To apply ***Lagrange Multipliers*** in Optimizing Problem

1.Theorem needed

Lagrange Multipliers

Calculate the maximum of $F(x_1, x_2, \ldots, x_n)$given that $f(x_1, x_2, \ldots, x_n) = 0$:

Define that

$$\mathcal{L} = F(x_1, x_2, \ldots, x_n) + \lambda f(x_1, x_2, \ldots, x_n)$$

Then

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \frac{\partial F}{\partial x_1} + \lambda \frac{\partial f}{\partial x_1} = 0 \\
\frac{\partial L}{\partial x_2} &= \frac{\partial F}{\partial x_2} + \lambda \frac{\partial f}{\partial x_2} = 0 \\
&\ldots \\
\frac{\partial L}{\partial x_n} &= \frac{\partial F}{\partial x_n} + \lambda \frac{\partial f}{\partial x_n} = 0 \\
\frac{\partial L}{\partial \lambda} &= f(x_1, x_2, \ldots, x_n) = 0
\end{aligned}$$

Calculate the equations above, then we will get the Maximum of F given the constraints f.

2.Optimizing in Comsumer's choice

A comsumer's utility function of a specific comsumer is

$$U(x_1, x_2) = x_1^\alpha x_2^\beta$$

given his budget constraint is

$$m = p_1 x_1 + p_2 x_2$$

Design (x_1, x_2) to maximaze his utility.

3.Solutions

Firstly, we apply a positive monototic transformation of $U(x_1, x_2)$:
Define

$$U_0 = logU = \alpha log x_1 + \beta log x_2$$

Then

$$\begin{aligned}\mathfrak{L} &= U_0 + \lambda(m - p_1x_1 + p_2x_2) \\ &= \alpha log x_1 + \beta log x_2 + \lambda(m - p_1x_1 - p_2x_2)\end{aligned}$$

Applying the Lagrange Multipliers:

$$\begin{aligned}\frac{\partial \mathfrak{L}}{\partial x_1} = \frac{\alpha}{x_1} - \lambda p_1 = 0 \\ \frac{\partial \mathfrak{L}}{\partial x_2} = \frac{\beta}{x_2} - \lambda p_2 = 0 \\ \frac{\partial \mathfrak{L}}{\partial \lambda} = m - p_1x_1 - p_2x_2 = 0\end{aligned}$$

Then, we can find that when

$$(x_1^*, x_2^*) = (\frac{m\alpha}{p_1(\alpha+\beta)}, \frac{m\beta}{p_2(\alpha+\beta)})$$

the comsumer's utility is maximazed.

Note that the notes above is equal to
A comsumer's budget function is

$$p_1x_1 + p_2x_2$$

given his maximazed utility is

$$U(x_1, x_2) = \overline{U}$$

Design (x_1, x_2) to minimaze his cost.