CS 375 – Analysis of Algorithms

Professor Eric Aaron

Lecture – M W 1:00pm

Lecture Meeting Location: Davis 117

Designing a *Reduction* Algorithm: A Simple Example

- Sometimes, we can design an algo to *reduce* a problem *A* to a problem *B*
  - Such an algorithm is called a *reduction* from *A* to *B*
  - A reduction from *A* to *B* is a kind of algo that solves *A* by using a subroutine that solves *B*—thus reducing *A* to *B*
- Here's a *very* simple example. Consider problems *A* and *B* with these specifications:

A:
- **Input: List L = [c_1 … c_n] of numbers.**
- **Output: *True* if the first element of L is 375; False otherwise.**

B:
- **Input: List M = [d_1 … d_k] of numbers.**
- **Output: *True* if the last element of M is 375; *False* otherwise.**

- What's an algorithm that would solve *A*, using a solution for *B* as a subroutine?

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- **Output: *True* if the last element of M is 375; *False* otherwise.**

- What's an algorithm that would solve *A*, using a solution for *B* as a subroutine?

Here's one possible reduction from *A* to *B*:
- **From input L to *A*, create list M = [c_1]**
- **Run a subroutine for *B* on input M. If it returns True, your algorithm for *A* should return True; if it returns False, your algorithm for *A* should return False.**

Can you analyze the complexity and explain the correctness of this reduction algorithm?

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You'll be designing a reduction for Project 2! Some Important notes:
- **This example may be much simpler than the reduction you design!**
- **Your write-up of your reduction will need to be more detailed than the notes on the previous slide are.**
- **As always, please feel free to ask me questions!**

Business

- Grading update:
  - SA2, SA3 in progress
- PS2 out, extended deadline: due Oct. 26
- Project 2 Lookahead out
  - Plenty to get started on!
  - First part due Oct. 24
  - Please read over instructions and let me know if there are any questions!
- The full Proj2 assignment document that I post will be lengthy
  - It is meant to be a *teaching* and *reference* document in places, with many hints and specific examples
  - I hope you find it useful!
- Please email me with Proj2 teams by end of day today!

Business: Project 2

- Project 2 out *very soon*
  - Multi-stage project, with final due date in early November
- **Project 2 is to be done in teams of 4**
  - If you'd like my help finding a team for you, please let me know!
- Parts of Project 2:
  1. **Design Exhaustive Search Algorithms**: Your team will collectively design exhaustive search algorithms for 8 problems.
  2. **Improve Time Efficiency**: Your team will pick one of the problems and make your exhaustive search algorithm more efficient.
  3. ***Reduction***: For the same problem chosen for part 2 above, you will *reduce* that problem to one of the other seven problems from part 1.
  4. **Create and Give a Presentation**: Your team will present work from the previous three parts of the assignment, *using loop invariants* where appropriate to explain correctness.

Hint: Your team may want to be strategic about which of the 8 problems you choose to focus on for your improvements, reduction, and presentation. Pick a problem for which you can do good work!

The green-shaded ones are examples of *polynomial time* classes—upper bounded by n^k for some constant *k*. Problems solvable in polynomial time are considered *tractable.* (More about this later in the semester!)

Time Complexity Classes Illustrated!

Complexity Class	*What we call it*	*Example algorithms / objects*
O(1)	Constant	Print "Hello, World!"; stack operations [and much, much more—be careful!]
O(lg n)	Log time	Binary search
O(n)	Linear	Exhaustive search of an array (linear search); Merge (as used in Mergesort)
O(n lg n)	n lg n	Mergesort; Heapsort [Recall: sorting can be done in θ(n lg n)]
O(n^2)	n-squared; quadratic	Insertion / selection / bubble sort; several graph algos
O(n^3)	n-cubed; cubic	*My favorite algorithm*! (a graph algo)
O(2^n)	Exponential	Number of *subsets* of a set of size n
O(n!)	Factorial	Number of orderings / permutations of elements of a list of length n

O(lg n): Binary Search

- *Binary search*: Divide-and-conquer search algorithm on arrays
  - Designed for *sorted* arrays—uses fact that array is sorted for more efficient algorithm
- Are you familiar with this algorithm?
  - How could search be made more efficient on a sorted array?

Problem:

Input: sorted array *A*,
value *v* for which to search

Output: index *i* such that *v* = *A*[*i*]
or the special value NIL if
v* does not appear in *A

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A quick vocabulary note:

Divide-and-conquer **refers to algorithms that break problems down into subproblems of the same type**

We'll go into this more when we look at recursion—recursive algos are often divide-and-conquer!

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  L = 0; R = n-1
  while L ≤ R do
    mid = (L+R)/2 # int division
    if v == A[mid]
      return mid
    elif v > A[mid]
      L = mid + 1
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What would a complexity argument be for this algorithm?

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What would a complexity argument be for this algorithm?

- How many iterations through the while loop?
- How much work done each iteration?

Worst case time complexity: O(lg n)

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```

What could a ***loop invariant*** be, for a correctness argument for algorithm?

Correctness: Binary Search

- *Binary search*: How would we explain its correctness?

(Recall the text in this text box from a previous slide…)

To (informally) use loop invariants to help explain algo correctness:

- **Explain how the invariant is true before the first iteration of the loop**
- **Explain how the invariant is true after each following iteration**
- **Explain how the invariant property shows that the algo meets its specifications**

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```

What could a ***loop invariant*** be, for a correctness argument for algorithm?

One possibility:

- **A is unchanged from original input**
- **v may occur in A[L..R], but not elsewhere in A**

<u>Recall problem specifications</u>:

- Input: sorted array *A*, value *v*
- Output: index *i* such that *v = A[i]*, or the NIL if *v* does not appear in *A*

O(lg n): Binary Search

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- Recursion warmup! What's a recursive algo for binary search?

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```
Algorithm: BinSrch(A,v,low,high)
 if low > high
  return False
 else
  mid = (low+high)/2 # int division
  if v == A[mid]
   return True
  elif v > A[mid]
   return BinSrch(A,v,mid+1,high)
  else # must be v < A[mid]
   return BinSrch(A,v,low,mid-1)
```

Problem (*modified!*):

Input: sorted array *A*,
value *v* for which to search,
integers *low* and *high* to specify range of *A* in which to search

Output: True if *v* is an element of *A[low.. high]*,
False otherwise

O(lg n): Binary Search

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```

What would the initial call to this function be, to find *v* in all of *A*?

You may have noticed the specification for this is different from the original spec'n for the search problem!

We could use a *wrapper* function to make this work with the original specification

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```

What would the initial call to this function be, to find *v* in all of *A*?

Note: It's the same sequence A each time. Copying or altering A (with, e.g., list slicing) would take extra time

Important: The recursive cases bring the sub-problems closer to the base case where low > high

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 return NIL
```

Is the complexity of this recursive version different from the complexity of the iterative version?

O(lg n): Binary Search

- *Binary search*: Divide-and-conquer search algorithm on arrays
  - Designed for *sorted* arrays—uses fact that array is sorted for more efficient algorithm
- Complexity analysis: In the worst case, for input A of size n, there are lg(n) recursive calls and O(1) work each time

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 L = 0; R = n-1
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Worst case time complexity: θ(lg n)

Are the ideas about complexity on this slide new to you?

We'll talk *much* more about them as the semester goes along!

This isn't Mergesort! It's just the Merge algo used in Mergesort

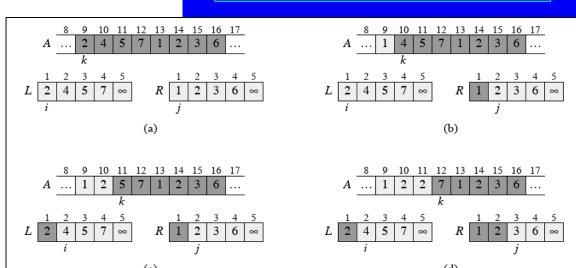
See CLRS, pg. 31

O(n): Merge

- See CLRS page 34 for specfications / loop invariant
  - In the function call *Merge(A,p,q,r)*, what are *A, p, q,* and *r*?

```
MERGE(A, p, q, r)
n1 = q - p + 1
n2 = r - q
let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
for i = 1 to n1
    L[i] = A[p + i - 1]
for j = 1 to n2
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
```

To me, one cool thing about this algo is how effective it can be to do a constant amount of work for each element in the input!
(Sorta like with stack operations—a powerful data structure with fast operations!)



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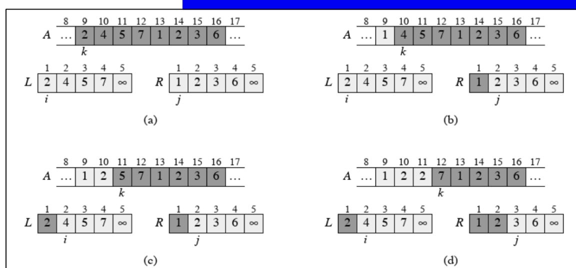
- How would you explain the time complexity of this algorithm?
- What is the algorithm's space complexity?



See CLRS, pg. 34

O(n lg n): Mergesort

- Mergesort is a *classic* example of an *n lg n* algorithm
  - Algo idea: Repeatedly split input list in half, sort each half separately, then Merge the two halves together
  - Uses the *Merge* function as a subroutine
  - Pretty fast algo, because *Merge* is O(n)
- Pseudocode, from CLRS:

```
MERGE-SORT(A, p, r)
  if p < r
      q = ⌊(p + r)/2⌋
      MERGE-SORT(A, p, q)
      MERGE-SORT(A, q + 1, r)
      MERGE(A, p, q, r)
```