ECS 122A Lecture 4

Divide + Conquer

Closest pair of points in 2D

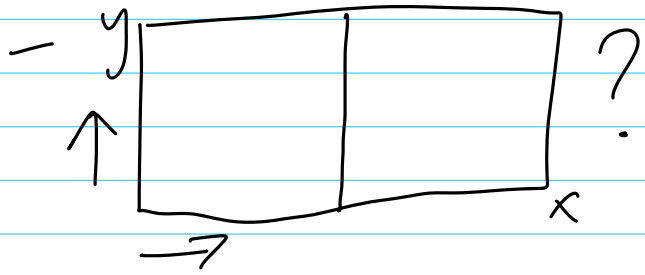
Trivial $O(n^2)$ alg

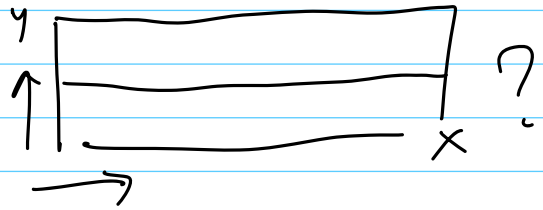
Given list of n points (x_i, y_i)

Find the pair closest to each other.

Want to improve to $O(n \log n)$ time.

How to divide?





- For $O(n \log n)$ time, want split into $\frac{n}{2}, \frac{n}{2}$ halves.

Split by median in x-coordinate
(can do y too)

Observation (case analysis):

Closest pair either
- both in left half ← recursion

- both in right half
- one left one right ~ needs work

Alg outline:

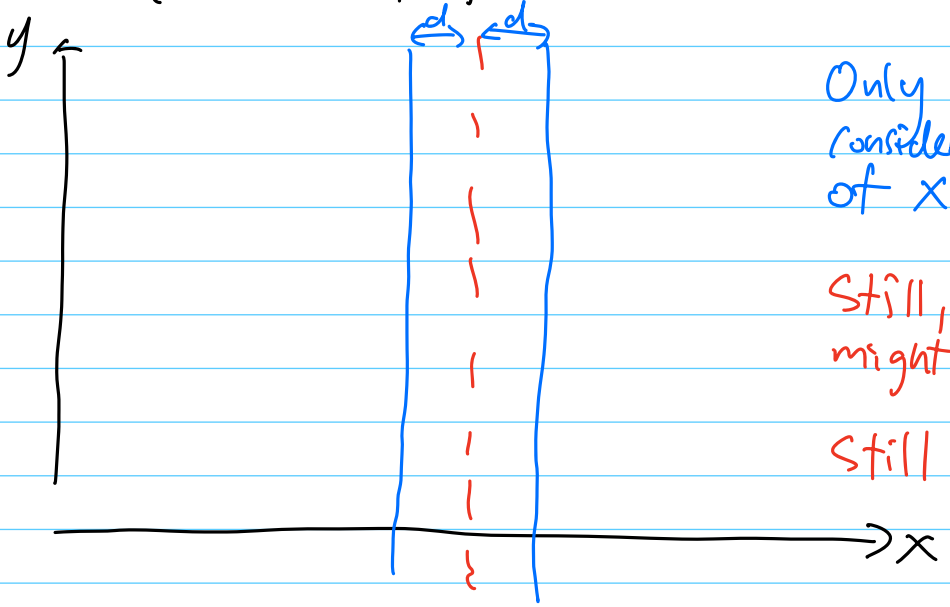
```
function closest(List A)
    Find median, split into left, right halves
    d_L = closest(left half)
    d_R = closest(right half)

    // Compute one left one right case
       as d_cross in O(n) time
return min(d_L, d_R, d_cross)
```

How to compute one left one right case?

Trivial: $O(n^2)$ ⇒ total runtime >> $O(n \log n)$

The alg is clever geometry. Good to see

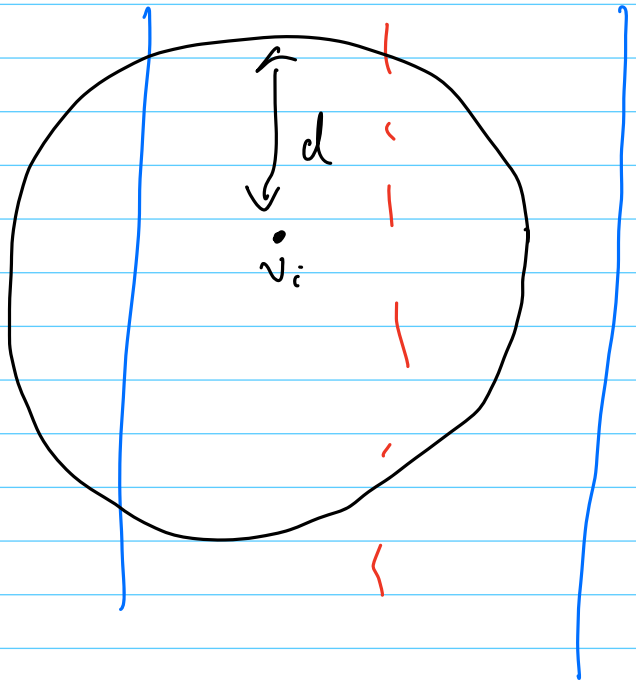
Let $d = \min(d_L, d_R)$



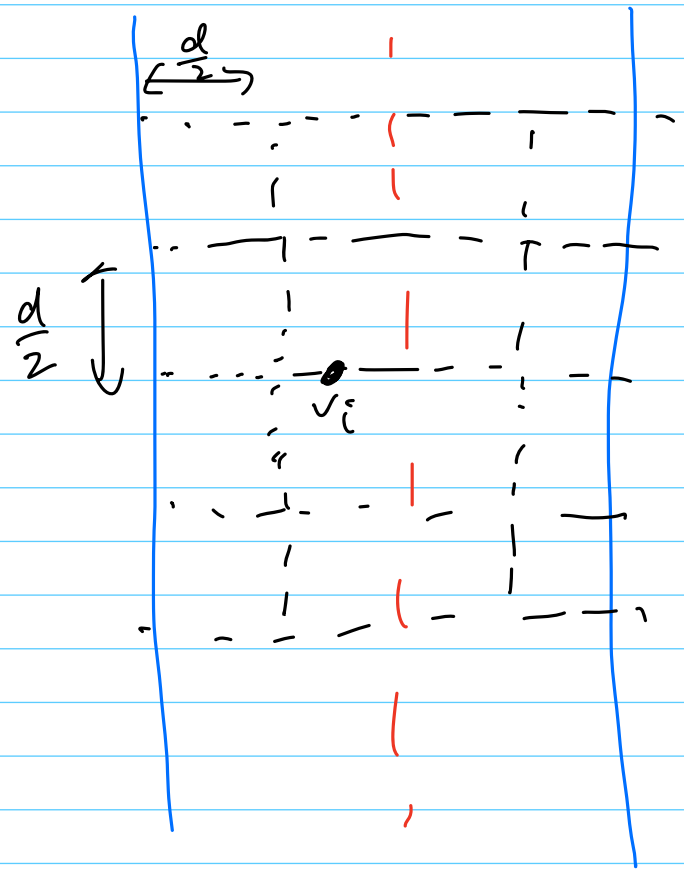
Only need to consider strip of X-median ± d

Still, every point might be in strip

Still $O(n^2)$...



Only need to consider radius-d ball around v_i



This grid covers the radius-d circle around v_i

At most 1 point in each square since square has diameter/diagonal $d/\sqrt{2}$

Only need to consider 15 closest points in y-coordinate in strip

Actually only 7 points, if we sweep from small to large y.

function closest(List A)
Find median, split into left, right halves
d_L = closest(left half)
d_R = closest(right half)
Sort points in A in y-coordinate, new list A' → filter to strip of $\min(d_L, d_R)$
for each v_i in A', compare w/ next 7 points in A'. ⇒ Compute min dist d_{cross}
return $\min(d_L, d_R, d_{cross})$.

$O(n \log n)$ by sort, though clever algs for $O(n)$

$O(n \log n)$

Runtime? $O(n \log^2 n)$... not quite.

If only we can sort in $O(n)$ time...

Observation: closest has exactly the same recursive structure as mergesort!

☆ Do both at the same time

(compute closest, merge in x-coord & y-coord)

Exercise to figure out details.

Strassen's Algorithm

Given 2 nxn matrices A, B, compute

$$C = AB$$

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad B = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}$$

$c_{ij} = \langle a_i, b_j \rangle$ ~O(n) time

Naive computation is $O(n^3)$

Matrix multiplication inherently has DfC structure

$$A = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \qquad B = \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}$$

$$= \begin{array}{|c|c|} \hline A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{22}B_{12} + A_{22}B_{22} \\ \hline \end{array} = C$$

C_{11} C_{12} C_{21} C_{22}

8 $\frac{n}{2} \times \frac{n}{2}$ multiplications

Runtime recurrence:

$$T(n) \approx 8\,T\left(\frac{n}{2}\right) + c_1 \cdot n^2$$

Can prove that $T(n) \geq \Omega(n^3)$

This doesn't help

Strassen's observation: (Magic)

Do 7 clever multiplication instead (plus more $O(n^2)$ arithmetic)

$$M_1 = (A_{11} + A_{22})(B_{11} + B_{22})$$

$$M_2 = (A_{21} + A_{22})\,B_{11}$$

$$M_3 = A_{11}\,(B_{12} - B_{22})$$

$$M_4 = A_{22}\,(B_{21} - B_{11})$$

$$M_5 = (A_{11} + A_{12})\,B_{22}$$

$$M_6 = (A_{21} - A_{11})(B_{11} + B_{12})$$

$$M_7 = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix}$$

$T(n) \leq 7\,T(\frac{n}{2}) + cn^2$

$\Rightarrow$ Can show that $T(n) \leq c' n^{\log_2 7} - cn^2$

↑ sufficiently large

$\log_2 7 < \log_2 8 = 3$

Fun exercise:

Given complex #s

$a+bi$, $c+di$

compute product using 3 real number multiplications.