Q1.

Machine	% of production	% Defective
M1	20%	1%
M2	30%	2%
M3	50%	3%

By Bayes' Theorem,

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{P(B)}$$

Then, taking A to be the probability that a part DID NOT come from machine M_3 ($\sim M_3$), and B to be the probability that a part is defective (B = Defective)

$$P(\sim M_3 \mid \text{Defective}) = \frac{P(\text{Defective} \mid \sim M_3)\,P(\sim M_3)}{P(\text{Defective})}$$

Further,

$$P(\text{Defective} \mid \sim M_3) = P(\text{Defective} \mid M_1) + P(\text{Defective} \mid M_2)$$
$$= 1\% + 2\%$$
$$= 3\% = 0.03$$

$$P(\sim M_3) = 1 - P(M_3) = 0.5 = 50\%$$

$$P(\text{Defective}) = P(\text{Defective} \mid M_1)P(M_1) + P(\text{Defective} \mid M_2)P(M_2) + P(\text{Defective} \mid M_3)P(M_2)$$
$$= (0.01)(0.2) + (0.02)(0.3) + (0.03)(0.5)$$

so

$$P(\sim M_3 \mid \text{Defective}) = \frac{(0.03)(0.5)}{(0.01)(0.2)+(0.02)(0.3)+(0.03)(0.5)}$$

$$= 0.3478$$

$$\sim 34.78\%$$

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2. $\{X_1, X_2, \ldots, X_n\}$ is i.i.d sequence of samples of pdf $P_X(x)$, with $\mathbb{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$.

The CLT states that

(1) $$\lim_{n\to\infty} P\left(a \leq \frac{(X_1 + X_2 + \ldots + X_n) - n\mu}{\sigma\sqrt{n}} \leq b\right) = \int_a^b \frac{1}{2\pi} e^{-\frac{1}{2}x^2} dx$$

i.e., that the normalized p.d.f. for the sample mean $\bar{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$ approaches a Gaussian distribution.

(2) If $$S = \sqrt{4n}\,(\bar{X}_n - \mu) = \frac{\sqrt{4}\,(n\bar{X}_n - n\mu)}{\sqrt{n}}$$

Then $$\lim_{n\to\infty} S = \lim_{n\to\infty} \frac{2\,(n\bar{X}_n - n\mu)}{\sqrt{n}}$$

(3) $$= \lim_{n\to\infty} (2\sigma)\left[\frac{(X_1 + X_2 + \cdots + X_n) - n\mu}{\sigma\sqrt{n}}\right],$$

which is 2σ times the random variable (1).

Using the fundamental transform theorem [1]

(4) $$f_Y(y) = f_X(g^{-1}(y))\left[\frac{\partial g^{-1}(y)}{\partial y}\right], \quad \text{for } y = g(x)$$

with $g(x) = (2\sigma)x$, and $f_X = \frac{1}{2\pi} e^{-\frac{1}{2}x^2}$

then the P.D.f of $S = g(x)$ is

(5) $$\frac{1}{4\sigma\pi} e^{-\frac{1}{2}\left(\frac{x}{2\sigma}\right)^2}$$

with mean $\underline{0}$ and variance $\sigma_S^2 = 2\sigma$.

[1] B.M. Ayyub, & R.H. McCuen, Probability, statistics, and reliability for engineers and scientists, 2nd ed. Chapman & Hall, CRC Press LLC, 2003, p. 184.

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Q2. b) Let X = outcome of fuel cell test with $Var(X) = 4h^2$

Then $\sigma = 2h$ is the standard deviation of one test.

If the scientist performs N tests, then the sample mean $\bar{X}_n$ has variance $\frac{\sigma}{\sqrt{N}}$.

On a normal bell curve (p.d.f $\frac{1}{2\pi} \exp(-\frac{1}{2} x^2)$), 95% of the probability mass lies within 2σ of the mean value μ. Since the samples X_i are i.i.d, their limiting p.d.f approaches a normal distribution

$$\bar{X}_n \sim \mathcal{N}(\mu, \sigma)$$

by the CLT. Then a 95% confidence interval for $\bar{X}_n$ will occur when

$$\frac{\sigma}{\sqrt{N}} = 0.2h$$

$$\sqrt{N} = 10$$

$\underline{N = 100}$ samples are required.