

To apply **Lagrange Multipliers** in Optimizing Problem

1.Theorem needed

Lagrange Multipliers

Calculate the maximum of $F(x_1, x_2, \dots, x_n)$ given that $f(x_1, x_2, \dots, x_n) = 0$:

Define that

$$\mathcal{L} = F(x_1, x_2, \dots, x_n) + \lambda f(x_1, x_2, \dots, x_n)$$

Then

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial F}{\partial x_1} + \lambda \frac{\partial f}{\partial x_1} = 0$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial F}{\partial x_2} + \lambda \frac{\partial f}{\partial x_2} = 0$$

...

$$\frac{\partial \mathcal{L}}{\partial x_n} = \frac{\partial F}{\partial x_n} + \lambda \frac{\partial f}{\partial x_n} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = f(x_1, x_2, \dots, x_n) = 0$$

Calculate the equations above, then we will get the Maximum of F given the constraints f .

2.Optimizing in Consumer's choice

A consumer's utility function of a specific consumer is

$$U(x_1, x_2) = x_1^\alpha x_2^\beta$$

given his budget constraint is

$$m = p_1 x_1 + p_2 x_2$$

Design (x_1, x_2) to maximize his utility.

3.Solutions

Firstly, we apply a positive monotonic transformation of $U(x_1, x_2)$:

Define

$$U_0 = \log U = \alpha \log x_1 + \beta \log x_2$$

Then

$$\begin{aligned} \mathcal{L} &= U_0 + \lambda(m - p_1 x_1 - p_2 x_2) \\ &= \alpha \log x_1 + \beta \log x_2 + \lambda(m - p_1 x_1 - p_2 x_2) \end{aligned}$$

Applying the Lagrange Multipliers:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\alpha}{x_1} - \lambda p_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{\beta}{x_2} - \lambda p_2 = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= m - p_1 x_1 - p_2 x_2 = 0 \end{aligned}$$

Then, we can find that when

$$(x_1, x_2) = \left(\frac{m\alpha}{p_1(\alpha + \beta)}, \frac{m\beta}{p_2(\alpha + \beta)} \right)$$

the consumer's utility is maximized.