Exam No. B210392

Q2. C

EXAM No. B210392 EM2B COURSEWORK 1.

Q1. (i) Show that the work of $\vec{F} = (x^3 - y^3)\hat{\imath} + (x^3 - y^3)\hat{\jmath}$ on c, the unit circle centered at the origin with anti-clockwise direction, is:

$$\text{Work} = \oint_c \vec{F} \cdot d\vec{r}$$

Parameterize in $x = r\cos\theta$, $y = r\sin\theta$, $d\vec{r} = \begin{bmatrix} -r\sin\theta \\ r\cos\theta \end{bmatrix} d\theta$

then the work is $\oint \vec{F} \cdot d\vec{r} =$ (taking all values of $r = 1$):

$$W = \int_0^{2\pi} \left\{ (\cos^3\theta - \sin^3\theta)(-\sin\theta) + (\cos^3\theta - \sin^3\theta)(\cos\theta) \right\} d\theta$$

$$= \oint_0^{2\pi} \left\{ \sin^4\theta + \cos^4\theta - \sin\theta\cos^3\theta - \cos\theta\sin^3\theta \right\} d\theta \quad (1)$$

Completing the square,

$$\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$$

$$= (1)^2 - 2(\sin\theta\cos\theta)^2$$

$$= 1 - 2\left(\tfrac{1}{2}\sin 2\theta\right)^2$$

$$= 1 - \tfrac{1}{2}\sin^2 2\theta$$

$$\Rightarrow \quad \sin^4\theta + \cos^4\theta = 1 - \tfrac{1}{2}\left(\frac{1 - \cos 4\theta}{2}\right) \quad (2)$$

By grouping terms,

$$-\sin\theta\cos^3\theta - \cos\theta\sin^3\theta = -\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$= -\sin\theta\cos\theta\,(1)$$

$$= -\tfrac{1}{2}\sin 2\theta \quad (3)$$

Substituting (2) and (3) into (1), we obtain

$$W = \oint_0^{2\pi} \left[1 - \tfrac{1}{2}\left(\frac{1 - \cos 4\theta}{2}\right) - \tfrac{1}{2}\sin 2\theta \right] d\theta \quad (4)$$

To evaluate the integral (4), split into parts:

$$W = \int_0^{2\pi} 1\, d\theta - \int_0^{2\pi} \tfrac{1}{4}\, d\theta + \int_0^{2\pi} \frac{\cos 4\theta}{4}\, d\theta - \int_0^{2\pi} \tfrac{1}{2}\sin 2\theta\, d\theta \quad (5)$$

All of the terms in (5) with trigonometric integrands evaluate to zero, as they each complete one or more full periods over the integration and their average value is 0. This leaves the constant terms, with value

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$$\int_0^{2\pi} 1 \, d\theta - \int_0^{2\pi} \frac{1}{4} \, d\theta$$

$= \boxed{\frac{3}{2}\pi}$ is the total work done on the line.

(ii) Verify your answer using Green's theorem.

Green's theorem for work: $\oint_c \vec{F} \cdot \vec{r} = \iint_R \vec{\nabla} \times \vec{F} \, dA$.

if $\vec{F} = \begin{bmatrix} x^3 - y^3 \\ x^3 - y^3 \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ then Green's theorem states

$$\oint \vec{F} \cdot \vec{r} = \iint_R \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \, dA$$

$$= \iint_R 3x^2 + 3y^2 \, dA \qquad (1)$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$, and taking our differential area $dA = r\,dr\,d\theta$, (1) becomes

$$W = \iint_R \left[3(r\cos\theta)^2 + 3(r\sin\theta)^2\right] r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[3r^3\cos^2\theta + 3r^3\sin^2\theta\right] dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^1 3r^3 \, dr\,d\theta$$, pulling out $\cos^2\theta + \sin^2\theta = 1$

$$= \int_0^{2\pi} \left[\frac{3}{4} r^4\right]_{r=0}^{r=1} d\theta$$

$$= \frac{3}{4}\theta \Big|_{\theta=0}^{\theta=2\pi}$$

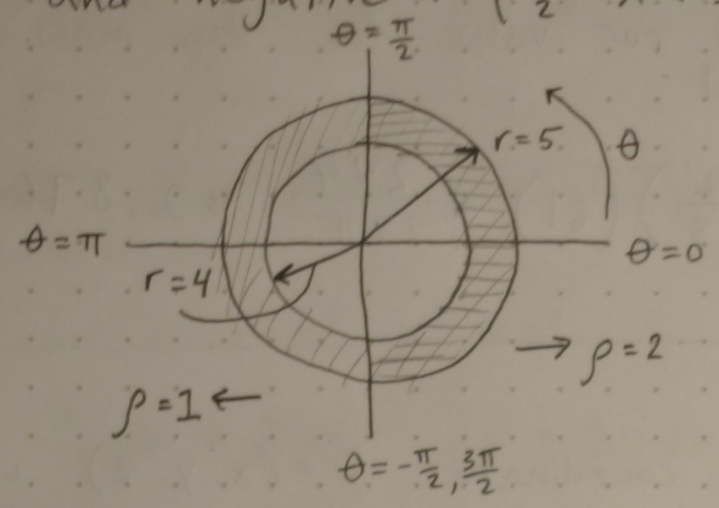
$$= \frac{3\pi}{2} \qquad (2)$$

Thus, the work is the same whether calculated explicitly or by using Green's theorem. This indicates that there are no discontinuities inside the region.

Q2. Consider a ring formed by subtracting a disk of rad. 4 from a disk of radius 5, both centered at (0,0). Compute the x-coordinate of the center of mass given the density

$$\rho = \begin{cases} 1, & x<0 \\ 2, & x \geq 0 \end{cases}$$

First we find the total mass. We will use polar coordinates; generally, the integrals will be split between positive x $\left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$ and negative x $\left(\frac{\pi}{2} < x < \frac{3\pi}{2}\right)$.



The total mass is

$$M = \iint_R \rho \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_4^5 2 \, r \, dr \, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_4^5 1 \, r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[r^2 \right]_{r=4}^{r=5} d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left[\frac{1}{2} r^2 \right]_{r=4}^{r=5} d\theta$$

$$= 9\theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} + \frac{9}{2} \theta \Big|_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}}$$

$$M = \frac{27\pi}{2} \qquad (1)$$

We use this mass estimate M to determine the coordinates $\bar{x}, \bar{y}$ of the center of mass. Since the object is symmetric about the y-axis, $\bar{y} = 0$. $\bar{x}$ is given by

$$\bar{x} = \frac{1}{M} \iint_R \rho \, x \, dA$$

$$= \frac{1}{M} \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_4^5 \left[(2) \, r \cos\theta \right] r \, dr \, d\theta + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_4^5 \left[1 \, r \cos\theta \right] r \, dr \, d\theta \right\}$$

$$= \frac{1}{M} \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \, d\theta \left[\frac{2}{3} r^3 \right]_{r=4}^{r=5} + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta \, d\theta \left[\frac{1}{3} r^3 \right]_{r=4}^{r=5} \right\}$$

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$$= \frac{1}{M}\left\{ \left[\sin\theta\right]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \left[\frac{2}{3}(61)\right] + \left[\sin\theta\right]_{\theta=\frac{\pi}{2}}^{\theta=\frac{3\pi}{2}} \left[\frac{1}{3}(61)\right]\right\}$$

$$= \frac{1}{M}\left\{ 2\left[\frac{2}{3}(61)\right] - 2\left[\frac{1}{3}(61)\right]\right\}$$

$$= \frac{1}{M}\left(\frac{2}{3}\right)61 \qquad (2)$$

Substituting into (2) our value for the mass of the object (1), we find

$$\bar{x} = \left(\frac{2}{27\pi}\right)\left(\frac{2}{3}\right)(61) = \frac{244}{27\pi} \approx 2.876$$

Q3. A point mass m at coordinates $\vec{P} = (x, y, z)$ with a force $\vec{f}$ of magnitude

$$\|\vec{f}\| = \frac{mg}{r^2}$$

(i) Derive an expression for the force field $\vec{f}$ in the 3D frame.

A vector has <u>magnitude</u> and <u>direction</u>. In this case the magnitude is given as $\frac{mg}{r^2}$ and the direction is towards the origin, i.e., $-\vec{P}$. To obtain the force field f we need only multiply the magnitude with the unit vector in the $-\vec{P}$ direction

$$-\hat{P} = \frac{-\vec{P}}{\|P\|} = \begin{bmatrix} -x/r \\ -y/r \\ -z/r \end{bmatrix} = -\frac{1}{r}\vec{P} \qquad (1)$$

Then our force field is

$$\|\vec{f}\|(-\hat{P}) = -\frac{mg}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \left(r = \sqrt{x^2+y^2+z^2}\right)$$

Fully expanded,

$$\vec{f}(x,y,z) = \frac{-mgx}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\imath} + \frac{-mgy}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{\jmath} + \frac{-mgz}{(x^2+y^2+z^2)^{3/2}}\hat{k}$$

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ii) Show that $\vec{f}$ is conservative and derive its potential function φ such that $\vec{f} = \vec{\nabla}\varphi$.

If $\vec{f}$ is conservative, $\vec{\nabla} \times \vec{f} = 0$

$$\vec{\nabla} \times \vec{f} = \det \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{mgx}{r^3} & -\frac{mgy}{r^3} & -\frac{mgz}{r^3} \end{pmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}\left(\frac{-mgz}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{-mgy}{r^3}\right)\right) - \hat{j}\left(\frac{\partial}{\partial x}\left(\frac{-mgz}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{-mgx}{r^3}\right)\right)$$
$$+ \hat{k}\left(\frac{\partial}{\partial x}\left(\frac{-mgy}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{-mgx}{r^3}\right)\right) \quad (1)$$

Since generally $\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)$

$$= \left(-\frac{3}{2}\right)\left(\frac{1}{(x^2+y^2+z^2)^{5/2}}\right)(2x)$$

$$= \frac{-3x}{r^5} \quad (2)$$

we can expand (1) by substituting appropriately with (2):

$$\vec{\nabla} \times \vec{f} = \hat{i}\left(\frac{3mgyz}{r^5} - \frac{3mgzy}{r^5}\right) - \hat{j}\left(\frac{3mgxz}{r^5} - \frac{3mgzx}{r^5}\right) + \hat{k}\left(\frac{3mgxy}{r^5} - \frac{3mgyx}{r^5}\right)$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k}$$

Since there is no curl, the force field is <u>conservative</u> away from the origin. ∎

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To determine a potential function φ such that $\vec{f} = \vec{\nabla}\varphi(x,y,z)$, we need to integrate:

$$\vec{f} = -\frac{mg}{r^3}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \partial\varphi/\partial x \\ \partial\varphi/\partial y \\ \partial\varphi/\partial z \end{bmatrix} \qquad (1)$$

Take $\partial\varphi/\partial x = -mgx/r^3$ as a representative case.

Then $\varphi = \int -mgx/r^3 \, dx = -mg\int \frac{x}{r^3}\,dx$ (2)

Recall that $r = \sqrt{x^2+y^2+z^2}$, and perform a u-substitution:

$$u = r^2 = x^2 + y^2 + z^2 \qquad (3)$$

$$\partial u/\partial x = 2x, \quad du = 2x\,dx \qquad (4)$$

Substituting (3) and (4) into (2):

$$\varphi = -mg\int \frac{1}{2}\frac{1}{u^{3/2}}\,du$$

$$= -mg\left(\tfrac{1}{2}\right)(-2)u^{-\frac{1}{2}} + F(y,z)$$

$u = r^2$ so $u^{-\frac{1}{2}} = \frac{1}{r}$, and since we can perform the integration equivalently with y and z as the integrated variables, the only permissible function of integration is the constant function c, leaving

$$\varphi = \frac{mg}{r} + c$$ as our potential function. ∎