Dimensional Analysis with Linear Algebra

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Source

Buckingham's Pi theorem states that any relations between natural quantities can be expressed in an equivalent form using *Pi groups*, dimensionless quantities formed between those quantities.

Assumptions:

The following assumptions must hold:

1. u, our quantity of interest, must equal some function $f(x_1, x_2, x_3, \ldots, x_n)$, that is, n measurable quantities expressed as independent variables & parameters x_i. It is further assumed that the equation

$$u = f(x_1, x_2, x_3, \ldots, x_n)$$

is dimensionally homogeneous.

2. The quantities $\{u, x_1, x_2, x_3, \ldots, x_n\}$ are measured in terms of m fundamental dimensions $\{L_1, L_2, L_3, \ldots, L_n\}$
3. If W is any quantity of $\{u, x_1, \ldots, x_n\}$, then

$$[W] = L_1^{p_1} \cdot L_2^{p_2} \cdot \ldots \cdot L_m^{p_m}$$

Then we can create $\mathbf{P} = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{bmatrix}$, the *dimension vector* of W.

This gives us the $m \times n$ dimension matrix

$$\mathbf{A} = \left[\mathbf{P}_1 | \mathbf{P}_2 | \cdots | \mathbf{P}_n\right] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ p_{m1} & p_{m2} & \cdots & p_{mn} \end{bmatrix}$$

Conclusions of the Buckingham Pi Theorem

1. The relation $u = f(x_1, x_2, \ldots, x_n)$ can be expressed in terms of dimensionless quantities.
2. The number of dimensionless quantities is

$$k + 1 = n + 1 - \texttt{rank}(A)$$

(The reason for k + 1 is that we pull out the original quantity u from the matrix **A**. Otherwise this term would not appear.)

3. Since **A** has $\texttt{rank}(A) = n - k$, there are k linearly independent solutions of $\mathbf{Az} = \mathbf{0}$ denoted as $z^1, z^2, \ldots, z^k$.

Let **a**, an m-column vector, be the dimension vector of u, and let **y**, an n-column vector, be a solution of

$$\mathbf{Ay} = -\mathbf{a}$$

Then the relation $u = f(x_1, x_2, \ldots, x_n)$ simplifies to $g(\Pi_1, \Pi_2, \ldots, \Pi_k)$.

There is one Π group for each linearly indepenent set of $\mathbf{Az} = \mathbf{0}$, plus one Π group for u. The parameters in each pi group are raised to the respective row of $z\prime$.

Why it Works:

Recall that the nullspace of a matrix **A** is the space of all vectors **z** for which $\mathbf{Az} = \mathbf{0}$. The multiplication **Az** is a linear combinations of the columns of **A**:

$$\mathbf{Az} = [z_1\mathbf{P}_1 | z_2\mathbf{P}_2 | \ldots | z_n\mathbf{P}_n]$$

This linear combination of the columns of **A** is the same thing that you get when you raise each of the parameters x_n to the respective element of **z**:

$$[x_i^{z_i}] = [W]^{z_i} = \left(L_1^{p_1} \cdot L_2^{p_2} \cdot \ldots \cdot L_m^{p_m}\right)^{z_i} = L_1^{p_1 z_i} \cdot L_2^{p_2 z_i} \cdot \ldots \cdot L_m^{p_m z_i}$$

Which corresponds to column i of **Az**. Finally, since **z** is in the nullspace of **A**, the sum of the powers on each of the base units L will be 0, resulting in an overall dimensionless quantity.