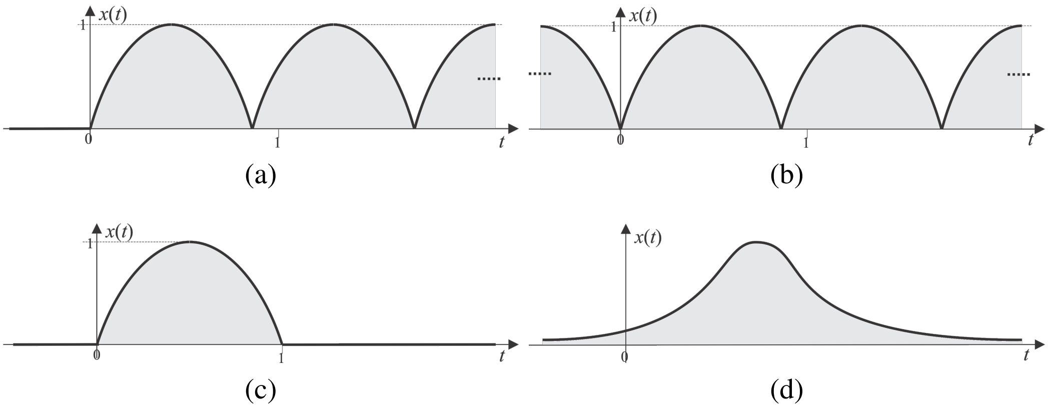


Figure 3.10: Which signal can be modelled using a Fourier Series?

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t\right) + B_n \sin\left(n\omega_0 t\right)\right] \tag{3.20}$$

This is known as a **Fourier Series**, and is graphically represented in Figure 3.9, and can be used for representing **periodic signals**; periodic signals are signals that repeat with a fixed period or repetition rate. The decomposition can be thought in terms of taking a very large number of signal generators (such as those found in the electronics lab), and setting them to multiples of the fundamental frequency of the waveform. Then the waveforms from these generators are each multiplied by some fixed coefficients, and finally all the waveforms are summed together.

KEYPOINT! (Solutions of ODEs). It is straightforward to find the solution to the ODE that describes a system when a sine and cosine wave is applied to the system input.
Therefore, using *linearity and signal decompositions*, it is straightforward to find the solution when a more complex signal is applied to the input of the system.

Two examples of Fourier series decompositions are shown in Figure 3.11 and Figure 3.12, where the original waveform is decomposed into weighted sinusoids of different frequencies, as shown in the graphs on the left. The graphs on the right show the approximation to the waveform when the first n sine waves and cosines are added together. We will see later how coefficients A_n and B_n in Equation 3.20, as well as the fundamental frequency ω_0, are calculated for a given waveform $x(t)$.

Example 3.1 (Multiple Choice). The Fourier Series is a powerful tool for decomposing signals into simpler building blocks.

Which of the signals in Figure 3.10 can be analysed using the Fourier Series expansion?

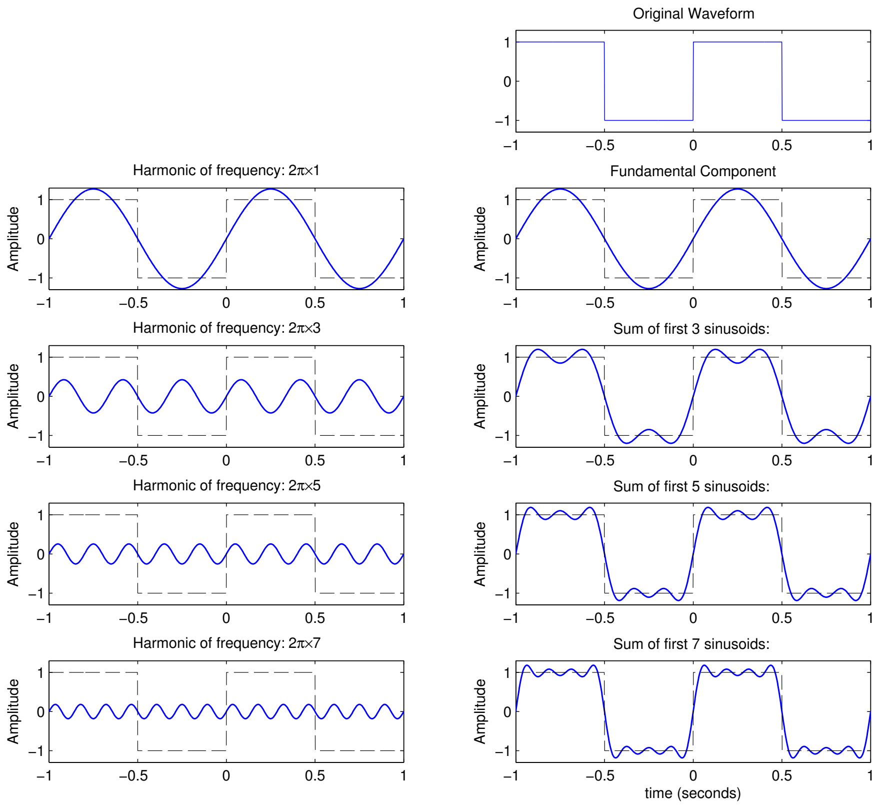


Figure 3.11: Two periods of a square wave and its Fourier series approximation with an increasing number of terms, from the fundamental, through to the summation of the first seven harmonics.

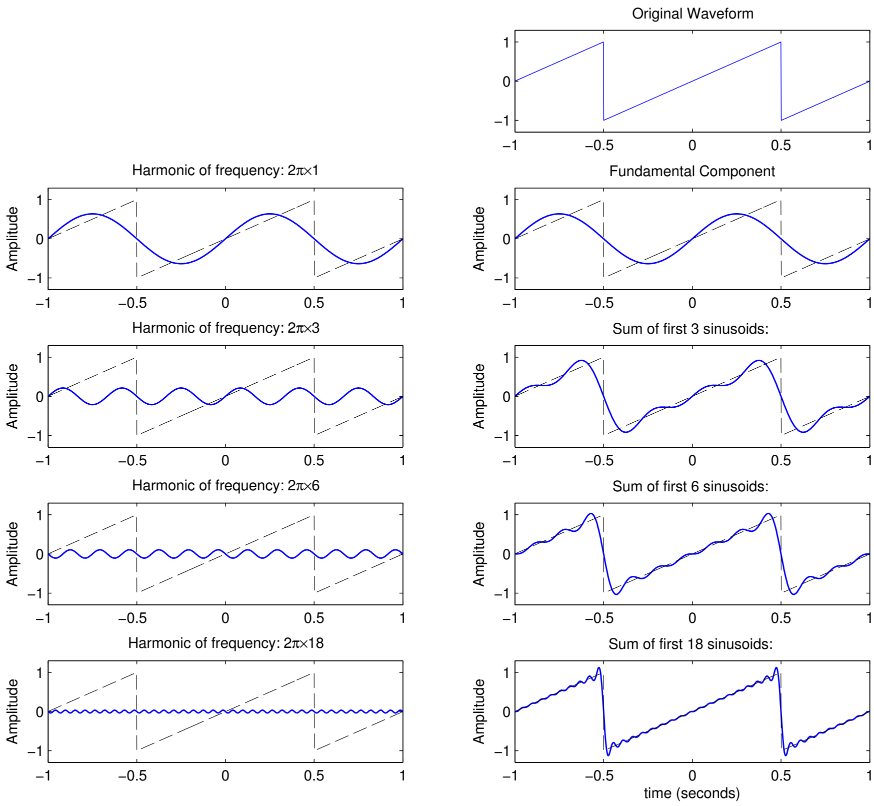


Figure 3.12: Two periods of a sawtooth wave and its Fourier series approximation with an increasing number of terms, from the fundamental, through to the summation of the first 18 harmonics.

– End-of-Topic 13: **Motivating the importance of signal decompositions, and the study of Fourier theory** –

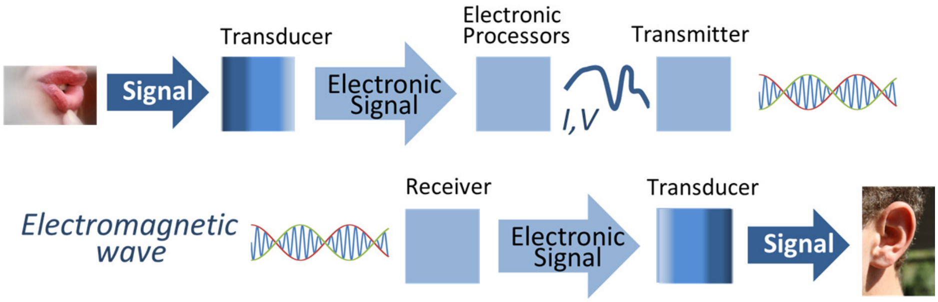
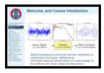


Figure 3.13: A communication system, highlighting the role of signals and systems (`https://commons.wikimedia.org/wiki/File:Signal_processing_system.png`).

3.2 What are Signals and Systems?



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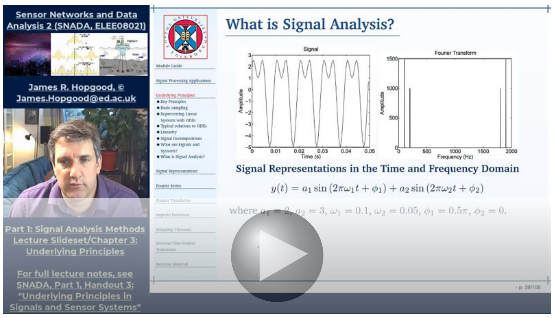
Topic Summary 14 Summary of Concepts in Signal and System Analysis

Topic Objectives:

- Summary of key underlying concepts in one place.
- Reminder of the signal processing and communications chain.
- Highlights further connections in signal and systems theory.
- Aims to finish the narrative journey of introducing conceps in signal and system analysis.

Topic Activities:

Type	Details	Duration	Progress
Watch video	06 : 20 minute video	3× video length	
Read Handout	Read page 85 to page 88 and reflect	8 mins/page	



http://media.ed.ac.uk/media/1_ydndnpa2

Video Summary: This Topic summarises a number of key ideas that have been mentioned in the Topics up-to this point, and aims to bring together a lot of the ideas with further narrative and technical connections. It aims to complete the picture of what you can expect in Signal and Systems analysis.

The first half of this course focuses on **signal and system analysis**, while the second half provides an introduction to communication theory. But what are signals and systems in the first instance?

Systems **Systems** represent mechanical devices, electrical circuits, computer processors, and so forth. Essentially, a system describes the behaviour of a physical system or device.

Signal A **signal** is an input to a **system**.

The **Signals** and **Systems** component of this course introduces techniques for analysing:

Properties of Systems for example modelling the capacitor microphone in Figure 3.14a; determining how a sound pressure wave translates into an electrical signal by modelling both the mechanical and electrical circuits.

One particular property of mechanical and electrical circuits called the **frequency response**, can be analysed using the **Fourier Transform** given by

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\ e^{-j\omega t}\, dt \tag{3.21}$$

where $h(t)$ is the so-called **impulse response** of the system. [5]

Output of System Calculating the output of a system given a particular input to the system; for example the output of the amplitude modulated (AM) radio demodulator in Figure 3.14b.

If the input to a system is $x(t)$, the output, $y(t)$, can be found via the **convolution integral** given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\ h(t-\tau)\ d\tau \tag{3.23}$$

where, again, $h(t)$ is the **impulse response** of the system.

New slide

3.3 What is Signal Analysis?

Approximately the first seven lectures in this course consider **signal analysis**, which discusses alternative representations of signals rather than just a graph of amplitude versus time. For example, the signal in Figure 3.15a is given by:

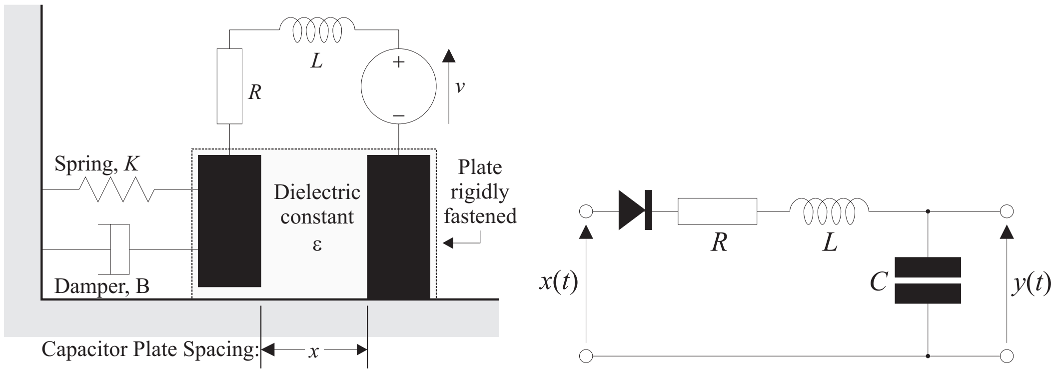
$$y(t) = a_1 \sin\left(2\pi\omega_1 t + \phi_1\right) + a_2 \sin\left(2\pi\omega_2 t + \phi_2\right) \tag{3.24}$$

where $a_1 = 2$, $a_2 = 3$, $\omega_1 = 0.1$, $\omega_2 = 0.05$, $\phi_1 = 0.5\pi$, $\phi_2 = 0$. Since this signal is represented by six parameters, it is possible to represent the signal in terms

[5] For information, there is a relationship between the Fourier transform and a more general transform called the Laplace transform that will be studied in the third year control course.

$$H(s) = \int_{0^-}^{\infty} h(t)\ e^{-st}\, dt \tag{3.22}$$

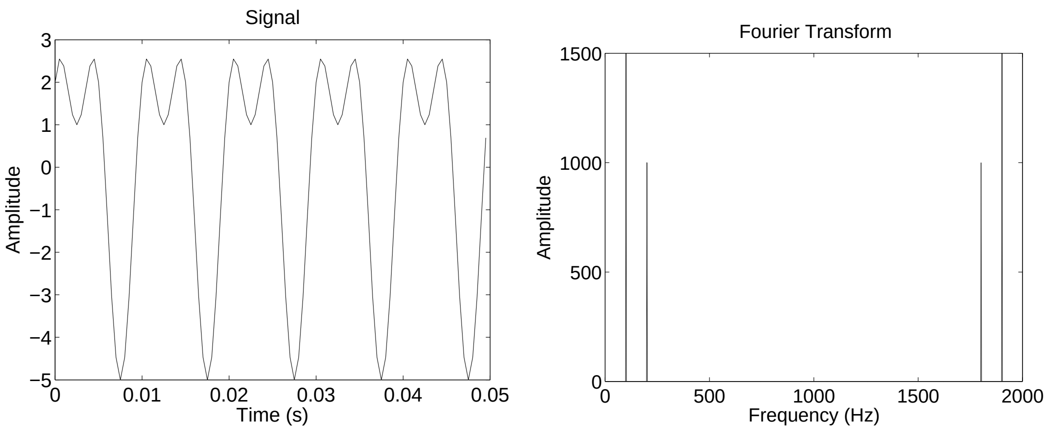
In its simplest form, the Fourier transform can be obtained by extending the limits to $(-\infty, \infty)$ and setting $s = j\omega$. However, this depends on some criteria such as whether $s = j\omega$ is in the so-called region of convergence (ROC) or not.



(a) Simplified Model of a Capacitor Microphone.

(b) A communications demodulator circuit, for simple AM radio.

Figure 3.14: Applications of signal and system theory.



(a) Signal waveform

(b) Spectral content (Fourier transform)

Figure 3.15: Signal Representations in the Time and Frequency Domain

of frequency, ω, in the so-called Fourier or frequency domain. This is shown in Figure 3.15b.

The Fourier Transform, $F(\omega)$, as a function of frequency, ω, can be obtained from the signal waveform, $f(t)$, which is a function of time, via the relationship:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{3.25}$$

The Fourier transform is a generalisation of the Fourier series in Equation 3.20 but can be applied to non-periodic (or aperiodic signals), whereas the Fourier series can only be used for analysing periodic signals.

– End-of-Topic 14: **An abstract view of signal and system analysis –**

3.4 Tutorial Exercises

There are currently no tutorial questions associated with this handout.

4

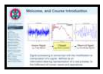
Signal and System Representations

Great acts are done by a series of small deeds.

Lao Tzu

This handout gives an introduction to the notion of signals and systems, considers different types of signals, classification of signals (energy and power), and signal decomposition theory.

4.1 Introduction



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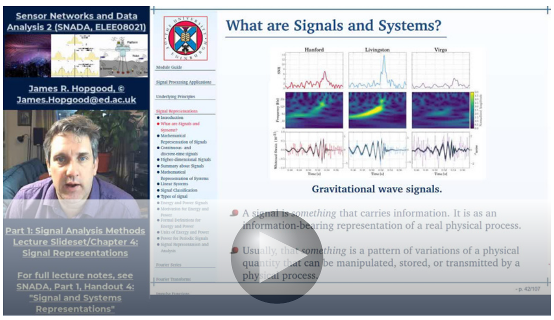
Topic Summary 15 Definition of Signals

Topic Objectives:

- Definition of a signal, and exemplar types of signals.
- Formally define a one-dimensional amplitude waveform, both in continuous- and discrete-time.
- Context of real-world signals and sampled signals.
- Introduction to higher-dimensional signals and multi-channel signals.

Topic Activities:

Type	Details	Duration	Progress
Watch video	14 : 33 minute video	3× video length	
Read Handout	Read page 92 to page 97	8 mins/page	
Practice Exercises	Exercises 4.1 and 4.2	30 mins	



http://media.ed.ac.uk/media/1_b1ifr5ld

Video Summary: This topic introduces the mathematical definitions of signals, and terminology used to describe signals, for example multi-dimensional and multi-channel signals. The topic discusses a variety of different types of signals, including in detail one-dimensional waveforms or time-series and two-dimensional images, and then discrete-time and continuous-time signals.

With mobile telephones, the Internet, digital audio and video entertainment systems, and other digital communication systems commonplace in modern society, the terms **signal** and **system** are frequently used in everyday language. Although these terms are used casually, what do they really mean in an Engineering sense? This handout will introduce the formal notion of **signals** and **systems**, and the idea that they can be represented by mathematical functions.

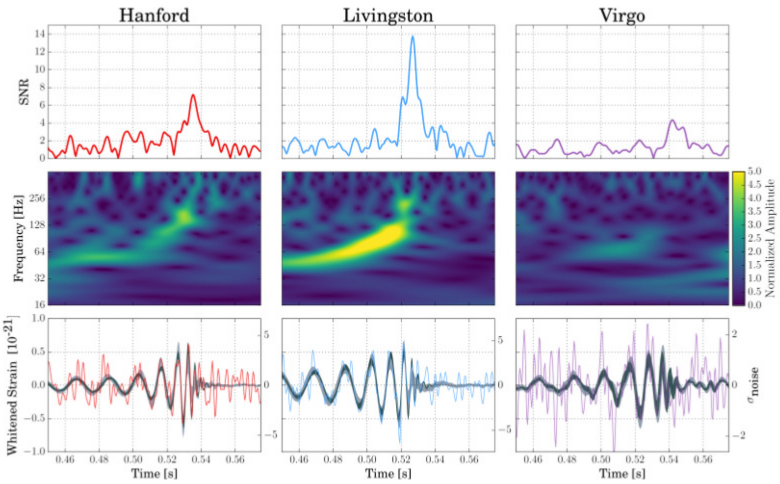
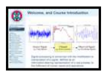


Figure 4.1: Gravitational wave signals (see `https://commons.wikimedia.org/wiki/File:GW170814_signal.png`).

More specifically, the following topics are covered in this handout:

- What are **signals** and **systems**?
- How are signals and systems represented mathematically?
- Consider the special case of linear systems, and how to test whether a system is linear or not.
- Consider different types of signals, for example periodic and non-periodic, continuous-time and discrete-time signals.
- How are signals classified further, for example, considering signal measures (energy and power)?

4.2 What are Signals and Systems?



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Common usage and understanding of the word *signal* is actually correct from an Engineering perspective within some rather broad definitions: a signal is thought of as *something* that carries information. Usually, that *something* is a pattern of variations of a physical quantity that can be manipulated, stored, or transmitted by a physical process. Examples include speech signals, general audio signals, video or image signals, biomedical signals, radar signals, and seismic signals, to name but a few.

So formally, a **signal** is defined as an information-bearing representation of a real physical process. It is important to recognise that signals can take many equivalent forms or representations. For example, a speech signal is produced as an acoustic signal, but it can be converted to an electrical signal by a microphone, or a pattern of

(a) A speech signal.

(b) An image.

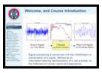
(c) Synthetic aperature radar signal.

Figure 4.2: Different types of signals.

magnetization on a magnetic tape, or even as a string of numbers as in digital audio recording.

The term *system* is a little more ambiguous, and can be subject to interpretation. [1] In Engineering terminology, however, a **system** is something that can manipulate, change, record, or transmit **signal**s. In general, **system**s operate on **signal**s to produce new signals or new signal representations. For example, a digital audio recording, whether on a compact disc (CD) or an MPEG-1 Audio Layer 3 (MP3), stores or represents a music signal as a sequence of numbers. A digital audio playback device, such as a CD player, is a **system** for converting the numerical representation of the signal stored on the disk to an acoustic signal that can be heard.

The goal of this course is to develop a framework wherein it is possible to make precise statements about both signals and systems. Specifically, it will be shown that mathematics is an appropriate language for describing and understanding signals and systems. The representation of signals and systems by mathematical equations facilitates an understanding of how signals and systems interact, and how systems can be designed and implemented to achieve a prescribed purpose.



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4.2.1 Mathematical Representation of Signals

A *signal* is defined as an information-bearing representation of a real process. It is a pattern of variations, commonly referred to as a waveform, that encodes, represents, and carries information.

Many signals are naturally thought of as a pattern of variations with time. For example, a speech signal arises as a pattern of changing air pressure in the vocal tract, creating a sound wave, which is then converted into electrical energy using a microphone. This electrical signal can then be plotted as a time-waveform, and an example is shown

[1]The word *system* can also correctly be understood as a process, and often the word *system* is used to refer to a large organisation that administers or implements some process!

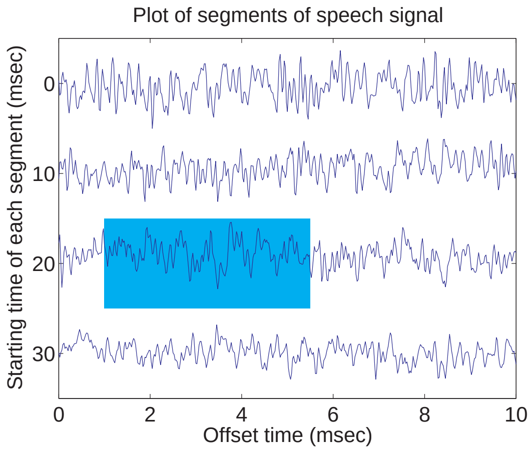
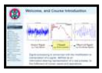


Figure 4.3: Plot of part of a speech signal. This signal can be represented by the function $s(t)$, where t is the independent variable representing time. The shaded region is shown in more detail in Figure 4.4.

in Figure 4.3. The vertical axis denotes air pressure or microphone voltage, and the horizontal axis represents time. This particular plot shows four contiguous segments of the speech waveform. The second plot is a continuation of the first, and so on, and each plot is vertically offset with the starting time of each segment shown on the left vertical axis.

4.2.1.1 Continuous-time and discrete-time signals



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The signal shown in Figure 4.3 is an example of a one-dimensional **continuous-time signal**. Such signals can be represented mathematically as a function of a single independent variable, t, which represents time and can take on any real-valued number. Hence, each segment of the speech waveform can be associated with a function $s(t)$. In some cases, the function $s(t)$ might be a simple function, such as a sinusoid, but for real signals, it will be a complicated function.

Generally, most *real world* signals are continuous in time and analogue: this means they exist for all time-instances, and can assume any value, within a predefined range, at these time instances. Although most signals originate as continuous-time signals, digital processors and devices can only deal with **discrete-time signals**. A discrete-time representation of a signal can be obtained from a continuous-time signal by a process known as **sampling**. There is an elegant theoretical foundation to the process of sampling, and this will be covered later in the course. In the meantime, it suffices to say that the result of sampling a continuous-time signal at isolated, equally spaced points in time is a sequence of numbers that can be represented as a function of an index variable that can take on only discrete integer values.

The sampling points are spaced by the **sampling period**, denoted by T_s. Hence, the continuous-time signal, $s(t)$, is *sampled* at times $t = nT_s$ resulting in the discrete-time

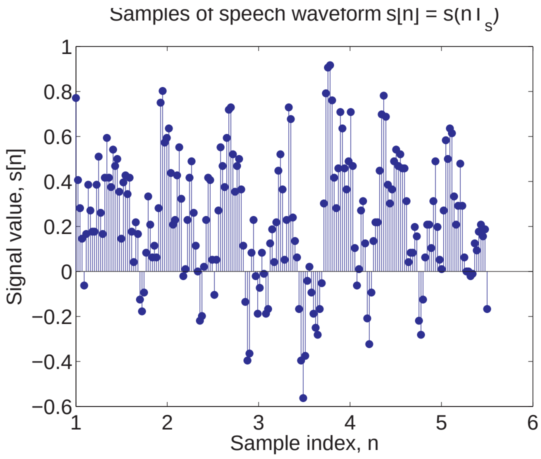


Figure 4.4: Example of a discrete-time signal, which is a sampled version of the shaded region shown in Figure 4.3.

waveform denoted by:

$$s[n] = s(nT_s)\,, \quad n \in \{0,\,1,\,2,\,\ldots\}. \tag{4.1}$$

where n is the index variable. A discrete-time signal is sometimes referred to as a discrete-time sequence, since the waveform $s[n]$ is a sequence of numbers. [2] Figure 4.4 shows an example of a short segment of the speech waveform from Figure 4.3, with a sampling period of $T_s = \frac{1}{44100}$ seconds. This corresponds to sampling at a rate of 44.1 kHz. It is not possible to evaluate the continuous-time function $s(t)$ for every value of t, only at a finite-set of points, which will take a finite time to evaluate. Intuitively, however, it is known that the closer the spacing of the sampled points, the more the sequence retains the shape of the original continuous-time signal. The question arises, then, regarding what is the largest **sampling period** that can be used to retain all or most of the information about the original signal.

New slide

4.2.1.2 Higher-dimensional Signals

While many signals can be considered as evolving patterns in time, many other signals are not time-varying patterns at all. For example, an image formed by focusing light through a lens onto an imaging array is a spatial pattern. Thus, an image is represented mathematically as a function of two independent spatial variables, x and y; thus, a picture might be denoted as $p(x, y)$. An example of a **gray-scale image** is shown in Figure 4.5; thus, the value $p(x_0, y_0)$ represents the particular shade of gray at position (x_0, y_0) in the image.

Although images such as that shown in Figure 4.5 represents a quantity from a physical two-dimensional (2-D) spatial continuum, digital images are usually discrete-variable 2-D signals obtained by sampling a continuous-variable 2-D signal. Such a 2-D

[2]Note, the convention that parenthesis ( ) are used to enclose the independent variable of a continuous-time function, and square brackets [ ] enclose the index variable of a discrete-time signal.

Figure 4.5: Example of a signal that can be represented by a function of two spatial variables.

discrete-variable signal would be represented by a 2-D sequence or array of numbers, and is denoted by:

$$p[m, n] = p(m\Delta_x, n\Delta_y), \quad m, n \in \{0, 1, \dots N-1\}. \tag{4.2}$$

where m and n take on integer values, and Δ_x and Δ_y are the horizontal and vertical sampling spacing or periods, respectively.

Two-dimensional functions are appropriate mathematical representations of still images that do not change with time; on the other hand, a sequence of images that creates a video requires a third independent variable for time. Thus, a video sequence is represented by the three-dimensional (3-D) function $v(x, y, t)$.

4.2.1.3 Summary about Signals

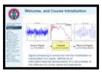
New slide

In summary, the purpose of this section is to introduce the idea that signals can:

- be found across numerous disciplines, from finance to astrophysics, in measurement devices from oscilloscopes to smartphones;
- be represented by mathematical functions in one or more dimensions, for example images, videos, or 3D video sequences;
- be functions of continuous or discrete variables (for example time, or space);
- use different notation depending on the nature of the signal;
- for the purposes of this course, the signals are considered to be deterministic, and therefore known for all time or space.

KEYPOINT! (Signals and functions). The connection between functions and signals is key to signal processing, and, at this point, functions serve as abstract symbols for signals. This is an important, but very simple, concept for using mathematics to describe signals and systems in a systematic way.

– End-of-Topic 15: **Introduction to Definition of a Signal, and Types of Signals –**



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4.2.2 Mathematical Representation of Systems

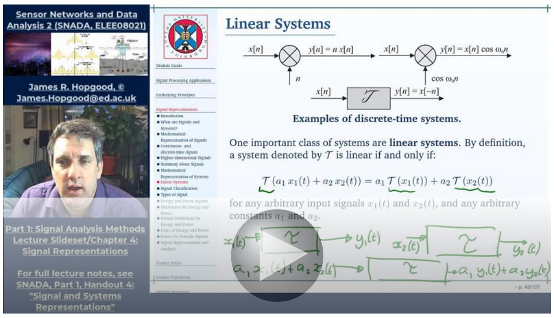
Topic Summary 16 Systems and Testing System Linearity

Topic Objectives:

- Definition of a system.
- Room acoustics as an example of systems operating on signals.
- Formal definition of a linear system.
- Process for testing linearity of a system.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 28 minute video	3× video length	
Read Handout	Read page 98 to page 101	8 mins/page	
Try Example	Try Example 4.1	8 mins	
Practice Exercises	Exercise 4.3	20 mins	



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Video Summary: This topic introduces a very basic introduction to the mathematical definitions of systems, in order to place the signal analysis component of the course in context. System theory is covered in detail in third year courses in Engineering. The main focus of this topic is the mathematical definition of linear systems, and how to test for linearity of a system. Two examples of testing for linearity is provided in the last third of this video.

A **system** manipulates, changes, records, or transmits **signal**s. To be more specific, a one-dimensional continuous-time system takes an input signal $x(t)$ and produces a corresponding output signal $y(t)$. This can be represented mathematically by the expression

$$y(t) = \mathcal{T}\left\{x(t)\right\} \tag{4.3}$$

Sidebar 4 Example of a non-linear system

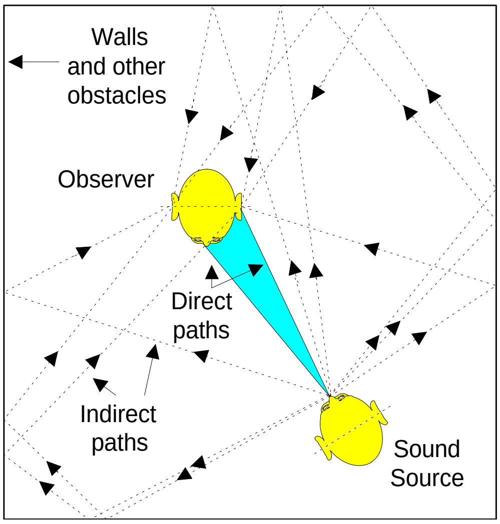
As an example of the functional relationship in Equation 4.3, consider a signal that is the square of the input signal; this is represented by the equation

$$y(t) = [\,x(t)\,]^2 \tag{4.4}$$

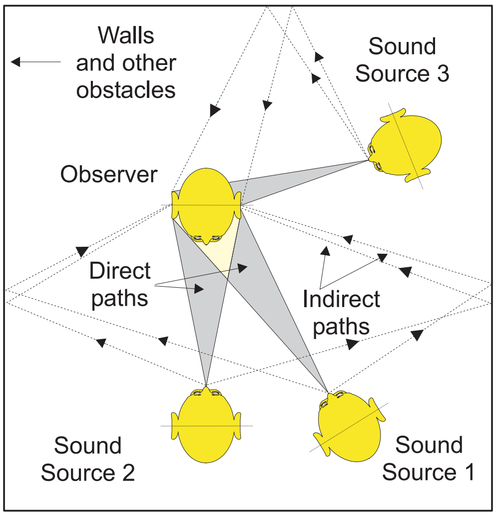
This means that the function $\mathcal{T}$ in Equation 4.3 is given by:

$$\mathcal{T}x = x^2 \tag{4.5}$$

This square-law function can be very useful in demodulation within a simple radio.



(a) Acoustic path from a sound source to a microphone.



(b) Many sound sources within a room.

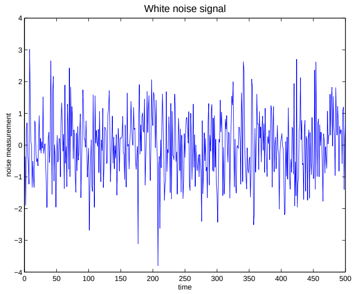
Figure 4.6: Observed signals in room acoustics.

which means that the input signal, $x(t)$, be it a waveform or an image, is operated on by the system, which is symbolised by the operator $\mathcal{T}$ to produce the output $y(t)$.

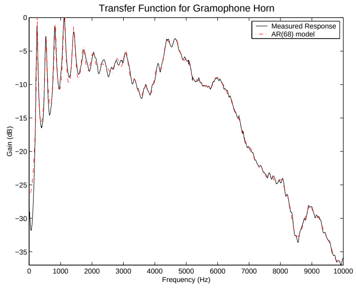
Figure 4.6 show how signals can be generated and observed in a real application. In Figure 4.6, the sound source and the information received by the observer, or microphone, are the **signals**; the room acoustics represent the **system**. Figure 4.7 shows the **input signal** to the system, a *characterisation of the system*, and the resulting **output signal**.

As mentioned in a previous handout, the subject of signals and systems is the basis of a branch of Engineering known as signal processing; this area is formally defined as follows:

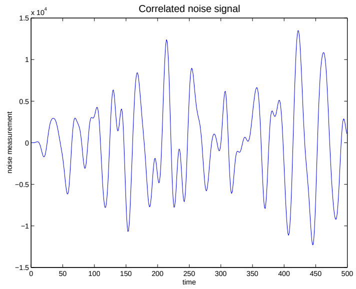
> *Signal processing* is concerned with the modification or manipulation of a signal, defined as an information-bearing representation of a real process,



(a) Source signal *into* a system.



(b) A frequency response *representing* the characteristics of the system.



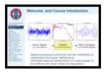
(c) The output of the system.

(d) Block diagram representation of signal paths.

Figure 4.7: The result of passing a signal through a system.

that has been passed through a *system*, to the fulfillment of human needs and aspirations.

4.2.3 Linear Systems



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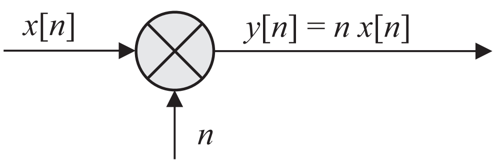
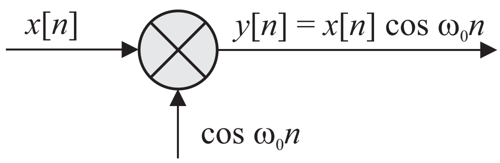
One important class of systems are **linear systems**, as discussed in the previous handout. By definition, a system denoted by $\mathcal{T}$ is linear if and only if:

$$\mathcal{T}\left(a_1\,x_1(t) + a_2\,x_2(t)\right) = a_1\,\mathcal{T}\left(x_1(t)\right) + a_2\,\mathcal{T}\left(x_2(t)\right) \tag{4.6}$$

for any arbitrary input signals $x_1(t)$ and $x_2(t)$, and any arbitrary constants a_1 and a_2. If the system is not linear, then it is generally called nonlinear, although there are other



(a) The *time* multiplier.



(b) The modulator

(c) The *folder* or flipping system.

Figure 4.8: Examples of discrete-time systems.

Figure 4.9: System Block Diagram

terms for special cases.

Example 4.1 (Linear Systems). Determine if the systems described by the following input-output equations are linear or nonlinear:

- $$y(t) = x(t) - 0.5x(t-1) \tag{4.7}$$
- $$y(t) = \ln x(t) \tag{4.8}$$

SOLUTION. The simple block diagram for a system is shown in Figure 4.9. Suppose an input $x(t)$ generates a particular output $y(t)$. Similarly, it follows that any input $x_k(t)$ produces $y_k(t)$. If a system is linear, then with the input $x_3(t) = a_1\, x_1(t) + a_2\, x_2(t)$, the output should be $y_3(t) = a_1\, y_1(t) + a_2\, y_2(t)$. If this relationship is not satisfied, then the system is not linear.

- For the first problem, $y_k(t) = x_k(t) - 0.5\, x_k(t-1)$. Setting $k = 3$, and *replacing* $x_3(t)$ with $x_3(t) = a_1\, x_1(t) + a_2\, x_2(t)$, it follows:

$$y_3(t) = x_3(t) - 0.5\, x_3(t-1) \tag{4.9}$$
$$= (a_1\, x_1(t) + a_2\, x_2(t)) - 0.5\, (a_1\, x_1(t-1) + a_2\, x_2(t-1)) \tag{4.10}$$

  Rearranging this expression accordingly, it follows:

$$y_3(t) = a_1\, \underbrace{(x_1(t) - 0.5\, x_1(t-1))}_{y_1(t)} + a_2\, \underbrace{(x_2(t) - 0.5\, x_2(t-1))}_{y_2(t)} \tag{4.11}$$
$$= a_1\, y_1(t) + a_2\, y_2(t) \tag{4.12}$$

  It is therefore recognised that this system is linear.

- In the second example, where $y(t) = \ln x(t)$, then following a similar approach, note that

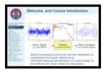
$$y_3(t) = \ln (a_1\, x_1(t) + a_2\, x_2(t)) \tag{4.13}$$

□

  There is no way to express this as $y_3(t) = a_1\, y_1(t) + a_2\, y_2(t)$, and therefore the system is nonlinear.

– End-of-Topic 16: **Linear Systems (and how to test)** –

4.3 Signal Classification



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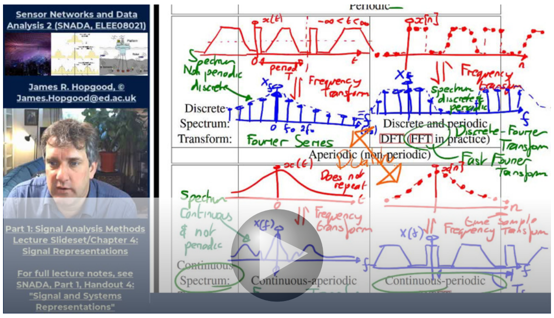
Topic Summary 17 Deterministic time-series signal classification

Topic Objectives:

- Distinguish periodic and non-periodic, discrete-time and continuous signals.
- Ability to distinguish different signal types.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 57 min video	3× length	
Read Handout	Read page 102 to page 106	8 mins/page	



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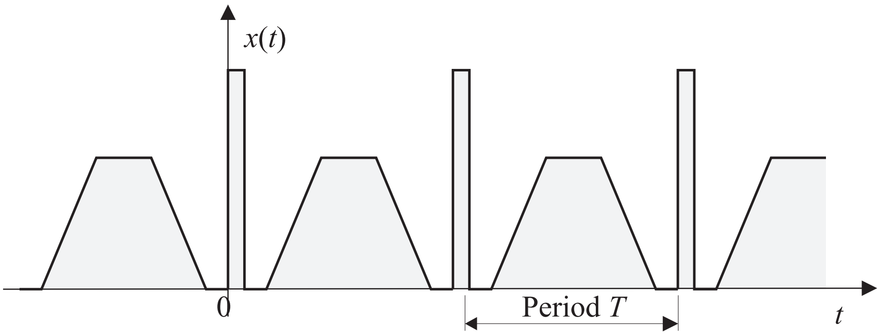
Video Summary: This Topic considers four different types of deterministic signals, namely continuous-time periodic signals and non-periodic signals, and discrete-time periodic and non-periodic signals. The Topic considers basic properties of each of these classes of signals, and places a special focus on the appropriate frequency transform method and spectral representation for each of these signal classes. Key observations of the spectral properties are highlighted, including the duality principle. This Topic covers background material that will be used throughout the rest of this course, and there are no associated self-study questions.

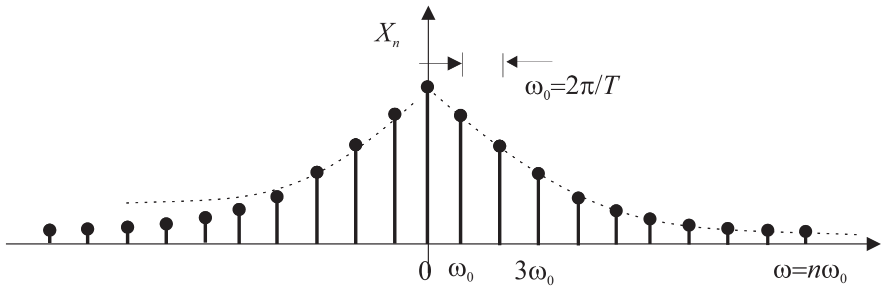
Before considering the analysis of signals and systems, it is necessary to be aware of the general classifications to which signals can belong, and to be aware of the significance of some subtle characteristics that determine how a signal can be analysed. Not all signals can be analysed using a particular technique.

Different types of **deterministic** signals include:

- continuous-time periodic signals;



(a) An example of a periodic signal with period T.

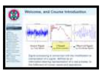


(b) The Fourier series of the periodic signal in Figure 4.10a with fundamental frequency $\omega_0 = {}^{2\pi}/_{T}$.

Figure 4.10: Example of a periodic signal and its spectral representation, found using the Fourier series.

- continuous-time non-periodic (or aperiodic) signals;
- discrete-time periodic signals;
- discrete-time aperiodic signals.

The variety of signal classes rapidly changes when the notion of **random** or **stochastic** signals are introduced (not until the fourth-year!).



4.3.1 Types of signal

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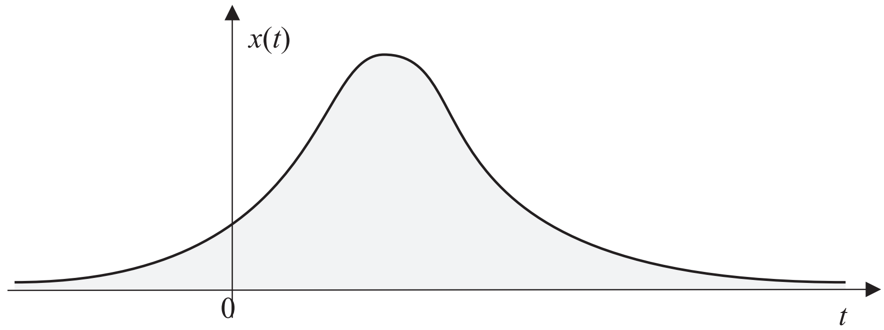
In general, there are four distinct types of deterministic signals that must be analysed:

Continuous-time periodic Such signals repeat themselves after a fixed length of time known as the period, usually denoted by T. This repetition continues ad-infinitum (i.e. forever).

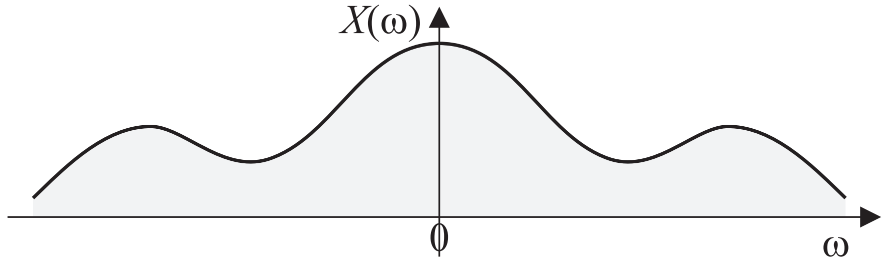
Formally, a signal, $x(t)$, is periodic if

$$x(t) = x(t + m\,T)\,,\ \forall\, m \in \mathbb{Z} \tag{4.14}$$

where the notation $\forall\, m \in \mathbb{Z}$ means that m takes on *all* integer values: in other-words, $m = -\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$. The



(a) An example of an aperiodic signal.



(b) The Fourier transform of the aperiodic signal in Figure 4.11a, where every possible frequency exists.

Figure 4.11: Example of an aperiodic signal and its spectral representation, found using the Fourier transform.

smallest positive value of T which satisfies this condition is the defined as the **fundamental period**.

These signals will be analysed using the **Fourier Series**, and are used to represent real-world waveforms that are near to being periodic over a sufficiently long period of time.

An example of a periodic signal is shown in Figure 4.10a. This kind of signal vaguely represents a line signal in analogue television, where the rectangular pulses represent line synchronisation signals.

Continuous-time aperiodic Continuous-time aperiodic signals are not periodic over all time, although they might be locally periodic over a short time-scale.

These signals can be analysed using the **Fourier transform** for most cases, and more often using the **Laplace transform**. Aperiodic signals are more representative of many real-world signals.

Again, real signals don't last for all time, although can last for a considerably long time. An example of an aperiodic signal is shown in Figure 4.11a.

Discrete-time periodic A discrete-time periodic signal is shown in Figure 4.12, which is essentially a *sampled* version of the signal shown in Figure 4.10a. Note in this case, the period is often denoted by N, primarily to reflect the fact the time-index is now n; in other words, $x[n] = x(nT_s)\,, \quad n \in$

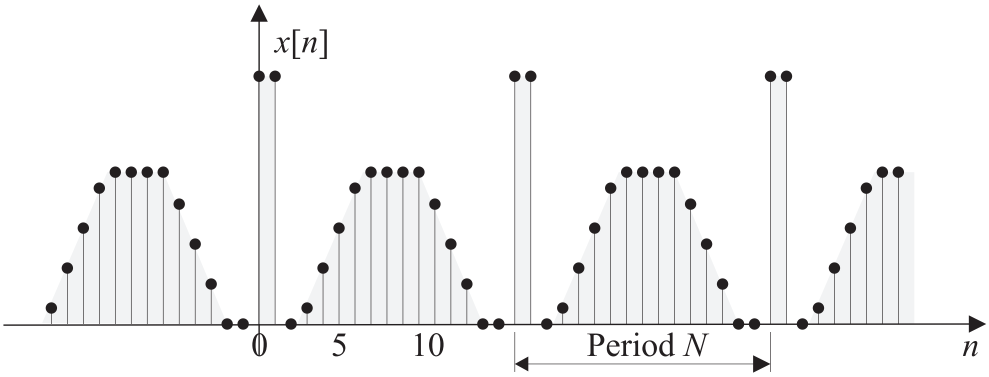


Figure 4.12: A discrete-time periodic signal.

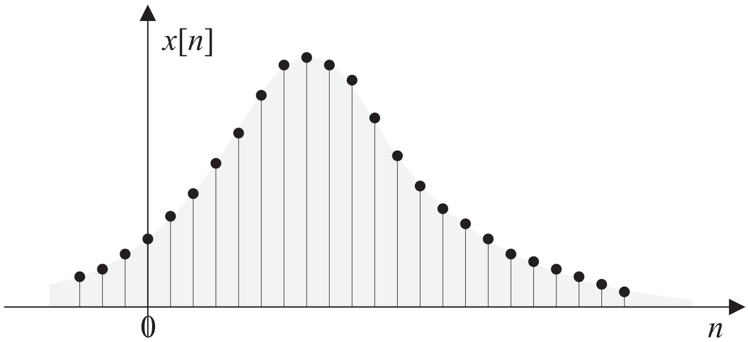


Figure 4.13: An example of a discrete-time aperiodic signal.

$\{0,\, 1,\, 2,\, \ldots\}$, where T_s is the sampling interval.

A discrete-time signal, $x[n]$, is periodic if:

$$x[n] = x[n + m\,N]\,,\; \forall\, m \in \mathbb{Z} \tag{4.15}$$

This is, of course, similar to Equation 4.14. Discrete-time periodic signals can be analysed using the discrete-time Fourier series or discrete Fourier transform (DFT) depending on whether the period is a multiple of the number of samples.

Discrete-time aperiodic Analogous to the continuous-time aperiodic signal in Figure 4.11a, a discrete-time aperiodic signal is shown in Figure 4.13.

Aperiodic discrete-time signals will be analysed using the discrete-time Fourier transform (DTFT). It can also be analysed using the so-called z-transform, which is the discrete-time version of the **Laplace transform**, although this will not be covered in complete detail until the third and fourth year courses, **Signals and Communications 3**, **Discrete-Time Signal Analysis**.

Finite-length discrete-time signals Discrete-time signals can also be classified as being finite in length. In other words, they are not assumed to exist for all-time, and what happens outside the **window** of data is assumed

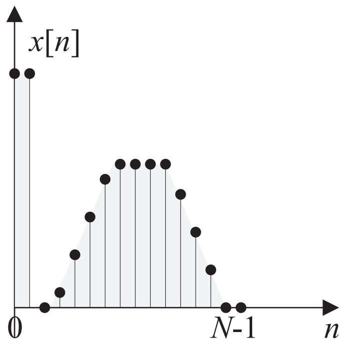


Figure 4.14: An example of a finite-duration signal.

unknown. These signals can be modelled using the so-called **DFT**, but again this is not covered until the fourth year course, **Discrete-Time Signal Analysis**. The Fast Fourier transform (FFT) is the well-known fast (low complexity) version of the DFT.

– End-of-Topic 17: **Summary of Different Types of Signals?** –

Summary Slide 3 Signal Representation and Analysis

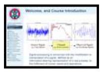
Signal Categories

- Signals are either **continuous-time** or **discrete-time**, and may be **periodic** or **aperiodic** (or non-periodic to avoid confusion).
- For each of the four possible types of signals, there is a particular frequency-domain method that should be used in its analysis.
-

<table>
<tr><td></td><td>Continuous-time</td><td>Discrete-time</td></tr>
<tr><td></td><td colspan="2">Periodic</td></tr>
<tr><td>Discrete
Spectrum:
Transform:</td><td></td><td>Discrete and periodic
DFT (FFT in practice)</td></tr>
<tr><td></td><td colspan="2">Aperiodic (non-periodic)</td></tr>
<tr><td>Continuous
Spectrum:
Transform:</td><td>Continuous-aperiodic</td><td>Continuous-periodic
DTFT</td></tr>
</table>

KEYPOINT! (Length of signals). These signal classes are assumed to be **infinite in duration** (exist for all time), whether in continuous- or discrete-time. We consider **finite-duration signals**, signals that last only for a known time-frame, and for which their characteristics outside that time-frame are unknown in the fourth year.

4.3.2 Energy and Power Signals



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Topic Summary 18 Measuring the size of a signal (and introduction to signal norms)

Topic Objectives:

- Understanding how to measure the size (or norm) of a signal.
- Motivation for Energy and Power.

Topic Activities:

Type	Details	Duration	Progress
Watch video	16 : 59 min video	$3\times$ length	
Read Handout	Read page 108 to page 113	8 mins/page	
Try Example	Try Example 4.2	15 mins	

http://media.ed.ac.uk/media/1_t1q1pzcc

Video Summary: This Topic considers the question of how to measure how big a signal is! This has applications from the simplest speech detection or voice activity detector, through to measuring the size of a residual error signal. The Topic motivates the concept of a signal norm by looking at measuring the size of other objects (expressions for volume), through to measuring the energy or power in physical components (through $P = {V^2}/{R}$). This leads to the L_2-norm or energy and power of a signal. These concepts will be formalised in the following topic, but this video first considers alternatives such as taking the sum of the magnitude of the signal (the L_1-norm), and the benefits of these different measures.

The are many applications, such as signal detection, where knowing the *size* of a signal is important. A large signal such as aircraft noise as it flies over a particular town might be more or less significant than a longer signal of lower amplitude, but it all depends on the application.

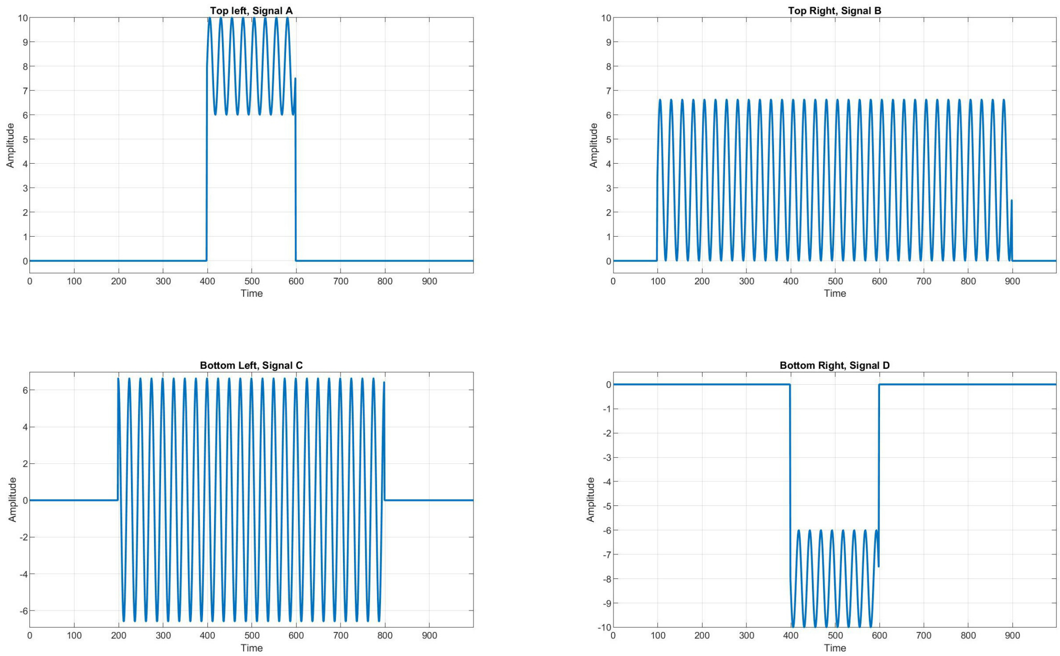


Figure 4.15: Which signal is the largest?

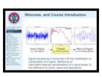
Example 4.2 (Multi-choice Question). Which of the signals shown in Figure 4.15 is the *largest*!?

Moreover, as stated in Section 4.3.1, signals can be analysed using a variety of frequency-domain transform methods, such as the **Fourier series**, **Fourier transform**, **Laplace transform**, and for discrete-time, the **z-transform**. For example, the Fourier transform is used to analyse aperiodic continuous-time signals.

However, not all aperiodic signals can be analysed using the Fourier transform, and the reason for this can be directly related to a fundamental property of a signal: a measure of *how much signal there is*.

Therefore it is relevant to consider the **energy** or **power** as a means for characterising a signal. The concepts of **power** and **energy** intuitively follow from their use in other aspects of the physical sciences. However, the concept of signals which exist for all time requires careful definitions, in order to determine when it has **energy** and when it has **power**.

Intuitively, energy and power measure *how big* a signal is. A motivating example of measuring the size of something is given in Sidebar 5, and in Figure 4.16. However, there are other possible signal measures, as discussed in Sidebar 6.



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4.3.2.1 Motivation for Energy and Power Expressions

Considering power from an electrical perspective, if a voltage $x(t)$ is connected across a resistance R, the dissipated power at time τ is given by:

Figure 4.16: What is the size of an object?

Sidebar 5 Size of a Human Being

Suppose we wish to devise a signal number V as a measure of the size of a human being. Then clearly, the width (or girth) must also be taken into account as well as the height. One could make the simplifying assumption that the shape of a person is a cylinder of variable radius r (which varies with the height h). Then one possible measure of the size of a person of height H is the person's volume, given by:

$$V = \pi \int_0^H r^2(h)\, dh \tag{4.16}$$

This can be found by dividing the object into circular discs (which is an approximation), where each disc has a volume $\delta V \approx \pi r^2(h)\ \delta h$. Then the total volume is given by $V = \int dV$.

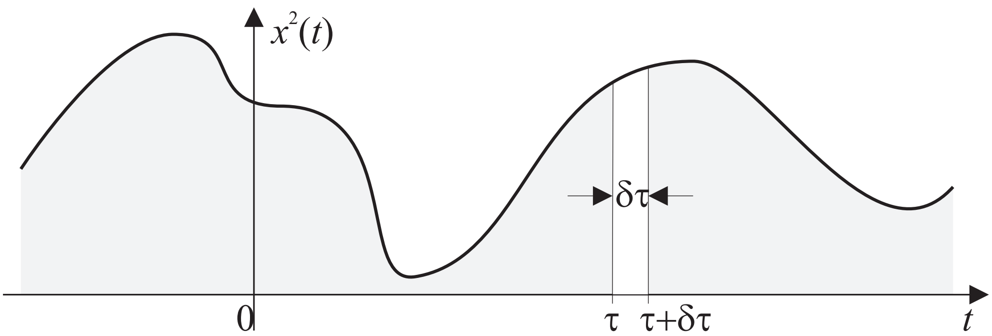


Figure 4.17: Energy Density.

Summary Slide 4 Power and Energy

Size of an Object

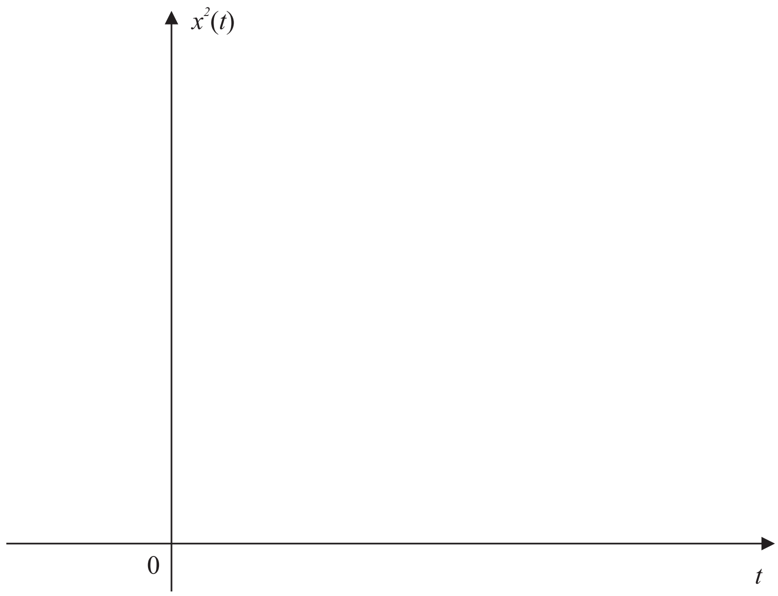
Summary Slide 5 Power and Energy

Strength of a Signal

What is the strength of a signal, or how large is it?

Magnitude of the waveform?

The squared magnitude is simpler to manage as it is differentiable.



KEYPOINT! (Size of a signal). A measure of signal strength must consider not only the signal amplitude, but also its duration.

$$P(\tau) = \frac{x^2(\tau)}{R} \propto x^2(\tau) \tag{4.19}$$

where $\propto$ denotes *proportional to*. In this case, the constant of proportionality is the inverse resistance. Since energy and power are related through the expression

$$\text{Energy} = \text{Power} \times \text{Time}, \tag{4.20}$$

the energy dissipated between times τ and $\tau + \delta\tau$, as indicated in Figure 4.17, is:

$$\delta E(\tau) \propto P(\tau)\, \delta\tau \propto x^2(\tau)\delta\tau \tag{4.21}$$

The total energy over all time can thus be found by integrating over all time:

$$E \propto \int_{-\infty}^{\infty} x^2(\tau)\, d\tau \tag{4.22}$$

This leads to the formal definitions of energy and power.

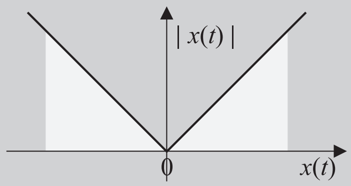
– End-of-Topic 18: **Introduction to Concepts of Energy and Power Signals for Measuring the Size of a Signal** –

Sidebar 6 Other signal measures

1. While the area under a signal $x(t)$ is a possible measure of its size, because it takes account not only of the amplitude but also of the duration, is defective since even for a very large signal, the positive and negative areas could cancel each other out, indicating a signal of a small size.

2. Using the sum of square values can potentially give undue weighting to any outliers in the signal, where an outlier is defined as an unusual signal variation that is not characteristic of the rest of the signal; an example might be a high-energy shot burst of interference.

3. Therefore, taking the integral of the absolute value, $|x(t)| \equiv \operatorname{abs} x(t)$, is a possible measure and in some circumstances can be used. The relationship between input and output for this signal measure is shown below.



Unfortunately, dealing with the absolute value of a function can be difficult to manipulate mathematically. However, using the area under the square of the function is not only more mathematically tractable but is also more meaningful when compared with the electrical examples and the volume in Sidebar 5.

4. These notions lead onto the more general subject of **signal norms**. The L_p-norm is defined by:

$$L_p(x) \triangleq \left(\int_{-\infty}^{\infty} |x(t)|^p \, dt\right)^{\frac{1}{p}}, \qquad p \geq 1 \tag{4.23}$$

In particular, the expression for energy is related to the L_2-norm, while using the absolute value of the signbal gives rise to the L_1-norm:

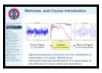
$$L_1(x) \triangleq \int_{-\infty}^{\infty} |x(t)| \, dt \tag{4.24}$$

which is the integral of the absolute value as described above in part 3.

5. While Parseval's theorem, described on later for the power of periodic signals, develops a relationship between the L_2-norms in the time-domain and frequency-domain, in general no relation exists for other values of p.

6. Note that the L_p-norm generalises for discrete-time signals as follows:

$$L_p(x) \triangleq \left(\sum_{-\infty}^{\infty} |x[t]|^p\right)^{\frac{1}{p}}, \qquad p \geq 1 \tag{4.25}$$



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4.3.2.2 Formal Definitions for Energy and Power

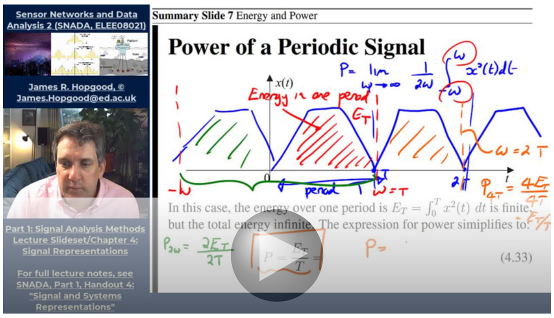
Topic Summary 19 Formal Energy and Power Definitions

Topic Objectives:

- Formal definitions for Energy and Power.
- Consider the energy and power of a step function.
- Units of Energy and Power.
- Consider simplification for the power of periodic signals.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 57 min video	3× length	
Read Handout	Read page 115 to page 121	8 mins/page	
Try Example	Try Examples 4.3 to 4.6	25 mins	
Practice Exercises	Exercises 4.4 and 4.5	20 mins	



http://media.ed.ac.uk/media/1_mbtys0ih

Video Summary: This Topic formally introduces the abstract definitions for energy and power for measuring the size of a signal. Energy is the measure or metric for non-periodic signals that decay away in the limit. Several examples of calculating energy are worked through, including what happens if you try and calculate energy for signals such as the step signal. This leads to the reason that power signals are defined, and this is the measure for signals that do not decay away, or are periodic. A derivation of the power formula is provided, so that the origin of the calculation is not a mystery. While energy and power are abstract concepts in the context of measuring the size of measuring an abstract signal, they can be converted to real units by considering the dimensions of the signals involved. Finally, there is a simplified expression for the power of a signal, and this is derived at the end of this Topic.

Based on the justification in Section 4.3.2.1, the formal abstract definitions for energy and power that are independent of how the energy or power is dissipated are defined below.

Energy Signals A continuous-time signal $x(t)$ is said to be an **energy signal** if the total energy, E, dissipated by the signal over all time is both *nonzero* and *finite*. Thus:

$$0 < E < \infty \quad \text{where} \quad E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{4.26}$$

where $|x(t)|$ means the magnitude of the signal $x(t)$. If $x(t)$ is a real-signal, this is just its amplitude. If $x(t)$ is a complex-signal, then $|x(t)|^2 = x(t)\, x^*(t)$ where $*$ denotes complex-conjugate. In this course, however, only real signals will be encountered.

A necessary condition for the energy to be finite is that the signal amplitude $|x(t)| \to 0$ as $|t| \to \infty$, otherwise the integral in Equation 4.26 will not exist. When the amplitude of $x(t)$ does not tend to zero as $|t| \to \infty$, the signal energy is likely to be infinite. A more meaningful measure of the signal size in such a case would be the time average of the energy if it exists. This measure is called the **power** of the signal.

Example 4.3 (Calculating Energy). Consider the signal:

$$x_1(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \tag{4.27}$$

1. What is the energy in this signal?
2. How does this compare with the size of the signal?

$$x_2(t) = \begin{cases} 0 & t < 0 \\ 7e^{-t/4} & t \geq 0 \end{cases} \tag{4.28}$$

SOLUTION. Note that in each case the functional form of the signal changes for $t < 0$ and $t \geq 0$, then as the signal decays away for $t \to \infty$, the signal is likely to be an Energy signal. For the first part of the question, then:

$$E_1 = \int_{-\infty}^{\infty} x_1^2(t)\, dt = \int_{-\infty}^{0} 0^2 \, dt + \int_0^{\infty} \left(e^{-t}\right)^2 dt \tag{4.29}$$

$$= \int_0^{\infty} e^{-2t} \, dt = \left[\frac{e^{-2t}}{-2}\right]_0^{\infty} = 0 + \frac{1}{2} = \frac{1}{2} \tag{4.30}$$

Similarly, in the second case:

$$E_2 = \int_{-\infty}^{\infty} x_2^2(t)\, dt = \int_0^{\infty} 47\, e^{-\frac{t}{2}} \, dt \tag{4.31}$$

$$= \left[\frac{49e^{-\frac{t}{2}}}{-\frac{1}{2}}\right]_0^{\infty} = 98 \tag{4.32}$$

□

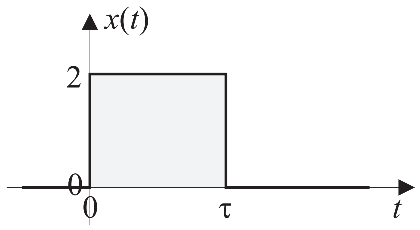


Figure 4.18: Rectangular pulse of length τ.

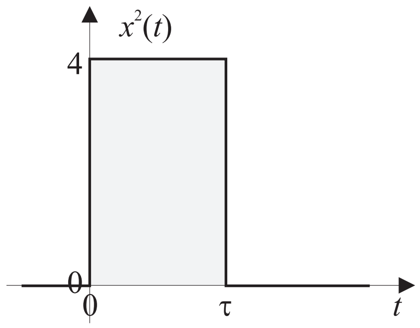


Figure 4.19: The total energy of the signal in Figure 4.18 can be found as the area under the curve representing the square of the rectangular pulse, as shown here.

So the second signal is certainly larger, mainly because it decays away more slowly than the first case, as well as the higher amplitude. The reader is encouraged to plot these signals to verify this carefully.

Power signals If the average power delivered by the signal over all time is both *nonzero* and *finite*, the signal is classified as a power signal:

$$0 < P < \infty \quad \text{where} \quad P = \lim_{W\to\infty} \frac{1}{2W} \int_{-W}^{W} |x(t)|^2 \, dt \qquad (4.33)$$

where the variable W can be considered as half of the width of a *window* that covers the signal and gets larger and larger.

Example 4.4. Name a type of signal which is not an example of an **energy signal**?

SOLUTION. A periodic signal has finite energy over one period, so consequently has infinite energy. However, as a result it has a finite average power and is therefore a power signal, and not an energy signal.

Example 4.5 (Rectangular Pulse). What is the energy of the rectangular pulse shown in Figure 4.18 as a function of τ, and for the particular case of $\tau = 4$?

Summary Slide 6 Energy and Power

Energy Signals

Define the total energy of a signal $x(t)$ as:

$$E = \int_{-\infty}^{\infty} x^2(t)\ dt \tag{4.34}$$

A signal $x(t)$ is said to be an **energy signal** if the total energy, E, dissipated by the signal between the beginning and end of time is *nonzero* and *finite*, such that:

$$0 < E < \infty \tag{4.35}$$

Generally **energy signals** satisfy: $|x(t)| \to 0$ as $|t| \to \infty$.

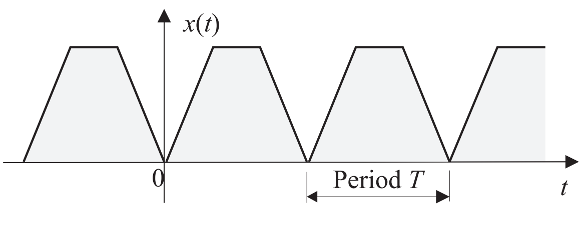
Power Signals

Define the average power as:

$$P = \lim_{W \to \infty} \frac{1}{2W} \int_{-W}^{W} x^2(t)\ dt \tag{4.37}$$

A signal $x(t)$ is said to be a **power signal** if the average power delivered by the signal from the beginning to the end of time is nonzero and finite, such that:

$$0 < P < \infty \tag{4.38}$$



Summary Slide 7 Energy and Power

Power of a Periodic Signal

In this case, the energy over one period is $E_T = \int_0^T x^2(t)\ dt$ is finite, but the total energy infinite. The expression for power simiplifies to:

$$P = \frac{E_T}{T} = \tag{4.39}$$

Power in a Step Function

Some non-periodic signals are **power signals**, often satisfying $|x(t)\,| \not\Rightarrow 0$ as $|t| \to \infty$ such that $E = \int_{-\infty}^{\infty} x^2(t)\,dt \to \infty$:

SOLUTION. The signal can be represented by

$$x(t) = \begin{cases} 2 & 0 \le t < \tau \\ 0 & \text{otherwise} \end{cases} \tag{4.43}$$

so that the square of the signal is also rectangular and given by

$$x^2(t) = \begin{cases} 4 & 0 \le t < \tau \\ 0 & \text{otherwise} \end{cases} \tag{4.44}$$

Since an integral can be interpreted as the area under the curve (see Figure 4.19), the total energy is thus:

$$E = 4\tau \tag{4.45}$$

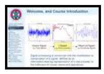
□

When $\tau = 4$, $E = 16$.

Example 4.6 (Multiple Choice). The signal $x(t) = \exp(-|t|)$ is:

1. an energy signal, but not a power signal;
2. a power signal, but not an energy signal;
3. both an energy and a power signal;
4. not an energy signal, nor a power signal?

4.3.2.3 Units of Energy and Power



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It is important to consider the physical units associated with energy and power, and therefore to determine how the abstract definitions of E and P in Equation 4.26 and Equation 4.33 can be converted into real energy and power.

Consider again power from an electrical perspective. When considering "direct current" (DC) signals, power is given by

$$P_{DC} = \frac{V^2}{R} = \frac{\text{Volts}^2}{\text{Ohms}} = \text{Watts} \tag{4.46}$$

where V is the signal voltage, and R is the resistance through which the power is dissipated. Consider now the units of the abstract definition of power, P in Equation 4.33:

$$P = \frac{1}{\text{time}} \times \text{Volts}^2 \times \text{time} = \text{Volts}^2 = \text{Watts} \times \text{Ohms} \tag{4.47}$$

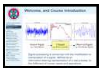
where the second unit of *time* comes from the integral term dt, and in which the integral may be considered as a summation. Therefore, by comparing Equation 4.46

and Equation 4.33, the abstract definition of power, P, can be converted to real power by **Ohms** for the case of electrical circuits.

Similarly, the units of energy in Equation 4.26 is given by

$$E = \text{volts}^2 \times \text{time} \tag{4.48}$$

Therefore, to convert the abstract energy to Joules, it is again necessary to divide by **Ohms** by noting that energy is power multiplied by time.



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4.3.2.4 Power for Periodic Signals

The expression for power in Equation 4.33 can be simplified for periodic signals. Consider the periodic signal in Figure 4.10a. Let $2W = T$ and define:

$$P_T = \frac{1}{2W} \int_{-W}^{W} |x(t)|^2 \, dt \tag{4.49}$$

Thus, the average power over two periods is $2P_T$, and the average power over N periods is P_{NT}. Then, it should becomes clear that:

$$P_T = P_{NT}, \, \forall N \in \mathbb{Z} \tag{4.50}$$

since the average in each period is the same. Consequently, **power** for a periodic signal with period T may be defined as:

$$P = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, dt \tag{4.51}$$

Note that the limits in Equation 4.51 may be over any period and thus can be replaced by $(\tau, \, \tau + T)$ for any value of τ.

– End-of-Topic 19: **Definitions of Energy and Power Signals** –

4.4 Tutorial Exercises
Signal and System Classification

Exercise 4.1 (Signal Classifications). [*Difficulty:* **3** ($\star\star\star$)] A **multi-channel signal** is one in which multiple *related* signals are measured simultaneously. For example, a *stereo microphone* gives a multi-channel (*stereo*) signal, as does the signal from an electrocardiogram (ECG) consisting of 12-electrodes.

Classify the following signals according to whether they are: (1) single- or multi-dimensional; (2) single or multi-channel; (3) continuous-time or discrete-time; (4) analogue or digital in amplitude. Give a brief explanation as to why for each.

1. Closing prices of utility stocks on the New York Stock Exchange.
2. A colour movie.
3. The position of the steering wheel of a car in motion, relative to the car's frame of reference.
4. The position of the steering wheel of a car in motion, relative to the ground reference frame.
5. Weight and height measurements of a growing human taken every month.

Exercise 4.2 (Simple Decompositions). [*Difficulty:* **2** ($\star\star$)] The **unit step** or **Heaviside step function** is given by the function:

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{4.52}$$

1. Sketch the function $u(t)$.
2. Write the function $x(t)$ shown in Figure 4.20 in terms of $u(t)$.

HINTS. Consider a linear decomposition for the waveform $x(t)$.

Exercise 4.3 (Linear versus nonlinear systems). [*Difficulty:* **3** ($\star\star\star$)] By definition, a system denoted by $\mathcal{T}$ is linear if and only if:

$$\mathcal{T}\left(a_1\, x_1(t) + a_2\, x_2(t)\right) = a_1\, \mathcal{T}\left(x_1(t)\right) + a_2\, \mathcal{T}\left(x_2(t)\right) \tag{4.53}$$

for any arbitrary input signals $x_1(t)$ and $x_2(t)$, and any arbitrary constants a_1 and a_2.

Determine if the systems described by the following input-output equations are linear or nonlinear:

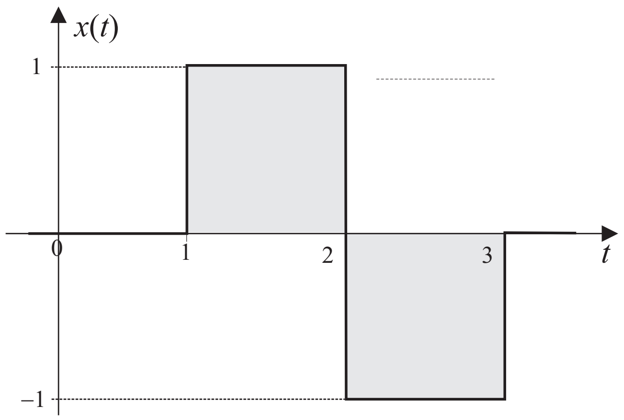


Figure 4.20: Pulse Waveform

1.
$$y(t) = t\,x(t) \tag{4.54}$$

2.
$$y(t) = x\big(t^2\big) \tag{4.55}$$

3.
$$y(t) = x^2(t) \tag{4.56}$$

4.
$$y(t) = A\,x(t) + B \tag{4.57}$$

5.
$$y(t) = e^{x(t)} \tag{4.58}$$

⋈

Exercise 4.4 (Energy and Power). [*Difficulty:* **2** ($\star\star$)] A voltage signal, $v(t)$, is connected across a $5\ \Omega$ resistor at $t = 0$. If the voltage is defined as:

$$v(t) = \begin{cases} 2\,\exp(-3t) + 4\exp(-7t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases} \tag{4.59}$$

what is the total energy dissipated in the resistor?

HINTS. The signal $v(t)$ has *units* of volts; hence, if using the expression for the **energy** of an *abstract* signal as given in the lecture notes, think carefully about the units of the resulting equations.

Final answer: $E = 0.68$ Joules.

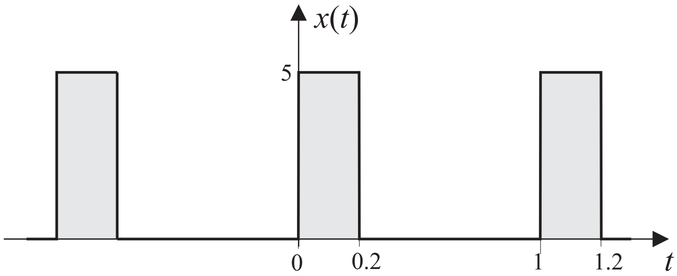


Figure 4.21: Pulse waveform.

Exercise 4.5 (Power of Periodic Waveform). [*Difficulty:* **2** ($\star\star$)] Calculate the power associated with the periodic waveform shown in Figure 4.21 (where two full periods are shown). Assume the signal is dissipated across a $1\ \Omega$ resistor.

Final answer: 5 Watts.

Exercise 4.6 (Energy and Power). [*Difficulty:* **3** ($\star\star\star$)]

Consider the following two signals:

- the two-sided exponential function,

$$x(t) = 2\,e^{-\frac{|t|}{2}}, \quad -\infty < t < \infty;$$

- the sawtooth function,

$$y(t) = \begin{cases} t & -1 < t \leq 1 \\ y(t+2) & \text{otherwise} \end{cases}. \qquad \bowtie$$

For each signal:

1. sketch the signal waveform;
2. determine whether **energy** or **power** provides the most suitable *measure* of the size of the signal, justifying your answer;
3. calculate the value of the measure chosen in part 2.

5

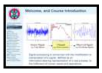
The Fourier Series for Spectral Analysis of Continuous-Time Periodic Signals

Great acts are done by a series of small deeds.

Lao Tzu

This handout introduces the Fourier Series for the analysis of continuous-time periodic signals. It assumes the reader has some familiarity with the trigonometric Fourier Series, although this is revised. The handout then moves onto the more general, and arguably useful, form called the complex Fourier Series. The complex form allows for a simple derivation of the Fourier transform in the next handout. This handout gives relationships between the trig and complex forms, some basic properties, and then develops Parseval's theorem for calculating Energy in the Frequency domain.

5.1 Trigonometric Fourier Series



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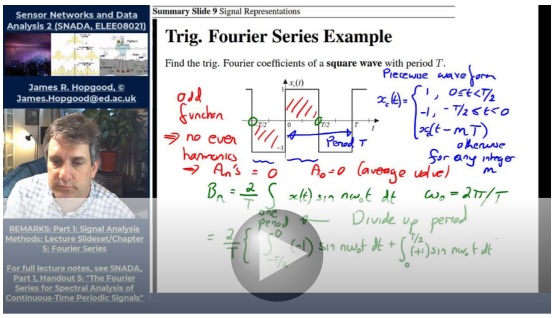
Topic Summary 20 Revision of Trigonometric Fourier Series

Topic Objectives:

- Revision/gentle introduction to the Trigonometric Fourier Series.
- Graphical visualisation of the Fourier Series.
- Example of calculating Fourier Series coefficients.
- Other forms of the Fourier Series from other Disciplines.

Topic Activities:

Type	Details	Duration	Progress
Watch video	28 : 21 min video	3× length	
Read Handout	Read page 126 to page 132	8 mins/page	
Try Code	Use MATLAB code to generate signals	20 minutes	
Try Example	Try Example 5.1	20 mins	



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Video Summary: This Topic gives a revision of Trigonometric Fourier Series. It builds on the motivation for using signal decompositions for analysing signals, and decomposes any finite power periodic signal into an infinite summation of sinusoids and cosinusoids, whose frequencies are integer multiples of a fundamental frequency. The Topic gives the forumulas for calculating the Fourier coefficient, as well as the Fourier series expansion itself. A graphical example is shown, through decomposing a sawtooth wave, and also provides an audio reconstruction. The second half of the video considers a simple example of calculating the trigonometric Fourier coefficients of a square wave.

The Fourier Series is a method for decomposing a period continuous-time signal into a sum of sinusoids and co-sinusoids. This decomposition helps analyse complex linear

Sidebar 7 Periodicity of Fourier Series

It is important to show that the waveform in Equation 5.1 is indeed a periodic signal with fundamental period T, regardless of the amplitudes of the coefficients A_n and B_n.

To prove this, it is sufficient to show that $x(t) = x(t + T)$. From Equation 5.1, note that:

$$x(t+T) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 \{t+T\}\right) + B_n \sin\left(n\omega_0 \{t+T\}\right)\right] \tag{5.3}$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t + n\omega_0 T\right) + B_n \sin\left(n\omega_0 t + n\omega_0 T\right)\right] \tag{5.4}$$

and using Equation 5.2, then

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t + 2\pi\, n\right) + B_n \sin\left(n\omega_0 t + 2\pi\, n\right)\right] \tag{5.5}$$

$$= \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t\right) + B_n \sin\left(n\omega_0 t\right)\right] \tag{5.6}$$

systems as sinusoids and co-sinusoids are particular solutions to ordinary differential equations (ODEs).

Any finite power periodic signal $x(t)$ with a period of T seconds can be represented as a summation of sine waves and cosine waves:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t\right) + B_n \sin\left(n\omega_0 t\right)\right] \tag{5.1}$$

The **fundamental frequency** of $x(t)$ is the inverse of the fundamental period, and is therefore given by:

$$f_0 = \frac{1}{T}\ \text{Hz} \qquad \text{or} \qquad \omega_0 = \frac{2\pi}{T}\ \text{rad/s} \tag{5.2}$$

Note this means that $\omega_0 T = 2\pi$, and this can be used to show that the Fourier Series is indeed a periodic function (see Sidebar 7).

KEYPOINT! (Trigonometric Fourier Series). Note that the signal decomposition in Equation 5.1 is known as the trigonometric Fourier series, since it is decomposed in terms of the trigonometric functions $\sin$ and $\cos$. This is in contrast to the mathematically equivalent but slightly more tractable complex Fourier series developed in Section 5.3, which decomposed a signal in terms of complex phasors.

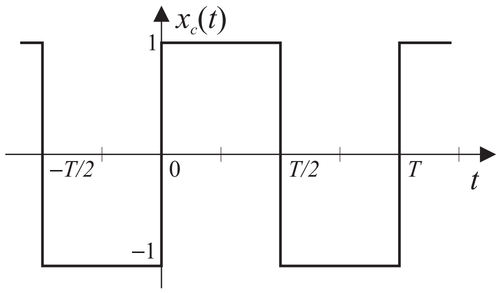


Figure 5.1: Square wave.

KEYPOINT! (Fourier Series Coefficients). The trigonometric **Fourier coefficients**, A_n and B_n, can be calculated directly from the signal using the identities (which will be in the data-sheet with the exam):

$$A_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos(n\omega_0 t)\, dt, \quad n \in \{0, 1, 2, \dots\} \tag{5.7}$$

$$B_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin(n\omega_0 t)\, dt, \quad n \in \{1, 2, \dots\} \tag{5.8}$$

□

Note that the limits in the integrals for the Fourier coefficients can be over any period whatsoever, and therefore it is sufficient to write:

$$A_n = \frac{2}{T} \int_{\mathcal{T}} x(t) \cos(n\omega_0 t)\, dt, \quad n \in \{0, 1, 2, \dots\} \tag{5.9}$$

$$B_n = \frac{2}{T} \int_{\mathcal{T}} x(t) \sin(n\omega_0 t)\, dt, \quad n \in \{1, 2, \dots\} \tag{5.10}$$

where $\mathcal{T}$ means over any period and is thus given by $\mathcal{T} = (a,\, a + T)$ for any value of a.

Example 5.1 (Fourier Series of a Square Wave). Show that the square wave in Figure 5.1 has Fourier series given by:

$$x_c(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega_0 t}{2n-1} \tag{5.15}$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency. Thus, it can be seen that there are only **odd harmonics**, as the coefficients can be explicitly written as $A_n = 0$ for all n, while:

$$B_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd} \end{cases} \tag{5.16}$$

⋈

Moreover, notice that not are there only odd harmonics, but the function also consists of sine waves only. The expression in Equation 5.15 is written more compactly than Equation 5.16 suggests by summing over all positive integers, and noting that $2n - 1$ is an odd number.

Summary Slide 8 Frequency Analysis

Trigonometric Fourier Series

- Any finite power periodic signal $x(t)$ with a period of T seconds can be represented as:

$$x(t) = \underbrace{\frac{A_0}{2}} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t\right) + B_n \sin\left(n\omega_0 t\right)\right] \quad (5.11)$$

  The **fundamental frequency** of $x(t)$ is $f_0 = \frac{1}{T}$ Hz or $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$ rad/s.

- The trigonometric **Fourier coefficients**, A_n and B_n, can be calculated directly from the signal using:

$$A_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos\left(n\omega_0 t\right) dt, \quad n \in \{0, 1, 2, \ldots\} \quad (5.12)$$

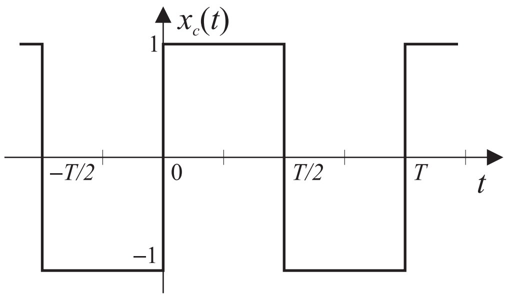
$$B_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \sin\left(n\omega_0 t\right) dt, \quad n \in \{1, 2, \ldots\} \quad (5.13)$$

KEYPOINT! (Building Blocks). The Fourier series decomposes a signal into basic building blocks, which are the trigonometric functions $\sin\left(n\omega_0 t\right)$ and $\cos\left(n\omega_0 t\right)$.

Summary Slide 9 Signal Representations

Trig. Fourier Series Example

Find the trig. Fourier coefficients of a **square wave** with period T.



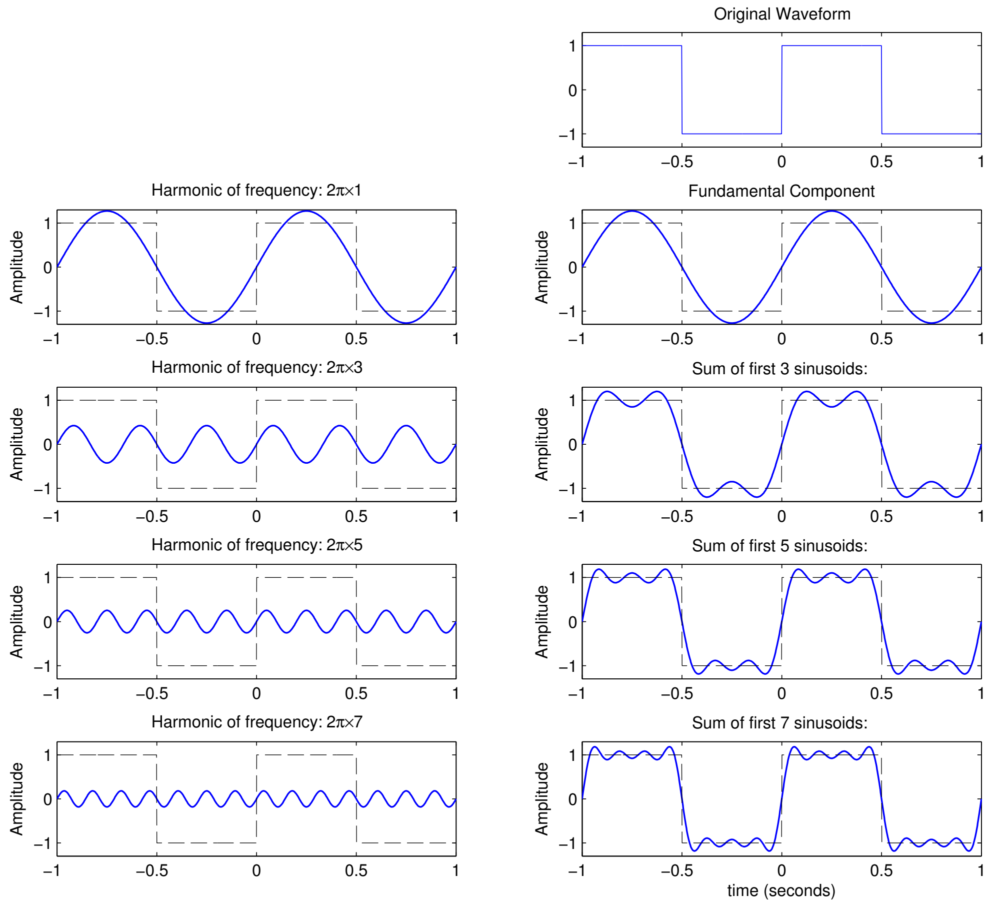


Figure 5.2: Two periods of a square wave and its Fourier series approximation with an increasing number of terms, from the fundamental, through to the summation of the first seven harmonics.

```
f0 = 220.5;
fs = 44100;

t = linspace(-1, 1, 2*fs);
numComponents = 18;

x_saw = zeros(1, length(t));
x_saw_audio = zeros(1, numComponents * fs);
x_saw_audio(1 : fs)  = 2 / pi * sin(2 * pi * f0*(0:fs-1)/fs);

for n = 2 : numComponents
  % Fourier Coefficients for the Sawtooth Wave
  Xm_saw = 2 / (pi * n) * (-1)^(n+1);

  xm_saw_audio = Xm_saw * sin(2 * pi * n * f0*(0:fs-1)/fs);
  x_saw_audio((n-1) * fs + 1 : n*fs)  = x_saw_audio((n-2) * fs + 1 : (n-1)*fs) + xm_saw_audio;
end

x_saw_audio = [zeros(1, 0.1*fs), x_saw_audio] / 0.95*max(x_saw_audio);


soundsc(x_saw_audio, fs)
```

Figure 5.3: Code for visualising Fourier series.

Note that Figure 5.2 shows the Fourier series approximation to this square wave. The coefficients in Equation 5.15 are used when plotting this calculation. The Fourier series approximation for a Sawtooth waveform (shown in Figure 5.4) is shown in Figure 5.5.

KEYPOINT! (Acoustic Visualisation). This is a great time to use a MATLAB demonstration to visualise the Fourier series of a sawtooth function, but this time using audio rather than graphs! This is achieved using the following code:

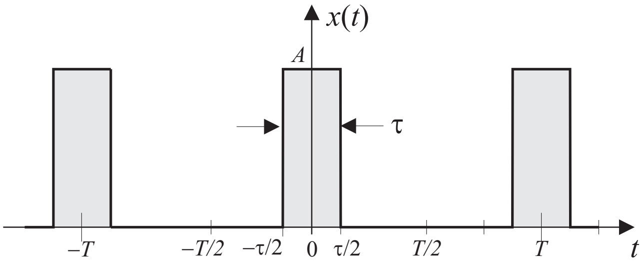
```
f0 = 220.5; fs = 44100; numComponents = 18;
t = linspace(-1, 1, 2*fs);

x_saw = zeros(1, length(t));
x_saw_audio = zeros(1, numComponents * fs);
x_saw_audio(1 : fs) = 2 / pi * sin(2 * pi * f0*(0:fs-1)/fs);

for n = 2 : numComponents
 % Fourier Coefficients for the Sawtooth Wave
 Xm_saw = 2 / (pi * n) * (-1)^(n+1);
 xm_saw_audio = Xm_saw * sin(2 * pi * n * f0*(0:fs-1)/fs);
 x_saw_audio((n-1) * fs + 1 : n*fs) = ...
 x_saw_audio((n-2) * fs + 1 : (n-1)*fs) + xm_saw_audio;
end

soundsc(x_saw_audio, fs)
```

– End-of-Topic 20: **Revision of Trigonometric Fourier Series** –

Summary Slide 10 Examples Class Week 3: Trigonometric Fourier Series

Trig. Fourier Series of Pulse Wave

Find the trigonometric Fourier series coefficients of a pulse wave with period T and width τ.



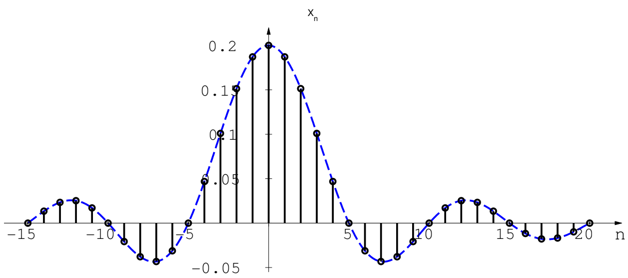
$$A_n = \tag{5.17}$$

$$=$$

$$=$$

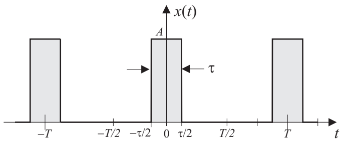
$$A_n =$$

The nth Fourier coefficients corresponds to the component with frequency $\omega_n = n\omega_0 = n\frac{2\pi}{T}$. What do these coefficients look like when plotted against coefficient index, n?



Summary Slide 11 Examples Class Week 3 Continued: Trigonometric Fourier Series

Trig. Fourier Series of Pulse Wave

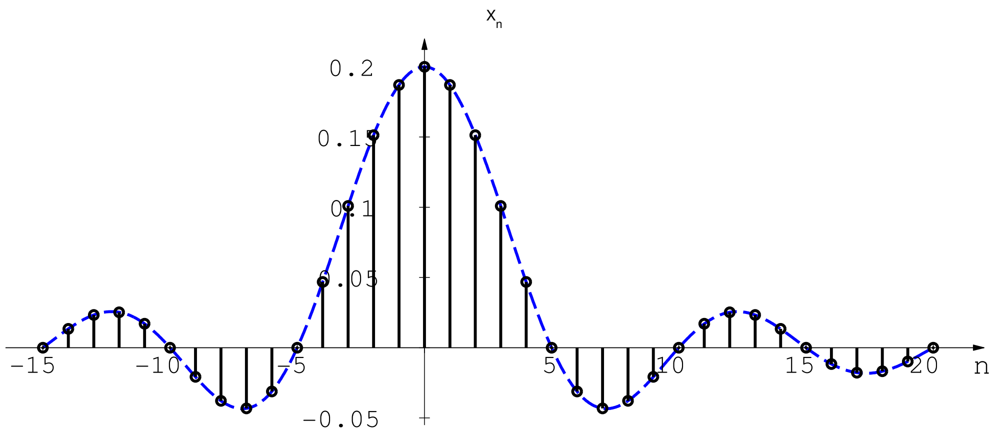


The Fourier coefficients for a pulse wave with period T and pulse width τ are given by $B_n = 0$, $\forall n$, and:

$$A_n = \frac{2A\tau}{T} \operatorname{sinc}\left(\frac{\pi n \tau}{T}\right) \tag{5.18}$$

KEYPOINT! (Sinc Function). This is the classic **sinc** function, and it is important to know how to plot this – need to draw it very often.

In particular, the zero-crossings occur when :



Consider case: $A = 1$, $T = 1$, and $\tau = \frac{1}{5}$.

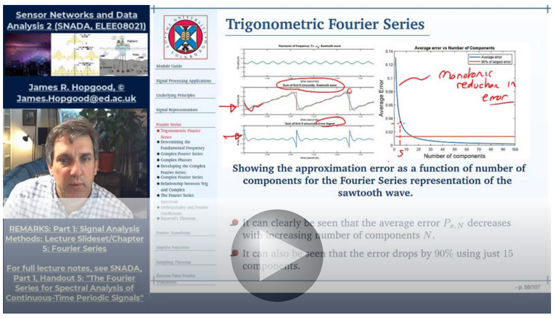
Topic Summary 21 Insight into the Trigonometric Fourier Series

Topic Objectives:

- Understanding the general form of the Trigonometric Fourier Series.
- Understanding Signal Approximation through the Truncated Fourier Series.
- Application of Signal Approximation to Loseless Image Compression.

Topic Activities:

Type	**Details**	**Duration**	**Progress**
Watch video	12 : 51 min video	3× length	
Read Handout	Read page 135 to page 138	8 mins/page	
Try Code	Use MATLAB code to generate signals	20 minutes	
Try Example	Try Example 5.2	4 mins	



http://media.ed.ac.uk/media/1_2ko939vt

Video Summary: This Topic seeks to expand our understanding of Fourier Series. First, through an example, the Topic considers the general form or structure of the trigonometric Fourier series, and what is a valid Fourier Series and what isn't. Second, the Topic considers the notion of signal approximation through the truncated Fourier Series, where only a finite number of harmonics are used rather than an infinite number. To quantify the goodness-of-fit of the approximation, the power of the error signal is considered as a function of the number of harmonics, where it is seen this power drops monotonically. The Topic introduces the idea of an acceptable level of error, and discusses the application of this in Image Compression and other signal compression algorithms.

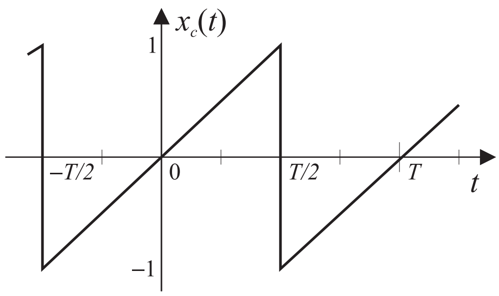


Figure 5.4: Sawtooth wave.

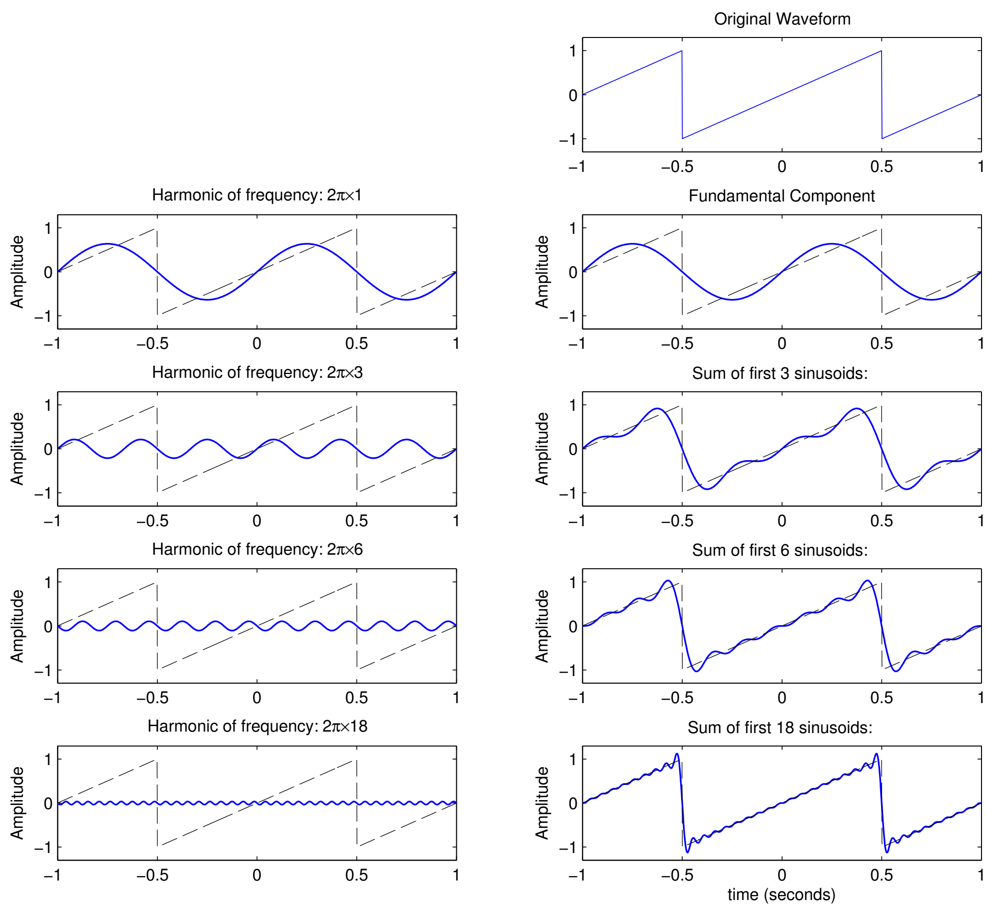
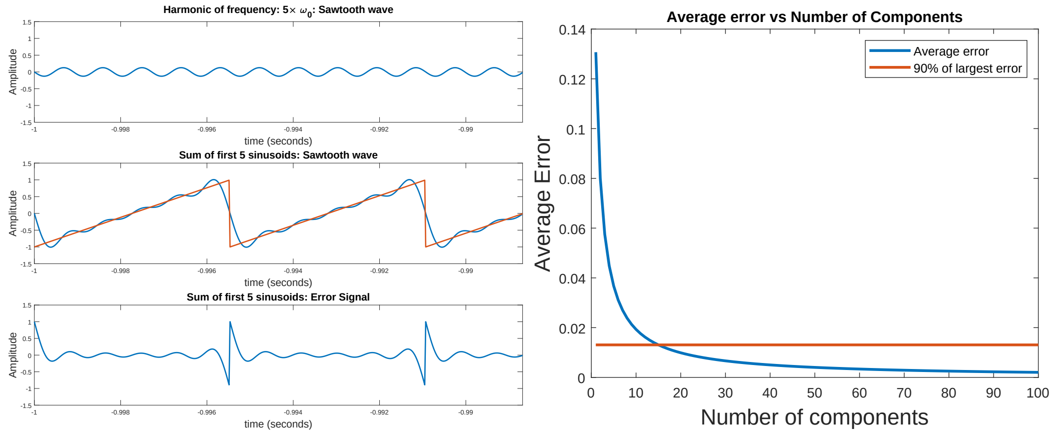


Figure 5.5: Two periods of a sawtooth wave and its Fourier series approximation with an increasing number of terms, from the fundamental, through to the summation of the first 18 harmonics.



(a) Truncated Fourier approximation. (b) Average error of the approximation.

Figure 5.6: Showing the approximation error as a function of number of components for the Fourier Series representation of the sawtooth wave shown in Figure 5.4.

Example 5.2 (Multi-choice: Fourier Series). Which of the following is the correct Fourier series expansion of the sawtooth signal shown in Figure 5.4:

1. $$x_c(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos n\omega_0 t}{n}$$

2. $$x_c(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\omega_0 t}{n}$$

3. $$x_c(t) = \frac{4}{\sqrt{\pi}}+\frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\omega_0 t}{n}$$

4. $$x_c(t) = \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sqrt{\sin n\omega_0 t}}{n}$$

⋈

You do not need to undertake detailed mathematical analysis, but use basic properties of the Fourier series.

Finally, it is interesting to consider what happens if a signal $x(t)$ is represented by a truncated Fourier series using just the first N harmonics:

$$\tilde{x}_N(t) = \frac{A_0}{2} + \sum_{n=1}^{N}\left[A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t)\right] \tag{5.20}$$

The error between this and the original signal can be written as:

$$e_{x,N}(t) = \tilde{x}_N(t) - x(t) \tag{5.21}$$

The size of the error can be measured using the expression for the power of a periodic signal, namely:

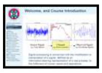
$$P_{x,N} = \frac{1}{T} \int_{-T/2}^{T/2} |e_{x,N}(t)|^2 \, dt \tag{5.22}$$

Clearly, the smaller the power in this error signal, the better the approximation.

The error signal and the approximation power is shown in Figure 5.6. It can clearly be seen that the average error $P_{x,N}$ decreases with increasing number of components N. It can also be seen that the error drops by 90% using just 15 components.

– End-of-Topic 21: **Understanding the Trigonometric Fourier Series –**

5.2 Determining the Fundamental Frequency and Period



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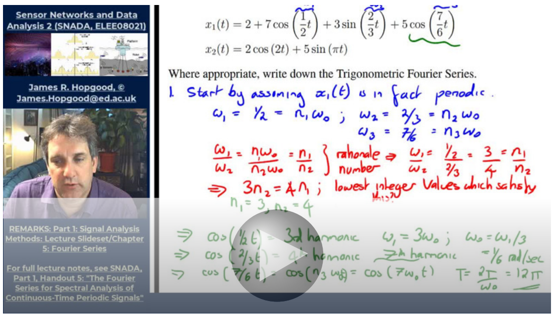
Topic Summary 22 Determining the Fundamental Frequency of a Signal

Topic Objectives:

- Method for calculating the fundamental frequency of a periodic signal, assuming it is not known in advance.
- Identifying when the sum of periodic signals is not periodic.
- Identifying Fourier Series coefficients by inspection.

Topic Activities:

Type	Details	Duration	Progress
Watch video	13 : 28 min video	3× length	
Read Handout	Read page 139 to page 142	8 mins/page	
Try Code	MATLAB code to generate signals	20 mins	
Try Example	Try 5.3 and 5.4	20 mins	
Practice Exercises	Exercises 5.1 and 5.2	30 mins	



http://media.ed.ac.uk/media/1_rmaghhqf

Video Summary: This Topic looks at the case of calculating the fundamental frequency or period of a periodic waveform, where this is not known in advance. The Topic also considers, as a consequence, whether the sum of periodic signals results in a periodic signal. The calculation is relatively straightforward, but needs some care. Several detailed examples are provided for finding the fundamental frequency, if it exists. If a finite arbitrary sum of sinusoids and co-sinusoids is provided, then it is demonstrated how the Trigonometric Fourier coefficients can be obtained by inspection.

It has been seen, through the Fourier series, that every periodic signal can be expressed as a sum of sinusoids of a fundamental frequency, $\omega_0 = \frac{2\pi}{T}$ where T is the fundamental period, and the harmonics at integer multiples of the fundamental.

One may ask whether a sum of sinusoids of *any* frequencies represents a periodic signal. If so, how does one determine the period? Consider the following three functions:

$$x_1(t) = 2 + 7\cos\left(\frac{1}{2}t + \phi_1\right) + 3\cos\left(\frac{2}{3}t + \phi_2\right) + 5\cos\left(\frac{7}{6}t + \phi_3\right) \tag{5.28}$$

$$x_2(t) = 2\cos(2t + \phi_1) + 5\sin(\pi t + \phi_2) \tag{5.29}$$

$$x_3(t) = 3\sin\left(2\sqrt{2}t + \phi\right) + 7\cos\left(6\sqrt{2}t + \phi\right) \tag{5.30}$$

Recall that every frequency in a periodic signal is an integer multiple of the fundamental frequency ω_0. Therefore, the ratio of any two frequencies is of the form m/n, where m and n are integers. This means that the ratio of any two frequencies is a rational number. When the ratio of two frequencies is a rational number, the frequencies are said to be **harmonically** related.

The largest number of which all the frequencies are integer multiples is the fundamental frequency. In other words, the fundamental frequency is the greatest common factor (GCF) of all the frequencies in the series. Thus:

1. the frequencies in the *spectrum* of $x_1(t)$ in Equation 5.28 are $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{7}{6}$ (the "direct current" (dc) frequency isn't considered). The ratios of the successive frequencies are $3 : 4$ and $4 : 7$, respectively. Because both of these numbers are rational, all the three frequencies in the spectrum are harmonically related, and the signal $x_1(t)$ is periodic. The GCF is $\frac{1}{6}$, as:

$$\frac{1}{2} = 3 \times \frac{1}{6} \tag{5.31}$$

$$\frac{2}{3} = 4 \times \frac{1}{6} \tag{5.32}$$

$$\frac{7}{6} = 7 \times \frac{1}{6} \tag{5.33}$$

   Therefore, the fundamental is $\omega_0 = \frac{1}{6}$ and the three frequencies present in the spectrum are the third, fourth, and seventh harmonics. Observe that the fundamental frequency component is absent in this Fourier Series.

2. the signal $x_2(t)$ in Equation 5.29 is not periodic because the ratio of the two frequencies in the spectrum is $\frac{2}{\pi}$ which is not a rational number.

3. the signal $x_3(t)$ is periodic because the ratio of the frequencies $2\sqrt{2}$ and $6\sqrt{2}$ is $\frac{1}{3}$ which is a rational number. The GCF of $2\sqrt{2}$ and $6\sqrt{2}$ is $2\sqrt{2}$. Therefore, the fundamental frequency $\omega_0 = 2\sqrt{2}$, and the period is: Determining the period

$$T_0 = \frac{2\pi}{(2\sqrt{2})} = \frac{\sqrt{2}}{2}\pi \tag{5.34}$$

Summary Slide 12 Determining the Fundamental Frequency

Frequency and Period Revisited

Determine the fundamental period of the following signals:

$$x_1(t) = 2 + 7\cos\left(\frac{1}{2}t\right) + 3\sin\left(\frac{2}{3}t\right) + 5\cos\left(\frac{7}{6}t\right)$$
$$x_2(t) = 2\cos(2t) + 5\sin(\pi t)$$

Where appropriate, write down the Trigonometric Fourier Series.

Example 5.3 (Multi-Choice: Fundamental Frequency). What is the fundamental frequency for the signal:

$$x(t) = 15\sin(30\pi t) - 8\cos(50\pi t) + 10\cos(70\pi t) \tag{5.35}$$

$\bowtie$

1. 5 Hz
2. 5 rad/s
3. 7.5 Hz
4. 7.5 rad/s
5. 10 Hz
6. 10 rad/s
7. 15 Hz
8. 15 rad/s

Example 5.4. Consider the signal:

$$x(t) = \cos\left(\frac{2}{3}t + \frac{\pi}{6}\right) + \sin\left(\frac{4}{5}t + \frac{\pi}{4}\right) \tag{5.36}$$

1. Determine if $x(t)$ is periodic.
2. If it is periodic, find the fundamental frequency and the period.
3. What harmonics are present in $x(t)$?
4. If $x(t)$ is a periodic signal, write down the Trigonometric Fourier series without any detailed calculations.

SOLUTION. Periodic with $\omega_0 = \frac{2}{15}$ and period $T_0 = 15\pi$. The fifth and sixth harmonics. Using the trigonometric identities that:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B \tag{5.37}$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B \tag{5.38}$$

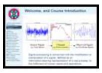
and noting that $\cos{}^{\pi}/_{6} = {}^{\sqrt{3}}/_{2}$, and so forth, the signal can be written as:

$$x(t) = \frac{\sqrt{3}}{2}\cos(5\omega_0 t) - \frac{1}{2}\sin(5\omega_0 t) + \frac{\sqrt{2}}{2}\sin(6\omega_0 t) + \frac{\sqrt{3}}{2}\cos(6\omega_0 t) \tag{5.39}$$

$\square$

This can be recognised to already be in a trigonometic Fourier series with $A_5 = \frac{\sqrt{3}}{2}$, $B_5 = -\frac{1}{2}$, $A_6 = \frac{\sqrt{2}}{2}$, $B_6 = \frac{\sqrt{2}}{2}$, and all other coefficients equal to zero.

– End-of-Topic 22: **How to calculate fundamental frequencies and Fourier Series from Observation** –



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5.3 Complex Phasors as a Fourier Basis Set

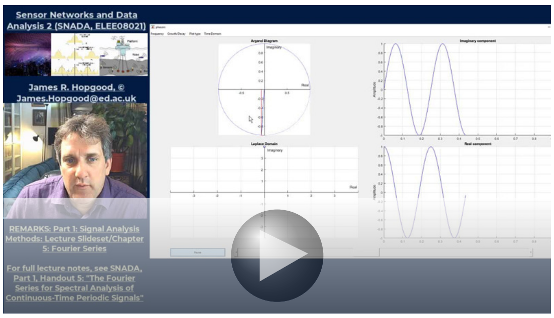
Topic Summary 23 Complex Phasors and their Application to Fourier Analysis

Topic Objectives:

- Motivating the Development of the Complex Fourier Series.
- Overview or revision of Complex Phasors.
- MALTAB demonstration of generating Complex Phasors.
- Euler's forumula for writing sinusoids and co-sinusoids as a linear combination of phasors and anti-phasors.

Topic Activities:

Type	Details	Duration	Progress
Watch video	09 : 29 min video	3× length	
Read Handout	Read page 143 to page 145	8 mins/page	
Try Code	MATLAB demo on complex phasors	10 mins	



http://media.ed.ac.uk/media/1_li6kl6cp

Video Summary: This Topic motivates the reasons for developing the complex Fourier series, with the aim of working towards the generalisation of spectral analysis methods with Fourier transforms. The Topic then introduces complex phasors as a generalisation of sinusoids and co-sinusoids, which can be used to represent signals as a complete basis set.

The trigonometric Fourier series in Equation 5.1 uses sinusoids and co-sinusoids to construct the periodic signal $x(t)$. However, the integrals in Equation 5.9 and Equation 5.10 for calculating the Fourier coefficients can often be somewhat tiresome and overly complicated because of the use of the sine and cosine terms. The trigonometric Fourier series can be algebraically simplified using the notion of complex phasors. This section will develop the complex Fourier series, which in turn will help develop the more general and flexible **Fourier transform**.

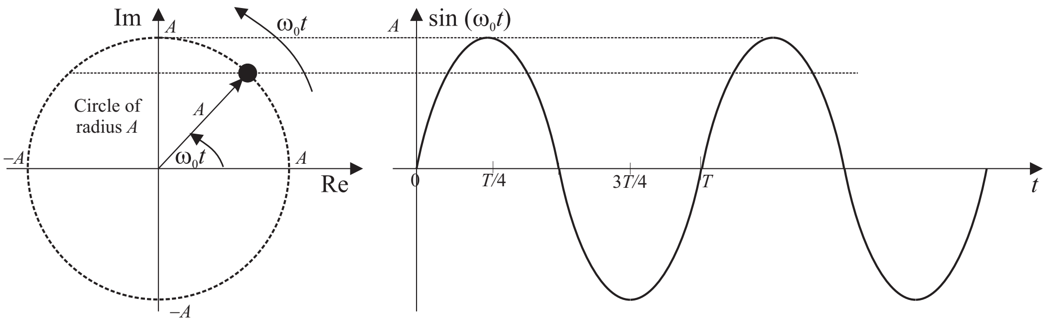
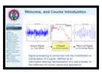


Figure 5.7: A sine wave as the projection of a complex phasor onto the imaginary axis.

To develop the complex Fourier series, it is important to introduce the concept of a phasor. A phasor is a **complex signal**, that has both a *real* and *imaginary* component, and is able to relay information about **amplitude**, **frequency**, and **phase**. A phasor is used in a variety of electrical engineering applications, including electromagnetics. Phasors can provide a more compact representation of signals, but they also simplify the analysis of many problems by explicitly using the wonderful properties of complex numbers.

While phasors are complex signals (in the mathematical sense), summing phasors can yield mathematically real signals, which are what we observe in everyday life. Or at least, what we think we observe – remember that amplitude only signals are often the result of constraints on transducer and sensor design. Microphones detect amplitude changes only, whereas other sensors can detect amplitude and phase.



5.3.1 Complex Phasors

New slide

Complex phasors result from **Euler's formula**. Consider the sum of a sinusoid and co-sinusoid both of peak amplitude A and frequency ω_0 rad/s. Then **Euler's formula** states that:

$$A\,\cos\left(\omega_0 t\right) + j\,A\,\sin\left(\omega_0 t\right) = A\,\exp\left(j\omega_0 t\right) \tag{5.40}$$

As a result, a sinusoid or cosine wave can be obtained by taking the real or imaginary component, respectively, of the complex phasor, such that

$$A\,\cos\left(\omega_0\,t\right) = \Re\left\{A\,\exp\left(j\omega_0 t\right)\right\} \tag{5.41}$$

$$A\,\sin\left(\omega_0\,t\right) = \Im\left\{A\,\exp\left(j\omega_0 t\right)\right\} \tag{5.42}$$

This is shown graphically for a sine wave in Figure 5.7, where the complex phasor is projected onto the imaginary axis. [1]

[1] Note the usual notation that $\Re\{\}$ denotes the real part of a complex number and $\Im\{\}$ denotes the imaginary component. Moreover, note that $\exp(\theta)$ and e^{θ} are used interchangeably; the former expanded version is often used when trying to make the text as clear as possible, while the latter more compact version is typically used when there isn't physically enough space available to print the longer version.

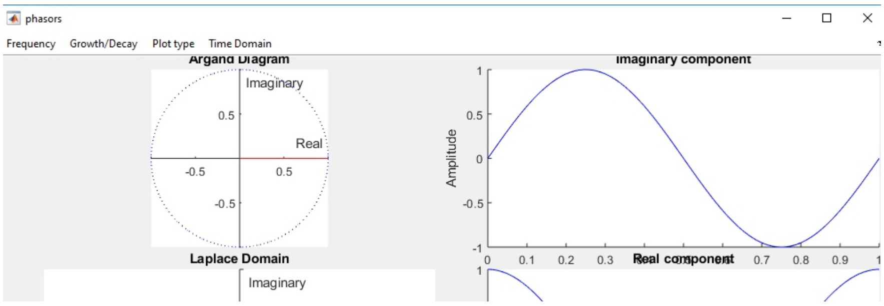


Figure 5.8: A screenshot of the MATLAB demo: *phasors*.

Using Equation 5.40, and the conjugate relationship:

$$A\,\cos\left(\omega_0 t\right) - j\,A\,\sin\left(\omega_0 t\right) = A\,\exp\left(-j\omega_0 t\right) \tag{5.43}$$

then by adding and subtracting Equation 5.40 and Equation 5.43, the following relationships are obtained:

$$\cos\left(\omega_0 t\right) = \frac{\exp\left(j\omega_0 t\right) + \exp\left(-j\omega_0 t\right)}{2} \tag{5.44}$$

$$\sin\left(\omega_0 t\right) = \frac{\exp\left(j\omega_0 t\right) - \exp\left(-j\omega_0 t\right)}{2j} \tag{5.45}$$

KEYPOINT! (MATLAB Demo). Available on LEARN is the MATLAB demo `phasors` which will help visualise the phasors. A screenshot is shown in Figure 5.8.

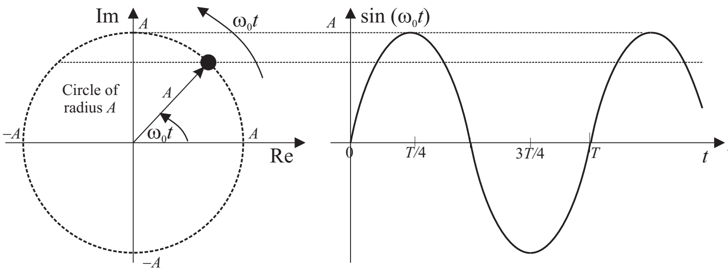
– End-of-Topic 23: **Overview of Complex Phasors** –

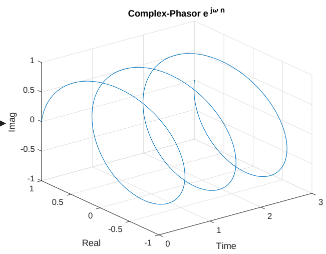
Summary Slide 13 Developing the Complex Fourier Series

Complex Phasors

A complex phasor of fixed amplitude A and frequency ω_0 rad/s, can be split into real and imaginary parts :

$$A\,\exp\left(j\omega_0 t\right) = A\,\cos\left(\omega_0 t\right) + j\,A\,\sin\left(\omega_0 t\right) \tag{5.46}$$





Note $\omega_0 = {}^{2\pi}/_{T}$ and these key points for the phasor $P = A\,e^{j\omega_0 t}$:

- $@t = 0,$
- $@\omega_0 t = \frac{\pi}{2} \Rightarrow t = \frac{T}{4},$
- $@\omega_0 t = \pi \Rightarrow t = \frac{T}{2},$
- $@\omega_0 t = \frac{3\pi}{2} \Rightarrow t = \frac{3T}{4},$

Cosine and sine waves are real, $\Re\{\}$, and imaginary, $\Im\{\}$, parts of P:

$$A\,\cos\left(\omega_0\,t\right) = \Re\left\{A\,\exp\left(j\omega_0 t\right)\right\} \tag{5.47}$$

$$A\,\sin\left(\omega_0\,t\right) = \Im\left\{A\,\exp\left(j\omega_0 t\right)\right\} \tag{5.48}$$

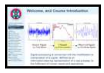
Sums of Complex Phasors

Equivalently,

$$\cos\left(\omega_0 t\right) = \frac{\exp\left(j\omega_0 t\right) + \exp\left(-j\omega_0 t\right)}{2} \tag{5.49}$$

$$\sin\left(\omega_0 t\right) = \frac{\exp\left(j\omega_0 t\right) - \exp\left(-j\omega_0 t\right)}{2j} \tag{5.50}$$

5.3.2 Developing the Complex Fourier Series



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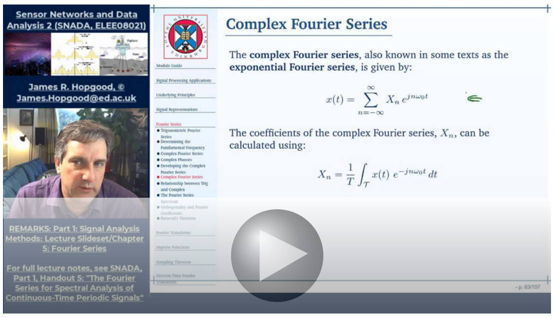
Topic Summary 24 Developing the Complex Fourier Series

Topic Objectives:

- Detailed development of the Complex Fourier series from the Trigonmetric Fourier Series.
- Conceptual insights into the use of Complex Phasors.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 53 min video	3× length	
Read Handout	Read page 147 to page 148	8 mins/page	



http://media.ed.ac.uk/media/1_zb1s6kw8

Video Summary: This Topic provides a derivation of the Complex Fourier series, given the Trigonometric Fourier series and the concept of complex phasors. This further topic is there for students interested in the derivation, and is not examinable.

Consider, then, substituting the identities in Equation 5.44 and Equation 5.44 into the **trigonometric Fourier series** in Equation 5.1:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \left\{ \frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right\} + B_n \left\{ \frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j} \right\} \right] \tag{5.51}$$

Notice that there are terms in $e^{jn\omega_0 t}$ and $e^{-jn\omega_0 t}$, which for different values of t have different functional values. These terms can be grouped together to give: [2]

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[\left(\frac{A_n - j\,B_n}{2} \right) e^{jn\omega_0 t} + \left(\frac{A_n + j\,B_n}{2} \right) e^{-jn\omega_0 t} \right] \tag{5.52}$$

[2] Note that the right hand term in Equation 5.51 contains the term $1/j = -j$; this is emplicitly used in the manipulation when going from Equation 5.51 to Equation 5.52.

While this is one summation, it can be written as two separate summations as follows:

$$x(t) = \frac{A_0}{2} + \left[\sum_{n=1}^{\infty} \left(\frac{A_n - j\,B_n}{2}\right) e^{jn\omega_0 t}\right] + \left[\sum_{n=1}^{\infty} \left(\frac{A_n + j\,B_n}{2}\right) e^{-jn\omega_0 t}\right] \tag{5.53}$$

Note that while both summations have the same variable of summation, n, the summations can be considered separate. In the second summation, consider substituting the variable $m = -n$. This means that rather than the summation being over $n = 1$ to $n = \infty$, it is now over $m = -\infty$ to $m = -1$. Hence, the second summation becomes:

$$\left[\sum_{n=1}^{\infty} \left(\frac{A_n + j\,B_n}{2}\right) e^{-jn\omega_0 t}\right] \equiv \left[\sum_{m=-\infty}^{-1} \left(\frac{A_{-m} + j\,B_{-m}}{2}\right) e^{jm\omega_0 t}\right] \tag{5.54}$$

Now, for notational convenience, define:

$$X_n = \begin{cases} \frac{A_n - j\,B_n}{2} & \text{for } n > 0 \\ \frac{A_0}{2} & \text{for } n = 0 \\ \frac{A_{-n} + j\,B_{-n}}{2} & \text{for } n < 0 \end{cases} \tag{5.55}$$

Then substituting Equation 5.54 and Equation 5.55 into Equation 5.53 gives:

$$x(t) = X_0\, e^{j0\omega_0 t} + \left[\sum_{n=1}^{\infty} X_n\, e^{jn\omega_0 t}\right] + \left[\sum_{m=-\infty}^{-1} X_m e^{jm\omega_0 t}\right] \tag{5.56}$$

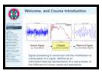
where it is noted $e^{j0\omega_0 t} = 1$. Since both summations involve the term $e^{jm\omega_0 t}$, and the summation indices don't overlap, they can be combined to give one summation over the full range of integers.

This gives the complex Fourier series, which means that any periodic signal $x(t)$ can be written as

$$x(t) = \sum_{n=-\infty}^{\infty} X_n\, e^{jn\omega_0 t} \tag{5.57}$$

– End-of-Topic 24: **Developing the Complex Fourier Series** –

5.3.3 Complex Fourier Series



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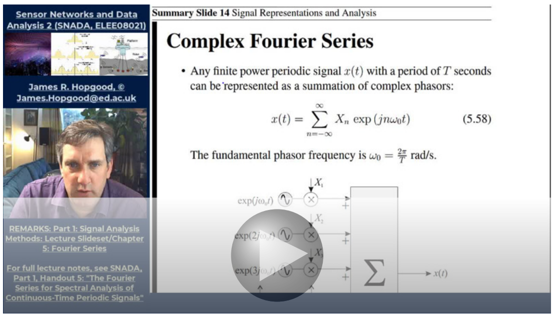
Topic Summary 25 Calculating Complex Fourier Coefficients

Topic Objectives:

- Review of the Complex Fourier Series Expressions and its physical intepretation.
- Example of calculating Complex Fourier Series Coefficients with single functional form over one period.
- Example of calculating Complex Fourier Series Coefficients with piece-wise functional form over one period.

Topic Activities:

Type	**Details**	**Duration**	**Progress**
Watch video	18 : 55 min video	3× length	
Read Handout	Read page 149 to page 152	8 mins/page	
Try Example	Try Example 5.5	10 mins	
Practice Exercises	Exercise 5.3	20 mins	



http://media.ed.ac.uk/media/1_kzdfo6ul

Video Summary: This Topic presents the Complex Fourier Series expressions, and provides a physical interpretation o what it means; namely, that a continuous-time period signal can be represented by an infinite summation of complex phasors, whose frequencies are integer multiples of a fundamental. The Topic briefly highlights the concept of a negative frequency, and how pairs of phasors with positive and negative frequencies combine together to give a real signal. The Topic then considers two examples of calculating the Complex Fourier coefficients. The first example has a single functional form over one period, whereas the second example requires the region of integration to be split up.

The **complex Fourier series**, also known in some texts as the **exponential Fourier**

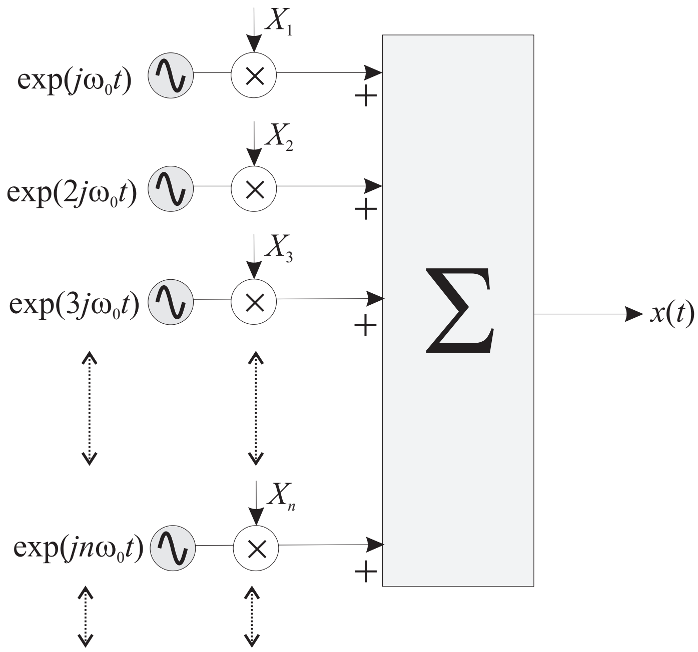
Summary Slide 14 Signal Representations and Analysis

Complex Fourier Series

- Any finite power periodic signal $x(t)$ with a period of T seconds can be represented as a summation of complex phasors:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \exp\left(jn\omega_0 t\right) \tag{5.58}$$

The fundamental phasor frequency is $\omega_0 = \frac{2\pi}{T}$ rad/s.



- The complex Fourier coefficients, X_n, can be calculated using:

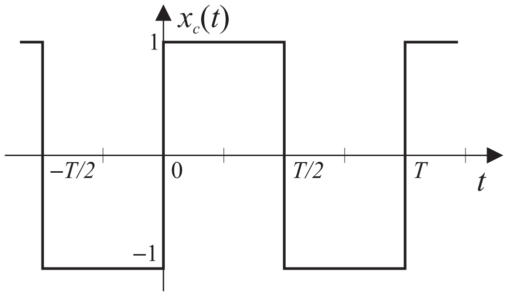
$$X_n = \frac{1}{T} \underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \exp\left(-jn\omega_0 t\right)\, dt}_{} \tag{5.59}$$

-

Summary Slide 15 Signal Representations

Complex Fourier Series Example

Find the **complex Fourier coefficients** of this **square wave**:



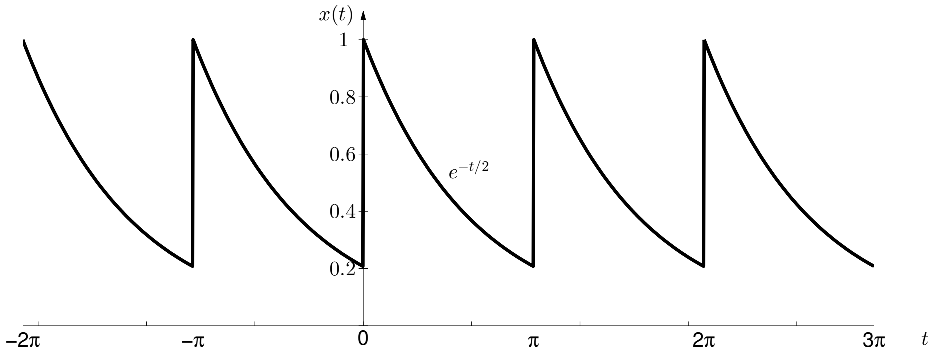


Figure 5.9: A periodic signal consisting of partial exponential decays.

series, is given by:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n\, e^{jn\omega_0 t} \tag{5.60}$$

The coefficients of the complex Fourier series, X_n, can be calculated using:

$$X_n = \frac{1}{T}\int_{\mathcal{T}} x(t)\; e^{-jn\omega_0 t}\, dt \tag{5.61}$$

where the notation $\int_{\mathcal{T}}$ is used to denote the integral over any period of the waveform.

Example 5.5 (Complex Fourier Series). Find the complex Fourier coefficients for the signal in Figure 5.9.

SOLUTION. The period of the waveform is $T = \pi$, and therefore $\omega_0 = \frac{2\pi}{T} = 2$. Hence, chosing the period over which to integrate as being $(0,\, \pi)$, the complex Fourier coefficients are given by:

$$X_n = \frac{1}{T}\int_{\mathcal{T}} x(t)\; e^{-jn\omega_0 t}\, dt \tag{5.62}$$

$$X_n = \frac{1}{\pi}\int_0^{\pi} e^{-\frac{t}{2}}\, e^{-j2nt}\, dt = \frac{1}{\pi}\int_0^{\pi} e^{-\left(\frac{1}{2}+j2n\right)t}\, dt \tag{5.63}$$

$$= \frac{1}{\pi}\left[\frac{e^{-\left(\frac{1}{2}+j2n\right)t}}{-\left(\frac{1}{2}+j2n\right)}\right]_0^{\pi} = \frac{1}{\pi}\left\{\frac{e^{-\left(\frac{1}{2}+j2n\right)\pi}-1}{-\left(\frac{1}{2}+j2n\right)}\right\} \tag{5.64}$$

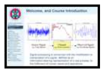
Noting that $e^{-\left(\frac{1}{2}+j2n\right)\pi} = e^{-\frac{\pi}{2}}\, e^{-j2\pi\, n} = e^{-\frac{\pi}{2}}$, this gives the final answer:

$$X_n = \frac{2}{\pi}\left\{\frac{1-e^{-\frac{\pi}{2}}}{1+j4n}\right\} \approx \frac{0.504}{1+j4n} \tag{5.65}$$

□

– End-of-Topic 25: **Examples of the Complex Fourier Series** –

5.4 Relationship between Trig and Complex Forms



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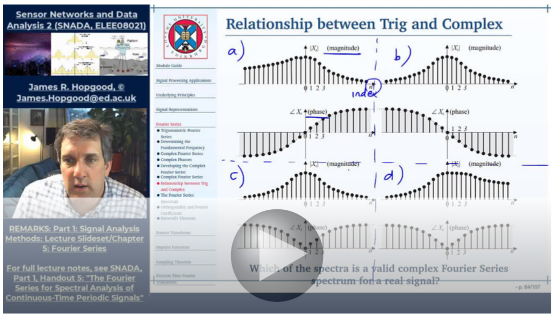
Topic Summary 26 Relationship between Trig and Complex Fourier Coefficients

Topic Objectives:

- Considers detailed relationship between Fourier coefficients.
- Example of demonstrating this relationship with a square wave.
- Conjugate symmetry property of complex Fourier coefficients.
- Plotting magnitude and phase of complex Fourier coefficients vs coefficient index.

Topic Activities:

Type	Details	Duration	Progress
Watch video	22 : 32 min video	3× length	
Read Handout	Read page 154 to page 158	8 mins/page	
Try Example	Try 5.6 and 5.7	10 mins	
Practice Exercises	Exercise 5.4	30 mins	



`http://media.ed.ac.uk/media/1_slq67us5`

Video Summary: This Topic investigates in detail the relationships between the trigonometric Fourier coefficients, and the complex Fourier coefficients. The relationships are demonstrated through the example of comparing the Fourier coefficients for the square wave periodic function. The Topic then moves onto the more general discussion of the properties of the complex Fourier coefficients, primarily conjugate symmetry resulting in symmetric amplitude and asymmetric phase for real valued signals. The Topic then moves onto the idea of plotting the complex Fourier coefficients as a function of the coefficient index, which is the basis for plotting the spectrum of the signal, as shown in future Topics. Since the Fourier coefficients are, in general, complex, then it becomes necessary to plot magnitude and phase on separate diagrams. Putting together this theory, the Topic then poses an example to consider, followed by a physical interpretation of the complex Fourier series.

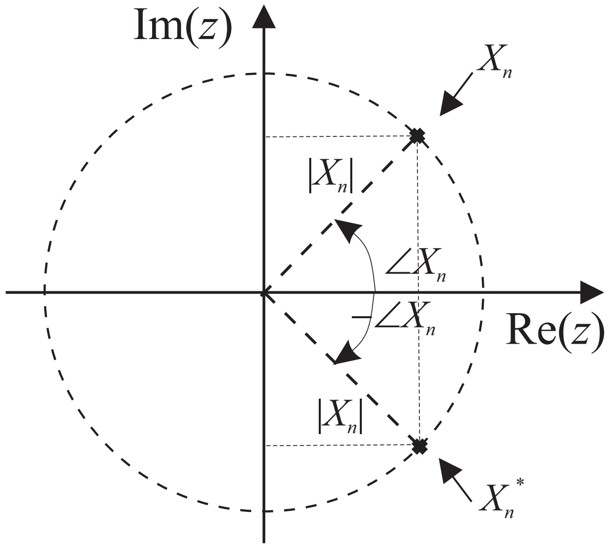


Figure 5.10: Conjugate symmetry properties of the complex Fourier coefficients.

Equation 5.55 provides a simple relationship between the complex and trigonometric Fourier series coefficients:

$$X_n = \begin{cases} \frac{A_n - j\,B_n}{2} & \text{for } n > 0 \\ \frac{A_0}{2} & \text{for } n = 0 \\ \frac{A_{-n} + j\,B_{-n}}{2} & \text{for } n < 0 \end{cases} \tag{5.68}$$

The relationship in Equation 5.68 can be further simplified by defining $A_n = A_{-n}$, $B_n = -B_{-n}$, and $B_0 = 0$. Hence, Equation 5.68 can alternatively be written as:

$$X_n = \frac{A_n - j\,B_n}{2} \quad \text{for all } n \tag{5.69}$$

As a result of the relationship in Equation 5.69, it is clear that the complex Fourier coefficients are complex conjugate symmetric, such that

$$X_{-n} = X_n^* \tag{5.70}$$

From this, the following important properties follow:

1. The amplitude is symmetric about the origin (even symmetry):

$$|X_{-n}| = |X_n| \tag{5.71}$$

2. The phase is asymmetric about the origin (odd symmetry):

$$\angle X_{-n} = -\angle X_n \tag{5.72}$$

This relationship is shown graphically in Figure 5.10.

Summary Slide 16 Complex Fourier Series

Relationship between Coefficients

- There is a simple relationship between the complex and trigonometric Fourier series coefficients :

$$X_n = \begin{cases} \frac{A_n - j\,B_n}{2} & \text{for } n > 0 \\ \frac{A_0}{2} & \text{for } n = 0 \\ \frac{A_{-n} + j\,B_{-n}}{2} & \text{for } n < 0 \end{cases}$$

- By defining $A_n = A_{-n}$ (even), $B_n = -B_{-n}$ (odd), and $B_0 = 0$, this can alternatively be written as:

$$X_n = \frac{A_n - j\,B_n}{2} \quad \text{for all } n$$

- Using this relationship, **conjugate (Hermitian) symmetry** follows, :

$$X_{-n} = X_n^*$$

  This implies

  :

  - The amplitude is symmetric about the origin (even symmetry):
  $$|X_{-n}| = |X_n|$$
  - The phase is asymmetric about the origin (odd symmetry):
  $$\angle X_{-n} = -\angle X_n$$

Summary Slide 17 Physical interpretation of the complex Fourier coefficients

Physical Interpretation

KEYPOINT! (Fourier Decomposition). By definition, the Fourier series is used to represent a periodic signal as a sum of cosine waves. The frequencies of these cosine waves are 0, ω_0, $2\omega_0$, $3\omega_0$, and so on. The magnitude of the complex Fourier coefficient $|X_n|$ is half the amplitude of the nth harmonic. The angle of the complex Fourier coefficient $\angle X_n$ is the phase shift associated with the nth harmonic. Thus, the nth harmonic might be written as:

$$x_n(t) = 2|X_n| \cos\left(n\omega_0 t + \angle X_n\right) \qquad \square$$

To show this, consider the nth harmonic in the trigonometric Fourier series:

$$x_n(t) = A_n \cos\left(n\omega_0 t\right) + B_n \sin\left(n\omega_0 t\right) \tag{5.66}$$

Recall the trigonometric identity:

$$r\ \cos\left(\alpha \pm \beta\right) = r\ \cos\alpha \cos\beta \mp r\ \sin\alpha \sin\beta$$

so that by setting :

$$r\ \cos\left(n\omega_0\, t + \beta\right) = \tag{5.67}$$

Comparing Equation 5.67 and Equation 5.66 gives:

Using the classic identity $\sin^2\beta + \cos^2\beta = 1$, this gives:

using the relationship of the trigonometric and complex Fourier coefficients from the previous slide, namely $X_n = \frac{A_n - jB_n}{2}$. Similarly,

which proves the desired expression.

Example 5.6 (Coefficient Relationships). Consider again the square wave shown in Figure 5.1. Earlier it was shown in Example 5.1 that the Trigonometric Fourier series coefficients were given by $A_n = 0$ for all n, while the B_n's were given by Unknown :

$$B_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi\, n} & n \text{ odd} \end{cases} \tag{5.73}$$

Calculate the Complex Fourier series coefficients.

SOLUTION. Using:

$$X_n = \frac{A_n - j\, B_n}{2} \quad \text{for all } n \tag{5.74}$$

using $A_n = 0$, this simplifies to:

$$X_n = \frac{B_n}{2j} = \begin{cases} 0 & n \text{ even} \\ \frac{2}{j\pi\, n} & n \text{ odd} \end{cases} \tag{5.75}$$

□

which is the calculation that was found earlier.

Example 5.7 (Multi-choice: Fourier Coefficient Relationships). Which of the following spectra shown in Figure 5.11 is a valid complex Fourier Series spectrum for a **real** signal?

– End-of-Topic 26: **Relationships between Trigonometric and Complex Fourier Coefficients –**

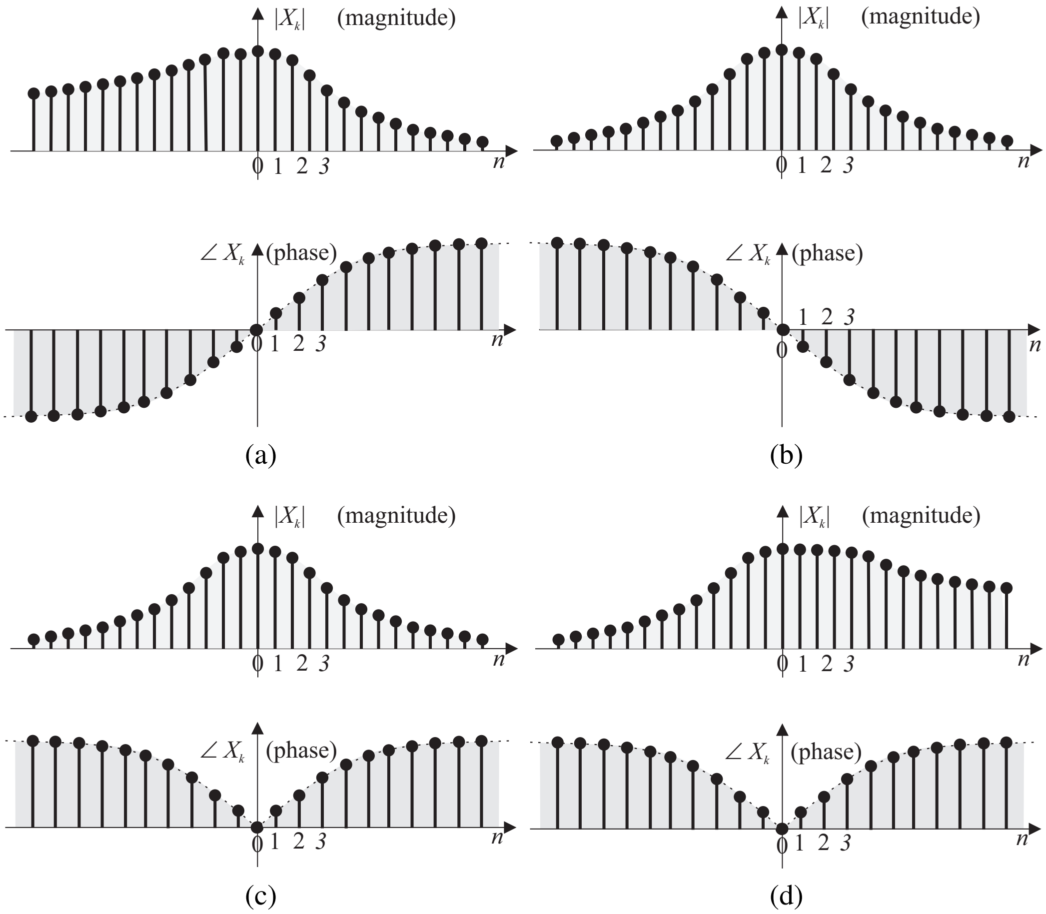
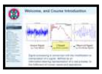


Figure 5.11: Which of the spectra is a valid complex Fourier Series spectrum for a **real** signal?

5.5 The Fourier Series Spectrum



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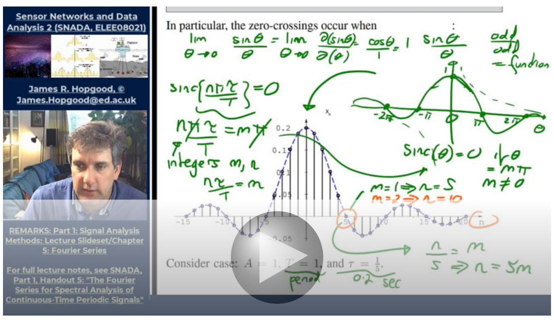
Topic Summary 27 Example of Plotting the Spectrum of a Periodic Signal

Topic Objectives:

- Calculates the Complex Fourier Coefficients of a Periodic Pulse sequence.
- Introduce the sinc function, and how to plot this function.
- Plotting the Complex Fourier Coefficients of the Periodic Pulse sequence.
- Discussion of plotting the magnitude and phase complex coefficients.

Topic Activities:

Type	**Details**	**Duration**	**Progress**
Watch video	27 : 29 min video	$3\times$ length	
Read Handout	Read page 160 to page 166	8 mins/page	
Practice Exercises	Exercise 5.5	30 mins	



http://media.ed.ac.uk/media/1_0qbxlgfw

Video Summary: This Topic considers how to plot the Spectrum of a periodic signal, by plotting the complex Fourier coefficient against frequency of the corresponding complex phasor. This Topic considers a very common example, the spectrum of a periodic pulse sequence. First, the complex Fourier coefficients are calculated, and it is shown that these can be written as a so-called sinc function. The video shows how to plot the sinc function, including what happens at the origin, and when the function intersects the independent variable axis. This enables the complex Fourier coefficients of the periodic pulse signal to be plotted against frequency. The video finishes by showing how to plot magnitude and phase of these coefficients separately, while meeting the conjugate symmetry constraints of the complex Fourier coefficient.

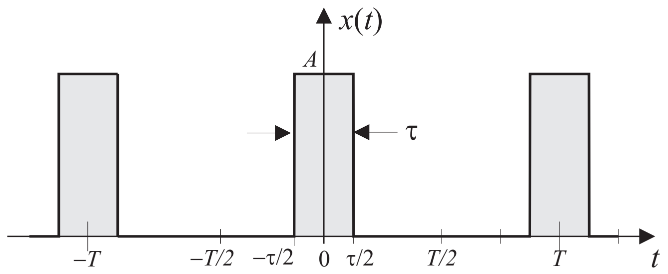


Figure 5.12: Pulse wave with period T and pulse width τ.

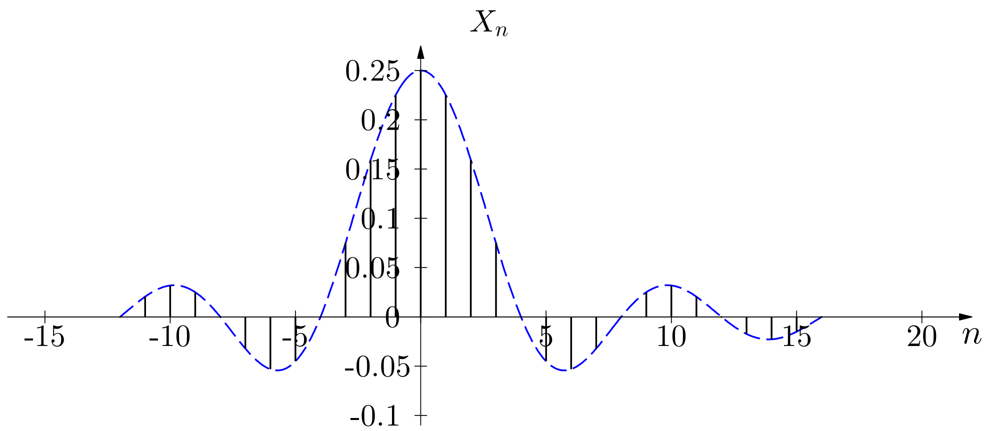


Figure 5.13: Fourier coefficients of the rectangular pulse wave shown in Figure 5.12, when $A = 1$ and $T = 4\tau$.

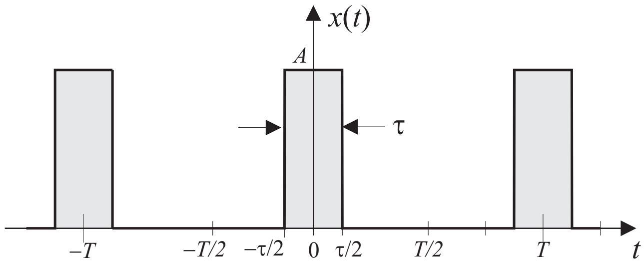
The nth coefficient of the complex Fourier series, X_n, is associated with a complex-phasor of frequency $\omega = n\omega_0$ where ω_0 is the fundamental frequency, $\omega_0 = \frac{2\pi}{T}$, with T being the fundamental period.

It is of interest to consider plotting the complex Fourier coefficients, X_n, against the index of the Fourier coefficients, n. This will help develop a visualisation of the contribution of each Fourier coefficient as a function of frequency.

Summary Slide 18 Complex Fourier Series of Pulse Wave

Another CFS Calculation Example

Find the CFS coefficients of a pulse wave with period T and width τ.



$$X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\exp\left(-jn\omega_0 t\right)\, dt \tag{5.76}$$

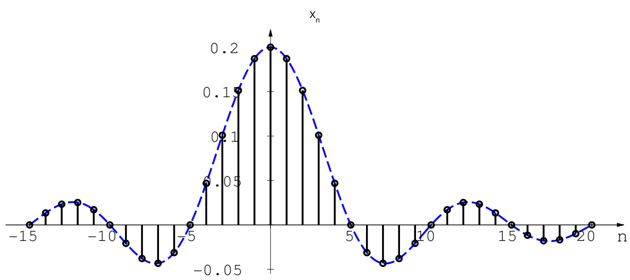
$$= \frac{1}{T}\left[\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\, e^{-jn\omega_0 t}\, dt + \qquad\qquad\qquad\qquad\qquad\qquad\right]$$

$$= \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right] \tag{5.77}$$

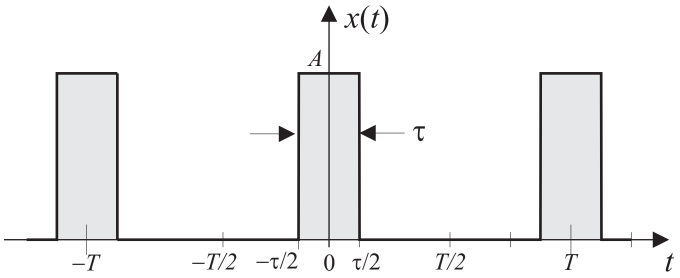
$$X_n = \tag{5.78}$$

$$X_n \triangleq \tag{5.79}$$

The nth complex Fourier coefficient corresponds to the component with frequency $\omega_n = n\omega_0 = n\frac{2\pi}{T}$. What do these coefficients look like when plotted against coefficient index, n?



Summary Slide 19 Complex Fourier Series of Pulse Wave (Continued)

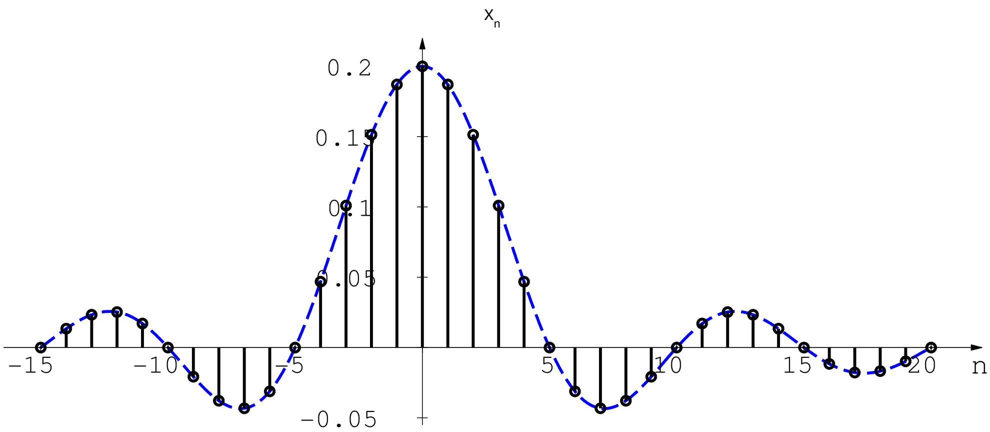


The complex Fourier series for a pulse wave with period T and pulse width τ are given by:

$$X_n = \frac{A\tau}{T}\,\mathrm{sinc}\left(\frac{\pi n \tau}{T}\right) \tag{5.80}$$

KEYPOINT! (Sinc Function). This is the classic **sinc** function, and it is important to know how to plot this – need to draw it very often.

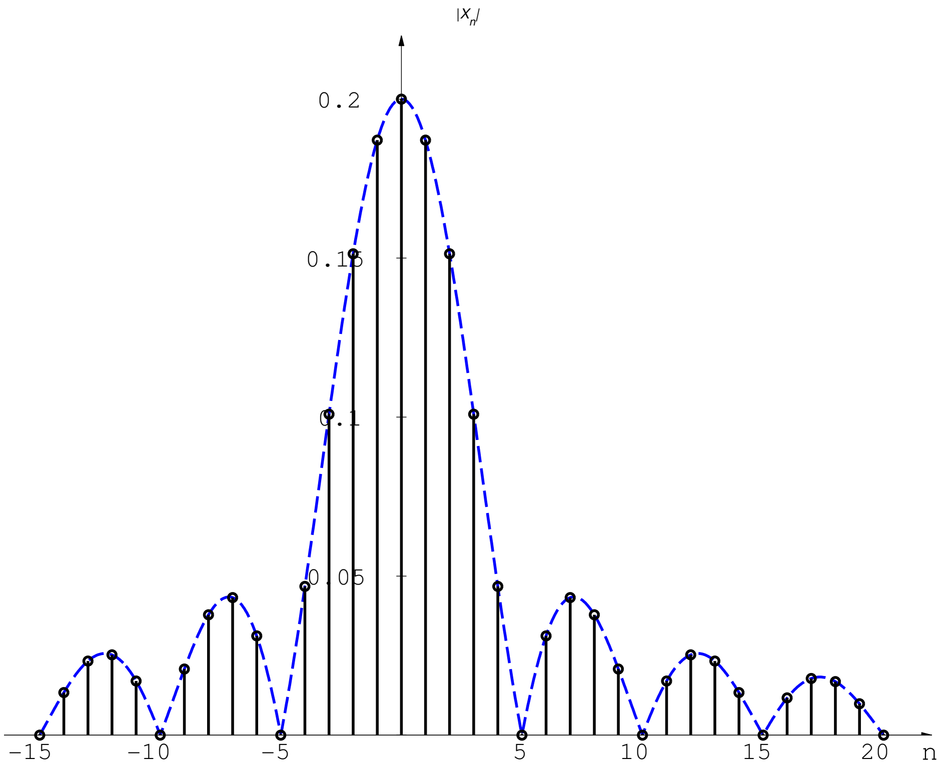
In particular, the zero-crossings occur when :

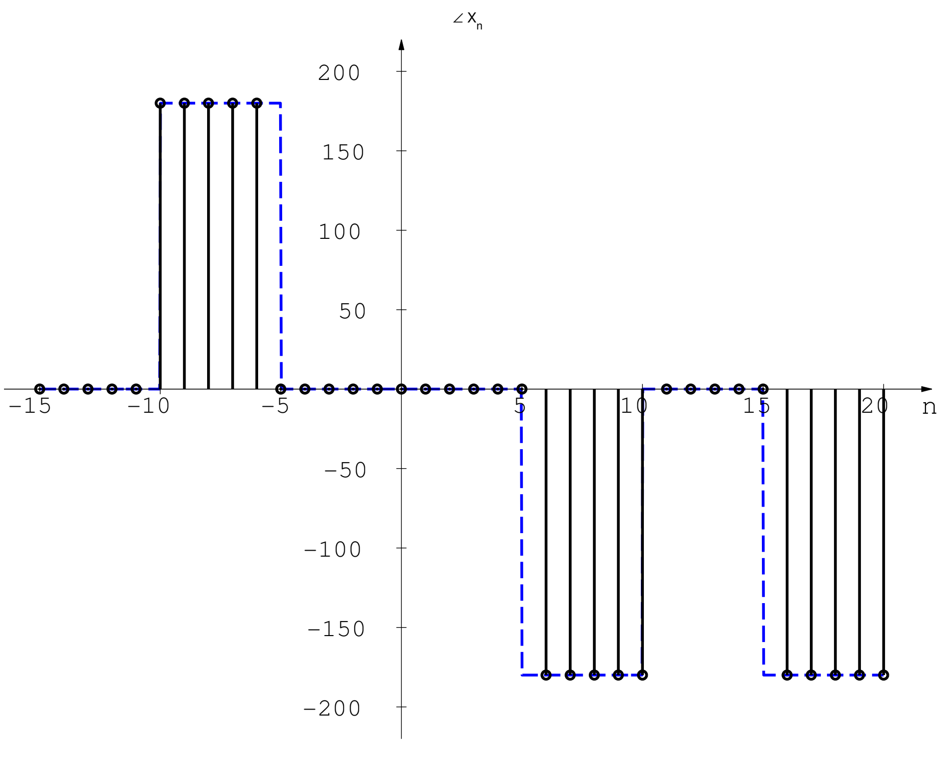


Consider case: $A = 1$, $T = 1$, and $\tau = \frac{1}{5}$.

Summary Slide 20 Complex Fourier Series of Pulse Wave (Continued)

Plotting Spectrum of Coefficients





Example 5.8 (Complex Fourier Series of Pulse Wave). Calculate the complex Fourier coefficients of a pulse wave with period T and pulse width τ, as shown in Figure 5.12.

SOLUTION.

$$X_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\exp\left(-jn\omega_0 t\right)\, dt \tag{5.82}$$

$$= \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\exp\left(-jn\omega_0 t\right)\, dt \tag{5.83}$$

$$X_n = \frac{A}{-jn\omega_0 T}\left[\exp\left(-jn\omega_0\frac{\tau}{2}\right) - \exp\left(jn\omega_0\frac{\tau}{2}\right)\right] \tag{5.84}$$

$$= \frac{A\tau}{T}\frac{\sin\left(n\omega_0\tau/2\right)}{n\omega_0\tau/2} \tag{5.85}$$

$$X_n = \frac{A\tau}{T}\operatorname{sinc}\left(\frac{n\omega_0\tau}{2}\right) \tag{5.86}$$

Note that when $n = 0$, $\operatorname{sinc}(0) = 1$, and $X_0 = \frac{A\tau}{T}$. Also, notice that when $\frac{n\omega_0\tau}{2}$ is a multiple of π, then $\operatorname{sinc}(m\pi) = 0$ for any integer $m \neq 0$.
Therefore, the sinc function crosses the zero-axis when:

$$\frac{n\omega_0\tau}{2} = m\pi \quad \Rightarrow \quad \frac{2\pi n\tau}{2T} = m\pi \quad \Rightarrow \quad n = m\frac{T}{\tau},\ m\in\mathbb{Z} \tag{5.87}$$

□

where $m \in \mathbb{Z}$ means that m is any integer. For the case shown in Figure 5.13, where $T = 4\tau$, when the zero crossings occur when $n = 5m$, or equivalently when $n = \ldots, -12, -8, -4, 0, 4, 8, 12, \ldots$, as shown in the figure.

For the rectangular pulse train shown in Figure 5.12, the coefficients are given by Equation 5.86 and are real-valued. They can therefore be plotted against index n for all values of n, and the sinc function gives the shape shown in Figure 5.13. This figure shows the case when $A = 1$ and $T = 4\tau$ for any value of τ. In the figure, the dotted continuous line represents the evaluation of Equation 5.86 as though n were continuous. However, n only takes on integer values, and therefore the vertical lines indicate the actual Fourier coefficient value. Note that the coefficients are zero at multiples of $\frac{T}{\tau} = 4$.

Of even more interest is that the complex Fourier coefficients can be plotted as a function of the frequencies in Hertz with $f = nf_0$, or in radians per second with $\omega = n\omega_0 = 2\pi n f_0$. In order to plot against an actual frequency, then the period must actually be defined. For the case in Figure 5.13 where $T = 4\tau$, suppose that in fact $\tau = 0.5$ such that $T = 2$. Thus, the fundamental frequency $f_0 = 0.5$ Hz or $\omega_0 = \pi$ rad/s. Hence, the frequency of the nth harmonic is given by $0.5n$ Hz or πn rad/s. For the case of Hertz, this produces the figure shown in Figure 5.14; note again the zero crossings at multiples of $f = 4f_0 = \frac{4}{2} = 2$ Hz. When the coefficients are

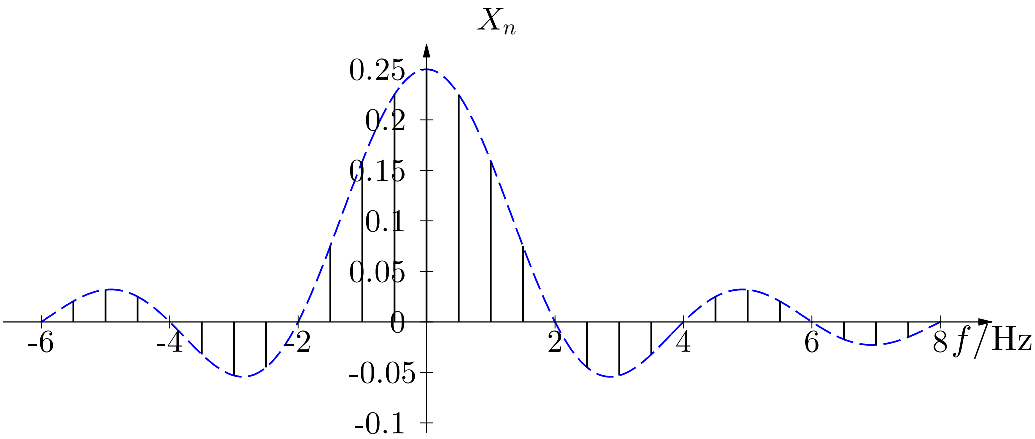


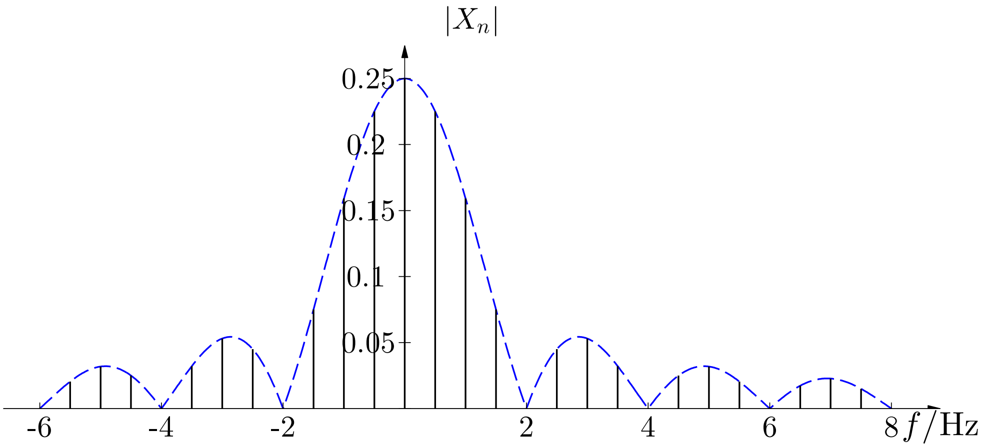
Figure 5.14: Fourier coefficients as function of frequency, f Hz, of the rectangular pulse wave shown in Figure 5.12, when $T = 2$ and $T = 4\tau$.

plotted as a function of frequency in the manner shown in Figure 5.14, this is referred to as the **complex Fourier Spectrum**.

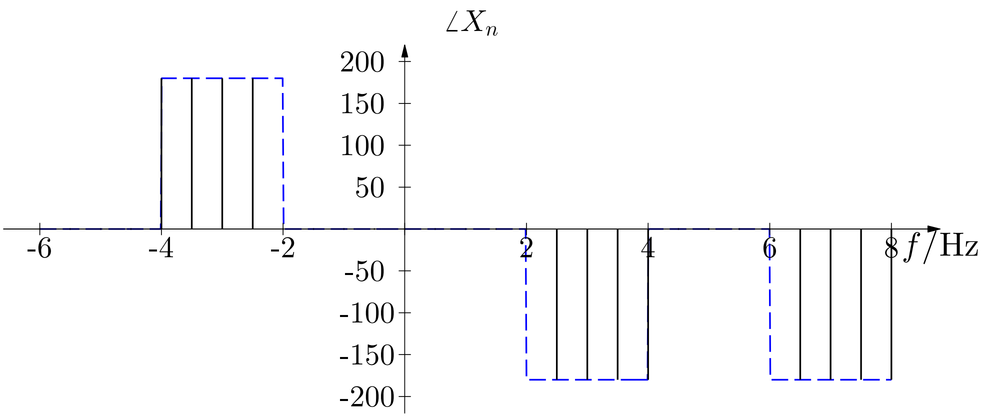
While the example in Example 5.8 yielded real coefficients, this is the exception rather than the rule, and the example in Example 5.5 showed that the coefficients can easily be complex. It is not possible to plot complex Fourier coefficients on a single plot, and therefore it is necessary to plot the magnitude and phase separately, as shown in Figure 5.15 for the waveform in Figure 5.12.

Note that the magnitude $|X_n|$ is always positive and symmetric with frequency, as indicated by Equation 5.71. The phase is asymmetric as given by Equation 5.72. Moreover, note that for real positive numbers, the phase is zero, while for real negative numbers, the phase is *either* π or $-\pi$. The convention used in these notes is that for positive frequencies the phase of a real negative number is $-\pi$ or $-180\,\mathrm{deg}$. Using the asymmetry property in Equation 5.71, the phase of negative real coefficients for **negative frequencies** is therefore π or $180\,\mathrm{deg}$.

– End-of-Topic 27: **Plotting a Fourier Series Spectrum** –



(a) Magnitude spectrum.



(b) Phase spectrum.

Figure 5.15: Fourier spectrum of the rectangular pulse wave shown in Figure 5.12, when $T = 2$ and $T = 4\tau$.

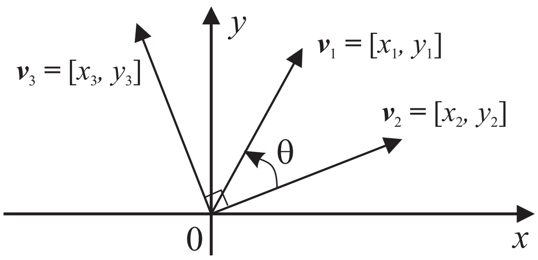
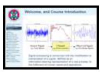


Figure 5.16: The orthogonality of vectors.



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5.6 Orthogonality and Fourier Coefficients

Topic Summary 28 Orthogonality and Fourier Coefficients

Topic Objectives:

- Understanding orthogonality and its applications.

Topic Activities:

Type	Details	Duration	Progress
Read Handout	Read page 168 to page 172	8 mins/page	
Try Example	Try Example 5.9	15 mins	
Practice Exercises	Exercise 5.6	30 mins	

A signal $x(t)$ can be decomposed into **basis functions** or *atoms* only if those basis functions cannot be further decomposed into simpler functions. In otherwords, it must *not* be possible to write any particular basis function as a linear combination of other basis functions. This requires the propery of orthogonality.

Different choices of basis functions can lead to improved *approximating* or *compression* properties. To determine if these basis functions are fundamental building blocks it is necessary to check that they are orthogonal.

The property of orthogonality for functions is an extension of the concept that is familiar in linear algebra. Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal if their **vector product** or **dot product** is equal to zero.

Consider the vectors in $\mathbf{v}_1$ and $\mathbf{v}_2$ in Figure 5.16 whose directions from the origin have an angular difference of θ. Then the vector product is given by:

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = |\mathbf{v}_1|\,|\mathbf{v}_2| \cos\theta \tag{5.88}$$

If the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal then $\theta = \frac{\pi}{2}$ and $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$. This then means that any point in the direction of $\mathbf{v}_1$ can only be reached by a scale multiple of $\mathbf{v}_1$, and not by any contribution from $\mathbf{v}_2$. In Figure 5.16, $\mathbf{v}_1$ and $\mathbf{v}_3$ are orthogonal. The vector product then simplifies in this case to:

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_1^H \mathbf{v}_2 = \sum_{k=1}^{N} x_k\, y_k^* = 0 \tag{5.89}$$

where in this case, $N = 2$, and $\mathbf{v}^H$ denotes the complex conjugate transpose of the vector $\mathbf{v}$ which is also known as **Hermitian**. Note that the reason for using the complex conjugation in the definition is to ensure that $\mathbf{u} \cdot \mathbf{u}$ is a positive number for $\mathbf{u} \neq \mathbf{0}$ or, in otherwords, that the inner product of a vector with iteself gives the total **power** in the signal. [3] This definition can be extended from a vector space to a function space by replacing the summation by an integral.

Two periodic signals $f_1(t)$ and $f_2(t)$ are said to be **orthogonal** if their product integrated over one period (T) is zero:

$$\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t)\ f_2^*(t)\ dt = 0 \tag{5.90}$$

This can be generalised to a set of functions $\{f_n(t)\}$ by writing the integral:

$$I_{nm} = \frac{1}{T} \int_{\mathcal{T}} f_n(t)\ f_m^*(t)\ dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{5.91}$$

where again $\int_{\mathcal{T}}$ stands for integration over any contiguous interval of T seconds. The principle of orthogonality is an important property fundamental to making Fourier decompositions work.

The **basis functions** used in the trigonometric Fourier series are orthogonal, since as shown in Sidebar 8:

$$\frac{2}{T} \int_{\mathcal{T}} \cos(n\omega_0 t)\ \cos(m\omega_0 t)\ dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \neq 0 \end{cases} \tag{5.98a}$$

$$\frac{2}{T} \int_{\mathcal{T}} \sin(n\omega_0 t)\ \sin(m\omega_0 t)\ dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \neq 0 \end{cases} \tag{5.98b}$$

$$\frac{2}{T} \int_{\mathcal{T}} \sin(n\omega_0 t)\ \cos(m\omega_0 t)\ dt = 0 \qquad \text{for all } n \text{ and } m \tag{5.98c}$$

This property is what makes the calculation of Fourier coefficients possible as shown in Sidebar 9, and often crops up in calculations such as those used in the example in Parseval's theorem (see Equation 5.129 on Summary Slide 22 on page 178).

KEYPOINT! (Use of Orthogonality). It is important to appreciate that orthogonality allows us to calculate any individual Fourier coefficient, A_n or B_n without having to evaluate any other coefficient first. This extends to other decompositions, such as the Fourier-Legendre transform.

[3] There are further reasons for the complex conjugate sign to ensure that the axioms for the definition of an inner product are satisfied. However, you don't really need to know this at this stage. As a further parallel, note that when dealing with complex voltage and complex current, power is defined as $P = V\,I^*$, again to ensure that power is a real positive value.

Sidebar 8 Orthogonality of Trigonometric Fourier Basis Functions

To show that the trigonometric Fourier basis functions $\cos(\omega_0 t), \sin(\omega_0 t)$, $\cos(2\omega_0 t), \sin(2\omega_0 t), \ldots, \cos(n\omega_0 t), \sin(n\omega_0 t), \ldots$ are mutually orthogonal, then taking the functions in turn:

$$I = \frac{1}{T}\int_{\mathcal{T}} \cos(n\omega_0 t)\,\cos(m\omega_0 t)\;dt \tag{5.92}$$

$$= \frac{1}{2T}\left[\int_{\mathcal{T}} \cos(\{n+m\}\omega_0 t)\;dt + \int_{\mathcal{T}} \cos(\{n-m\}\omega_0 t)\;dt\right] \tag{5.93}$$

Since $\cos(\omega_0 t)$ executes one complete cycle during any interval of duration T, then $\cos(\{n+m\}\omega_0 t)$ executes $n+m$ cycles during any interval of duration T. Therefore, the first integral in Equation 5.93 is equal to zero, as the area under complete cycles of sines and cosines is zero. The same is true for the second integral in Equation 5.93, except when $n = m$, so that the function becomes a constant. Hence, the integral in Equation 5.92 is zero for all $n \neq m$, and when $n = m$, the integral is equal to:

$$I = \frac{1}{2T}\int_{\mathcal{T}} dt = \frac{1}{2} \tag{5.94}$$

Hence,

$$I = \frac{1}{T}\int_{\mathcal{T}} \cos(n\omega_0 t)\,\cos(m\omega_0 t)\;dt = \begin{cases} \frac{1}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \neq 0 \end{cases} \tag{5.95}$$

Using similar arguments (and trigonometric identities), it can also be shown that:

$$I = \frac{1}{T}\int_{\mathcal{T}} \sin(n\omega_0 t)\,\sin(m\omega_0 t)\;dt = \begin{cases} \frac{1}{2} & \text{if } m = n \\ 0 & \text{if } m \neq n \neq 0 \end{cases} \tag{5.96}$$

and

$$I = \frac{1}{T}\int_{\mathcal{T}} \sin(n\omega_0 t)\,\cos(m\omega_0 t)\;dt = 0 \qquad \text{for all } n \text{ and } m \tag{5.97}$$

These properties can be used in deriving the identities for the trigonometric Fourier series.

Sidebar 9 Calculating Trigonometric Fourier Coefficients, Revisited

The formula for the Fourier series coefficients in Equation 5.7 and Equation 5.8 on page 128 can now be derived using orthogonality and Equation 5.98. The Fourier series is given by Equation 5.1 on page 127:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left[A_n \cos\left(n\omega_0 t\right) + B_n \sin\left(n\omega_0 t\right)\right] \tag{5.1}$$

To find the coefficient A_m, consider multiplying through by the term $\cos\left(m\omega_0 t\right)$ and integrating over one period: [a]

$$\begin{aligned}\int_{\mathcal{T}} x(t)\, \cos\left(m\omega_0 t\right)\, dt = \frac{A_0}{2} \int_{\mathcal{T}} &\cos\left(m\omega_0 t\right)\, dt \\ &+ \sum_{n=1}^{\infty} \Big[A_n \int_{\mathcal{T}} \cos\left(n\omega_0 t\right)\, \cos\left(m\omega_0 t\right)\, dt \\ &\quad + B_n \int_{\mathcal{T}} \sin\left(n\omega_0 t\right)\, \cos\left(m\omega_0 t\right)\, dt\Big]\end{aligned} \tag{5.99}$$

Note then that the first integral on the right hand side (RHS) is always equal to zero, except when $m = 0$, because it is an area under m cycles of a sinusoid which cancel out. Moreover, using orthogonality, it follows that the third integral is always equal to zero (using Equation 5.98c). This leaves the middle term which is equal to $\frac{T}{2}$ when $m = n \neq 0$ and zero otherwise, using Equation 5.98a. Therefore, noting the summation gives an infinite number of terms on the RHS, only one term is non-zero when $m = n \neq 0$, and this gives:

$$\int_{\mathcal{T}} x(t)\, \cos\left(m\omega_0 t\right)\, dt = A_m \frac{T}{2} \tag{5.100}$$

which is indeed the expression in Equation 5.7 on page 128. It can be seen that this is true for all $m \geq 0$. The expression in Equation 5.8 can be derived in a similar manner.

[a] In this formula, it has been noted that the order of integration and summation can be interchanged since the limits of the summation and integral do not depend on one another.

Similarly, it is straightforward to show that the basis functions $\exp(j\omega_0 t)$, $\exp(2j\omega_0 t), \ldots, \exp(jn\omega_0 t), \ldots$ are mutually orthogonal: [4]

$$\frac{1}{T}\int_{\mathcal{T}} \exp(jn\omega_0 t)\, \exp^*(jm\omega_0 t)\, dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \tag{5.102}$$

Thus, if the complex Fourier series is given by Equation 5.60 on page 152:

$$x(t) = \sum_{n=-\infty}^{\infty} X_n\, e^{jn\omega_0 t} \tag{5.60}$$

then multiplying through by $e^{-jm\omega_0 t}$ and integrating over one period gives:

$$\int_{\mathcal{T}} x(t)\, e^{-jm\omega_0 t}\, dt = \sum_{n=-\infty}^{\infty} X_n \int_{\mathcal{T}} e^{jn\omega_0 t}\, e^{-jm\omega_0 t}\, dt = X_m\, T \tag{5.103}$$

which is the expression for calculating the complex Fourier coefficients given in Equation 5.61.

The Trigonometric Fourier basis functions $\{\sin(n\omega_0 t), \cos(\omega_0 t)\}_{n=0}^{\infty}$ and the complex Fourier basis functions $\{e^{jn\omega_0 t}\}_{n=0}^{\infty}$ are not the only basis functions that satisfy the orthogonality principle, and can therefore be used as a way of decomposing signals into simpler building blocks. There are plenty of other basis functions, from Walsh functions, to Wavelets. This will be considered later in this course. In the meantime, Fourier is the primary basis function of interest because they are solutions to ordinary differential equations (ODEs).

Example 5.9 (Legendre Polynomials). Legendre polynomials are an alternative to complex exponentials, and form an orthogonal basis set. The first few *unnormalised* Legendre polynomials are shown graphically in Figure 5.17 and for the range $-1 \leq t \leq 1$ are given by:

$$P_0(t) = 1,\ P_1(t) = t,\ P_2(t) = \frac{1}{2}\left(3t^2 - 1\right),\ P_3(t) = \frac{1}{2}\left(5t^3 - 3t\right), \ldots \tag{5.110}$$

$\bowtie$

– End-of-Topic 28: **Understanding Orthogonality and Why it Matters!** –

Summary Slide 21 Theoretical Aspects of Fourier Decomposition

Orthogonality

Two periodic signals $f_1(t)$ and $f_2(t)$ are said to be orthogonal if their product integrated over one period (T) is zero:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t)\ f_2^*(t)\ dt = 0$$

Fourier Basis

The basis functions $\exp\left(j\omega_0 t\right)$, $\exp\left(2j\omega_0 t\right), \ldots, \exp\left(jn\omega_0 t\right), \ldots$ are mutually orthogonal for $(m,\, n) \neq 0$:

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \exp\left(jm\omega_0 t\right)\ \exp^*\left(jn\omega_0 t\right)\ dt = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

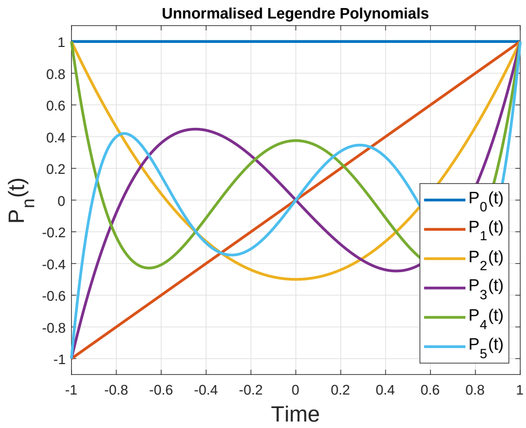
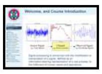


Figure 5.17: First few *unnormalised* Legendre polynomials.



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5.7 Parseval's Theorem

Topic Summary 29 Parseval's Theorem for Periodic Signals

Topic Objectives:

- Understanding Parseval's Theorm and its applications.

Topic Activities:

Type	Details	Duration	Progress
Read Handout	Read page 174 to page 177	8 mins/page	
Try Example	Try Examples 5.10 and 5.11	15 mins	

For periodic signals, the **average power** can be used as a measure for the size of the signal in the time-domain. A natural question is to ask whether the Fourier coefficients can be used to determine the power in a signal.

The answer is *yes* using Parseval's Theorem, which states that if a signal $x(t)$ with fundamental period T has complex Fourier coefficients X_n, then as shown in Sidebar 10:

$$P = \frac{1}{T}\int_{\mathcal{T}} |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |X_n|^2 \tag{5.125}$$

Using the relationship between the trigonometric and complex Fourier series coefficients in Equation 5.69, then as shown in Sidebar 12, Parseval's theorem can

[4]The orthogonality in Equation 5.102 can be readily shown. When $m = n$, the exponentials cancel, the integrand is equal to T, and therefore the left hand side (LHS) is equal to one. When $m \neq n$, the LHS of Equation 5.102 can be written as:

$$\int_{\mathcal{T}} \exp\left(j(n-m)\omega_0 t\right)\, dt = \int_{\mathcal{T}} \cos\left((n-m)\omega_0 t\right)\, dt + j\int_{\mathcal{T}} \sin\left((n-m)\omega_0 t\right)\, dt \tag{5.101}$$

Both integrals on the RHS represent the area under $n-m$ cycles of either $\sin$ or $\cos$. Since $n-m$ is an integer, both areas are zero, and hence Equation 5.102 follows.

Sidebar 10 Proof of Parseval's Theorem

Parseval's theorem for periodic signals can be proved by using the formula for average power of a periodic signal, as given by the expression on page 121 along with the complex Fourier series decomposition. Thus:

$$P = \frac{1}{T}\int_{\mathcal{T}} |x(t)|^2 \, dt = \frac{1}{T}\int_{\mathcal{T}} x(t)\ x^*(t)\ dt \tag{5.111}$$

$$= \frac{1}{T}\int_{\mathcal{T}} x(t) \left(\sum_{n=-\infty}^{\infty} X_n \, e^{jn\omega_0 t}\right)^* dt \tag{5.112}$$

where $x^*(t)$ has been **replaced** by the complex Fourier series expansion. There are two important properties of complex numbers that will be applied here:

1. the conjugate of the sum of complex numbers is the sum of the conjugates of complex numbers; thus, for two complex numbers a and b:

$$(a+b)^* = a^* + b^* \tag{5.113}$$

2. similarly, the conjugate of the product of complex numbers is the product of the conjugates, so that:

$$(a \times b)^* = a^* \times b^* \tag{5.114}$$

Hence, applying these to Equation 5.112 gives:

$$P = \frac{1}{T}\int_{\mathcal{T}} x(t) \left(\sum_{n=-\infty}^{\infty} X_n^* \, e^{-jn\omega_0 t}\right) dt \tag{5.115}$$

Since the limits of the summation and the integral are independent of one another, the order of integration and summation can be interchanged (or **rearranged**) to give:

$$P = \frac{1}{T}\sum_{n=-\infty}^{\infty} \int_{\mathcal{T}} x(t)\ X_n^* \, e^{-jn\omega_0 t} \, dt \tag{5.116}$$

$$= \sum_{n=-\infty}^{\infty} X_n^* \underbrace{\frac{1}{T}\left(\int_{\mathcal{T}} x(t)\ e^{-jn\omega_0 t} \, dt\right)}_{X_n} \tag{5.117}$$

$$= \sum_{n=-\infty}^{\infty} X_n^* \, X_n = \sum_{n=-\infty}^{\infty} |X_n|^2 \tag{5.118}$$

which is equivalent to the energy in the Fourier coefficients. Note that in the final stages, in Equation 5.117, the identity for the Fourier series coefficients has been **recognised**.

In this proof, three key stages have been identified: **replace**, **rearrange**, and **recognised**. This process is described further in Sidebar 11.

Sidebar 11 The 3 R's: A general approach to some proofs

In this course, there are a number of proofs that can be obtained by using the following three key stages:

Replace	Generally identify *one term* that can be replaced by an identity.
Rearrange	Look for a general rearrangement, such as changing orders of summations and integrations.
Recognise	Generally recognise *a term* that can be simplified substantially.

Sidebar 12 Equivalent Form of Parseval's Theorem

To show Equation 5.126, consider Equation 5.69 which states that

$$X_n = \frac{A_n - j\,B_n}{2} \quad \text{for all } n \tag{5.119}$$

where $A_{-n} = A_n$, $B_{-n} = -B_n$, and $B_0 = 0$. This also means that $X_{-n} = X_n^*$ and hence $|X_{-n}|^2 = |X_n|^2$. Hence, the RHS of Equation 5.126 gives:

$$P = \sum_{n=-\infty}^{\infty} |X_n|^2 = |X_0|^2 + \sum_{n=-\infty}^{-1} |X_n|^2 + \sum_{n=1}^{\infty} |X_n|^2 \tag{5.120}$$

Setting $m = -n$ in the second summation on the RHS only gives:

$$\begin{aligned}
P &= |X_0|^2 + \sum_{m=1}^{\infty} |X_{-m}|^2 + \sum_{n=1}^{\infty} |X_n|^2 && (5.121)\\
&= |X_0|^2 + 2\sum_{n=1}^{\infty} |X_n|^2 && (5.122)\\
&= \frac{A_0^2}{4} + 2\sum_{n=1}^{\infty} \left|\frac{A_n - j\,B_n}{2}\right|^2 && (5.123)\\
&= \frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} \left(A_n^2 + B_n^2\right) && (5.124)
\end{aligned}$$

be written in terms of the trigonometric Fourier series as:

$$P = \frac{A_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left(A_n^2 + B_n^2 \right) \tag{5.126}$$

This theorem can be used for calculating the power of a signal when its Fourier coefficients are given rather than the time-domain signal, and in some cases can make the calculation of power substantially more easy.

Example 5.10 (Calculating Identities). Recall the square-wave shown in Figure 5.1 on page 128. It was shown that the Fourier coefficients were given by the expression $X_n = {}^2/_{jn\pi}$ for n odd, and zero otherwise.

Using Parseval's theorem, the power of the signals in the time and frequency domains can be expressed as:

$$\frac{1}{T} \int_T |x(t)|^2 \; dt = \sum_{n=-\infty}^{\infty} |X_n|^2 \tag{5.131}$$

$$\frac{1}{T} \left[\int_{-\frac{T}{2}}^{0} (-1)^2 \, dt + \int_{0}^{\frac{T}{2}} (1)^2 \, dt \right] = 1 = 2 \sum_{m=1}^{\infty} \frac{4}{(2m-1)^2 \pi^2} \tag{5.132}$$

$$\frac{\pi^2}{8} = \sum_{m=1}^{\infty} \frac{1}{(2m-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \tag{5.133}$$

$\bowtie$

Fourier series analysis often leads to a number of surprising formulae such as this.

Example 5.11 (Another example). Find the Fourier Series of $x(t) = t^2$ for $0 < t \leq 2$, and otherwise $x(t) = x(t - 4k)$ for integer k.

Hence, using Parseval's theorem, show that:

$$\sum_{r=1}^{\infty} \frac{1}{r^4} = \frac{\pi^4}{90} \tag{5.134}$$

$\bowtie$

– End-of-Topic 29: **An Introduction to Parseval's Theorem and Some of its Applications!** –

Summary Slide 22 Developing Parseval's Theorem

Power of Periodic Signal

Using a simple example, consider the periodic signal:

$$x(t) = B_1 \sin(\omega_0 t) + B_2 \sin(2\omega_0 t) \tag{5.127}$$

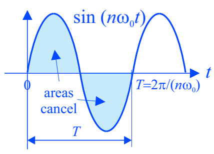
The fundamental frequency is $\omega_0 = \frac{2\pi}{T}$ where T is the fundamental period. Therefore, the power in the waveform is given by:

$$P = \frac{1}{T}\int_0^T |x(t)|^2 \; dt = \frac{1}{T}\int_0^T x(t) \; x^*(t) \; dt \tag{5.128}$$

$$\begin{aligned} P = & \frac{1}{T}\int_0^T |B_1|^2 \sin^2(\omega_0 t) \; dt \\ & + \underbrace{\frac{2}{T}\int_0^T B_1 \, B_2^* \sin(\omega_0 t) \sin(2\omega_0 t) \; dt} \\ & + \frac{1}{T}\int_0^T |B_2|^2 \sin^2(2\omega_0 t) \; dt \end{aligned} \tag{5.129}$$

By orthogonality, the middle term equals zero, and therefore

$$P = \frac{1}{T}\int_0^T |B_1|^2 \sin^2(\omega_0 t) \; dt + \frac{1}{T}\int_0^T |B_2|^2 \sin^2(2\omega_0 t) \; dt$$



$$P = \underbrace{\frac{|B_1|^2}{2} + \frac{|B_2|^2}{2}} \tag{5.130}$$

This indicates that the power of the waveform is related to the sum of the amplitudes of the individual sine and cosine waves.

Summary Slide 23 Developing Parseval's Theorem

Power of Periodic Signal (Continued)

Using the relationship between trig and complex coefficients:

$$X_n = \frac{A_n - j\,B_n}{2} \quad \text{for all } n$$

then for the problem considered on the previous slide

$$P = \frac{|B_1|^2}{2} + \frac{|B_2|^2}{2} = |X_1|^2 + |X_2|^2$$

Note the need for using the modulus sign rather than square signs when calculating power.

Parseval's Theorem

The power in a periodic signal $x(t)$ with fundamental period T and complex Fourier coefficients X_n is:

$$\begin{aligned} P &= \frac{1}{T}\int_T |x(t)|^2 \; dt \\ &= \sum_{n=-\infty}^{\infty} |X_n|^2 \\ &= \frac{A_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right) \end{aligned}$$

5.8 Tutorial Exercises

Exercise 5.1 (Periodic and Aperiodic Signals). [*Difficulty:* **2** ($\star\star$)] Determine whether or not each of the following continuous-time signals is periodic. For the signals which are periodic, specify its fundamental period in seconds and fundamental frequency in Hertz:

1. $$x_1(t) = 3\cos\left(5t + \pi/6\right) \tag{5.135}$$

2. $$x_2(t) = \cos\left(\frac{t}{6}\right)\cos\left(\pi\frac{t}{6}\right) \tag{5.136}$$

3. $$x_3(t) = \cos\left(\frac{\pi}{2}t\right) - \sin\left(\frac{\pi}{8}t\right) + 3\cos\left(\frac{\pi}{4}t + \frac{\pi}{3}\right) \tag{5.137}$$

KEYPOINT! (Periodic Discrete-time Signals). In contrast, a discrete-time sinusoid is periodic only if its frequency f is a rational number. More formally, a discrete-time signal $x[n]$ is periodic with period N where $N > 0$ if and only if:

$$x[n+N] = x[n]\,, \quad \text{for all } n \tag{5.138}$$

□

The smallest N for which Equation 5.138 is true is called the **fundamental period**. Prove this result by considering the sinusoidal signal $x[n] = \cos\left(2\pi f_0 n + \theta\right)$, and consequently determine whether or not each of the following discrete-time signals is periodic.

For the signals which are periodic, specify its fundamental period and fundamental frequency:

1. $$x_4[n] = 3\cos\left(5n + \pi/6\right) \tag{5.139}$$

2. $$x_5[n] = 2\exp\left[j\left(\frac{n}{6} - \pi\right)\right] \tag{5.140}$$

3. $$x_6[n] = \cos\left(\frac{\pi}{2}n\right) - \sin\left(\frac{\pi}{8}n\right) + 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right) \tag{5.141}$$

⋈

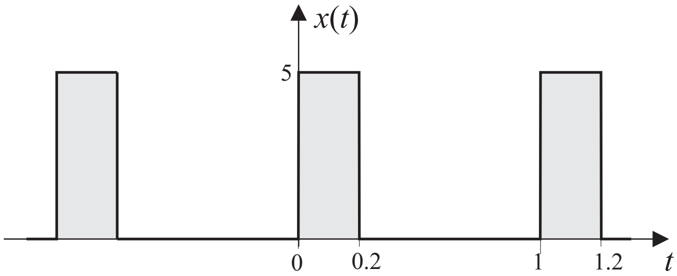


Figure 5.18: Pulse waveform.

Exercise 5.2 (Trigonometric Fourier Series from Observation). [*Difficulty:* **3** ($\star\star\star$)] Develop an expression for the trigonometric Fourier series of the following signals:

$$x_1(t) = 3\cos(5t) + 4\sin(10t) \tag{5.142}$$

$$x_2(t) = \cos(\omega_0 t) + \sin^2(2\omega_0 t) \tag{5.143}$$

Using these results, derive the complex form of the Fourier series.

Final answer:

1. The trigonometric coefficients for the first signal $x_1(t)$ are $A_1 = 3$ and $B_2 = 4$, and all the other coefficients are zero. The complex coefficients are $X_1 = X_{-1} = \frac{3}{2}$, $X_2 = -2j$, and $X_{-2} = 2j$.

2. The complex coefficients for the second signal $x_2(t)$ are $X_{-4} = X_4 = -\frac{1}{4}$, $X_1 = X_{-1} = \frac{1}{2}$, and $X_0 = \frac{1}{2}$.

Exercise 5.3 (Trigonometric and Complex Fourier Series). [*Difficulty:* **4** ($\star\star\star\star$)] Consider the periodic signal shown in Figure 5.18.

1. Develop an expression for the trigonometric Fourier series representations of the periodic signal in Figure 5.18.

2. Derive from *first principles*, using the appropriate Fourier integral, an expression for the complex Fourier coefficients.

HINTS. The following manipulation might be useful; frequently, the term $1 - e^{-j\theta}$, for some value of θ, might occur in an algebraic manipulation, and can be more usefully written as:

$$1 - e^{-j\theta} = e^{-j\frac{\theta}{2}}\left(e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}\right) = 2j\, e^{-j\frac{\theta}{2}} \sin\left(\frac{\theta}{2}\right) \tag{5.144}$$

□

Final answer:

1. The trigonometric coefficients are:

$$A_n = 2\,\mathrm{sinc}\left(\frac{2\pi n}{5}\right) = \begin{cases} 2 & \text{for } n = 0 \\ 2\frac{\sin\left(\frac{2\pi n}{5}\right)}{\frac{2\pi n}{5}} & \text{for } n > 0 \end{cases} \tag{5.145}$$

and

$$B_n = 2\frac{\left[1 - \cos\left(\frac{2\pi n}{5}\right)\right]}{\frac{2\pi n}{5}} \quad \text{for } n > 0 \tag{5.146}$$

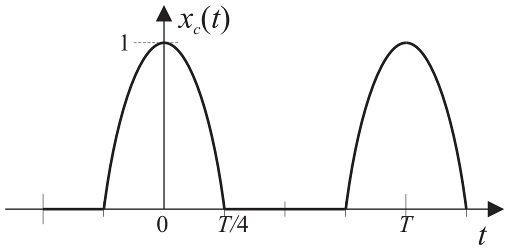


Figure 5.19: Rectified cosine wave.

2. The complex Fourier coefficients are given by:

$$X_n = e^{-j\frac{\pi n}{5}} \operatorname{sinc}\left(\frac{\pi n}{5}\right) \tag{5.147}$$

□

Exercise 5.4 (Relationship of Trigonometric and Complex Fourier Coefficients). [*Difficulty:* **4** $(\star\star\star\star)$] Continuing Exercise 5.3 and the periodic signal shown in Figure 5.18:

1. Check your answers to Exercise 5.3 by using the relationship between the trigonometric and Fourier coefficients.

2. Plot graphs of the magnitude and phase of the complex coefficients.

HINTS. The following manipulation might be useful; frequently, the term $1-e^{-j\theta}$, for some value of θ, might occur in an algebraic manipulation, and can be more usefully written as:

$$1 - e^{-j\theta} = e^{-j\frac{\theta}{2}}\left(e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}}\right) = 2j\, e^{-j\frac{\theta}{2}} \sin\left(\frac{\theta}{2}\right) \tag{5.148}$$

□

Exercise 5.5 (Complex Fourier Series). [*Difficulty:* **4** $(\star\star\star\star)$] Develop from first principles an expression for the trigonometric and complex Fourier series representations of the periodic signal shown in Figure 5.19.

Check your answers by using the relationship between the trigonometric and Fourier coefficients.

HINTS. The following trigonometric identity might be useful:

$$\cos A\, \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right] \tag{5.149}$$

□

Final answer: The trigonometric Fourier series representation can be expressed as:

$$x_c(t) = \frac{1}{\pi} + \frac{1}{2}\cos\omega_0 t + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(2n\omega_0 t)}{4n^2-1} \tag{5.150}$$

□

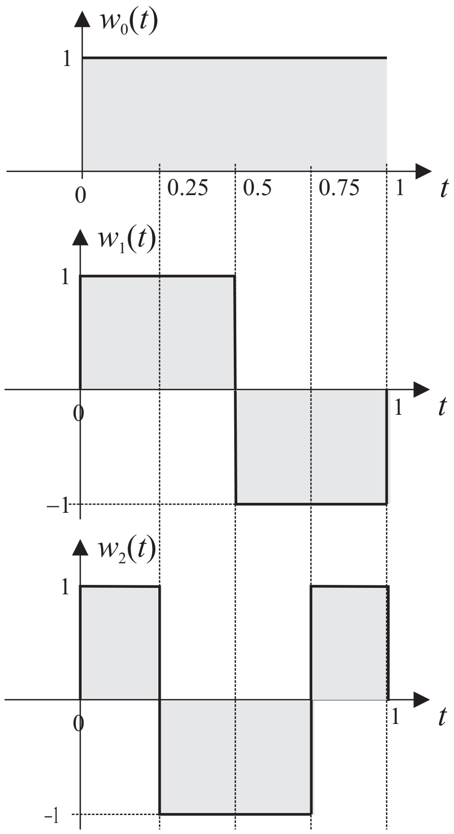


Figure 5.20: One period of the first three Walsh functions, $w_0(t)$, $w_1(t)$, and $w_2(t)$, for $0 \leq t < 1$.

Exercise 5.6 (Orthogonality). [*Difficulty:* **4** ($\star\star\star\star$)] A waveform can be represented as a linear combination of basis functions. In the case of Fourier analysis, the basis functions are the set of complex phasors $\exp{(j\omega t)}$ (or sine waves and cosine waves).

An alternative set of basis functions are the set of Walsh functions, which take on the values -1 and $+1$ only. One period of the first three Walsh functions, $w_0(t)$, $w_1(t)$, and $w_2(t)$, are shown in Figure 5.20, where the period of the basis functions is 1.

Moreover, since Walsh functions form an orthogonal set, any periodic waveform can be expressed as a weighted sum of an infinite number of them, as given by:

$$x(t) = \sum_{n=0}^{\infty} A_n \, w_n(t) \tag{5.151}$$

where the Walsh coefficients A_n can be calculated using:

$$A_n = \int_0^1 x(t) \; w_n(t) \; dt \tag{5.152}$$

1. Show that the three Walsh functions, $w_0(t)$, $w_1(t)$, and $w_2(t)$ form an **orthogonal set** of basis functions.

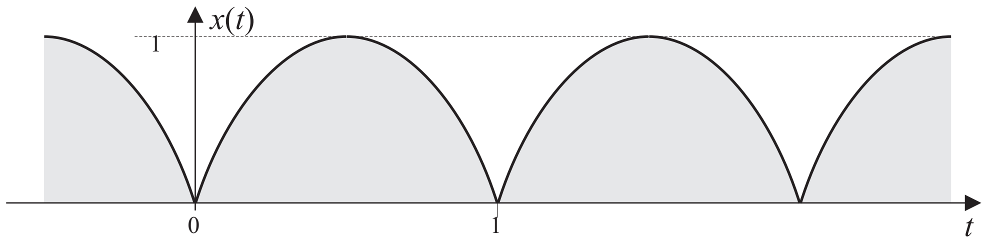


Figure 5.21: Full-wave rectified sine-wave of amplitude 1 V and period 1 seconds.

2. It is desired to approximate the full-wave rectified sine-wave shown in Figure 5.21 using the three Walsh functions, such that:

$$x(t) \approx \sum_{n=0}^{2} A_n \, w_n(t) \tag{5.153}$$

   Calculate the Walsh coefficients A_n, and sketch the waveform that results from this approximation.

3. How might the quality of an approximation of the waveform in Figure 5.21 change if the first three Fourier basis functions were used in instead?

Final answer: The Walsh coefficients are given by $A_0 = 0.637$, $A_1 = 0$, and $A_2 = -0.264$.

6

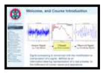
Fourier Transform for Spectral Analysis of Continuous-Time Aperiodic Signals

Great acts are done by a series of small deeds.

Lao Tzu

This handout builds on the Fourier series to develop the Fourier transform for analysing the spectral content on non-periodic signals. The handout considers the properties of the Fourier transform, and some simple applications.

6.1 Introduction to The Fourier Transform



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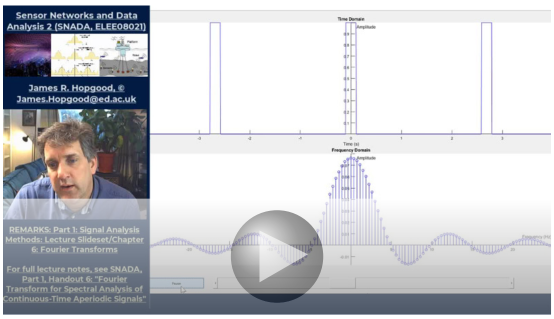
Topic Summary 30 Intuitive Derivation of the Fourier Transform from Complex Fourier Series

Topic Objectives:

- Discuss the derivation of the Fourier transform through a limiting operation.
- Consider the shape of the spectrum of the periodic pulse signal as function of period.
- Use a MATLAB simulation to see the same effect on a variety of signals.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 59 min video	3× length	
Read Handout	Read page 186 to page 193	8 mins/page	
Try Code	Use MATLAB demonstraton	20 minutes	



http://media.ed.ac.uk/media/1_sz4cs9tb

Video Summary: This Topic introduces the Fourier transform as the limiting operation of taking a periodic signal, and sending the period to infinity, so as to represent a non-periodic signal. The Topic considers this by looking at a particular example, namely the periodic pulse wave, where the spectrum has an elegant shape that is easy to analyse. The properties of this spectrum are then considered as the period is increased. It is noticed that the shape of the spectrum doesn't change, but the gaps between the frequency components are filled in with more frequencies. In the limit, the discrete-spectrum becomes continuous. This phenomena is then investigated further using some MATLAB code that can simulate this process for different periodic waveforms. The limiting spectrum is the Fourier transform of the non-periodic signal.

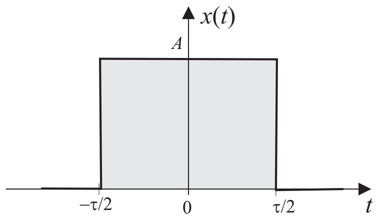


Figure 6.1: Simple Rectangular Pulse

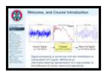
Periodic signals, which are themselves everlasting signals, can be decomposed as a sum of everlasting sinusoids or complex exponentials. The trigonometric and complex Fourier series allows us to analyse the spectral content of periodic signals, such as the one in Figure 6.2a. The natural question is to ask what is the frequency content of aperiodic signals, such as the the rectangular pulse shown in Figure 6.1, which is described by the equation:

$$x(t) = \begin{cases} 0 & t < -\tau/2 \\ A & \text{if } -\tau/2 \leq t < \tau/2 \\ 0 & t \geq \tau/2 \end{cases} \tag{6.1}$$

This section will introduce the Fourier transform that allows us to perform such a spectral analysis. The Fourier Series is, in fact, a special case of the Fourier transform, although the latter which can be used to analyse a much wider variety of signals including the non-periodic or aperiodic signals.

While the Fourier Series is indeed a special case of the so-called Fourier transform, this handout will actually develop the Fourier Transform from the Fourier series, by a limiting operation. This is an *Engineering* approximation, that is easy to explain, and does not need any advanced mathematics.

6.1.1 Conceptual Development of the Fourier Transform



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To motivate the Fourier transform, it can be derived from the Fourier series by applying a *limiting process*. This involves considering a periodic waveform with period T, and investigating how the Fourier coefficients change as $T \to \infty$. To this end, consider the periodic waveform in Figure 6.2a which, as shown in the previous handout on page 165, has Fourier coefficients given by:

$$X_n = \frac{A\tau}{T} \operatorname{sinc}\left(\frac{n\omega_0\tau}{2}\right) \tag{6.2}$$

which is the sinc function. This function is plotted in Figure 6.2b. The nth coefficient has a corresponding frequency $f = nf_0$.

For the particular waveform shown in Figure 6.2a, the pulse width $\tau = \frac{1}{20}$ and the period $T = 0.25$ seconds. This means that the frequency spacing in Figure 6.2b is

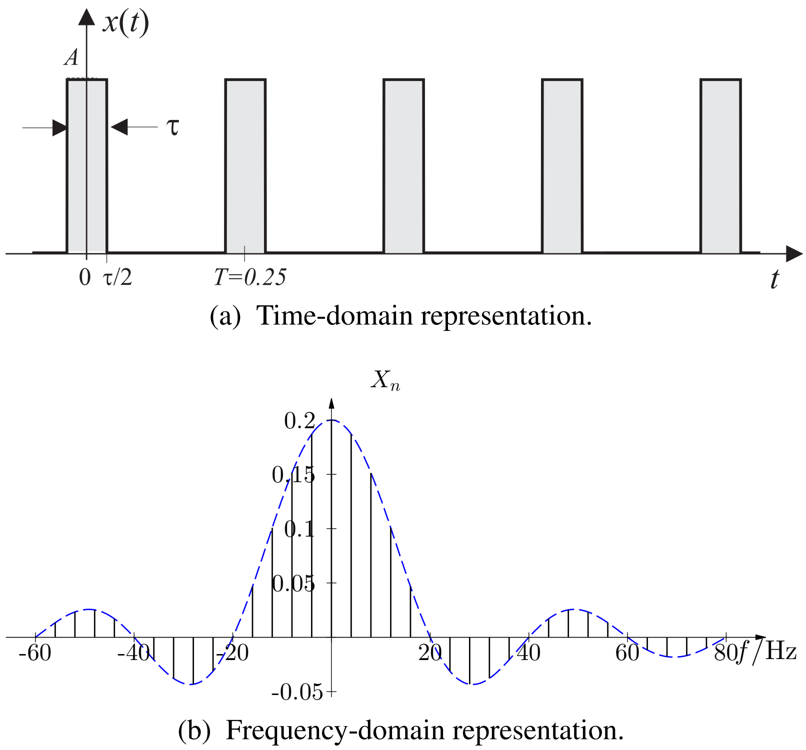


Figure 6.2: Pulse waveform with pulse width $\tau = \frac{1}{20}$ seconds, and period $T = 0.25$ seconds.

given by the fundamental frequency $f_0 = \frac{1}{T} = 4$ Hz. It also means that, using the results from page 165, the *zero-crossings* of Equation 6.2 occur at: [1]

$$n = m\frac{T}{\tau} = m\frac{\frac{1}{4}}{\frac{1}{20}} = 5m \quad 0 \neq m \in \mathbb{Z} \tag{5.87}$$

The nth coefficient corresonds to $f = nf_0$, or alternatively, the zero-crossings in Hertz are at $f = \ldots, -60, -40, -20, 20, 40, 60, 80, \ldots$ Hz. This is indeed the case as shown in Figure 6.2b.

In fact, it turns out that the zero-crossings as a function of frequency are always at these frequencies! This is because $f = nf_0 = m\frac{T}{\tau} \times \frac{1}{T} = \frac{m}{\tau} = 20m$ for $0 \neq m \in \mathbb{Z}$.

Consider now the case when the pulse width τ remains the same, but the period of the waveform is doubled to $T = 0.5$ seconds as shown in Figure 6.3a. The zero-crossings occur at multiples of 20 Hz, but now the frequency spacing is halved because $f_0 = \frac{1}{T} = 2$ Hz. Thus, the Fourier coefficients are now closer, as shown in Figure 6.3b.

If the period of the pulse waveform is extended to $T = 2$ seconds, then the frequency spacing is now $f_0 = 0.5$ Hz, and the spectrum is slowly being filled up. In the limit, as $T \to \infty$, the spacing of the frequencies $f_0 = \frac{1}{T} \to 0$ becomes infinitesimally small, and the spectrum becomes a continuous function of frequency.

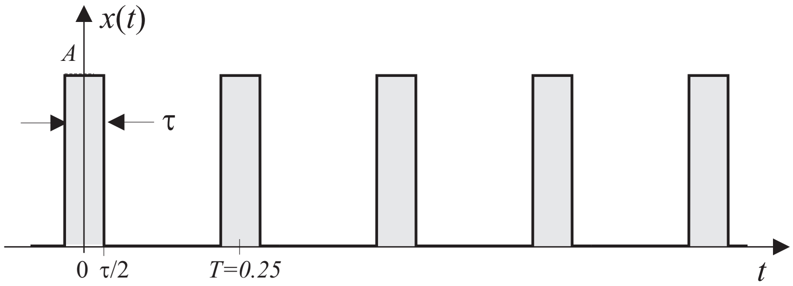
[1] The notation $m \in \mathbb{Z}$ means that m can take on any integer value, $m = \ldots, -3, -2, -1, 0, 1, 2, \ldots$. Therefore, the notation $0 \neq m \in \mathbb{Z}$ means all the integers excluding 0.

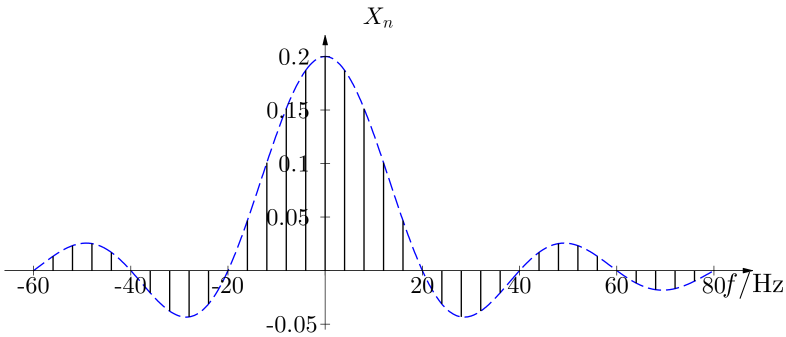
Summary Slide 24 Developing the Fourier Transform

What happens as $T \to \infty$?

- Consider a periodic pulse with fixed pulse width $\tau = \frac{1}{20}$ seconds, and variable period T. The Fourier coefficients are:

$$X_n = \frac{A\tau}{T}\,\mathrm{sinc}\left(\frac{n\omega_0\tau}{2}\right), \quad \omega_n = n\omega_0 = \frac{2\pi n}{T} \quad \text{or} \quad f_n = \frac{n}{T}$$





Recall zero-crossing positions; these values help plot the spectrum as a function of frequency, rather than by CFS coefficient index:

$$n\omega_0\tau/2 = m\pi \quad \Rightarrow$$

The frequencies of the zero-crossing points occur at:

$$f_n = n\,f_0 = n/T = m/\tau, \quad m \in \mathbb{Z} \text{ (set of integers)} \tag{6.3}$$

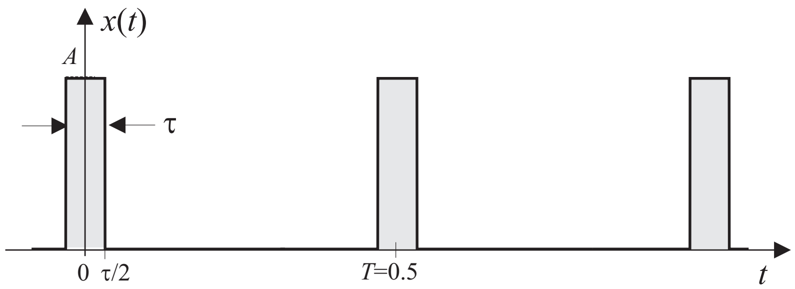
Note the zero-crossing frequencies do not depend on the period T.

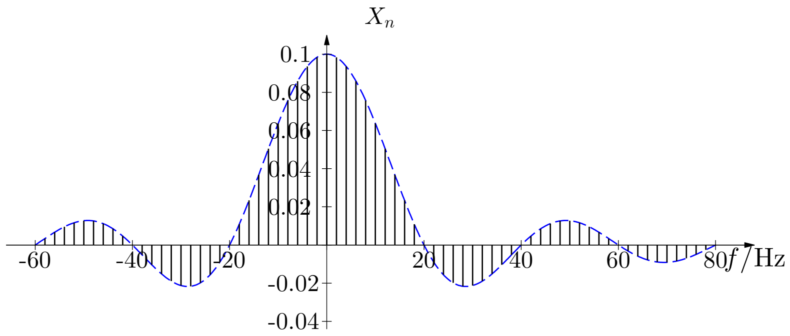
Summary Slide 25 Developing the Fourier Transform

What happens as $T \to \infty$?

- Consider a periodic pulse with fixed pulse width $\tau = \frac{1}{20}$ seconds, and variable period T. The Fourier coefficients are:

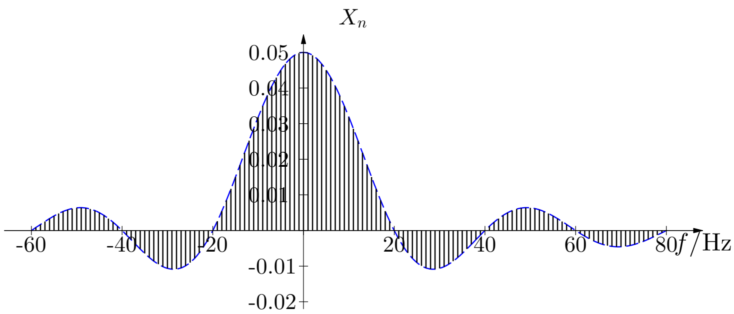
$$X_n = \frac{A\tau}{T}\,\mathrm{sinc}\left(\frac{n\omega_0\tau}{2}\right), \quad \omega_n = n\omega_0 = \frac{2\pi n}{T} \quad \text{or} \quad f_n = \frac{n}{T}$$

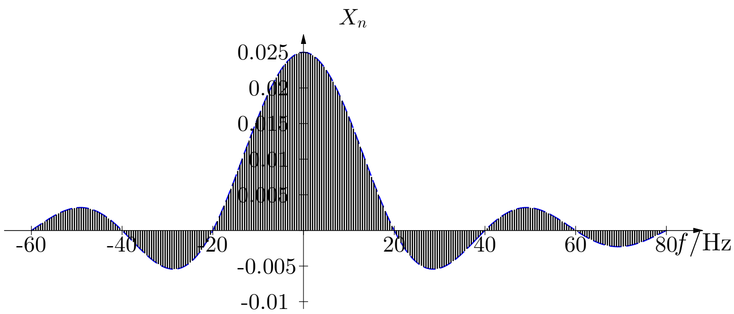


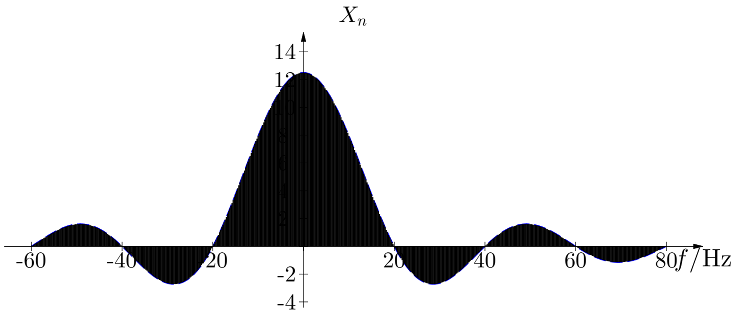


Summary Slide 26 Developing the Fourier Transform

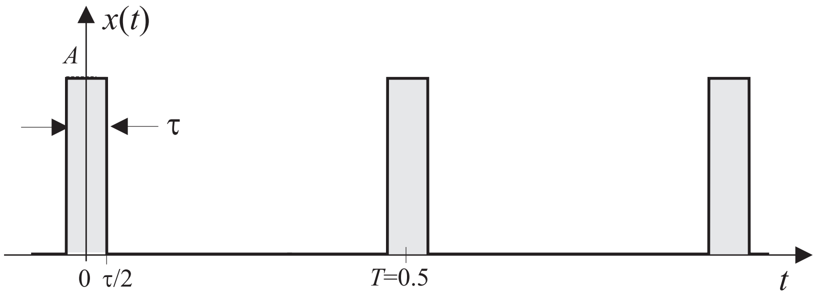
What happens as $T \to \infty$?



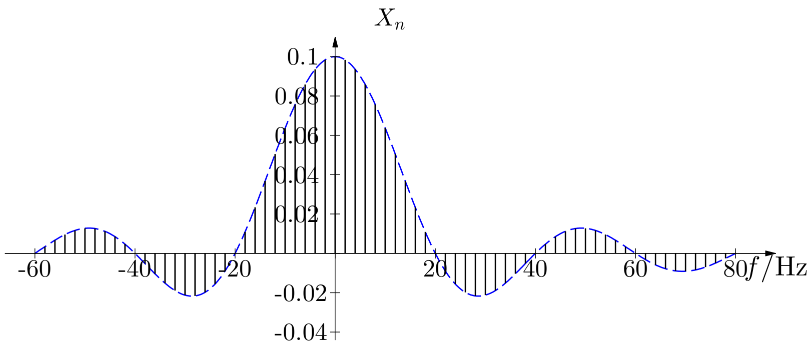




- Spectral shape doesn't change! Fourier coefficients get smaller as T gets larger (A and τ fixed), since $X_n = \frac{A\tau}{T}\,\mathrm{sinc}\left(\frac{n\omega_0\tau}{2}\right)$.



(a) Time-domain representation.



(b) Frequency-domain representation.

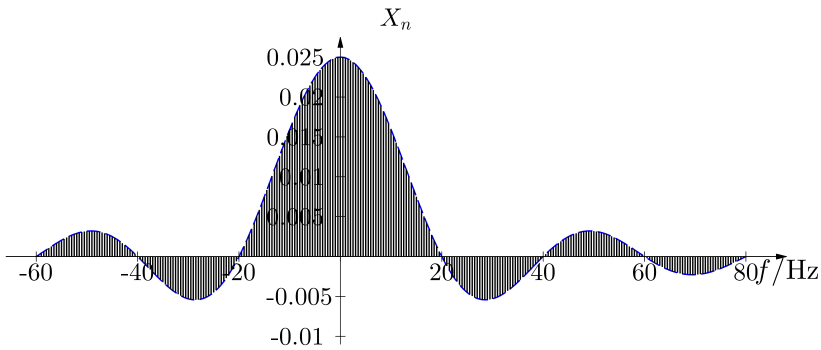
Figure 6.3: Pulse waveform with pulse width $\tau = \frac{1}{20}$ seconds, and period $T = 0.5$ seconds.

Note however, that the coefficients X_n are also getting smaller, due to the factor $\frac{1}{T}$ in Equation 6.2.Nevertheless, these observations with this particular example lead us to the mathematical development shown in the next section.

KEYPOINT! (MATLAB Demo). Available on LEARN is the MATLAB demo **FS-to-FT-demo** which will help visualise the limiting case of the Fourier series becoming the Fourier transform.

– End-of-Topic 30: **Intuitive Derivation of the Fourier Transform –**

(a) Time-domain representation.



(b) Frequency-domain representation.

Figure 6.4: Pulse waveform with pulse width $\tau = \frac{1}{20}$ seconds, and period $T = 2$ seconds.

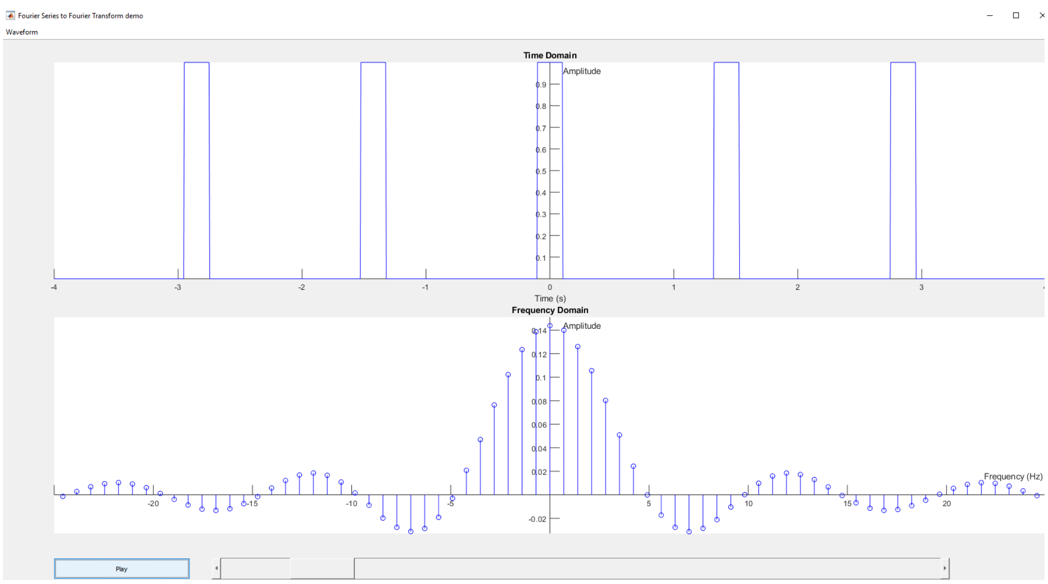
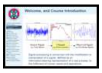


Figure 6.5: A screenshot of the MATLAB demo: *FS-to-FT-demo*.



6.1.2 Mathematical Development of the Fourier Transform

New slide

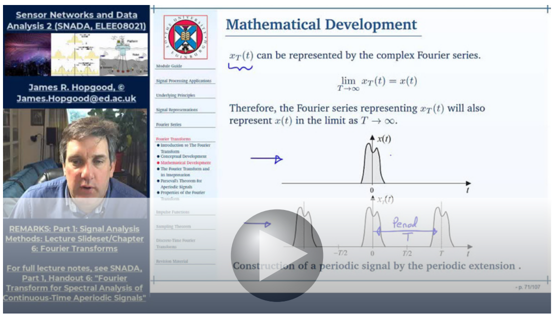
Topic Summary 31 Mathematical Development of the Fourier Transform from Complex Fourier Series

Topic Objectives:

- Consider the calculus of the derivation of the Fourier transform from the complex Fourier series.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 50 min video	$3\times$ length	
Read Handout	Read page 194 to page 199	8 mins/page	



http://media.ed.ac.uk/media/1_dquz3dvd

Video Summary: In this optional Topic, a mathematical derivation is presented for developing the Fourier transform as the limiting operation of taking the complex Fourier series, and letting the period of the waveform approach infinity. Therefore, the non-periodic signal is considered as a special case of a period signal but with infinite period. This idea was conceptually introduced in the previous Topic, but this video works through the calculus of this derivation.

Using the observations in Section 6.1, consider constructing a periodic signal, $x_T(t)$, with period T by repeating a non-periodic signal $x(t)$ at intervals of T seconds, as illustrated in Figure 6.6. The period T is made long enough to avoid overlap between repeated pulses. The periodic signal $x_T(t)$ can be represented by the complex Fourier series using the exponential basis functions$e^{j\omega_0 nt}$. In the limiting case, as the period $T \to \infty$, the pulses in the periodic signal repeat after an infinite period and therefore:

$$\lim_{T\to\infty} x_T(t) = x(t) \tag{6.6}$$

Therefore, the Fourier series representing $x_T(t)$ will also represent $x(t)$ in the limit as $T \to \infty$. Note that this approach to developing the Fourier transform is rather informal, and a thorough proof is substantially more involved than presented here.

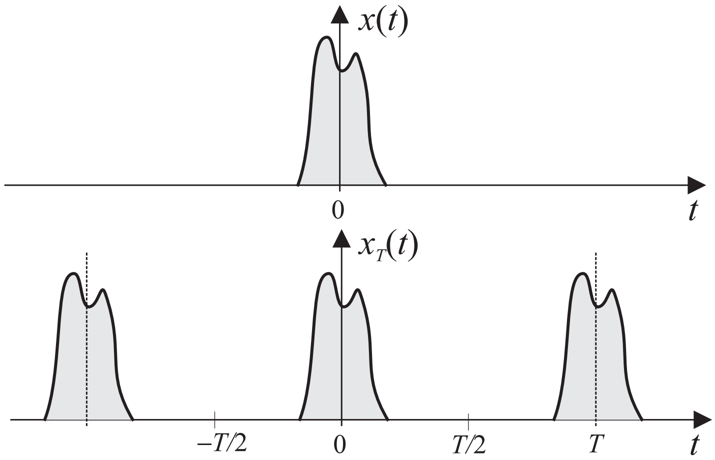


Figure 6.6: Construction of a periodic signal by the periodic extension of a non-periodic signal $x(t)$.

The periodic signal $x_T(t)$ has complex Fourier series:

$$x_T(t) = \sum_{n=-\infty}^{\infty} X_{T,n}\, e^{jn\omega_0\, t} \tag{6.7}$$

where $\omega_0 = \frac{2\pi}{T}$ and

$$X_{T,n} = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\, e^{-jn\omega_0 t}\, dt \tag{6.8}$$

Observe that, due to the factor of $\frac{1}{T}$ in Equation 6.8, the coefficients $X_{T,n}$ will tend to zero as $T \to \infty$. Therefore, it is necessary to define the modified Fourier coefficients as: [2]

$$\hat{X}_{T,n} \triangleq T\, X_{T,n} = \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\; e^{-jn\omega_0 t}\, dt \tag{6.9}$$

Now consider the limit $T \to \infty$, then the spacing of the frequency in radians per second, $\omega_0 = \frac{2\pi}{T} \to \delta\omega \to 0$, is infinitesimally small, and the frequency of the nth component $\omega_n = n\omega_0 \to \omega$ becomes a continuous value rather than integer multiples of the fundamental.

Hence, the complex Fourier coefficients can be written as a function of a continuous frequency variable ω, and noting that the limits in Equation 6.9 tend to $\pm\infty$, then:

$$X(\omega) \triangleq \lim_{T\to\infty} \hat{X}_{T,n} \tag{6.10}$$

$$= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_T(t)\; \exp\left(-jn\omega_0 t\right)\; dt \tag{6.11}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\; \exp\left(-j\omega t\right)\; dt \tag{6.12}$$

This is known as the **Fourier transform**.

Summary Slide 27 Developing the Fourier Transform

The Fourier Series as $T \to \infty$ ...

The complex Fourier coefficients are given by:

$$X_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, \exp\left(-jn\omega_0 t\right)\, dt$$

- Need to redefine the Fourier coefficients since $X_n \to 0$ as $T \to \infty$ because of the $\frac{1}{T}$ term:

$$\hat{X}_n = T X_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, \exp\left(-jn\omega_0 t\right)\, dt$$

- Recall again that $\omega_0 = \frac{2\pi}{T}$.
- In the limit as $T \to \infty$, :

$$\begin{aligned} X(\omega) &\triangleq \lim_{T\to\infty} \hat{X}_n \\ &= \lim_{T\to\infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\, \exp\left(-jn\omega_0 t\right)\, dt \end{aligned}$$

$$\underbrace{X(\omega)}_{} = \int_{-\infty}^{\infty} \underbrace{x(t)}_{}\, \underbrace{\exp\left(-j\omega t\right)}_{}\, dt$$

This is known as the *forward* **Fourier transform**, and is a continuous function of frequency.

Summary Slide 28 Developing the Fourier Transform

The Fourier Series as $T \to \infty$ ...

- Similarly, the Fourier series decomposition becomes

:

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\hat{X}_n}{T} \exp\left(jn\omega_0 t\right) \quad \text{where } \omega_0 = \frac{2\pi}{T}$$

$$= \sum_{n=-\infty}^{\infty} \hat{X}_n \exp\left(jn\omega_0 t\right) \underbrace{\frac{\omega_0}{2\pi}}_{\frac{1}{T}}$$

and in the limit as $T \to \infty$:

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \hat{X}_n \exp\left(jn\omega_0 t\right) \frac{\omega_0}{2\pi}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X\left(\omega\right) \exp\left(j\omega t\right)\, d\omega$$

This is known as the **inverse Fourier transform**.

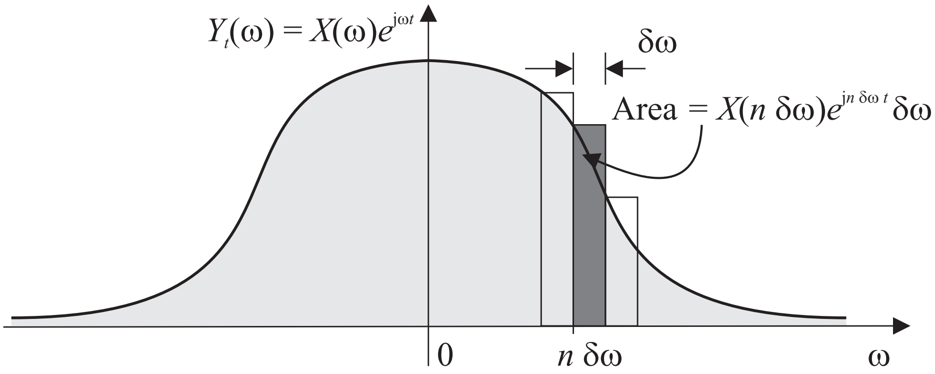


Figure 6.7: The Fourier series becomes the Fourier integral in the limit as $T \to \infty$.

Similarly, the **Fourier series** representation on page 148 can be expressed using the re-defined Fourier coefficients in Equation 6.9 as:

$$x_T(t) = \sum_{n=-\infty}^{\infty} X_n\, e^{jn\omega_0 t} \tag{6.14}$$

$$x_T(t) = \sum_{n=-\infty}^{\infty} \frac{\hat{X}_{T,n}}{T}\, e^{jn\omega_0 t} \tag{6.15}$$

$$= \sum_{n=-\infty}^{\infty} \hat{X}_{T,n}\, e^{jn\omega_0 t}\, \frac{\omega_0}{2\pi} \tag{6.16}$$

In the limit, as $T \to \infty$, then $\omega_0 \equiv \delta\omega \to 0$ and $\hat{X}_{T,n} \to X\,(n\delta\omega)$, such that:

$$x(t) = \lim_{T\to\infty} x_T(t) \tag{6.6}$$

$$x(t) = \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} \hat{X}_{T,n}\, e^{jn\omega_0 t}\, \frac{\omega_0}{2\pi} \tag{6.17}$$

$$x(t) = \frac{1}{2\pi} \lim_{T\to\infty} \sum_{n=-\infty}^{\infty} X\,(n\delta\omega)\; e^{jn\delta\omega t}\, \delta\omega \tag{6.18}$$

In the limit, this summation becomes an integral, and can be seen by considering the function $Y_t(\omega) = X\,(n\delta\omega)\; e^{jn\delta\omega t}$ as shown in Figure 6.7. The value $Y_t(\omega)\,\delta\omega$ is the area of a small rectangle of width $\delta\omega$ and height $Y_t(\omega)$. As $\delta\omega \to 0$, which occurs as $T \to \infty$, the infinite summation of these narrow rectangles becomes the area under the function $X\,(\omega)\; e^{j\omega t}$. Therefore, the limit in Equation 6.18 becomes the **inverse Fourier transform**:

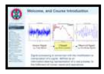
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X\,(\omega)\; e^{j\omega t}\, d\omega \tag{6.19}$$

[2]The symbol $\triangleq$ means *defined as*, meaning that this is the definition rather than an equivalence in an algebraic sense. The difference is subtle, and its use is not important here.

Together, Equation 6.12 and Equation 6.19 form the direct Fourier transform and **inverse Fourier transform** respectively. In particular, the **Fourier integral** in Equation 6.12 is a representation of an aperiodic signal in terms of a linear combination of complex phasors, but rather than the phasors have frequencies at an integer multiple of the fundamental, all frequencies are represented. [3]

– End-of-Topic 31: **Mathematical Derivation of the Fourier Transform** –

6.2 The Fourier Transform and its Interpretation



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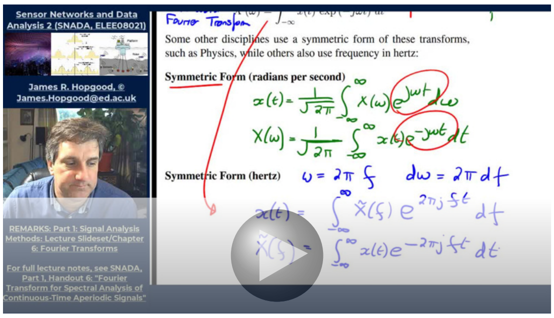
Topic Summary 32 The Fourier Transform and Example Calculations

Topic Objectives:

- Definition of the Fourier transforms, and alternative forms.
- Example of calculating the Fourier transform of a rectangular pulse.
- Plotting the Fourier transform.
- A physical interpretation of the Fourier Transform.

Topic Activities:

Type	Details	Duration	Progress
Watch video	17 : 42 min video	$3\times$ length	
Read Handout	Read page 200 to page 206	8 mins/page	
Try Example	Try Example 6.1	10 mins	
Practice Exercises	Exercises 6.1 and 6.2	30 mins	



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Video Summary: This Topic provides the definition of the Fourier transform, for analysing continuous-time non-periodic signals. The Topic provides the standard Engineering definition, alongside definitions found in other disciplines which aim to use a more symmetric form of the transform. The Topic then moves on to an example of calculating the Fourier transform of a rectangular pulse, as well as plotting the spectrum. Finally, the Topic considers a physical interpretation of what the Fourier spectrum corresponds to.

As developed in Section 6.1.2, any aperiodic (non-periodic signal) $x(t)$ can be represented as an integral of complex phasors $e^{j\omega t}$ of all frequencies $-\infty < \omega < \infty$,

[3]Note that this derivation should not be considered a rigorous proof of Equation 6.19, as the situation is a little more complicated than it might appear.

Sidebar 13 Shorthand Notation!

A short-hand functional or symbolic notation for the Fourier transform pair that is sometimes used is as follows:

- $X(\omega) = \mathcal{F}(x(t))$ means that $X(\omega)$ is the Fourier transform of $x(t)$ as given by Equation 6.21a;
- $x(t) = \mathcal{F}^{-1}(X(\omega))$ means that $x(t)$ is the inverse Fourier transform of $X(\omega)$ as given by Equation 6.21b.

  Alternatively, $x(t) \rightleftharpoons X(\omega)$ means that $x(t)$ and $X(\omega)$ are Fourier transform pairs.

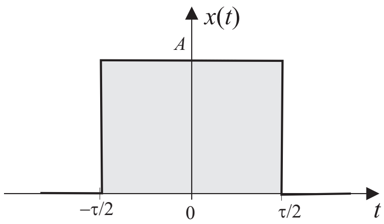


Figure 6.8: Simple Rectangular Pulse

with weightings of $\frac{X(\omega)}{2\pi}$. This gives rise to the following relations known as the **Fourier Transform pair**: [4]

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{6.21a}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \tag{6.21b}$$

Example 6.1 (Rectangular Pulse). Find the Fourier transform and the spectrum of the signal shown in Figure 6.8, which is given by:

$$x(t) = \begin{cases} 0 & t < -\tau/2 \\ A & \text{if } -\tau/2 \le t < \tau/2 \\ 0 & t \ge \tau/2 \end{cases} \tag{6.22}$$

[4] Note the symmetry of the Fourier transform integrals in Equation 6.21. In fact, in some areas of science such as Physics, the Fourier transforms are defined slightly differently to exploit this symmetry by placing a factor of $\frac{1}{\sqrt{2\pi}}$ in the direct Fourier transform, and removing the same factor from the inverse Fourier transform to give:

$$X(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{6.20a}$$

$$x(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \tag{6.20b}$$

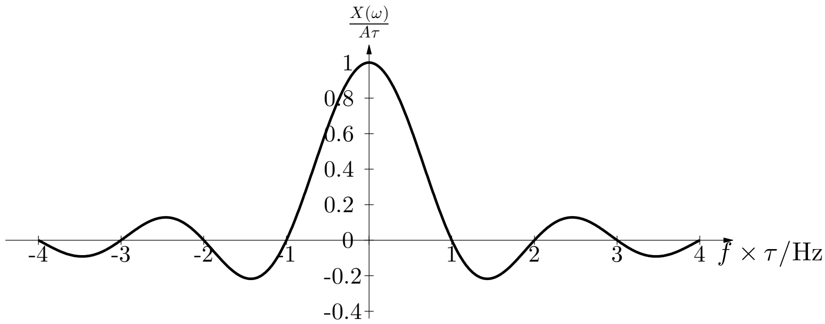


Figure 6.9: Fourier Spectrum for Rectangular Pulse

SOLUTION. Using the Fourier transform equation of the aperiodic finite energy signal $x(t)$:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, \exp(-j\omega t)\, dt \tag{6.23}$$

$$= \int_{-\tau/2}^{\tau/2} A\, \exp(-j\omega t)\, dt \tag{6.24}$$

$$+ \underbrace{\int_{-\infty}^{-\tau/2} 0\, \exp(-j\omega t)\, dt}_{=0} + \underbrace{\int_{\tau/2}^{\infty} 0\, \exp(-j\omega t)\, dt}_{=0} \tag{6.25}$$

$$= -\frac{A}{j\omega}\Big[\exp(-j\omega t)\Big]_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = -\frac{A}{j\omega}\left\{e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right\} \tag{6.26}$$

$$= \frac{2A}{\omega}\frac{e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}}{2j} = A\tau\, \frac{\sin\left(\omega\frac{\tau}{2}\right)}{\omega\frac{\tau}{2}} = A\tau\, \operatorname{sinc}\left(\omega\frac{\tau}{2}\right) \tag{6.27}$$

□

This is the infamous sinc function we have seen before. Plot $X(\omega)$ as continuous function of ω, or against f where $\omega = 2\pi f$. The zero-crossing at $\omega\tau/2 = m\pi$ or, as $\omega = 2\pi f$, at $f\tau = m \neq 0$, m int.

Let τf be **normalised linear frequency**: $\hat{X}(f) = A\tau\, \operatorname{sinc}(\pi f \tau)$. Note that the spectrum is also non-periodic. It is shown in Figure 6.9.

A physical interpretation of the Fourier transform is that it is used to represent a signal as a *sum* of cosine waves at all possible frequencies. It is also fair to consider that the Fourier transform can be considered as the Fourier series but with fundamental frequency $\delta\omega$ approaching zero. However, it is better to think that all possible frequencies are needed because the signal is *not periodic*, and therefore it cannot have **harmonics**.

Since a signal $x(t)$ is said to contain all possible frequencies, then a physical interpretation is to say that $x(t)$ has a *frequency component* in a small frequency band ω to $\omega + d\omega$ rad/s which can be written approximately as:

$$dx_\omega(t) = \frac{|X(\omega)|\, d\omega}{2\pi} \cos(\omega t + \angle X(\omega)) \tag{6.32}$$

Summary Slide 29 Developing the Fourier Transform

Continuous-Time Fourier Transform

Any non-periodic signal $x(t)$ can be represented as a continuum of sine waves and cosine waves using the **inverse Fourier transform** and **Fourier transform** respectively :

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t)\, d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, \exp(-j\omega t)\, dt$$

Some other disciplines use a symmetric form of these transforms, such as Physics, while others also use frequency in hertz:

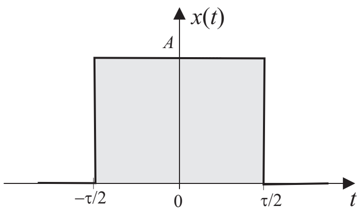
Symmetric Form (radians per second)

Symmetric Form (hertz)

Finally, note that the Fourier series has a very similar structure to the Fourier transform, a point that will be revisited later in this course:

Summary Slide 30 The Fourier Transform

Fourier Transform Example



Find the Fourier transform of the aperiodic finite energy signal $x(t)$:

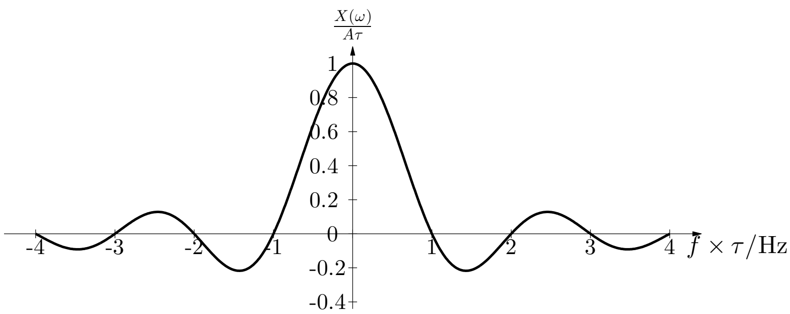
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, \exp(-j\omega t)\, dt \tag{6.28}$$

$$= \int \quad A \exp(-j\omega t)\, dt \tag{6.29}$$

$$= -\frac{A}{j\omega}\Big[\exp(-j\omega t)\Big] \tag{6.30}$$

Fourier Spectrum

Plot $X(\omega)$ as continuous function of ω, or against f where $\omega = 2\pi f$.

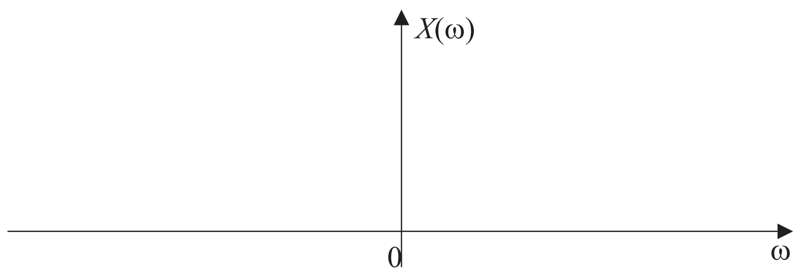


Summary Slide 31 The Fourier Transform

Physical Interpretation

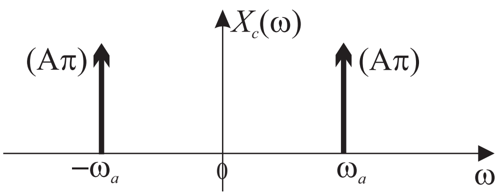
- The Fourier Transform represents a signal as the *sum* of complex phasors (or cosines) at all possible frequencies.
- A signal $x(t)$ is said to have a *frequency component* in a small frequency band ω to $\omega + d\omega$ rad/s given approximately by:

$$\frac{|X(\omega)|\, d\omega}{2\pi} \cos(\omega t + \angle X(\omega)) \tag{6.31}$$



1. The magnitude of the Fourier transform $\frac{|X(\omega)| d\omega}{2\pi}$ is the amplitude of a cosine wave with frequency ω rad/s.
2. The angle of the Fourier transform $\angle X(\omega)$ is the phase shift associated with the cosine wave.

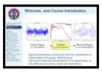
- All possible frequencies are required because the signal is not periodic and hence cannot have harmonics.
- Intuitively, the Fourier transform of the simplest periodic signal, $\cos \omega_a t$, should only have frequencies at $\pm\omega_a$, and be zero elsewhere. Will investigate this at end of the handout.



In other-words, Equation 6.32 can be interpreted as saying that the:

1. magnitude of the Fourier transform, $\frac{|X(\omega)|d\omega}{2\pi}$, is the amplitude of a cosine wave with frequency ω rad/s;
2. angle of the Fourier transform, $\angle X(\omega)$, is the phase shift associated with the cosine wave.

– End-of-Topic 32: **Interpretation of the Fourier Transform, with an Example –**



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6.3 Properties of the Fourier Transform

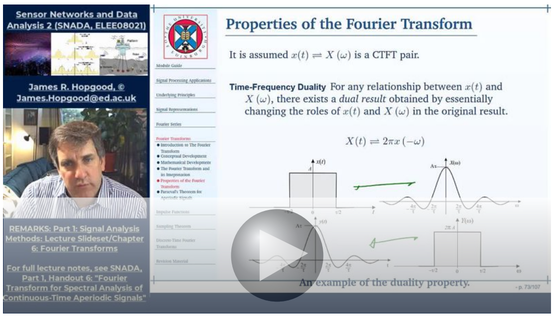
Topic Summary 33 Fourier Transform Properties: Conjugation, Linearity, Duality

Topic Objectives:

- Understanding benefits of using Fourier transform properties.
- Fundamental basic Properties, such as Conjugation and Linearity.
- Conceptual and useful property of the duality Theorem, and Physical Interpretation.

Topic Activities:

Type	Details	Duration	Progress
Watch video	22 : 48 min video	$3\times$ length	
Read Handout	Read page 207 to page 211	8 mins/page	



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Video Summary: This Topic introduces the benefits of understanding and using the various properties of the Fourier transform, including simplifying the analysis of signals, as well as conceptual insights that can help give a deeper understanding of the theory. This, and the next few videos, cover a number of properties of the Fourier transform. This Topic focuses on some elemental properties, such as the conjugation property of the continuous-time Fourier transform (CTFT), and conjugate symmetry for real signals. The topic next considers linearity of the Fourier transform, and then the important property of time-frequency duality.

There are a number of properties of the Fourier transform that can help simplify the analysis of the spectral content of the signal. In fact, in practical theoretical analysis, *tables of Fourier transforms* are used in which the spectrum of a number of basic signals are listed, and then other signals are derived from using the basic properties of the Fourier transform.

In addition to the basic symmetrical properties of the Fourier transform, take note that the most important properties are **duality**, **shift in time**, **scale in time**, **frequency shift**, **convolution in time**, and **multiplication in time**. The latter two properties will be investigated later in the course when convolution is introduced.

In each of the following properties, it is assumed that $x(t) \rightleftharpoons X(\omega)$ is a CTFT pair.

Conjugation Property The CTFT of the conjugate of a complex signal, $x^*(t)$, is the reversed conjugate of the CTFT of $x(t)$: [5]

$$x^*(t) \rightleftharpoons X^*(-\omega) \tag{6.39}$$

Conjugate Symmetry for Real Signals For a real signal, $x(t) = x^*(t)$, then it follows that:

$$X(-\omega) = X^*(\omega) \tag{6.40}$$

This is the **conjugate symmetry property** of the Fourier transform, applicable only to a real signal $x(t)$. This can alternatively be written in the form:

$$|X(-\omega)| = |X(\omega)| \tag{6.41a}$$

$$\angle X(-\omega) = -\angle X(\omega) \tag{6.41b}$$

Thus, for real $x(t)$, the amplitude spectrum $|X(\omega)|$ is an even function, and the phase spectrum $\angle X(\omega)$ is an odd function of ω. These results were derived earlier on page 155 for the Fourier series coefficients of a periodic signal.

Linearity of the Fourier transform The Fourier transform is linear: that is if $x_1(t) \rightleftharpoons X_1(\omega)$ and $x_2(t) \rightleftharpoons X_2(\omega)$, then:

$$a_1\, x_1(t) + a_2\, x_2(t) \rightleftharpoons a_1\, X_1(\omega) + a_2\, X_2(\omega) \tag{6.42}$$

The proof is essentially trivial (compared to other proofs), and the result can be extended to any finite number of terms. Thus, if $x_n(t) \rightleftharpoons X_n(\omega)$ for $n = 1, 2, \dots, N$, then: [6]

$$\sum_{n=1}^{N} a_n\, x_n(t) \rightleftharpoons \sum_{n=1}^{\infty} a_n\, X_n(\omega) \tag{6.43}$$

[5]Equation 6.39 can easily be proved by setting $\omega \to -\omega$ in Equation 6.21a, such that:

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt \tag{6.38}$$

Taking the conjugate of both sides and using the properties of complex numbers used on page 175 gives the required result.

[6]Equation 6.43 can be extended to an infinite number of terms only if the conditions required for the interchangeability of the operators of summation and integration are satisfied. However, this is beyond the scope of this course (in other-words, don't worry about it).

Summary Slide 32 Properties of the Fourier Transform

Elementary Properties

- Generally, plot the **Spectrum** $X(\omega)$ as a function of *radial* frequency ω in radians per second, where $\omega = 2\pi f$ with f in Hz.

- In general, $X(\omega)$ is complex, and therefore has both amplitude (**magnitude**) and angle (**phase**):

$$X(\omega) = |X(\omega)|\, e^{j\angle X(\omega)} \tag{6.33}$$

  in which $|X(\omega)|$ is the magnitude, and $\angle X(\omega)$ is the phase of $X(\omega)$.

- Can also plot the spectrum as a function of frequency in Hertz by setting $\omega = 2\pi f$, such that:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-2\pi j f t}\, dt \tag{6.34}$$

Equivalent Forms

Summary Slide 33 Properties of the Fourier Transform

Duality

If

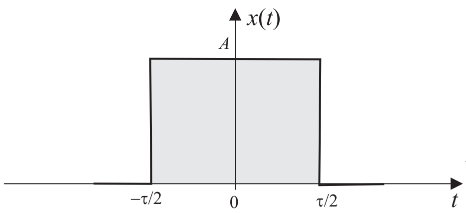
$$x(t) \rightleftharpoons X(\omega) \tag{6.35}$$

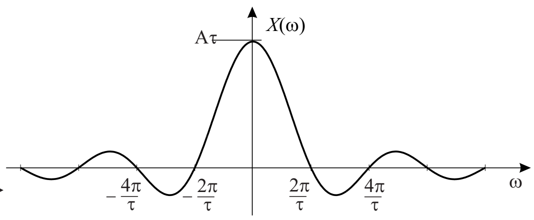
then,

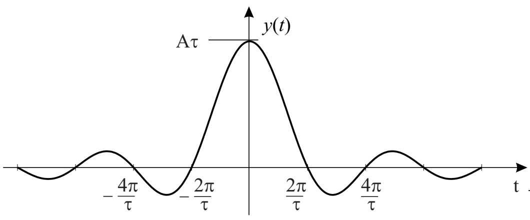
$$X(t) \rightleftharpoons 2\pi x(-\omega) \tag{6.36}$$

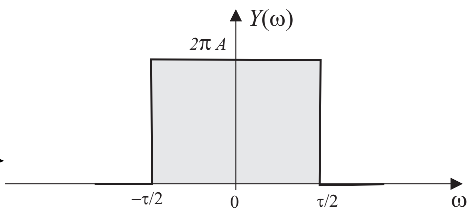
Example

The rectangular pulse has the sinc function as its Fourier transform:









A system with rectangular **frequency response** (above), is described by the **impulse response** in the time-domain, here the sinc function.

Time-Frequency Duality The Fourier transform relationships in Equation 6.21 are remarkably similar, other than a change of sign and a factor of 2π in the inverse Fourier transform of Equation 6.21b. [7] The inverse Fourier transform in Equation 6.21b can be obtained from the direct Fourier transform in Equation 6.21a by replacing $x(t)$ with $X(\omega)$, t with ω, ω with $-t$, and including a factor of 2π. In a similar way, Equation 6.21a can be obtained from Equation 6.21b.

This observation is the basis of the so-called duality of time and frequency, which means that for any result or For any relationship between $x(t)$ and $X(\omega)$, there exists a *dual result* or relationship obtained by essentially interchanging the roles of $x(t)$ and $X(\omega)$ in the original result (along with some minor modifications arising because of the factor 2π and a sign change). [8] Thus, in summary the **duality principle** states that if $x(t) \rightleftharpoons X(\omega)$, then:

$$X(t) \rightleftharpoons 2\pi x(-\omega) \tag{6.45}$$

The **duality property** can be demonstrated by the Fourier transform of a rectangular pulse, as shown in Figure 6.10. Note that the rectangular pulse is an even function. The *brick-wall* spectrum in Figure 6.10d represents a low-pass filter, and the relationship between the sinc function in the time domain and the low-pass filter in the frequency domain will become very important in the next handout.

KEYPOINT! (Duality property). The **duality principle** also highlights the fact that the *time domain* is no more or less important than the *frequency domain*, and vice-versa. While Engineers might be introduced to signal analysis in the time domain, the frequency domain is just as important as will be seen later in this course.

– End-of-Topic 33: **Conjugation, Linearity, and Duality: Properties of the Fourier Transform** –

[7]The factor of 2π that makes one difference in the Fourier transform relationships of Equation 6.21 on page 201 can be eliminated by a change of variable from ω to f (in Hertz). Hence, with $\omega = 2\pi f$ and $d\omega = 2\pi df$, and defining $\hat{X}(f) = X(2\pi f)$, then Equation 6.21 becomes:

$$\hat{X}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt \qquad \text{and} \qquad x(t) = \int_{-\infty}^{\infty} \hat{X}(f)\, e^{j2\pi f t}\, df \tag{6.44}$$

[8]The **duality principle** may be obtained with a photograph and its negative; a photograph can be obtained from its negative, and using an identical procedure, a negative can be obtained from the photograph.

Sidebar 14 Proof of the Duality Principle

It's useful to know where some of the results quoted in this section come from, rather than just using them as given formula. Therefore, this handout will include a few proofs, and the first one is the result in Equation 6.45. The inverse Fourier transform relation in Equation 6.21b can be written using a different variable of integration as:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(u)\, e^{jut}\, du \tag{6.46}$$

Setting $t \to -t$ and rearranging slightly gives:

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(u)\, e^{-jut}\, du \tag{6.47}$$

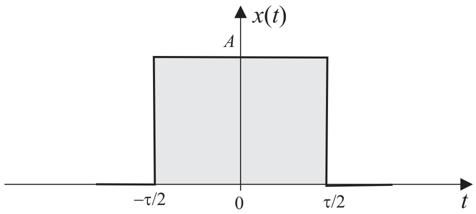
Similarly, replacing the variable t by ω gives:

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(u)\, e^{-ju\omega}\, du \tag{6.48}$$

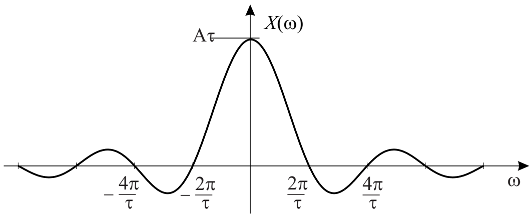
Finally, since u is just a variable of integration, this can be replaced by any letter, so it can be replaced by $u \to t$:

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt \tag{6.49}$$

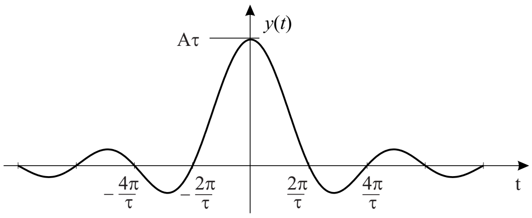
and the right hand side (RHS) can be seen to be the direct Fourier transform of the function $X(t)$.



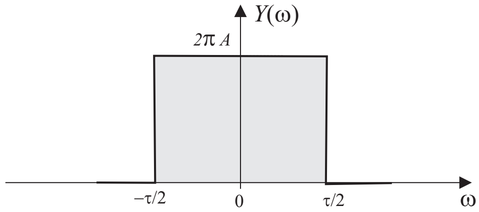
(a) A rectangular pulse, described by $x(t)$ as a function of time, t.



(b) The CTFT of the rectangular pulse, described by $X(\omega)$ as a function of frequency ω.



(c) A sinc function, described by $y(t) = X(t)$ as a function of t.



(d) The CTFT of $y(t) \rightleftharpoons Y(\omega)$ is determined using the duality principle, so since $y(t) = X(t)$, then $Y(\omega) = 2\pi\, x(-\omega)$.

Figure 6.10: An example of the duality property of the Fourier transform.

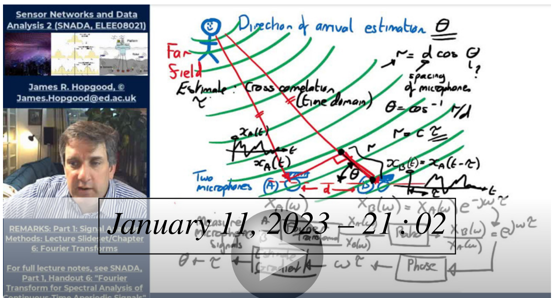
Topic Summary 34 Fourier Transform Properties: Scale-in-time and Shift-in-time

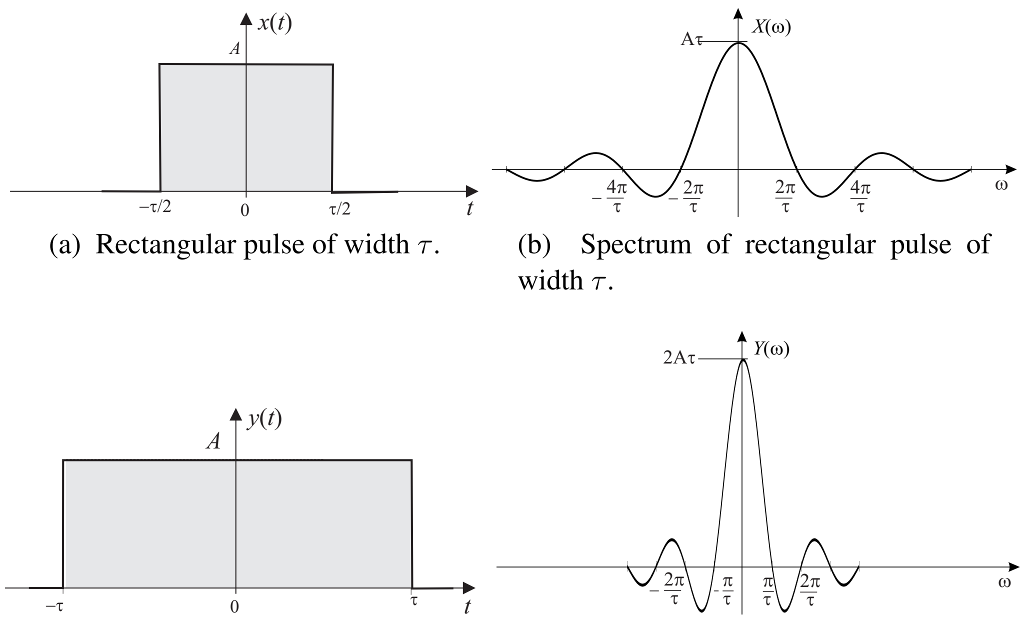
Topic Objectives:

- Introduction to Scale-in-Time and Shift-in-Time theorems.
- Application of Scale-in-Time to spectral analysis of variable-length signals.
- Importance of linear phase and its relationships to time-delays.
- Application of Shift-in-Time theorem to direction-of-arrival estimation.

Topic Activities:

Type	Details	Duration	Progress
Watch video	22 : 57 min video	3× length	
Read Handout	Read page 213 to page 219	8 mins/page	
Try Example	Try Example 6.2	10 mins	
Practice Exercises	Exercises 6.3 and 6.4	30 mins	





(a) Rectangular pulse of width τ.

(b) Spectrum of rectangular pulse of width τ.

(c) Rectangular pulse of width 2τ.

(d) Spectrum of rectangular pulse of width 2τ.

Figure 6.11: The scaling property of the Fourier Transform.

Scale in Time The **scale in time** property states that *compression in time* results in *spectral expansion*, and that *temporal expansion* of a signal results in *spectral compression*. Mathematically, if $x(t) \rightleftharpoons X(\omega)$, then:

$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \tag{6.56}$$

The proof of this formula will be left as one of the tutorial questions. Intuitively, *compression* in time by a factor of a means that the signal is varying faster by a factor of a. [9] To synthesis a faster varying signal, the frequencies of the sinusoidal components must increase by a factor of a, which is equivalent to saying that the frequency spectrum is *expanded* by a factor of a. Similarly, a signal *expanded* in time varies more slowly, hence the frequency components are lowered, implying that its frequency spectrum is *compressed*.

An example of the **scale-in-time** property is shown in Figure 6.11, where the standard spectrum of the time-centered rectangular pulse of width τ is shown, along with a rectangular pulse of length 2τ. As since in Figure 6.11d, *expansion* of the pulse in the time-domain has led to *compression* of the pulse in the frequency domain.

Shift in Time This important property is frequently used, and says that the Fourier transform of a delayed version of a signal, delayed by t_0 seconds, does not change the amplitude spectrum, but does however change the phase

[9] It is usually assumed that $a > 0$, although the argument still holds if $a < 0$.

Summary Slide 34 Properties of the Fourier Transform

Shift in Time

If

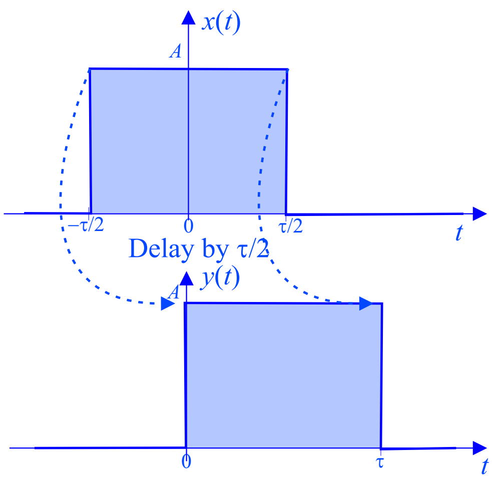
$$x(t) \rightleftharpoons X(\omega) \tag{6.50}$$

then, for any real constant t_0

$$x(t - t_0) \rightleftharpoons X(\omega)\, e^{-j\omega t_0} \tag{6.51}$$

This result shows that delaying a signal by t_0 seconds:

- does not change its amplitude spectrum;
- the phase spectrum, however, is changed by $-\omega\, t_0$.



Summary Slide 35 Properties of the Fourier Transform

Scale in Time

If

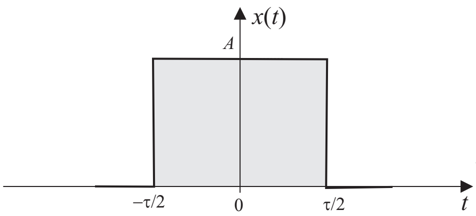
$$x(t) \rightleftharpoons X(\omega) \tag{6.54}$$

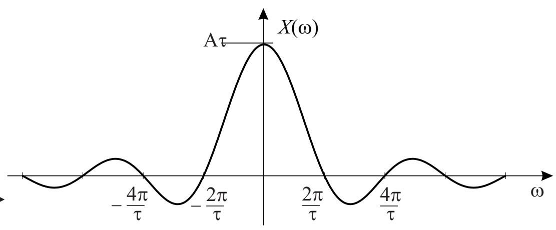
then, for any real constant a,

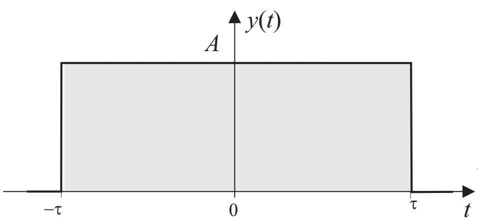
$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \tag{6.55}$$

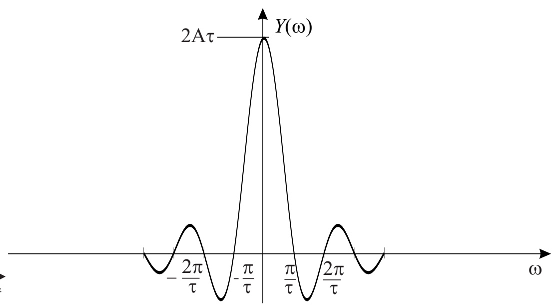
Example

Using the earlier rectangular pulse:









If a function is squashed in time, its Fourier transform stretches out in frequency. It is not possible to arbitrarily concentrate a function in both time and frequency. This is the **uncertainty principle**.

Sidebar 15 Proof of Shift in time Property

By definition

$$\mathcal{F}\left(x(t-t_0)\right) = \int_{-\infty}^{\infty} x(t-t_0)\; e^{-j\omega t}\, dt \tag{6.58}$$

Setting $\tau = t - t_0$ means that $d\tau = dt$ and when $t = \pm\infty$, $\tau = \pm\infty$. Substituting into the Fourier integral gives:

$$\mathcal{F}\left(x(t-t_0)\right) = \int_{-\infty}^{\infty} x(\tau)\; e^{-j\omega(\tau+t_0)}\, d\tau \tag{6.59}$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\tau)\; e^{-j\omega\tau}\, d\tau = e^{-j\omega t_0} X\left(\omega\right) \tag{6.60}$$

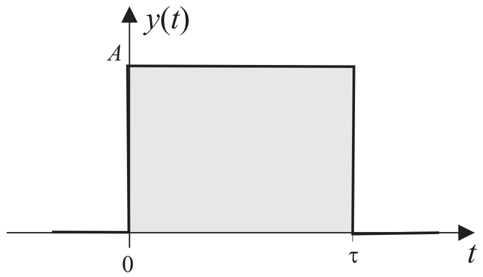


Figure 6.12: Rectangular pulse starting at $t = 0$.

spectrum by a term linear in frequency, $-\omega t_0$. Thus, if $x(t) \rightleftharpoons X\left(\omega\right)$, then for any real constant t_0:

$$x(t-t_0) \rightleftharpoons X\left(\omega\right)\; e^{-j\omega t_0} \tag{6.57}$$

Example 6.2 (Rectangular pulse). Calculate the Fourier transform for the pulse, $y(t)$ shown in Figure 6.12.

SOLUTION. Using the result from the lectures in Summary Slide 30, the Fourier transform of a pulse centered on zero and of width τ is:

$$x(t) \rightleftharpoons X\left(\omega\right) = A\tau\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right) \tag{6.62}$$

The pulse in Figure 6.12 is a delayed version of $x(t)$ by an amount $\frac{\tau}{2}$, and therefore $y(t) = x\left(t - \frac{\tau}{2}\right)$. Thus, using the shift-in-time theorem:

$$y(t) = x\left(t - \frac{\tau}{2}\right) \rightleftharpoons Y\left(\omega\right) \tag{6.63}$$

$$Y\left(\omega\right) = A\tau\, e^{-j\omega\frac{\tau}{2}}\,\mathrm{sinc}\left(\frac{\omega\tau}{2}\right) \tag{6.64}$$

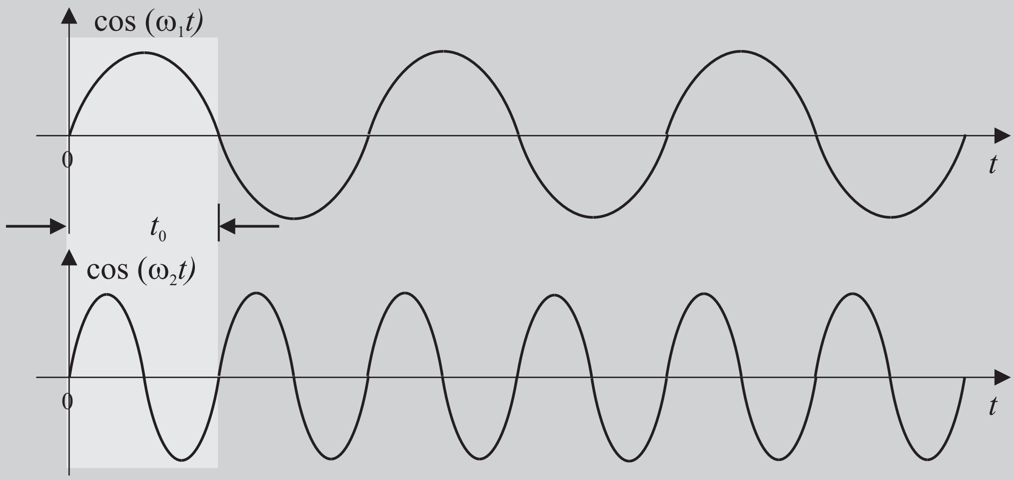
□

Sidebar 16 Physical Explanation of Linear Phase

Time delay in a signal causes a linear phase shift in its spectrum. This result can also be derived by *heuristic reasoning*. Imagine a signal $x(t)$ being synthesised by its Fourier components, which are sinusoids of certain amplitudes and phases. The delayed signal $x(t - t_0)$ can also be synthesised by the same sinusoidal components, each delayed by t_0 seconds. The amplitudes of these components remain unchanged, and therefore the amplitude spectrum of $x(t - t_0)$ is identical to that of $x(t)$. However, the time delay in each sinusoid does change the phase of each component by an amount ωt_0 since:

$$\cos \omega \left(t - t_0\right) = \cos \left(\omega t - \omega t_0\right) \tag{6.61}$$

This phase shift, as compared with $\cos \omega t$, is a linear function of frequency ω, meaning that higher-frequency components must undergo proportionately higher phase shifts to achieve the same delay. This effect is shown in the figure below, with two sinusoids, the frequency of the lower one being twice the frequency of the upper.



The same time-delay amounts to a phase shift of π in the upper sinusoid, and a phase shift of 2π in the lower sinusoid. This verifies the fact that *to achieve the same time delay, higher-frequency sinusoids must undergo proportionately higher phase shifts*.

– End-of-Topic 34: **Scale-in-time and Shift-in-time: Properties of the Fourier Transform** –

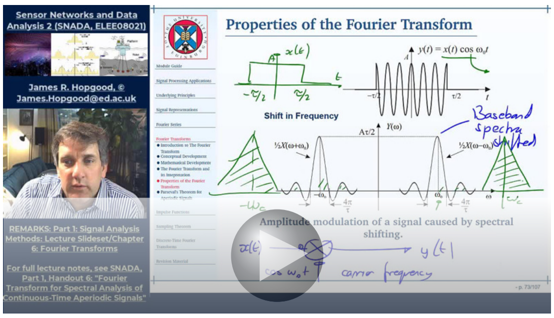
Topic Summary 35 Fourier Transform Properties: Shift-in-frequency theorem

Topic Objectives:

- Understand the Shift-in-Frequency Theorem.
- Aware of use of theorem in sampling theory and communications systems.
- Application to amplitude modulation techniques.

Topic Activities:

Type	Details	Duration	Progress
Watch video	11 : 12 min video	3× length	
Read Handout	Read page 219 to page 221	8 mins/page	



http://media.ed.ac.uk/media/1_gdyw8q2i

Video Summary: In this Topic, the shift-in-frequency theorem is investigated, and its importance in communication theory as well as sampling theory is highlighted. The duality between the shift-in-time and shift-in-frequency property is emphasised. The Topic considers the spectrum of a signal that is multiplied by a cosine wave, in terms of the spectrum of the original signal. Finally, the Topic highlights how this theorem is used in amplitude modulation, as the simplest communication system that enables multiple users to communicate simultaneously.

Summary Slide 36 Properties of the Fourier Transform

Shift in Frequency

If

$$x(t) \rightleftharpoons X(\omega) \tag{6.65}$$

then, for any real constant ω_0

$$x(t)\ e^{j\omega_0 t} \rightleftharpoons X(\omega - \omega_0) \tag{6.66}$$

The multiplication of a signal by a factor $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$.

Example: Amplitude Modulation

Modulation is used in communication systems to map a **baseband signal** to another part of the spectrum. Suppose a modulated signal is defined as:

$$y(t) = x(t)\ \cos(\omega_c t) \tag{6.67}$$

What is the spectrum of $y(t)$ in terms of (i. t. o.) $x(t)$?

Shift in Frequency The final property covered in this handout is the **frequency shift theorem**, which is fundamental for understanding the basics of communication schemes, as well as the sampling theorem in the next handout.

The frequency-shift theorem states that, if $x(t) \rightleftharpoons X(\omega)$, then for any real constant ω_0

$$x(t)\ e^{j\omega_0 t} \rightleftharpoons X(\omega - \omega_0) \tag{6.70}$$

This property means that the multiplication of a signal by a factor $e^{j\omega_0 t}$ shifts the spectrum of that signal by $\omega = \omega_0$. Note the duality between the time-shifting and the frequency shifting property. The proof of the frequency-shifting property is left as an exercise to the reader.

The most important application of the frequency-shift theorem is to calculate the spectrum of a signal when it is multiplied by a $\sin$ or $\cos$ term. This is necessary because $e^{j\omega_0 t}$ is a complex function and therefore cannot be generated in practice. To develop this result, notice that by setting $\omega_0 \to -\omega_0$ in Equation 6.70 gives:

$$x(t)\ e^{-j\omega_0 t} \rightleftharpoons X(\omega + \omega_0) \tag{6.71}$$

Hence, using the result that $\cos\omega_0 t = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$, it follows that:

$$x(t)\ \cos(\omega_0 t) \rightleftharpoons \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right] \tag{6.72}$$

This result shows that the multiplication of a signal $x(t)$ by a co-sinusoid of frequency ω_0 shifts the spectrum $X(\omega)$ by $\pm\omega_0$.

An example of application of this theorem is shown in Figure 6.13, where the rectangular pulse in Figure 6.10a and Figure 6.10b has been multiplied by a cosine function. This is known as **amplitude modulation** and is fundamental to amplitude modulated (AM) radio. Further discussion of AM will occur in the communications part of this course, and is discussed briefly in Sidebar 17.

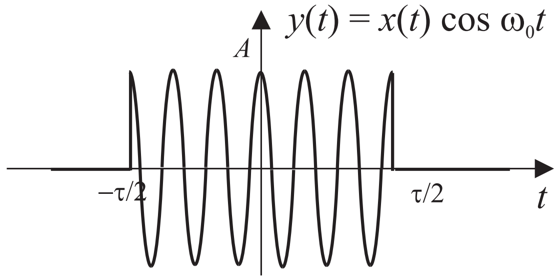
– End-of-Topic 35: **Shift-in-Frequency Theorem: Properties of the Fourier Transform** –

Sidebar 17 Amplitude Modulation

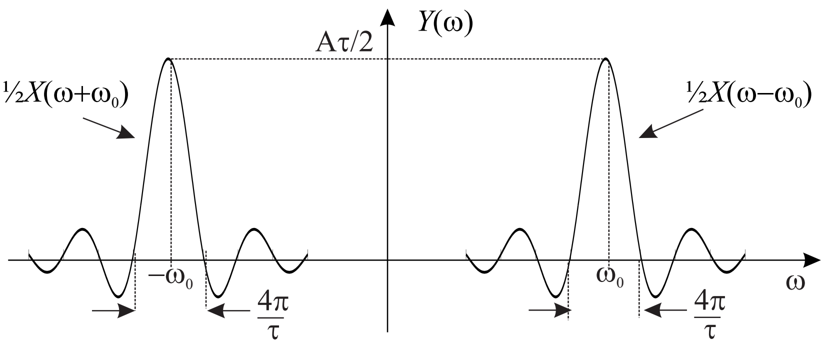
Modulation is used to shift signal spectra. This sidebar describes some situations that call for spectral shifting:

1. If several signals, all occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere; it is impossible to separate or retrieve the individual signals at the receiver. For example, if all analogue radio stations decide to broadcast audio signals simultaneously, a receiver cannot separate them. This problem is solved by using modulation, whereby each radio station is assigned a distinct **carrier frequency**. Each station transmits a modulated signal by multiplying the original signal by a cosine wave at the carrier frequency, using the result in Equation 6.72. This procedure shifts the signal spectrum to its allocated band, centered on the carrier frequency, which is not occupied by any other station. An analogue radio receiver can pick up any station by tuning to the band of the desired station. The receiver must now **demodulate** the received signal to undo the effect of modulation. Demodulation consists of another spectral shift required to restore the signal to its original band. The method of transmitting several signals simultaneous;y 0 over a channel by sharing its frequency band is known as frequency division multiplexing (FDM).

2. For effective radiation of power over a radio link, the antenna size must be of the order of the wavelength of the signal to be radiated. Audio signal frequencies are so low, with the wavelength so large, that impractically large antennas would be required for radiation. Hence, shifting the spectrum to a higher frequency, or a smaller wavelength, by modulation solves the problem.



(a) Rectangular pulse of width τ multiplied by $\cos\omega_0 t$.



(b) Resulting spectrum

Figure 6.13: Amplitude modulation of a signal caused by spectral shifting.

Topic Summary 36 Example of using Fourier Transform Properties

Topic Objectives:

- Example of solving a complex problem using the properties of the Fourier transform.

Topic Activities:

Type	Details	Duration	Progress
Read Handout	Read page 223 to page 225	8 mins/page	
Try Example	Try Example 6.3	30 mins	

Example 6.3 (Application of Fourier Transform Properties). This example follows the one given on Summary Slide 37, where it is required to find the Fourier transform of the signal shown in Figure 6.14. While this signal may seem arbitrary, the oscillating decaying nature is characteristic of a number of signals seen in nature, usually in response to a short large amplitude signal at the input of a system.

Using the properties of the Fourier transform, find the spectrum of the signal $x(t)$ shown in Figure 6.14.

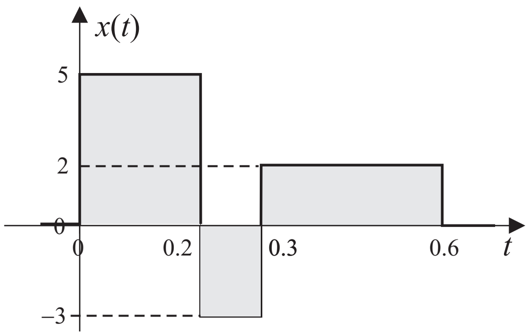


Figure 6.14: An example signal, that bears an approximation to a real signal that might appear at the output of a linear system.

SOLUTION. There are numerous ways of answering this question, including finding the transform from first principles, but this solution will show how to use a basic *atomic signal*, $x_a(t)$, as the starting point, from which the signal $x(t)$ is derived.

Let the *atomic building block* be that given by the *rectangular pulse*in Example 6.2 and shown in Figure 6.12 on page 217, in which the rectangular pulse $x_a(t) = y(t)$ is of width $\tau = 2$ and amplitude $A = {}^1\!/_2$.

The spectrum of $x_a(t)$ is given by Equation 6.64, with the parameters modified to give the simplified expression:

$$x_a(t) \rightleftharpoons X_a(\omega) = e^{-j\omega}\,\mathrm{sinc}(\omega) \tag{6.73}$$

The signal $x(t)$ consists of three pulses scaled in time, shifted in time, and scaled in amplitude by appropriate amounts. Note, however, that the amount of scaling-in-time and shifting depends on which operation is undertaken first; this issue is covered in more detail in Sidebar 18 on page 226, although generally scaling and then shifting is easier to visualise.

Hence, using linearity:

$$x(t) = x_1(t) + x_2(t) + x_3(t) \quad \Rightarrow \quad X(\omega) = X_1(\omega) + X_2(\omega) + X_3(\omega) \tag{6.74}$$

Recall that $x_a(t)$ is of width 2 and height 0.5. Hence, to get the three individual pulses, note:

1. The first pulse $x_1(t)$ is obtained by scaling in amplitude by $\alpha = {}^5\!/_{0.5} = 10$, and stretching $x_a(t)$ in time by a factor of $a = {}^2\!/_{0.2}$ which amounts to a scaling factor of or $a = 10$:

   $$x_1(t) = 10x_a(10t) \tag{6.75}$$

   so that when $t = \frac{1}{5}$, $x\left(\frac{1}{5}\right) = 10x_a(2)$ corresponding to the right edge of the pulse.

   Using the Fourier transform relationship that if $x(t) \rightleftharpoons X(\omega)$ then $x(at) \rightleftharpoons \frac{1}{|a|}X\left(\frac{\omega}{a}\right)$, it follows:

   $$X_1(\omega) = 10 \times \frac{1}{10} \times e^{-j\frac{\omega}{10}}\,\mathrm{sinc}\left(\frac{\omega}{10}\right) \tag{6.76}$$

2. Similarly, the second pulse $x_2(t)$ is obtained by scaling $x_a(t)$ in time by $a = {}^{2}/_{0.1} = 20$, then shifting to the right by $t_0 = {}^{1}/_{5}$, and finally scaling in amplitude by ${}^{-3}/_{0.5} = -6$.

   Noting that if $x(t) \rightleftharpoons X(\omega)$, then $x(t - t_0) \rightleftharpoons e^{-j\omega t_0} X(\omega)$, it then follows:

$$X_2(\omega) = -6 \times \frac{1}{20} \times e^{-j\frac{\omega}{5}} \times e^{-j\frac{\omega}{20}} \operatorname{sinc}\left(\frac{\omega}{20}\right) \tag{6.77}$$

3. Finally, setting the third pulse $x_3(t)$ is obtained by setting the scaling-in-time term to $a = \frac{2}{0.3} = {}^{20}/_{3}$, shifting by $t_0 = \frac{3}{10}$, and scaling in amplitude by ${}^{2}/_{0.5} = 4$, giving:

$$X_3(\omega) = 4 \times \frac{3}{20} \times e^{-j\frac{3\omega}{10}} \times e^{-j\frac{3\omega}{20}} \operatorname{sinc}\left(\frac{3\omega}{20}\right) \tag{6.78}$$

Simplifying slightly, this gives the final result:

$$X(\omega) = e^{-j\frac{\omega}{10}} \operatorname{sinc}\left(\frac{\omega}{10}\right) - \frac{3}{10} e^{-j\frac{\omega}{4}} \operatorname{sinc}\left(\frac{\omega}{20}\right) + \frac{3}{5} e^{-j\frac{9\omega}{20}} \operatorname{sinc}\left(\frac{3\omega}{20}\right) \tag{6.79}$$

This result is as would be expected by direct application of the result in Equation 6.64 with appropriately chosen A and τ and using the shift-in-time theorem.

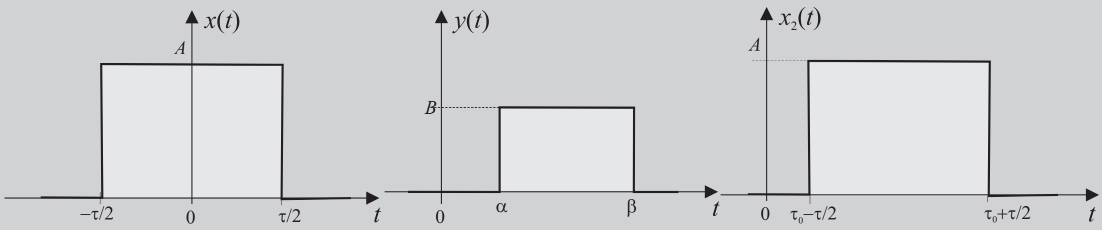
Table 6.1: Parameters for the Fourier Transform

Parameter	$x_1(t)$	$x_2(t)$	$x_3(t)$
Scale in amplitude, α	${}^{5}/_{0.5} = 10$	${}^{-3}/_{0.5} = -6$	${}^{2}/_{0.5} = 4$
Scale in time factor, a	${}^{2}/_{0.2} = 10$	${}^{2}/_{0.1} = 20$	${}^{2}/_{0.3} = {}^{20}/_{3}$
Shift in time value, t_0	0.1	0.25	0.45

$\square$

– End-of-Topic 36: **Application of Fourier Transform Properties** –

Sidebar 18 Order of Scaling and Shifting



Different amounts of *shifting* are required depending on the order in which the scaling and shifting are applied. Consider scaling and shifting (bth in time) the rectangular pulse $x(t)$ shown above in order to obtain the signal $y(t)$, also shown.

Scaling then Shifting Suppose $x(t)$ is first scaled by an amount a to obtain the signal $x_1(t)$, and then $x_1(t)$ is shifted by t_0 to obtain $y(t)$. Then,

$$y(t) = \frac{B}{A}x_1(t - t_0) = \frac{B}{A}x(a\,[t - t_0]) \tag{6.80}$$

Note that scaling in amplitude does not depend on the order of scaling and shifting.

Shifting then Scaling If $x(t)$ is first shifted by τ_0 giving $x_2(t)$, which is then scaled, the amount of shift needed changes. Let $x_2(t) = x(t - \tau_0)$, then:

$$y(t) = \frac{B}{A}x_2(at) = \frac{B}{A}x(at - \tau_0) \tag{6.81}$$

Comparing Equation 6.80 and Equation 6.81, it is seen the scaling factors are the same in each case and the shifts are related by $\tau_0 = at_0$.

Application to the rectangular pulse To reinforce this theory, consider the pulses $x(t)$ and $y(t)$ above. The width of $y(t)$ is $\beta - \alpha$, while the width of $x(t)$ is τ. Consider scaling then shifting, the required scaling is:

$$a = \frac{\tau}{\beta - \alpha} \qquad \Rightarrow \qquad x_1(t) = x\left(t\left[\frac{\tau}{\beta - \alpha}\right]\right) \tag{6.82}$$

such that when $t = \pm\frac{\beta-\alpha}{2}$, $x_1(t) = x(\pm{}^{\tau}\!/_{2})$ as expected, since evaluating the scaled pulse $x_1(t)$ at the extremes should give $x(t)$ at the extremes. The pulse, $x_1(t)$, centered on zero, is then shifted so the center of $x_1(t)$ coincides with the center of $y(t)$ giving $t_0 = \frac{\alpha+\beta}{2}$.

Suppose instead that $x(t)$ is first shifted by τ_0 to give $x_2(t)$ above. The width of $x_2(t)$ is still τ, and using the result $\tau_0 = at_0$, then:

$$\tau_0 = \frac{\tau}{2}\left[\frac{\beta + \alpha}{\beta - \alpha}\right] \tag{6.83}$$

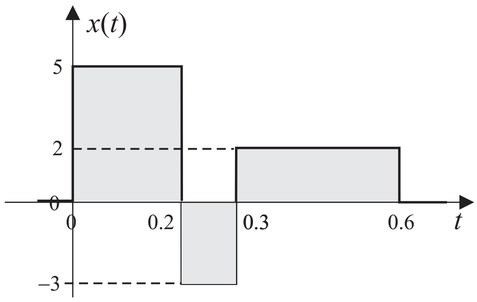
Summary Slide 37 Properties of the Fourier Transform

Linearity

The Fourier transform is linear, that is if $x_1(t) \rightleftharpoons X_1(\omega)$ and $x_2(t) \rightleftharpoons X_2(\omega)$, then:

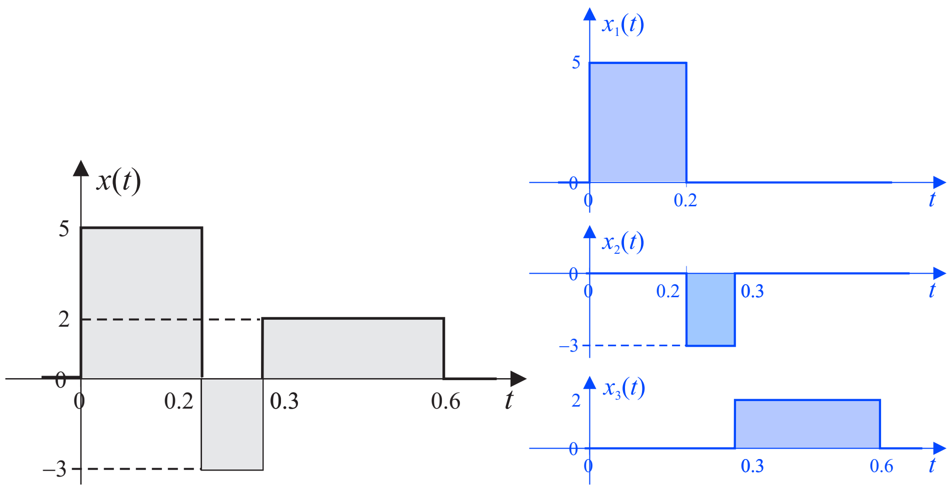
$$a_1\, x_1(t) + a_2\, x_2(t) \rightleftharpoons a_1\, X_1(\omega) + a_2\, X_2(\omega)$$

Example of using these Properties

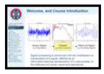


Summary Slide 38 Example of using Properties of the Fourier Transform

Linearity Example Continued



6.4 Parseval's Theorem for Aperiodic Signals



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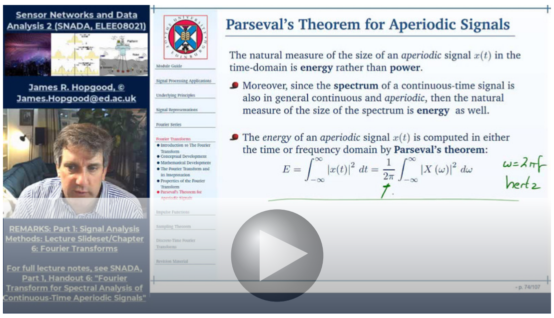
Topic Summary 37 Fourier Transform Properties: Parseval's Theorem for non-periodic Signals

Topic Objectives:

- Measuring the size of a signal in the frequency domain.
- Equivalence of energy calculation in time and frequency domains.
- Proof of Parseval's theorem.

Topic Activities:

Type	Details	Duration	Progress
Watch video	14 : 10 min video	3× length	
Read Handout	Read page 229 to page 230	8 mins/page	
Practice Exercises	Exercises 6.5, 6.6, and 6.7	30 mins	



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Video Summary: This Topic revisits measuring the size of a signal, but rather than doing this in the time-domain, it is now considered in the frequency-domain. By considering the natural size of a signal's spectrum, it is shown that energy in the time-domain is equal to the energy in the frequency-domain. This has applications for analysing signals in both domains, but also dealing with the case where the theoretical calculation of energy in the one-domain is difficult, but is simpler in the other-domain. Using this method, it is possible to create a number of useful identities, as shown in the associated self-study questions.

The natural measure of the size of an *aperiodic* signal $x(t)$ in the time-domain is **energy** rather than **power**, the latter of which was used to model periodic signals on page 174. Moreover, since the **spectrum** of a continuous-time signal is also in general continuous and *aperiodic*, then the natural measure of the size of the spectrum is **energy** as well. Hence, extending Parseval's theorem for **Fourier series** on page 174 indicates that the energy in the time-domain should equal the energy in the frequency-domain.

The *energy* of an *aperiodic* signal $x(t)$ is computed in either the time or frequency domain by **Parseval's theorem**:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \tag{6.84}$$

The function $|X(\omega)|^2 \geq 0$ shows the distribution of energy of $x(t)$ as a function of frequency, ω, and is called the **energy spectrum** of $x(t)$.

PROOF. The derivation of Parseval's theorem for Fourier transforms follows a similar line to the derivation of Parseval's theorem for Fourier series; it proceeds as follows:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} x(t)\; x^{\star}(t)\; dt = \int_{-\infty}^{\infty} x(t)\; \frac{1}{2\pi} \int_{-\infty}^{\infty} X^{\star}(\omega)\; e^{-j\omega t}\, d\omega\; dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^{\star}(\omega) \int_{-\infty}^{\infty} x(t)\; e^{-j\omega t}\; dt\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_c^{\star}(\omega)\, X(\omega)\; d\omega \end{aligned} \tag{6.85}$$

$\square$

– End-of-Topic 37: **Parseval's Theorem for Non-periodic Signals** –

Summary Slide 39 The Fourier Transform

Parseval's Theorem Revisited

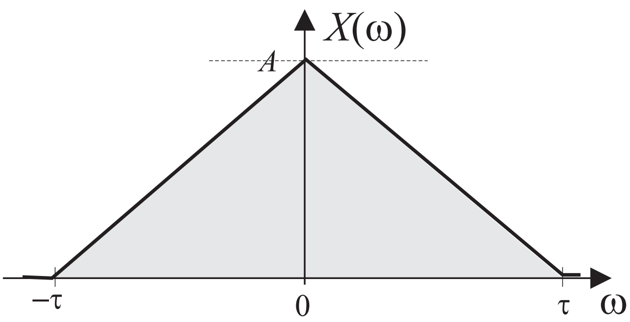
The **energy** in a signal $x(t)$ is given by:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 \, dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \end{aligned} \tag{6.86}$$

- In words: $\frac{|X(\omega)|^2}{2\pi}$ defines how the energy in the signal is distributed with frequency.
- Thus, $\frac{|X(\omega)|^2}{2\pi}$ is known as the energy spectral density (ESD).

Example

A signal $x(t)$ has Fourier transform shown in the figure below. Calculate the energy of the signal.



The time-domain signal $x(t)$ is actually the square of a sinc function, and therefore attempting to calculate the integral $\int \mathrm{sinc}^4 t \, dt$ is hard.

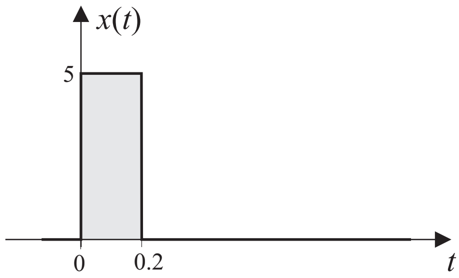


Figure 6.15: Rectangular pulse.

6.5 Tutorial Exercises

Exercise 6.1 (Fourier Transform). [*Difficulty:* **3** ($\star\star\star$)] Develop an expression for the Fourier transform of the rectangular pulse shown in Figure 6.15, which is defined by the equation:

$$x(t) = \begin{cases} 5 & \text{if } 0 \leq t < 0.2 \\ 0 & \text{otherwise} \end{cases} \tag{6.87}$$

Plot the magnitude and phase of the Fourier transform as a function of frequency ω.

HINTS. The following manipulation might be useful; frequently, the term $1 - e^{-j\theta}$, for some value of θ, might occur in an algebraic manipulation, and can be more usefully written as:

$$1 - e^{-j\theta} = e^{-j\frac{\theta}{2}} \left(e^{j\frac{\theta}{2}} - e^{-j\frac{\theta}{2}} \right) = 2j\, e^{-j\frac{\theta}{2}} \sin\left(\frac{\theta}{2}\right) \tag{6.88}$$

□

Final answer: The Fourier transform can be written as:

$$X(\omega) = e^{-\frac{j\omega}{10}} \operatorname{sinc}\left(\frac{\omega}{10}\right) \tag{6.89}$$

□

Exercise 6.2 (Fourier Transforms: More difficult example). [*Difficulty:* **3** ($\star\star\star$)] Calculate the Fourier transform of the half-sinewave pulse shown in Figure 6.16.

Final answer:

$$X(\omega) = \frac{1}{2} e^{-j\frac{\omega}{2}} \left[\operatorname{sinc}\left(\frac{\pi - \omega}{2}\right) + \operatorname{sinc}\left(\frac{\pi + \omega}{2}\right) \right] \tag{6.90}$$

□

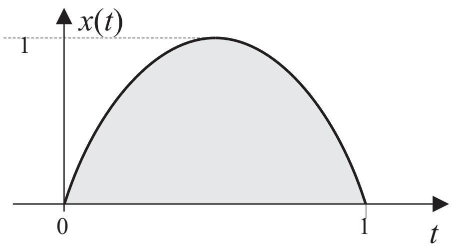


Figure 6.16: Sinewave pulse.

Exercise 6.3 (Properties of Fourier Transform). [*Difficulty:* **2** ($\star\star$)] Using the definition of the Fourier transform, prove the **scale in time** property of the Fourier transform, as well as the **frequency-shift property**. Thus, if $x(t) \rightleftharpoons X(\omega)$, then for any real constants a and ω_0:

$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \tag{6.91}$$

$$x(t)\ e^{j\omega_0 t} \rightleftharpoons X(\omega - \omega_0) \tag{6.92}$$

⋈

Exercise 6.4 (Fourier Transform Properties). [*Difficulty:* **3** ($\star\star\star$)] A function $f(t)$ has a Fourier transform $F(\omega)$ which equals $1 + \cos\omega$ over the range $-\pi < \omega \leq \pi$ and is zero elsewhere.

1. Calculate the function $f(t)$.
2. Find the Fourier transform of $f(t)$ when it is first delayed by τ and then scaled in time so that it lasts twice as long.

Final answer: 1.

$$f(t) = \frac{1}{1-t^2} \frac{\sin \pi t}{\pi t} \tag{6.93}$$

2.

$$\hat{F}(\omega) = \begin{cases} 2\,(1 + \cos 2\omega)\, e^{-2j\omega\tau} & -\frac{\pi}{2} < \omega \leq \frac{\pi}{2} \\ 0 & \text{elsewhere} \end{cases} \tag{6.94}$$

□

Exercise 6.5 (Parseval's Theorem). [*Difficulty:* **4** ($\star\star\star\star$)] Use Parseval's theorem to show that

$$\int_{-\infty}^{\infty} \operatorname{sinc}^2(kx)\ dx = \frac{\pi}{k} \tag{6.95}$$

⋈

Exercise 6.6 (Energy). [*Difficulty:* **4** ($\star\star\star\star$)]

1. Find the Fourier transform of the signal

$$f(t) = e^{-\alpha|t|},\ t \in \mathbb{R} \tag{6.96}$$

assuming that $\alpha > 0$.

2. For the signal:

$$x(t) = \frac{2a}{t^2 + a^2} \tag{6.97}$$

determine the essential bandwidth B (in hertz) of $x(t)$ such that the energy contained in the spectral components of $x(t)$ of frequencies below B Hz is 99% of the signal energy E_x.

Final answer: 1.

$$F(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2} \tag{6.98}$$

2.

$$B = \frac{1}{4\pi\alpha}\ln 100 \approx \frac{0.367}{\alpha} \tag{6.99}$$

□

Exercise 6.7 (Fourier Transform and Energy). [*Difficulty:* **3** ($\star\star\star$)] The Gaussian pulse signal is given by

$$x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}} \tag{6.100}$$

1. Find the Fourier transform $X(\omega)$ of the Gaussian pulse $x(t)$.
2. Show that the energy of the Gaussian pulse is given by

$$E_x = \frac{1}{2\sigma\sqrt{\pi}} \tag{6.101}$$

3. Verify the result in part 2 by using Parseval's theorem to derive the energy E_x from the spectrum $X(\omega)$.

HINTS. You may use the idenfity:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\, dt = \sqrt{2\pi} \tag{6.102}$$

□

7

Impulses and Other Fundamental Signals

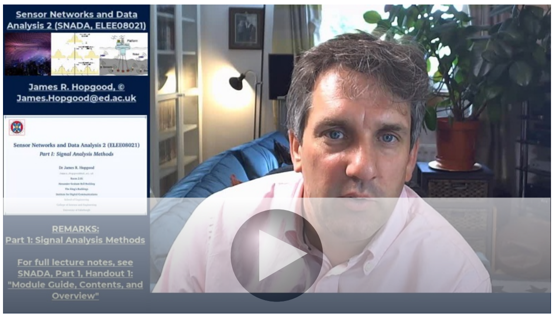
Great acts are done by a series of small deeds.

Lao Tzu

This handout covers the all important impulse function or Dirac Delta function, and its uses. The handout studies important results involving impulses, and its use in Fourier analysis.

7.1 Mid-course recap

At this point in the course, it is useful to reflect on what we have looked at, and where we are heading.



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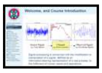
Video Summary: This video provides a recap of what this course is considering, why it considered particular techniques, and what is coming up in terms of the signal analysis. The video describes the importance of spectral analysis techniques, such as Fourier series and Fourier transforms, as well as key conceptual properties of those techniques. The video then introduces the notion of the sampling process for moving between the continuous-time and discrete-time world.

7.2 Some Simple Signals



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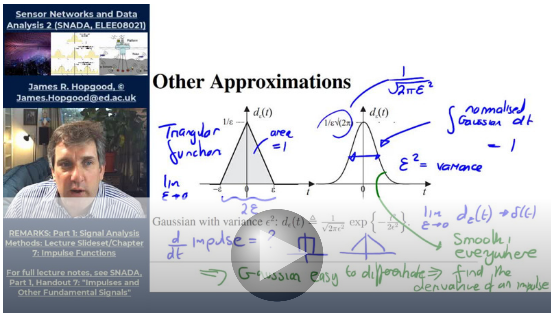
Topic Summary 38 Step and Impulse Functions, and their Relationship

Topic Objectives:

- Introduce step functions and their applications.
- Consider the derivative of the step function.
- Formalise the idea of the Dirac delta impulse function.
- Formalise the relationship between step and impulse functions.

Topic Activities:

Type	Details	Duration	Progress
Watch video	19 : 45 min video	$3\times$ length	
Read Handout	Read page 237 to page 243	8 mins/page	



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Video Summary: This Topic starts this Chapter on the theoretical glue that is needed to fully understand Nyquist sampling theory, that follows in the next Chapter. Sampling theory requires the understanding of the Dirac delta impulse function. This Topic introduces this impulse function by first considering the unit step function. The application of the step signal for representing causal functions is first presented. The gradient of the step function is then considered, and used to motivate the development of the impulse function. The relationship between the step and impulse functions is explained.

There are some simple signals that are extremely important in the analysis of signals and systems. The section covers the signals that will be relevant to the rest of this course.

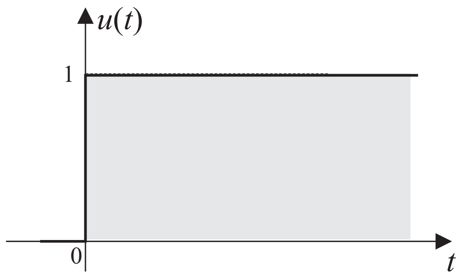
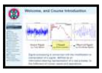


Figure 7.1: The unit step function, $u(t)$.



7.2.1 The Step Function

New slide

The step function essentially represents the result of turning on a *switch*. The unit step function, $u(t)$, is simply defined as:

$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{7.1}$$

The unit step function is also known as the **Heaviside step function**, and notice that there is a discontinuity at $t = 0$. The function is graphically represented in Figure 7.1.

Note that any causal function can be conveniently described in terms of the unit step by a simple multiplication. A causal signal, $x(t)$, can be described by the form:

$$x(t) = \begin{cases} x_+(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{7.2}$$

where $x_+(t)$ represents some function that can be defined for all values of time. Therefore, using the unit step function, $x(t) = x_+(t)\, u(t)$. Hence, the one-sided exponential-decay can be written as $x(t) = e^{-at}\, u(t)$.

The unit step function also proves very useful in specifying a different mathematical function over different intervals. Consider first the function shown in Figure 7.2a. This can be expressed as the sum of two delayed and weighted unit step functions. Hence, this rectangular pulse can be represented as:

$$x(t) = u(t-a) - u(t-b) \tag{7.3}$$

7.2.2 Dirac Delta Impulse

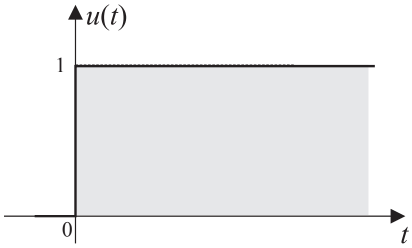
New slide

The unit impulse function $\delta(t)$ is one of the most important functions in the study of signals and systems. In its simplest terms, the unit impulse can be considered as the derivative of the unit step function, or alternatively, the unit step function is the integral of the impulse function over all time:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\; d\tau \tag{7.5}$$

Summary Slide 40 Basic Building Block Signals

Heaviside Step Function



The unit step, or "*switch signal*", has previously been defined as:

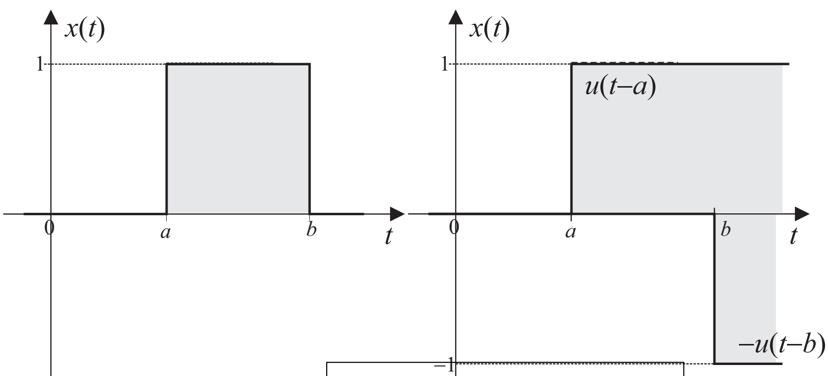
$$u(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \tag{7.4}$$

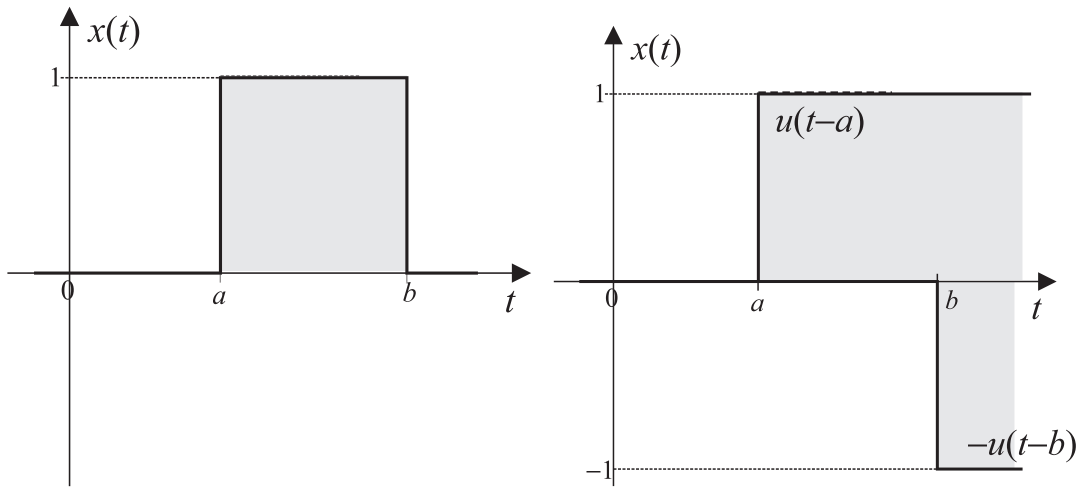
Useful for describing causal signals:

$$x(t) = \begin{cases} x_+(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad \text{or} \qquad x(t) = x_+(t)\ u(t)$$

Piecewise-Varying Functions

$$x(t) = u(t-a) - u(t-b)$$





(a) Rectangular Pulse (b) Difference of two step functions

Figure 7.2: Representation of a rectangular pulse by step functions.

This follows because the derivative of the step function is zero everywhere, except at $t = 0$, where there is a discontinuity. The derivative of the step function at $t = 0$ *might* be considered as being infinite, whereas really it is simply undefined.

The impulse was first defined by Dirac, and is hence sometimes known as the **Dirac delta function**, and is given by:

$$\delta(t) = 0 \qquad \text{if } t \neq 0 \tag{7.6a}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{7.6b}$$

An approximation to the impulse can be visualised as a tall, narrow, rectangular pulse of unit area, as illustrated in Figure 7.3a. The rectangular pulse is defined by $d_\epsilon(t)$:

$$d_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{if } |t| < \frac{\epsilon}{2} \\ 0 & \text{otherwise} \end{cases} \tag{7.7}$$

In other-words, $d_\epsilon(t)$ is a rectangular pulse of width ϵ and height $\frac{1}{\epsilon}$ such that its total area is unity. The impulse can be considered as the limiting case of $d_\epsilon(t)$ as $\epsilon \to \infty$, in which case $d_\epsilon(t)$ has become infinitesimally narrow but of infinitely large height, but with an overall area that has been maintained at unity. The impulse is generally denoted by the spear-like function shown in Figure 7.3b.

The unit impulse is useful for *sampling a signal*, and also generating an input that lasts for an infinitely short period of time, but is non-zero at the point the impulse occurs, which is why the unit-area property in Equation 7.6b is so important.

7.2.2.1 Relationship of Impulse and Step Functions

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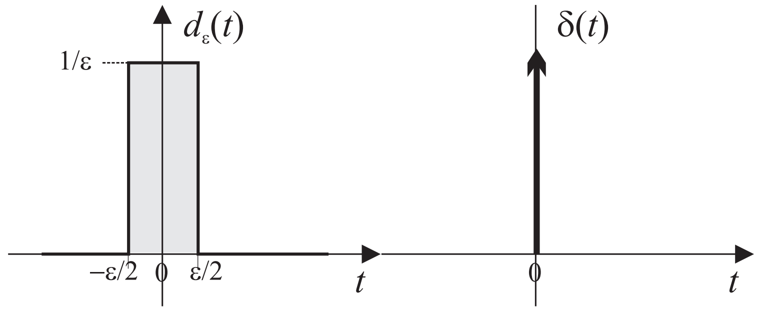
It should be apparent, from the discussion above, that the impulse function is the derivative of the step function. Similarly, the step function is the integral of the

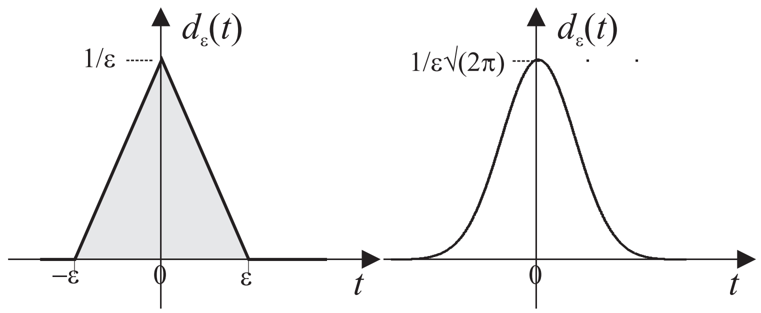
Summary Slide 41 Basic Building Block Signals

The Unit Impulse Function

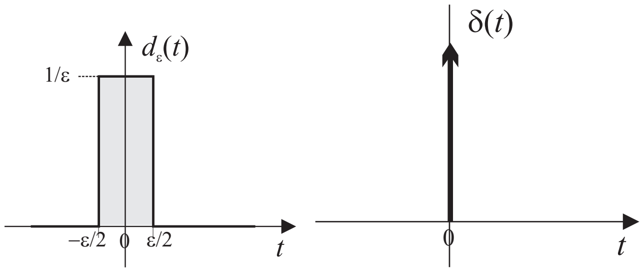


$$\delta(t) = 0 \qquad \text{if } t \neq 0$$
$$\int_{-\infty}^{\infty} \delta(t)\ dt = 1$$

Other Approximations



Gaussian with variance ϵ^2: $d_\epsilon(t) \triangleq \frac{1}{\sqrt{2\pi\epsilon^2}} \exp\left\{-\frac{t^2}{2\epsilon^2}\right\}$.



(a) An approximation to a unit impulse.

(b) Graphical notation for unit impulse.

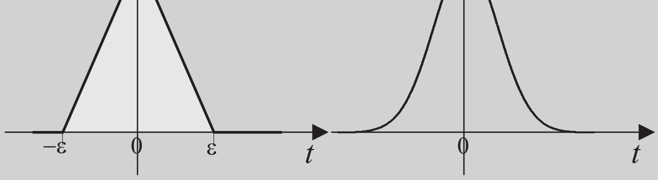
Figure 7.3: The unit impulse function.

Sidebar 19 Other Impulse Approximations

The rectangular approximation to an impulse is not the only approximation that can be used, and others are actually much better. For example, the triangular and Gaussian pulses shown below. In each case, the area under the curves is equal to one, and as $\epsilon \to 0$, both functions become taller and narrower. The Gaussian function is given by:

$$d_\epsilon(t) = \frac{1}{\epsilon\sqrt{2\pi}} \exp\left(-\frac{-t^2}{2\epsilon^2}\right) \tag{7.9}$$

The Gaussian approximation is useful because it is differentiable everywhere, whereas the rectangular approximation has discontinuities. Because the Gaussian function is differentiable, it is possible to define the derivative of an impulse as the limiting case of the derivative of a Gaussian! Which is quite cool!



impulse:

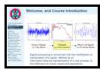
$$\delta(t) = \frac{du(t)}{dt} \tag{7.10}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\ d\tau \tag{7.11}$$

Note the variable of integration is essentially a dummy variable, and so it is important to distinguish between t and τ.

– End-of-Topic 38: **Key signals: the continuous-time step function, the Dirac-delta impulse function, and their relationships –**

7.2.2.2 Multiplication of a function by an impulse



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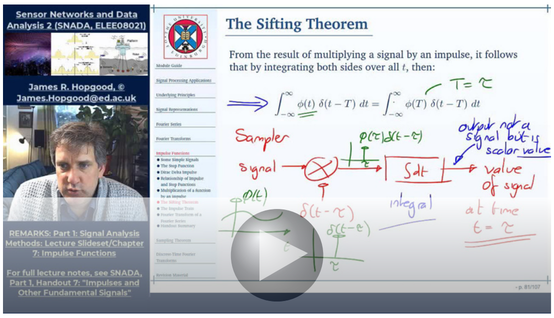
Topic Summary 39 Multiplying by an Impulse, and the Sifting Theorem

Topic Objectives:

- The result of multiplying a signal by an impulse.
- The concept of a sampler for extracting signal values.
- The sifting theorem.

Topic Activities:

Type	Details	Duration	Progress
Watch video	11 : 17 min video	3× length	
Read Handout	Read page 244 to page 245	8 mins/page	



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Video Summary: This Topic continues discussing important signals that are needed to complete the mathematical framework of Fourier analysis and sampling theory. After revising the limiting operation to obtain an impulse, this Topic considers the result of multiplying a signal by an impulse. This product produces another impulse, but where the weight is the signal value. The Topic the discusses the concept of a signal sampler in terms of multiplying by an impulse and integrating, and then extends the result by introducing the infamous sifting theorem. These important concepts are the basis of Nyquist sampling theory.

It is important to consider what happens when a function $\phi(t)$ is multiplied by the unit impulse $\delta(t)$. It is assumed that the function $\phi(t)$ is continuous at $t = 0$. Since the impulse $\delta(t)$ has non-zero value only at $t = 0$, and the value of $\phi(t)$ at $t = 0$ is $\phi(0)$, then:

$$\phi(t)\ \delta(t) = \phi(0)\ \delta(t) \tag{7.12}$$

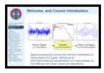
Thus, multiplication of a continuous-time function $\phi(t)$ with an unit impulse located at $t = 0$ results in an impulse, also located at $t = 0$, and has strength or weight $\phi(0)$

(as opposed to having unit weight for the unit impulse).

Use of the same argument leads to a generalisation of this result, stating that provided $\phi(t)$ is continuous at $t = T$, then if $\phi(t)$ is multiplied by the unit impulse centered on T, $\delta(t - T)$, this results in an impulse centered on T but with weight $\phi(T)$:

$$\phi(t)\ \delta(t - T) = \phi(T)\ \delta(t - T) \tag{7.13}$$

7.2.2.3 The Sifting Theorem



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From the result of multiplying a signal by an impulse, as explained in Equation 7.13, it follows that by integrating both sides over all t, then:

$$\int_{-\infty}^{\infty} \phi(t)\ \delta(t - T)\ dt = \int_{-\infty}^{\infty} \phi(T)\ \delta(t - T)\ dt \tag{7.14a}$$

$$= \phi(T) \underbrace{\int_{-\infty}^{\infty} \delta(t - T)\ dt}_{=1} \tag{7.14b}$$

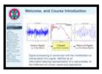
which yields the **sifting theorem**

$$\int_{-\infty}^{\infty} \phi(t)\ \delta(t - T)\ dt = \phi(T) \tag{7.15}$$

This means that the *area under the product of a function with an impulse is equal to the value of that function at the instance at which the impulse is located.* [1]

– End-of-Topic 39: **Manipulating signals using impulse functions** –

7.2.2.4 The Impulse Train



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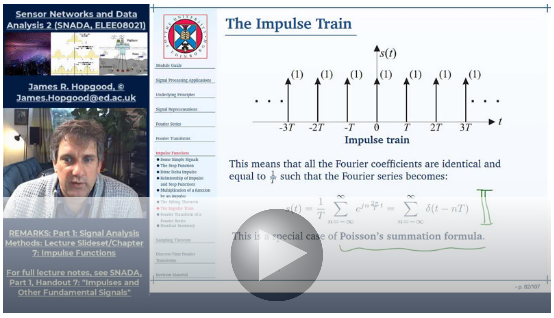
Topic Summary 40 The Impulse Train and its Fourier Spectrum

Topic Objectives:

- Introducing the Impulse Train for regular Signal Sampling.
- The Fourier Series of an Impulse Train.
- The Fourier Spectrum of an Impulse Train.
- Deriving Poisson's Summation Formula.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 11 min video	$3\times$ length	
Read Handout	Read page 246 to page 250	8 mins/page	



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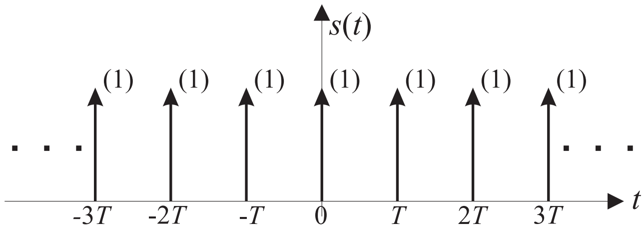
Video Summary: This Topic introduces the concept of the impulse train, a sequence of Dirac delta impulses, which can be used to perform regular sampling of a signal. The impulse train is fundamental to Nyquist sampling theory, and has a number of powerful and interesting properties. First, it is a periodic signal, and therefore can be analysed using the Fourier series. Second, the complex Fourier coefficients are all constant value, therefore effectively having a flat spectrum. This leads to the infamous Poisson's summation formula. Consequently, it can be shown that the Fourier transform of an impulse train is itself an impulse train, and therefore an eigenfunction of the Fourier transform operator.

The impulse train is a periodic signal with impulses centered at integer multiples of a **sampling period**, T, as shown in Figure 10.1.

[1]In this derivation, it is assumed that the function $\phi(t)$ is continuous at the instant where the impulse is located.

Summary Slide 42 Basic Signal Building blocks

Impulse Train



Its functional form is given by:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{7.16}$$

What is its Fourier series?

Summary Slide 43 Basic Building Block Signals

Poisson's Summation Forumla

$$\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi}{T}t} = \sum_{n=-\infty}^{\infty} \delta(t-nT) \tag{7.19}$$

Multiplication by an impulse

Since $\delta(t-T)$ has non-zero value only at $t = T$, and the value of $\phi(t)$ at $t = T$ is $\phi(T)$, then:

$$\phi(t)\ \delta(t-T) = \phi(T)\ \delta(t-T)$$

The Sifting Theorem

Integrating both sides over all t, then:

$$\int_{-\infty}^{\infty} \phi(t)\ \delta(t-T)\ dt = \int_{-\infty}^{\infty} \phi(T)\ \delta(t-T)\ dt \tag{7.20}$$

$$= \phi(T) \underbrace{\int_{-\infty}^{\infty} \delta(t-T)\ dt}_{=1} \tag{7.21}$$

$$\int_{-\infty}^{\infty} \phi(t)\ \delta(t-T)\ dt = \phi(T) \tag{7.22}$$

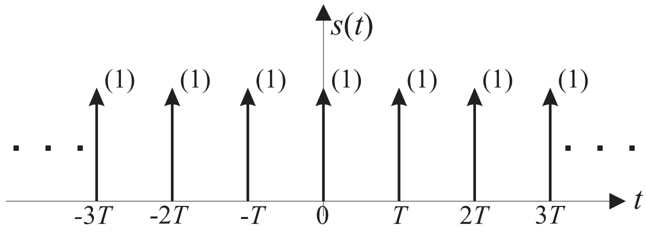


Figure 7.4: Impulse train

Its functional form is given by:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{7.23}$$

Note that the impulses have unit weight, and that impulses shouldn't be considered to have amplitudes. Note that the period of the waveform is T, and therefore Equation 7.23 can be written in the alternative form:

$$s(t) = \begin{cases} \delta(t) & -\frac{T}{2} < t < \frac{T}{2} \\ s(t - mT) & \text{for any integer m} \end{cases} \tag{7.24}$$

This function will be extremely useful in the next handout when developing the **sampling theorem**. However, as will be seen, the Fourier series of the impulse train proves to be invaluable. Therefore, consider writing Equation 7.23 by a Fourier series given as:

$$s(t) = \sum_{n=-\infty}^{\infty} S_n \, e^{jn\omega_0 t} \tag{7.25}$$

For the purposes of calculating the Fourier coefficients, it is preferable to choose limits that do not coincide with any points of discontinuity. Therefore, the limits are chosen to be $-\frac{T}{2}$ to $\frac{T}{2}$, and therefore the Fourier coefficients are given by:

$$S_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) \; e^{-jn\omega_0 t} \, dt \tag{7.26}$$

Using the **sifting theorem** in Equation 7.15, then the delta function is centered on $t = 0$ and therefore:

$$S_n = \frac{1}{T} \left[e^{-jn\omega_0 t} \right]_{t=0} = \frac{1}{T} \tag{7.27}$$

Sidebar 20 Fourier Series of an Impulse Train

An alternative approach to deriving the Fourier series of the impulse train is to consider a limiting case of the pulse wave on page 161 with $\tau = \epsilon$ and $A = \frac{1}{\epsilon}$. The Fourier series coefficients for this example were given on page 165:

$$X_n = \frac{A\tau}{T} \operatorname{sinc}\left(\frac{n\omega_0 \tau}{2}\right) \tag{7.29}$$

Note that the zero crossings occur at $n = m\frac{T}{\tau}$. Hence, as $\tau = \epsilon \to 0$, the rectangular pulses tend to impulses as described by the limiting in Section 7.2.2 on page 238, $A\tau \to 1$ in the Fourier coefficient expression on page 165, and also $\operatorname{sinc}\left(\frac{n\pi\tau}{T}\right) \to 1$. Hence, $X_n \to S_n = \frac{1}{T}$. Hence, the derivation of the Fourier series is entirely consistent using either derivation of the coefficients.

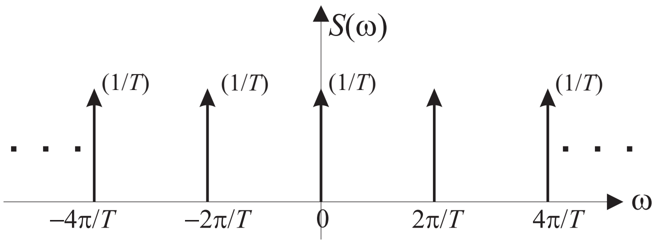


Figure 7.5: Spectrum of an Impulse train.

KEYPOINT! (Fourier Series of an Impulse Train). This means that all the Fourier coefficients are identical and equal to $\frac{1}{T}$ such that the Fourier series in Equation 7.25 becomes:

$$s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\frac{2\pi}{T}t} = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{7.28}$$

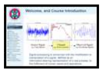
$\square$

This is a special case of **Poisson's summation formula**, and relates an impulse train to an infinite summation of complex exponentials with frequencies which are integer multiples of a fundamental.

Note that if the Fourier coefficients were plotted against frequency, then the fundamental is at $\omega_0 = \frac{2\pi}{T}$, so that a plot of the *spectral content* against frequency ω is also an impulse train as shown in Figure 7.5.

– End-of-Topic 40: **Key Signals: The impulse train and its spectrum –**

7.3 Fourier Transform of a Fourier Series



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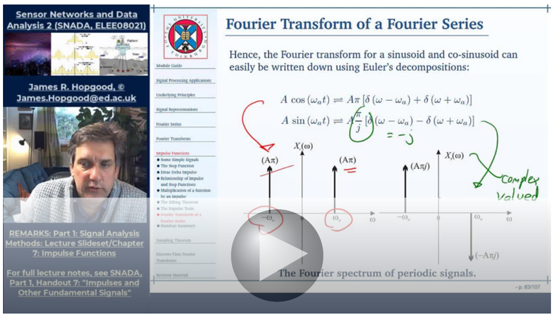
Topic Summary 41 Fourier Transform of a Fourier Series

Topic Objectives:

- The Fourier transform of a complex phasor.
- The Fourier transform of sinusoids and co-sinusoids.
- The Fourier transform of a Fourier Series.
- Key results and summary of Fourier analysis.

Topic Activities:

Type	Details	Duration	Progress
Watch video	$14:04$ min video	$3\times$ length	
Read Handout	Read page 251 to page 257	8 mins/page	

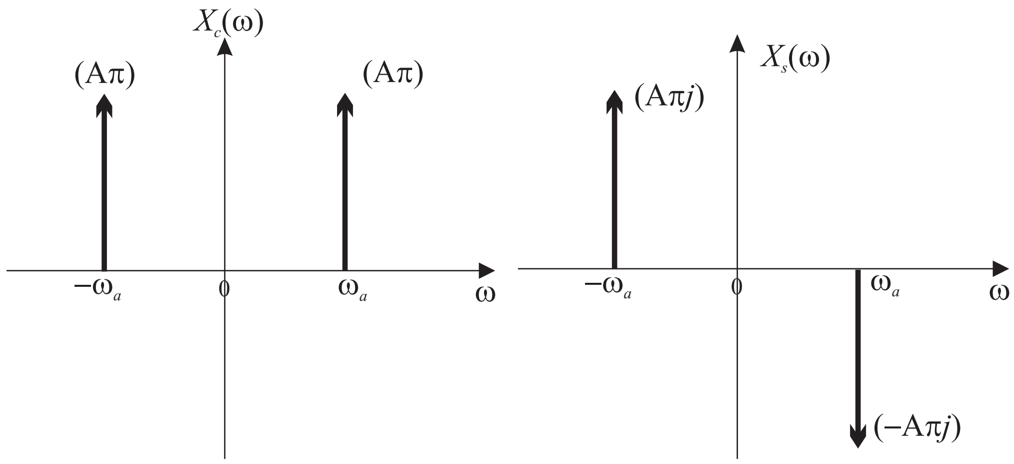


http://media.ed.ac.uk/media/1_gyv6ul46

Video Summary: This Topic reconciles the Fourier Series and Fourier Transform, by bringing the analysis techniques together through considering the Fourier Transform of a Fourier Series, using the definition of the Dirac Delta impulse functions. The Topic also considers the Fourier transform of the complex phasor, as well as the Fourier transform of a sinusoid and co-sinusoid. This Topic provides the theory needed to properly understand some other elements in the course, but is not needed for anyone who is just interested in the key concepts. The Topic also includes a summary of the concepts around Fourier transform theory elsewhere, although this summary is also available in the mid-course summary video.

This final section, which is added for completeness, addresses the question of unifying Fourier series and Fourier transforms. So far in this handout, we have learned that:

1. The Fourier series is used as an analysis tool for decomposing a **periodic signal**



(a) Spectrum of a Cosine wave. (b) Spectrum of a sine wave.

Figure 7.6: The Fourier spectrum of periodic signals.

into a linear combination of sine and cosine waves or complex phasors.

2. The Fourier transform is used as an analysis tool for **non-periodic** signals.

A natural question to ask is what is the Fourier transform of a **periodic signal**? This can be achieved using the notion of the impulse function, as introduced in Section 7.2.2 on page 238, and the Fourier transform pair derived in Sidebar 21 which states that the Fourier transform of a complex phasor is an impulse centered on the frequency of the phasor. Thus: $e^{j\omega_a t} \rightleftharpoons 2\pi\,\delta\left(\omega - \omega_a\right)$. The actual Fourier transform of the complex phasor is be better written using the functional notation:

$$\mathcal{F}\left(e^{j\omega_a t}\right) = 2\pi\,\delta(\omega - \omega_a) \tag{7.30}$$

Hence, the Fourier transform for a sinusoid and co-sinusoid can easily be written down using Euler's decompositions on page 145:

$$A\,\cos\left(\omega_a t\right) \rightleftharpoons A\pi\left[\delta\left(\omega - \omega_a\right) + \delta\left(\omega + \omega_a\right)\right] \tag{7.36a}$$

$$A\,\sin\left(\omega_a t\right) \rightleftharpoons A\frac{\pi}{j}\left[\delta\left(\omega - \omega_a\right) - \delta\left(\omega + \omega_a\right)\right] \tag{7.36b}$$

The Fourier spectrum of a sine wave and cosine wave are shown in Figure 7.6b and Figure 7.6a respectively. Note that while the spectral amplitudes of the sine-wave are complex, they are usually drawn as shown for simplicity, even though they have a non-zero phase term.

The Fourier transform of a periodic signal can thus be found by first writing a periodic signal, $x(t)$, as its complex Fourier series:

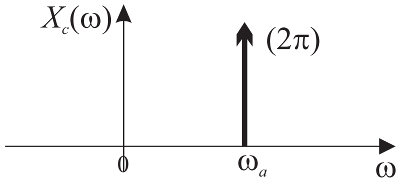
$$x(t) = \sum_{n=-\infty}^{\infty} X_n\, e^{jn\omega_0 t} \tag{7.37}$$

Summary Slide 44 Fourier Theory

Fourier Transform of Phasor

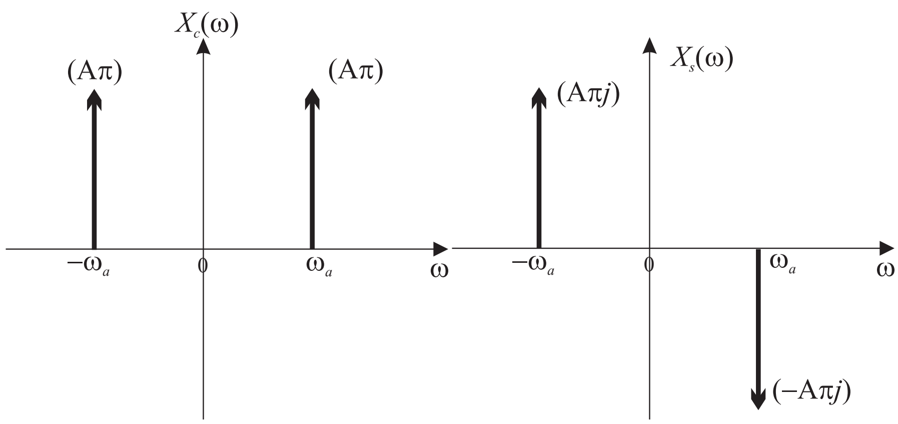
See handout for proof:

$$\mathcal{F}\left(e^{j\omega_a t}\right) = 2\pi\,\delta(\omega - \omega_a)$$



Fourier Transform of Sinusoids and Co-sinusoids

$$A\,\cos(\omega_a t) \rightleftharpoons A\pi\left[\delta(\omega-\omega_a)+\delta(\omega+\omega_a)\right]$$
$$A\,\sin(\omega_a t) \rightleftharpoons A\frac{\pi}{j}\left[\delta(\omega-\omega_a)-\delta(\omega+\omega_a)\right]$$



Sidebar 21 Fourier transform of a complex phasor

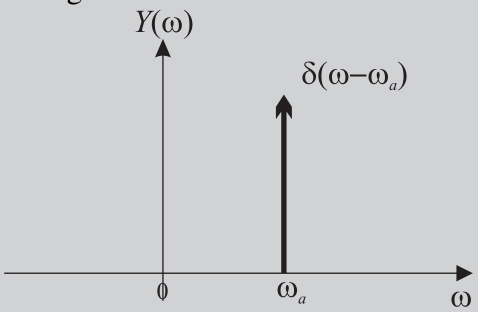
What is the Fourier transform of:

$$x(t) = e^{j\omega_a t} \tag{7.31}$$

To calculate this using the Fourier transform integral proves difficult, and involves a non-converge integral. However, let us consider the following Fourier spectrum:

$$Y(\omega) = \delta(\omega - \omega_a) \tag{7.32}$$

This is an impulse centered at frequency $\omega = \omega_a$, and is shown below. The time-domain signal corresponding to this spectrum can be obtained by taking the inverse Fourier transform of the spectrum in Equation 7.32. Thus:



$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\, e^{j\omega t}\, dt \tag{7.33}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_a)\; e^{j\omega t}\, dt \tag{7.34}$$

and hence using the **sifting theorem** in Equation 7.15 on page 245 gives:

$$y(t) = \frac{1}{2\pi}\left[e^{j\omega t}\right]_{\omega=\omega_a} = \frac{1}{2\pi}e^{j\omega_a t} = \frac{1}{2\pi}x(t) \tag{7.35}$$

and therefore it follows that $e^{j\omega_a t} \rightleftharpoons 2\pi\,\delta(\omega - \omega_a)$ are transform pairs.

so that the Fourier transform becomes:

$$X(\omega) = \mathcal{F}(x(t)) = \mathcal{F}\left(\sum_{n=-\infty}^{\infty} X_n\, e^{jn\omega_0 t}\right) \tag{7.38}$$

and since the limits of the infinite summation are independent of the limits of the Fourier integrand implicit in the notation $\mathcal{F}(\cdot)$, the order of integration and summation can be rearranged due to the linearity of the Fourier transform:

$$X(\omega) = \sum_{n=-\infty}^{\infty} X_n \mathcal{F}\left(e^{jn\omega_0 t}\right) \tag{7.39}$$

where it is noted that the Fourier coefficients X_n are independent of the variable of integration in the Fourier integrand in $\mathcal{F}(\cdot)$. Hence, using Equation 7.30, it follows that:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{\infty} X_n\, \delta(\omega - n\omega_0) \tag{7.40}$$

KEYPOINT! (Fourier Transform of a Fourier Series). The Fourier transform of a Fourier Series is an infinite summation of weighted impulses centered at integer multiples of the fundamental frequency.

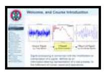
- The weight of the impulse centered at frequency $\omega = n\omega_0$ is $2\pi X_n$ which is the Fourier coefficient associated with the harmonic at frequency $n\omega_0$ scaled by 2π.
- Note then that the Fourier spectrum of a periodic signal is **discrete** in nature, and simply reflects the convention for plotting Fourier coefficients used up until this point. Hence, the vertical lines in plot of the Fourier coefficients on page 161 can now formally be considered as impulses.
- The relationship in Equation 7.39 is fundamental to a complete understanding of Fourier series and Fourier transforms.

Finally, note the Fourier transform of the impulse train indicates the following curious relationship:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \rightleftharpoons \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - n\frac{2\pi}{T}\right) \tag{7.41}$$

The Fourier transform of an impulse train is an impulse train as shown in Figure 7.5.

7.4 Handout Summary



The last three handouts have covered Fourier analysis of continuous-time signals in reasonable detail, covering: *New slide*

Trigonometric Fourier Series for modelling periodic signals in terms of sine and cosine waves;

Summary Slide 45 Fourier Theory

Fourier Transforming Fourier Series!

$$x(t) = \sum_{n=-\infty}^{\infty} X_n \, e^{jn\omega_0 t}$$

Taking Fourier transforms

$$\begin{aligned} X(\omega) &= \mathcal{F}\left(\sum_{n=-\infty}^{\infty} X_n \, e^{jn\omega_0 t}\right) \\ &= \sum_{n=-\infty}^{\infty} X_n \mathcal{F}\left(e^{jn\omega_0 t}\right) \\ X(\omega) &= 2\pi \sum_{n=-\infty}^{\infty} X_n \, \delta\left(\omega - n\omega_0\right) \end{aligned}$$

- The Fourier transform of a Fourier Series is an infinite summation of weighted impulses centered at integer multiples of the fundamental frequency.
- The weights of the impulses are 2π multiplied by the Fourier coefficients.
- Hence, the vertical lines in the plots of Fourier coefficients are actually impulses!
-

Complex Fourier Series for modelling periodic signals in terms of complex exponential phasors;

Relationship between Trigonometric and Complex Fourier Series and expression for calculating one set of coefficients given the corresponding other set;

Fourier transform for modelling non-periodic signals;

Properties of the Fourier transform for simplifying Fourier calculations;

Fourier transform of a Fourier Series for completing the picture and showing the direct relationship between the two.

Furthermore, these handouts have covered the basics of signal classification, strength of a signal in the time and frequency domain using energy and power calculations, as well as some basic signal types such as complex phasors, step and impulse functions.

– End-of-Topic 41: **The Fourier Transform of a Fourier Series, and summary of Topics on Continuous-Time Signal Analysis –**

7.5 Tutorial Exercises

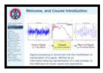
There are currently no tutorial questions associated with this handout.

8

Sampling: The Bridge from Continuous-Time to Discrete-Time

This handout introduces sampled data systems, the concept of sampling and the typical signals likely to be seen in the process, the mathematical concept of perfect sampling, and the Nyquist Criterion

8.1 Introduction



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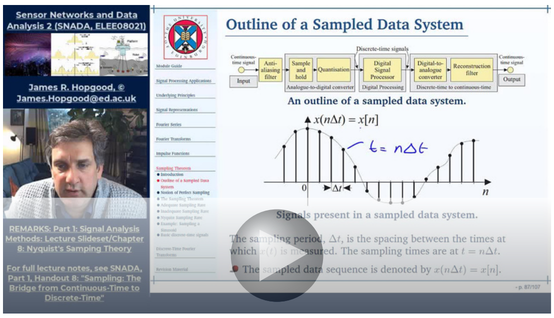
Topic Summary 42 Introduction to Nyquist Sampling Theory

Topic Objectives:

- Introduction to the purpose of sampling theory.
- Overview of a sampled data system and the signals at each stage.
- Reminder of the Nyquist sampling rate.
- Demonstration of the effect of aliasing.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 25 min video	3× length	
Read Handout	Read page 260 to page 266	8 mins/page	
Try Code	Use MATLAB demonstraton	15 minutes	



`http://media.ed.ac.uk/media/1_oveyqege`

Video Summary: This Topic introduces the concept of a sampled data system, which aims to measure and store values from a continuous-time signal to produce a discrete-time representation. The video emphasis that perfect sampling is possible for band-limited signals, even though it may appear that signal values in the time-domain are discarded. This introductory video reminds the viewer of the Nyquist sampling rate, but then considers whether this value is reasonable in practice. The Topic highlights the importance of the anti-aliasing filters and the reconstruction filter, for ensuring that the original signal can be recreated as faithfully as possible. The full sampled data system is explained in detail, and an example of aliasing presented through a MATLAB demonstration.

A continuous-time signal cannot easily be stored, as the signal exists for all time instances. Digital-based memory and processors are restricted by a finite-clock speed, meaning that they can only store samples at discrete intervals in time, and to a finite

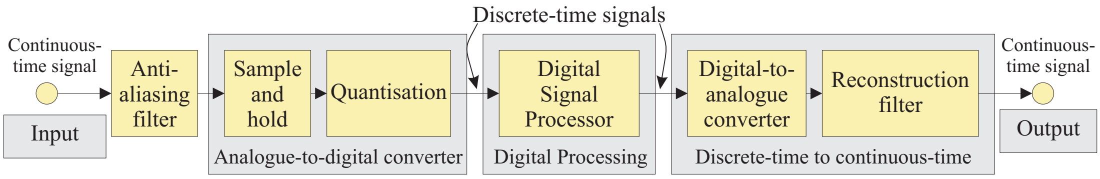


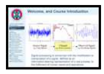
Figure 8.1: An outline of a sampled data system.

resolution, due to the quantisation of analogue-to-digital converters (ADCs). In this course, we shall not be concerned with the issue of quantisation, as the number of bits used in modern ADCs gives a sufficient representation of the analogue value of a signal at a particular point in time. However, the restriction that only a finite number of values representing a signal will be considered in detail.

A continuous-time signal can be stored and processed using a discrete-time system. For this purpose, it is important to maintain the sampling rate sufficiently high to permit the reconstruction of the original signal from these samples without error, or at least with an error to a given tolerance. The necessary quantitative framework for this purpose is provided by the Nyquist-Shannon sampling theorem. [1] Sampling theory is the bridge between the continuous-time and the discrete-time world.

A sampled continuous-time signal is a sequence of **impulses**, while a discrete-time signal is a sequence of numbers. Nevertheless, the information inherent in a sampled continuous-time signal is equivalent to that of a discrete-time signal. Thus, sampled continuous-time signals and discrete-time signals are essentially two different ways of presenting the same data. Therefore, all the concepts in the analysis of sampled signals will apply to discrete-time signals.

8.1.1 Outline of a Sampled Data System



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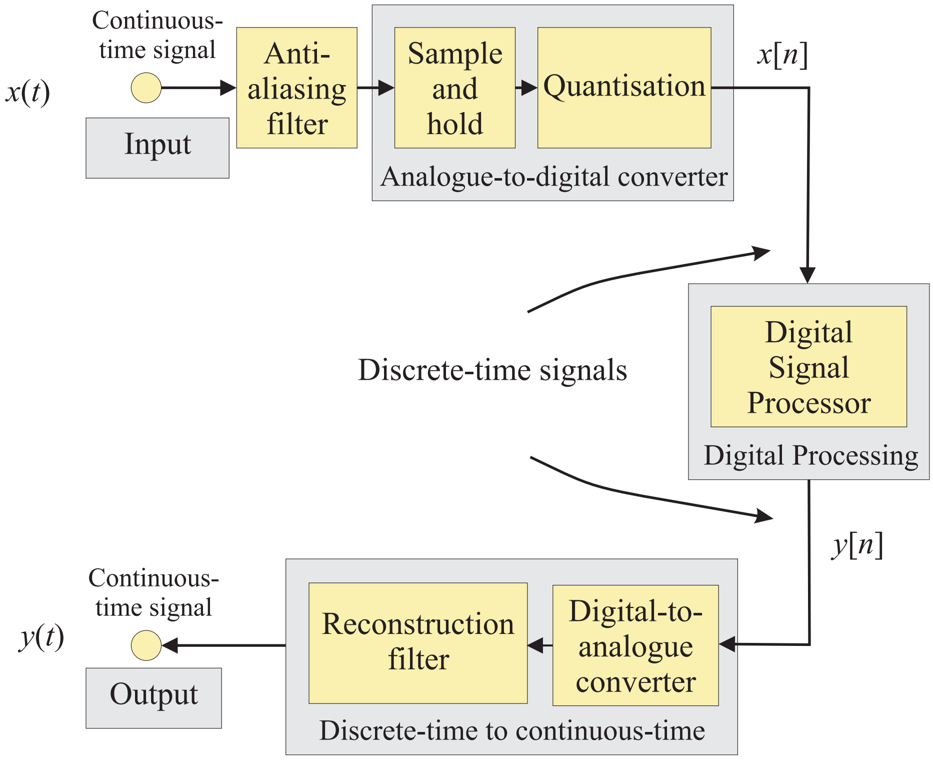
It is interesting to consider how one might attempt to store a continuous-time signal on a digital device. A practical sampled data system is shown in Figure 8.1, and each component is crucial to implementing the system. The **anti-aliasing** filter and **reconstruction filter** are absolutely crucial to sampling signals, and these will be discussed later in this handout.

Consider trying to *sample* the continuous-time signal $x(t)$ shown in Figure 8.2a. Intuitively, it is clear that one would want to measure the signal at discrete points in time and convert the analogue value to one that can be stored in digital memory using a analogue-to-digital converter (ADC).

[1] The theorem is commonly called the **Nyquist sampling theorem**, after Harry Nyquist and Claude Shannon who developed the theory. However, since it was also discovered independently by E. T. Whittaker, by Vladimir Kotelnikov, and by others, it is also known as the Nyquist-Shannon-Kotelnikov, Whittaker-Shannon-Kotelnikov, **Whittaker-Nyquist-Kotelnikov-Shannon**, WKS, etc., sampling theorem, as well as the Cardinal Theorem of Interpolation Theory. It is often referred to simply as the **sampling theorem**.

Summary Slide 46 Sampling Basics

Introduction



KEYPOINT! (Digital Processing of an Analogue World). A sampled data system processes real world signals in the continuous-domain by converting them to the discrete-domain.

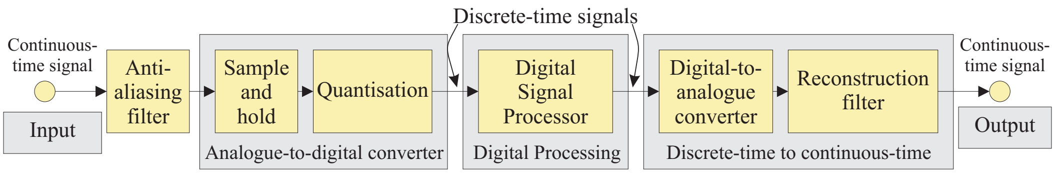
The digital signal processing (DSP) block could represent:

- Communication or Storage;
  –
  –
- Enhancement or signal analysis

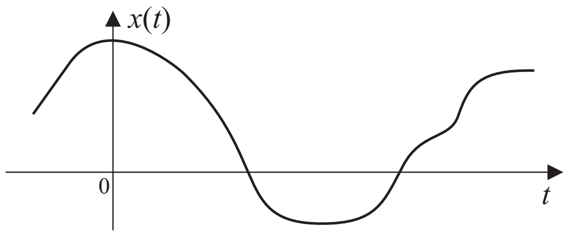
KEYPOINT! (DSP). The DSP is really a **computer**; it is effectively a CPU optimised for certain signal processing applications.

Summary Slide 47 Introduction to the Sampling Theorem

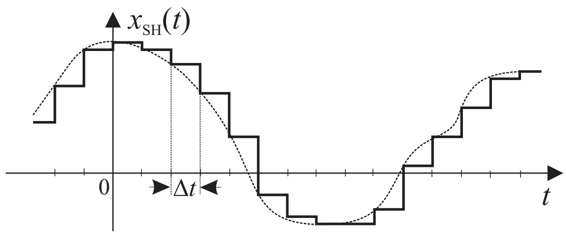
Signals in a Sampled Data System



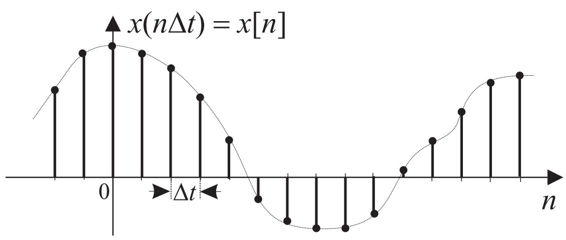
Original input waveform:



Output of the **sample & hold**:



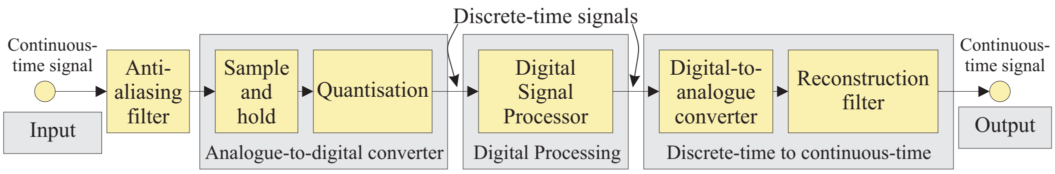
Perfectly sampled signal:



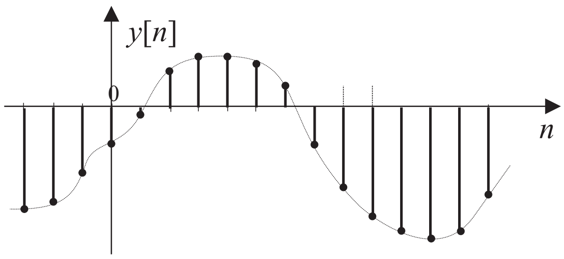
KEYPOINT! (Notation). The continuous-time signal $x(t)$ when evaluated at $t = n\Delta t$ (integer n), is the n-th sample with value $x(n\Delta t)$. The resulting descrete-time signal is $x[n] \triangleq x(n\Delta t)$.

Summary Slide 48 Introduction to the Sampling Theorem

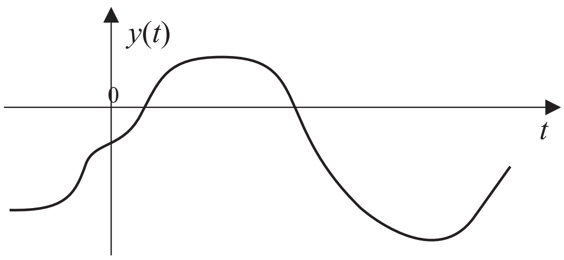
Sampled Data System (Continued)



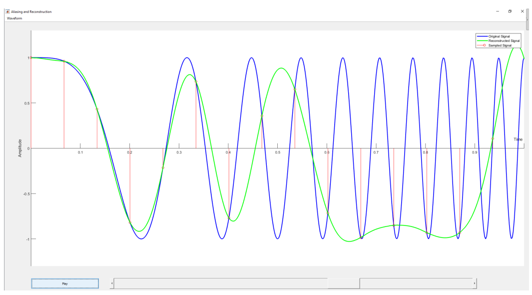
Digital Signal Processing!! e.g. interference cancellation in communications systems.

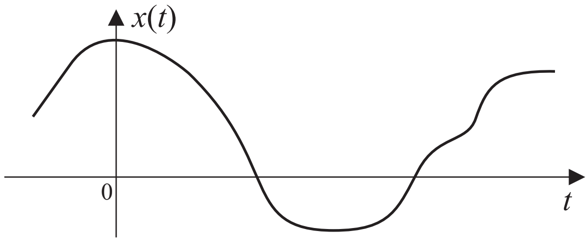


Signal reconstruction:

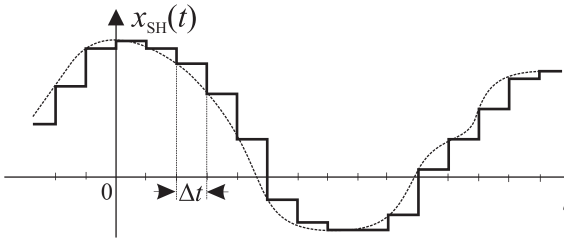


Aliasing Demo

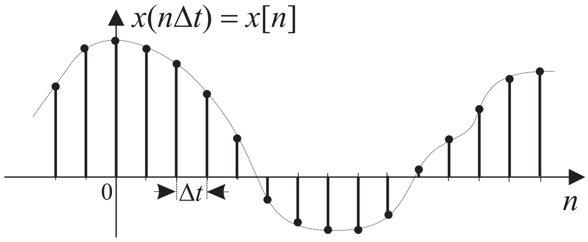




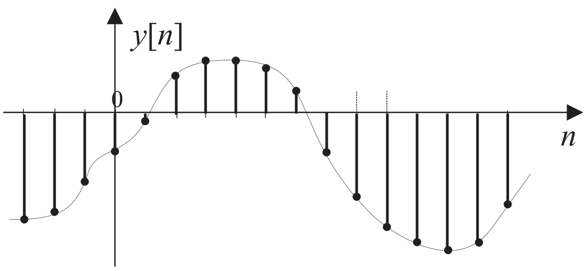
(a) Analogue input signal, $x(t)$.



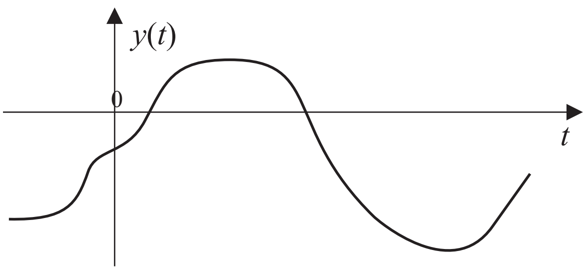
(b) Output of a sample-and-hold (S/H), $x_{SH}(t)$.



(c) The input data sequence $\{x(n\Delta t)\}$.



(d) The output sequence of a digital signal processing (DSP), $\{y(n\Delta t)\}$.



(e) The analogue output of a perfect digital-to-analogue converter (DAC).

Figure 8.2: Signals present in a sampled data system.

Since ADCs have a finite response time, then it is necessary to ensure that the signal at the input of the ADC is not varying with time. Therefore, once the signal has been sampled, the signal is *held* over the **sampling interval** or **sampling period** to give the ADC chance to perform. This step is known as **sample and hold**, and the result of the process on the signal in Figure 8.2a is shown in Figure 8.2b.

Clearly, however, the value stored is that at the beginning of the sampling period, and therefore a data sequence is generated as shown in Figure 8.2c. The so-called sampling period, Δt, is the spacing between the times at which the analogue signal $x(t)$ is measured. The sampling times are integer multiples, n, of the sampling period, Δt, and therefore $x(t)$ is measured at times $t = n\Delta t$. The sampled data sequence is therefore denoted by $x(n\Delta t) = x[n]$. [2] Once a discrete-time signal $x[n]$ is stored on a digital device, it can be manipulated as needs be by any **signal processing** algorithm. This is known as **digital signal processing (DSP)**, and manipulates one sequence of numbers, $x[n]$, into a new set of numbers $y[n]$, as shown in Figure 8.2d.

The final requirement is to *reconstruct* an analogue signal from the sequence of numbers, which essentially involves using a digital-to-analogue converter (DAC). However, to reconstruct the signal fully, the signal must be filtered using a **reconstruction filter**. This aims not only to remove artifacts from the DAC, but also to remove the so-called *spectral images* which arise due to the inherent nature of the sampling process itself. The reconstructed signal is shown in Figure 8.2e.

In this course, the effect of quantisation in the ADC and DAC will be ignored, and in effect the ADC and DAC will be considered as perfect *samplers*.

KEYPOINT! (MATLAB Demo). As will be seen, **aliasing** is the result of not sampling a signal at an adequate sampling rate. The effect of sampling can be seen using the MATLAB demo available on LEARN called **aliasing**.

– End-of-Topic 42: **Introduction to Sampled Data Systems** –

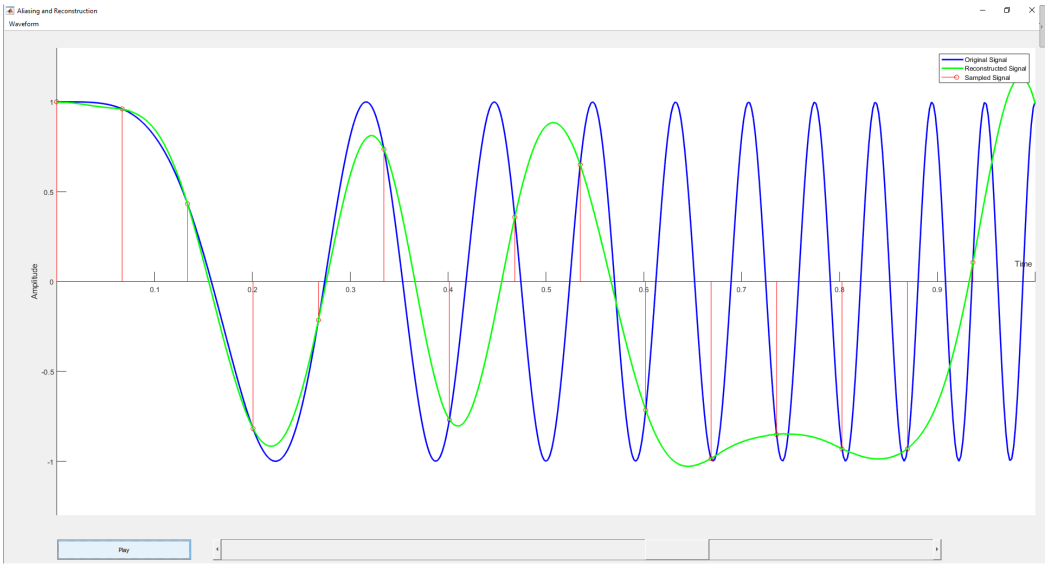
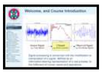


Figure 8.3: A screenshot of the MATLAB demo: *aliasing*.



New slide

8.1.2 Notion of Perfect Sampling

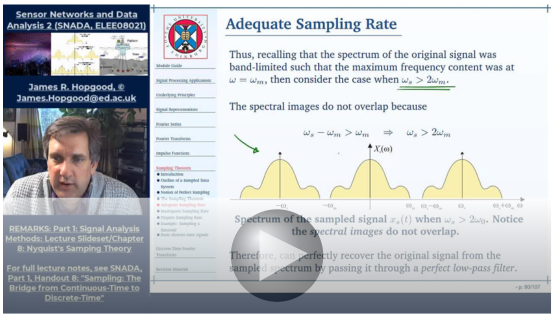
Topic Summary 43 Perfect Sampling, Derivation of Sampling Theorem, and Adequate Sampling

Topic Objectives:

- Introduction to the purpose of sampling theory.
- Overview of a sampled data system and the signals at each stage.
- Reminder of the Nyquist sampling rate.
- Demonstration of the effect of aliasing.

Topic Activities:

Type	Details	Duration	Progress
Watch video	18 : 55 min video	3× length	
Read Handout	Read page 268 to page 273	8 mins/page	
Try Code	Use MATLAB demonstraton	15 minutes	
Practice Exercises	Exercises 8.1 and 8.2	20 mins	



`http://media.ed.ac.uk/media/1_dk5mfftd`

Video Summary: This Topic begins by discussing the mathematical model of perfect sampling, namely multiplying the signal (that is to be sampled) by an impulse train. Using this model, the Topic discusses how to consider what the impact of sampling a signal is, and proposes to consider the resulting change in the spectrum of the sampled signal compared with the spectrum of the original signal. Undertaking this analysis, the relationship between these spectra is derived, which therefore leads to the standard Nyquist sampling rate. The origin of this result is described in detail using the mathematical tools developed so far in this course.

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[2] As a reminder, a discrete-time signal is denoted $x[n]$ where n is enclosed in square brackets instead of parentheses, as parentheses are reserved for enclosing a continuous-time variable, such as t.

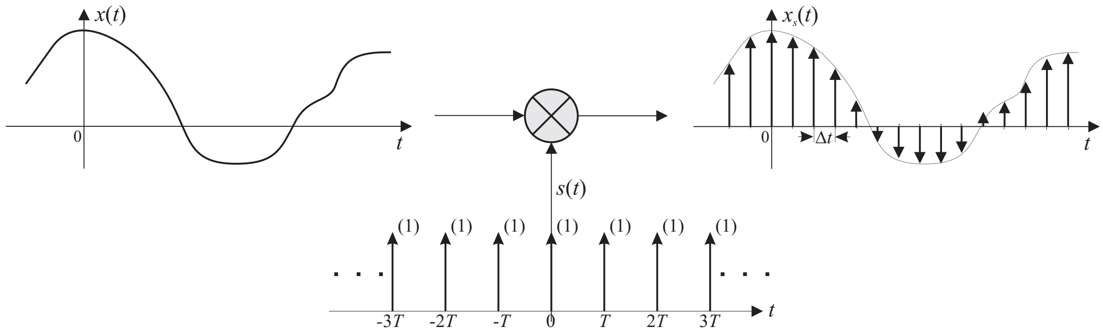


Figure 8.4: Mathematical modelling of perfect sampling.

Since a discrete-time signal $x[n]$ can be considered as a sequence of numbers, we do not yet have all the tools to analyse them as we did with continuous-time signals. However, it is possible to create a continuous-time signal which contains no more and no less information than the discrete-time counterpart, and therefore any analysis of this equivalent continuous-time signal is highly relevant.

This is achieved by the notion of perfect sampling, as shown in Figure 8.4. The original continuous-time signal $x(t)$ is multiplied by an impulse train, to yield another *continuous-time signal*, denoted by $x_s(t)$, but which consists only of impulses at the sampling points. The signal $x_s(t)$ in-between the impulses is well-defined, and is exactly zero. The weights of the impulses correspond to the numbers that would be stored in an equivalent discrete-time signal $x[n]$.

KEYPOINT! (Fourier Transform of a Sampled Signal). An interesting question is what is the Fourier transform of $x_s(t)$ given the Fourier transform of $x(t)$?

8.2 The Sampling Theorem

Consider a signal $x(t)$ shown in Figure 8.5 whose spectrum is *band-limited* such that it has a maximum frequency component at $\omega = \omega_m$, and that for all frequencies above this value, the spectrum is zero. Therefore, $x(t)$ is such that its spectrum $X(\omega)$ can be written as: *New slide*

$$X(\omega) = \begin{cases} \bar{X}(\omega) & |\omega| \leq \omega_m \\ 0 & |\omega| > \omega_m \end{cases} \tag{8.5}$$

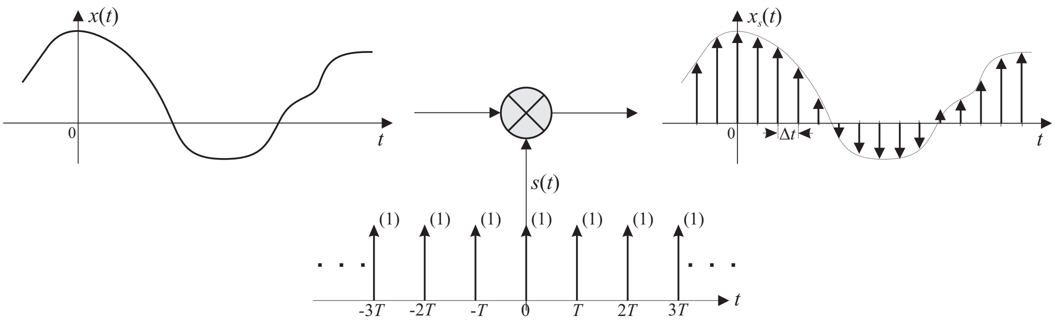
The sampled signal is given by the product of the original signal and an impulse train:

$$x_s(t) = s(t)\; x(t) \quad \text{where} \quad s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{8.6}$$

where T, sometimes called T_s, is the sampling interval.

Summary Slide 49 Sampling Theorem Developed

Perfect Sampling



Model sampled signal as product of original signal and impulse train:

$$x_s(t) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{8.1}$$

$$= x(t) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{8.2}$$

$$x_s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(t)\ e^{jn\omega_s t} \tag{8.3}$$

where $\omega_s = \frac{2\pi}{T}$ is the sampling frequency in radians per second.

KEYPOINT! (Recall frequency shift theorem!). If $x(t) \rightleftharpoons X(\omega)$, then $x(t)\, e^{jn\omega_s t} \rightleftharpoons X(\omega - n\omega_s)$.

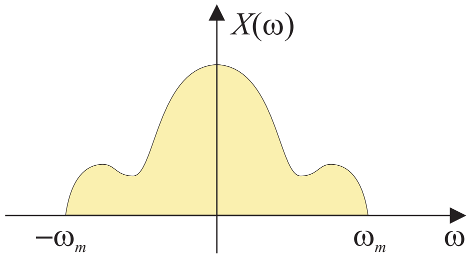
$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \tag{8.4}$$

KEYPOINT! (Fourier Transform of Sampled Signal). Spectrum of sampled signal is a linear combination of **spectral images**, where an **image** is a shifted version of the original spectrum centered on integer multiples of the sampling frequency.

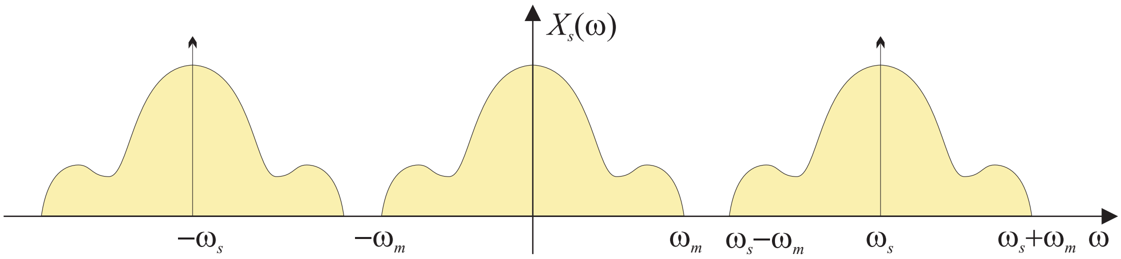
Summary Slide 50 Interpreting the sampling theorem

Adequate Sampling Rate

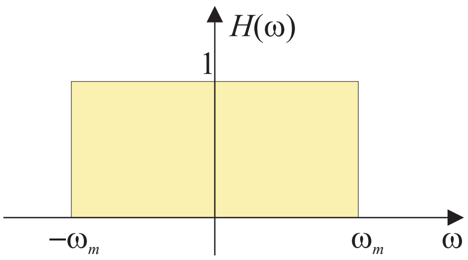
Spectrum of the original signal:



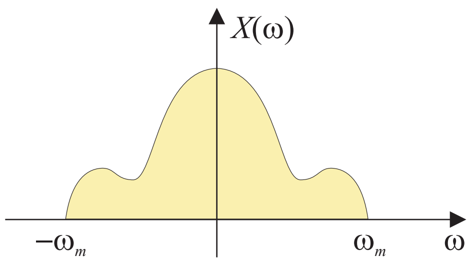
Spectrum of the sampled signal :



Frequency response of ideal low-pass filter



Spectrum of reconstructed signal $X(\omega) = H(\omega)\,X_s(\omega)$:



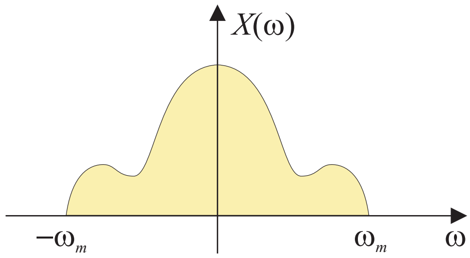


Figure 8.5: Original spectrum with maximum frequency content ω_m.

To consider the Fourier transform of $x_s(t)$, a more useful formula than the infinite summation of impulses is needed. Using the Fourier series expression for the impulse train from the last handout, it follows that:

$$x_s(t) = x(t) \times \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{8.7}$$

where $\omega_s = \frac{2\pi}{T}$ is the sampling frequency in radians per second. This expression can be rearranged slightly by taking $x(t)$ inside the summation to give:

$$x_s(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} x(t)\, e^{jn\omega_s t} \tag{8.8}$$

It is possible to take Fourier transforms of both sides using the frequency shift theorem, such that if $x(t) \rightleftharpoons X(\omega)$, then $x(t)\, e^{jn\omega_s t} \rightleftharpoons X(\omega - n\omega_s)$. Hence, writing $x_s(t) \rightleftharpoons X_s(\omega)$, it follows that:

$$X_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \tag{8.9}$$

KEYPOINT! (Fourier Transform of Sampled Signal). The expression in Equation 8.9 has a very simple interpretation: the spectrum of the sampled signal is a linear combination of **spectral images** of the original spectrum, where an image is a shifted version of the original spectrum such that it is centered on an integer multiple of the sampling frequency in radians per second.

New slide

8.2.1 Adequate Sampling Rate

The sampling result in Equation 8.9 should be interpreted graphically to see its effect. Thus, recalling that the spectrum of the original signal was band-limited such that the maximum frequency content was at $\omega = \omega_m$ as given by Equation 8.5, then consider the case when $\omega_s > 2\omega_m$.

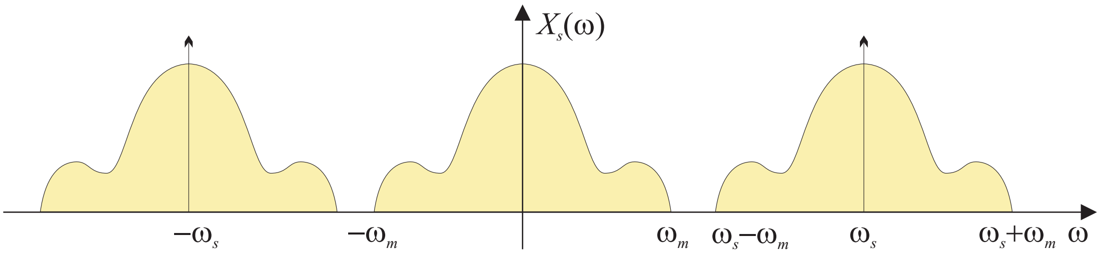


Figure 8.6: Spectrum of the sampled signal $x_s(t)$ when $\omega_s > 2\omega_0$. Notice the *spectral images* do not overlap.

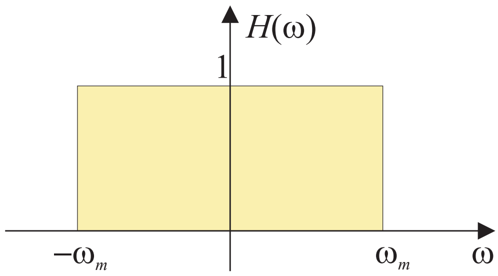


Figure 8.7: The **frequency response** of an ideal low-pass filter.

In this case, the *spectral images* which are shifted and centered on integer multiples of the sampling frequency ω_s do not overlap. Notice that the spectrum is periodic with period ω_s. The spectral images do not overlap because

$$\omega_s - \omega_m > \omega_m \quad \Rightarrow \quad \omega_s > 2\omega_m \tag{8.10}$$

as previously assumed.

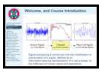
What is the relevance of this result, you might wonder? The key point is that the original spectrum centered on zero-frequency remains untouched by the spectral images that have resulted due to the sampling process. Therefore, it is possible to perfectly recover the original signal from the sampled spectrum by passing it through a *perfect low-pass filter*.

While we haven't yet covered exactly what happens as signals pass through linear systems or filters, it doesn't take too much imagination that the spectrum at the output of the filter is the product of the filters *frequency response*, denoted by $H(\omega)$, and that of the spectrum of the input signal.

Thus, the original signal can be recovered by the operation:

$$X(\omega) = H(\omega)\ X_s(\omega) \tag{8.11}$$

– End-of-Topic 43: **Notion of Perfect Sampling, Derivation of Nyquist's Sampling Theorem, and Adequate Sampling –**



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8.2.2 Inadequate Sampling Rate

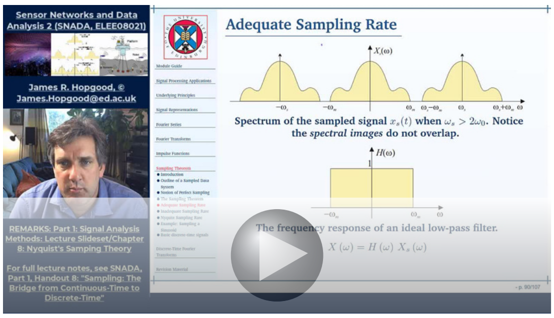
Topic Summary 44 Undersampling and Practical Data Systems

Topic Objectives:

- Understand the reason for, and effect, of aliasing.
- Understand how to avoid aliasing.
- Understand the practical limitations of sampled data systems.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 25 min video	3× length	
Read Handout	Read page 274 to page 278	8 mins/page	
Practice Exercises	Exercise 8.3	20 mins	



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Video Summary: This Topic considers the concept of aliasing, which is signal distortion due to the spectral overlap of the spectral images that result from the sampling process. This Topic discusses the condition under which aliasing occurs, and investigates what happens when the spectral images are summed together, and the resulting signal after the reconstruction filter. The Topic then considers strategies for how to avoid aliasing in practice, including the notion of anti-aliasing filters, and oversampling. The video also considers the limitations of practical data systems, such as imperfect low-pass filters in the anti-aliasing and reconstruction filters, as well as imperfect sample-and-hold and DACs. Finally, the video considers the exemplar case of high-resolution audio.

Now consider the case when $\omega_s < 2\omega_m$. In this case, $\omega_s - \omega_m < \omega_m$, and this means that the edges of the spectral images are overlapping. The resulting spectrum is the summation of the spectral images, remembering that the spectrum is in general a complex function. For the signal in Figure 8.5, if the sampling rate is not sufficiently

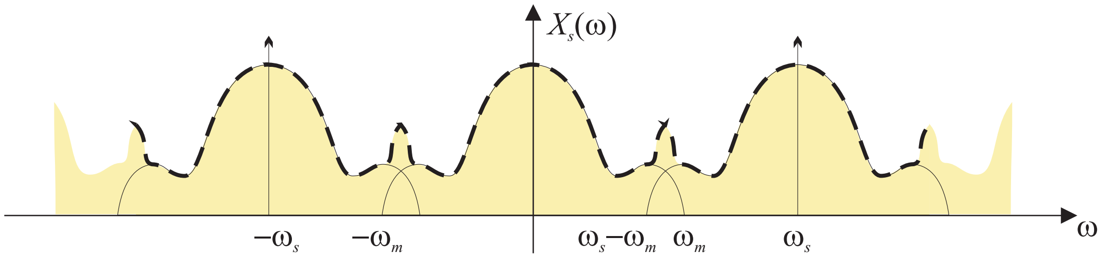


Figure 8.8: Spectrum of the sampled signal $x_s(t)$ when $\omega_s < 2\omega_0$. Notice the *spectral images* are now overlapping.

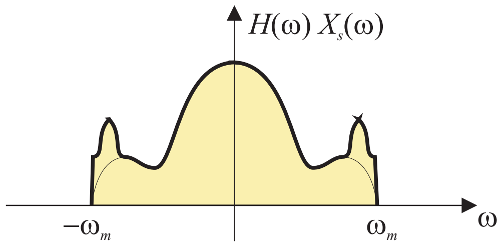


Figure 8.9: The reconstructed signal $\hat{X}(\omega) = H(\omega)H_s(\omega)$ is not the same as the original spectrum $X(\omega)$.

high, then the resulting spectrum is shown in Figure 8.8.

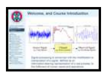
If the sampled signal is now passed through the low-pass filter shown in Figure 8.7, then the resulting signal is given by:

$$\hat{X}(\omega) = H(\omega)\, X_s(\omega) \neq X(\omega) \tag{8.12}$$

for all frequencies, as shown in Figure 8.9.

The effect of under-sampling has resulted in an effect called **aliasing**. This will be investigated in much more detail later in the handout.

8.2.3 Nyquist Sampling Rate



New slide

In order to prevent **aliasing**, sampling must occur at a frequency greater than twice the highest frequency present.

> **KEYPOINT! (Formally).** To prevent aliasing distortion, a band-limited signal of **bandwidth** ω_m must be sampled at a rate of at least $2\omega_m$ rad/s, where a band-limited signal is of the form in Equation 8.5.

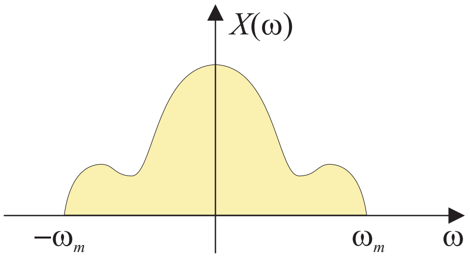
Unfortunately, it is impossible to build a perfect practical sampled data system for the following reasons:

- Perfect reconstruction is not possible for two reasons:

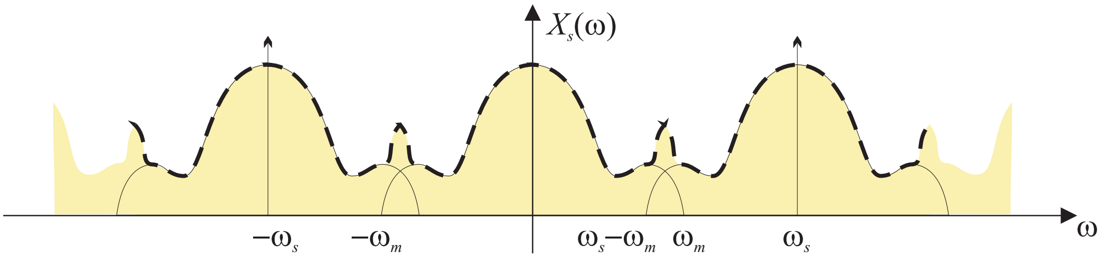
Summary Slide 51 Interpreting the sampling theorem

Inadequate Sampling Rate

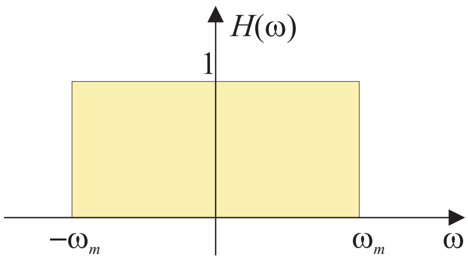
Spectrum of the original signal:



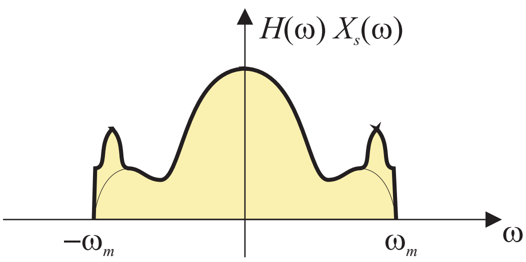
Spectrum of the sampled signal :



Frequency response of ideal low-pass filter



Spectrum of reconstructed signal $X(\omega) = H(\omega)\, X_s(\omega)$:



Summary Slide 52 Conclusion of the sampling theorem

The Nyquist Criterion

In order to prevent aliasing, sampling must occur at a frequency greater than twice the highest frequency present.

KEYPOINT! (Formally). To prevent aliasing distortion, a band-limited signal of **bandwidth** ω_m must be sampled at a rate of at least $2\omega_m$ rad/s, where a band-limited signal is such that:

$$X(\omega) = \begin{cases} \bar{X}(\omega) & |\omega| \leq \omega_m \\ 0 & |\omega| > \omega_m \end{cases} \tag{8.13}$$

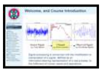
□

  - it is impossible to construct an ideal low-pass filter from a finite number of components;
  - if an analogue signal is required, then a DAC is always needed, which cannot recreate perfect impulses, and uses a zero-order-hold which introduces distortion.

- It is not always possible to ensure that the Nyquist criterion is met. This can be achieved by using an **anti-aliasing filter** prior to the sample-and-hold stage, whose function is to remove or attenuate all frequencies above half the sampling frequency. Again, because the **anti-aliasing filter** requires a low-pass filter, then it will not be perfect due to using a finite number of components.

– End-of-Topic 44: **Aliasing, Anti-Aliasing, and Limitations of Practical Sampled Data Systems –**

8.2.4 Example: Sampling a Sinusoid



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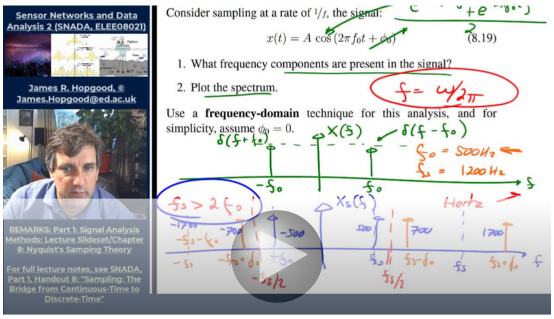
Topic Summary 45 Sampling a Sinusoid

Topic Objectives:

- Using Nyquist Sampling theory to plot spectrum of sampled sinusoid.
- Understanding the output of the reconstuction filter in the cases of adequate and inadequate sampling.
- Considering the sampling of a sampled signal in the time-domain.

Topic Activities:

Type	Details	Duration	Progress
Watch video	15 : 41 min video	3× length	
Read Handout	Read page 279 to page 283	8 mins/page	
Practice Exercises	Exercise 8.4	20 mins	



`http://media.ed.ac.uk/media/1_yhle27hv`

Video Summary: This Topic uses the sampling theory developed in the previous topics to analyse what happens when a pure sinusoid is sampled. The Topic considers two approaches: a time-domain and a frequency-domain approach. The frequency-domain approach considers the spectrum of the signal and seeks to plot the baseband spectrum, and then the spectral images at integer multiples of the sampling frequency. The components that are then between plus/minus half the sampling frequency are then those which will be seen at the output of the sampled data system after the reconstruction filter. The Topic considers the case of adequate and inadequate sampling. The video then moves on to consider a time-domain approach, which is somewhat less elegant than the frequency-domain approach, although of course it gives the sample result.

It is of interest to consider the simple case of sampling a sinusoid of frequency f_0 at a sampling frequency of f_s. Suppose the continuous-time signal is given by:

$$x(t) = A\cos(2\pi f_0 t + \phi_0) \tag{8.14}$$

and this is sampled at intervals $t = nT_s$ where $T_s = {}^1\!/_{f_s}$. Then, the resulting discrete-time signal is given by:

$$x[n] = A\cos\left(2\pi n\frac{f_0}{f_s} + \phi_0\right) \tag{8.15}$$

Since $\cos\theta = \cos(\theta + 2\pi\ell)$ for any integer ℓ, then choosing $\ell = mn$ where n and m are also integers, then Equation 8.15 can be written as:

$$x[n] = A\cos\left(2\pi n\frac{f_0}{f_s} + 2\pi nm\frac{f_s}{f_s} + \phi_0\right) \tag{8.16}$$

$$= A\cos\left(2\pi n\frac{f_0 + mf_s}{f_s} + \phi_0\right) \tag{8.17}$$

This is equivalent to sampling the signal$x(t) = A\cos\left(2\pi\hat{f}_0 t + \phi_0\right)$ where $\hat{f}_0 = f_0 + mf_s$ at a sampling frequency of f_s. In otherwords, sinusoids of frequencies that differ by an integer multiple of the sampling frequency result in an identical set of samples in the time domain, and are therefore indistinguishable.

If, however, the input frequency f_0 is restricted to the range $-{}^{f_s}\!/_2$ to ${}^{f_s}\!/_2$ (also called the **fundamental band**), then sinusoids of different frequencies are unique. This corresponds to the results given by the sampling theorem, in which no aliasing occurs.

Aliasing occurs when $f_0 > {}^{f_s}\!/_2$, and in such a case the resulting samples in Equation 8.17 would appear to be samples from a continuous-time sinusoid of frequency f_a in the **fundamental band**, where:

$$f_a = f_0 - mf_s, \qquad -\frac{f_s}{2} \le f_a \le \frac{f_s}{2}, \quad m \in \mathbb{Z} \tag{8.20}$$

The relationship between f_0 and f_a is shown in Figure 8.10. This relationship can be simplified further by noting that a (co)-sinusoid with a negative frequency is the same as a (co)-sinusoid with a positive frequency but with a different phase. This is since $\cos(-2\pi f_a t + \phi) = \cos(2\pi f_a t - \phi)$ and similarly $\sin(-2\pi f_a t + \phi) = -\sin(2\pi f_a t - \phi) = \sin(2\pi f_a t - \phi + \pi)$. Hence, this means that the *apparent frequency* of any sinusoid of frequency f_0 when sampled at frequency f_s is in the range from 0 to ${}^{f_s}\!/_2$ Hz. Therefore, Equation 8.20 becomes:

$$|f_a| = |f_0 - mf_s|, \qquad f_a \le \left|\frac{f_s}{2}\right|, \quad m \in \mathbb{Z} \tag{8.21}$$

The plot of the apparent frequency $|f_a|$ versus f_0 is shown in Figure 8.11, where the shaded areas denote the frequency bands in which a phase change also occurs.

Consider for example, the sinusoid in Equation 8.19 with $f_0 = 8000$ Hz sampled at a rate of $f_s = 3000$ Hz. The fundamental band is in the range $-1500\text{ Hz} \le f_a \le$

Summary Slide 53 Nyquist's Sampling Theorem

Sampling a Sinusoid

Consider sampling at a rate of $1/f_s$ the signal:

$$x(t) = A\,\cos\left(2\pi f_0 t + \phi_0\right) \tag{8.18}$$

What frequency components are present in the signal? Plot the spectrum. Use a **time-domain** technique for this analysis.

Consider sampling a sinusoid with $f_0 = 500$ Hz with $f_s = 600$ Hz. What is the observed frequency after the reconstruction filter?

Summary Slide 54 Nyquist's Sampling Theorem

Sampling a Sinusoid (Revisited)

Consider sampling at a rate of $1/f_s$ the signal:

$$x(t) = A\,\cos\left(2\pi f_0 t + \phi_0\right) \tag{8.19}$$

1. What frequency components are present in the signal?
2. Plot the spectrum.

Use a **frequency-domain** technique for this analysis, and for simplicity, assume $\phi_0 = 0$.

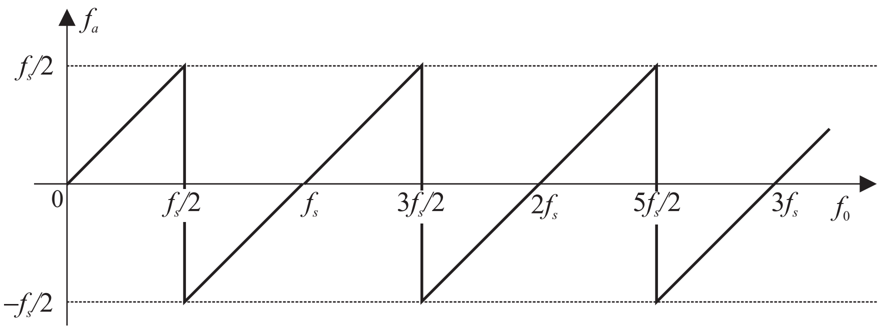


Figure 8.10: Apparent frequencies of a sampled sinusoid, f_a versus f_0.

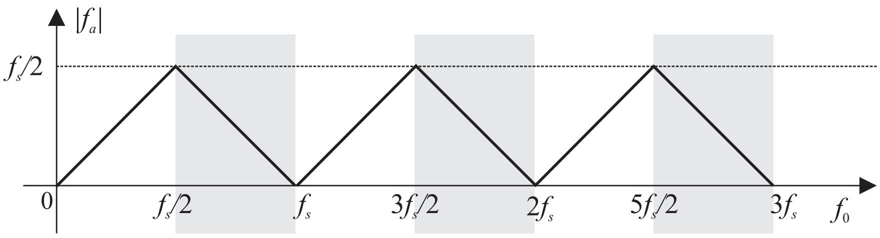
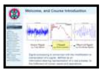


Figure 8.11: Apparent frequencies of a sampled sinusoid, $|f_a|$ versus f_0.

1500 Hz , and therefore the apparent frequency is $f_a = 8000 - 3 \times 3000 = 1000$, or $|f_a| = 1000 \leq 1500$. This result is predicted by Nyquist's sampling theorem.

– End-of-Topic 45: **Example: Sampling a sinusoidal Signal –**

8.3 Basic discrete-time signals



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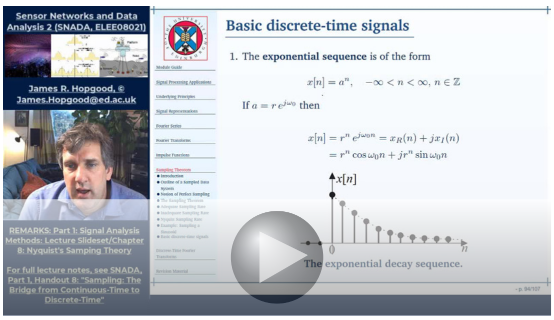
Topic Summary 46 Introduction to Discrete-Time Signals

Topic Objectives:

- Introduction to native discrete-time signals.
- Definition of discrete-time impulse, step, and exponential functions.
- Introduction to creating discrete-time signals in MATLAB.

Topic Activities:

Type	Details	Duration	Progress
Watch video	07 : 02 min video	$3\times$ length	
Read Handout	Read page 284 to page 287	8 mins/page	
Try Code	Reproduce MATLAB demonstraton	10 minutes	



http://media.ed.ac.uk/media/1_9gh5jpin

Video Summary: This Topic is a very brief introduction to the idea of native discrete-time signals, created as sequence of numbers, rather than viewed as a discrete-time representation of an underlying continuous-time signal. Three very simple signals are introduced, such as the unit impulse, unit step, and expontential decay (as well as a decaying cosine or sinusoid). Finally, the video gives a brief overview of creating discrete-time finite-duration signals in MATLAB. The viewer is then asked to consider how to create other signals in MATLAB, for example a sinusoid of a specific frequency.

There are some basic discrete-time signals that commonly occur in discrete-time signal analysis:

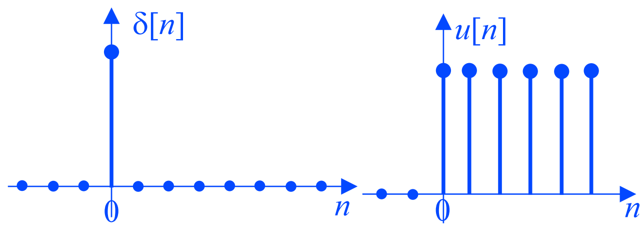
Summary Slide 55 Discrete-time Signal Anaylsis

Basic Discrete-Time Signals

1. The **unit sample** or **unit impulse**:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{8.22}$$

$$= \tag{8.23}$$

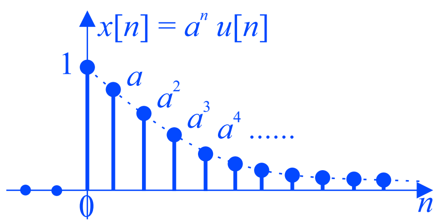


2. The **unit step** sequence:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{8.24}$$

3. The **exponential sequence**:

$$x[n] = a^n\, u[n] \tag{8.25}$$



For complex $a = r\, e^{j\omega_0}$ for $r > 0$, $\omega_0 \neq 0,\ \pi$, then:

$$x[n] = r^n\, e^{j\omega_0 n} = r^n \cos \omega_0 n + j r^n \sin \omega_0 n \tag{8.26}$$

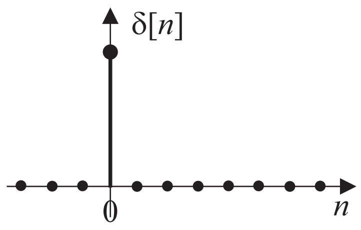


Figure 8.12: The unit pulse sequence, which is the equivalent of the Dirac delta function in discrete-time.

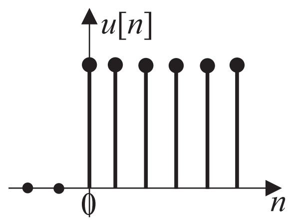


Figure 8.13: The unit step sequence, which is the equivalent of the Heaviside-step function.

1. The **unit sample** or **unit impulse** sequence $\delta[n]$ is defined as:

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{8.27}$$

$$= \{\dots, 0, 0, \underset{\uparrow}{1}, 0, 0, \dots\} \tag{8.28}$$

2. The **unit step** sequence, $u[n]$ is defined as:

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{8.29}$$

3. The **exponential sequence** is of the form

$$x[n] = a^n, \quad -\infty < n < \infty, n \in \mathbb{Z} \tag{8.30}$$

If a is a complex number, such that $a = r\,e^{j\omega_0}$ for $r > 0$, $\omega_0 \neq 0, \pi$, then $x[n]$ is complex valued and given by:

$$x[n] = r^n\,e^{j\omega_0 n} = x_R(n) + jx_I(n) \tag{8.31}$$

$$= r^n \cos\omega_0 n + jr^n \sin\omega_0 n \tag{8.32}$$

where $x_R[n]$ and $x_I[n]$ are real sequences given by:

$$x_R[n] = r^n \cos\omega_0 n \quad \text{and} \quad x_I[n] = r^n \sin\omega_0 n \tag{8.33}$$

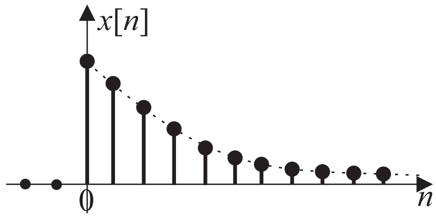


Figure 8.14: The exponential decay sequence.

– End-of-Topic 46: **Introduction to basic discrete-time signals** –

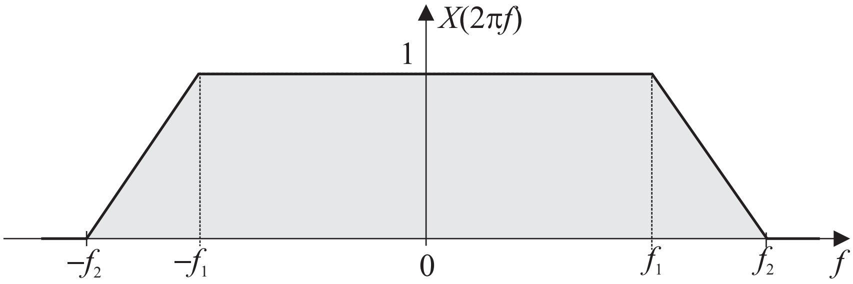


Figure 8.15: The spectrum of the signal in part 2 of Exercise 8.2.

8.4 Tutorial Exercises Sampling Theorem

Exercise 8.1 (Fourier Transforms and Nyquist). [*Difficulty:* **3** ($\star\star\star$)] Determine the Nyquist sampling rate and sampling interval for the signal:

$$x(t) = 0.01 \operatorname{sinc}(100\pi t) \tag{8.34}$$

⋈

Exercise 8.2 (Sampling Theory). [*Difficulty:* **2** ($\star\star$)]

1. Find the Fourier transform of the signal $v(t) = \delta(t)$, the unit impulse function.
2. A signal $x(t)$ whose spectrum is shown in Figure 8.15 is sampled at a frequency of $f_s = f_1 + f_2$ Hz. Find all the sample values of $x(t)$ merely by inspection of $X(2\pi f)$ or the sampled spectrum.

Exercise 8.3 (Application of the sampling theorem). [*Difficulty:* **3** ($\star\ \star\ \star$)] A communications signal, $x(t)$, has spectrum, $X(f)$, shown in Figure 8.16.

1. Write down the highest frequency present in the signal in Figure 8.16, and hence write down the minimum sampling frequency needed to sample $x(t)$ without distortion.

   Plot the spectrum of the signal when sampled at this frequency, as a function of frequency in Hertz, and explain how $x(t)$ can be reconstructed from these samples.

2. A busy Engineer misreads the figure in Figure 8.16, concludes the bandwidth of the signal is really 10 Hz, and incorrectly decides the minimum adequate frequency for sampling $x(t)$ is 20 Hz.

   Sketch the spectrum $X_s(f)$ of the sampled signal when $x(t)$ is sampled at a rate of 20 Hz.

   Can the signal $x(t)$ be reconstructed from these samples?

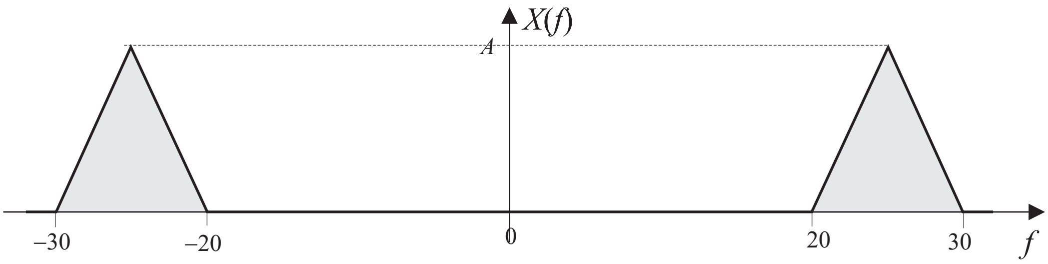


Figure 8.16: Spectrum of a signal, $X(f)$, as a function of frequency in Hertz.

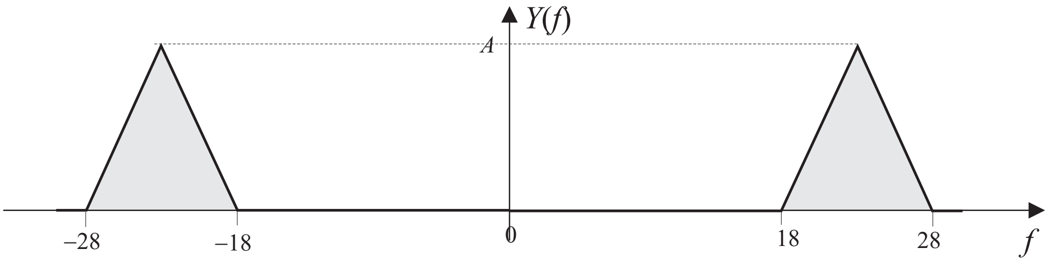


Figure 8.17: Spectrum of a signal, $Y(f)$, as a function of frequency in Hertz.

3. The same Engineer, using the same reasoning as in part 2, looks at another bandpass signal, $y(t)$, with spectrum $Y(f)$ as shown in Figure 8.17, and concludes that a sampling rate of 20 Hz can be used.

   Sketch the spectrum of the resulting signal when $y(t)$ is sampled at a rate of 20 Hz. Can the signal $y(t)$ be reconstructed from these samples?

Exercise 8.4 (Sampling a sinusoid). [*Difficulty:* **3** ($\star\star\star$)] A co-sinusoid of frequency f_0 is sampled at a rate of $f_s = 20$ Hz. Find the **apparent frequency** of the sampled signal if f_0 is:

1. 8 Hz
2. 12 Hz
3. 20 Hz
4. 22 Hz
5. 32 Hz

Derive your results using two separate methods, namely:

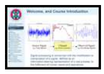
1. The time-domain method which considers what co-sinusoid $\cos(2\pi f_a t + \phi)$ yields the same samples as the sinusoid $\cos(2\pi f_0 t + \phi)$ when sampled at f_s.
2. A frequency-domain method in which the spectrum of the co-sinusoid (assume $\phi = 0$ for simplicity) when sampled at frequency f_s.

9

Fourier Analysis of Discrete-Time Aperiodic Signals

This handout introduces the discrete-time Fourier transform (DTFT) for analysing infinite-duration discrete-time signals.

9.1 Fourier Analysis of Discrete-Time Signals



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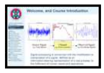
There are various ways of developing tools for the analysis of (infinite-duration) discrete-time signals in the frequency domain. The simplest starting point is to again reconsider the Fourier transform of a sampled data sequence.

9.1.1 Developing the discrete-time Fourier transform



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A sampled signal is given by the product of the original continuous-time signal, $x(t)$, and an impulse train, such that:

$$x_s(t) = s(t)\ x(t) \quad \text{where} \quad s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{9.1}$$

Using the result that multiplication of a function, $\phi(t)$, by an impulse centered on T, gives $\phi(t)\ \delta(t-T) = \phi(T)\ \delta(t-T)$, then Equation 9.1 may be written as:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT)\ \delta(t - nT) \tag{9.2}$$

Taking the Fourier transform gives:

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)\; e^{-j\omega t}\, dt \tag{9.3}$$

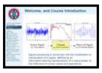
$$= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x(nT)\;\delta(t-nT) \right) e^{-j\omega t}\, dt \tag{9.4}$$

Since the limits of the integral and the summation are independent of one another, the order of integration and summation can be interchanged without any difficulty, and thus:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT) \underbrace{\left(\int_{-\infty}^{\infty} \delta(t-nT)\, e^{-j\omega t}\, dt \right)}_{\text{apply sifting theorem}} \tag{9.5}$$

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT)\; e^{-jn\omega T} \tag{9.6}$$

This is known as the **discrete-time Fourier transform (DTFT)**, and is the tool for analysing infinite-duration discrete-time signals.



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9.1.2 Properties of the DTFT

A key property of the DTFT is that it is periodic, with period $\frac{2\pi}{T}$, where T is the sampling period. This is because, if m is an integer,

$$X_s\left(\omega + m\frac{2\pi}{T}\right) = \sum_{n=-\infty}^{\infty} x[n]\; e^{-jn\left(\omega + m\frac{2\pi}{T}\right)T} = \sum_{n=-\infty}^{\infty} x[n]\; e^{-jn\omega T - j2\pi m} \tag{9.29}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\; e^{-jn\omega T}\, e^{-j2\pi m} = X_s(\omega) \tag{9.30}$$

since $e^{-j2\pi m} = 1$ for any integer m. This is hence a periodic spectrum. However, this isn't surprising, as this has been seen in the sampling theorem expression given by:

$$X_s\left(\omega\right) = \frac{1}{T} \sum_{n=-\infty}^{\infty} X\left(\omega - n\omega_s\right) \tag{9.31}$$

This expression can easily be shown to be periodic. Because of the periodicity of the spectrum, then it is usual to write the discrete-time Fourier transform (DTFT) as follows. The DTFT of the discrete-time sequence $x[n]$ for $n \in \mathbb{Z}$ is given by:

$$X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\; e^{-jn\omega T} \tag{9.32}$$

Summary Slide 56 Fourier Analysis of Discrete-Time Signals

Developing the DTFT

$$x_s(t) = s(t)\ x(t) \quad \text{where} \quad s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Using the result $\phi(t)\ \delta(t-T) = \phi(T)\ \delta(t-T)$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} \underbrace{x(nT)}_{x[n]}\ \delta(t - nT) \tag{9.7}$$

Taking transforms:

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)\ e^{-j\omega t}\, dt \tag{9.8}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x(nT)\ \delta(t-nT) \right) e^{-j\omega t}\, dt \tag{9.9}$$

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT) \underbrace{\left(\int_{-\infty}^{\infty} \delta(t-nT)\, e^{-j\omega t}\, dt \right)}_{\text{apply sifting theorem}} \tag{9.10}$$

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x(nT) \left[e^{-j\omega t} \right]_{t=nT}$$

$$X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-jn\omega T} \tag{9.11}$$

This is the **discrete-time Fourier transform (DTFT)**, denoted by $X\left(e^{j\omega T}\right)$ for reasons that will become apparent. Compare the DTFT with the complex Fourier series – note they are the same in reverse.

Summary Slide 57 Fourier Analysis of Discrete-Time Signals

How does the DTFT relate to Nyquist

- We have two expressions for the discrete-time signal spectrum:

$$X_s(\omega) = X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-jn\omega T} \tag{9.12}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \tag{9.13}$$

How are these expressions related?

$$X_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \tag{9.14}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\ e^{-j(\omega - n\omega_s)t}\,dt \tag{9.15}$$

$$= \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\ \underbrace{\left\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\right\}}\ dt \tag{9.16}$$

$$= \int_{-\infty}^{\infty} x(t)\ e^{-j\omega t}\ \underbrace{\sum_{n=-\infty}^{\infty}\delta(t - nT)}\ dt \tag{9.17}$$

$$= \sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\{x(t)\ e^{-j\omega t}\}\delta(t - nT)\ dt \tag{9.18}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\ e^{-j\omega nT} \tag{9.19}$$

This analysis gives us a second approach for deriving the DTFT.

Summary Slide 58 Fourier Analysis of Discrete-Time Signals

Example of the DTFT

Find the DTFT of the signal:

$$x[n] = a^n\, u[n] \tag{9.20}$$

Under what conditions is the expression valid?

1. Note the range of ω is bounded by half the sampling frequency, $\omega_s = \frac{2\pi}{T}$ since the spectrum is periodic; i.e. $-\frac{\omega_s}{2} < \omega \leq \frac{\omega_s}{2}$ or $-\frac{\pi}{T} < \omega \leq \frac{\pi}{T}$ or $-\pi < \omega T \leq \pi$.

2. Defining the **normalised frequency** $\hat{\omega} = \omega\, T$, it follows that $-\pi < \hat{\omega} \leq \pi$.

Summary Slide 59 Fourier Analysis of Discrete-Time Signals

Developing the DTFT

- The DTFT is periodic with period equal to the sampling frequency, $\omega_s = \frac{2\pi}{T}$, where T is the sampling period.
- Since:

$$X_s(\omega) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-jn\omega T} \tag{9.21}$$

it follows

:

$$\begin{aligned} X_s\left(\omega + m\frac{2\pi}{T}\right) &= \sum_{n=-\infty}^{\infty} x[n]\ e^{-jn\left(\omega+m\frac{2\pi}{T}\right)T} \\ &= \sum_{n=-\infty}^{\infty} x[n]\ e^{-jn\omega T}\, e^{-2\pi jnm} = X_s(\omega) \end{aligned}$$

since mn is an integer, and therefore $e^{-2\pi jnm} = 1$. This result is expected given the sampling theorem.

- Due to this periodicity of the DTFT, with period $\frac{2\pi}{T}$, it is common notation to write the DTFT as:

$$X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\ e^{-jn\omega T} \tag{9.22}$$

- Is there an IDTFT? Yes, of course, but what does it look like? Moreover, how would you analyse a periodic signal?

Summary Slide 60 Fourier Analysis of Discrete-Time Signals

The inverse-DTFT

Since the spectrum of $x[n]$ is periodic with period $\frac{2\pi}{T}$, it can be analysed using Fourier series! Defining $\Omega = \frac{2\pi}{T}$ and $\omega_0 = \frac{2\pi}{\Omega} = T$:

$$\hat{x}_n = \frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} X\left(e^{j\omega T}\right)\, e^{jn\omega_0 \omega}\, d\omega \tag{9.23}$$

$$= \frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \left(\sum_{m=-\infty}^{\infty} x[m]\, e^{-jm\omega T} \right) e^{jn\omega T}\, d\omega \tag{9.24}$$

interchanging the order of integration and summation

$$= \sum_{m=-\infty}^{\infty} x[m]\, \underbrace{\frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{-jm\omega T} e^{jn\omega T}\, d\omega}_{\text{use orthogonality}} \tag{9.25}$$

Using othogonality, then

$$\frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{-jm\omega T} e^{jn\omega T}\, d\omega = \delta[n-m] \tag{9.26}$$

giving $\hat{x}_n = x[n]$. In other-words, the inverse-DTFT (IDTFT) is given by:

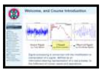
$$x[n] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X\left(e^{j\omega T}\right)\, e^{jn\omega T}\, d\omega \tag{9.27}$$

KEYPOINT! (Fourier Analysis of Discrete-time Signals). The DTFT and IDTFT are Fourier transform pairs for analysing infinite-duration discrete-time signals. With sampling period T, they are:

$$X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega T}$$

$$x[n] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X\left(e^{j\omega T}\right)\, e^{jn\omega T}\, d\omega \tag{9.28}$$

□



9.1.3 The inverse-DTFT

New slide

Since the spectrum of the discrete-time signal $x[n]$, given by $X\left(e^{j\omega T}\right)$, is periodic with *period* $\frac{2\pi}{T}$, then an interesting question to ask is what does a Fourier series analysis of the spectrum give? Since the spectrum is periodic, even though in the frequency domain, taking the Fourier series is a perfectly valid approach. However, in order to ensure orthognality of the basis functions used, it is necessary to modify a sign in the Fourier series expression.

Defining $\Omega = \frac{2\pi}{T}$, the modified Fourier series coefficients of the spectrum of $X\left(e^{j\omega T}\right)$ are given by:

$$\hat{x}_n = \frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} X\left(e^{j\omega T}\right) e^{jn\omega_0\omega} \, d\omega \tag{9.33}$$

where the term $e^{jn\omega_0\omega}$ doesn't have the minus sign that you would usually expect in the expression for calculating Fourier series coefficients; in otherwords, this is equivalent to $e^{-jn\omega_0 t}$ with t replaced by $-\omega$. Note that the *fundamental* $\omega_0 = \frac{2\pi}{\Omega}$ following the usual definition that ω_0 is proportional to the *inverse* of the *period* of the waveform. Note, then, that (interestingly) $\omega_0 = T$, which is nice and simple.

Substituting Equation 9.32 gives, noting one must be careful about the variables used in the summation:

$$\hat{x}_n = \frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} \left(\sum_{m=-\infty}^{\infty} x[m] \, e^{-jm\omega T} \right) e^{jn\omega T} \, d\omega \tag{9.34}$$

interchanging the order of integration and summation noting that the limits are independent,

$$= \sum_{m=-\infty}^{\infty} x[m] \underbrace{\frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{-jm\omega T} e^{jn\omega T} \, d\omega}_{\text{use orthogonality}} \tag{9.35}$$

Using orthogonality, (see the Handout on Fourier Series but with T replaced by Ω and t replaced by ω) then

$$\frac{1}{\Omega} \int_{-\frac{\Omega}{2}}^{\frac{\Omega}{2}} e^{-jm\omega T} e^{jn\omega T} \, d\omega = \delta[n-m] \tag{9.36}$$

which means that Equation 9.35 becomes

$$\hat{x}_n = \sum_{m=-\infty}^{\infty} x[m] \, \delta[n-m] = x[n] \tag{9.37}$$

by use of the *sifting theorem*, or just direct expansion of the summation in Equation 9.37 by considering the index n to be a particular given value. Hence, with $\hat{x}_n = x[n]$, this means that the time-domain samples can be recovered by taking the Fourier series of the DTFT.

In other-words, the inverse-DTFT (IDTFT) is given by:

$$x[n] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X\left(e^{j\omega T}\right) e^{jn\omega T}\, d\omega \tag{9.38}$$

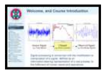
KEYPOINT! (DTFT and IDTFT transform pairs). A discrete-time signal $x[n]$ can be analysed in the frequency domain using the discrete-time Fourier transform (DTFT), and reconstructed from the spectral representation using the inverse-DFT (IDFT), given by:

$$X\left(e^{j\omega T}\right) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jn\omega T} \tag{9.39a}$$

$$x[n] = \frac{T}{2\pi} \int_{-\frac{\pi}{T}}^{\frac{\pi}{T}} X\left(e^{j\omega T}\right) e^{jn\omega T}\, d\omega \tag{9.39b}$$

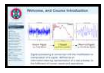
□

9.1.4 Duality of the DTFT



The duality principle discussed earlier in the course provides some interesting insights into Fourier analysis. *New slide*

9.2 Developing the Discrete Fourier Transform

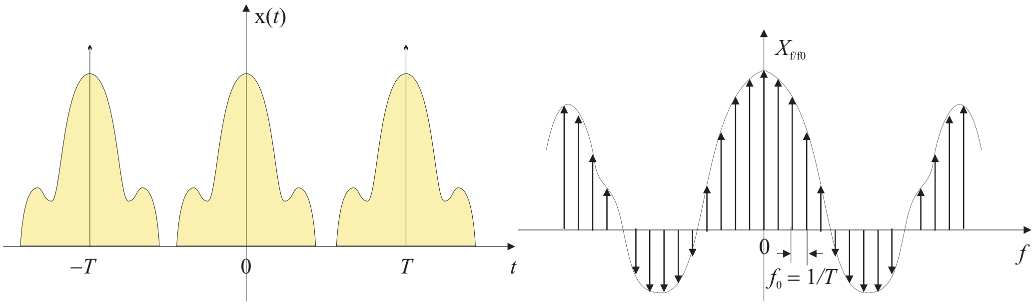


To compute Fourier transforms on a computer, none of the existing transforms suffice. The discrete Fourier transform (DFT) is needed. *New slide*

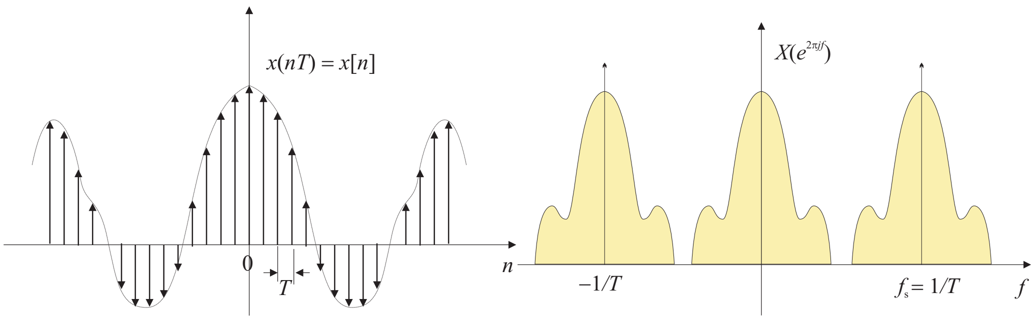
Summary Slide 61 Fourier Analysis of Discrete-Time Signals

Duality of the DTFT

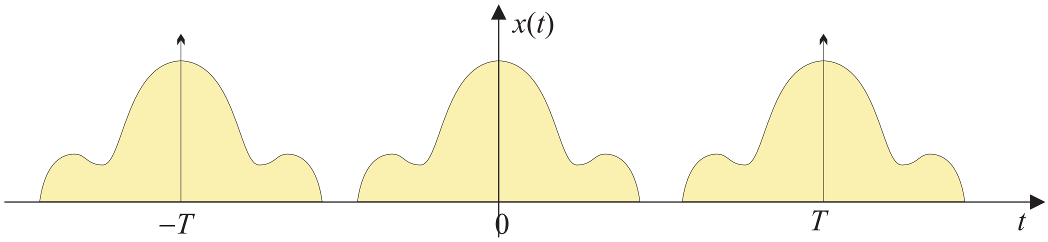
1. The **Fourier analysis** of a continuous-time periodic signal is the **Fourier Series**, which is a discrete-collection of Fourier coefficients X_n, although each coefficient has associate with it a frequency $f = nf_0$.



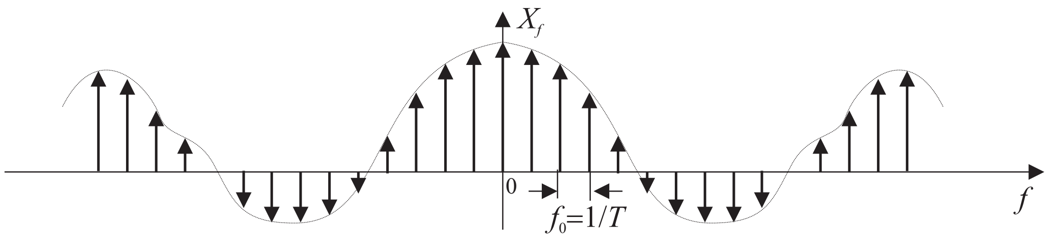
2. The **Fourier analsysis** of a discrete-time signal is the **discrete-time Fourier transform**, whose spectrum is continuous and periodic.



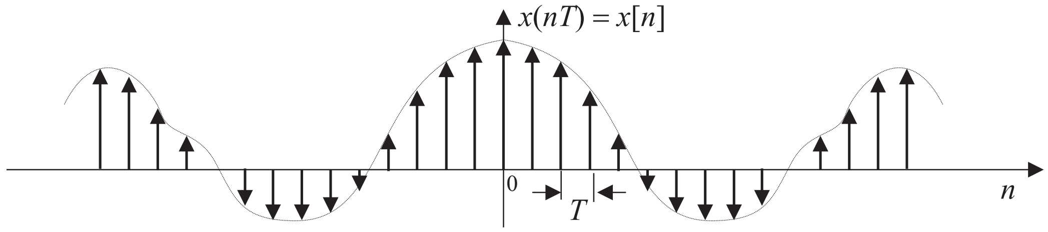
KEYPOINT! (Duality Revisited). These results could have been deduced using the principle of **duality**.



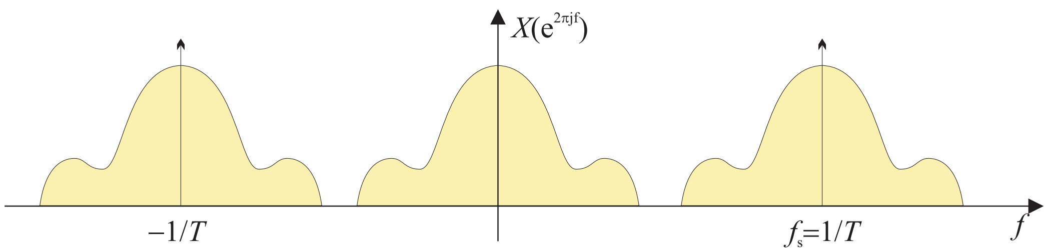
(a) Periodic time signal, $x(t)$.



(b) Spectrum, consisting of delta functions weighted by Fourier coefficients, X_f.



(c) Discrete-time input signal, $x[n]$.

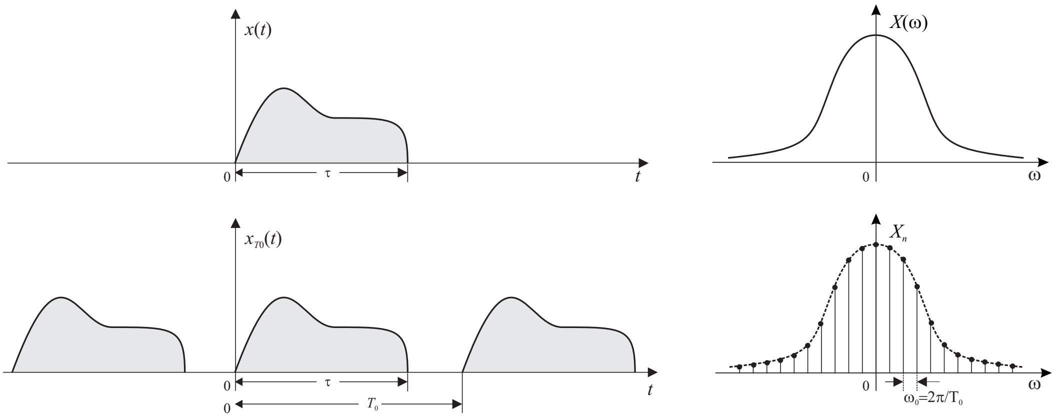


(d) Periodic spectrum, $X\left(e^{j\omega T}\right)$.

Figure 9.1: Duality of the Fourier Series and the discrete-time Fourier transform.

Summary Slide 62 Fourier Analysis of Discrete-Time Signals

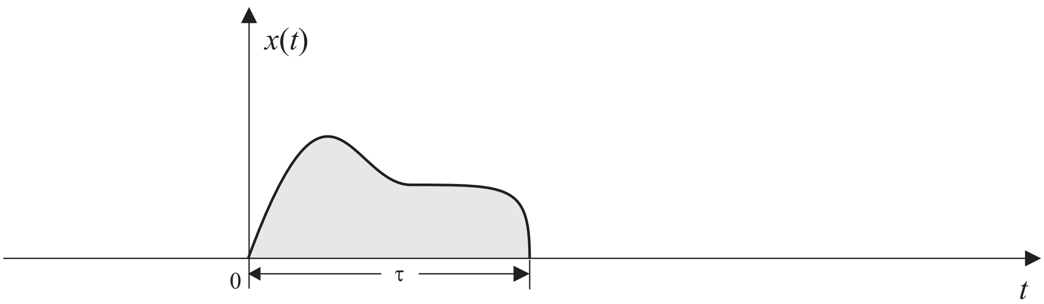
Spectral Sampling



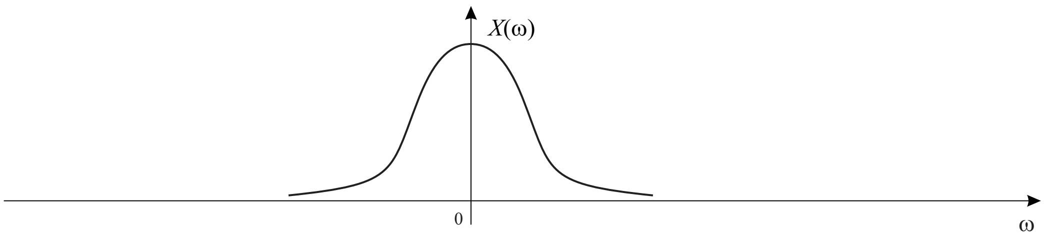
Although spectral sampling is difficult to implement electronically, it could be performed optically using an **optical prism** and *measurement slits*. However, it is the concept that is crucial!

Summary Slide 63 Fourier Analysis of Discrete-Time Signals

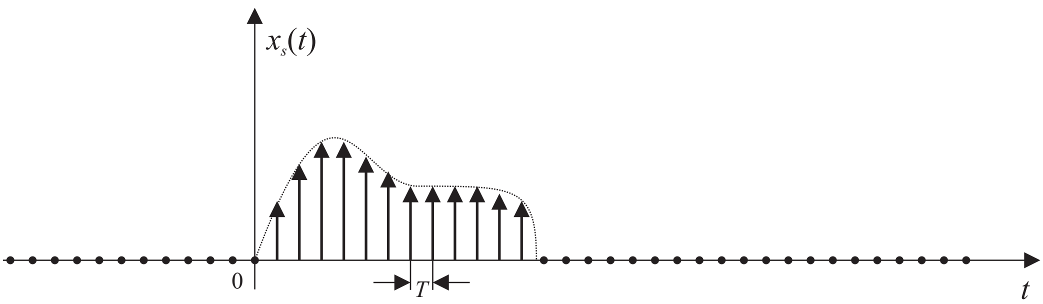
Temporal-Spectral Sampling!



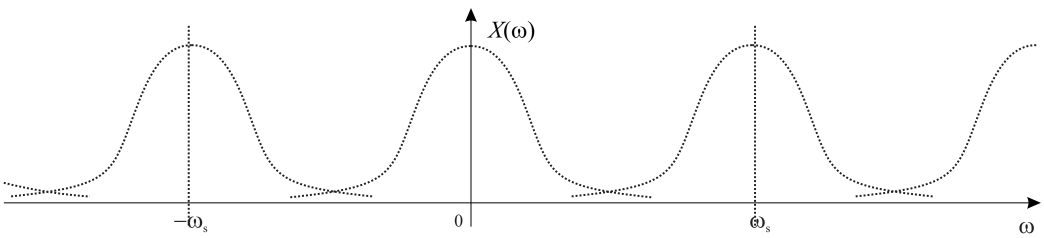
Time domain signal



Spectrum



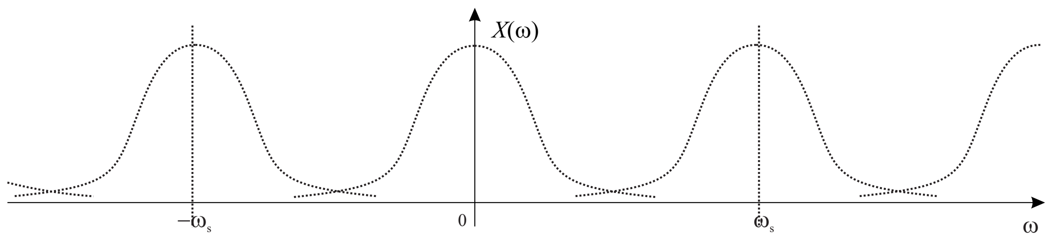
Sample time domain signal



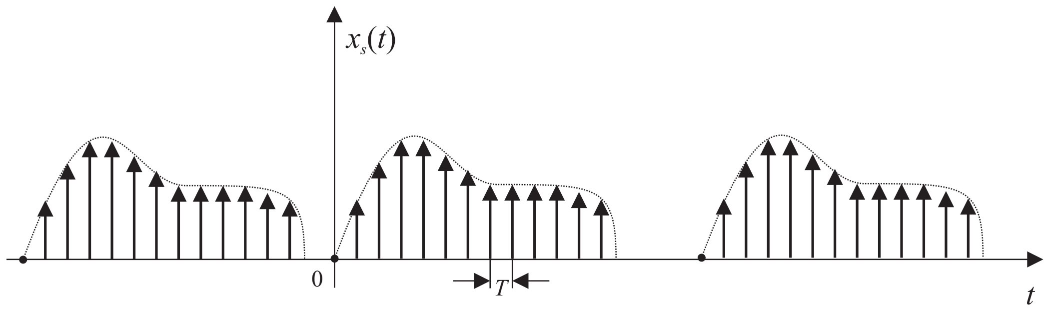
Resulting Spectrum

Summary Slide 64 Fourier Analysis of Discrete-Time Signals

Temporal-Spectral Sampling!



Sample spectrum (not shown)



Resulting time-domain spectrum

9.3 Tutorial Exercises

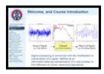
There are currently no tutorial questions associated with this handout.

10

Revision Material and Revision Tutorials

This handout covers some information about the format of the exam, some general knowledge revision questions related to the Signals aspect of the course, as well as a list of past related exam questions.

10.1 Format of the Examination May 2022



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The format of the exam consists of two sections, for which all questions are compulsory and must be answered.

1. Section A which consists of ONE question covering broad aspects from the entirety of the course.
2. Section B which consists of TWO questions; these questions are in-depth.

All questions are worth 20-marks, with the exam being marked out of a total of 60. This is the same format as the May 2021 exam.

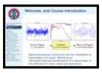
However, note that this format differs from the exam format used in previous versions of the course (Signals and Communications 2, SCEE08007), where Section B consisted of more questions and gave you a choice. The change from optional to all compulsory questions for the new version of this course arose due to the open-book nature of the examination.

Generally, the distribution of content within the questions corresponds to the distribution of content within the lecture course. Therefore, you can expect:

- around half of the paper to cover questions related to signal analysis (power, Fourier, sampling theory etc);

- around a third of the course related to communications theory;
- one sixth of question course on machine learning.

A datasheet is available for the signal analysis and communication components, and this has been added to LEARN under **Assessment**>**Datasheet for SNADA**. However, since the exam is open book, you are of course not restricted to just using this datasheet and can use your lecture notes. It is provided as this datasheet is usually available in the exam.



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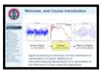
10.2 General Knowledge Revision Questions

A good start to being prepared for the examinations for the signal analysis component of the course is to ensure that you are able to answer the following questions without consulting your lecture notes or tables. If you need to regularly consult with taught materials in the exam, this will cost you valuable time.

It is also strongly advised that you have worked through all the tutorial questions and their solutions, which are available on LEARN.

The following questions cover general knowledge in signals and communications system theory part of the course which you should have on the tip of your tongue in order to tackle the examination questions. There is no particular order to the questions and they by no means form an exhaustive set. Solutions are not provided, but each of the questions do not need long mathematical answers; many of them can be answered with a short statement.



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10.2.1 Signal Analysis Related Questions

1. Write down the expressions for the energy and power of a signal, $x(t)$.
2. Write down the key characteristics that distinguish the following types of signals:
   (a) Discrete-time
   (b) Continuous-time
   (c) Periodic
   (d) Non-periodic
3. Write down the defining equations for the trigonometric Fourier series, the complex Fourier series, the Fourier transform and the discrete-time Fourier transform.
4. What is the relationship between the complex and trigonometric Fourier series coefficients?
5. How would you work out the fundamental frequency of a periodic-waveform?

6. If a signal can be decomposed as a linear combination of sinusoids and co-sinusoids, is it certain to result in a periodic waveform? Explain your answer.

7. What is the duality theorem for Fourier transforms?

8. What is the sifting theorem?

9. What is:

   (a) the Fourier transform of a unit impulse?
   (b) the Fourier series of an impulse train?

10. What is the Fourier transform of a sinusoid and a co-sinusoid?

11. Write down Parseval's theorem for continuous-time periodic-signals and also for non-periodic signals.

12. If the Fourier transform of a signal, $x(t)$, is given by $X(\omega)$, what are the Fourier transform of the signals:

    (a) $g(t) = x(t - \tau)$
    (b) $h(t) = x(t)\ e^{j\omega_0 t}$

13. A complicated waveform $x(t)$ can be decomposed into simpler functions given by:

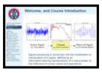
$$x(t) = \sum_{p=1}^{P} \alpha_p \, x_p(t) \tag{10.1}$$

    where α_p are known constants, and the simpler waveforms have Fourier transforms $x_p(t) \rightleftharpoons X_p(\omega)$, which are also known.

    Write down an expression for the Fourier transform of $x(t)$, denoted $X(\omega)$ in terms of the α_p's and $X_p(\omega)$'s.

14. (a) Sketch and label a block diagram of a practical sampled data system.
    (b) State Nyquist's sampling theorem.
    (c) How does the spectrum of a sampled signal relate to the spectrum of the original continuous-time signal?
    (d) A sine-wave of frequency 29.5 kHz is applied to a sampled data system with a sampling rate of 10 kHz.
        i. What frequencies will be present at the output of the D/A converter?
        ii. What frequencies will be present at the output of the reconstruction filter?

15. Explain why, in practical analogue-to-digital and digital-to-analogue conversion applications, it is impractical to sample a signal perfectly. Why is perfect signal reconstruction impossible even when a perfectly sampled signal is available?



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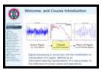
10.3 Past Related Exam Questions



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10.3.1 Signals and Communications 2 Questions

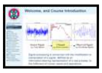
> Past exam papers related to the ”Sensor Networks and Data Analysis” course can be found in the course ”Signals and Communications 2” with course code SCEE08007. You can find these using by going to the University Exam Papers Online website (`https://exampapers.ed.ac.uk/` and searching for code SCEE08007.

Papers from academic year 2015/16 to 2018/19 are available, so the past four years of papers in total. Note that the paper for 2019/20 is not available, as the exam was cancelled due to the pandemic. Please also note that the following questions are not relevant due to a slight change in the course curriculum (due to introducing the machine learning component):

- 2015/16 Paper (May 2016): Question A1 c)ii) on orthogonality (5 marks).
- 2016/17 Paper (May 2017): Question A1 b) on orthogonality (1 mark)
- 2018/19 Paper (May 2019): Question B2 is “doable” in the sense that the question is self-contained with the relevant theory described as part of the question. However, it is a variation of a tutorial question that was not explicitly covered this year. Please feel free to try this question as a challenge, but note that as we did not explicitly cover orthogonality this year, this question should not be considered examinable.

If you identify any other questions which you think are not relevant, or you are concerned were not adequately covered in the course, please check with the teaching team first on the discussion boards.

Although orthogonality was not covered this year, it is an important concept that you are likely to come across in other courses in future.



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10.3.2 Signals Analysis Related Questions (Pre-2012)

There are a few past exam questions from the third year **Signals and Systems** course that relate to this course, but they are few and far between. Moreover, as time passes, it will become increasingly difficult to access the orginal copy of these papers. However, I have attempted to list them below as a record of the available questions. The course titles are **U00439 Electronic Engineering 3** for the years 2005 to 2009 and **ELEE09007 Electronic Engineering 3: Signal and Systems** for 2010 and 2011.

December 2005 Question B2

December 2006 Question A1, part a) and part b)

December 2007 Two questions cover aspects of the course:

1. Question A1, part a)
2. Question B2, same as December 2005 Question B2

December 2008 Question B2 – note this is a difficult question!

December 2009 Three questions cover aspects of the course:

1. Question A1, part b)
2. Question B2 – this is one of the tutorial questions in Handout 3
3. Question B3, part a)

December 2010 Question A1, part a), part d) and and part e)

December 2011 Question A1, part d)

These questions are listed below as a set of exercises.

Example 10.1 (December 2005, Question B2, Sampling Theory).

1. The **sampling function** is a periodic function composed of impulses. This function is shown in Figure 10.1, and is represented by the expression:

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Show that $s(t)$ can be written as a Fourier series whose complex coefficients are all identical, such that it is of the form:

$$s(t) = K \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$$

Indicate clearly the values of K, and ω_0, in terms of T.

2. A signal $f(t)$, with Fourier transform $F(\omega)$, is sampled every T seconds using the sampling function. The sampled signal, $f_s(t)$, is given by $f_s(t) = s(t)\, f(t)$. Using the result in part 1, show that the Fourier transform of $f_s(t)$ is given by: 6

$$F_s(\omega) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(\omega - n\omega_0) \qquad \bowtie$$

You may use the **frequency shift theorem** which states that if $f(t) \rightleftharpoons F(\omega)$, then $e^{j\omega_0 t} f(t) \rightleftharpoons F(\omega - \omega_0)$.

3. With reference to the result in part 3, explain the meaning of the terms **aliasing** and **Nyquist frequency** when applied to a sampled signal. How can aliasing be avoided? Use diagrams where appropriate.

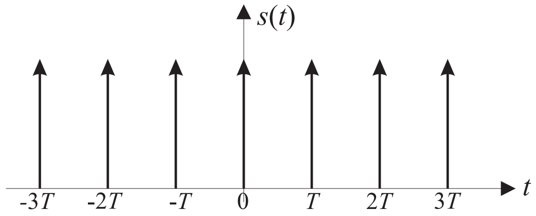


Figure 10.1: The sampling function – an impulse train.

4. Explain why, in practical analogue-to-digital and digital-to-analogue conversion applications, it is impractical to sample a signal perfectly. Why is perfect signal reconstruction impossible even when a perfectly sampled signal is available?

Example 10.2 (December 2006, Question A1, parts a) and b)). 1. Describe the primary difference between **periodic** and **aperiodic** signals.

Give an example of a **periodic** signal, and an example of an **aperiodic** signal. In each case, you may either sketch the signal, taking care to indicate any key characteristics, *or* write down a functional representation.

2. Write down the expressions for the **energy** and **power** of a signal, $x(t)$.

   Give an example of a signal that should be classified as an **energy signal**. Give an example of a **power signal**. Justify your answers.

Example 10.3 (December 2007, Question A1, part a)). 1. A sampled data system for analogue-to-digital and digital-to-analogue conversion applications is shown in Figure 10.2.

(a) Name the three main subsystems, $\{S_1,\, S_2,\, S_3\}$, of the analogue-to-digital block. Name the two main subsystems, $\{R_1,\, R_2\}$, of the discrete-time to continuous-time converter.

The subsystems should be named in the order found in a sampled data system.

(b) With reference to the subsystems $\{S_1, S_2\}$ in Figure 10.2, give one reason why it is impractical to *sample* a signal perfectly.

(c) Explain, with reference to subsystem R_2, why perfect reconstruction of an analogue signal is impossible even when a perfectly sampled signal is available?

Example 10.4 (December 2009, Question A1, part b)).

Consider the following two signals:

- the two-sided exponential function,

$$x(t) = 2\,e^{-\frac{|t|}{2}}, \quad -\infty < t < \infty;$$

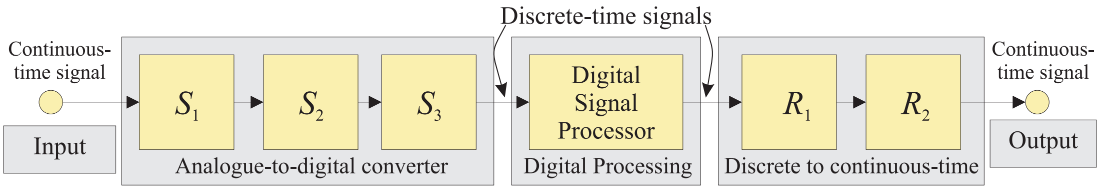


Figure 10.2: A sampled data system.

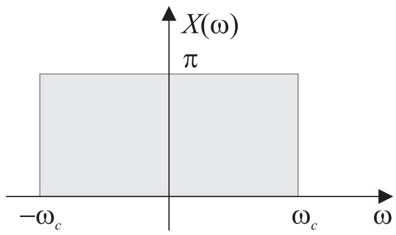


Figure 10.3: Fourier transform of $x(t)$.

- the sawtooth function,

$$y(t) = \begin{cases} t & -1 < t \leq 1 \\ y(t+2) & \text{otherwise} \end{cases} . \qquad \bowtie$$

For each signal:

1. sketch the signal waveform;
2. determine whether **energy** or **power** provides the most suitable *measure* of the size of the signal, justifying your answer;
3. calculate the value of the measure chosen in part 2.

Example 10.5 (December 2009, Question B3, part a)).

A signal, $x(t)$, has Fourier transform, $X(\omega)$, as shown in Figure 10.3.

1. Find an expression for the signal, $x(t)$, as a function of time, t.
2. Using Parseval's theorem, or otherwise, calculate the total energy in the signal $x(t)$.

Example 10.6 (December 2010, Question A1, parts a), d), and e)). 1. Consider the following two signals:

$$x_a(t) = \sin(12t) + \cos(4t)$$
$$x_b(t) = (1 + 2\cos(40\pi t))\,\cos(10\pi t) \qquad \bowtie$$

(a) In each case, find the fundamental frequency of the periodic waveform in hertz.

(b) For the signal $x_a(t)$ only, find the complex exponential Fourier series coefficients without performing any integration.

2. Write down expressions for the **energy** and **power** of a continuous-time signal, $x(t)$. Pay particular attention to defining the limits of any integrals used in your expressions.

3. A band-limited continuous-time analogue signal, $x_c(t)$, has non-zero spectral components up to a frequency of f_m. The signal is sampled at a frequency $f_s \geq 2f_m$ and digitised to produce the discrete-time signal $x[n]$.

   (a) State clearly the steps involved in reconstructing the analogue signal, $x_c(t)$, from the sampled signal, $x[n]$.

   (b) Explain why perfect reconstruction of $x_c(t)$ from $x[n]$ is impossible even when the sampling process from $x_c(t)$ to $x[n]$ was perfect.

Example 10.7 (December 2011, Question A1, part d)).

By considering the fundamental frequencies of the following signals, state if they are periodic or non-periodic:

$$\begin{aligned} x_a(t) &= \sin(6t) + \cos(15t) \\ x_b(t) &= \sin(2t) - \cos(\pi t) \end{aligned} \qquad \bowtie$$

If the signal is periodic, write down its fundamental frequency in hertz.

11
Errata and Recent Changes

To err is human, but when the eraser wears out ahead of the pencil, you're overdoing it.

J Jenkins

This Handout is a list of the changes made to the course notes over the years, and gives an indication of the history of this document.

11.1 Errata, Recent, and Major Changes in Notes

These notes have evolved since a major course revision in 2013, where the entire course was entirely rewritten. The structure of these notes built on existing material based on the textbook:

> Mulgew B., P. M. Grant, and J. S. Thompson, *Digital Signal Processing: Concepts and Applications*, Palgrave, Macmillan, 2003.
>
> **IDENTIFIERS** – *Paperback*, ISBN10: 0333963563, ISBN13: 9780333963562
>
> See `http://www.homepages.ed.ac.uk/pmg/SIGPRO/`

The notes have undergone minor and major corrections each year, based on feedback from students, and the notes are far from static. In fact, as the course is delivered each year, a number of corrections and changes will have been made since the published version of these lecture notes were printed, as well as future amendments. The corrected versions will be posted on the course web-site and updated continuously. Some changes are not simply corrections, but exist for numerous reasons, including making explanations clearer, providing more detail and so forth.

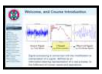
Previously, a list of these corrections and changes were presented here. However, after experience, it has seemed more appropriate to publish the entire **change-log**, and this is included in reverse chronological order in the final handout (available on LEARN). The changes are listed by year, and then by Chapter, and changes to Tutorial Questions or Solutions are also included.

Very minor updates will be added to Twitter `@HopgoodTeaching` on `http://twitter.com/HopgoodTeaching` rather than sending large numbers of emails out via LEARN. The hashtag `UoE_SCEE08007` will be used for comments relating to this course.

Note that all page numbers and equations references refer to the *published version* of this document. Currently, it is not possible to give the new page numbers and equation references due to the ever-changing re-pagination and equation numbering.

11.2 Recent Changes by Year

This section includes major recent changes by previous years. The list is by far from complete, but helps give an indication to interested students how the course has developed over the last few years.



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11.2.1 Major Changes in 2019 to 2020

Some changes this year include:

- *End-of-topic* markers, which are used in semi-automatic segmentation of the lecture recordings.

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11.2.2 Major Changes in 2018 to 2019

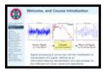
There have been a number of major changes this year, in addition to numerous minor chanages. The key major changes include:

- The addition of multiple choice questions for class discussion.
- The implicit inclusion of three major MATLAB demos used to highlight key principles.
- The separation of a *background* and *motivational* chapter in two more focused handouts on **signal processing applications**, and **underlying principles** of signals and systems.

- The separation of the chapter on Discrete-Time analysis into handouts on **sampling thoery** and the **discrete-time Fourier transform (DTFT)**. This should help separate these concepts, and make for a smoother transition into the third year course.
- Fixed a bug relating to the numbering of the summary slides.

Further additions include:

11.2.2.1 Handout Three



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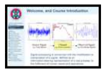
Added Linear System Test In the mathematical representation of systems, added examples of how to test whether a system is liner or not. Added Wednesday 30th January, 2019

11.2.2.2 Handout Four



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Added Parseval's Theorem Examples Added some examples, albeit theoretical, for application of Parseval's theorem for periodic signal. Added Thursday 07th February, 2019

Added Orthogonality Application Showed the use of Legendre polynomials (work in progress). Added Thursday 7th February, 2019

Added Examples Class Slides Find Trig. Fourier Coefficients of a rectangular pulse. Added Wednesday 30th January, 2019

11.2.3 Major Changes in 2017 to 2018

- Handout 2 on Signal Representation and Analysis has been restructured into three new handouts. This should make it easier when navigating the document.

11.2.4 Major Changes in 2015 to 2016

11.2.4.1 Handout One

Update Monday 26th September, 2016 Three minor typos corrected.

11.2.5 Major Changes in 2013 to 2014

11.2.5.1 Handout One

Update Thursday 06th March, 2014: Section 0.1 – Coure Preface Minor update to the first paragraph, making the split between signal analysis and communications more explicit.

11.2.5.2 Handout Two

Updated Section 2.2.3.1 Updated paragraphs describing discrete-time signals, since the DTFT is in fact covered in the second year; it is the z-transform which isn't covered until the third year.

Updated Sidebar 6 Includes comments on more general signal norms.

Updated Section 2.3.2.4 *Relationship between Trig and Complex Forms*: added Figure 2.19 on Page 71, which shows a graphical representation of the conjugate symmetry properties of the complex Fourier coefficients.

Updated Section 2.3.2.5 Minor update to *The Fourier Series Spectrum* on Page 72, adding a commment about why one wants to plot the Fourier coefficient X_n against coefficient index n.

Update Saturday 22nd February, 2014: Example 2.6 This example has been updated slightly.

Update Saturday 22nd February, 2014: Sect 2.3.2.7 The sections on Orthogonality and Parseval's theorem have been reordered (these were Sections 2.3.2.7 and 2.3.2.8). Moreover, the second on orthogonality has been revised to extend the concept of orthogonality of vectors to basis functions.

Update Thursday 27th March, 2014: Sect 2.4.2.3 Minor fixes to the section on the Fourier series of an impulse Train.

Update Tuesday 15th April, 2014: Sidebar 6 Added a diagram to Sidebar 6 which covers the absolute value as a signal norm. This diagram will have been included in lectures.

Update Tuesday 15th April, 2014: Section 2.3.3.4 Inserted main document text on *Parseval's Theorem for Aperiodic Signals* – this material was covered in one of the summary slides, but has been included in the main body of the text.

11.2.5.3 Chapter Three

Update Wednesday 12th March, 2014: Fixed Summary Slide 41 There were incorrect minus signs in Equations 3.2, 3.2, and the frequency shift theorem on this slide. They have now been fixed.

Update Wednesday 07th May, 2014: Fixed Equation 3.20 There was a problem with the apparent frequency in Equation 3.20 which was inconsistent with the rest of the text. This has now been corrected.