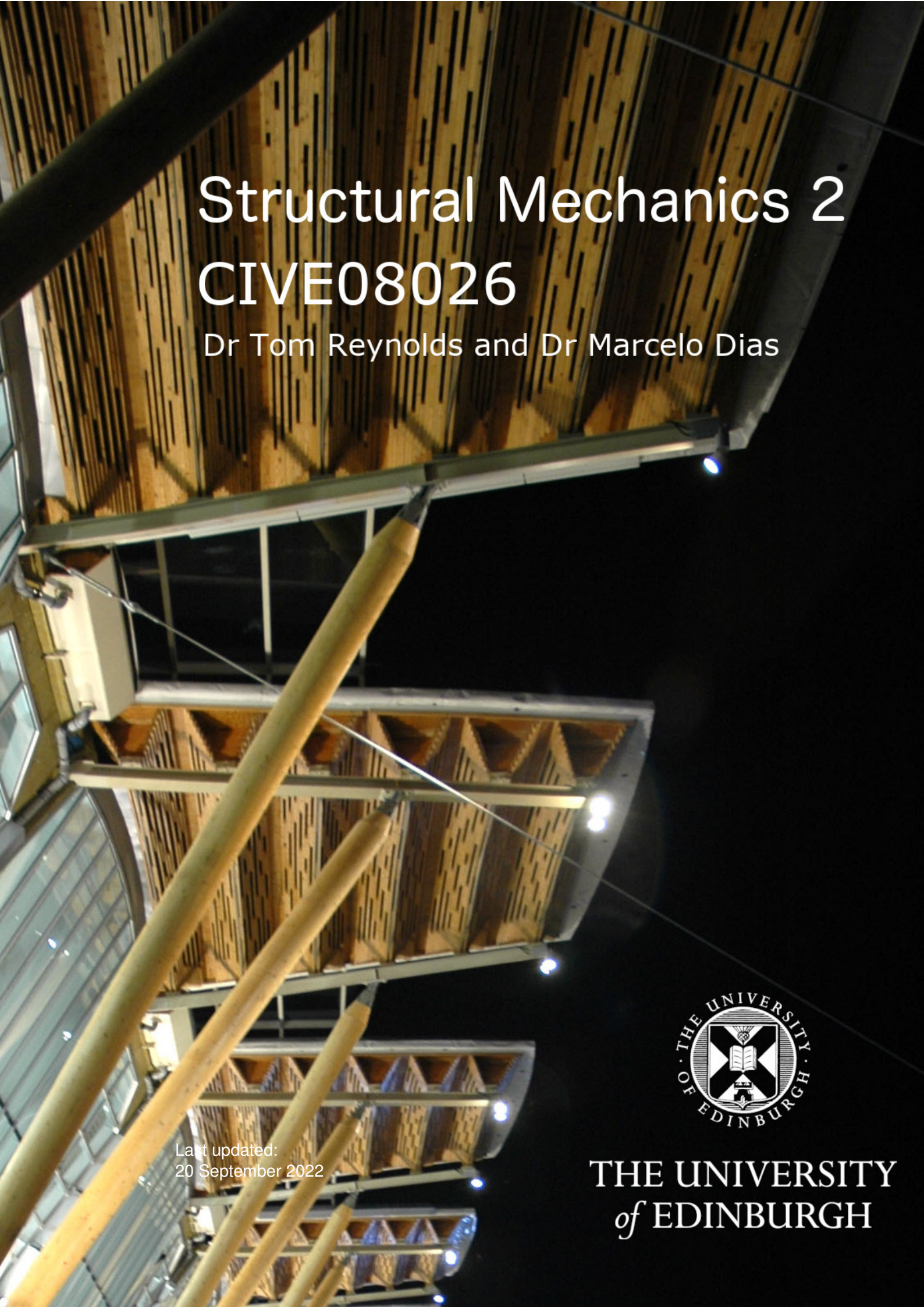

Structural Mechanics 2
CIVE08026

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Course overview

This course covers the fundamental principles of structural engineering which underpin the design of any element to resist an applied force.

Learning outcomes

On completion of this course, you will be able to:

1. Describe and manipulate fundamental concepts of stress, strain, and deformation in members carrying axial, bending, shear, and torsional loads;
2. Determine how statically determinate trusses and beams carry load; for beams using diagrams of bending moment and shear force, and evaluate the resulting elastic deflections of the beams;
3. Analyse structural cross sections, so as to determine the elastic stress and strain distributions, as well as the deformations, resulting from axial, bending and torsional actions; and
4. Describe and manipulate relevant concepts of combined loadings and stress and strain transformation.

References

These notes should be read in conjunction with lecture material and the referenced sources. The most important references for this course are available as ebooks through the library's subscriptions:

- M.S. Williams and J.D. Todd, "Structures, Theory and Analysis" MacMillan Press (covers the fundamental theory you will learn in this course).

https://discovered.ed.ac.uk/permalink/f/gfso8q/44UOE_ALMA51201478300002466

- J.E. Gordon, "Structures, or why things don't fall down", Plenum Press (Gives an intuitive and interesting view on the structural mechanics that is all around us).
  https://discovered.ed.ac.uk/permalink/f/gfso8q/44UOE_ALMA51248640630002466

The text by Williams and Todd has chapters which align with each topic in the course, and contains further worked examples and practice questions which may be useful to you.

You may wish to consult other texts for different angles and approaches on the concepts we introduce. Here are a few other texts you might find useful:

- T. H. G. Megson, "Structural and Stress Analysis", Butterworth-Heinemann. (includes further worked examples and practice questions)
- J.M. Gere, "Mechanics of Materials", 6th Edition, Thomson (Based on American practice, and some of the problems are in non-SI units).
- R.C.Hibbeler, "Statics and Mechanics of Materials" SI Edition, Pearson Education (Includes the basic materials in both traditional Statics and Strength of Materials. It is also based on American practice but SI units are used throughout this edition).

How to use these notes

At the end of most sections there are questions. It is strongly recommended that you attempt these questions to aid your learning throughout the course and as revision. We will be available to discuss your problems with and insights into these questions in tutorials.

There are generally two types of question: those that reinforce ideas presented in the class and those that are intended to build upon the taught material. Both are equally important.

In solving these questions, it is strongly recommended that a spreadsheet or other numerical method (perhaps Python) be developed as familiarity with such tools is essential to your development as an engineer.

1 Introduction

1.1 What is structural mechanics?

Structural mechanics calculations are required in the design of most things around us. Think of something that might break, and you have an application where structural mechanics calculations were probably used in design. We will study how forces are transferred through a structure (for example, the weight of a vehicle in the middle of a bridge transferred to the ground at either edge of the bridge span, or the aerodynamic forces on an aeroplane wing to the fuselage), and we will investigate the stresses and strains that occur in the material of the structure as it is loaded.

Structural mechanics goes under lots of other names: ‘Mechanics of materials’, ‘Strength of materials’, ‘Theory of Structures’, ‘Analysis of structures’, and so on. Structural Mechanics 2 covers primarily the basic contents of classic textbooks on Statics and Mechanics of Materials. Structural Mechanics can be used to analyse engineering structures, e.g. bridges, buildings, tunnels, wind turbines, and so on.

We can usually interpret these structures as beams, columns, plates, struts, shafts, shells etc (described in more detail in the next section). For any of these structures, structural mechanics can be used to predict:

- the effects of forces (loads) on pieces of the structure;
- the stresses that develop in the structure as a result of the loads (possibly causing failure);
- the strains that those stresses cause;

- the deformations of the structure (possibly deforming too much).

In many cases, the deformations and forces are interlinked so that there is a feedback loop in the analysis process.

2 Structural Forms and Terminology

2.1 Structural elements and examples

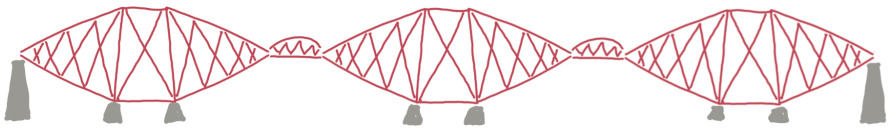
The easiest and most convenient way of approaching structural analysis is to consider standard structural forms. These forms can be identified in most structures. In the descriptions below, an efficient structure is one that uses the full strength of the material throughout the member to carry the loads. An inefficient one is one that does not use the material strength effectively.

Struts, ties and columns

Member subjected mainly to axial forces in compression or tension (often vertical). Stiff and strong (carries loads comparable with material strength).

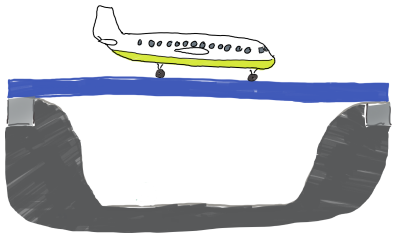
Trusses

An assembly of struts and ties (referred to as bars) providing a stiff frame with material used efficiently: stiff and strong (each bar carries loads comparable with material strength). The Forth Road Bridge incorporates truss elements:



Beams

Member subjected to bending loads (usually horizontal): flexible, weak and inefficient, but crucially needed for flat surfaces, such as floors, roadways.

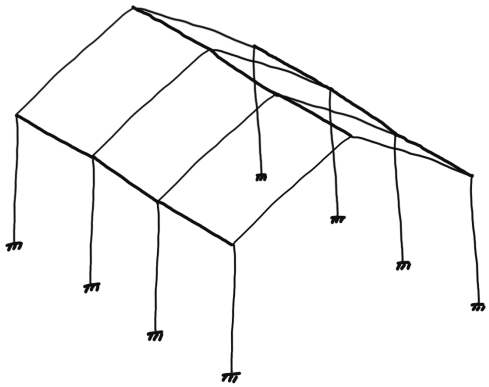


Shafts

Members carrying torsion (a load that twists the shaft about its length), for example in a drive shaft: can be stiff and strong (carries loads comparable with material strength).

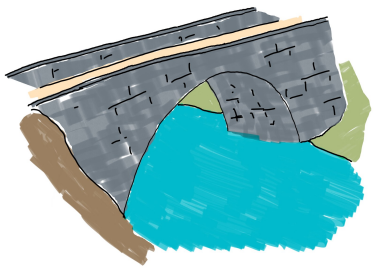
Frames

An assembly of beams and columns: good for functionality, but the material is not used efficiently; flexible and weak (each member is usually in bending); convenient for buildings, since you have large open spaces for windows, and getting in and out.



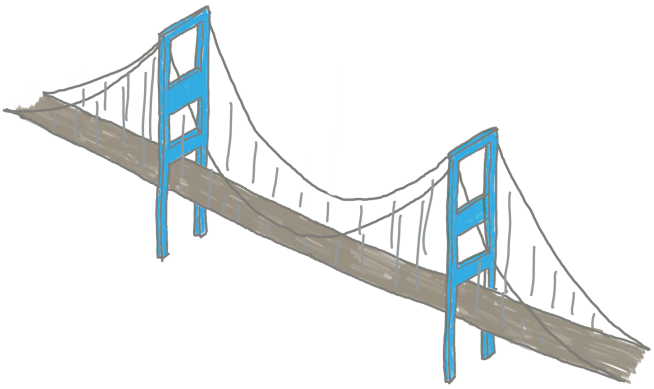
Arches

A structure curved in the vertical plane, so that gravity loads applied to it cause compression forces to develop in the structure. Stiff and strong (whole structure in compression up to material strength).



Suspension and cable-supported structures

A structure draped in the vertical plane, so that gravity loads applied to it cause tension forces to develop in the cables: stiff and strong (cable structures are ties: achieves material strength)



Plates and slabs

A two-dimensional beam, giving a complete floor or similar surface. Slab refers to concrete construction: plate to metal and other construction (eg. car floor). Relatively flexible and relatively weak (bending in two directions is better than one direction). A table can be thought of as a plate (the tabletop) supported by columns (the legs).

Shells and pressure vessels

A three-dimensional curved thin structure, usually used for containing or storing things (plastic bottles and glasses, Millennium dome, aircraft and spacecraft, cars, tanks, silos, chimneys). Stiff and strong (a shell structure is like an arch in two directions).

Combinations

These forms may be combined in real structures. For example, a pipe might be pressurised (as a pressure vessel) and carry an axial load (as a tie). In this course we will start by understanding how structural members (struts, shafts and beams...) work. In later lectures we will look at combinations of structural form.

2.2 Structural diagrams

Structural engineers have ways of describing and drawing structures which convey the way they transfer forces. The terminology does not always have the same meaning to engineers as it does in wider society, and a structural drawing does not always look like the structure itself, so learning this verbal and graphical language is an important part of your training.

Once you understand how to translate a building or structure in the real world into a well-described structural diagram, you will be able to apply the whole range of structural analysis and design techniques we teach you here, and translate it back into a real structure which improves people's lives and makes the world a better place.

Here's an example of translating a real structure into its representation as an engineering diagram. The main structural function of the shelf shown below can be described by the following diagram alongside it:

This shelf is a very simple example of a truss, which is an assembly of struts and ties to create a stiff frame.

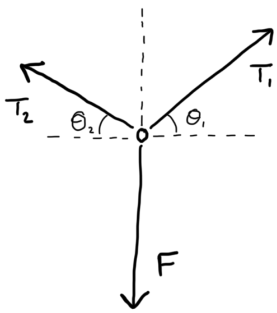
3 Static Equilibrium

3.1 Basic Structural Mechanics Concepts

In this section we take a quick overview of some basic structural mechanics concepts, including forces and moments, point and distributed loads, support conditions, global equilibrium of structures, structural determinacy, and limit states.

3.1.1 Forces; Moments, Torques and Couples

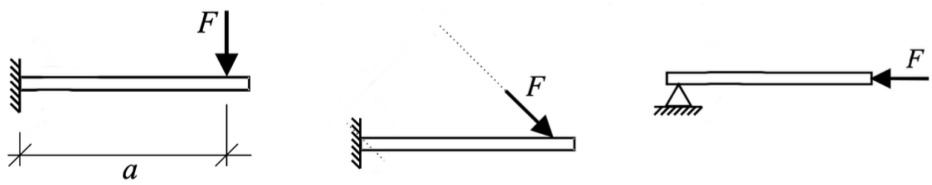
Forces are vectors, so need to be combined by vector addition.



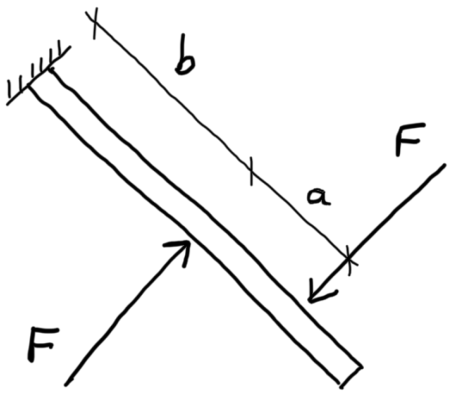
A moment describes the tendency of a force to cause rotation of a body. Moment induced by a force is defined as:

$$\text{Moment} = \text{Force} \times \text{Lever arm}$$

$$M = Fa$$



A couple is a pure moment that is applied as a pair of equal and opposite forces. Taking moments about a general point X:

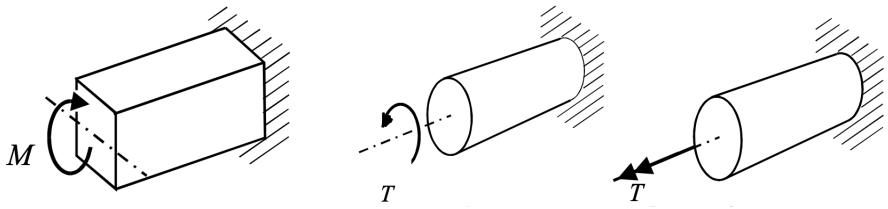


$M =$ (independent of b)

Therefore, the couple applies a pure moment.

Moment *versus* Torque

A couple can cause two types of stress resultant in a structural element: bending and torsion, also called torque. A bending moment (M in the image below) is a moment about a perpendicular axis of a structural element to cause bending. A torque (T in the image below) is a moment applied about the axis of a structural element to cause twisting, for example in a drive shaft.



Note the “double headed arrow” symbol for a torque moment. The same symbol may also be used to represent a bending moment; think how this may be drawn for the moment acting on the beam shown on the left).

3.1.2 Applied Loads

There are many actions that can apply load to a structure:

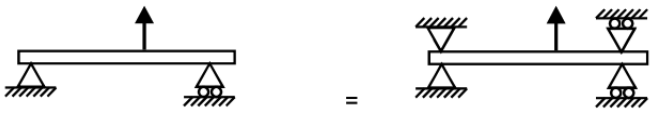
- Live loads
- Dead loads
- Wind load; Snow load (environmental loads)
- Fire load (high temperatures affect material properties, so a bit different, but still an action)
- Earthquake loads
- Dynamic loads in vehicles
- Impact loads

3.1.3 Support Conditions and Reactions

A structure carries the load applied to it through it’s ***body*** to its ***supports***. For example, a bridge carries the weight of a lorry to the abutments or piers, where the load is transferred into the ground. A tree carries wind load through its trunk to its roots, where the load is transferred into the ground. The forces imposed on the structure by the ground are the ***reactions***.

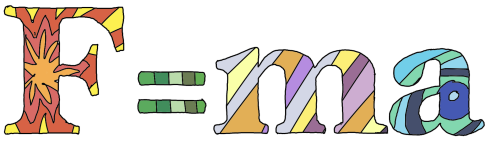
The transfer of loads through the structure causes stresses, and the resultant of such stresses over a section is called Stress Resultants, or more conveniently Internal Forces. The calculation of the Reactions (at supports) and the Internal forces (within structure body) constitutes a fundamental task of structural analysis. The type of structural support determines the reaction forces that can be transferred through it.

These support conditions are used repeatedly in structural mechanics: it's worth making sure you recognise them! NB: Supports are usually assumed to take both upwards and downwards loads.

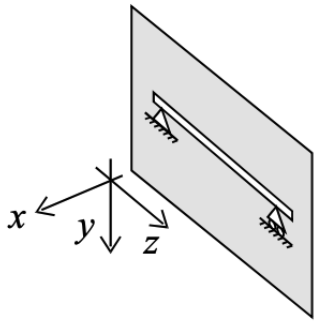


3.2 Static Equilibrium in 2D

For a structure to be static (not accelerating), the sum of forces and moments resolved in any direction on any part (cut-out) or the whole of the structure must be zero.



In this course, we define the geometry of our 2D structures in the y-z plane (In some textbooks you may see different use of the axes; but this is just a matter of choice).



Hence equilibrium of a body in 2D space requires:

Horizontal equilibrium

Vertical equilibrium

Moment equilibrium

3.2.1 Support Requirements

Every structure in 2D space must have at least 3 support reactions in order to satisfy all 3 equilibrium equations under any applied loads.

The three support reactions can come from a combination of:

- Built-in end:
- Simple support:
- Simple support on rollers:

(NB: Torsion requires treatment in 3D space, which will be dealt with in Chapter 15)

Note: where the direction of a reaction (force or moment) is not apparent in the first place, we can start by assuming a certain direction, and let the final result tell if it is indeed in the assumed direction (“positive” sense) or otherwise (“negative” sense). For uniformity we can start by assuming “rightward”, “upward” and “anticlockwise” for the horizontal, vertical and moment reaction, respectively.

3.3 Examples

Draw appropriate supports and reactions for the following structures.

3.3.1 Tied arch bridge

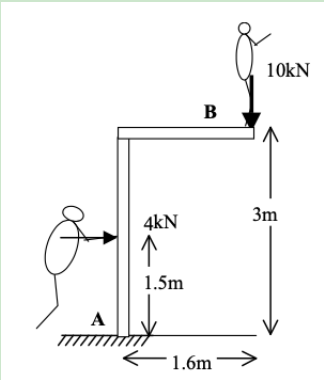
3.3.2 Motorway sign

3.3.3 Sign hanging from cables

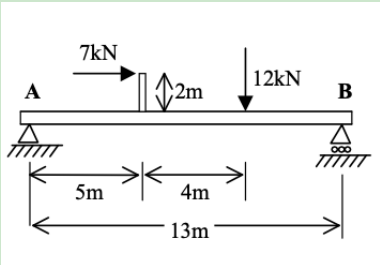
(for this example, also calculate the reactions in terms of the weight of the sign)

Questions

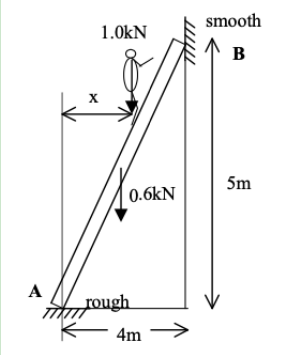
1. The bus shelter shown below is being designed. The extreme design situation for the foundation has been identified as a group of vandals attempting to knock it over by climbing onto the edge of the roof whilst their mates attempt to push it over. Determine the forces and moment that the foundation must be able to sustain without damage to resist this design situation.



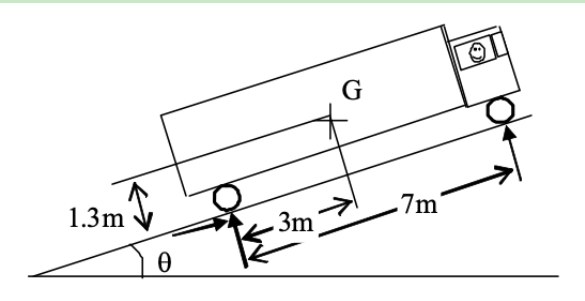
2. A simply supported beam is loaded as shown below. Calculate the support reactions at the two supports.



3. The uniform ladder below weighs 0.6kN. It is leaning against a smooth vertical wall and rests on rough ground at the bottom. a) Calculate the vertical and horizontal reactions at the top and bottom of the ladder when no one is on it. b) A person weighing 1.0kN climbs up the ladder until they are located at a horizontal distance x from the foot of the ladder. Determine the horizontal and vertical reactions at the ladder base in terms of x. c) If x = 4m, how large must the coefficient of friction against the ground be for the ladder not to slip down?



4. A heavy truck of weight 400kN is shown below, parked on an extremely steep hill of slope θ with its hand brake applied to the rear brakes only. Determine the reactions between the road and the truck's wheels in terms of θ. Find the steepest value of θ that the truck can manage without sliding if the coefficient of friction between the tyres and the slippery road is μ = 0.8. Would it be better if the truck were parked facing downhill?



Numerical answers. Q1: 10kN, -4kN, 22kNm; Q2: -7kN, 2.62kN, 9.38kN; Q3: 0.6kN, 0.24kN, 0.24kN; 0.24+0.2x, 1.6kN, -(0.24+0.2x); 0.65. Q4: 228.6cosθ+74.3sinθ, 171.3cosθ-74.3sinθ, 400sinθ, 28.2°; no 21.7°.

4 Statical Determinacy

There are two important classes of structures that you need to be aware of:

- Statically Determinate or Determinate structures,
- Statically Indeterminate or Redundant structures.

This classification determines how easy it is to analyse a structure. You'll have seen in lectures there are consequences for how the structure behaves as well, particularly under accidental load and thermal loading.

We will also mention systems which, due to their support conditions and arrangement of members, cannot resist a force without moving. These are referred to as:

- Mechanisms.

Here we focus on external determinacy. That is, we're looking at whether a structure is statically determinate in terms of its support reactions; whether the support reactions can be determined from equilibrium alone.

4.1 Determinate structures

A structure is Statically Determinate if the forces inside it are controlled only by equilibrium. Failure occurs when the material strength is reached at the weakest point. Externally determinate structures have the minimum number of support reactions to resist the loads applied to them.

If a 2D structure has 3 reactions, the reactions can be determined from equilibrium alone, and the structure is statically determinate in terms of its support reactions (note that it may still be statically indeterminate in terms of its internal forces).

4.2 Indeterminate or Redundant structures

A structure is Statically Indeterminate, or Redundant if the forces inside it are controlled by both equilibrium and compatibility. As well as satisfying equilibrium, we need to understand how the structure fits together (compatibility) and how it deforms.

Redundancy is desirable in a structure, since it makes the structure more robust. Collapse does not follow from failure of the material in a single member. However, the structural analysis is more complex.

If a 2D structure has more than 3 reactions, we cannot find them from the 3 equilibrium equations alone, and the structure is statically indeterminate, or

redundant. In such cases, the solution will require additional equations from the deformation compatibility.

Practically this has consequences for structural behaviour:

- Indeterminate structures can have states of internal stress and force with no external load. These can develop due to imperfect fit between structural members, or changes in length due to thermal effects.
- Indeterminate structures can lose some part of the structure due to accidental damage and still remain stable overall:

Questions

1. For what force should each of the three legs of this bench be designed? Analyse the bench in two dimensions, assume it is 2m long with a self weight of 0.2 kN/m. How does this compare to the force if there were two legs?

Numerical answers. It's complicated! If you've got a numerical answer, look again.

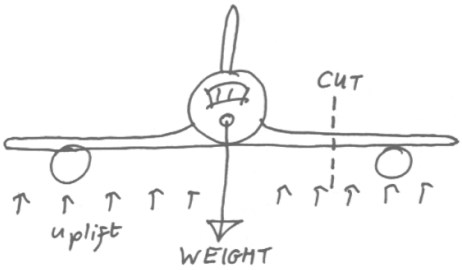
5 Free Body Diagrams and Stress Resultants

Global equilibrium only tells us the loads carried through the supports of a structure. We also need to know how the structure carries load from the points at which loads are applied to the supports. This lecture shows how these “internal forces”, or more precisely “stress resultants” can be found by making imaginary cuts in the structure and satisfying equilibrium of the remaining structure.

5.1 Free body diagram

A very useful concept in structural mechanics is the Free Body Diagram. We use this to determine the forces acting inside a structure.

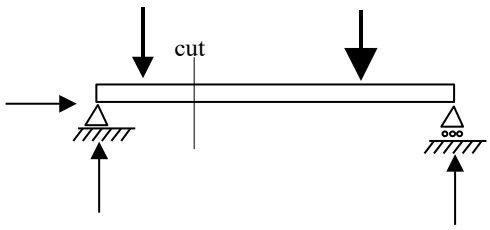
- Imagine cutting the structure into two parts.
- Consider one of the pieces of structure that you have isolated. This is a free body.
- Before we cut the structure, there would have been stresses within the cut section.
- Cutting “exposes” these stresses, and we can find the resultants of them, i.e. the internal forces, by satisfying static equilibrium of the free body.



5.2 Internal Stress Resultants (also called internal forces)

What is going on inside a beam when it is loaded?

We cut the member to determine the stress resultants required to keep the remaining portion in equilibrium.



Here is the free body diagram for this beam cut at a point along its length:

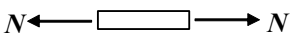
- N is the axial force (tensile +ve)
- V is the shear force (clockwise +ve)
- M is the bending moment (sagging +ve)

A “cut” of a section exposes two opposite faces (the right-side face of the cut is not shown in the above figure), and the same internal forces but viewed on the opposite faces act against each other and in opposite directions.

Therefore, the sign of an internal force cannot be specified by a unique direction (e.g. leftward or rightward), but by a “paired” direction giving rise to “tension” or “compression” and so forth. The “paired” internal forces can be conveniently illustrated in the diagrams below.

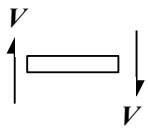
If one is consistent in the way stress resultants, then it doesn’t matter in which orientation they are drawn. However, it can help to avoid errors (and we’ll try to keep to this throughout the course) if one uses the same sign convention throughout one’s calculations. Here are the orientations of positive stress resultants that we will adopt throughout this course:

Axial force



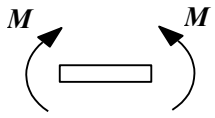
Tension positive

Shear force



Clockwise positive

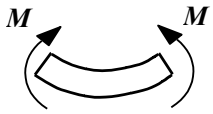
Bending moment



Sagging positive

There is a particular terminology for directions of bending. In sagging, there is compression at the top of the beam and tension at the bottom, and in hogging those are reversed.

Sagging bending (positive)



Hogging bending (negative)

5.3 Stress resultants and failure

Once we have determined the stress resultants in an element, we can compare them with the strength of the beam section to determine whether it will fail. Often, we might be interested in choosing a beam section that is stronger than the required stress resultant.

If any one of the internal stress resultants (Moment M, shear V or axial force N) exceed the strength of the structure, the structure has reached the ultimate limit state, and it will fail. If more than one of M, V and N are present, the stress resultants will interact. The interaction of the stress

resultants must then be taken into account in determining the strengths M_{max}, V_{max} and N_{max}.

If the member strength is reached at some point within a structure, it does not necessarily mean that the structure will collapse. Remember the difference between determinate and indeterminate beams: equilibrium alone can be used to find the stress resultants in determinate structures. An important consequence is that in a determinate structure, collapse will occur if the stress resultants at a single location reach the cross-section maximum values.

In a redundant structure, if the stress resultants at a single location in the structure reach the cross-section maximum values, it does not necessarily mean that collapse of the structure occurs. However, this “local strength” assessment of the strength of the whole structure does give a safe (lower bound) estimate of collapse, so it is often used in conjunction with an elastic analysis.

6 Stress Resultants in Beams

The internal forces (stress resultants) vary along the length of a beam. Using the concept of the free body, introduced in Chapter 5, we can calculate this variation, and use it to look for salient features, such as the maximum bending moment, shear or axial force. During the design process, we might compare the maximum values with the strength of the cross-section to decide when and where the Ultimate Limit State will be reached.

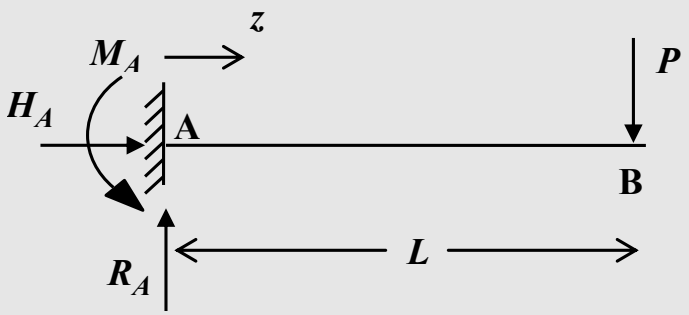
6.1 Shear Force and bending Moment Diagrams

A plot of the variation in shear force along the beam is known as the *Shear Force Diagram*. A plot of the variation in bending moment along the beam is known as the *Bending Moment Diagram*. These diagrams are extremely useful design tools. Since they are relatively simple diagrams, engineering students often think that it must be possible to “see” the shape of the diagram just by inspection of the structure, and when they can’t do that they pronounce that they are “not good” at drawing them.

The reality is that drawing bending moment and shear force diagrams requires careful examination of different free bodies based on cuts in different parts of the structure. Once you do that, drawing the diagrams is straightforward.

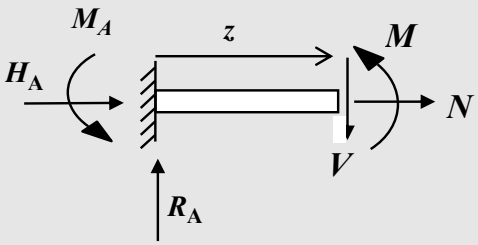
6.2 Example

6.2.1 Point load on a cantilever



From global equilibrium of the whole structure:

Cut the beam at any coordinate z and determine the internal stress resultants (internal forces) $N(z)$, $V(z)$ and $M(z)$ from equilibrium:



Resolving horizontally:

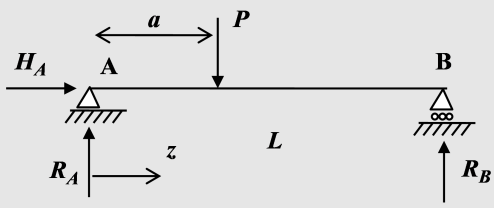
Resolving vertically:

Moments about the location of “cut”:

Plot the variation of V with z to give the Shear Force Diagram (SFD)

Plot the variation of M with z: this is called the Bending Moment Diagram (BMD)

6.2.2 Single point load on a beam



From global equilibrium:

Cut the beam before the load:
$z < a$

Resolving vertically:

V does not vary with z.

Taking moment about cut=0, then rearranging:

Cut the beam after the load:
$z > a$

Resolving vertically:

V does not vary with z.
Moments about cut=0, then rearranging:

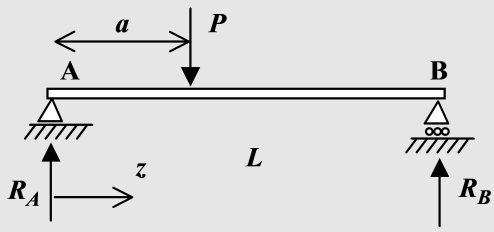
The largest value of M occurs when and is given by

.

The largest value of M occurs when $z = a$ and is given by

Note that we could have considered equilibrium of the beam to the right of the cut (and obtained the same results).

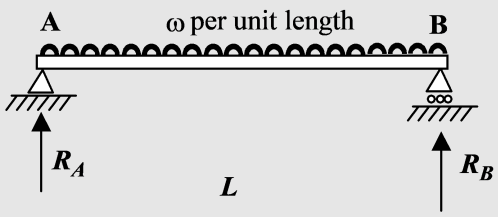
Plot the shear force and bending moment underneath the image of the beam:



Plot the variation of *V* with *z* to give the *Shear Force Diagram (SFD)*.

Plot the variation of *M* with *z* to give the *Bending Moment Diagram (BMD)*.

6.2.3 Uniformly distributed load on a beam



From global equilibrium:

Cut the beam at some general point z and write equilibrium equations for the free body:

Resolving vertically:

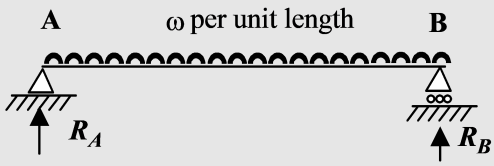
So V varies linearly with z, from at the left, to at the right.

Moments about the cut at coordinate z:

So M varies parabolically with z.

The maximum moment in the beam is at $z =$ and is $M_{max} =$.

Plot the shear force and bending moment underneath the image of the beam:



Plot the variation of *V* with *z* to give the *Shear Force Diagram (SFD)*.

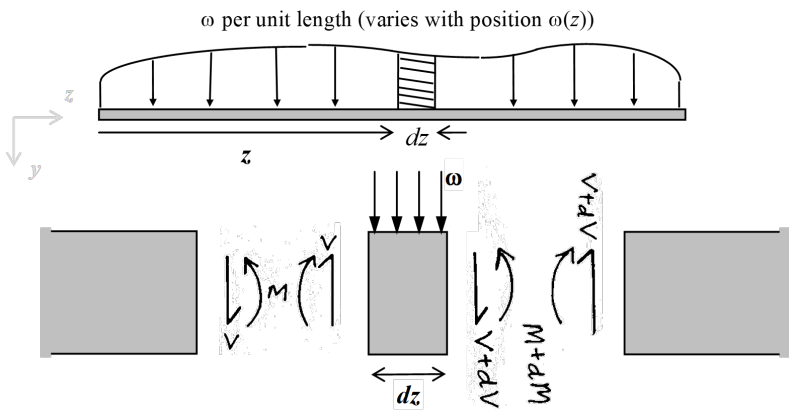
Plot the variation of *M* with *z* to give the *Bending Moment Diagram (BMD)*.

The bending moment diagram and shear force diagram for a beam can be constructed by making a series of cuts along a beam, and considering free-body equilibrium (as in the previous lecture).

Here we discuss the inter-relationship between distributed load, shear force and bending moment, and subsequently we introduce a quicker method for constructing bending moment and shear force diagrams.

6.3 Relationship between distributed load, shear force and bending moment

Examine a small slice cut out of the middle of a beam with a distributed load:



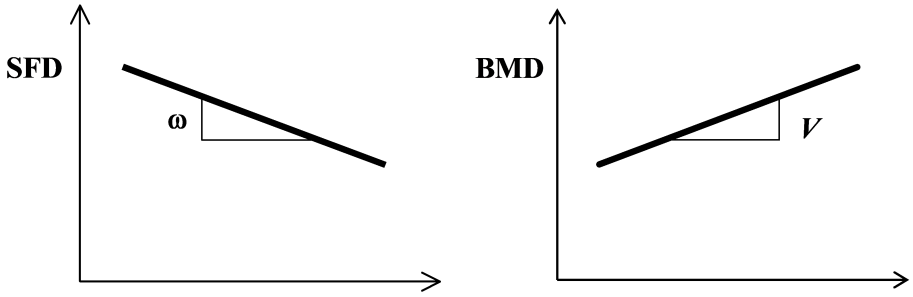
$$\sum F = 0 \uparrow\downarrow$$

$$\sum M \circlearrowleft = 0 \Rightarrow$$

So:

In words:

- the slope of the Shear Force Diagram is equal to minus the distributed load at that point (assuming downward ω, or w, as positive);
- the slope of the Bending Moment Diagram is equal to the shear force at that point.



From this, we can also establish that between any two points A and B along the z-axis:

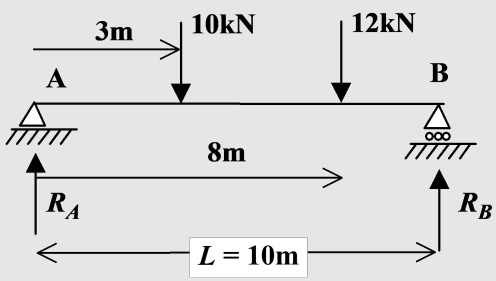
$$V_B = V_A - \int_A^B \omega dz \qquad M_b = M_a + \int_A^B V dz$$

where the integrals are the "area" of the distributed load (ω) and the "area" of the shear force between points B and A, respectively.

We can often use these relations to assist in drawing the BMDs and SFDs without the need to use equilibrium of free bodies all the time.

6.4 Example

6.4.1 Simply supported beam with two point loads on it



Global equilibrium to find the reactions:

$$R_B = 12.6\,\text{kN}$$

$$R_A = 9.4\,\text{kN}$$

We will compare the two methods for constructing the shear force diagram and bending moment diagram:

1. Use equilibrium equations derived from free body diagrams of the cut beam.
2. Deduce the diagrams from the loading.

1. Equilibrium equations for cut beam: Cut the beam at each section between loads. For each cut, draw the free body diagram:

For z <3m

$$V =$$

$$M =$$

At z = 3m, M =

For 3m< z <8m

$$V =$$

$$M =$$

At z = 8m, M =

For 8m< z

$$V =$$

$$M =$$

At z = 10m, M =

2. Direct deduction:

SFD:

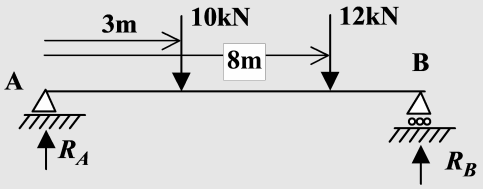
-
-

BMD:

-

•

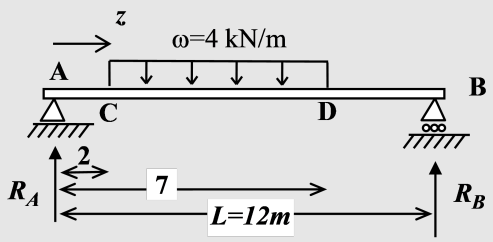
Draw the SFD and BMD under the structure:



Plot the variation of *V* with *z* to give the *Shear Force Diagram (SFD)*

Plot the variation of *M* with *z* to give the *Bending Moment Diagram (BMD)*

6.4.2 Simply supported beam with partial uniformly distributed load



Global equilibrium for support reactions:

Horizontal equilibrium: H_A =

Moments about B:

R_A =

Moments about A:

R_B =

Check:

Free body diagram construction from equations:

For $z < 2$:

$V =$

$M =$

At $z = 2\,\text{m}$, $M =$.

For $2 < z < 7$:

$V =$

At $z = 2\,\text{m}$, $V =$

At $z = 7\,\text{m}$, $V =$

$M =$.

Values of z must be substituted in to find the BMD.

For $7 < z$:

$V =$

$M =$

At $z = 12\,\text{m}$, $M =$

Drawing diagrams from direct deduction:

SFD:

-
-
-
-

BMD:

-
-
-
- Find key values of the bending moment:

- At $z = 2$, .
- At $z = 7$, $M =$. (or
  $M_B =$,
  $M_D =$)
-
-

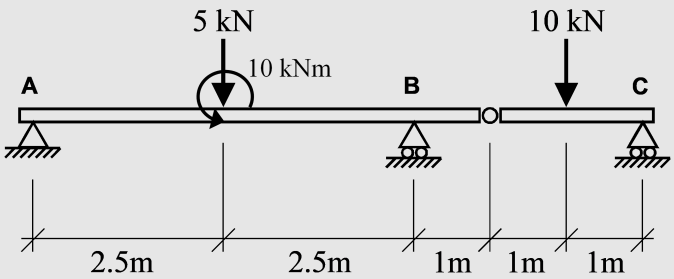
$M_{max} =$

$=$

(or $M_{max} =$)

Thus, the SFD and BMD can be constructed without equations.

6.4.3 Pinned beams carrying point loads and concentrated moment



Support Reactions

We need to find four support reactions (H_A, R_A, R_B, R_C).

- We have three equations for global equilibrium.

- For the fourth equation, note that:

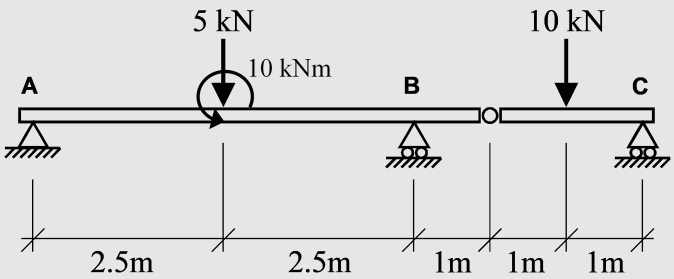
Horizontal equilibrium $\Rightarrow H_A =$.

Moments about the internal pin (take free body from the hinge rightward) $\Rightarrow R_C =$.

Moments about A $\Rightarrow$

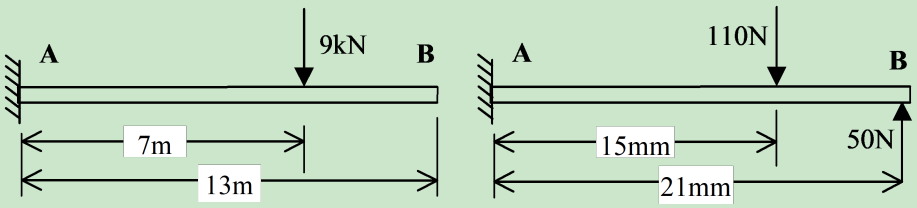
Vertical equilibrium $\Rightarrow$

Construct the *shear force diagram (SFD) and bending moment diagram (BMD)*

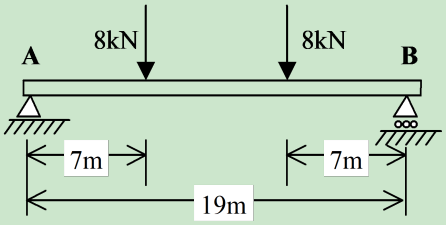


Questions

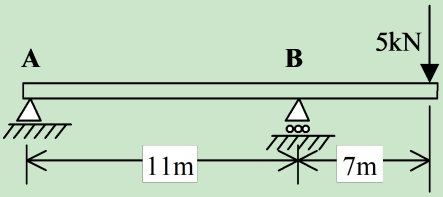
1. For each of the structures shown below, perform the following steps, marking key values on all diagrams:
   a) Calculate the support reactions.
   b) Draw the shear force diagram.
   c) Draw the bending moment diagram.
   d) Draw the axial force diagram, if axial force is present.



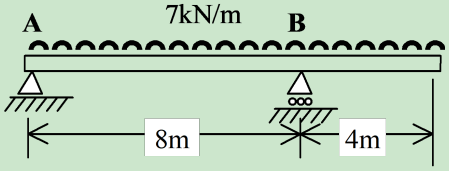
1) Point loaded cantilever



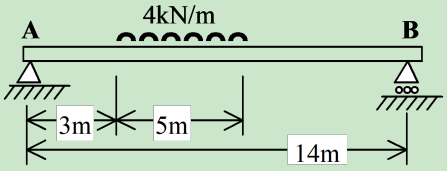
2) Cantilever with two loads



3) Symmetrically loaded SS beam

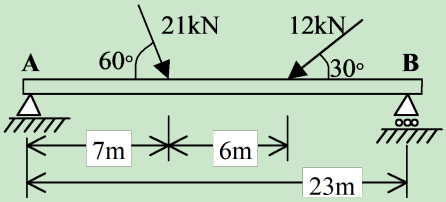


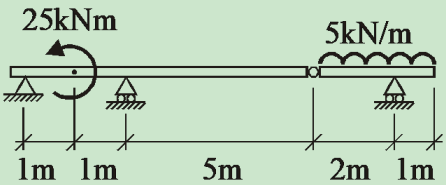
4) SS beam with cantilever



5) Beam with distributed load

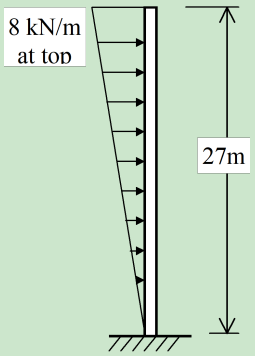
6) Beam with partial distributed load

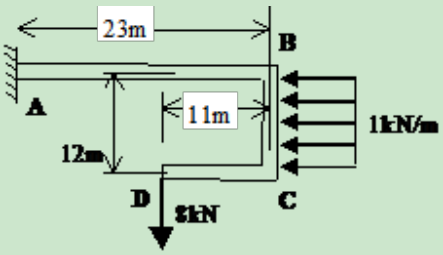




7) Beam with inclined loads

8) Pinned beam





9) Tower with linearly increasing wind load

10) Elevated walkway support structure

Answers (selected values). Q1: +9kN, -63kNm. Q2:-50,+60N,+300,-600Nmm. Q3: +8, 0, -8kN, +56,+56kNm. Q4: 3.18, +8.18kN, -3.18, +5kN, -35kNm. Q5: 21, 63kN, +21, 35, +28kN, +31.5, -56kNm. Q6: 12.14, 7.86kN, +12.14, -7.86kN, +36.4, +54.9, +47.1kNm. Q7: 15.26, 8.93kN, 0.11kN, +15.26, +2.93, 8.93kN, +106.8, +89.3kNm. Q8: 3.125, 0.625, 11.25kN, +3.125, -21.875kNm. Q9: 108kN, 1944kNm. Q10: 8, 12, 0, -8kN; 168kNm, -12, +8, 0kN; 168, +16, +88, 0kNm.

7 Axially Loaded Elements

7.1 Simple Mechanical Behaviour of Materials Under Axial Load

The traditional test for the behaviour of a material under load is the tensile test, in which a bar of the material is stretched in tension. The load is called an axial load because it acts down the axis of the bar.

The applied load N and the extension of the bar δ are measured. These are related to the cross-sectional area of the bar A and the original length of the bar L.

7.1.1 Direct Stress

We can cut the bar in tension at any point along its length, and the stress resultant on the free body will be the same.

The direct stress σ in the bar is the force per unit area:

$$N = \sigma \times A$$

This stress acts in the same direction as the force.

7.1.2 Direct strain

Stresses cause the bar to change length. The direct strain ϵ is the extension e divided by the original length of the bar L:

$$\epsilon_a = \frac{e}{L}$$

(The subscript a indicates that this is the strain in the axial direction).

Consider two bars, one of which is twice as long as the other. The long one will extend twice as much as the shorter one (Hooke's Law). However, the strain in the two specimens is the same.

7.1.3 Young's modulus

A direct stress σ induces a direct strain ϵ in the bar. The ratio between these two quantities is constant during elastic deformation, and is known as Young's Modulus E.

$$E = \frac{\sigma}{\epsilon}$$

This ratio is named after Thomas Young (1773-1829), the English scientist who discovered it. The general linear elastic relationship was discovered earlier and is known as Hooke's Law after Robert Hooke, who was a contemporary of Isaac Newton.

7.1.4 Lateral strains

When a bar is stretched, it not only gets longer, but also gets thinner.

The lateral shrinking of the bar is called the lateral strain, ϵ_l. The lateral strain relates to the lateral deformations in the same way as axial deformation:

$$\epsilon_l = \frac{\Delta}{a}$$

7.1.5 Poisson's ratio

The French mathematician Siméon Denis Poisson (1781-1840) discovered that the lateral strain, ϵ_l, is a fixed proportion of the direct strain ϵ_a, so his name is given to Poisson's ratio ν :

$$\nu = \frac{\epsilon_l}{\epsilon_a}$$

Poisson's ratio cannot be less than $\nu = 0$ and cannot be greater than $\nu = 0.5$ for isotropic elastic materials. Typical values are:

-
-
-

7.1.6 Strength of axially loaded structures

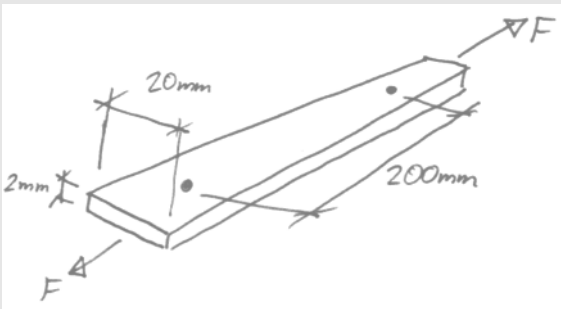
Most structural calculations are to prevent failure or collapse. They therefore need to use the material strength as the measure of reaching the Ultimate Limit State.

The strength of a ductile material is usually characterised by the yield stress, σ_y. For brittle materials, the ultimate strength, σ_u, is used.

7.2 Example

7.2.1 Tension test

A flat sheet of steel is to be tested in a tension test. It is 2mm thick, 20mm wide and has a gauge length of 200mm. (Gauge length = length over which extension measured, ignoring the end arrangements where the specimen is held in the testing machine).



At an extension of δ = 0.060mm the bar carries a load of F = 2.41 kN, and the width of a = 20 mm has been reduced by Δ = 0.001 74 mm

The bar yields at 11.72 kN.

Find Young's modulus E, Poisson's ratio ν and the Yield Stress σ_y. (Do calculations in MPa and mm).

[Note: Pa $\equiv$ N/m^2, so MPa $\equiv 1 \times 10^6$ N/m^2 $\equiv$ 1 N/mm^2. So if we work in N and mm, stress will be in MPa]

First calculate Young's modulus:

Cross-sectional area,

$$A =$$

Stress in bar,

$$\sigma =$$

Direct strain in the bar,

$$\epsilon_a =$$

Young's modulus,

$$E =$$

Now calculate Poisson's ratio:

Lateral strain,

$$\epsilon_l =$$

Poisson's ratio,

$$\nu =$$

Then the yield stress

$$\sigma_y =$$

Summary:

Young's modulus, $E =$

Poisson's ratio, $\nu =$

Yield stress, $\sigma_y =$

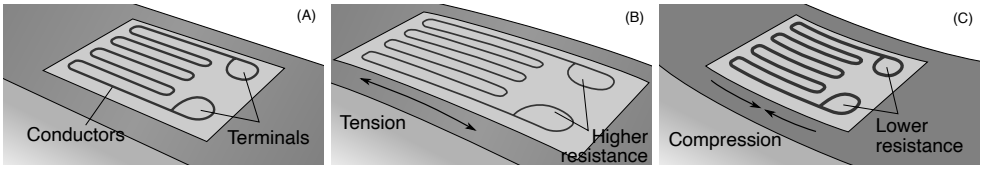
7.3 Strain Guages

In this course, you will measure strain in various structures to challenge and verify the theoretical principles you've learned. To do this, you will use strain guages.

A strain guage works by being fully adhered to the material in which strain is being measured. Therefore when the material stretches, the strain guage stretches, and when the material compresses, the strain guage compresses. The strain guage has a pattern of metallic conductor, laid out so that strain in one direction acts to increase or reduce its electrical resistance, while

strain in the perpendicular direction has very little effect.

This is illustrated by these images of a strain guage on the surface of a beam bending to induce either tension or compression:



https://commons.wikimedia.org/wiki/File:StrainGaugeVisualization.svg

7.4 Members Carrying Axial Load

In this lecture we consider the strength and stiffness of axially-loaded members by examining how they carry load and how they deform under this load. The concept of the stress resultants will be employed to relate the applied load and the resulting stresses within the member.

7.4.1 Uniform Members Carrying Axial Load

There are a number of names for members that carry axial load:

- Tension members are generally called ties (or hangers).
- Compression members are usually called struts in aircraft, columns in buildings, piers in bridges or stanchions (old word).
- Members of a truss may be in tension or compression and are sometimes called bars.

These single members are determinate. Failure occurs when the material strength is reached, but compression members may fail by buckling. Buckling is beyond the scope of SM2, but will be covered later in your degree.

In the previous example we considered the simple tension test, which is an example of a uniform member carrying axial load. We have discussed:

- how it carries load (the stresses inside it);
- how it deforms in terms of the direct and lateral strains;
- how material properties are determined by experiment (Young's modulus, Poisson's ratio, material strength.)

We will return to a uniform member carrying axial load to complete the discussion on member deformation under a given axial load, and the member stiffness.

7.4.2 Axial stress and strain

Recall under an axial load F,

Axial stress

$$\sigma = \frac{F}{A}$$

Axial strain

$$\epsilon = \frac{\sigma}{E}$$

7.4.3 Axial extension

Strain is defined as

$$\epsilon = \frac{\delta}{L}$$

Combining the above three equations gives an expression for the extension of the bar in terms of the applied load:

$$\delta$$

(This relationship is only valid for members with a uniform cross-sectional area that carry uniform axial load).

The deformation of a structure usually controls the serviceability limit state of a structure.

7.4.4 Elastic stiffness

Re-arranging the expression for the total extension of the bar gives:

$$k =$$

This is the elastic stiffness of the bar, k, defined as the force divided by the overall deformation.

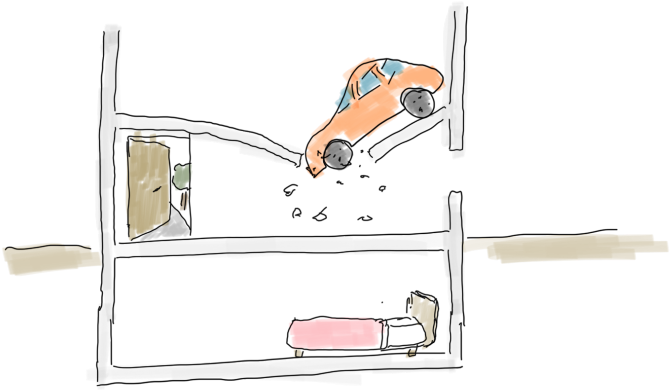
The concept of stiffness can be applied to any elastic structure, carrying any sort of load. In general,

$$k =$$

7.4.5 Strength of axially loaded structures

Often, the purpose of structural calculations is to design against failure or collapse. They therefore need to use the material strength as the measure

of reaching the Ultimate Limit State.



The strength of an axially loaded member is the force in it when yielding occurs. A ductile bar will yield when $\sigma = \sigma_y$. Hence, the strength of an axially loaded member is given by $F_y = A\sigma_y$. The load is often called the "squash load" when in compression.

The yield condition is the simplest measure of failure. However, we will see later in the course that some structural members can carry further load after they start to yield.

The ultimate limit state has not been reached if:

[Stress resultants caused by loads, F] $\leq$ [Strength of the structure, FY]

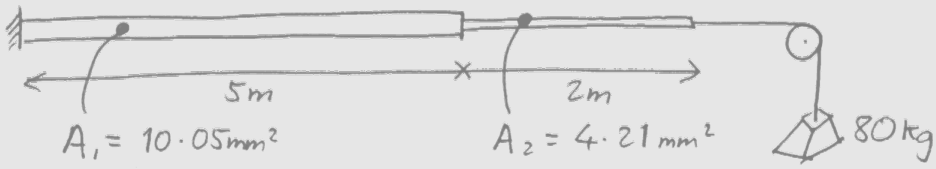
7.4.6 Varying Cross-sections

The previous section considered uniform (prismatic) members, in which the cross-section did not vary along the member. However, we can use the same method to analyse members with varying cross-sections.

7.5 Example

7.5.1 Joined steel cables

A steel cable of length 5 m and cross sectional area 10.05 mm^2 is attached end-to-end to a second steel cable of length 2 m and cross sectional area 4.21 mm^2. If an 80 kg mass is hung from the end of the two joined cables, what is the total elongation? The Young's modulus of the cables is 200×10^3 MPa.



Both of the cables carry the same axial load, $F = 80 \times 9.81 = 785$ N.

Consider the extension of the two cables separately:

Extension of first cable,

$$\delta_1 =$$

Extension of second cable,

$$\delta_2 =$$

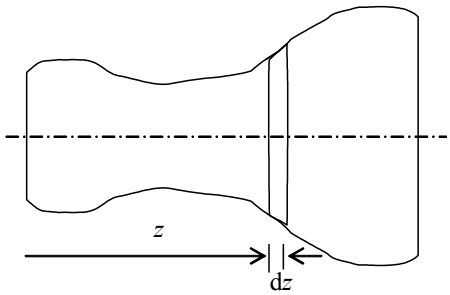
The total elongation of the cables is

7.5.2 Varying section along the length: general

In the above example, there were two distinct cables with different cross-sections. Due to the different cross-sections, the stress inside each cable was different, and hence the strain and elongation of each cable was different. To find the total elongation, we added the elongation of the two cables.

We can apply the same method to a general member, in which the cross-section of a member can vary continuously along its length (but the stress resultant stays the same). To indicate that the cross-sectional area, A, varies with position, we write it as $A(z)$.

Image from the RIBA journal



Behaviour of a short piece of the member

Consider a very short element of the bar (of length δz) - this is analogous to one of the cables in the example.

The stress in the element is:

$$\sigma(z) =$$

The strain in the element is:

$$\epsilon(z) =$$

The change in length of the element is:

$$d\Delta =$$

(analogous to the elongation of one of the cables).

Integrate with respect to length

To find the total change in length of the member, we need to sum the change in length of each element, by integrating:

$$\Delta(z) =$$

Substituting for the strain

$$\Delta(z) =$$

Total elongation over length L:

$$\delta = \Delta(L) =$$

(Note that if A is constant, this gives the previous expression, $\delta = \frac{FL}{AE}$)

7.6 Varying Axial Load

In addition to varying cross-section, the above technique can be used to find the deformation and stress distribution within a bar that carries distributed axial loads.

If the load is applied to the member along its length (axially distributed load), by equilibrium the internal axial force can be found to change along the member, written as $N(z)$.

The stress in the element is:

$$\sigma(z) =$$

The strain in the element is:

$$\epsilon(z) =$$

The deformation is given by:

$$\Delta(z) = \int_0^z \epsilon(z)dz =$$

This technique for determining the deformation of a member is widely used in structural mechanics, and we will meet it again. The process is summarised as:

- Consider the deformation of a short element of a member.
- Integrate to find the deformed shape of the whole member.
- Apply boundary condition constants according to the support conditions.

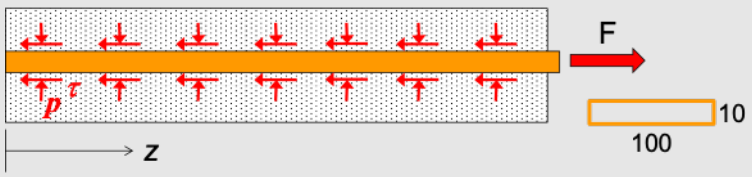
7.7 Example

7.7.1 Bar embedded in an embankment

A flat steel bar is embedded in an embankment. We need to calculate the pull-out resistance of the bar. The solids exert a pressure of 200 kPa all around the bar. The friction coefficient between the bar and the solid is $\mu = 0.61$.

The bar has length $L = 8$ m, width $b = 100$ mm and thickness $t = 10$ mm. It is made from material with $E = 2 \times 10^5$ MPa and $\sigma_y = 240$ MPa.

- Find the stress everywhere in the bar.
- Decide if the bar will yield before it can be pulled out.
- Find the extension of the bar as it is pulled out.



Stress distribution in bar (Working in MPa and mm)

Pressure on the bar,

$$p =$$

Frictional drag acts on all the perimeter of the bar to try to stop it from coming out, and this actually forms the distributed axial load here. At the impending pull-out:

Frictional drag stress,

$$\tau = \mu p =$$

Perimeter of bar section

$$s =$$

Frictional drag force per unit (mm) length

$$f =$$

The internal axial force (stress resultant) N varies along the bar.

At $z = 0$, the force in the bar,

At any position z,

Stress in the bar at position z,

$$\sigma(z) =$$

Will the bar yield?

Stress at the loaded end of the bar ($z = 8000mm$)

$$\sigma_{max} =$$

The yield stress $\sigma_y = 240$ MPa

Extension of bar

Stress at z,

$$\sigma(z) =$$

Strain in the bar,

$$\epsilon(z) =$$

Extension of bar,

$$\delta =$$

The bar extends 4.29 mm and has a max stress in it of 215 MPa. Notice that this 8 m long bar extends only 4 mm when it is almost at the yield stress.

7.7.2 Summary

For a non-uniform member carrying axial load (where the cross-section varies or it carries distributed load), consider the deformation of a short length of the member, and integrate along its length to find the total deformation.

Questions

1. A brass tube has external diameter 50 mm, internal diameter 40 mm and length 300 mm. The tube is placed in a compression testing machine. After an axial compression of 25 kN has been applied, the reduction is length is measured as 0.203 mm. Calculate the elastic modulus, E, of the brass.
2. A circular steel rod of diameter 10mm is stretched in a testing machine to determine its properties. A gauge is attached to the rod to measure changes of length and diameter. When a force of 10kN has been applied to the rod, the original gauge length of 50mm has extended by 3.14×10^{-2} mm, and the diameter has reduced by 1.90×10^{-3} mm.
   a) Calculate the direct stress in the bar, the direct strain and the lateral strain.
   b) Find the elastic modulus of the steel. Determine also Poisson's ratio.
   c) The bar is stretched further until it begins to yield at a force of 21.3 kN. The bar breaks when the force reaches 27.4 kN. Determine the yield stress and the ultimate tensile stress of the steel.
3. Electrical instrumentation is to be used at the bottom of an old mine. To bring the signals to the surface, an insulated solid copper wire of constant cross-section is lowered down, hanging from the top. The wire is 460m long and the unit weight of copper is $\rho = 43.7\,\mathrm{kN/m^3}$. Find the maximum stress in the wire under its own weight. Can this stress be reduced by increasing the cross-sectional area of the wire?
4. For the wire in the previous question, draw a diagram of the wire with a vertical coordinate whose origin is at the bottom of the wire.
   a) Determine the stress distribution in the wire. (Make a cut to the wire at some general height and find the axial force in the wire as

a function of "z" from the equilibrium of the free body, and then the axial stress).

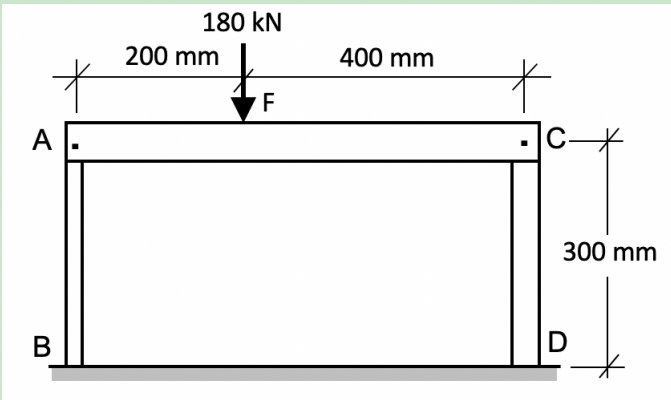
b) Show that the strain distribution in the wire is given by $\epsilon = \frac{\gamma z}{E}$ at height z above bottom (where γ is the unit weight of the wire).

c) Use the strain distribution to find an expression for the total extension of the wire.

d) If the modulus of the wire is $E = 1.32 \times 10^5$ MPa, what is the extension?

e) If the ultimate tensile stress of the wire is only $\sigma_U = 29$ MPa, how deep a mine shaft can the wire be lowered down without breaking it?

5. A rigid bar AC is pinned to two solid circular posts as shown in the figure below. AB is made of steel and has a diameter of 20mm, and CD is made of aluminium and has a diameter of 40mm. A vertical load of 180 kN is applied at point F. Determine the vertical displacement of point F on AC. Take the Young's modulus for steel to be 200 GPa, and for aluminium 80 GPa.



Numerical Solutions: Q1: 5.23×10^4 MPa Q2: (a) 127.3 MPa, 6.28×10^{-4}, 1.90×10^{-4}; (b) 2.03×10^5 MPa, 0.303; (c) 271 MPa, 349 MPa; Q3: 20.1 MPa; No, explain why!; Q4: (d) 35.0 mm, (e) 664 m; Q5: 0.45 mm (downwards)

8 Trusses

8.1 Truss Analysis

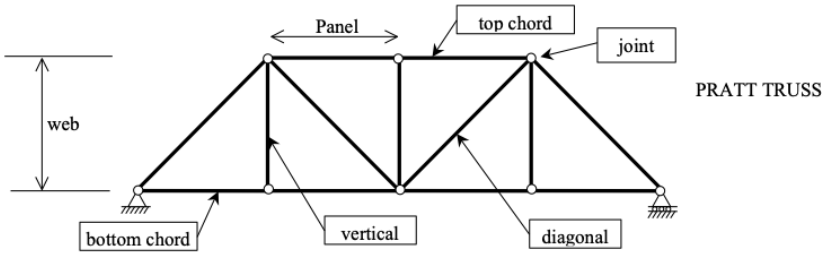
8.1.1 Introduction

Trusses arranged in different classical forms were (usually) named after their inventors or proponents. For example, the Pratt, Howe, Warren, and

Bowstring trusses.

8.1.2 Assumptions and Terminology in Truss Analysis

In a classical truss, the joints are pinned to ensure that they cannot transmit bending moment, and hence the bars carry only axial load. In practice, the joints are bolted, riveted, or welded, however, the bending moments are still very small compared to the axial load, and classical truss analysis can be applied.



A tension member is called a tie and a compression member is called a strut.

We will only consider 'plane' trusses in this course: trusses in which all the members lie in a single 2D plane. This is not very realistic, but many 3D structures can be reduced to a series of 2D structures in different planes to allow analysis. (Truss analysis can be extended to 3D, but the equilibrium equations become more complex and it is more convenient to use computer packages to analyse trusses in 3D).

We use statics to determine the forces in the bars of a truss under given loading.

The following assumptions are made:

- All bars or members meet only at the joints.
- All loads are applied to the truss at the joints.
- Each joint contains an ideal frictionless pin that allows free movement. The pin does not transmit bending moments and only forces are transmitted.
- All bars or members carry only axial loads in tension or compression.
- There is no change in geometry due to loading (i.e.: displacements are small).

Sign convention: Tensile bar forces are positive.

8.1.3 Methods of Analysis

There are lots of methods of finding the forces in the bars of trusses, which fall into three categories:

- Graphical (which look rather dated now, although can give rise to design methods through "Graphic Statics*")
- Computational (taught in 3rd year)
- Analytical

* See http://dx.doi.org/10.1007/s00158-013-1002-x if you'd like to read more about Graphic Statics (this is entirely optional!).

In this course, we will cover the two key analytical methods:

- the Method of Joints; and
- the Method of Sections

Each of these two methods is useful for particular kinds of problem, as illustrated in the tutorial exercises.

8.1.4 Method of Joints

USED: To find all the bar forces in a truss.

METHOD: Apply 2 equilibrium equations to each joint in succession.

STEPS:

1. Draw a free body diagram of the whole truss, and find the support reactions.
2. At each joint in turn, write the two joint equilibrium equations.
3. Solve the set of resulting simultaneous equations in the bar forces.

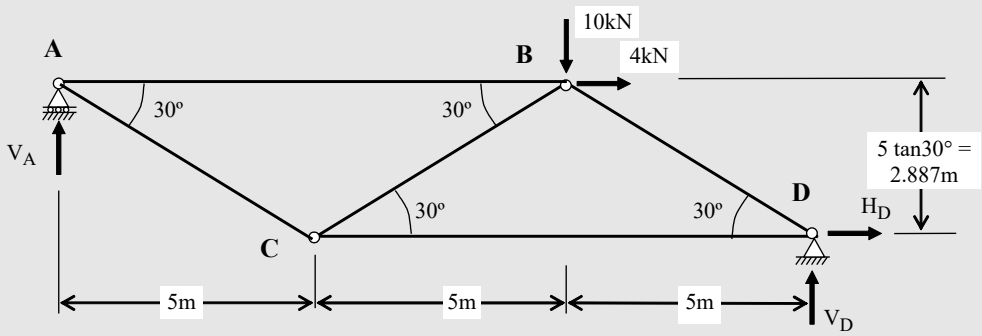
NOTES:

- The work involved depends very much on the sequence in which the joints are analysed.
- The sequence in which the joints are analysed does not affect the forces in the bars.
- Take care over correct signs.

8.2 Example

8.2.1 Method of Joints

Find the forces in all the bars of the following truss:



All the bar forces are needed, so use the Method of Joints.

Step (1) - Global equilibrium

Horizontal equilibrium:

$$H_D =$$

Moments about D:

$$V_A =$$

Vertical equilibrium:

$$V_D =$$

Steps (2) and (3) - Write two equilibrium equations at each joint in turn and solve the resulting equations.

Joint A

Vertical equilibrium

Horizontal equilibrium

Joint D

Vertical equilibrium

Horizontal equilibrium

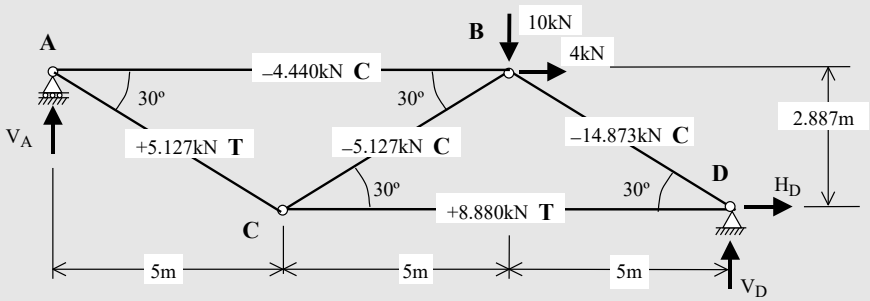
Joint C

Vertical equilibrium

Horizontal equilibrium: Not needed, as we now know all bar forces, however, check that we haven't made any mistakes.

Summary

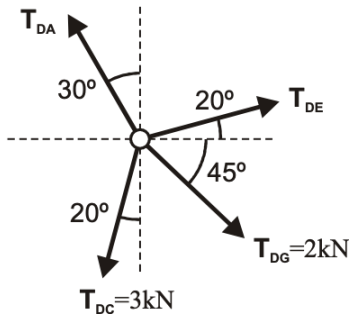
Bar	Bar tension	Value of force (kN)	Tension (T) or compression (C)
AB	TAB	4.440	C
BC	TBC	5.127	C
AC	TAC	5.127	T
BD	TBD	14.873	C
CD	TCD	8.880	T



Note that it was not necessary to write equations at Joint B; all bar forces are now known.

8.2.2 A note about inclined bars

The two equilibrium equations do not need to be written in the horizontal and vertical directions. Careful choice of the direction in which you write the equilibrium equations can avoid the need for simultaneous equations. For example:



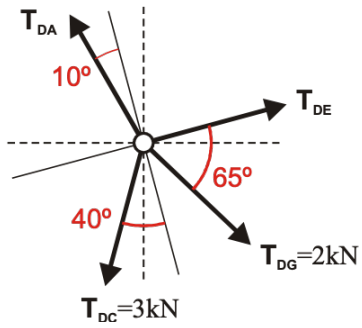
Resolving horizontally:

$$T_{DA}\sin 30 + 3\sin 20 = T_{DE}\cos 20 + 2\cos 45$$

Resolving vertically:

$$T_{DA}\cos 30 + T_{DE}\sin 20 = 3\cos 20 + 2\sin 45$$

i.e. two simultaneous equations for T_{DA} and T_{DE}. In this case, it is better to resolve perpendicular to TDE:



(i.e. the forces are uncoupled.)

8.3 Method of Sections

USED: To find a single bar force in a truss.

METHOD: Apply 3 equilibrium equations at a cut across the structure.

STEPS:

1. Draw a free body diagram of the whole truss, and find the support reactions.
2. Knowing the bar in which you wish to find the force, devise a cut through all the bars, such that only three bars are cut through (there will be three equations of equilibrium).
3. Write the three equilibrium equations (usually it is best to write 2 moment equilibrium equations, since these reduce the number of unknowns present).

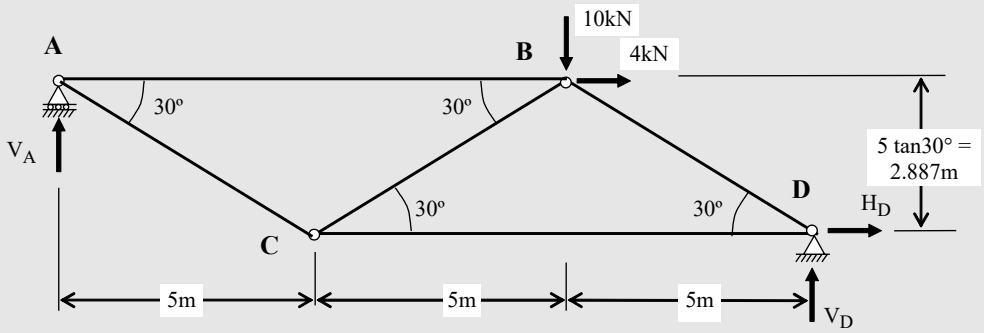
Select the cut with care to eliminate unwanted unknown forces.

The Method of Sections is a much faster way of finding the forces in particular bars, provided it is carefully thought out.

8.4 Example

8.4.1 Method of Sections

For the same truss as before, find only the forces in bars *BC* and *CD*. Only two bar forces are needed, so use the Method of Sections.



Choose a cut that passes through the full depth of the truss, and through bars *BC* and *CD*. The piece cut off must not have three forces that are simply convergent at a point.

Step (1) - Global equilibrium

Horizontal equilibrium:

$$H_D =$$

Moments about D:

$$V_A =$$

Vertical equilibrium:

$$V_D =$$

Step (2) - Cut the truss to reveal the required unknowns

This is a free body diagram, so make sure that you include all the forces acting on it.

Step (3) - Write the equations of equilibrium

Bar *BC*

Vertical equilibrium

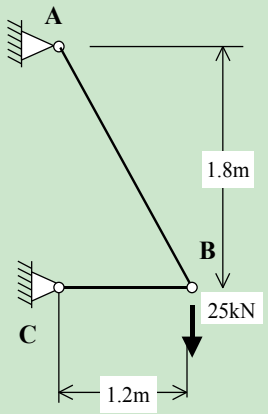
Bar *CD*

Moments about B

Questions

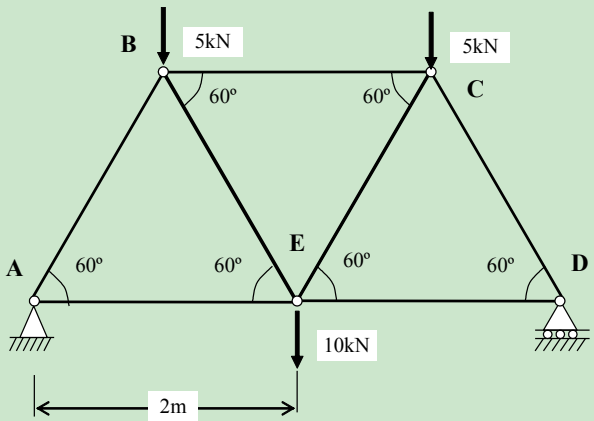
Part 1: Truss Analysis by the Method of Joints

1. A load of 25 kN is supported by the frame ABC as shown in the figure below.



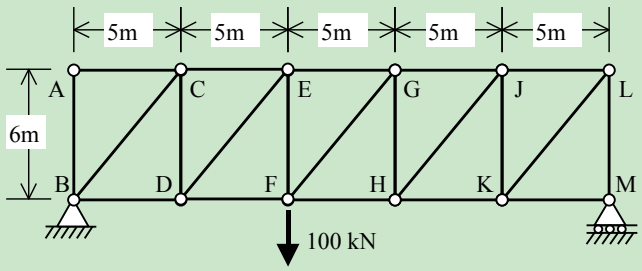
   a) Calculate by the method of joint equilibrium the forces in members AB and BC, and state whether these members are in tension or compression.
   b) Calculate the magnitude of the reactions at the Joints A and C.

2. Calculate the magnitude of the forces in the members of the pin-jointed frame ABCDE shown below and state whether the member is in tension or compression. The support at D is on rollers. Take advantage of the symmetry of the frame. All members of the frame are of equal length.

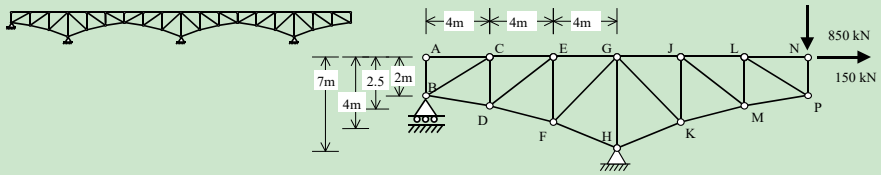


Part 2: Truss Analysis by the Method of Sections

3. The truss bridge shown below carries loads which move across the bottom chord. Using the method of sections, investigate the force in the member DE as the load of 100 kN moves from B to D, F, H, K and M. Plot the force in DE against the position of the load. (Ignore the self-weight of the truss).

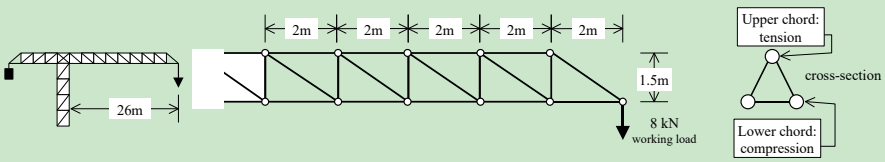


4. A bridge is constructed from repeated sections as shown below. The three most critical members are FH, GH and KH. Find the forces in these three bars when horizontal and vertical loads act at joint N as shown.



5. A construction site crane boom is built in modular form, with a triangular cross-section 1.5 m high and 1 m wide as shown in the figure below. Each cell of the crane boom is 2 m long and is composed of three longitudinal members: a single tension member at the top and two compression members at the bottom. A pair of diagonals is also placed on the sides.

   The crane to be erected on your site must be able to support an 8 kN working load to a maximum reach of 26 m. A safety factor of 1.8 is to be used, so the design failure load is $8 \times 1.8 = 14.4$ kN.



Each bar must be chosen from three different sizes. Smaller sizes are cheaper and have less weight but are weaker.

Size	Cross-sectional area mm^2	Tensile strength kN	Compressive strength kN
A	500	100	30
B	1000	200	100
C	1500	300	250

Using the method of sections, write general expressions for the force in the tension, compression and diagonal members at any location in the boom. (Remember that there are two compression members). From these forces select the lightest member which may be used in each location, and deduce the number of each size needed. Ignore the weight of the boom.

For this exercise, the boom may be treated as a 2D truss: ignore the inclination of the diagonals into the page.

Numerical solutions: Q1: (a) 30.05kN (T), -16.67kN (C); (b) 25.0, -16.67kN; 0, +16.67kN. Q2: -11.55 C; -8.66 C; 5.77 T; 5.77kN T. Q3: 0, 26.0, -78.1, -52.1, -26.0, 0 kN. Q4: Reactions: HH = -150kN, VB = -938kN, VH = +1788kN; TFH = -2009kN, THK = -1821kN, TGH = +511kN tension. Q5: TT = 9.6x kN, TB = -9.6(x+2) kN, TD = 24.0 kN; Top chord CCC BBBBB AAAA: Bottom chord CCCC BBBBBBB AA: Diagonal all A.

9 Euler Beam Theory

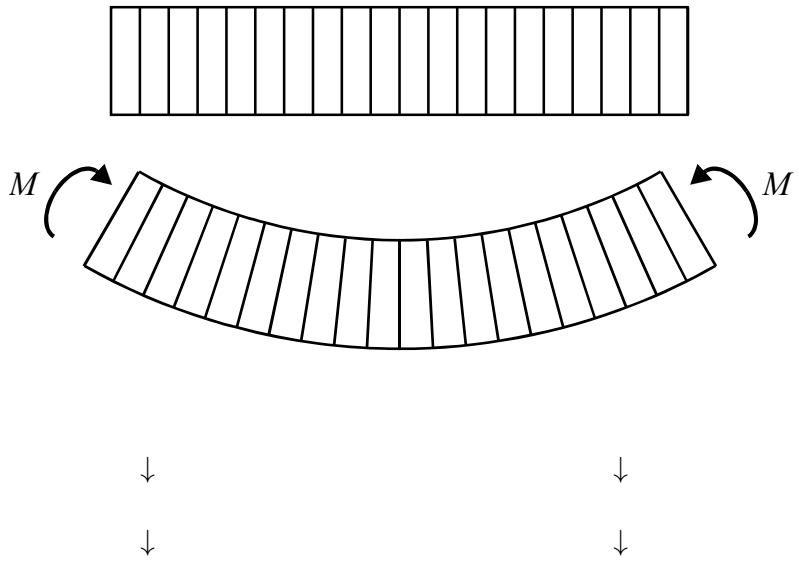
We have investigated the strength and stiffness of members carrying axial loads, and trusses, which we can use to form many useful structures from axially-loaded elements. However, in almost any real structure, there will be elements for which we can not, or choose not to, maintain purely axial loads. Often this means elements are also subjected to bending moments, and a structural member predominantly carrying bending moments is called a beam.

9.1 Elastic Bending of Beams

The way in which beams carry load is described by *Euler's Theory* (commonly known as *Engineering Bending Theory*). This theory describes the deflections and strengths (which are closely interlinked) of beams under a variety of loading conditions.

9.1.1 Qualitative Description Of Beam Behaviour

Before looking at the details of Euler's Theory, it is worth exploring qualitatively how a beam behaves when a sagging bending moment is applied to it. The figure below shows a beam with a series of straight lines on it, vertical in its undeformed shape. These straight lines represent planes in 3D. Note that they are shown as remaining straight (and so remaining planar) when a moment is applied. This is a key assumption of Euler's theory: plane sections remain plane.



There is some point in the beam where the fibre length doesn't change. We call this the "neutral axis". The material will first yield at either the top or bottom of the beam.

The beam is curved. We describe the amount it is curved as the "curvature" *(If we know the curvature, we can integrate along the length of the beam to find the deflected shape, in much the same way as we integrated to find the extension of a rod under axial load in Chapter 7)*.

This is (in outline) the reasoning behind Euler's Theory, which we will now examine in detail. (Note that Euler's beam theory works from deformations to find the stress distribution, unlike our approach for axially loaded members where we started with stress and ended with deflected shape).

9.1.2 Assumptions of Euler's Engineering Bending Theory

The Euler Engineering Bending Theory is based on the following assumptions:

- The deflections of the beam are small compared with its geometry (i.e. deflections are typically less than the beam thickness, though this should really be expressed in terms of the slope of the beam: see later).
- Sections through the beam that are plane and normal to the beam axis remain plane after bending: "plane sections remain plane". This is also

called “the plane strain assumption”.

- The beam is slender (the ratio of length to depth is greater than about 5: beams that do not satisfy this condition are called “deep beams” and require more complicated analysis).
- The beam is subject to bending alone (no axial forces present).

The above assumptions apply whether the material of the beam is elastic or plastic, and can be used to analyse beams right up to plastic collapse. They can also be used for nonlinear materials like concrete.

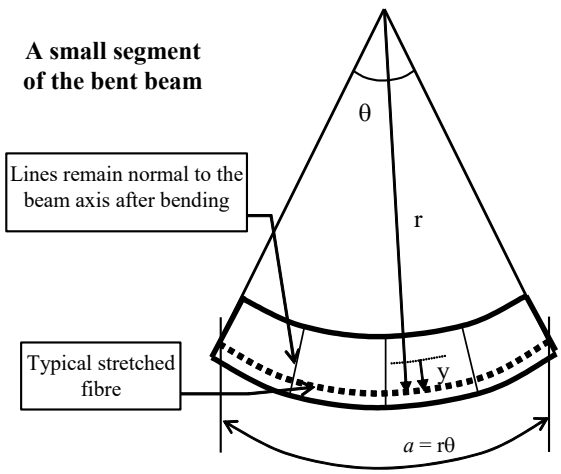
However, *linear bending theory* also makes the additional assumption:

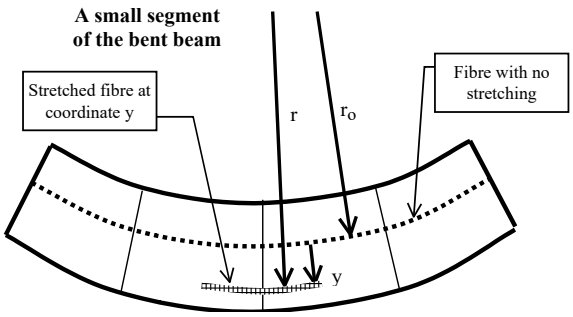
- The beam is made of homogeneous linear elastic material.

9.1.3 Strain Distribution through the Beam Depth

The assumption that “plane sections remain plane” leads to the strain distribution that must occur in the beam.

We have observed that the top fibres are being compressed and becoming shorter, while the bottom fibres are being stretched and becoming longer. At any point, the beam is locally bent in a small arc (which subtends an angle θ) of a circle centred some distance from the beam:





The length of a fibre in the beam at a general position (distance r from the centre of the circle) is given by:

$$a = r\theta$$

Where θ is in radians.

The *Neutral Axis* is the position (defined by r_0) in the beam where the fibre is neither stretched nor compressed, and hence its length is equal to the original length, a_0, of the unloaded beam:

$$a_0 = r_0\theta$$

r_0 is the radius of curvature of the beam.

The strain (parallel to the beam axis, z) at a general position in the beam (distance r from the centre of the circle) is given by:

$$\varepsilon_z = \frac{\text{new length} - \text{original length}}{\text{original length}} = \frac{a - a_0}{a_0} = \frac{r\theta - r_0\theta}{r_0\theta}$$

To simplify the equations, define the transverse coordinate y, which has its origin at the neutral axis:

$$r = r_0 + y$$

The strain can then be written instead as:

$$\varepsilon_z = \frac{(r_0 + y)\theta - r_0\theta}{r_0\theta} = \frac{y}{r_0}$$

Defining $\kappa = 1/r_0$, i.e., the reciprocal of the radius of the local arc, which is called "curvature", we have:

$$\varepsilon_z = \frac{y}{r_0} = \kappa y$$

This equation shows that:

- the longitudinal strain in the beam varies linearly with the distance y from the axis on which there is no strain;
- as the beam bends more, the radius of the local arc, r_0, becomes smaller, its reciprocal (i.e. "curvature") becomes larger, and the strain increases; and
- the strain is at its most positive (tensile) on the bottom fibre (y is max) when the curvature is positive.

The plane strain assumption is valid whether the beam is elastic, or yielded, or materially nonlinear.

9.1.4 Bending Stress from Curvature

In the last section we determined the strain distribution in a beam. The strains are assumed to be only along the length of the beam (direct strain), and are given by:

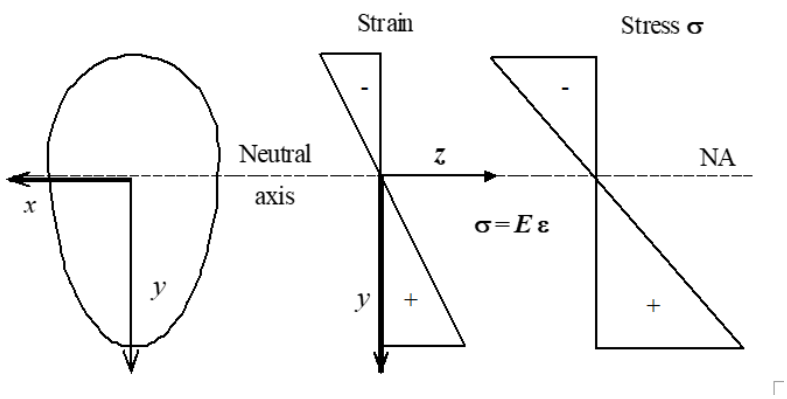
$$\varepsilon_z = \kappa \cdot y$$

If the material is linear elastic, then the stresses in the longitudinal direction (direct stress) are:

$$\sigma_z = E \cdot \kappa \cdot y$$

(We will drop the subscript z, since the stresses are only in one direction).

Both the strain and stress vary linearly across the section, and are zero on the Neutral Axis. We can illustrate this for an arbitrary cross section:



9.1.5 Position of the Neutral Axis

Euler's Theory describes beams that only carry bending moments, and no axial load. We can use this to determine the position of the neutral axis.

For the axial force carried by the section to be zero, the net axial stress across the section must sum to zero:

$$N = \int \sigma dA = 0$$

where dA indicates an integration over the cross-sectional area A.

$$N = \int E\kappa y dA = 0$$

E and κ are constant throughout the section, so:

$$N = \int y dA = 0$$

The coordinate y has its origin on the Neutral Axis, so for the axial force to be zero, the position of the neutral axis must satisfy this equation.

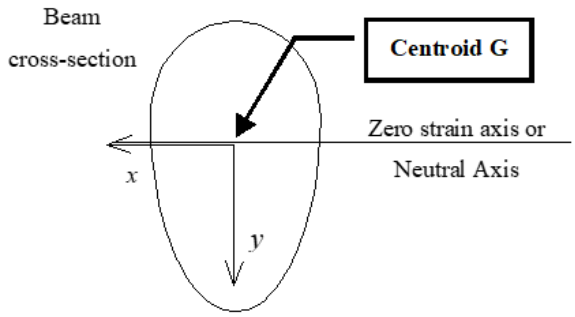
This is the same condition that defines the centroid or centre of mass of a body. Thus the Neutral Axis (NA) (for pure bending) passes through the section centroid.

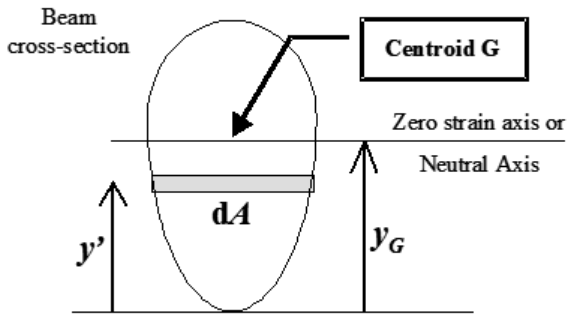
To find the position of the centroid (hence the neutral axis position), start by defining a coordinate y' at a convenient point in the section. e.g. use the bottom of the section.

The position of the centroid is given by:

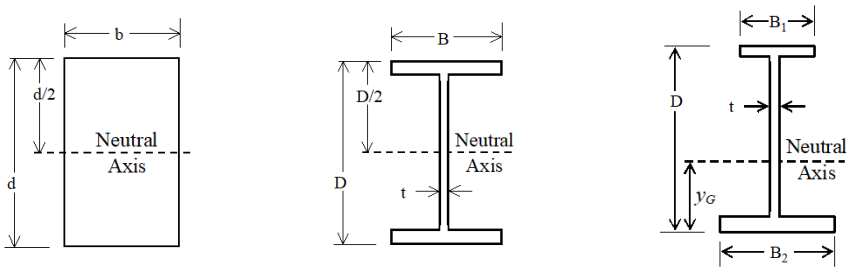
$$y_G = \frac{\int y' dA}{A} = \frac{\sum y_i' A_i}{A}$$

Once this is known, we have also defined the origin of the y coordinate (which defines the strain distribution).





Sometimes the neutral axis can be seen to be at the mid-height of the beam by symmetry, but if there is no horizontal line of symmetry, it will not be.



Rectangular section Symmetrical I section Monosymmetric beam

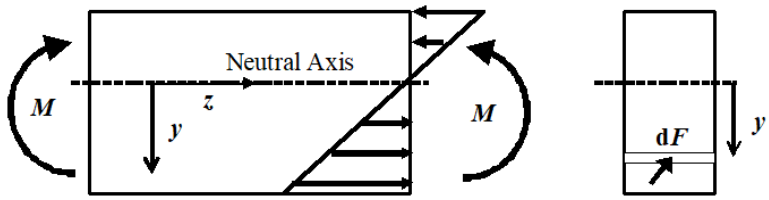
9.2 Elastic Bending of Beams: Bending Moment and Bending Stress Relationship

9.2.1 Bending Stresses and Moments

We have seen that the stresses in the beam at any section are:

$$\sigma = E\kappa y$$

We can evaluate the bending moment from this stress distribution.



The elemental force dF on a small element of area dA at distance y from the NA is:

$$dF = \sigma dA = E\kappa y dA$$

Taking moments about the neutral axis:

$$dM = ydF = E\kappa y^2 dA$$

The total bending moment carried by the section is found by integrating:

$$M = \int E\kappa y^2 dA = \kappa E \int y^2 dA = EI_{xx}\kappa$$

Here we have defined $I_{xx} = \int y^2 dA$, which is a key property of a structural cross section. It is called the *second moment of area*, in this case about the x-x axis. Hence in general:

$$\kappa = \frac{M}{EI}$$

(from here we can see that "EI" is a stiffness measure at the section level, and it is referred to as the "flexural rigidity").

This allows us to write a set of equations which describes (very concisely) the relationship between moment M, stress σ and curvature κ in a beam, related by the geometric properties I and y and the material property E. This is a good equation to memorise.

$$\frac{\sigma}{y} = \frac{M}{I} = E\kappa$$

The Second Moment of Area

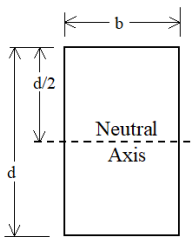
The second moment of area, $I_{xx} = \int_A y^2 dA$, is a geometric property of the section. It has units (Length)^4 eg: mm^4.

I_{xx} is called the *second moment of area* about the x-x axis.

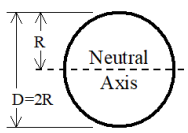
EI_{xx} is called the beam's *flexural rigidity* about the x-x axis.

For simplicity and generality, the subscript $x - x$ will be omitted for the rest of this section.

Solving the integral equation to find I is something you'll be readily able to do, but in reality it's not often required. For standard sections, I can be found from tables of section properties. For example:



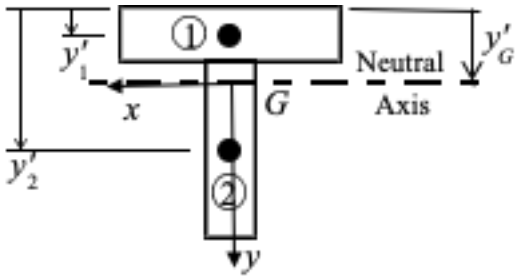
$A = bd$
$I_{xx} = \frac{bd^3}{12}$
Rectangular section



$A = \pi R^2$
$I = \frac{\pi}{4} R^4$
Circular bar

Geometric properties by combination of shapes

Many structural elements are formed by combinations of rectangular shapes. Take this T section as an example:



Using a set of rules to combine the properties of these simple shapes, we can find the neutral axis and second moment of area of most engineering cross-sections.

The position of the neutral axis in a beam is the same as the centroid of the cross-section, which is given by:

$$y'_g = \frac{\sum y'_i A_i}{\sum A_i} = \frac{y'_1 A_1 + y'_2 A_2}{A_1 + A_2}$$

where y'_i is the distance of the centre of rectangle i below the top of the section A_i = breadth $\times$ depth is the cross-sectional area of rectangle i.

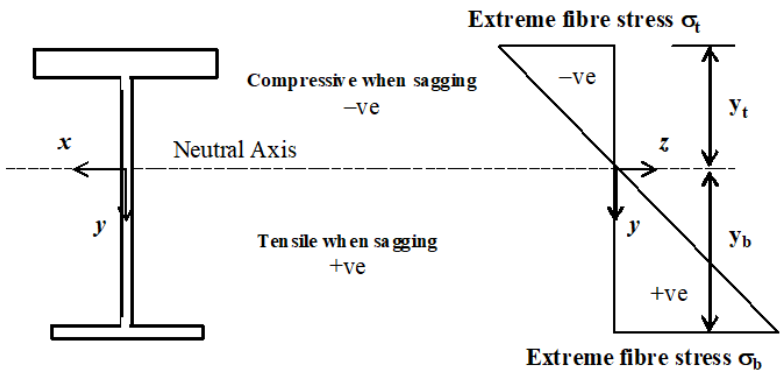
The Second Moment of Area has units of (length)^4 and is given by:

$$I = \sum \left\{ \frac{b_i d_i^3}{12} + A_i \left(y'_i - y'_G\right)^2 \right\}$$

where: b_i is the breadth (i.e. the dimension parallel to the neutral axis) of rectangle i; d_i is the depth (perpendicular to the neutral axis) of rectangle i. $((y'_i - y'_G))$ gives the y-distance between the centre of rectangle i to the neutral axis (centroid) G. This equation is known as the “parallel axis theorem”.

9.2.2 Failure of Beams

A beam will fail when the maximum stress in the beam section reaches the material strength. This occurs at the extreme fibres.



For the bottom extreme fibre,

$$\sigma_b = \frac{My_b}{I}$$

For the top extreme fibre,

$$\sigma_t = \frac{My_t}{I}$$

The extreme fibre stress is commonly called the bending stress.

Note that sagging moment (M positive) gives:

- tensile stress (σ positive) in the bottom fibre (y_b positive)
- compressive stress (σ negative) in the top fibre (y_t negative)

If the beam is made from a *brittle* material, failure occurs when the bending stress reaches the *fracture stress*.

If the beam is made from a *ductile* material, then the material in the beam will first start to yield at the extreme fibre. The maximum moment that the beam can carry without any yielding is the *First Yield Moment*, M_Y, given by:

$$\sigma_Y = \frac{My}{I}$$

and rearranging,

$$M_Y = \frac{I}{y}\sigma_Y$$

The ultimate capacity of a ductile beam is higher than the first yield moment. However, M_Y is often used to give a safe estimate of failure.

The Elastic Section Modulus

The maximum stress occurs either at the bottom or top of the section. The section modulus is introduced to enable a simpler relationship between the maximum (longitudinal) stress and the moment,

$$\sigma_b = \frac{My_b}{I} = \frac{M}{Z_b}$$

$$\sigma_t = \frac{My_t}{I} = \frac{M}{Z_t}$$

where

$Z_b = \frac{I}{y_b}$ is the elastic section modulus for the bottom of the section

$Z_t = \frac{I}{y_t}$ is the elastic section modulus for the top of the section

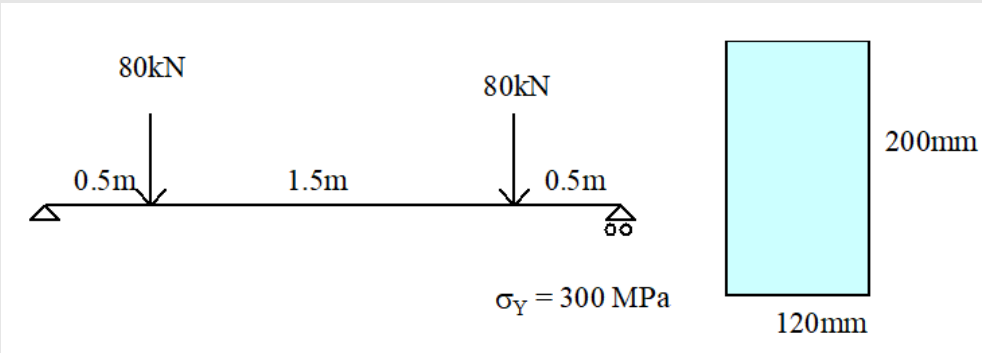
Tables of section properties often include values of Z.

This gives a simpler expression for the maximum bending stress,

$$\sigma_{max} = \frac{M}{Z}$$

9.3 Examples

9.3.1 Example 1.



Find the:

- Maximum bending stresses.
- First yield moment if the material has a strength of σ_Y = 300 MPa

Work in N, mm and MPa (consistent set of units).

Section properties

For a rectangular beam:

$$I_{xx} =$$

Distance from the centroid (in the middle) to the extreme fibre either way is:

So the elastic section modulus is:

$$Z_e =$$

Maximum bending stress

The maximum bending moment has a constant value between the point loads, it is “sagging”:

And the maximum bending stress takes place at the extreme fibre of the cross-section

$$\sigma =$$

The section is symmetric, so the stress has the compressive stress at the top fibre is the same value as the tension at the bottom fibre.

The load factor against first yield is

$$\lambda = \frac{\sigma_Y}{\sigma_{max}} =$$

The bending stress (50 MPa) is much lower than the yield stress of 300 MPa, so this beam could take a lot more load.

First yield moment

The first yield moment is given by:

$$M_Y =$$

This also gives a load factor against first yield of

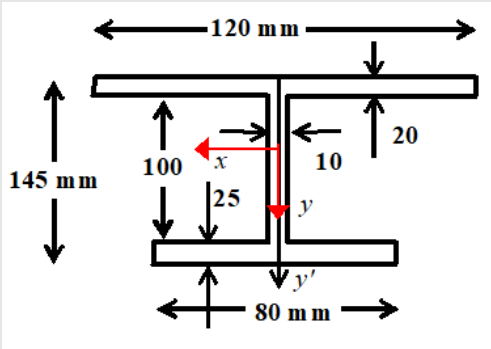
$$\lambda =$$

9.3.2 Example 2.

A beam in bending has a cross-section as shown below. The centroid (neutral axis, or N.A.) of the section is found to be 66.5 mm below the top

edge, and the 2nd moment of area of the section is found to be $I = 17.4 \times 10^6$ mm^4.

1. Calculate the section moduli.
2. If the section is subjected to a sagging bending moment M = 50kNm, find the maximum tensile and compressive stresses of the section.



Section moduli

The distance of the top fibre to N.A. y_t = 66.5mm, and the distance of the bottom fibre to N.A.y_b = 78.5mmmm.

Z_t =

Z_b =

Maximum stresses

Maximum stresses take place at the extreme fibres. For sagging moment, max. compressive stress takes place at the top fibre, and its value is:

σ_t =

Max. tensile stress takes place at the bottom fibre, and its value is:

σ_b =

It can be seen from the above procedure that, using the section modulus can make the calculation simpler, especially if we are only interested in the magnitude of the max. stress, which is often the case in the design of the metal structures (where tensile and compressive strength of the material is the same).

However, if the stresses need be identified as being tensile or compressive, a judgement needs to be made in accordance with the direction of the bending moment, as also seen above.

On the other hand, if we use the general equation for the bending stresses, $\sigma = \frac{My}{I}$, and observe the sign convention, we will know tensile or compressive stress from the sign of the result directly.

Questions

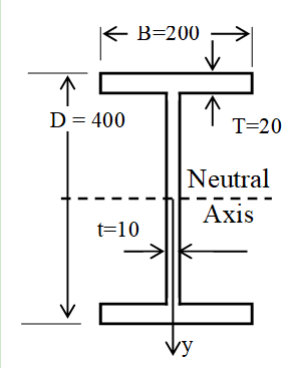
1. A doubly symmetric I cross-section has the dimensions (in mm) as shown in the figure below.

   The 2nd moment of the area can be calculated by:

$$I_{xx} = \frac{1}{12}\left[BD^3 - (B-t)(D-2T)^3\right]$$

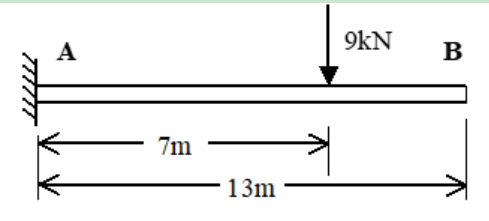
   The cross-section is subjected to a hogging moment of 250 kNm.

   Calculate the bending stresses (direct stresses) at the top and bottom of the section, and at the top and bottom positions of the web, and state if the stresses are tension or compression. Then sketch the distribution of the bending stresses along the depth of the section. Mark the above key stress values.



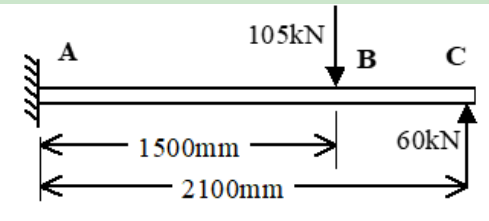
2. For each of the beams below, find a) the maximum bending stress in the beam, and b) the load factor against first yielding, λ = (yield stress / maximum bending stress). The material in all the beams has a yield stress of σ_Y = 250 MPa. To find the maximum stress:
   - Draw the bending moment diagram
   - Identify the moments that could cause the maximum bending stress in the beam (draw and use shear force diagram if needed)
   - Identify the bending stresses at these positions.

a) Point loaded cantilever



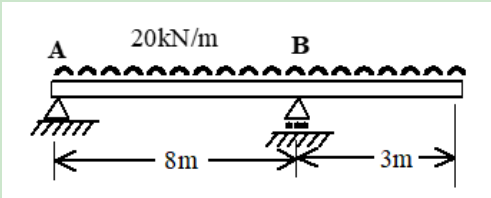
Uniform section beam with $Z_e = 3.64 \times 10^5$ mm^3.

b) Cantilever with two loads



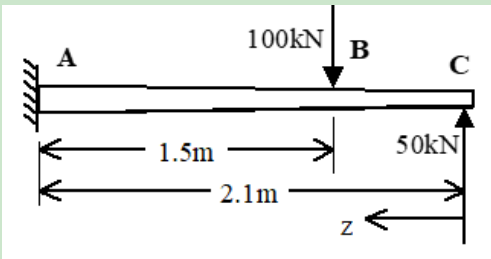
Uniform section beam with $Z_e = 2.50 \times 10^5$ mm^3.

c) Beam with distributed load



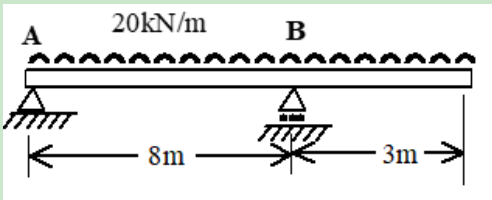
Uniform section beam with $Z_e = 4.40 \times 10^5$ mm^3 .

d) Non-uniform cantilever



Non-uniform beam with linearly varying section modulus:
$Z_e = (1 + z)1 \times 10^5$ mm^3 with z in metres.

e) Beam with unsymmetrical supports



Uniform beam of unsymmetrical section:

$$Z_{top} = 5.60 \times 10^5 \text{ mm}^3$$

$$Z_{bottom} = 7.60 \times 10^5 \text{ mm}^3$$

Find the maximum tensile stress and the maximum compressive stress.

Answers. Q1 : +152.46, -152.46, +137.22, -137.22MPa Q2: a): -63kNm; 173.1MPa, 1.44; b): 36, -31.5kNm, 144.0MPa, 1.74; c): -90.0, +118.16kNm; 268.5MPa, 0.93 (beam yields); d): +30, -45kNm; 1.6×10^5, 3.1×10^5 mm^3, 145.2, 187.5 MPa, 1.33; e): -3.4375m, +118.16, -90kNm, +156.1, +160.7, 211.0, 118.4MPa, t 1.556, c 1.185;

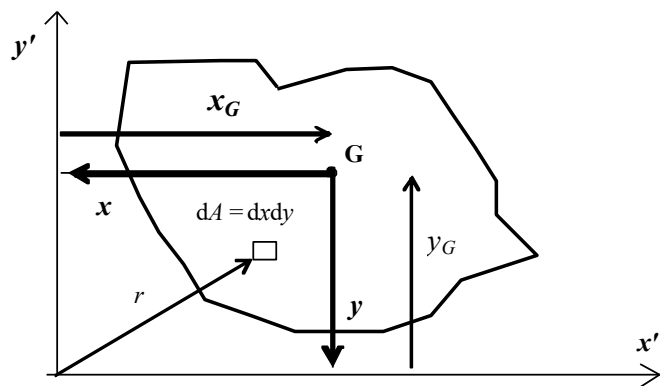
10 Composite Beam Sections

10.1 Geometric Properties of Cross Sections

The behaviour of structural members is dependent upon their cross-sectional shape. The preceding lectures have introduced a variety of section properties without spending any time investigating how we derive these properties and calculate their values. In this lecture, we will look at these section properties in detail.

10.1.1 Definition of axes

The position of the centroid, *G*, of a section is crucial to its response in bending. One of the first steps is thus to determine the position of the centroid of a section. Take a generic cross section shape:



Two parallel axes are needed in the process of determining the centroid. First, (x', y') has its origin at any convenient point (eg: the corner of the section). (x', y') is used to find the position of the centroid.

(x, y) has its origin at the centroid of the section (which is initially unknown). y pointing downwards to be consistent with stress analysis.

The position of the centroid is defined as (xG, yG).

10.1.2 Area

Finding the area, you will commonly use the area eqations you're used to. Mathematically:

$$A = \int dA = \int \int dxdy = \int \int dx'dy'$$

If you know the cross-sectional areas of the constituent parts of the section (e.g. if they are rectangular), you can, of course, sum them:

$$A = \sum_i A_i$$

10.1.3 First moment of area

The first moment of area $H_{x'}$ about the x' axis, and $H_{y'}$ about the y' axis are defined as follows:

$$H_{x'} = \int y' dA = \int \int y' dx' dy'$$
$$H_{y'} = \int x' dA = \int \int x' dx' dy'$$

Again, known (e.g. rectangular) areas can be used as the basis to compute the first moment of area for the whole section:

$$H_{x'} = \sum_i y'_{Gi} A_i$$

$$H_{y'} = \sum_i x'_{Gi} A_i$$

where y'_{Gi} and x'_{Gi} are the coordinates of the centroids of each of the known areas.

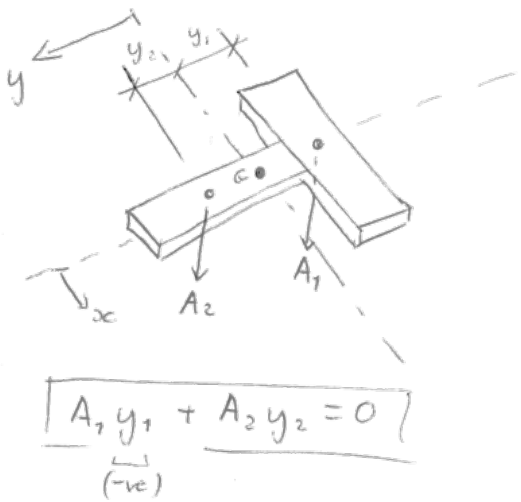
10.1.4 Centroid

The centroid (which is also the centre of mass if the section has uniform density) is defined from the first moment of area. The first moment of area about an axis through the centroid is zero.

So the centroid is defined by:

$$H_x = \int y dA = 0 \qquad or \qquad H_x = \sum_i y_{Gi} A_i = 0$$

This sketch shows how the concept works for a slice from a beam formed from two rectangular cross-sections:



However, we cannot define y (or y_1 and y_2 in the sketch) until we have found the position of the centroid. Initially, we must work in a general coordinate, y' :

The first moment of area about the x' axis is:

$$H_{x'} = \int y' dA = \int (y_G - y) dA = \int y_G dA - \int y dA = y_G \int dA - 0 = y_G A$$

So, the first moment of area can be used to find the centroid of a section:

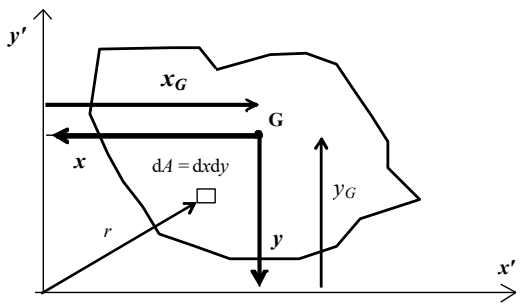
$$\int y' dA = y_G A \qquad \therefore \qquad y_G = \frac{\int y' dA}{A}$$

and a similar argument holds for the x' coordinate of the centroid x_G.

The equations to find the centroid of a cross section are:

$$x_G = \frac{\int x' dA}{A} = \frac{\sum_i x_{Gi} A_i}{\sum_i A_i}$$

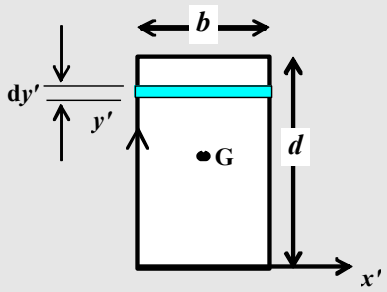
$$y_G = \frac{\int y' dA}{A} = \frac{\sum_i y_{Gi} A_i}{\sum_i A_i}$$



10.2 Examples

10.2.1 Centroids for simple shapes

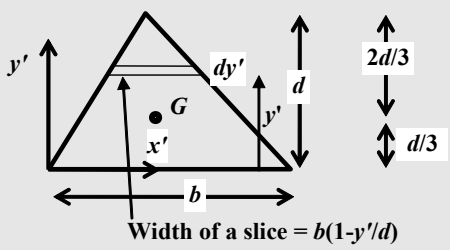
Rectangle



$H_{x'}$ =

y_G =

Triangle



$H_{x'}$ =

=

=

y_G =

10.3 Second moment of area

It is important to remember that the second moment of area is evaluated about the section's centroid, using the axes (x, y).

Second moment of area about the x axis:

$$I_{xx} = \int y^2 dA = \int \int y^2 dxdy$$

Second moment of area about the y axis:

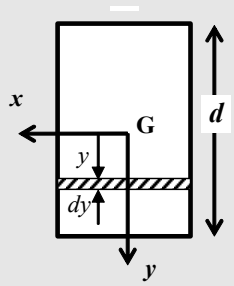
$$I_{yy} = \int x^2 dA = \int \int x^2 dxdy$$

The second moment of area is sometimes called the 'moment of inertia' because of its use in dynamics. There's no inertia component in these bending calculations, but the term is still used, I suppose because it's shorter.

10.4 Examples

10.4.1 Second moments of area for simple shapes

Rectangle

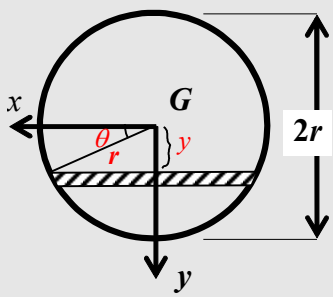


$I_{xx} =$

$I_{xx} =$

Similarly : $I_{yy} =$

Circular section



I_{xx} =

Substitute y =

dy = and

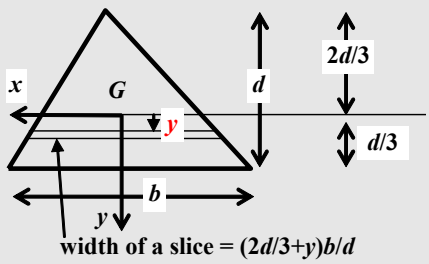
I_{xx} =

=

=

=

10.4.2 Triangle



I_{xx} =

=

=

Similarly I_{yy} =

Note that the rectangle has

10.4.3 Radius of gyration

The radius of gyration is a characteristic dimension of the body that relates the area and the inertia (i.e. 2nd moment of area). It is taken from the analysis of rotating bodies.

Definition: $I = Ar^2$.

so: $I_{xx} = Ar_x^2$ and $I_{yy} = Ar_y^2$

Thus: $r_x = \sqrt{\frac{I_{xx}}{A}}$ and $r_y = \sqrt{\frac{I_{yy}}{A}}$

The radius of gyration is a measure of how effective the cross-section is in resisting bending. A high radius of gyration means that the area has been given a high second moment of area.

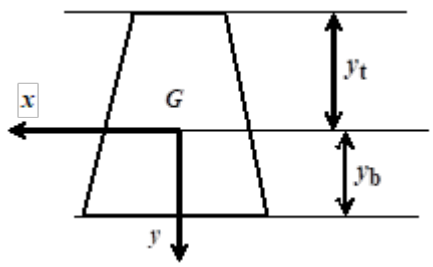
10.4.4 Elastic section modulus

The elastic section modulus Z_e has been defined before in the context of maximum bending stresses. It is the ratio of second moment of area I to the distance from the centroid to the extreme fibre y_e.

If the distance from the centroid to the extreme fibres of the section are y_t and y_b then the elastic section moduli Z_e for bending about x-axis are:

$Z_{e,t} = \frac{I_{xx}}{y_t}$ and $Z_{e,b} = \frac{I_{xx}}{y_b}$

Thus for calculating the magnitude of maximum bending stresses over a cross-section (area), we can use the simple form of $\sigma_t = \frac{M}{Z_{e,t}}$, $\sigma_b = \frac{M}{Z_{e,b}}$



NB: however, for the detailed stress distribution over a cross-section, including the maximum stresses and whether they are tension or

compression, we can always use the general expression of bending stress to do the calculation: $\sigma = \frac{My}{I}$.

10.4.5 Polar second moment of area

The polar second moment of area is defined as:

$I_{zz} = \int r^2 dA$ but $r^2 = x^2 + y^2$, so

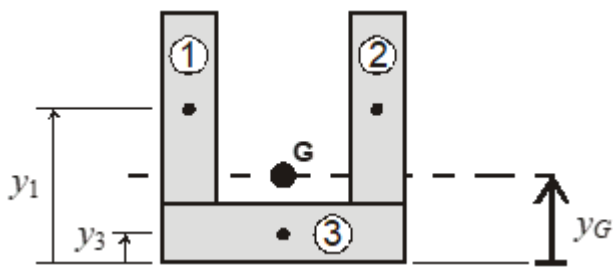
$I_{zz} = \int (x^2 + y^2) dA = \int y^2 dA + \int x^2 dA = I_{xx} + I_{yy}$

Thus the polar second moment I_{zz} is the sum of the second moments about the axes.

10.4.6 Sections that can be subdivided into rectangles

Common beam sections such as I sections, T sections, channels, or angles can be subdivided into rectangles. This makes it easier to find their geometric properties.

First moment of section, centroid position



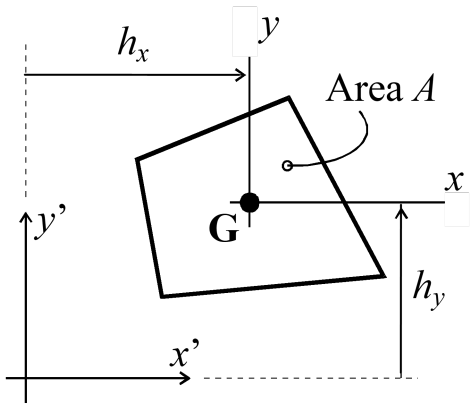
The integral in the equation for the position of the centroid can be replaced by summing over all the rectangles:

$$y_G = \frac{\int y' dA}{A} = \frac{\sum (y'_i \times A_i)}{\sum A_i}$$

Or

$$y_G = \frac{y_1 A_1 + y_2 A_2 + y_3 A_3 + \ldots}{A_1 + A_2 + A_3 + \ldots}$$

Parallel axes theorem



The *parallel axes theorem* is very useful for finding the second moment of area for a section that can be sub-divided into separate standard shapes e.g. rectangles. The theorem allows the second moment of area to be calculated about an axis that does not pass through the centroid of an individual area or a sub-section:

$I_{x'x'}$ = the 2nd moment of area about a general x' axis.

I_{xx} = the $2^n d$ moment of area about the x axis (through the centroid of the particular area A).

The second moment of area about an axis that does not pass through a section's centroid, $I_{x'x'}$, is given by:

$$I_{x'x'} = \int y'^2 dA = \int (y + h_y)^2 dA = \int y^2 dA + 2h_y \int y dA + h_y^2 \int dA$$

$$I_{x'x'} = \int y^2 dA + Ah_y^2$$

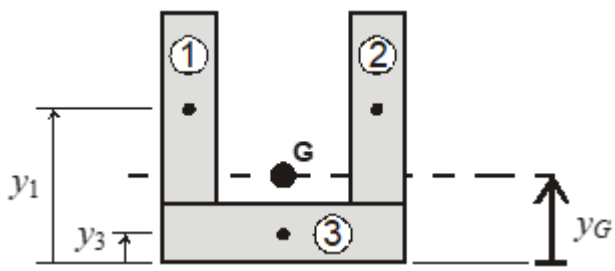
i.e: $I_{x'x'} = I_{xx} + Ah_y^2$ where I_{xx} is the second moment of the individual area about its own centroidal axis.

Thus the second moment of an area about an axis that is displaced by h from its own centroid can be found as the sum of:

- the second moment of area about a parallel axis through its centroid and
- the product of the area and the square of the distance h.

$$\boxed{I_{x'x'} = I_{xx} + Ah^2}$$

Second moment of area of a ”Combined” section

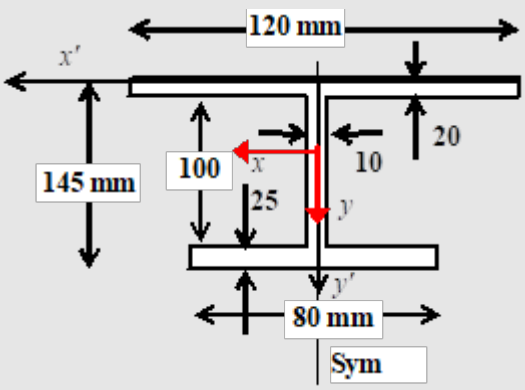


The second moment of the whole section about its (overall) centroidal axis can be found by summing the second moment of each separate area, calculated using the parallel axis theorem:

$$I_{xx} = \sum_i (I_i + A_i h_{iy}^2) = \sum_i \left(\frac{b_i d_i^3}{12} + A_i h_{iy}^2 \right)$$

$$I_{xx} = \left(\frac{b_1 d_1^3}{12} + A_1 (y_1 - y_G)^2 \right) + \left(\frac{b_2 d_2^3}{12} + A_2 (y_2 - y_G)^2 \right) + \ldots$$

10.4.7 Example: I-section beam



Calculate the following geometric properties for the I-beam shown:

- the area;
- the position of the centroid;
- the second moment of areas;
- the radii of gyration; and
- the elastic section moduli.

<u>Area</u>

$A =$

<u>Centroid</u>

From symmetry, $x_G =$

$y_G =$

$=$

$=$

<u>Second moment of area about centroid</u>

Evaluate each rectangular segment for its own I and use the parallel axis theorem to evaluate the I about the section neutral axis.

$$I_{XX} =$$

$$=$$

The section is symmetrical about an axis parallel to the y direction, which simplifies the calculation for I_{yy}.

$$I_{yy} =$$

$$I_{yy} =$$

$$I_{yy} =$$

$$I_{yy} =$$

Radii of gyration

$$r_x =$$

$$r_y =$$

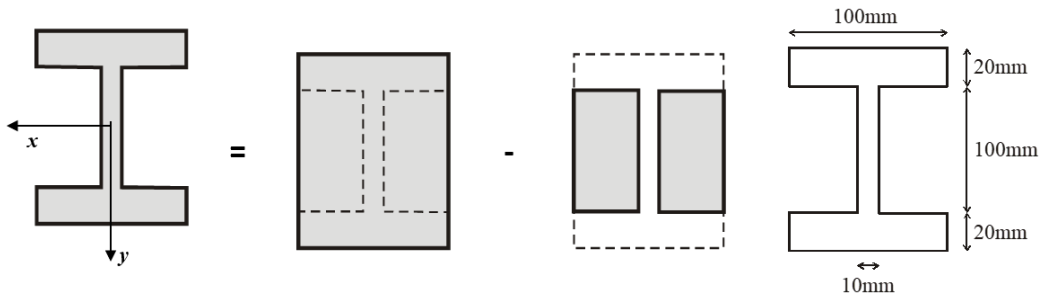
Elastic section modulii (magnitude)

$$Z_{et} =$$

$$Z_{eb} =$$

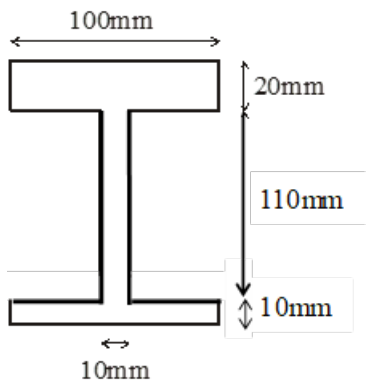
10.5 Short cut scenarios

We can subtract second moments of area. For special section geometry this could make the calculation simpler. One such scenario is a symmetric I section. For the calculation of I_{xx} it can be done by simple subtraction as shown without using the parallel axis theorem; this is made possible because the centres of the "negative" areas also lie on the x-x axis:



$$I_{xx} = \frac{100 \times 140^3}{12} - 2 \times \frac{45x100^3}{12} = 15.4 \times 10^6 \text{ mm}^4$$

NB: Observe the condition for the application of simple subtraction. For example it cannot be used for the calculation of I_{xx} for the section shown below. This is because for this section the centres of the "negative" areas are off the x-x axis and therefore parallel axis theorem would have to be involved in the subtraction.



In such a case there will be no advantage in using the subtraction approach as compared to the usual method of summation of separate parts using the parallel axis theorem.

11 Composite Beams (Elastic Stresses)

It is often desirable to combine two materials in a single member. In this lecture, we investigate how 'composite' beams can be analysed.

11.1 Introduction

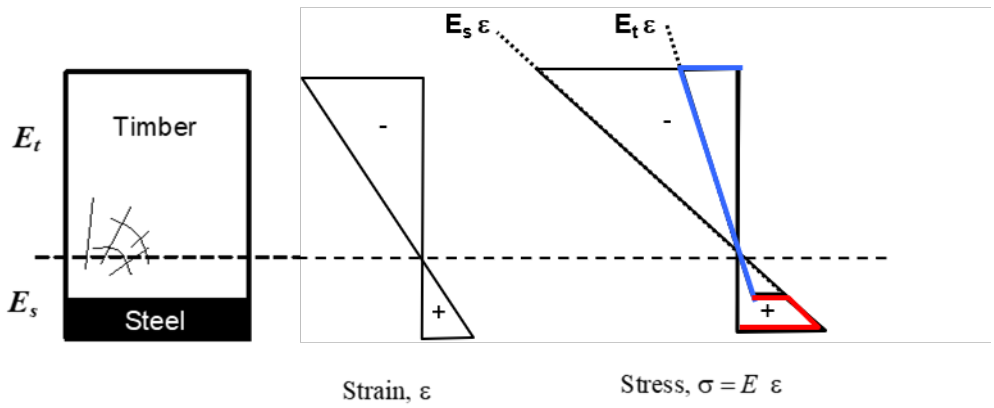
Composite beams are beams that are made of more than one material. The different materials are assumed to be 'fully bonded', so that the assumption that plane sections remain plane remains valid.

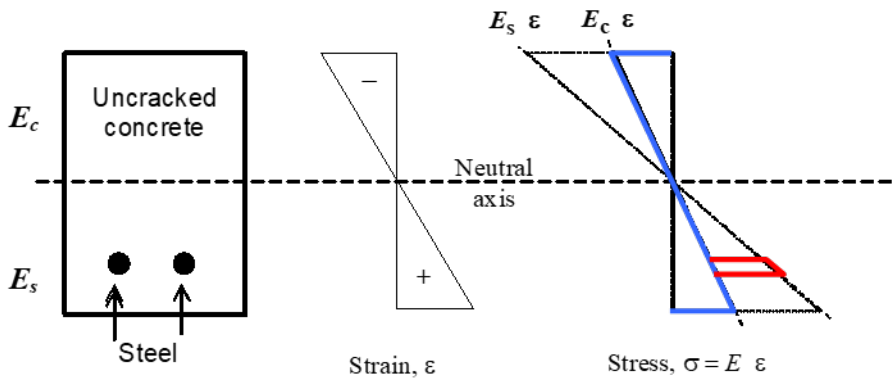
Advantages: Efficient placing of material.

- weaker, lightweight material can be used for core of beam
- stronger, heavier, expensive material forms skins at the edge of the beam
- steel reinforcement can be placed near the tensile face of a concrete beam

The following is an elastic treatment, valid for materials that are not cracked or yielded. (Note that the term 'composite beam' is often used to refer to a composite steel-concrete beam, where cracking makes the assumptions used here somewhat inaccurate).

11.2 Curvatures, strains and stresses





Plane sections remain plane, so the axial strain at distance y from the neutral axis is given by:

$\varepsilon = \kappa y$ where κ is the common curvature for both parts of the section.

Recall $\kappa = -\frac{d^2v}{dz^2} = \frac{M}{EI}$, where EI is the flexural rigidity of the section. For a composite section, the total EI is made up from two (or more) parts with different material properties.

The strain depends only on the curvature of bending, and not on the material being bent. However, the stress at distance y from the neutral axis is given by $\sigma = E\varepsilon$, so this depends on the material.

- In the steel: $\sigma = E_s\varepsilon$

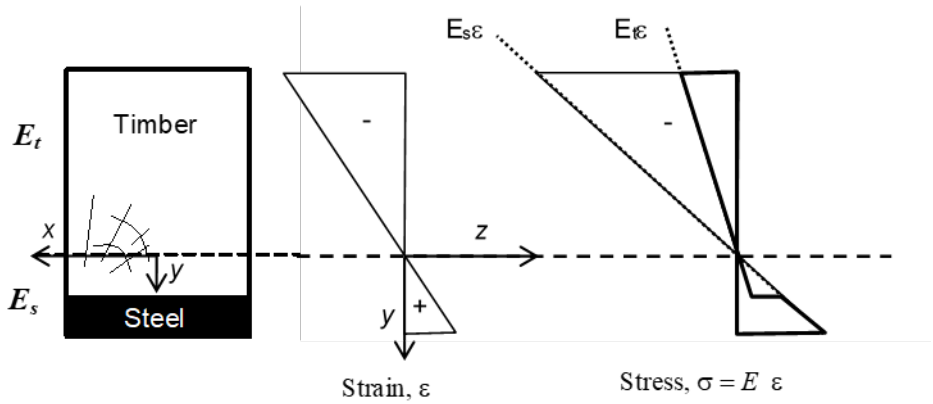
- In the timber or concrete part: $\sigma = E_t\varepsilon$

11.3 Transformed section

11.3.1 Derivation

The difference between the two materials is in the stresses induced by a given curvature.

As $\kappa = \frac{M}{EI}$, or $EI = \frac{M}{\kappa}$, the total flexural rigidity EI can be evaluated by examining the bending moment resultant from the stresses under a given curvature, i.e., a given strain state. This is illustrated in the figures below.



For the example timber-steel composite section, consider Timber as the base (1st) material. To evaluate the change of the stress resultants due to the composite section, we shall check the stress-force state in the 2nd material, i.e. steel here.

The force in a thin sliver of steel (at location y, and depth = dy) within the steel part of the beam is

$dF = b_s dy E_s\varepsilon$ where b_s is the breadth of the steel at this level.

If this sliver were to be replaced by a sliver of timber, whose modulus is E_t, the same force could be developed if the equivalent timber has a width b_{st} satisfying:

$dF = b_s dy E_s\varepsilon = b_{st} dy E_t\varepsilon$

i.e.

$$b_s E_s = b_{st} E_t$$

The width of this equivalent sliver of timber is therefore

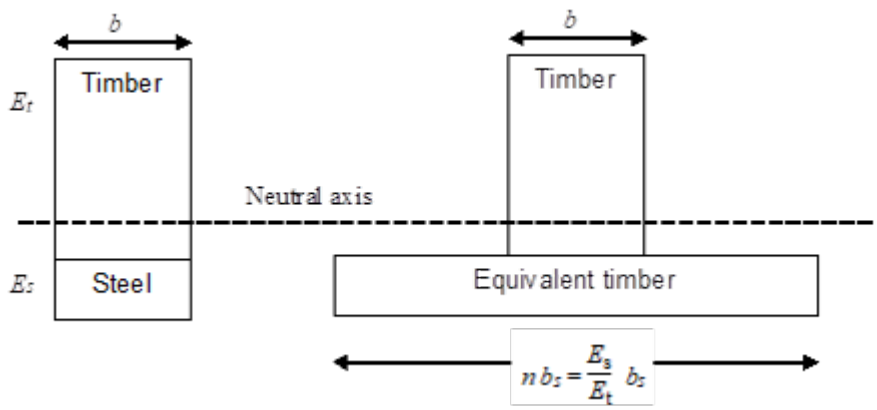
$$b_{st} = b_s \frac{E_s}{E_t}$$

which shows that the equivalent width (of timber) is related to the real width (of steel) by the ratio of the elastic moduli, called the *Modular Ratio*, *n*:

$n = \frac{E_s}{E_t}$, and $b_{st} = b_s n$

As the equivalent sliver of timber produces the same force as the original sliver of steel, at the same location *y*, the effect of the equivalent timber completely represents the effect of the steel in terms of both the force (which affects the neutral axis) and the bending moment for a given curvature (and hence the flexural rigidity).

This means the above composite beam can be transformed into an equivalent beam made of a single (base) material of timber:



The transformed section is equivalent to the original section for bending about a horizontal axis. This section can be substituted for the original to find the flexural rigidity, *EI*, and other properties.

(* The section could be just as easily transformed to an equivalent steel section using the reciprocal of the modular ratio.)

11.3.2 Strains in the transformed material

The effective second moment of area, I_{eff}, of the transformed section can be calculated as normal, in this example case as an inverted T section.

Note that I_{eff} depends upon which material you choose to use for the transformed section. For this reason it is best to quote the flexural rigidity, $(EI)_{total}$, because this correctly represents the original bending stiffness for a composite section and will be the same value irrespective of which material is chosen for the transformed section.

For the timber-steel section transformed into timber,

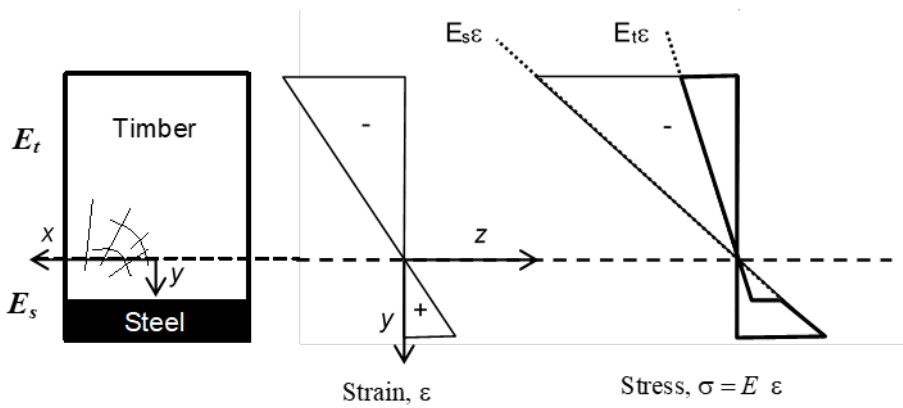
$$(EI)_{total} = E_t I_{eff}$$

The curvature of the beam is given by:

$$\kappa = \frac{M}{E_t I_{eff}}$$

The strain at any position y, including in the transformed material, can be calculated by:

$$\epsilon = \kappa y = \frac{My}{E_t I_{eff}}$$



11.3.3 Stresses in the transformed material

In the *transformed section* (one made entirely from timber), the stress distribution, as if all timber, would be:

$$\sigma_t = E_t \varepsilon = \frac{My}{I_{eff}}$$

Clearly this stress distribution will be valid for the original timber part of the section.

In the steel part, however, the stress distribution (for the same strain distribution from the transformed section) will depends upon the Young's modulus of the material, i.e. steel:

$$\sigma_s = E_s \epsilon = E_s \frac{My}{E_t I_{eff}} = n \frac{My}{I_{eff}}$$

So: the real stress in the steel part is equal to the equivalent timber stress multiplied by the Modular ratio n.

In summary, the stress distribution in the original composite section:

In the timber:

$$\sigma_t = \frac{My}{I_{eff}}$$

...which is the stress in the transformed timber section.

In the steel:

$$\sigma_s = n \frac{My}{I_e ff}$$

. . . which is the (stress in the transformed timber section) $\times$ (the modular ratio of steel to timber).

11.4 Example

A timber/steel composite beam has the cross-section shown. The elastic moduli of the timber and steel are

$E_s = 2 \times 10^5$ MPa and $E_t = 1 \times 10^4$ MPa

1. Find the flexural rigidity of the composite beam
2. If the section is subjected to a sagging moment of 20 kNm, find the maximum stresses in timber and steel, respectively. Plot the stress distribution over the whole section.
3. Find the first failure moment if the limiting stresses are given by σ_s = 120 MPa and σ_t = 8 MPa in either tension or compression

1. Transform to timber, Modular Ratio, n =

   Find the transformed width over the steel part:
   b_{eff} =

Flexural rigidity

Neutral axis: Distance down from top surface:

y_G =

I_{eff} =

=

The flexural rigidity is then found as an equivalent value for timber, since it has all been transformed into timber:

$(EI)_{total}$ =

=

2. The y-coordinate of the top extreme fibre of timber $y_{t,top} =$.

   The y-coordinate of the bottom extreme fibre of timber

   $$y_{t,bot} =$$

   The y-coordinate of the top extreme fibre of the steel

   $$y_{s,top} =$$

   The y-coordinate of the bottom extreme fibre of the steel

   $$y_{s,bot} =$$

   So the maximum stress in timber will take place at the timber top fibre, and in steel it will be at the bottom fibre.

   $$\sigma_{t,max} =$$

   $$\sigma_{s,max} =$$

   To plot the stress distributions, two more stress values are needed; stress at bottom fibre of timber, and stress at the top fibre of steel.

   $$\sigma_{t,bot} =$$

   $$\sigma_{s,top} =$$

   Plot. . .

3. <u>Stresses and maximum allowable moments</u> Based on timber (top fibre):

   $$M_y =$$

   (for checking, timber bottom fibre):

   $$M_y =$$

   Based on steel (bottom fibre):

   $$M_y =$$

$M_y =$

Each of these assessments is a limit on the maximum moment that can be induced in the beam without material failure. Clearly the most restrictive limit will control, so

-
-

11.4.1 Composite section under axial load

From the above discussion on bending of a composite beam, it can be deduced that a composite section subjected to an axial load N can be analysed using the concept of the transformed section as well. Taking the timber-steel composite section as an example, in the simple case with the axial load applied at the centroid position of the transformed section, the axial (direct) stress due to the axial force can be calculated as:

In timber:

$$\sigma_{t,N} = \frac{N}{A_{eff}}$$

In steel:

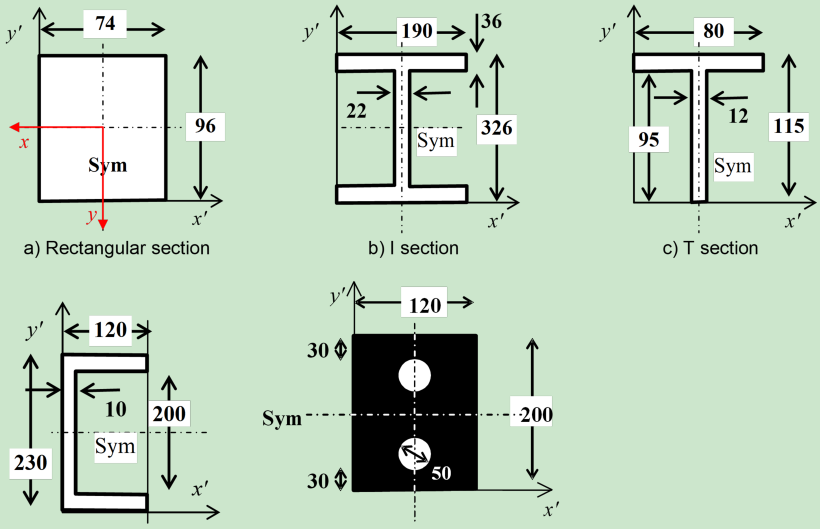
$$\sigma_{s,N} = n\frac{N}{A_{eff}}$$

If the axial load is not applied at the centroid of the transformed section, it will become an eccentric axial loading situation, and the effect will be a combination of the pure axial load (as shown above), and a bending moment equal to the axial force multiplied by the eccentricity (measured from the centroid of the transformed section). The stress due to the bending moment part can be analysed in the same way as the composite beam under pure bending. The total stresses will be the superposition of the two parts of the stresses.

Questions

1. For the following structural sections, find:
   a) the cross-sectional area A;
   b) the location of the centroid;
   c) the second moment of area about each centrodial axis, I_x and I_y;
   d) the elastic section modulus about each centrodial axis, Z_{ex} and Z_{ey} (different values for each extreme fibre where appropriate); and
   e) the radius of gyration about each centrodial axis, r_x and r_y.

   All dimensions are in mm.



a) Rectangular section
b) I section
c) T section
d) C section
e) Extruded rectangular section with two holes

2. A rectangular timber beam section (shown overleaf on left) has Young's modulus $E = 7$ GPa and an ultimate strength of $14\,\mathrm{N/mm^2}$ in both tension and compression.
   a) the beam is required to carry a sagging bending moment about its major axis of $M_x = 34$ kNm. Confirm that the beam will fail under this moment;
   b) the beam is strengthened by bonding a 120×15 mm aluminium plate to the bottom face of the timber section (as overleaf on right), with Young's modulus 70 GPa and a yield strength

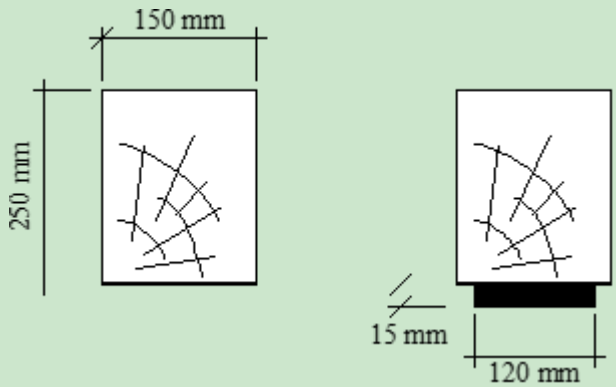
150 N/mm^2. Sketch an equivalent section made entirely from timber;

c) calculate the vertical position of the centroid of the strengthened beam, and its effective second moment of area, I_x;

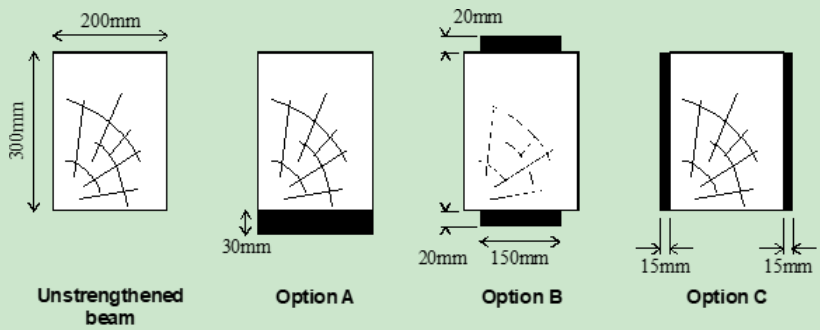
d) calculate the stresses at critical points in the strengthened section;

e) calculate the load factor against collapse, state where failure will first occur; and

f) comment on whether the aluminium strengthening plate is large enough and suggest how the proposed strengthening method could be improved.



3. A timber beam in the floor of a warehouse deflects excessively when it is loaded. Three options are being considered to stiffen the beam, as shown below. In each case, one or more steel plates (Es = 200 kN/mm^2) are bonded to the timber beam (E_t = 10 kN/mm^2). For each option, sketch an equivalent section made entirely from one material. Use this transformed section to calculate the flexural rigidity of the section (EI) for bending about a horizontal axis. Which section would give the greatest reduction in deflection?



Q1: a) 7104 mm^2, 37, 48 mm, 5.46×10^6, 3.24×10^6 mm^4, 1.14×10^5, 8.76×10^4 mm3, 27.7, 21.4 mm.

b) 19 268 mm^2, 95,163 mm, 3.19×10^8 , 4.14×10^7 mm^4, 1.96×10^6, 4.36×10^5 mm^3, 128.7, 46.3 mm.

c) 2740 mm^2, 40, 81.08 mm, 3.11×10^6, 8.67×10^5 mm^4, 3.84×10^4, -9.17×10^4, 2.17×10^4 mm^3, 33.7, 17.8 mm.

d) 5600 mm^2, 40.36, 115 mm, 4.83×10^7 , 8.23×10^6 mm^4, 4.20×10^5, 1.03×10^5, -2.04×10^5 mm^3, 92.9, 38.3 mm.

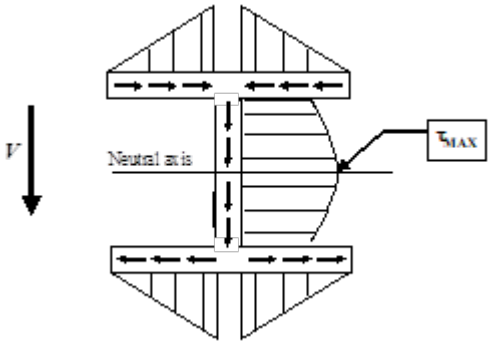
e) 20 073 mm^2, 60, 100 mm, 7.143×10^7, 2.819×10^7 mm^4, 7.143×10^5, 4.698×10^5 mm^3, 59.65, 37.47 mm.

Q2: 195×10^6 mm^4, 21.8 MPa, λ = 0.64, (b) 1200 mm, (c) 168 mm below top, 409×10^6 mm^4, (d) timber: -13.96 MPa, 6.82 MPa, aluminium: 68.2 MPa, 80.6 MPa, (e) λ = 1.00, timber in compression.

Q3: 15.48×10^{12}, 35.26×10^{12}, 18.00×10^{12} Nmm^2

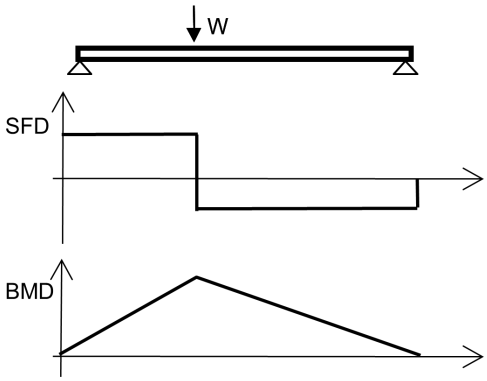
12

Shear Stresses in Beams (1)



We have used Euler beam theory to look at the distribution of direct stress in a beam subjected to bending. Beams also carry shear. In the next two lectures we will determine the distribution of shear stress within a beam.

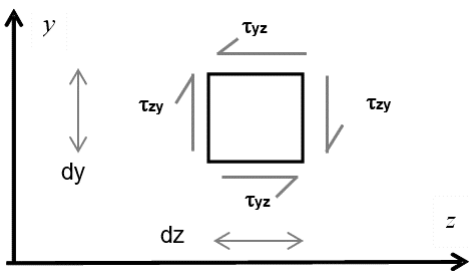
12.1 Introduction



We know from overall statics that a beam under load must support shear forces. Consequently, shear stresses must act on a vertical section cut through the beam.

How are these shear stresses distributed over the cross-section, and what implications does it have for failure of beams with weak shear resistance?

12.2 Complementary Shear Stresses



Consider a small element carrying shear stresses of:

- τ_{zy} on a vertical face.
- $\tau_y z$ on a horizontal face.

Moment by the two couples must balance out:

$$\textit{Vertical shear stress} \times \textit{side length} \times \textit{lever arm} = \textit{Horizontal shear stress} \times \textit{side length} \times \textit{lever arm}$$

$$\tau_{zy} dydz = \tau_{yz} dzdy$$

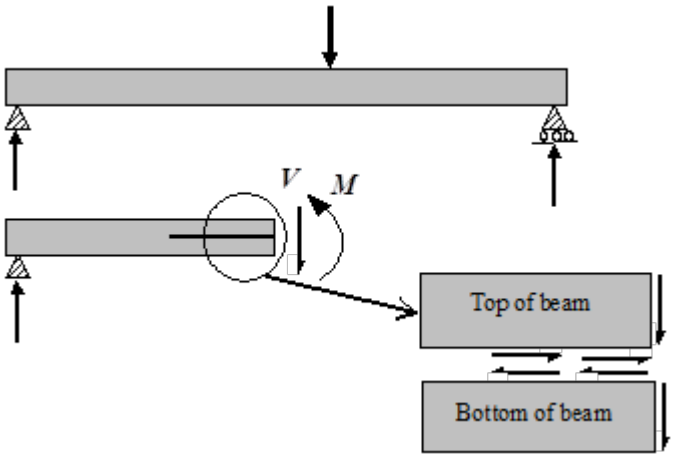
So,

$$\tau_{zy} = \tau yz$$

These are called *Complementary Shears*. A vertical shear cannot exist without an equal horizontal shear also being present.

- the shear stresses in perpendicular directions have equal magnitude
- these shear stresses act in the opposite direction (anticlockwise / clockwise)

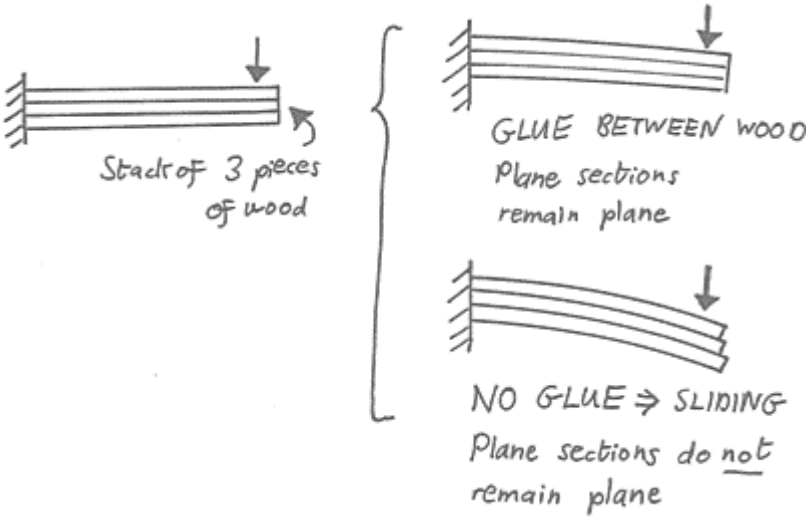
Clockwise shears are conventionally positive; anticlockwise shears are conventionally negative.



As a result, we cannot have shear stresses on a vertical cross-section face without inducing complementary shear stresses on a horizontal plane. To determine the distribution of shear stress on a vertical section of a beam, we will use complementary shear, and start by finding the shear stresses on a horizontal section along the beam.

12.2.1 Evidence of horizontal shear stress

The presence of horizontal shear stress can be illustrated by considering a laminated timber beam, made by gluing several layers of material together.



If a stack of three pieces of wood is used as a cantilever beam the four pieces of wood will bend separately. When loaded the end of the stack of wood will distort, because the axial load in each piece of wood must be zero, so it does not change length.

If the same four pieces of wood are glued together, horizontal shear stress can be transferred across the bonded surfaces. The four pieces of wood will act as a single beam: the top piece of wood will be in compression, the bottom piece of wood will be in tension, and the beam will be much stiffer than without the glue. This is not possible without a shear connection across a horizontal joint.

12.3 Derivation of the shear stress distribution

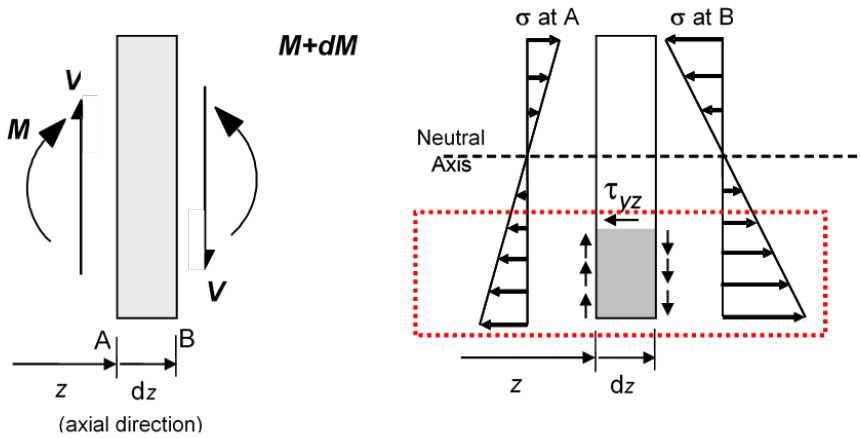
12.3.1 Variation in bending stresses

From previous lectures we understand that along the length of the beam, $V = \frac{dM}{dz}$.

i.e. if a beam carries shear force, the bending moment will change along its length (with the slope of the BMD equal to the shear force).

The change in bending moment leads to a change in *bending stresses* along the beam. If we consider a short length dz of the beam (between

section A and section B), the bending moment changes from M to $M + dM$. The bending stress distribution due to bending increases to match.



Now consider a free body isolated from the bottom of the beam. The direct stress at section A is smaller than the direct stress acting on section B. Hence, equilibrium requires shear stress to occur across the horizontal cut.

We can thus find the shear stress distribution on a vertical section through the beam by:

Finding the variation in bending stress (σ) from the change in bending moment (M).

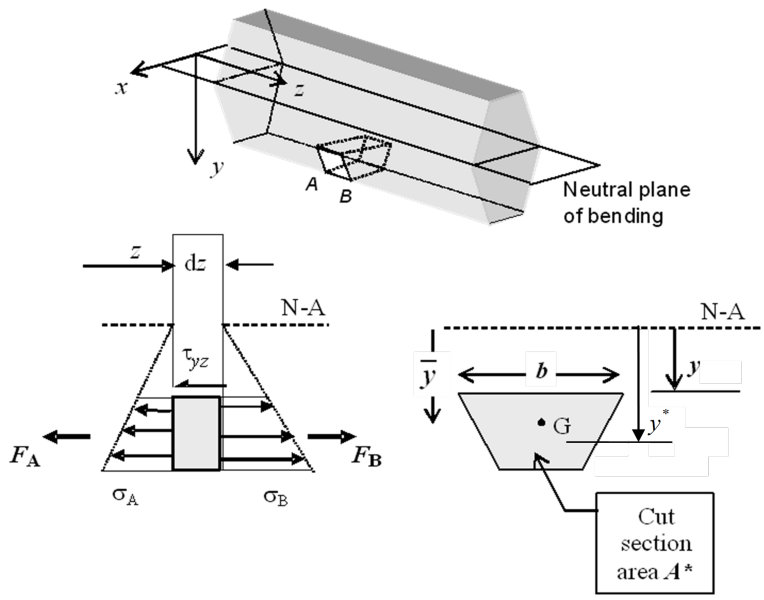
$\downarrow$

Using horizontal equilibrium to find the horizontal shear stress (τ_{yz})

$\downarrow$

Using complementary shear to find the vertical shear stress (τ_{zy})

12.3.2 Horizontal equilibrium to find the horizontal shear stress (τ_{yz})



Consider a beam of a generic cross-section. The equilibrium of a small section cut from the bottom of the beam is:

$$F_B - F_A = \tau_{yz} b dz$$

i.e.:

Total force due to direct stresses σ on B - Total force due to direct stresses σ on A = Horizontal shear force at the top of the free body

The force FA due to direct stresses at section A is:

$$F_A = \int_{Area,A*} \sigma_A dA$$

The force FB due to direct stresses at section B is:

$$F_B = \int_{Area,A*} \sigma_B dA$$

So, equilibrium requires that:

$$\int_{A^*} \sigma_B dA - \int_{A^*} \sigma_A dA = \tau_{yz} b dz$$

$$\int_{A^*} (\sigma_B - \sigma_A) dA = \tau_{yz} b dz$$

From the bending stress formula, and denote the y-coordinate within the cut-area A^* as y^*, we have:

$$\sigma_A = \frac{My^*}{I}, \sigma_B = \frac{(M + dM)y^*}{I}$$

Substituting,

$$\int_{A^*} \frac{dM}{I} y^* dA = \tau_{yz} b dz$$

$$\frac{dM}{dz} \int_{A^*} y^* dA = \tau_{yz} b I$$

$V = \frac{dM}{dz}$, and $\int_{A^*} y^* dA$ is the 1st moment of area A^* about the N.A., so:

$$V(A^* \bar{y}) = \tau_{yz} b I$$

where: A^* is the area in the cross-section between the horizontal cut to the bottom edge, and $\bar{y}$ is the distance from the Neutral Axis to the centroid of this area A^*.

Rearranging gives:

$$\tau_{yz} = V \frac{A^* \bar{y}}{Ib} \quad \text{(a)}$$

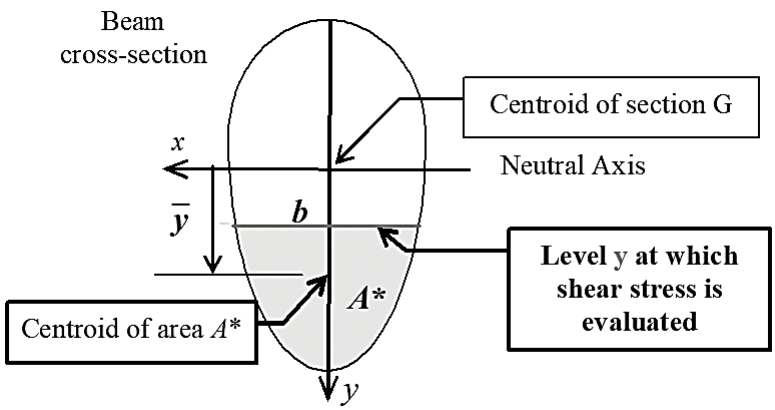
12.3.3 Complementary shear to find the vertical shear stress (τ_{zy})

Finally (by complementary shear), the vertical stress must be equal to the horizontal shear stress ($\tau_{zy} = \tau_{yz}$) at the level of the horizontal cut, i.e., location of "y" from the N.A., giving:

$$\tau_{zy} = V \frac{A^* \bar{y}}{Ib} \quad \text{(b)}$$

Thus, the shear at any level y in the beam can be found by:

- making a cut at y through the beam cross section;
- evaluating the area from the cut, A^* ;
- finding the distance $\bar{y}$, between the centroid of the area A^* and the centroid of the beam cross-section. NB: $\bar{y}$ here is the absolute value of the distance defined above.



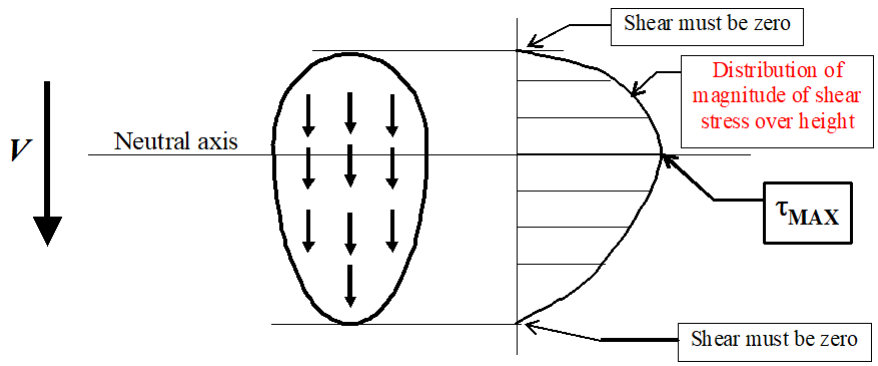
Further remarks about the shear stress calculation

- The formula (b) calculates the shear stress at a distance of y from the centroid of the cross section. For locations which have the same y value, i.e. on the horizontal line (level) shown in the above figure, the shear stress is the same.
- y does not explicitly appear in the formula; it is involved through the area A^* and its centroidal distance .
- The shear force V and 2nd moment of area I are constants for the whole section.
- For sections with variable width, b is a variable and is the width measured at level y.
- The formula calculates the absolute value of the shear stress; the direction of the shear stress on a solid cross section follows the direction of the shear force.

12.4 Maximum shear stress

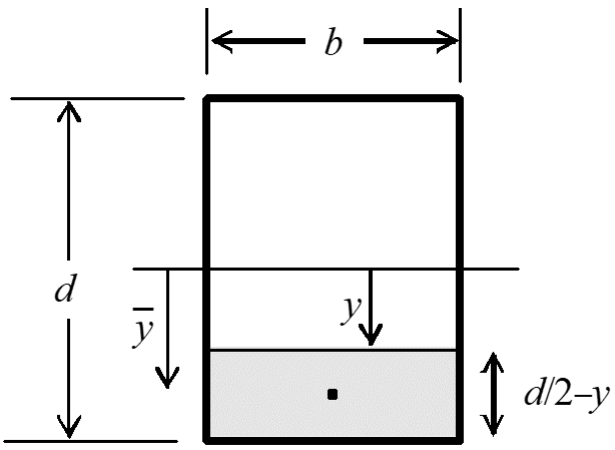
The first moment of the "cut-off" area, $A^*\bar{y}$, always reaches its maximum value when the "cut-off" line is at the neutral axis, so the shear stress in the cross-section is always maximum at the neutral axis.

Thus, if it is only the maximum shear that is being sought, it is enough to evaluate $\tau_{zy} = V\frac{A^*\bar{y}}{Ib}$ at the neutral axis.



12.5 Shear stresses in rectangular sections

The simplest example of a shear stress distribution is in a rectangular cross-section.



The distribution of shear stress in the cross-section is given by:

$$\tau_{zy} = V\frac{A^*\bar{y}}{Ib}$$

1. Let $b = 100\,\text{mm}$, $d = 200\,\text{mm}$, $V = 100\,\text{kN}$. Find the shear stress at:
   - $y = 50\,\text{mm}$
   - $y = 0$ (Neutral axis)
2. Obtain a general expression of the shear stress as a function of y (and b, d, V)

1. i) Make a cut at $y = 50\,\text{mm}$:

$$A^* = 5000\,\text{mm}^2$$
$$b = 100\,\text{mm}$$
$$\tau = 5.625\,\text{MPa}$$
$$\bar{y} = 75\,\text{mm}$$
$$I = 6.67 \times 10^7\,\text{mm}^4$$

   ii) Similarly make a cut at $y = 0$:

$$\ldots$$
$$\ldots$$
$$\tau = 7.5\,\text{MPa}$$

2. For a general expression, make a cut at height y, the items in the shear stress equation can be identified:

$$A^* = b\left(\frac{d}{2} - y\right) = \frac{bd}{2}\left(1 - 2\frac{y}{d}\right)$$
$$\bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = \frac{d}{4}\left(1 + 2\frac{y}{d}\right)$$
$$I = \frac{bd^3}{12}$$
$$b = b$$

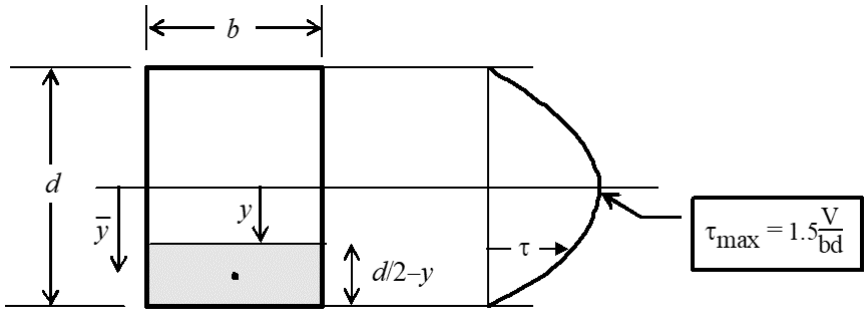
   The shear stress at any level is therefore related to the shear force V at this section as:

$$\tau_{zy} = V\frac{A^*\bar{y}}{Ib} = V\frac{bd}{2}\left(1 - 2\frac{y}{d}\right)\frac{d}{4}\left(1 + 2\frac{y}{d}\right)\frac{12}{bd^3 \times b}$$
$$= \frac{V}{bd}\frac{3}{2}\left(1 - 4\left(\frac{y}{d}^2\right)\right)$$

   in which A is the area of the cross-section.

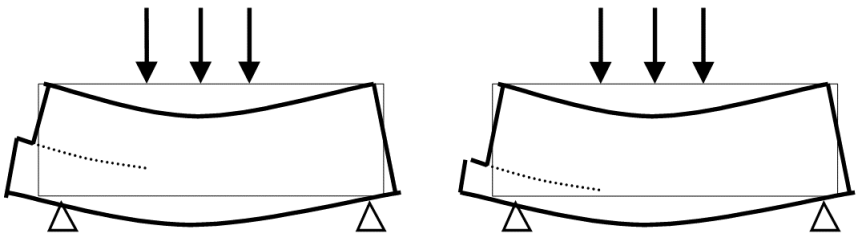
NB:

- V/bd is the mean shear stress ($\tau_{ave} = V/A$)
- The maximum shear stress takes place at midheight (neutral axis $y = 0$), and for a rectangular section $\tau_{max} = 1.5\tau_{ave}$
- Distribution of the shear stress over the section is a parabolic function with vertical position (y).
- At $y = \pm d/2$, the shear is zero.



12.5.1 Shear cracking in wooden beams

This shear distribution has considerable importance for wooden (timber) beams, because the shear strength for transfer of shear stresses between the fibres of the wood is often low. The horizontal (complementary) shears are equal to the vertical shears and must be transferred. This leads to shear failures of short span timber beams in a shear mode.



The maximum shear occurs at the neutral axis, so one might expect that this is where shear cracks would occur. However, because the shear is relatively uniform in the central zone of the beam, but the shear strength of different parts of a piece of wood can vary quite substantially, the crack does not always coincide precisely with the maximum stress location.

* Next time you are in an old pub with old wooden beams, look for shear cracks extending from one end towards the middle of the beam. They are quite common. This type of shear crack would “unglue” the section above and below the crack, causing an increase of the stresses and an increase in the deflection.

13

Shear stresses in beams (2)

In the previous chapter we derived a formula for the shear stress in a general beam section carrying shear force, and applied the formula to a rectangular section. In this lecture we will look at how to apply the formula to thin walled I- and T-sections, as well as box sections.

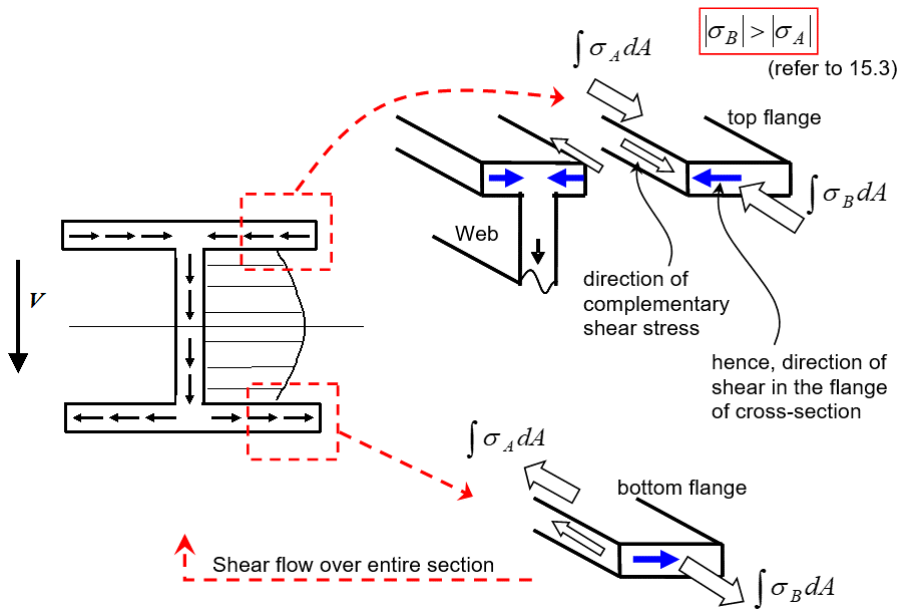
13.1 Shear stresses in I - T sections

A class of members in which shear stresses frequently need to be found is thin walled I and T sections. The shear distribution has some features in common with rectangular sections but there are also important differences.

The derivation of the shear stress formula follows the same idea as what has been applied for an arbitrary solid section, i.e., draw a cutting line on the section through the location where the shear stress is to be evaluated, establish the complementary shear stress from the equilibrium with the direct stresses. The stress on the cross-section at the cutting location is the same as that complementary shear stress. This leads to the same shear stress formula:

$$\tau_{zy} = V\frac{A^*\bar{y}}{Ib}$$

The shear stress in the web follows the direction of the shear force. However, the shear stress in the thin flange goes in the horizontal direction. This can be understood by cutting off a portion of the flange, as illustrated below.

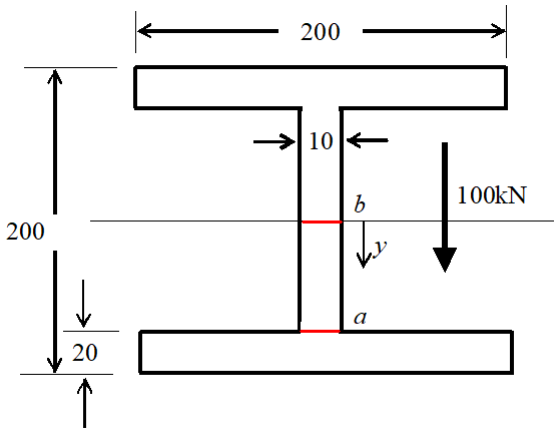


Hence, the direction of the shear stress throughout the section forms a "flow":

- Each side of the flange may be examined separately, and the shear flows come inwards (top flange) or outwards (bottom flange) with respect to the flange centre.
- The forces in the flange contribute nothing to carrying the overall vertical shear force V acting on the cross-section. Only the web carries vertical shear stresses; hence the shear force V is carried by the web alone

The detailed distribution of the magnitude of the shear stress will be discussed in the sections that follow.

13.1.1 Shear stresses in web



Calculate the shear stress at i) $y = 80$ (location a) , and ii) $y = 0$ (neutral axis, b). Let $V = 100\,\text{kN}$.

$$\tau = \tau_{zy} = V\frac{\sum(A^*\bar{y})}{Ib}$$

$$V = 100\,\text{kN}$$

$$I = 68.48 \times 10^6\,\text{mm}^4$$

(calculation of I here may be simplified by considering double symmetric section).

i) At "a": make a horizontal "cut", so the area below a (i.e. the entire bottom flange) becomes A^*,

$$A^* = 4000\,\text{mm}^2$$

$$\bar{y} = 90\,\text{mm}$$

$$b = 10\,\text{mm}$$

$$\tau_a = V\frac{A^*\bar{y}}{Ib} = 52.6\,\text{N/mm}^2 = 52.6\,\text{MPa}$$

ii) At "b": make a horizontal cut at b; the whole area below b will be A^*, for which the 1st moment of area can be calculated by dividing A^* into two rectangles:

$$A^*_I = 4000\,\text{mm}^2$$

$$\bar{y}_I = 90\,\text{mm}$$

$$A^*_{II} = 800\,\text{mm}^2$$

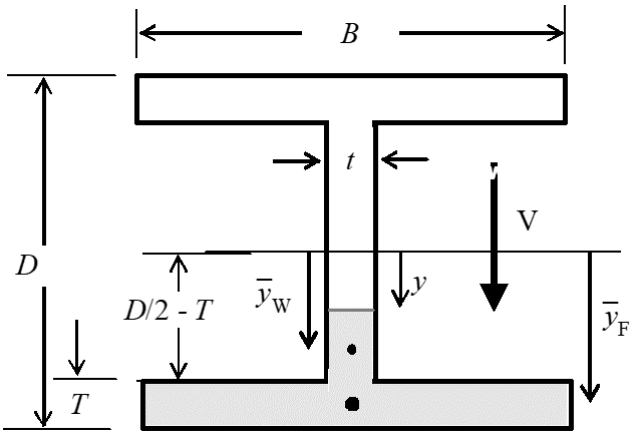
$$\bar{y}_{II} = 40\,\text{mm}$$

$$\sum(A^*\bar{y}) = 392\,000\,\text{mm}^3$$

$$b = 10\,\text{mm}$$

$$\tau_b = V\frac{\sum(A^*\bar{y})}{Ib} = 57.2\,\text{N/mm}^2 = 57.2\,\text{MPa}$$

A general expression can be obtained for the shear stress within the web as a function of y (and B, D, T, t)



Consider a cut at a height y, within the web of an I-beam.

Note that where A^* can be split into separate rectangular areas, we can write $\tau = V\frac{\sum(A^*\bar{y})}{Ib}$.

For the flange: $A^*_F = BT$, $\bar{y}_F = \frac{D-T}{2}$

For the web:

$$A^*_W = t\left(\frac{D}{2} - T - y\right)$$

$$\bar{y}_W = y + \frac{1}{2}\left(\frac{D}{2} - T - y\right)$$

Hence,

$$\bar{y}_W = \frac{1}{4}(D - 2T + 2y)$$

$$I = I$$

$b = t$ at the cut in the web where the shear stress is to be evaluated.

The shear stress at height y in the web is therefore related to the shear force V at this section as:

$$\tau = V\frac{\sum(A^*\bar{y})}{Ib}$$

$$\tau = \frac{V}{It}\left(BT\frac{(D-T)}{2} + t\left(\frac{D}{2} - T - y\right)\left(\frac{D-2T+2y}{4}\right)\right)$$

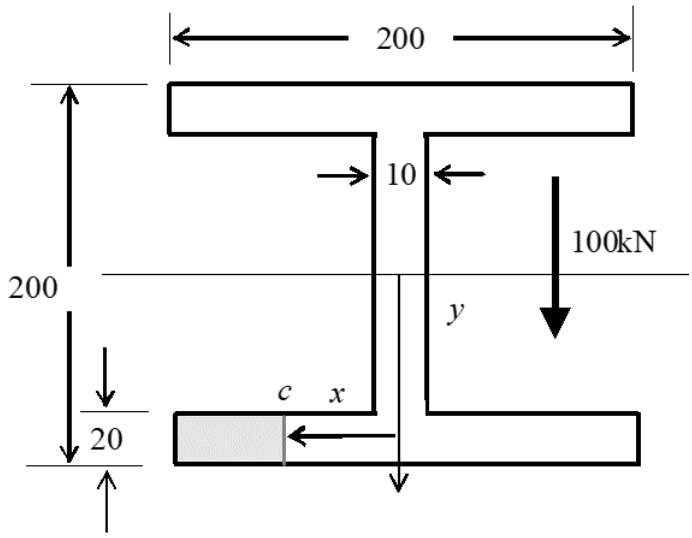
$$\tau = \frac{V}{It}\left(\frac{BT(D-T)}{2} + \frac{t}{8}\left[(D-2T)^2 - 4y^2\right]\right)$$

13.1.2 Shear stresses in the flanges

For the shear stresses in the (thin) flanges of the I-section, we need to take a cut vertically at the position where shear stress is to be evaluated.

The shear stress is still given by: $\tau = V\frac{A^*\bar{y}}{Ib}$

As an example, let's find the shear stress in the flange at $x = 50$ (location "c"):



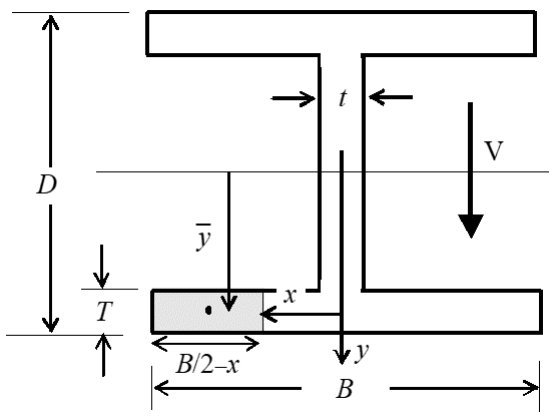
$$A^* = 1000\,\text{mm}^2$$

$$\bar{y} = 90\,\text{mm}$$

$$b = 20\,\text{mm}$$

$$\tau_c = V\frac{A^*\bar{y}}{Ib} = 6.57\,\text{N/mm}^2 = 6.57\,\text{MPa}$$

Also, a general expression can be obtained for the shear stress within the flange as a function of x (and B, D, T, t).



For a cut at position x, the items in this equation can be simply identified:

$$A^* = T\left(\frac{B}{2} - x\right) = \frac{BT}{2}\left(1 - 2\frac{x}{b}\right)$$

$$\bar{y} = \frac{1}{2}D - \frac{1}{2}T = \frac{D-T}{2}$$

$$I = I$$

(the second moment of area for the whole section)

$$b = T$$

(because the cut is through the flange)

The shear stress on a vertical cut is therefore related to the vertical shear force V at this section as:

$$\tau = V\frac{A^*\bar{y}}{Ib}$$

$$\tau = V\frac{BT}{2}\left(1 - 2\frac{x}{b}\right)\left(\frac{D-T}{2}\right)\frac{1}{IT}$$

$$\tau = V\frac{B(D-T)}{4I}(1 - 2\frac{x}{B})$$

Thus the shear stress in the flange varies linearly with x.

13.1.3 Shear stress distribution

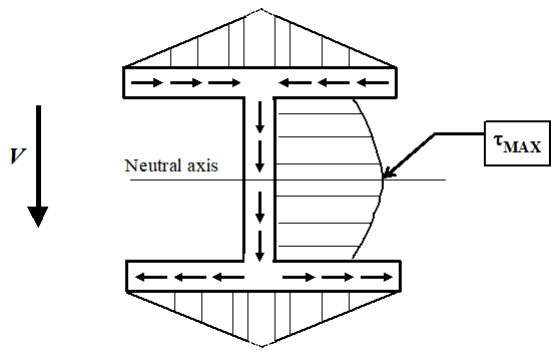
According to the general expressions, in the web the shear stress distribution is quadratic with y:

$$\tau = \frac{V}{It}\left(\frac{BT(D-T)}{2} + \frac{t}{8}\left[(D-2T)^2 - 4y^2\right]\right)$$

The direction of the shear stress in the web is vertical (the resultant is "V"). Thus together with the magnitude shown above, we can sketch the shear stress distribution for the web as shown in the figure below.

In the flanges, the shear stress is in the horizontal direction as discussed earlier. Its magnitude is zero at the tip of the flanges, and increases linearly with distance from the flange tip, to a maximum at the flange / web junction:

$$\tau = V\frac{B(D-T)}{4I}\left(1 - 2\frac{x}{B}\right)$$



13.1.4 Maximum shear stress

The maximum shear stress occurs at the neutral axis ($y = 0$). For the generic I section:

$$\tau_{MAX} = \frac{V}{2It}\left(BT(D-T) + \frac{t(D-2T)^2}{4}\right)$$

It is easiest for practical usage to express the maximum shear in terms of the average shear on the web only. For this purpose, the "web" is generally assumed to run from the midpoint of one flange/web intersection to the

other, giving it a height of $(D - T)$ and an area of $A_W = t(D - T)$. The mean shear stress in the web is then:

$$\tau_{mean} = \frac{V}{A_W} = \frac{V}{t(D - T)}$$

The maximum shear stress, occurring at the NA is given by:

$$\tau_{MAX} = \alpha\tau_{mean} = \alpha\frac{V}{A_W} = \alpha\frac{V}{t(D - t}$$

in which $\alpha = \frac{(D-T)^2}{2I}\left(BT + \frac{t}{4}\frac{(D-2T)^2}{D-T}\right)$ which can be tabulated for sections of given dimensions.

For most sections, α is only slightly bigger than 1.0, typically around 1.2, so the average shear stress in the web is not very different from the maximum value.

It is often sufficient to use this constant, α, to determine the maximum shear stress, τ_{MAX}, stress from the average shear stress, V/A_w:

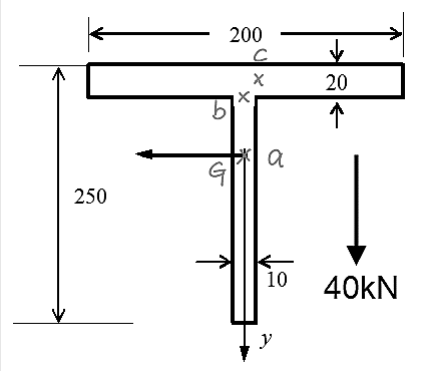
$$\tau_{MAX} = \alpha\frac{V}{A_W}$$

where A_W is the area of the web, and α is a coefficient.

- For a rectangular section, $A_W = A$, $\alpha = 1.5$.
- For an I-section beam, α will be around 1.1, although this varies a bit according to the proportions of the section.

13.1.5 Further example

Find the shear stress at points "*a*", "*b*" and "*c*" on the T section shown below.



1) Find location of centroid, and calculate I

$y_G =$

$I =$

2) Shear stress at "*a*" (centroid)

Use A^* below the centroid:

Use A^* above the centroid:

– Note that the direction of the shear will be identified according to the shear flow, therefore when calculating we only need to get the absolute values –

Hence,

$\tau_a =$

3) Shear stress at "*b*"

$b =$

$A^* \bar{y} =$

$\tau_b =$

4) Shear stress at "*c*" Make a vertical cut through "*c*", $b = 20\,\text{mm}$

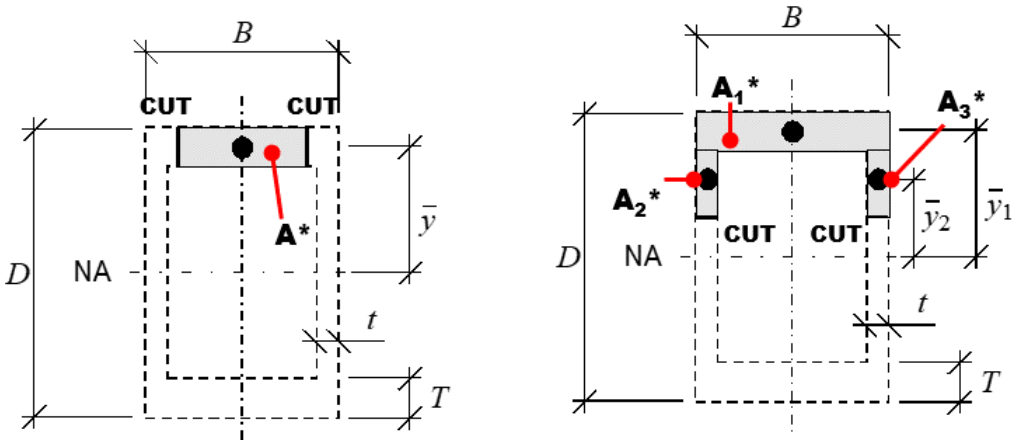
$$A^* \bar{y} =$$

$$\tau_c =$$

The shear stress in the flange of the cross section "flows" horizontally inwards from the tip to the centre.

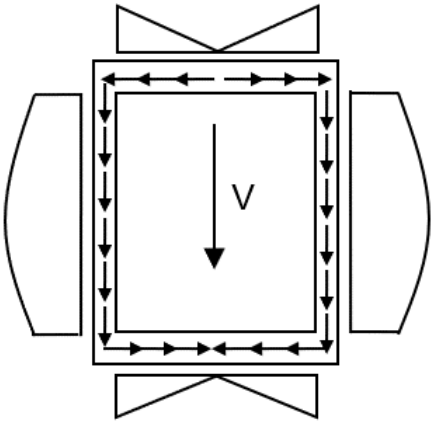
13.2 Shear stresses in box sections

The above method can be used to find the shear stress in other sections. For example, we can use it to find the shear stress distribution in the flanges and webs of a box section.



Note that in this case we need to apply symmetry, since we need two cuts to take out a "free body". Consequently, the parameter "*b*" in the shear stress formula will be equal to $2T$ (for flange) or $2t$ (for web) when calculating the shear stress at the symmetrically cut locations.

With the above understanding and the concept illustrated in the previous sections, it can be easily established that the shear flow and shear stress distribution in a box section will look like:



13.3 Shear stresses in a composite section

For a composite beam, such as the timber-steel beam discussed in the composites chapter, failure often occurs at the "bonded" interface between the two materials, and this is determined by the shear stress at the interface on the cross-section as compared to the bond strength (keep in mind the complementary shear phenomenon).

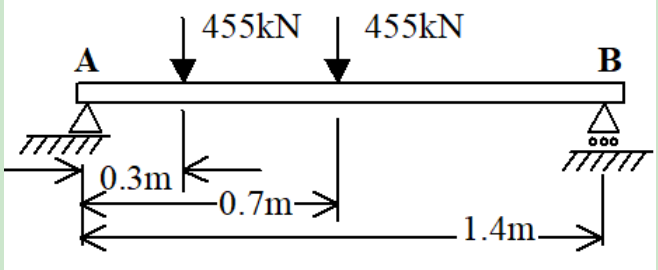
The shear stress at the bonded interface can be analysed using the concept of transformed cross section. Recall the example beam in the composites chapter. If the section is subjected to a vertical shear force of 50 kN, the shear stress at the timber-steel interface can be found as:

$$\tau_{t-s} = V\frac{A^*\bar{y}}{Ib} = 50000\frac{(1500 \times 10) \times 116}{(608 \times 10^6) \times 75} = 1.91\,\text{MPa}$$

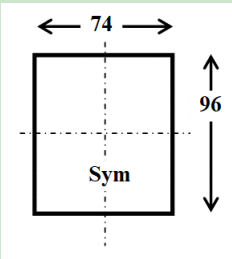
(Note the value for "*b*" here is 75 mm which is the width of the actual bonded interface).

Questions

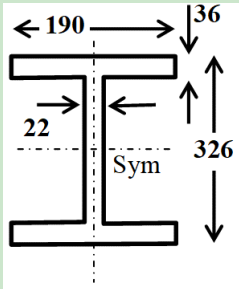
1. In the beam below, shear controls the design because the beam is very short. Draw the shear force diagram and identify the maximum value of shear V. What direction is the maximum shear (shearing clockwise or anticlockwise)?



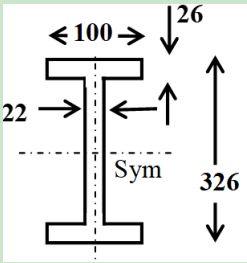
Five different cross-sections, bent about the major axis, are being proposed for the beam design:



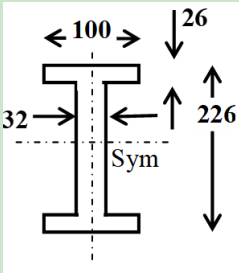
(a) Rectangular section



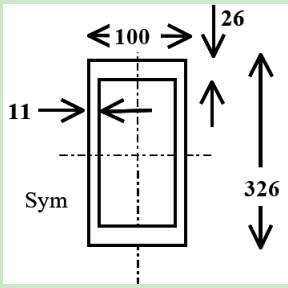
(b) I section No 1



(c) I section No 2



(d) I section No 3



(e) Rectangular hollow section

(All dimension are in mm)

For each cross-section subjected to the maximum shear, V:

- Find the area (A) and second moment of area (I).
- Determine the shear stress at key points in the cross-section by making suitable cuts in the cross-section (for the rectangular hollow section, use symmetry).
- Plot the distribution of shear stress in the section.
- Find the maximum value of shear stress.
- Determine the load factor against first yield in shear. The uniaxial yield stress of the sections is $\sigma_y = 250\,\text{MPa}$, and the yield stress in shear can be found from $\tau_y = (\sigma_y/\sqrt{3})$. (This is from von Mises criterion, which you will meet in a later course).
- Using the web area alone, find the average shear stress in the web and compare it with the peak value found accurately above.

Finally decide which section produces the lowest shear stress or the highest load factor against failure and comment on why you think this is.

585kN, shearing clockwise.

a): $A = 7104\,\text{mm}^2$, 123.5 MPa, 1.169, 82.3 MPa:

b): $A = 19\,268\,\text{mm}^2$, $I_x = 3.191 \times 10^8\,\text{mm}^4$, 25.3 MPa, 82.6 MPa, 97.4 MPa, 1.482, $A_{w,eff} = 6380\,\text{mm}^2$, 91.7 MPa

c): $A = 11\,228\,\text{mm}^2$, $I_x = 1.550 \times 10^8\,\text{mm}^4$, 28.3 MPa, 66.9 MPa, 102.3 MPa, 1.411, $A_{w,eff} = 6600\,\text{mm}^2$, 88.6 MPa

d): $A = 10\,768\,\text{mm}^2$, $I_x = 6.634 \times 10^7\,\text{mm}^4$, 44.1 MPa, 71.6 MPa, 105.0 MPa, 1.374, $A_{w,eff} = 6400\,\text{mm}^2$, 91.4 MPa

e): $A = 11\,228\,\text{mm}^2$, $I_x = 1.550 \times 10^8\,\text{mm}^4$, 22.1 MPa, 66.9 MPa, 102.3 MPa, 1.411, $A_{w,eff} = 6600\,\text{mm}^2$, 88.6 MPa

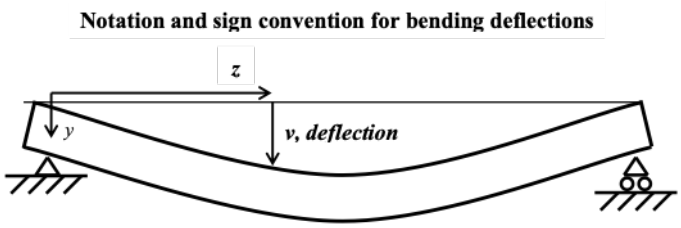
14 Beam deflection

In the previous chapter, we started from a representation of the deformed shape of a beam, in terms of its curvature, to define the stress and strain distributions within it. While curvature was a good representation of the deformed shape for that purpose, what we will often want to know for engineering design is how far each part of the beam has moved from its original position, its deflection. In this chapter we will relate curvature to deflection and show how to calculate the deformed shape of a beam.

14.1 Relating curvature and deflection of a beam

We define the beam coordinate system by (y, z) with z along its axis with the cross-section in the $x - y$ plane.

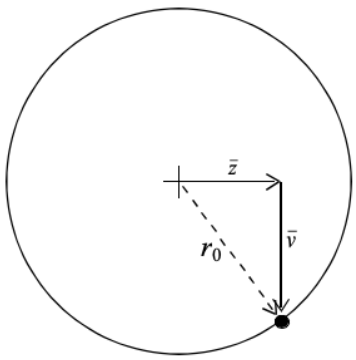
The beam deflection is in the y direction and is defined as v, and the deflected shape is defined as a function of position: $v = v(z)$. (We use a consistent notation for displacements: (u, v, w) in the (x, y, z) directions respectively).



We have shown that the strain in the beam (ϵ_z) is related to the radius of curvature ($r0$) by $\epsilon_z = y/r_0$. The radius of curvature ($r0$) is related to the deflected shape $v(z)$ of the beam, which can be established from geometry. By this relationship we can subsequently derive the deflection.

We saw that the curvature of the beam can be represented locally by a circular arc. The relationship between radius r_0, and the (increments of) horizontal and vertical positions can be established using an analogy to a circle:

$$\bar{z}^2 + \bar{v}^2 = r + 0^2$$



Differentiate once w.r.t $\bar{z}$:

$$2\bar{z} + 2\bar{v}\frac{d\bar{v}}{d\bar{z}} = 0$$

so,

$$\frac{d\bar{v}}{d\bar{z}} = -\frac{\bar{z}}{\bar{v}}$$

Differentiate a second time w.r.t $\bar{z}$ (use the quotient rule):

$$\frac{d^2\bar{v}}{d\bar{z}^2} = -\frac{\bar{v} \times 1 - \bar{z} \times \frac{d\bar{v}}{d\bar{z}}}{\bar{v}^2} = -\frac{\bar{v} - \frac{\bar{z}^2}{\bar{v}}}{\bar{v}^2} = -\frac{\bar{v}^2 + \bar{z}^2}{\bar{v}^3}$$

$$\frac{d^2\bar{v}}{d\bar{z}^2} = -\frac{\left[\bar{v}^2 + \bar{z}^2\right]^{\frac{3}{2}}}{\bar{v}^3\left[\bar{v}^2 + \bar{z}^2\right]^{\frac{1}{2}}} = -\frac{\left[1 + \frac{\bar{z}^2}{\bar{v}^2}\right]^{\frac{3}{2}}}{\left[\bar{v}^2 + \bar{z}^2\right]^{\frac{1}{2}}} = -\frac{\left[1 + \left(\frac{d\bar{v}}{d\bar{z}}\right)^2\right]^{\frac{3}{2}}}{r_0}$$

The actual z and v in the beam are not exactly $\bar{z}$ and $\bar{v}$ but their increments are the same, i.e., $dz = d\bar{z}$, $dv = d\bar{v}$, therefore the above differential equation is also applicable to z and v.

We have defined the curvature, κ, of the beam as the inverse of the radius of curvature, r_0, so rearranging gives:

$$\kappa = \frac{1}{r_0} = -\frac{\frac{d^2\bar{v}}{d\bar{z}^2}}{\left[1 + \left(\frac{d\bar{v}}{d\bar{z}}\right)^2\right]^{\frac{3}{2}}}$$

This is the relationship between the beam curvature and the deflected shape, v.

We can use the small deformations assumption mentioned previously to simplify our expression for the curvature. If the slope of the beam is small, then the denominator is close to 1. e.g.: a slope of $6° = 0.1$ radians gives:

$$\left[1 + \left(\frac{d\bar{v}}{d\bar{z}}\right)^2\right]^{\frac{3}{2}} = 1.015$$

Hence the curvature, radius of curvature and deflected shape are related by:

$$\kappa = \frac{1}{r_0} = -\frac{d^2\bar{v}}{d\bar{z}^2}$$

We already know

$$EI\kappa = M$$

Hence,

$$-EI\frac{d^2v}{dz^2} = M$$

(Note that the minus sign here is because of the definition of positive y-axis (hence v) pointing downwards, thus a positive (sagging) moment results in a negative curvature in the defined axis-system. In some texts, you may find this equation written without the minus sign, due to different choices of the axis-system).

This equation gives a relationship between the bending moment, M, and the second derivative of the deflection, v (with respect to position along the beam, z). EI is called the flexural rigidity. This equation can be used to find the deflection, v.

14.1.1 Finding the Deflected Shape of a Beam

To find the deflected shape of the beam:

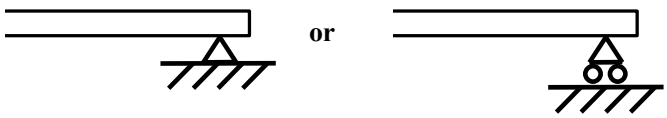
1. Find the bending moment distribution along the beam, $M = M(z)$.
2. Use the governing equation, $-EI\frac{d^2v}{dz^2} = M$ to write an expression for $\frac{d^2v}{dz^2}$ as a function of z.

3. Integrate twice to give the deflected shape, $v = v(z)$
4. Apply boundary conditions to find the constants of integration.

14.2 Support Conditions

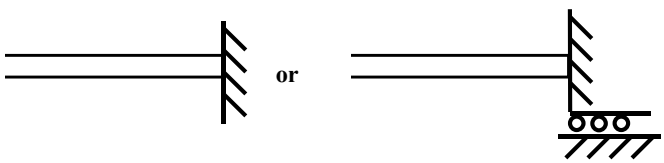
To find the deflected shape by integration of the above equation, suitable integration constants must be found from the boundary conditions at the supports.

'Simple support' or 'Pinned support'



There's no deflection at the support, so $v = 0$. Rotation is free, so dv/dz can be non-zero.

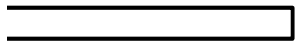
Fixed support or 'Built-in' or 'Encastré' or 'Encastered'



Again, there's no deflection at the support, so $v = 0$, but now the rotation is also zero, so $dv/dz = 0$.

Note, however, there is a moment at the fixed end and so the curvature is not zero

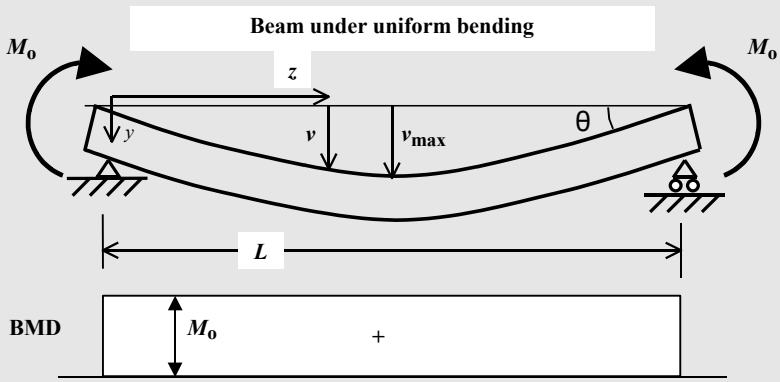
Free end



No boundary conditions are needed. In other words, at a free end it does not provide known slope or deflection condition which may be used for the determination of the deflection integration constants.

14.3 Examples

14.3.1 Constant bending moment



We often draw our beams with large deflections as this aids visualisation, however, always remember that Euler Bending Theory assumes small deflections, as is usually the case in engineering applications.

The bending moment is constant in the beam:

$$M(z) =$$

Use the equation:

Integrating once:

(1)

(note that the first derivative dv/dz is also the slope (rotation) of the deflected curve at z).

Integrating again:

This gives the deflections v as a function of z (i.e. the deflected shape). We apply boundary conditions to determine the constants of integration, C_1 and C_2.

Boundary conditions:

1. At $z = 0$,
2. At $z = L$,

Substituting C_1 and C_2 into the general expression (2) for deflection v:

$$EIv =$$

or

$$v =$$

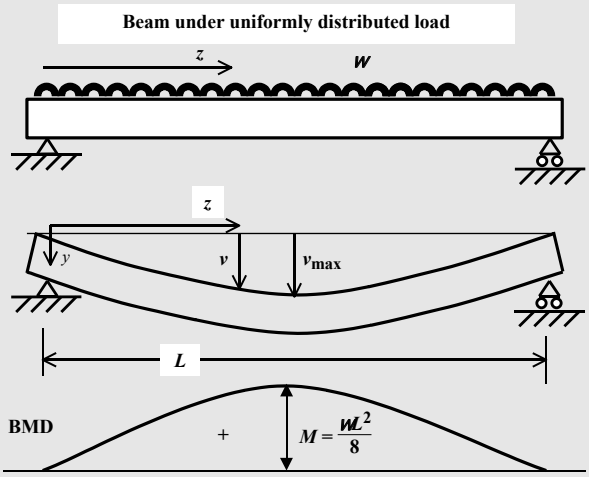
This is the deflected shape of the beam. It is parabolic (a quadratic function), which is always the case when the beam is under a constant moment.

The maximum deflection occurs where

So the maximum deflection is at the midspan, which might be expected from symmetry. The maximum deflection at $z =$ is then found as

$$v_{max} =$$

14.3.2 Beam with uniformly distributed load



To calculate the bending moment distribution, make a cut at a distance z along the beam.

Taking moments about the cut:

$$M =$$

Apply the general equation of bending deflection:

$$-EI\frac{d^2v}{dz^2} =$$

Integrate once to obtain the slope of the beam

$$-EI\frac{dv}{dz} = \qquad (1)$$

Integrating again to obtain the deflection v:

$$-EIv = \qquad (2)$$

Apply boundary conditions:

Using (2), at $z = 0$, $v =$ $\qquad C_2 =$

Using (2), at $z = L$, $v =$ $\qquad C_1 =$

Substitute C_1 and C_2 into (2):

$$-EIv =$$

$$v =$$

This is the general equation for the deflected shape of a simply supported beam under u.d.l. The maximum deflection occurs when $dv/dz = 0$, or when

which can be solved to find the location z of the maximum deflection (there will be three solutions). On the other hand, symmetry suggests that the maximum deflection will be at the centre, $z = L/2$, which can be verified to satisfy the equation above. The other two solutions are off the ends of the beam, so are not relevant.

The maximum deflection is then:

$$v_{max} =$$

$$v_{max} =$$

14.4 Stiffness

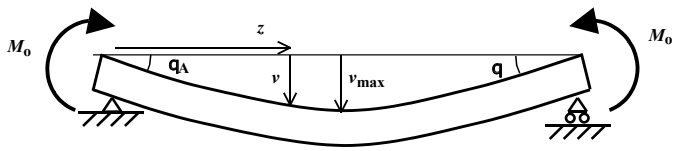
The global stiffness of a beam is defined as the ratio of the applied load to the corresponding deflection. It is a property of the structure, and depends on the particular load applied. Thus we might say “estimate the stiffness under a point load at midspan”, or sometimes “what is the stiffness seen by a point load at midspan”. Here are the stiffnesses for the structures in the examples above.

Simply supported beam under uniformly distributed load

The stiffness k is the ratio of the applied load to the central deflection.

$$k = \frac{\omega}{v_{max}} = \frac{P}{\left(\frac{5\omega L^4}{384EI}\right)} = \frac{384EI}{5L^4}$$

Beam carrying constant moment



The stiffness is the ratio of the applied moment M_0 to the rotation θ that it causes.

The end rotation is given by $\theta = dv/dz$ (a small angle approximation) at $z = 0$.

From §14.3.1

$$-EI\frac{dv}{dz} = M_0 z - \frac{1}{2}M_0 L$$

$$\theta_A = \left.\frac{dv}{dz}\right|_{z=0} = \frac{M_0 L}{2EI}$$

So the stiffness is

$$k = \frac{M_0}{\theta_A} = \frac{2EI}{L}$$

14.5 Superposition for Elastic Structures

Linear superposition allows us to combine the results of different applied loads to find the total response of a structure. In this section we take advantage of superposition to determine the deflections of beams under non-standard or combined loads.

14.5.1 The Principle of Superposition

Superposition is the formal name for a principle that says that: If a body is elastic and subject to small displacements then, when any two load cases

are applied to it, the total deflection at any point in a given direction is the sum of the deflections in that direction from the two cases, and the total stress in a given direction at any point is the sum of the stresses in that direction from the two cases. This principle effectively states that elastic bodies are linear systems. If the effects of any *two* load cases can be added, then any number of loads can be added in the same way.

This principle allows us to use standard results to solve a large number of more complicated beam and structure problems without the need for double integration (which can rapidly become complicated).

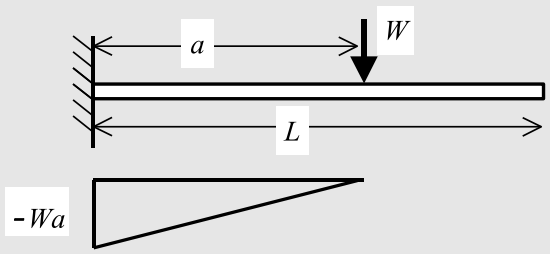
14.5.2 Standard Results of Deflections and Slopes for Simple Cases

In many situations, the superposition of standard deflection results gives a very quick method for determining the deflection of a beam at specific locations (far quicker than double integration of the curvature and applying the boundary conditions).

The most useful of these standard cases are given in your datasheet for this course. You will be given these standard cases in a datasheet for exams in later structures courses. However, it may help with quick early-stage design calculations if you learn a few of them by heart (for example, the tip deflection of a point-loaded cantilever, and the central deflection of a centrally loaded simply supported beam).

14.6 Examples

14.6.1 Tip deflection of a cantilever *not* loaded at the end



Consider the deflected shape: no bending moments occur beyond the load, so the beam must be straight from that point on.

From the standard results (θ in radians):

$$\delta_1 = \qquad\qquad \theta_1 =$$

Since the part to the right of the applied load is a straight line,

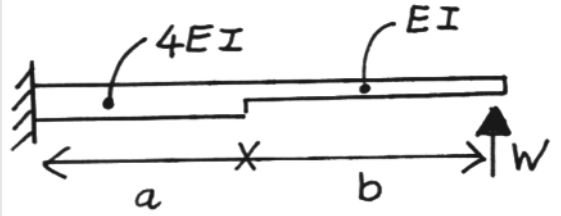
$$\delta_2 =$$

So the total tip deflection is

$$\delta_t =$$

14.6.2 Cantilever with change in stiffness

What is the tip deflection of this cantilever, due to the tip load, W?



Split this beam into two parts, at the change in section thickness. First, establish the internal forces (including moments) at the cut:

The deflection of these two pieces of beam can be found using the standard cases:

$\delta_1 =$ $\delta_2 =$

$\theta =$ $\delta_3 =$

The total deflection at the tip of the beam is

$$\delta_t =$$

14.7 Symmetric and Antisymmetric Structures

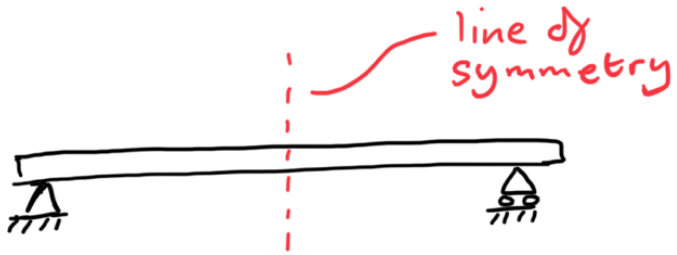
If the geometry and material properties of a structure are symmetrical, then it can be possible to simplify its analysis. For beam deflections, we may be able to either

- split the structure at a line of symmetry to leave a standard case for

analysis, or

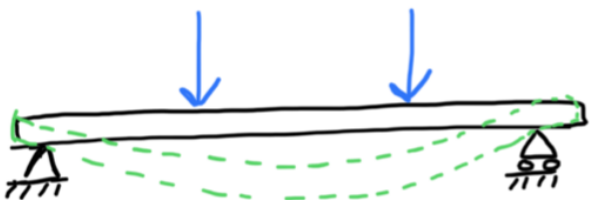
- use the constraints imposed by symmetry to give extra information for our calculation.

If we have a symmetric structure (if we're looking at a beam structure with only vertical loads, then the roller doesn't affect the symmetry),



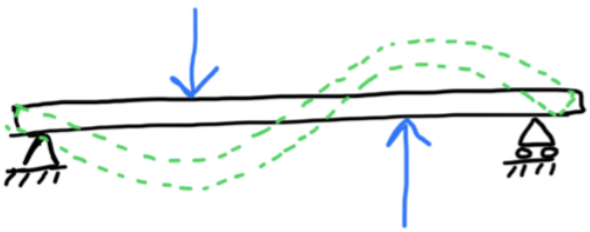
then if the applied loads are either symmetric, or antisymmetric (having opposite signs on either side of the line of symmetry), we gain useful information about the way the structure behaves.

Looking at the deformed shape of a structure with a symmetric load,



we can see that a symmetric load must have a symmetric displacement, and so the slope must be zero at the line of symmetry.

For an antisymmetric load,



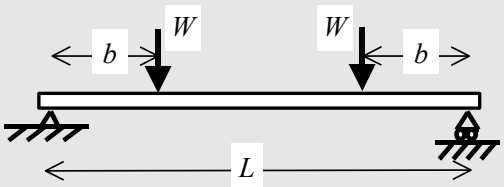
the displacement must have opposite values either side of the line of symmetry, so to avoid a discontinuity, the displacement must be zero at the line of symmetry.

As well as giving information about the deformed shape, symmetry can tell us about the internal forces. If we make a cut along the line of symmetry, we can show the stress resultants on both sides of the cut. The value of the moment M and shear force S will depend on the applied load, but whatever the loads, the stress resultants must be equal in magnitude on each side of the cut, and opposite in direction, as shown below:

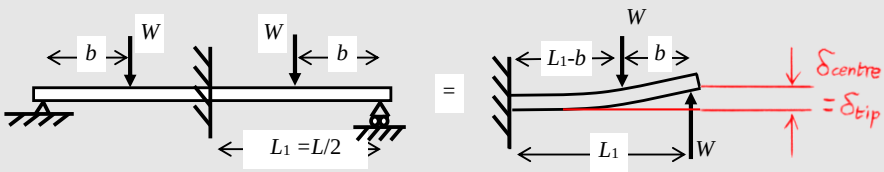
We can see that the moment is symmetric about the line of symmetry, so in a structure under antisymmetric loading this moment must be zero. Similarly, the shear force is antisymmetric and so must be zero under symmetric loading.

14.8 Examples

14.8.1 Pair of loads on a beam, find the central deflection



By noting the symmetry, cutting the beam in half, and treating the half beam as a cantilever, this becomes a cantilever subjected to two point loads.



Let L =4 m, W =100 kN, b =1 m and E =200 GPa= 200×10^6 kN/m^2,
I =20×10^6 mm^4 = 20×10^{-6} m^4, EI = 40 000 kNm^2

For the cantilever with downwards W at b, and taking downwards deflection as positive. From §14.6.1:

$$\delta_1 =$$

Where a = and L_1 = . For the cantilever with upwards W at the tip, from standard solution:

$$\delta_2 =$$

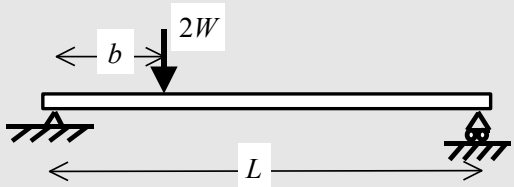
$$\delta_{tip} =$$

The actual deflection at the centre of the simply supported beam is therefore

Following the same procedure, a general expression can be worked out.

14.8.2 Unsymmetrical load on a beam

Find the central deflection of the beam under the unsymmetrical point load:



Here, superposition can be used to turn the problem into the sum a symmetric and an antisymmetric result:

= +

= +

So the central deflection of the asymmetrically loaded beam with load $2W$ is the same as that of the symmetrically loaded beam with two point loads W. From above:

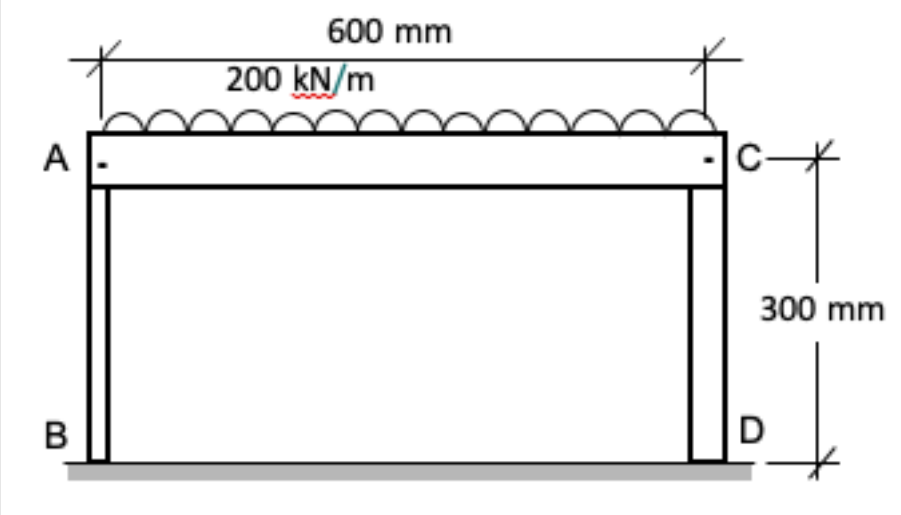
$$\delta =$$

Where $a =$ and $L_1 =$.

This simplifies to:

$$\delta_t =$$

14.8.3 Beam on a deformable support



Beam *AC* is pinned to two solid circular posts as shown in the figure below. A uniformly distributed load of 200 kN/m is applied on *AC*. Beam *AC* is made of steel and its cross section has a second moment of area I =106 mm^4. *AB* is made of steel and has a diameter of 20 mm, and *CD* is made of aluminium and has a diameter of 40 mm. Take the Young's modulus for steel to be 200 GPa, and for aluminium 80 GPa. Discuss the steps in order to find the vertical displacement at the centre (mid-span) of *AC*. (Note: detailed analysis is to be carried out in the tutorial questions.)

The vertical displacement at the centre of *AC*, denoted as point *E*, comprises two parts:

- ;
-

$\delta_{E,1}$ can be determined as a result of axial deformation (shortening) of posts *AB* and *CD*, according to the geometric relationship.

The axial deformations in *AB* and *CD* can be calculated from their axial forces, which can be found from the vertical support force for beam *AC* at *A* and *C*, respectively.

For the deflection at the centre of beam AC due to bending alone, this can be calculated using the standard formula for simply supported beam under the uniformly distributed load, i.e.

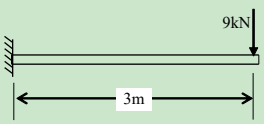
$$\delta_{E,2} =$$

Finally, by superposition,

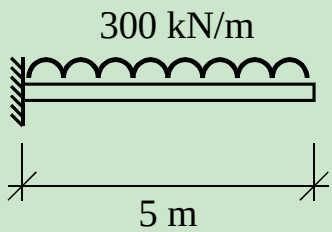
$$\delta_E =$$

Questions

1. For the two cantilever beams shown below, determine:
   a) the bending moment distribution (variation of M with z, from the bending moment diagram),
   b) the deflected shape, v (by integrating the moment and applying suitable boundary conditions),
   c) the maximum deflection.

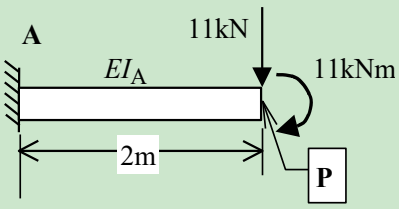


Steel beam ($E = 2 \times 10^5$ MPa)
with $I = 4 \times 10^6$ mm^4

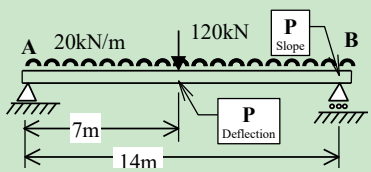


$EI = 1.6 \times 10^5$ kNm^2

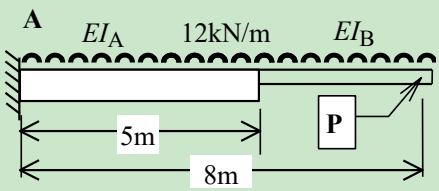
2. For each of the beams below, use superposition to find the deflection and the slope at the points marked P. The results of standard solutions given in the datasheet may be taken as known.



$EI_A = 5 \times 10^3$ kNm^2

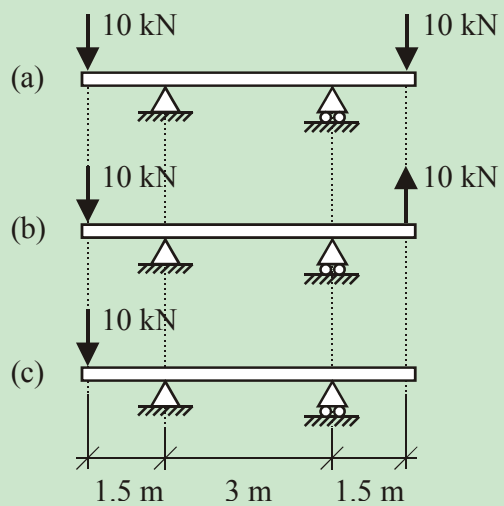


$EI = 3 \times 10^5$ kNm^2



$EI_A = 7 \times 10^4$ kNm^2 and $EI_B = 2 \times 10^4$ kNm^2

3. A 6 m long over-hang beam rests on two simple supports, and has flexural stiffness EI =4900 kNm^2. The figure shows the beam subjected to three different combinations of point loads acting at the beam's ends.



   a) Two 10 kN loads, both acting down
   b) Two 10 kN loads, one up and one down
   c) A single 10 kN load, acting down on the left end of the beam

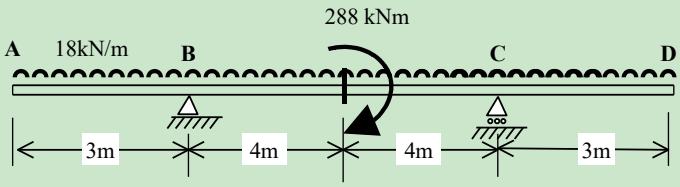
   For each combination of loads, find the vertical deflection at the centre of the beam and state its direction.

   Hint:

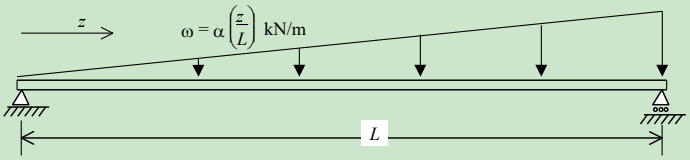
   - consider reducing the beam to simply-supported case with only the main span
   - Use symmetry and anti-symmetry where appropriate.

4. For the beam shown below, use superposition to find the central deflection of the beam, given that $EI = 3.12 \times 10^4$ kNm^2. Sketch the deflected shape of the beam. (Hint: break and reduce the beam into simply supported case with only the middle span. Note also that a

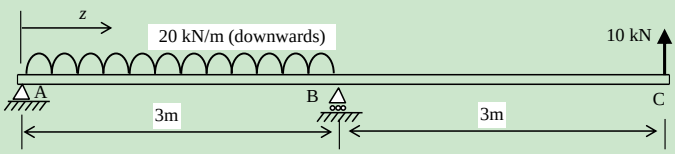
symmetric beam subjected to a concentrate moment at the middle results in an antisymmetric deflection shape).



5. A uniform beam is subject to a linearly increasing distributed load $\omega = \alpha(z/L)$, as shown below. Express the bending moment as a function of z, and subsequently find the deflected shape of the beam (using the method of integration). Show that the maximum deflection is at $z = 0.519L$ and is given by $0.006522\omega L^4/(EI)$.

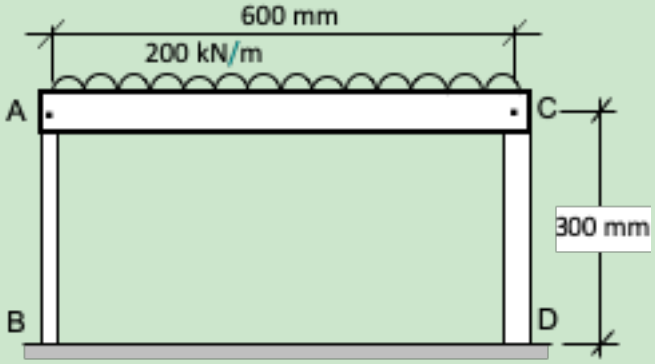


6. For the uniform beam shown in the figure below,
   a) derive the deflected shape for the span AB (as a function of z),
   b) based on the above result and using the Datasheet (if needed), determine the deflection at point C.



EI =5000 kNm^2

7. A beam AC is pinned to two solid circular posts as shown in the figure below. A uniformly distributed load of 200 kN/m is applied on AC. Beam AC is made of steel and its cross section has a second moment of area $I = 1 \times 10^6$ mm^4. AB is made of steel and has a diameter of 20 mm, and CD is made of aluminium and has a diameter of 40 mm. Determine the vertical displacement at the mid-span of AC. Take the Young's modulus for steel 200 GPa, and for aluminium 80 GPa.



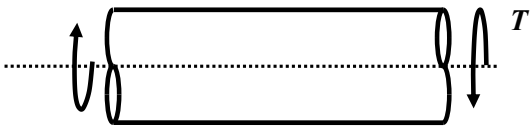
Numerical answers. Q1: (a) (iii) 101.3 mm, (b) (iii) 146 mm; Q2: (a) δ = 10.3 mm. θ = 0.0088 radians (b) δ =56.2 mm. θ = 0.0125 radians (c) δ = 13.4 + 9.6 + 21.4 + 10.7 + 11.6 + 19.3 + 6.1 = 92.1 mm, θ = 0.00357 + 0.00386 + 0.00643 + 0.00270 = 0.0166 radians. Q3: (i) 3.44 mm (upwards), (ii) 0 mm, (iii) 1.72 mm (upwards); Q4: $\delta_1 = 5\omega L^4/(384EI)$, $\delta_2 = 0$, $\delta_3 = -(\omega L^2/2)(L^2/2EI)$, $\delta = 312/EI$ = 10.0 mm; Q5: $R_A = \omega L/6$, $C_1 = -(7/360)\alpha L^3$; Q6: (b) 49.5 mm (upwards) ; Q7: 1.92 mm (downwards) .

15 Torsion

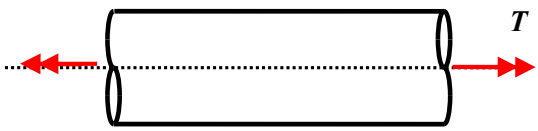
15.1 Uniform Members Carrying Torsion

Where a shaft is used to transmit torque, torsion will be an important stress resultant to consider.

Torsion, (normally denoted T), causes a shaft to twist: the two ends of the shaft twist relative to each other around its axis. The deformation of a torsion member is described by the angle of twist, θ, between two (end) sections.

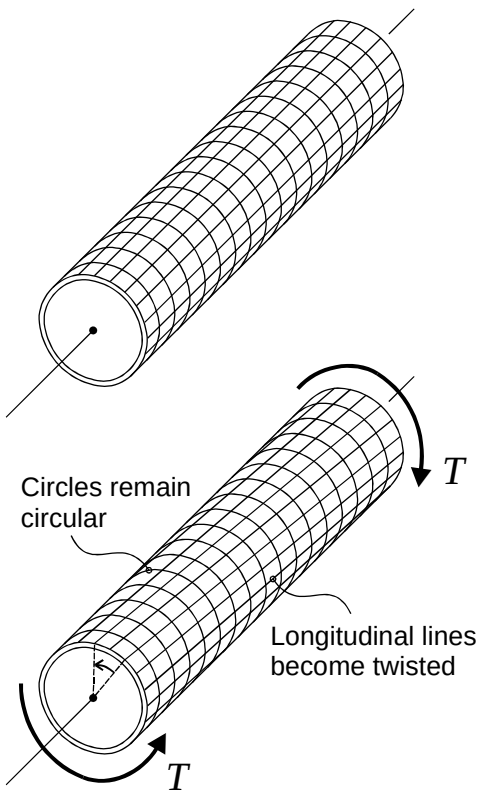


For convenience we shall adopt the double-head arrow as representation of the torsional moment. The direction of the arrow for torsion will be the same as that for axial force in the element, and the equilibrium operation for members carrying torsion can actually be carried out in a similar fashion as an axially load member.



15.1.1 Shear deformation

Picture a grid can be drawn on the outside of a circular shaft. After the shaft has been loaded in torsion, the lines of the grid will still be straight, but they will not be perpendicular; the axial lines will rotate due to twist along the shaft. Sections across the shaft remain flat during twisting (this is the same "plane sections remain plane" assumption we used in analysis of bending).

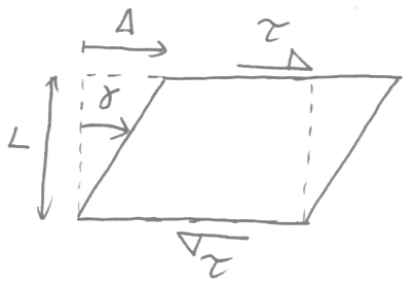


A shaft under torsion carries the load in shear, so as we consider this type of loading, we can explore more about shear behaviour. Note that the treatment of torsion here assumes that the material is linear elastic, homogenous and isotropic.

15.1.2 Shear strain

Shear strain describes the shearing deformation of a material carrying shear stress. The engineering shear strain γ is the change of angle (in radians)

between two initially perpendicular sides of an element of material under shear stress τ, as shown in this sketch:



$$\gamma = \frac{\Delta}{L}$$

A more general definition will be developed in later years.

15.1.3 Shear Modulus

The shear modulus G describes the stiffness of a material in shear, and is given by:

$$G = \frac{\text{Shear stress}}{\text{Shear strain}} = \frac{\tau}{\gamma}$$

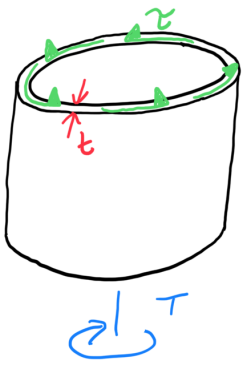
In an incompressible material, there is a relationship between The Young's modulus, shear modulus, and Poisson's ratio are related by:

$$G = \frac{E}{2(1+\eta)}$$

15.2 Torsion in a hollow circular shaft

15.2.1 Shear stress

A thin-walled, hollow circular shaft is the simplest torsion member, because the shear stress in the shaft is constant everywhere.



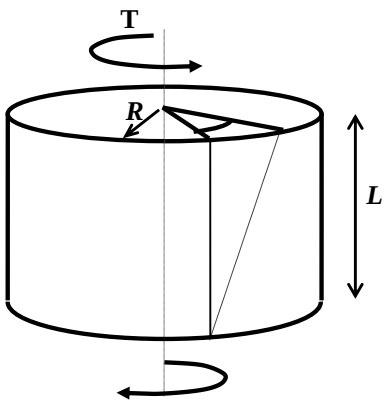
Resultant moment from the shear stress about the section centre makes up the total internal torque, and consider the constant distance r:

$$\tau \times (2\pi r t) \times r = T$$

$$\tau = \frac{T}{2\pi r^2 t}$$

t denotes the thickness of the wall, r is the radius to centre of wall.

15.2.2 Shear strain



Hence, the shear strain in a hollow circular shaft is

$$\gamma = \frac{\tau}{G} = \frac{T}{2\pi r^2 tG}$$

15.2.3 Deformation

From the figure shown on the right, it can be established that the angle of twist, ϕ, of the shaft is related to the shear strain, γ, by:

$$\phi \times r = \gamma \times L \qquad \gamma = \frac{\phi r}{L}$$

Hence, the angle of twist is given by:

$$\phi = \frac{\gamma L}{r} = \frac{TL}{2\pi r^3 tG} \text{ inradians}$$

The general expression for torsion in a uniform member can be written as

$$\phi = \frac{TL}{GJ} \qquad T = GJ\frac{\phi}{L}$$

where J is known as the torsion constant. It is equal to the polar second moment of area (which will be discussed later). For a thin-walled hollow circular section:

$$J = 2\pi r^3 t$$

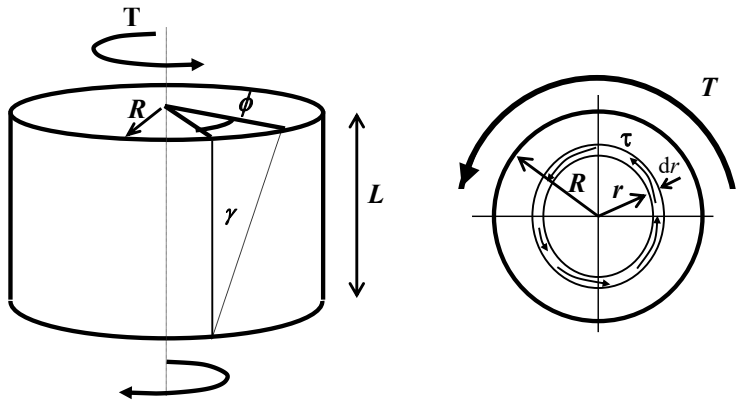
15.2.4 Torsional stiffness

The stiffness of a structure in general is the ratio of applied force to deformation. In the case of torsion of a shaft, the appropriate force is the applied torque, T, and the deformation is the angle of twist ϕ.

$$k = \frac{T}{\phi} = \frac{GJ}{L}$$

15.3 Torsion in a solid circular shaft

A solid circular shaft can be regarded as being made from a series of nested cylinders (i.e.: hollow circular sections). We can use this to find the torsional behaviour of a solid circular shaft.



The cylinders do not slide past one another, since they are in reality forming one solid piece of material. This is satisfied by assuming radial lines remain straight (hence the whole section it twisted by the same angle ϕ). Thus, the compatibility equation is written as:

$$\phi_r = \phi \left(= \frac{\gamma L}{R} \right)$$

Cylinder at radius r

Consider one of the cylinders, at radius r. Let the thickness of this cylinder be $t = dr$, and use the previous analysis. The torque carried by this cylinder is:

$$dT = G \left(2\pi r^3 dr \right) \frac{\phi}{L}$$

Combine all the cylinders

Sum the torques carried by all the nested cylinders that make up the shaft (by integration). The torque is given by:

$$T = \int_0^R dT = \int_0^R G\left(2\pi r^3 dr\right)\frac{\phi}{L} = G\frac{\phi}{L}\int_0^R 2\pi r^3 dr = G\frac{\phi}{L}J$$

or

$$\phi = \frac{TL}{GJ}$$

where

$$J = \int_0^R 2\pi r^3 dr = \frac{\pi R^4}{2}$$

for a circular solid shaft.

15.3.1 Shear stress

The shear stress at radius r in the shaft is related to the shear strain by $\tau_r = G\gamma$, where τ_r is the shear stress at radius r. From above:

$$\gamma_r = \frac{\phi r}{L}$$

$$\tau_r = G\left(\frac{\phi r}{L}\right)$$

The maximum shear stress takes place on the surface of the shaft, at $r = R$:

$$\tau_{max} = \frac{TR}{J}$$

Note the similarity between the equations for torsion in circular shafts and axial load:

Torsion:

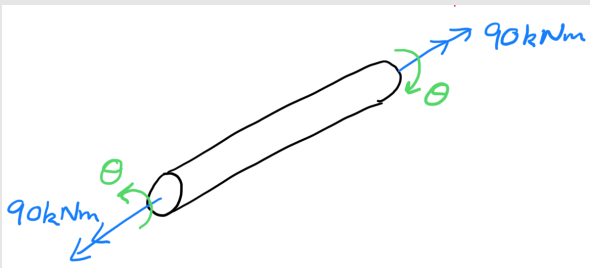
$$\phi = \frac{TL}{GJ} \qquad \tau_{max} = \frac{TR}{J}$$

Axial load:

$$\delta = \frac{FL}{EA} \qquad \sigma = \frac{F}{A}$$

15.4 Example

A ship's drive shaft is to be made from a solid circular-section. It must withstand a torque of 90 kNm.



- Find the shaft's radius if the design shear stress is τ_{max} = 60 MPa.
- Find the angle of twist if G =70 GPa and the shaft is 3 m long.
- What is the shaft's torsional stiffness?

Minimum radius of shaft

$$\tau_{max} =$$

$$\tau_{max} =$$

R_{min} =

Angle of twist

J =

ϕ =

Stiffness of shaft

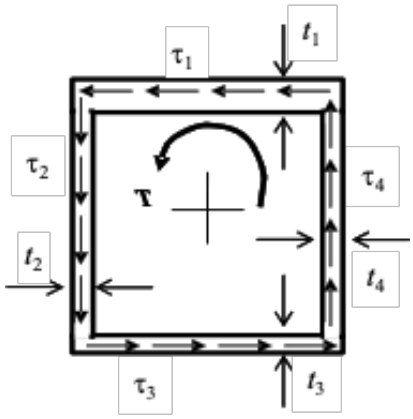
Stiffness =

15.5 Torsion in Other Sections

So far we have only considered circular sections. However, the torsional behaviour of any closed section can be analysed in a similar way. This analysis applies to thin-walled closed sections. Torsion of open sections (e.g. U, Z, I, T shapes) is more complicated and beyond SM2A.

15.5.1 Shear flow in closed sections under torsion

The shear around a closed section forms a closed loop, and so we use the concept of shear “flowing” around the section to help visualise and understand it. In a general section with varying wall thickness,



internal equilibrium requires that “shear flow” is constant around the section:

$$\text{shear stress} \times \text{thickness} = \text{constant}$$

$$\tau \times t = \text{constant}$$

6.1.2. Shear Stress To find the shear relationship between torque and shear stress for a general closed section, we need to sum up the contribution of the shear stresses around the section. This was straightforward for a hollow circular section, but is slightly more difficult for a general section.

Consider a short length of the wall, *ds*.

The contribution of shear in this length to the torque is:

$$dT = \tau \times t \times ds \times R$$

But, $1/2 \times ds \times R$ = the area of the shaded triangle, dA.

$$dT = (\tau \times t) \times dA \times 2$$

$(\tau \times t)$ is the shear flow and is constant.

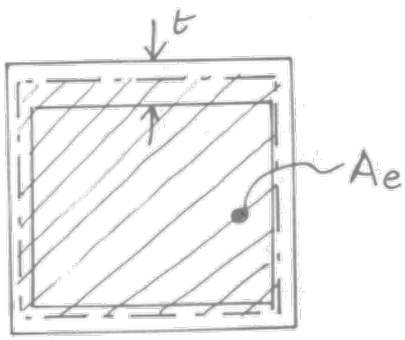
Integrate around the section to find the total torque, T:

$$T = \oint dT = 2(\tau \times t) \times \oint dA$$

$$T = 2\tau t Ae$$

$$\tau = \frac{T}{2A_e t}$$

Where T is the applied torque, t is the local thickness of the section and A_e is the area enclosed by the section.



A_e is strictly defined by a line through the middle of the wall of the section, but for a thin-walled section the error may be small if either the outer or inner line of the wall is used.

15.5.2 Twist

The angle of twist, ϕ, can be found using the general uniform torsion equation (as for cylindrical sections):

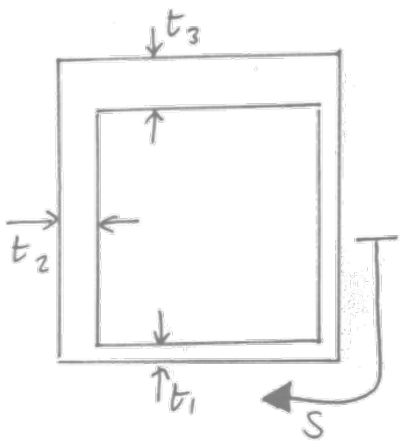
$$\phi = \frac{TL}{GJ}$$

For a general closed hollow section, it can be shown (proof omitted) that the torsion constant, J, can be found from:

$$J = 4\frac{A_e^2}{\oint \frac{ds}{t}}$$

Where t is the local thickness of the section, A_e is the area enclosed by the section and s is a coordinate that follows around the wall of the box section.

If the section can be split into regions with constant t, as in this sketch:



then the integral can be discretised:

$$\oint \frac{ds}{t} = \sum_i \frac{s_i}{t_i}$$

15.5.3 Special Case – Circular Hollow Section

Check that for a circular hollow section, the above analysis agrees with the equations for a circular hollow section:

$$A_e = \pi r^2$$

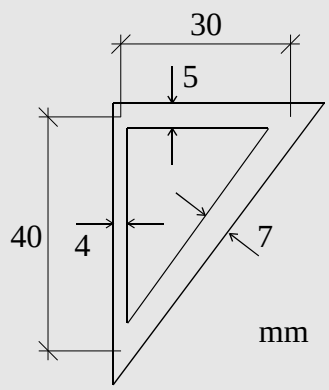
$$\oint \frac{ds}{t} = \frac{s}{t} = \frac{2\pi r}{t}$$

$$J = 4\frac{A_e^2}{\oint \frac{ds}{t}} = 4\frac{\pi^2 r^4}{\frac{2\pi r}{t}} = 2\pi r^3 t$$

$$\tau = \frac{T}{2A_e t} = \frac{T}{2\pi r^2 t} = \frac{Tr}{J}$$

15.6 Example

15.6.1 Hollow section



For the closed triangular hollow section shown, find:

- the distribution of shear stress, τ, in the walls; and

- the angle of twist, ϕ.

The member is 1000 mm long, made from steel with $G = 81\,\text{kN/mm}^2$ and carries a torque of 500 Nm.

Shear stress

First calculate the shear flow:

$$A_e =$$

$$\tau \times t =$$

This can then be used to calculate the stresses in each part of the member:

In the top wall: $\tau =$

In the left wall: $\tau =$

In the diagonal wall: $\tau =$

Twist

$$\oint \frac{ds}{t} =$$

$$J =$$

$$\phi =$$

15.7 Non-uniform torsion of shaft

If the torque carried by a shaft varies along its length, or the cross-section of the shaft changes with z, then consider a short length of the shaft and integrate to find the total angle of twist of the shaft. This is exactly the same method as we used to analyse a member carrying axial load with non-uniform cross-section or loading.

The incremental version of the torsion equation (for a short length dz of the shaft) is

$$d\phi = \frac{T}{GJ} dz$$

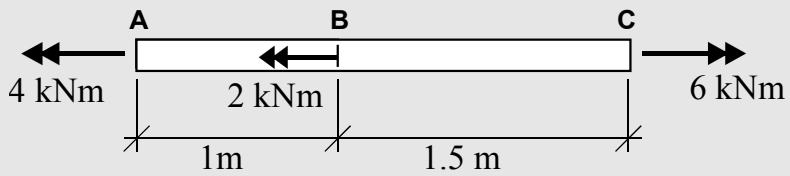
If the shaft consists of multiple uniform segments,

$$\phi = \sum \frac{T_i L_i}{GJ_i}$$

For the convenience of analysing the torques in individual segments, it is useful to represent the torques using the double-headed arrows, as can be seen from the example below.

15.8 Example

A shaft is made from a solid circular section of 60 mm in diameter. Lengths and torques acting on the shaft are as shown. The shaft is made from steel with a shear modulus of $G = 81$ GPa.



- Find the torques carried through Part AB and BC (i.e., internal torques), respectively.
- Calculate the angle of twist at C relative to A.

By just inspection we can see the set of torques are in balance (in equilibrium).

Make a "cut" between A and B, as shown, we get $T_{AB} =$.

Noting "+ve" here follows the double-headed arrow direction at the cut, which indicates that the torque is anticlockwise when viewed from the cut towards A, and so will be the angle of twist.

Similarly (making a "cut" between B and C, and balancing), $T_{BC} =$.

For a solid circular section:

$$J =$$

Hence we can calculate the twist of AB (B relative to A):

$$\phi_{BA} =$$

the twist of BC (C relative to B):

$$\phi_{CB} =$$

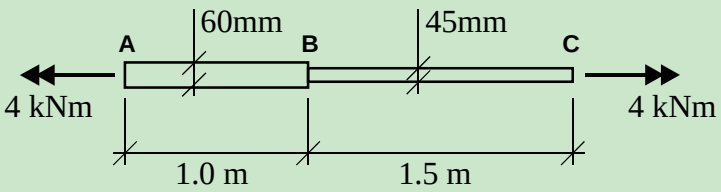
and the twist of C relative to A:

$$\phi_{CA} =$$

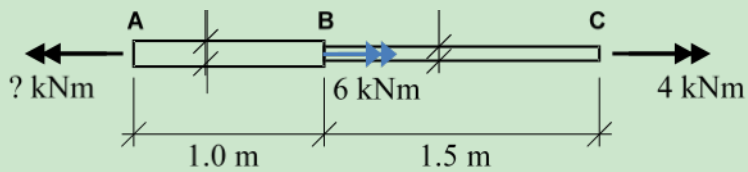
(anticlockwise viewing from C towards A)

Questions

1. A brass tube has external diameter 50 mm, internal diameter 40 mm and length 300 mm. The tube is placed in a torsion testing machine. A torque of 480 kNmm is applied to the tube, and the relative rotation of the two ends of the tube is found to be 0.02 rad. Calculate the shear modulus, G, of the brass.
2. A drive shaft is made from two solid circular sections, joined end to end at B. Portion AB is 60 mm in diameter and 1 m long; portion BC is 45 mm in diameter and 1.5 m long. The shaft carries a torque of $T = 4$ kNm, and is made from steel with a shear modulus of $G = 81$ GPa.



   a) Calculate the relative rotation of the two ends of the shaft (assuming zero rotation at end A of the shaft).
   b) Calculate the maximum shear stress in the shaft. Where is this? Sketch the variation in shear stress along the shaft.
   c) Sketch the variation of the magnitude of the shear stress across the diameter of the section at C.
   d) If a torque of 6 kNm is also applied on the shaft at B, while the torque of 4 kNm at C remains, as shown below, what will be the torque at A so as to maintain the equilibrium? What are the internal torques within part AB and part BC, respectively?



3. If the shaft shown in q.2(d) is made of two hollow square sections with a side length (centre-to-centre of opposite walls) of 120 mm for *AB* and 80 mm for *BC*, and the wall thickness is uniformly 5 mm throughout,

   a) calculate the shear stress in part *AB* and part *BC*. Sketch the shear stress distribution along the section walls (indicate the direction and the magnitude).

   b) calculate the relative rotation of the two ends of the shaft.

Q1: 2.00×10^4 MPa; Q2: (a) 12.8 °, (b) 223 MPa, d) −10 kNm; 10 kNm, 4 kNm; Q3: (a)69.4 MPa, 62.5 MPa; (b) 0.0432 rad/2.48 °

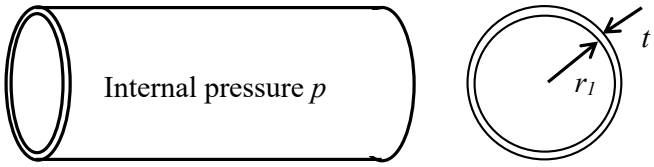
16 | Pressure Vessels

Pressure vessels are used to contain things, and are examples of shell structures. For example, pressure vessels are used to store liquids and solids under pressure, to contain chemical reactions or as pipes to transport materials. Other examples are tanks and silos, aircraft and spacecraft, chimneys and submarines.

Here, we consider the membrane theory for shells, which is satisfactory for simple purposes. This simple theory assumes that the stress is uniform through the thickness of the vessel wall (there are no bending stresses) and the pressure vessel is assumed to rupture when the stresses reach the yield stress of the wall material. In real shells, there are bending stresses near their edges, which cause yield at lower pressures than predicted by membrane theory, and it should therefore by used with caution.

Homestake Mining Locomotive, Fireless, Arizona. Built by the Porter Locomotive Company in 1923.

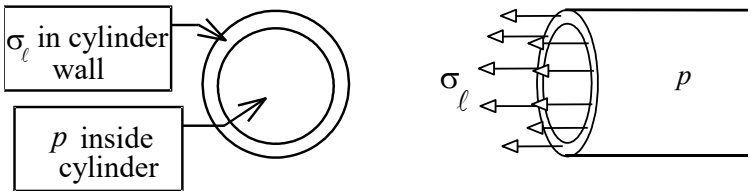
16.1 A Cylinder under Internal Pressure



Longitudinal stresses and stress resultant

Cut the cylinder across its cross-section:

Membrane theory assumes that the walls of the cylinder are thin, so $t << R$



From horizontal (longitudinal) equilibrium, the longitudinal stress, σ_l, is:

Stress in shell wall $\times$ area of shell wall = pressure $\times$ area of interior

$$A_{wall} = \pi r_2^2 - \pi r_1^2 = \pi\left(r_2^2 - r_1^2\right) = \pi\left(r_2 + r_1\right)\left(r_2 - r_1\right) = \pi(2r)(t)$$

where r_2 is the exterior radius and $r = 0.5(r_1 + r_2)$ is the radius to the centreline of the wall.

$$\sigma_l \times 2\pi r t = p \times \pi r_1^2$$

using the thin wall assumption $r_1 \approx r$

$$\sigma_l = \frac{pr}{2t}$$

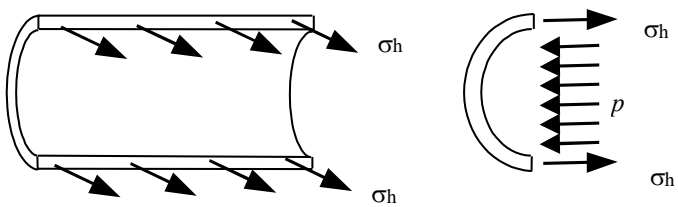
The longitudinal stress resultant, N_l, is:

$$N_l = p \times \pi r^2$$

These are important equations for cylinders. Note that if $t << r$, then $p << \sigma_l$

Circumferential or hoop stresses and stress resultant

Now cut the cylinder along its length:



The circumferential, or hoop stress is σ_h, and is found from transverse equilibrium:

Stress in shell wall $\times$ area of shell wall = pressure $\times$ area of interior

$$\sigma_h \times L \times 2t = p \times 2r_1 \times L$$

$$\sigma_h = \frac{pr_1}{t} \approx \frac{pr}{t}$$

This is the most important equation for cylinders.

The circumferential stress resultant, N_h, can be defined over a unit length of the cylinder:

$$N_h = pr$$

16.2 Example

A cylinder gas container has length 0.2 m, interior diameter 0.1 m, thickness 0.5 mm. Internal pressure is 10 kPa. Find the hoop and longitudinal stresses in the cylinder wall.

First we can convert the values into a common set of units. Newtons and millimetres are good because N/mm^2 is the same as MPa, which we commonly use for material strength.

The length becomes , the interior diameter becomes
and the thickness is already . The internal pressure becomes
.

Then we can apply the equations derived above:

$$\sigma_h =$$

$$\sigma_l =$$

17 Superposition of Stress

Linear superposition can be used to find the net stresses within a structure subjected to combined loads.

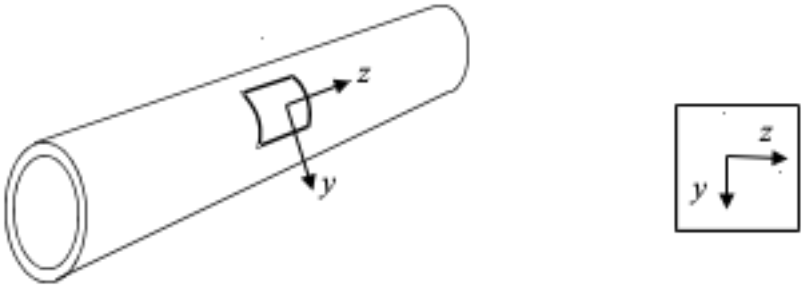
We used the principle of superposition in an earlier lecture to find the deflection of a beam carrying complicated loads by summing the results of simpler load cases. The same principle applies to stresses.

If a body is elastic and subject to small displacements, then when any two load cases are applied to it:

- the total deflection at any point in a given direction is the sum of the deflections in that direction from the two cases, and
- the total stress in a given direction at any point is the sum of the stresses in that direction from the two cases.

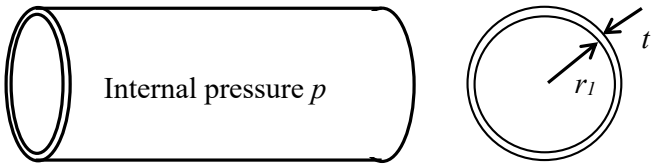
17.1 Hollow cylinders under combined loads

To visualise the stress states in the hollow cylinder it is helpful to draw the stresses acting on a small element of the cylinder wall:



17.1.1 Thin-walled cylinders under internal pressure

Previously, we have considered a thin-walled cylindrical vessel with closed ends under internal pressure:



Equilibrium of suitable free bodies (Chapter 16) gave the membrane stresses within the cylinder:

Longitudinal stress

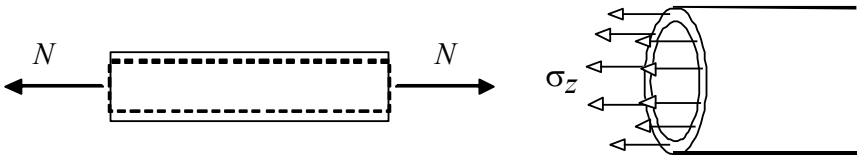
$$\sigma_l = \frac{pr}{2t}$$

Circumferential or hoop stress

$$\sigma_h = \frac{pr}{t}$$

17.1.2 Membrane stresses in a thin-walled cylinder under axial load

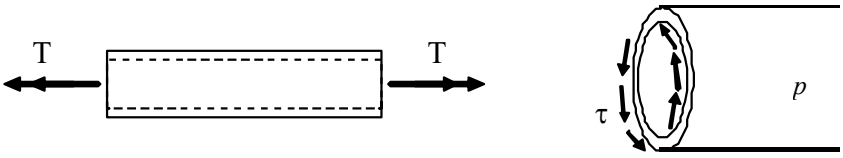
The same cylinder can also carry an externally-applied axial tension (Chapter 7):



$\sigma_z = N/(2\pi rt)$ $\sigma_y = 0$ $\tau = 0$

17.1.3 Membrane stresses in a thin-walled cylinder under torsion

The same cylinder can also carry an externally-applied torque (Chapter 15):



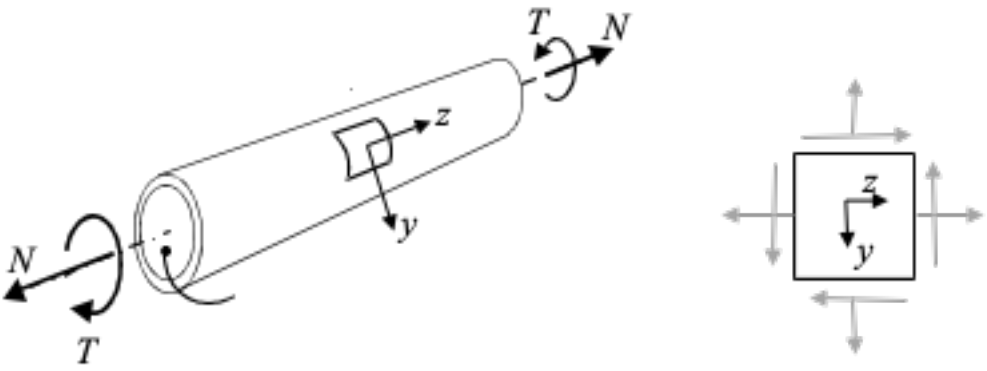
$\sigma_z = 0$

$\sigma_y = 0$

$\tau = T/(2\pi r^2 t)$

17.1.4 Combined stress

If a thin-walled cylinder is subjected to a combination of (i) internal pressure, (ii) axial load and (iii) torque, we can find the total stress state in the wall of the cylinder by applying linear-elastic superposition.



total stress = pressure stress + axial load stress + torsion stress

$$\sigma_z = \frac{pr}{2t} + \frac{N}{2\pi rt} + 0$$

$$\sigma_y = \frac{pr}{t} + 0 + 0$$

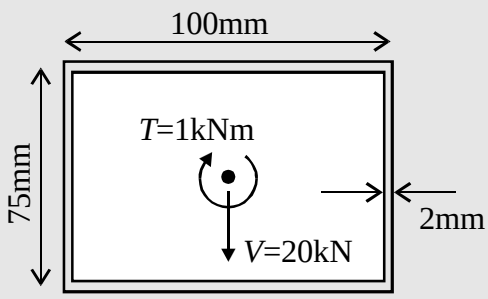
$$\tau = 0 + 0 + \frac{T}{2\pi r^2 t}$$

17.2 Combined shear and torsion in box sections

Applied torque and shear force are both carried by shear stress. Superposition is used to find the total shear stress within the section.

17.3 Example

Find the shear stress distribution for a box section subjected to a combination of a torque 1 kNm and a vertical shear force of 20 kN. (Note the dimensions shown are from centre of wall to centre of wall).



Shear stress due to torque, T (Chapter 15):

$\tau_T =$

$A_e =$

$\tau_T =$

Shear stress due to shear force V (Chapter 12):

$\tau_V =$

$I =$

At the neutral axis, where the maximum shear occurs,

$$b =$$

$$\sum A * \bar{y} =$$

$$\tau_V =$$

At top of web / end of flange:

$$A * y =$$

$$\tau_V =$$

17.4 Combined bending and axial force in beams

Direct stress (along the axis of a beam) can be due to:

- axial load
- bending moments about the two principal axes of the section

Under elastic conditions, the stresses due to the axial forces and bending about each of the principal axes can be summed to give the total direct stress.

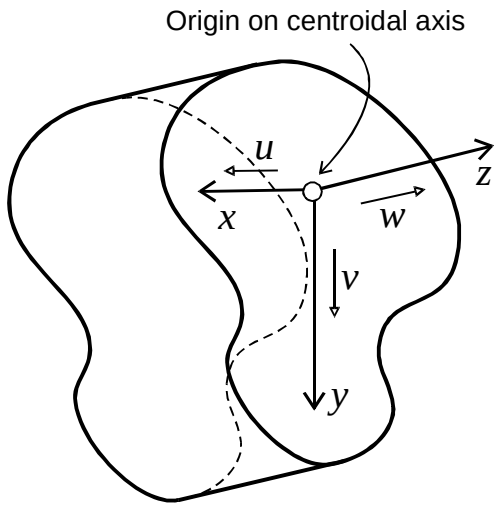
(The formulae that follow are only valid for elastic conditions. They are useful for the assessment of the Serviceability Limit State, or for Fatigue Assessment. After yield, structural engineering is more concerned with the ultimate strength of the cross-section, for which elastic stress evaluation is not so relevant).

To avoid confusion, it is important to be clear about signs.

Coordinate system, deflection and loading:

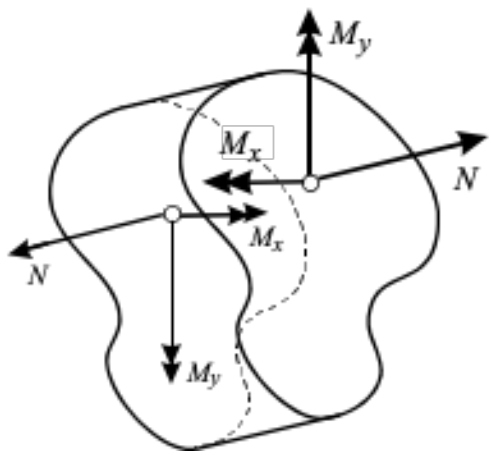
- Axes are defined so that z is along the axis of the beam with y pointing downwards.
- The origin of the axes is taken on the centroid of the beam.

- Beam deformations in the x, y, z directions are u, v, and w respectively. Positive deformations occur in the direction of positive coordinate axes.



Bending Moments and Axial Load

- Direct stress along beam (axial stress) is tensile positive
- Bending moments are defined as sagging positive. Positive moments induce tensile (positive) stresses at a positive x or y coordinate.

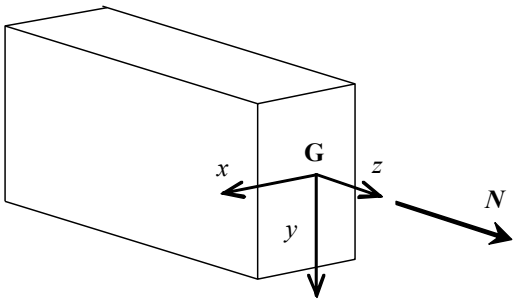


Axial forces

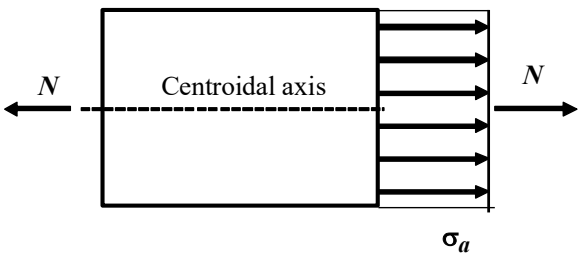
For a beam with cross-section area A subjected to a tensile force N, the direct stress in the z direction is uniform across the cross-section and is given by:

$$\sigma_a = \frac{N}{A}$$

where both stress and force are tensile positive. The notation σ_a is used to indicate that this is the component of stress from axial force.

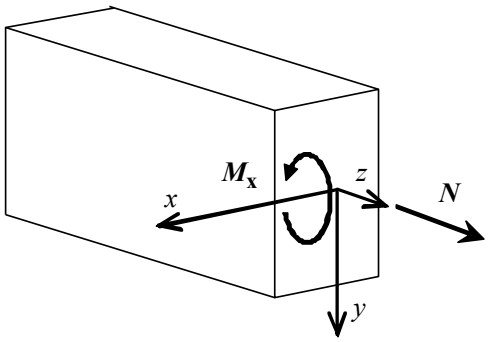


The line of action of the force P must be through the centroid of the cross-section. Otherwise, this is not a case of pure axial loading.



Bending about the x-axis: Uniaxial bending

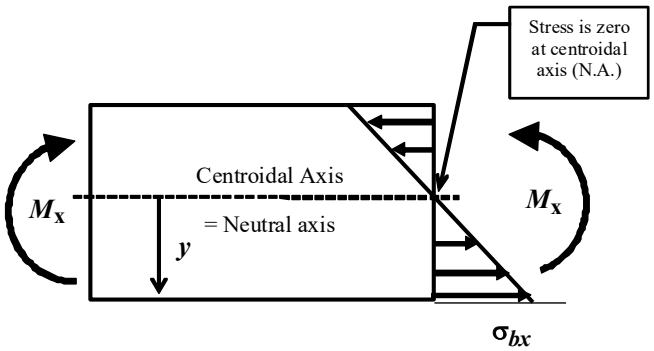
The moment M_x is conventionally sagging positive. A sagging moment causes tensile stresses in the positive coordinate in the orthogonal (y) direction.



The direct stress (tensile positive) in the z direction caused by this moment M_x can be found using:

$$\sigma_b x = \frac{M_x y}{I_x}$$

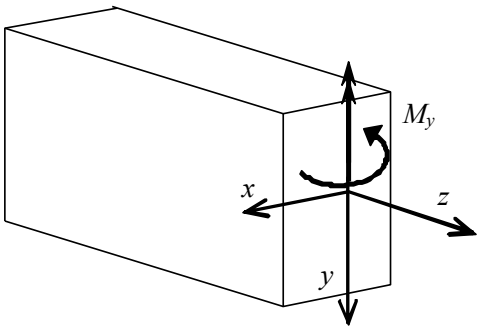
where I_x is the second moment of area about the x-axis.



The notation σ_{bx} is used to indicate that this is the component of stress from bending about the x-axis.

Bending about the y-axis: Uniaxial bending

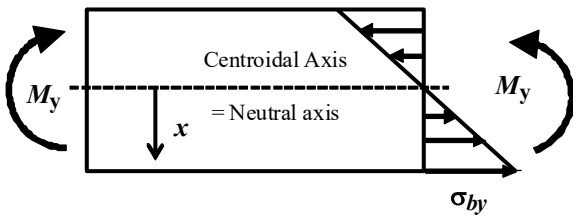
The moment M_y is also sagging positive. A sagging moment causes tensile stresses in the positive coordinate in the orthogonal (x) direction.



The direct stress (tensile positive) in the z direction caused by this can be found using:

$$\sigma_{by} + \frac{M_y x}{I_y}$$

where I_y is the second moment of area about the y-axis.

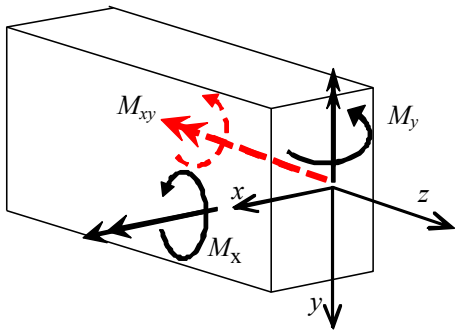


The notation σ_{by} is used to indicate that this is the component of stress from bending about the y-axis.

The above equations are only valid when the x- and y-axes are the principal axes. If a moment is applied about an axis that is not principal, it can be resolved into its two components about the two principal axes and treated as biaxial bending, as defined below.

Bending about both the x- and y-axes simultaneously: Biaxial bending

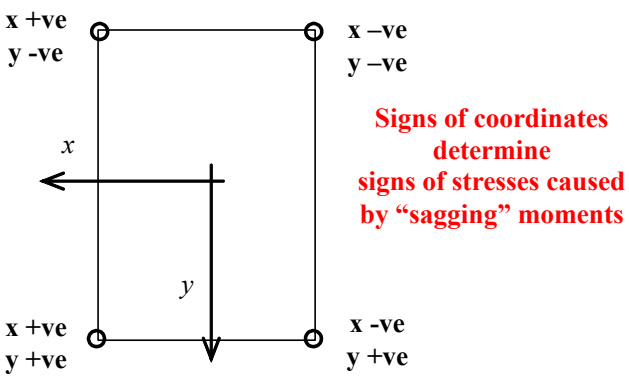
A biaxial bending moment can be resolved into its two components about the two principal axes.



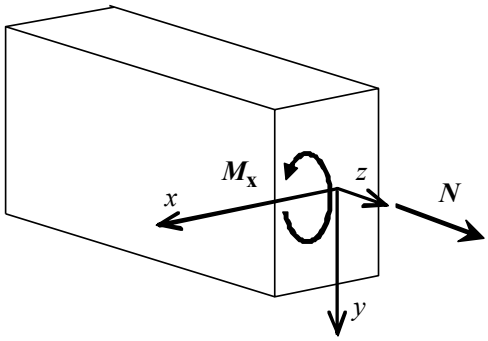
The direct stresses in the *z* direction from bending about the two axes (tensile positive) can then be added directly:

$$\sigma_b = \sigma_{bx} + \sigma_{by} = \frac{M_x y}{I_x} + \frac{M_y x}{I_y}$$

This produces a linear variation of stress across the section, but the maximum occurs in one corner, instead of on one edge.



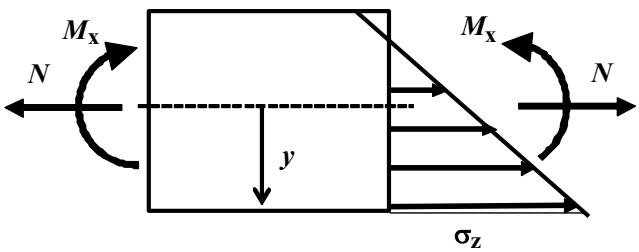
17.4.1 Combined Uniaxial bending and Axial loading



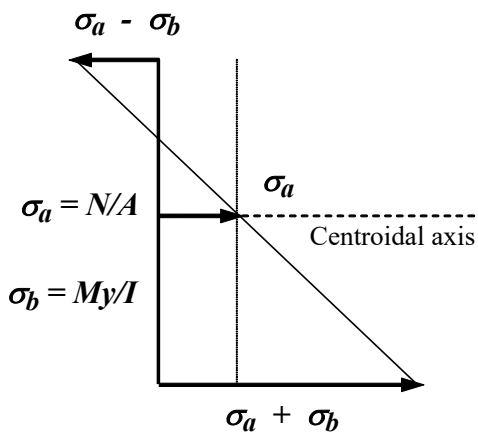
(N = tension positive)

When axial force occurs together with bending about one axis:

$$\sigma = \sigma_a + \sigma_{bx} = \frac{N}{A} + \frac{M_x y}{I_x}$$



The individual stress components can be identified separately:



Note the neutral axis under combined N+M no longer coincides with the centroidal axis (of geometry).

17.4.2 Combined Biaxial bending and Axial loading

When axial force and biaxial bending all occur together:

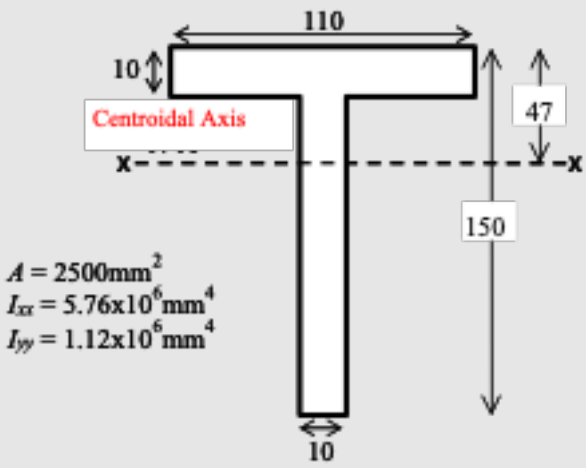
$$\sigma = \sigma_a + \sigma_{bx} + \sigma_{by} = \frac{N}{A} + \frac{M_x y}{I_x} + \frac{M_y x}{I_y}$$

NB: the above equation assumes axial tension and “sagging” moments as +ve.

17.5 Example

A T section beam carries stress resultants of:

- Axial tension $N = 5\,\text{kN}$,
- Major axis moment (sagging) $M_x = 10\,\text{kNm}$, and
- Minor axis moment (sagging) $M_y = 2\,\text{kNm}$.

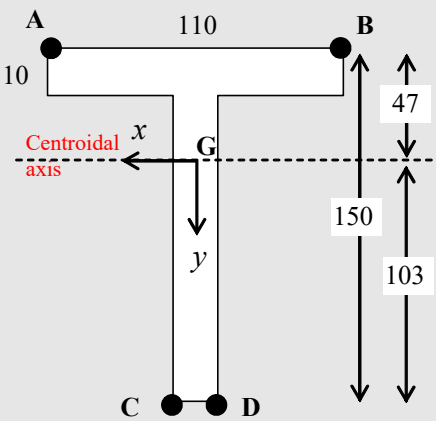


Find the maximum direct stress (most tensile or most compressive) and its location.

The section has one axis of symmetry (vertical) so the principal axes are aligned with the x and y directions. We can use the complete biaxial bending stress formula directly.

$$\sigma =$$

We need to check each of the extreme points on the section to determine the location of the highest stress:



Coordinates of the corners are:

	x	y
A		
B		
C		
D		

At Point A:

$\sigma_A =$

$\sigma_A =$

At Point B:

$\sigma_B =$

At Point C:

$\sigma_C =$

At Point D:

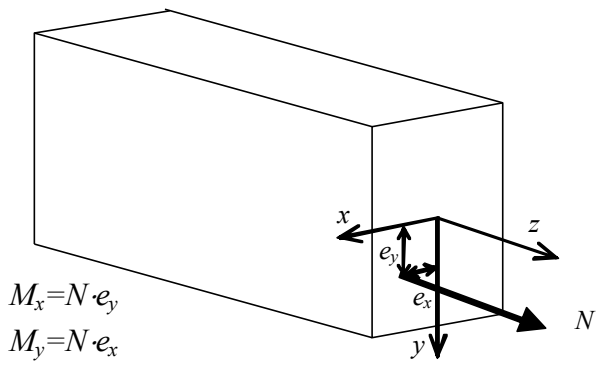
$\sigma_D =$

The highest stress is therefore at with a tensile value of .
The most compressive stress is at with a value of .

17.5.1 Eccentric axial load

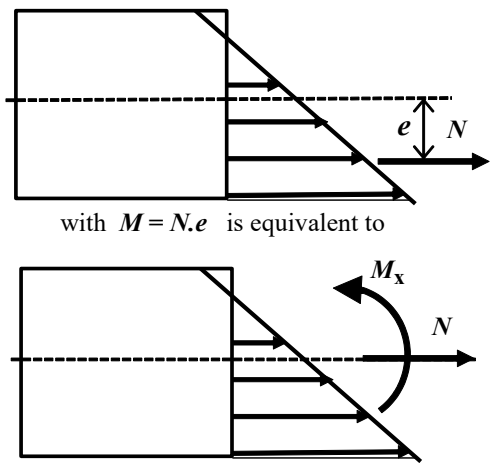
Consider a member under an eccentric axial load with eccentricity of e_y or e_x, or both.

This is equivalent to the member being subjected to a combination a centric axial load of N with bending moment of $M_x = Ne_y$ and $M_y = Ne_x$.



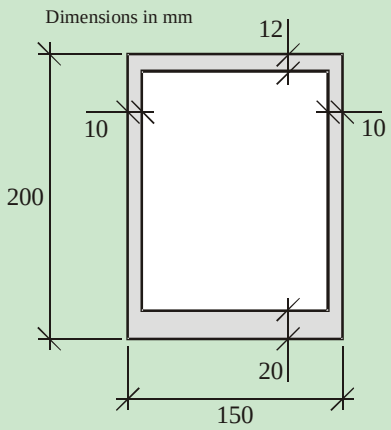
Be careful to identify the direction of the moments (being “sagging” or “hogging”) by inspecting both the direction of the axial force and the eccentricity. For example, both moments in the case shown in the figure below are “sagging” moments.

Subsequent analysis of the stresses can be carried out in the same way as the member being subjected to a combination of $N + Mx + My$.



Questions

1. A shaft is made as a thin cylindrical shell. It has a radius of 100 mm and a wall thickness of 8 mm.

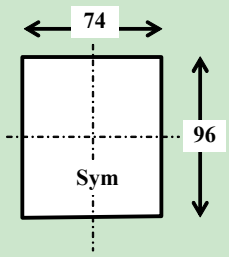
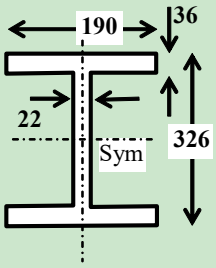
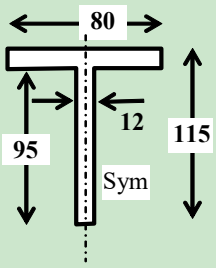


   Calculate the membrane stresses in the cylinder wall due to each of the following load cases, and sketch the stresses on a small element of the wall:-

   a) a tensile axial load of 930 kN;
   b) a torque of 40 kNm; and
   c) the above axial load and torque acting at the same time.

2. A short thin cylindrical aluminium tube with closed ends has a diameter of 600mm and wall thickness 5 mm. It is internally pressurised to a pressure of 2.7 MPa. It carries an axial compressive load of 1000 kN and a torque of 180 kNm. Calculate the membrane stresses in the cylinder wall, and sketch the state of stress on a small element of the cylinder wall.

3. The sections in the table below are subjected to combinations of axial force and bending moment.

	Rectangular section	I section	T section
			
A	7104 mm^2	19 268 mm^2	2740 mm^2
x_G	37 mm	95 mm	40 mm
y_G	48 mm	163 mm	81 mm
I_x	5.46×10^6 mm^4	3.19×10^8 mm^4	3.11×10^6 mm^4
I_y	3.24×10^6 mm^4	4.14×10^7 mm^4	8.67×10^5 mm^4

a) Rectangular section under axial tension of P = 500 kN and a major axis moment M_x = 11 kNm.

b) I section under axial compression of P = $(-)$2000 kN and a major axis momentM_x = 150 kNm.

c) I section under axial tension of P=1500kN and a minor axis moment My=70kNm.

d) I section under axial tension P=2000kN, major axis moment Mx=90kNm and minor axis moment My=40kNm

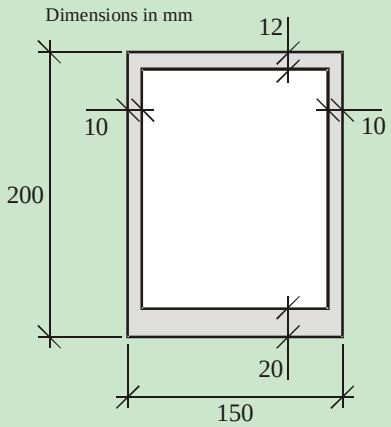
e) T section under axial compression of P = (-)200kN and a major axis moment Mx=10kNm.

f) T section under axial compression of P = (-)200kN acting eccentrically on the y-axis at 50mm above the centroid.

For each section:

i. calculate the maximum tensile and compressive stress;
ii. plot the stress distribution across the section (this is not required for bi-axial bending cases); and
iii. find the load factor against yield, if the yield stress is σ_Y = 250 MPa throughout.

4. The rectangular hollow section shown below is 2m long, and made from steel with shear modulus G = 81 GPa and yield strength $\sigma_Y = 250 N/mm^2$.



a) A clockwise torque T = 65 kNm is applied to the member.
   i. Calculate the distribution of shear stress, τ. On a sketch of the cross-section, draw the distribution of shear stress, and use arrows to show how the shear stress flows around the section.
   ii. Determine the angle of twist, γ, along the member.

b) A shear force V = 300 kN is applied to the member (without the torque). Calculate the distribution of shear stress, τ, resulting from this shear force. On a sketch of the cross-section, draw the distribution of shear stress, and use arrows to show how the shear stress flows around the section. Show the values of shear stress at significant points in the section.

c) Sketch the distribution of shear stress around the section if both the torque and shear force act at the same time. Annotate your sketch with values of shear stress.

Numerical solutions: Q1: (a) 185.02, 0, 0 MPa; (b) 0, 0, 79.58 MPa; (c) 185.02, 0, 79.58 MPa; Q2: -25.10, +162.0, 63.66; Q3: (a) 167.2, -26.4 MPa, 1.496 ; (b) -27.2, -180.4 MPa, 1.39; (c) 238.6, -82.9 MPa, 1.05 ; (d) 57.9, 241.6, -34.0, 149.7 MPa, 1.04; (e) -182.0, 187.6 MPa, 1.33; (f) same as (e); Q4: (a) (i) 1262 N/mm, 126, 105, 63 MPa; (ii) 1.9º; (b) 59.8, 98.2, 75.5 MPa; (c) 224.2, 27.8 MPa.

18 Stress Transformation

So far, we have only considered uniaxial stress and strain states within a structure. This is appropriate for many engineering structures, such as beams, but many other structures carry 2-dimensional or 3-dimensional states of stress. These lectures will examine the stresses and strains within 2D structures, constitutive (stress-strain) relationships in 2D, and failure criteria in 2D.

18.1 Stress in 2D

18.1.1 Background

A large amount of structural analysis and design can be based on uniaxial stress states. However, in most engineering situations multi-axial stress analysis is essential to answer questions such as:

- When will material loaded in 2D/3D yield?
- How can we visualise stresses?
- What are the multi-axial stress-strain relationships?
- How will material deform under multi-axial loading?

More complicated structures require the analysis of stresses and strains in 3-dimensions.

18.1.2 Stresses in 3D

Local values of force on an elemental area within a beam:
Direct or normal stress:

$$\sigma_z = \lim_{\Delta A \to 0} (\frac{P_z}{\Delta A})$$

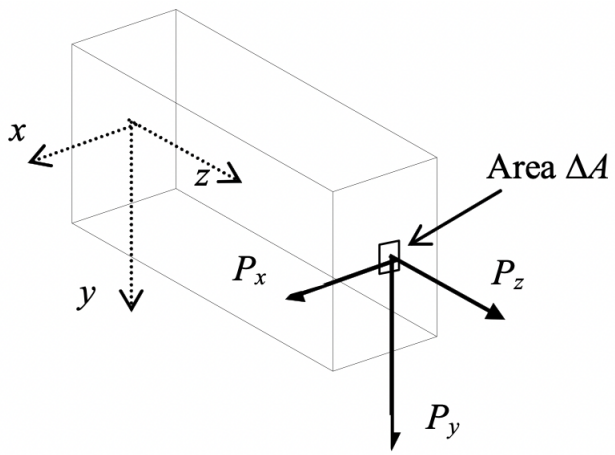
Shear stresses: τ_{zy}=

$$\lim_{\Delta A \to 0} (\frac{P_y}{\Delta A})$$

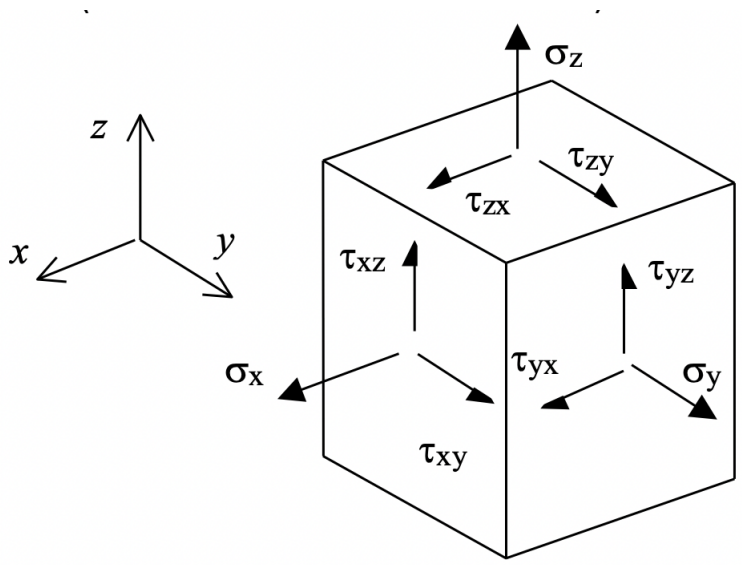
and τ_{zx}=

$$\lim_{\Delta A \to 0} (\frac{P_x}{\Delta A})$$

Units are N/m^2 = Pa or N/mm^2 = MPa



(Forces act on face cut across z axis) In the general case, both direct stresses and shear stresses can occur on each of the six faces of a cube.



For
textbfdirect or normal stresses, a subscript defines the direction in which it acts.

σ_x acts in the x direction.

For shear stresses, two subscripts are used. The first describes the face on which it acts, the second the direction it acts in.

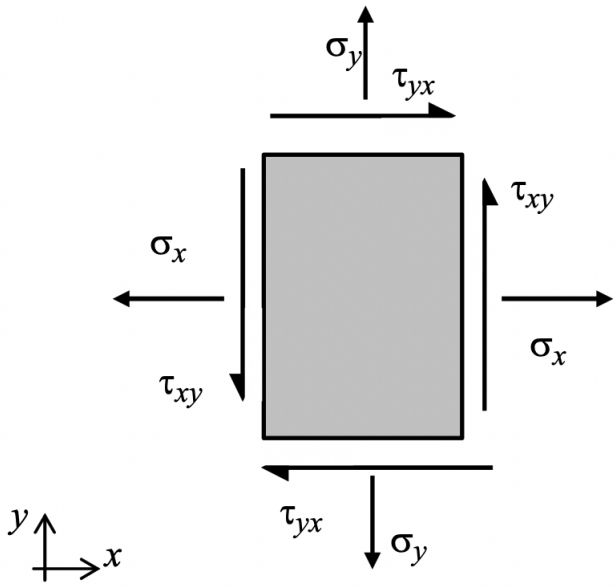
tau_{xy} acts on a face cut across the x axis and acts in the y direction.

18.1.3 Positive Values of 2D Stress

σ is positive in tension.

The positive direction of shear stress is more complicated:

τ is positive where it acts in a positive axis direction on a positive face.



(A 'positive' face has its outward normal in the positive direction).

By moment equilibrium of this element (Section 12.2), we can say:

$$\tau_{xy} = \tau_{yx}$$

NB: The sign conventions used in structures are the opposite of those used in geotech.

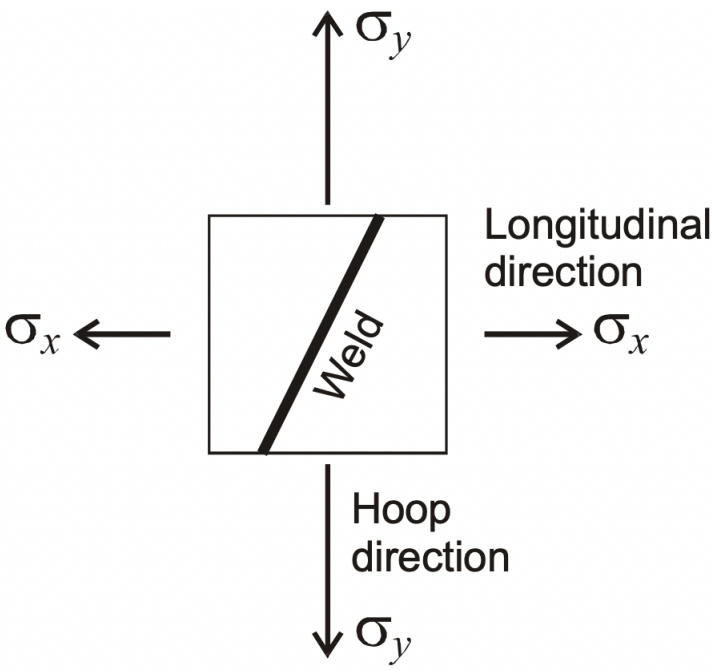
18.2 Relationships between stresses in two directions

18.2.1 Stresses in Different Orientations

It is often necessary to carry out a *Stress Transformation* to find the stresses that act in a different direction.

For example, pressure vessel are often made from helically wound strips of steel, which are welded together to form a cylinder. To find out whether the weld is strong enough, we need to find the direct stress and shear stress that cross it, which are perpendicular and parallel to the weld direction. We

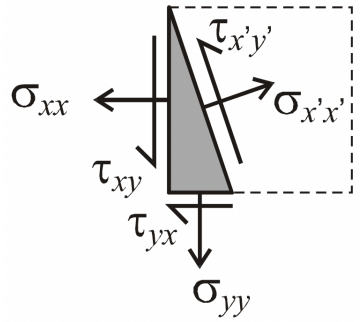
use a stress transformation to convert the longitudinal and hoop stresses in the (x, y) directions into stresses in the weld direction.



Stress transformations are also fundamental to material failure criteria for stresses acting in multiple directions (covered in a later lecture), and are an essential part of finite element analysis using computers.

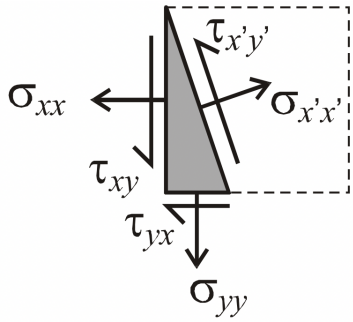
Derivation of the Stress Transformation Equations

Isolate two different free bodies from the original element by making cuts perpendicular to the x' and y' directions:



Cut perpendicular to x'

- Resolve perpendicular to the cut to find $\sigma_{x'x'}$
- Resolve parallel to the cut to find $\tau_{x'y'}$

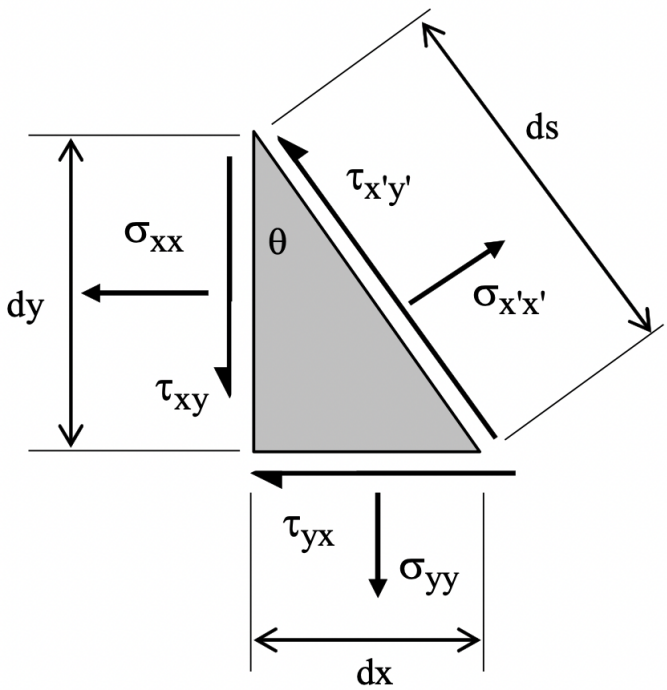


Cut perpendicular to y'

- Resolve perpendicular to the cut to find $\sigma_{y'y'}$
- Resolve parallel to the cut to find $\tau_{y'x'}$

Finding x'x' from equilibrium

Apply force equilibrium to the first free body:



Resolve forces in the x' direction (normal to the inclined cut surface):

$$\sigma_{x'x'} ds = \sigma_{xx} dy \cos\theta + \tau_{xy} dy \sin\theta + \sigma_{yy} dx \sin\theta + \tau_{yx} dx \cos\theta$$

where:

$$ds = \frac{dy}{\cos\theta} \quad and \quad dx = \frac{dy}{\tan\theta}$$

$$\therefore \quad \sigma_{x'x'} \frac{dy}{\cos\theta} = \sigma_{xx} dy \cos\theta + \tau_{xy} dy \sin\theta \quad + \sigma_{yy} dy \tan\theta \sin\theta + \tau_{yx} dy \tan\theta \cos\theta$$

Solving for $\sigma_{x'x'}$:

$$\sigma_{x'x'} = \sigma_{xx} \cos^2\theta + \tau_{xy} \cos\theta \sin\theta \quad + \sigma_{yy} \sin^2\theta + \tau_{yx} \sin\theta \cos\theta$$

$$\sigma_{x'x'} = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + \quad 2\tau_{xy} \sin\theta \cos\theta$$

where:

$$\cos^2 = \frac{1 + \cos 2\theta}{2}, \quad \sin^2 = \frac{1 - \cos 2\theta}{2}, \quad and \quad \cos\theta \sin\theta = \frac{\sin 2\theta}{2}$$

$$\therefore \quad \sigma_{x'x'} = \sigma_{xx} \frac{1 + \cos 2\theta}{2} + \sigma_{yy} \frac{1 - \cos 2\theta}{2} + \tau_{xy} \sin 2\theta$$

$$\sigma_{x'x'} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} + \frac{(\sigma_{xx} - \sigma_{yy})}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

The 2D Stress Transformation Equations

$\sigma_{y'y'}$ and $\tau_{x'y'}$ can be found in the same way as above, giving the complete 2D stress transformations for the stresses in any orientation:

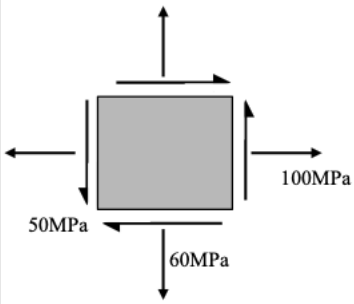
$$\sigma_{x'x'} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} + \frac{(\sigma_{xx} - \sigma_{yy})}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

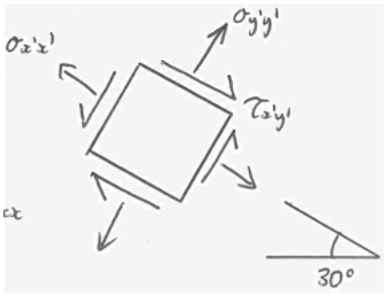
$$\sigma_{y'y'} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} - \frac{(\sigma_{xx} - \sigma_{yy})}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{(\sigma_{xx} - \sigma_{yy})}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

where θ is an anticlockwise rotation.

Example





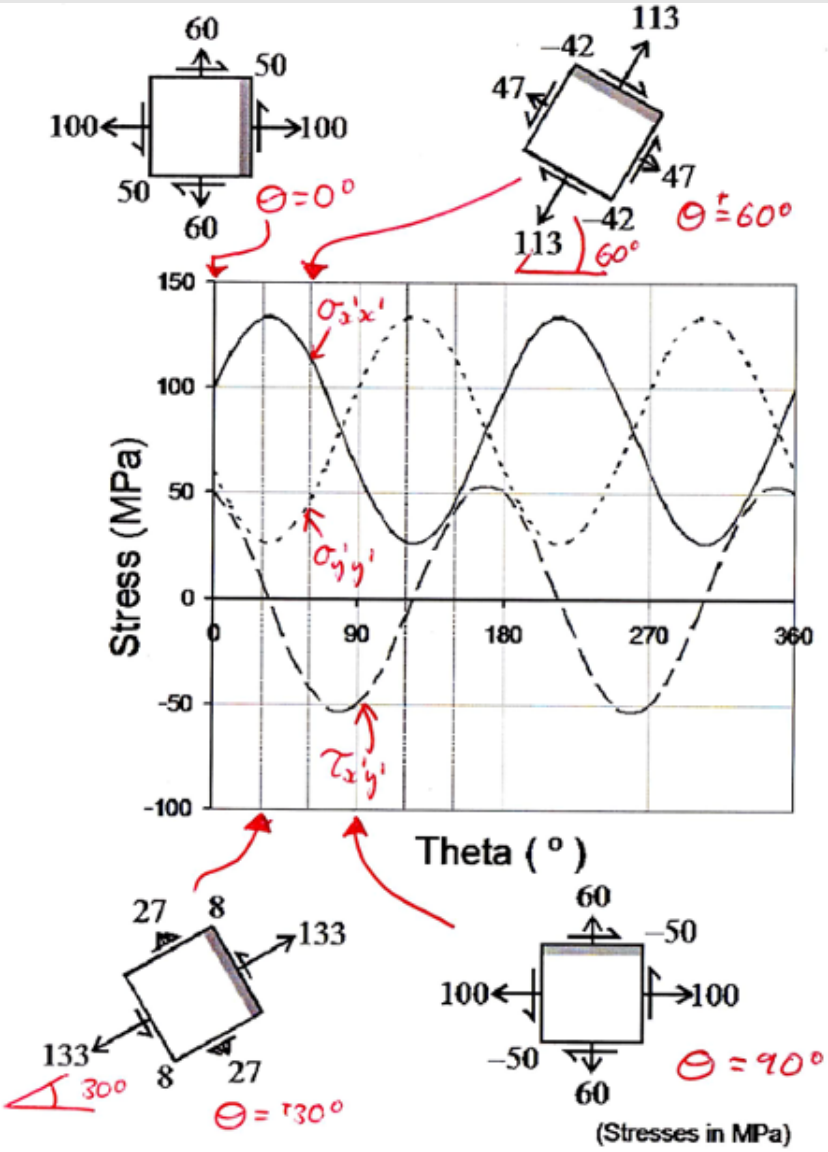
Calculate the stresses on an element that is rotated $\theta = -30^o$ clockwise from the x, y directions shown.

$\theta =$ (negative because θ is defined as anticlockwise positive):

$\sigma_{x'x'} =$

$\sigma_{y'y'} =$

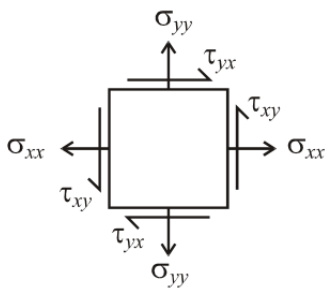
$\tau_{x'y'} =$


60
50
100
100
50
60
θ=0°
113
−42
47
47
−42
113
60°
θ=60°
150
100
50
0
-50
-100
Stress (MPa)
0
90
180
270
360
Theta (°)
σx'x'
σy'y'
τx'y'
27
8
133
133
8
27
30°
θ=30°
60
−50
100
100
−50
60
θ=90°
(Stresses in MPa)

18.3 Mohr's Circle for Stress

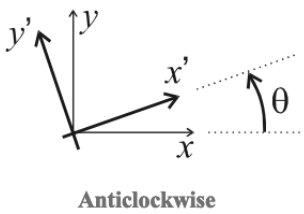
The known stress state is defined in the (x,y) axes by:

σ_{xx} σ_{yy} σ_{xy} σ_{yx}



Original element (known stress state)

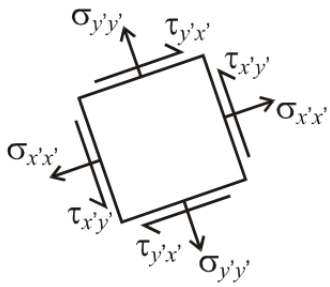
The original element is rotated through an <u>*anticlockwise*</u> angle of θ to get the transformed element. These elements are both the same piece of material in the structure, but with different orientations.



Axis rotation

The transformed stress state axes are (x', y') and the stresses acting on the transformed element are:

σ_{xx} σ_{yy} σ_{xy} σ_{yx}



Transformed element (required stress state)

$$\sigma_{x'x'} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} + \frac{(\sigma_{xx} - \sigma_{yy})}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'y'} = \frac{(\sigma_{xx} + \sigma_{yy})}{2} - \frac{(\sigma_{xx} - \sigma_{yy})}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

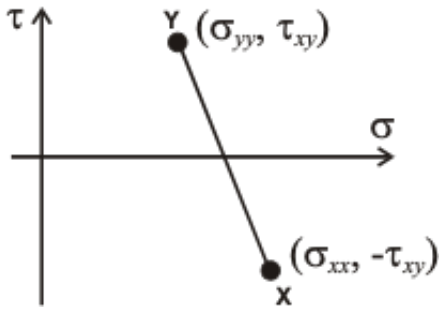
$$\tau_{x'y'} = -\frac{(\sigma_{xx} - \sigma_{yy})}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

where θ is an anticlockwise rotation.

We can plot the stress state on the side of the element on a graph of (σ, τ). In this plot (the Mohr's circle space), we define *clockwise* shear as positive (ie: positive on the σ_{yy} face), so we can plot two points to define the known stress state in an element of material:

X $(\sigma_{xx}, -\tau_{xy})$

Y $(\sigma_{yy}, +\tau_{xy})$



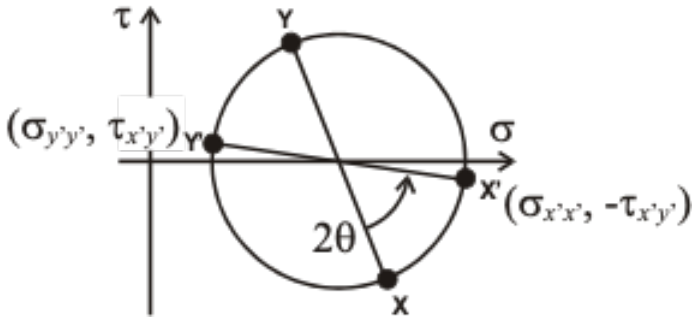
We can plot the transformed stress state on the same graph:

X' $(\sigma_{x'x'}, -\tau_{x'y'})$

Y' $(\sigma_{y'y'}, +\tau_{x'y'})$

The $\cos 2\theta$ and $\sin 2\theta$ terms mean that these points always lie on a circle in the (σ, τ) plot. This is *Mohr's Circle*.

If the element is rotated by θ in *real space*, the transformed stress state is found by rotating by an angle 2θ (in the same direction) around the Mohr's Circle.



18.3.1 Derivation of Mohr's circle from the stress transformation equations

Mohr's circle can be derived algebraically by substituting...

$$C = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) \quad \alpha = \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right) \quad \beta = \tau_{xy}$$

... into the stress transformation equations, giving:

$$(\sigma_{x'x'} - C) = \alpha \cos 2\theta + \beta \sin 2\theta$$

$$(\sigma_{y'y'} - C) = -\alpha \cos 2\theta - \beta \sin 2\theta$$
$$\tau_{x'y'} = -\alpha \sin 2\theta + \beta \cos 2\theta$$

The 1st and 3rd (or the 2nd and 3rd) equations describe a circle when plotted in (σ, τ) space, with radius $R = \sqrt{\alpha^2 + \beta^2}$:

$$(\sigma_{x'x'} - C)^2 + \tau^2_{x'y'} = (C + \alpha \cos 2\theta + \beta \sin 2\theta - C)^2 + (-\alpha \sin 2\theta + \beta \cos 2\theta)$$
$$= \alpha^2 \cos^2 2\theta + 2\alpha\beta \sin 2\theta \cos 2\theta + \beta^2 \sin^2 2\theta + \alpha^2 \sin^2 \theta - 2\alpha\beta \sin 2\theta \cos 2\theta + \beta^2 \cos^2 2\theta$$
$$= \alpha^2(\cos^2 2\theta + \sin^2 2\theta) + \beta^2(\sin^2 2\theta + \cos^2 2\theta)$$
$$= \alpha^2 + \beta^2$$

So $(\sigma - C)^2 + \tau^2 = R^2$

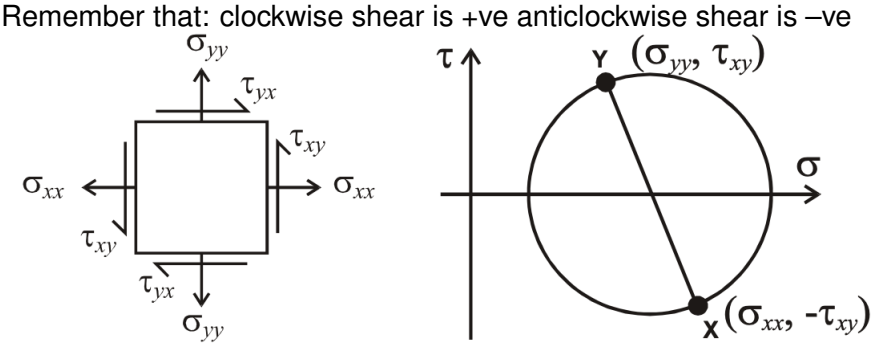
Note that the stress transformation equations are in terms of 2θ -this is why a rotation of θ gives a rotation of 2θ on the Mohr's Circle.

18.3.2 Using Mohr's Circle

Mohr's Circle is a far more convenient and reliable method for stress transformation than applying the equations directly. The graphical construction means that it is easier to spot calculation errors.

Plot the known stress state

Remember that: clockwise shear is +ve anticlockwise shear is –ve



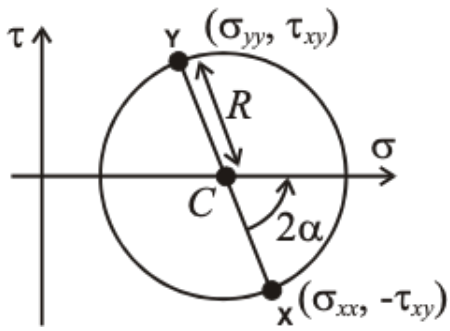
Draw the diameter **X - Y**, and sketch the Mohr's Circle.

Define the geometry of the circle

The geometry of the Mohr's Circle is described by:

- the centre, C
- the radius, R
- the angle of the diameter **X-Y** to the horizontal, 2α

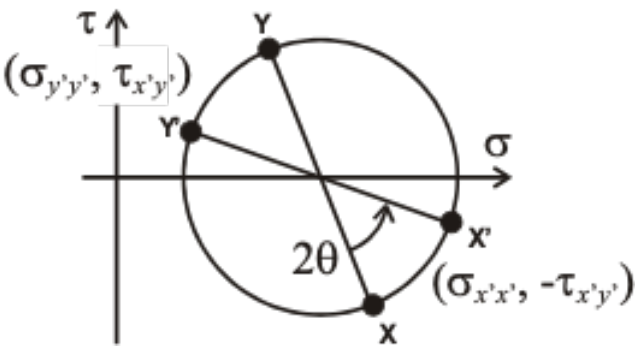
You do not need to memorise equations for these. Sketch the Mohr's circle, then use simple trigonometry to find them.



<u>Find the transformed stress state</u>

The transformed stress state is found by rotating through an angle 2 in the same direction as the rotation in real space. Mark the points **X'** and **Y'** on Mohr's circle.

The transformed stresses $\sigma_{x'x'}, \sigma_{y'y'}, \tau_{x'y'}$ are again found from the geometry of the circle.



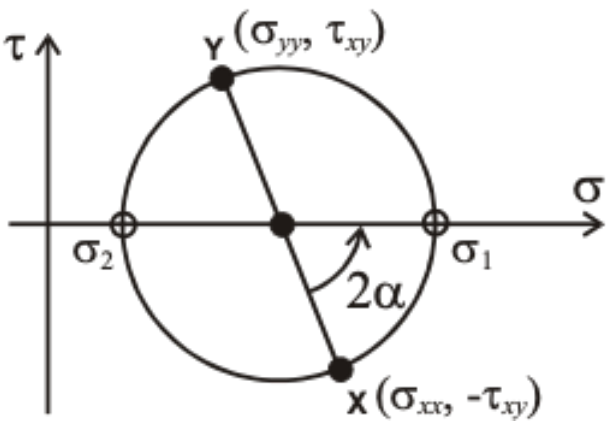
18.3.3 Summary of sign conventions for drawing Mohr's circles of stress

σ is +*ve* tensile τ is +*ve* clockwise (i.e. usual shear on x-faces is plotted negative) θ is +*ve* anticlockwise **A rotation of θ in real space gives a rotation of 2θ on the Mohr's circle.**

The sign convention for shear (clockwise positive in Mohr's circle space) ensures that the rotation in real space and Mohr's circle are in the same direction. These conventions are used in almost all engineering (except for Soil Mechanics and Geology, where compression is positive).

18.4 Principal Stresses

There are two points on a Mohr's Circle where the shear stress, τ, is zero, and these correspond to the maximum and minimum values of direct stress, σ.

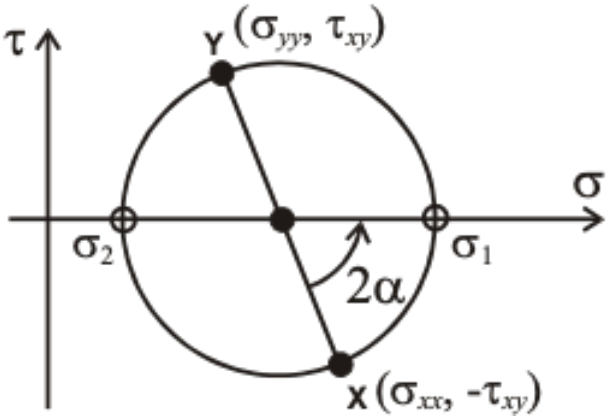


These direct stresses are called the Principal Stresses.

- σ_1 is the most tensile principal stress
- σ_2 is the most compressive principal stress

(σ_1 might be compressive, but is always more tensile than σ_2; σ_2 might be tensile, but is always more compressive than σ_1).

The stresses σ_1 and σ_2 occur in two directions, at anticlockwise rotations α from the x and y axes.



These special directions are the *Principal Directions*, and the planes on which the principal stresses act are the *Principal Planes*.

The angle α is found using trigonometry (see above):

$$\tan 2\alpha = (\frac{\tau_{xy}}{\sigma_{xx} - C}) = (\frac{2\tau_{xy}}{\sigma_{xx} - \sigma_{yy}})$$

Note that $\tan 2\alpha$ has multiple solutions, at intervals of $2\alpha = 90^o$.

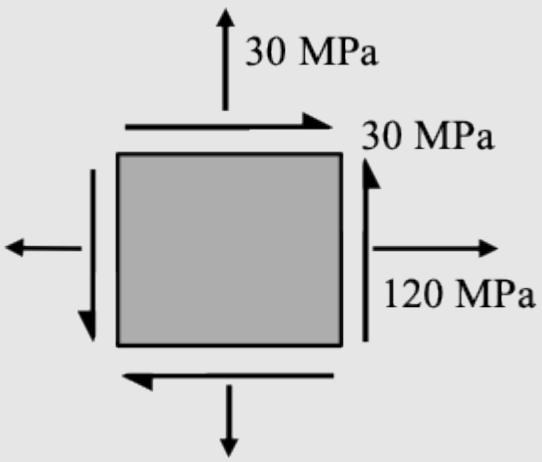
Thus, the equation finds the direction of one of the principal stresses relative to the *x* direction. To find out *which* of the principal stresses this corresponds to, it is always best to check the result of the calculation by inspection of Mohr's Circle.

Check the direction of α using Mohr's Circle.

Example

For the stress state shown in the figure below, determine:

(a) The principal stresses and directions
(b) The maximum shear and its coexistent direct stress and orientation
(c) The stresses on a plane rotated 15° clockwise from the x, y directions.



Principal stresses and their directions

First sketch the Mohr's Circle, then use trigonometry to define its geometry.

Centre of circle, **C =**

Radius of circle, R =

Most tensile principal stress,

σ_1 =

Most compressive principal stress,

σ_2 =

Direction of σ_1,

α =

Maximum shear stress, its coexistent direct stress and orientation

τ_{max} =

Coexistent direct stress $\sigma_{\tau max}$ =

Orientation from x is the angle

β =

β=

The stresses on planes at an angle of 15° clockwise from *x*, *y*

Sketch the circle: 15° clockwise in space is 30° clockwise on the circle or θ= -15°.

α =

$\sigma_{x'x'}$ =

$\sigma_{y'y'}$ =

$\tau_{x'y'}$ =

18.5 Failure of materials in 2D and 3D

18.5.1 Yield Criteria in 2D and 3D

We have so far only considered the idea of failure of a material by yield in a 1D tension test, with only direct stress applied. The yield stress is the "uniaxial tension yield stress". It is often necessary to know when a material will yield if there are stresses acting in more than one direction. We need theories of material failure under *multi-axial stresses*. In this lecture we will consider two multi-axial failure criteria: von Mises and Tresca.

18.5.2 von Mises Criterion for Yield in 2D and 3D

The most widely used criterion for metals is the von Mises Yield Criterion, also attributed to Hencky and to Huber (so their names are sometimes used instead).

Yield occurs under 2D stresses when:

$$\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2 = \sigma_Y^2$$

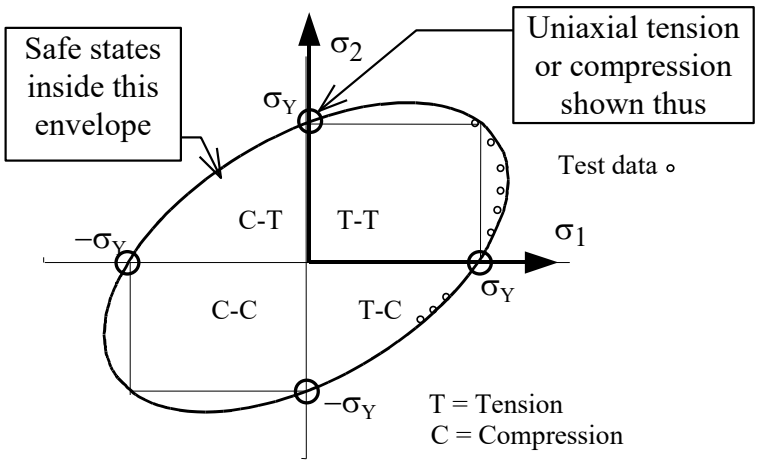
σ_{eq} is called the von Mises equivalent stress, where:

$$\sigma_{eq} = \sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}$$

Given a set of stresses σ_1 and σ_2, the load factor λ required to produce yield is given by:

$$\lambda = \frac{\sigma_Y}{\sigma_{eq}} = \frac{\sigma_Y}{\sqrt{\sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2}}$$

Yield criteria can be plotted on axes of σ_1 and σ_2, to create a "yield surface". Any stress state in 2D can be plotted as a single point on this graph, represented by its principal stresses, so test data describing the point at which yield occurred can be plotted here.



Test data tends to fall just slightly inside the von Mises envelope, making it very slightly unsafe, but it is the most accurate general 2D yield criterion.

von Mises criterion is based upon the assumption that a material has a certain capacity to absorb energy by distortion (and not by being hydrostatically compressed).

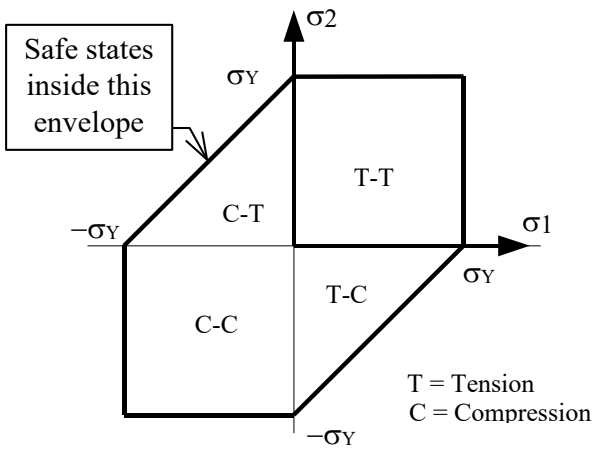
In general, stresses can act in three dimensions. If the principal stresses in the three principal directions are σ_1, σ_2 and σ_3, then yield occurs when:

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_Y^2$$

When $\sigma_3 = 0$, this reduces to the 2D equation above. The yield surface can be drawn in three dimensions with axes of σ_1, σ_2 and σ_3.

18.5.3 Tresca's Yield Criterion for yield in 2D and 3D

An alternative (sometimes simpler) criterion for yield of the material was devised by Tresca. This criterion assumes that yield under 2D stresses occurs when the shear stress on any plane reaches a critical value, which is the same as that for uniaxial tension.



Yield occurs under 2D stresses when

$$\max\left(|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\right) = \sigma_Y$$

The calculated value σ_{eq} is called the Tresca equivalent stress:

$$\sigma_{eq} = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\right)$$

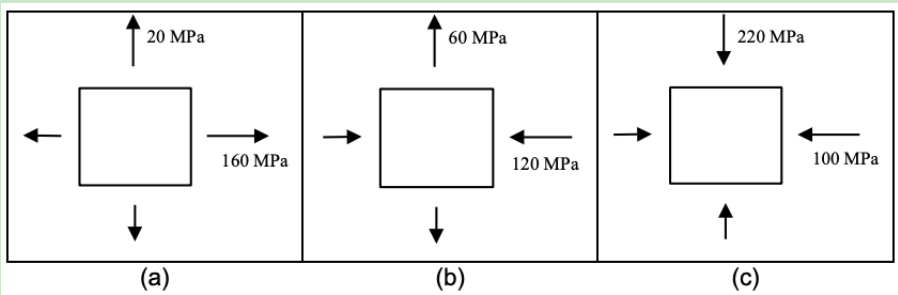
Given a set of stresses σ_1 and σ_2, the load factor λ required to produce yield is given by:

$$\lambda = \frac{\sigma_Y}{\sigma_{eq}} = \frac{\sigma_Y}{\max\left(|\sigma_1|, |\sigma_2|, |\sigma_1 - \sigma_2|\right)}$$

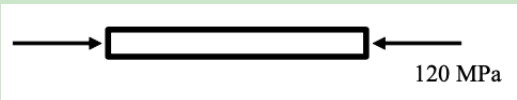
This criterion is not as accurate as von Mises for metals, but it can be useful because it is sometimes simpler (if we know that both stresses are tensile, then yield is simply when the larger one reaches σ_Y); and it is conservative (i.e. a safe assumption) since it normally underestimates the real strength.

Questions

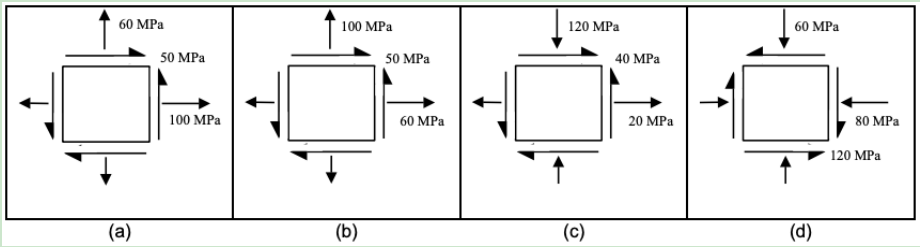
1. For each of the stress states shown below:
   - sketch Mohr's circle;
   - find the maximum (most tensile) principal stress σ_1;
   - find the minimum (most compressive) principal stress σ_2;
   - find the maximum shear τ_{max}; and
   - identify the direction of the most tensile principal stress.



2. A bar carries an axial compressive stress of 120 MPa, as shown below. Sketch Mohr's circle for the bar, and determine the maximum shear and the orientation of the plane on which it acts. (This is a typical failure mode for a brittle material such as cast iron carrying compression).

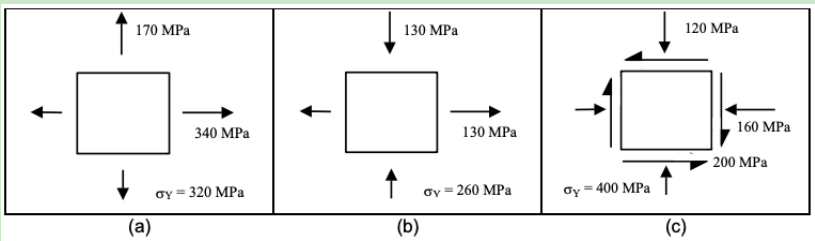


3. For each of the stress states shown below:
   - sketch Mohr's circle;
   - find the maximum (most tensile) principal stress σ_1 and its orientation in space relative to the x direction (clockwise or anticlockwise?);
   - find the minimum (most compressive) principal stress σ_2 and its orientation in space relative to the x direction;
   - deduce the maximum shear τ_{max} and its orientation in space relative to the x direction.



4. For each of the following stress states, find the load factor on first yield, λ, according to:
   (i) von Mises criterion
   (ii) Tresca's criterion

   State whether the criterion suggests that yield has occurred or not.

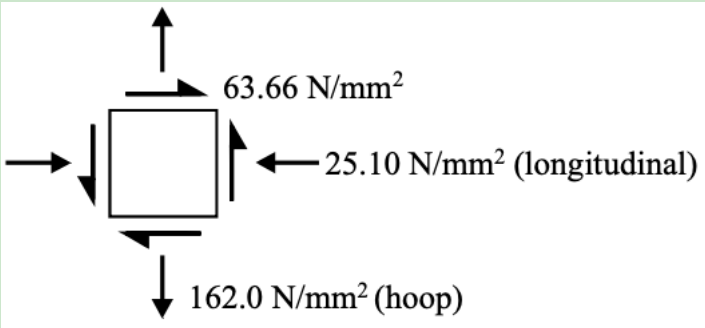


5. A thin-walled spherical vessel contains a gas pressurised to 800 kPa. The spherical vessel has a radius of 3 m, and a wall thickness *t*. It is made from steel with a yield stress of σ_Y = 275 MPa.
   (a) Determine the membrane stresses acting in the wall of the pressure vessel in terms of the wall thickness, *t*. (Consider equilibrium of a suitable portion of the pressure vessel).
   (b) Sketch the stress state in a small element of the wall.
   (c) Use von Mises yield criterion to determine the wall thickness, *t*, that will give a load factor against yield of λ= 2.0.

6. A cylindrical vessel is being designed with spherical end closures. The cylinder radius is 3600mm and it is to be made from steel with a yield stress of σ_Y = 260 MPa. The design pressure is 250 kPa and a load factor of λ= 1.8 is required against first membrane yield according to von Mises. Calculate the required thickness of the cylinder, and the required thickness of the spherical ends.

Note that unless the radial strain in the cylinder and end caps is matched (as in question 1), bending stresses will develop at the junction between the cylinder and sphere, and these will cause first yield at a lower pressure than the membrane theory calculation predicts.

7. In SM2 tutorial 7, question 3, you found the stress state in the wall of a cylindrical aluminium tube subject to combined internal pressure, axial compression and torque. This is shown below. For this stress state, sketch the Mohr's circle, and deduce the principal stresses and their orientation relative to the longitudinal axis. If the yield stress is 230MPa, find the load factor against failure, λ, according to von Mises criterion. Is the safety margin large enough?



Q1: (a) +160,+20,+70 MPa, x; (b) +60,-120,+90 MPa, y; (c) -100,-220, 60 MPa, x. **Q2**: 60 MPa, 45°; **Q3**: (a) +133.9MPa, 34.1^{o}a, +26.1MPa, 124.1^{o}a=55.9^{o}c; 53.9MPa, 79.1^{o}a (b)+133.9MPa, 55.9^{o}a, +26.1MPa, 34.1^{o}c; 53.9MPa, 100.9^{o}a; (c) +30.6MPa, 14.9^{o}a 130.6MPa, 75.1^{o}c, 80.6MPa, 59.9^{o}a; (d) +50.4MPa, 47.4^{o}c, -190.4MPa, 42.6^{o}a, 120.4MPa, 2.4^{o}c; **Q4**: (a) 1.087 n, 0.941 y; (b) 1.155 n, 1.000 y; (c) 1.066 n, 0.995 y; **Q5**: (a) σ =pr/(2t), (c) 8.7mm; **Q6**: 5.4, 3.1mm; **Q7**: 25.10, +162.0, 63.66, 181.61, -44.71MPa, +17.12^o, 1.108, No;