To apply ***Lagrange Multipliers*** in Optimizing Problem

1.Theorem needed

Lagrange Multipliers

Calculate the maximun of $F(x_1, x_2, ..., x_n)$given that $f(x_1, x_2, ..., x_n) = 0$:

Define that

$$\mathcal{L} = F(x_1, x_2, ..., x_n) + \lambda f(x_1, x_2, ..., x_n)$$

Then

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= \frac{\partial F}{\partial x_1} + \lambda \frac{\partial f}{\partial x_1} = 0 \\
\frac{\partial L}{\partial x_2} &= \frac{\partial F}{\partial x_2} + \lambda \frac{\partial f}{\partial x_2} = 0 \\
&\cdots \\
\frac{\partial L}{\partial x_n} &= \frac{\partial F}{\partial x_n} + \lambda \frac{\partial f}{\partial x_n} = 0 \\
\frac{\partial L}{\partial \lambda} &= f(x_1, x_2, ..., x_n) = 0
\end{aligned}$$

Calculate the equations above, then we will get the Maximum of F given the constraints f.

2.Optimizing in Comsumer's choice

A comsumer's utility function of a specific comsumer is

$$U(x_1, x_2) = x_1^\alpha x_2^\beta$$

given his budget constraint is

$$m = p_1 x_1 + p_2 x_2$$

Design (x_1, x_2) *to maximalize his utility.*

3.Solutions

Firstly, we apply a positive monototic transformation of $U(x_1, x_2)$:
Define

$$U_0 = logU = \alpha logx_1 + \beta logx_2$$

Then

$$\begin{aligned}\mathfrak{L} &= U_0 + \lambda(m - p_1x_1 + p_2x_2) \\ &= \alpha logx_1 + \beta logx_2 + \lambda(m - p_1x_1 - p_2x_2)\end{aligned}$$

Applying the Lagrange Multipliers:

$$
\begin{aligned}
\frac{\partial \mathfrak{L}}{\partial x_1} &= \frac{\alpha}{x_1} - \lambda p_1 = 0 \\
\frac{\partial \mathfrak{L}}{\partial x_2} &= \frac{\beta}{x_2} - \lambda p_2 = 0 \\
\frac{\partial \mathfrak{L}}{\partial \lambda} &= m - p_1 x_1 - p_2 x_2 = 0
\end{aligned}
$$

Then, we can find that when

$$(x_1, x_2) = (\frac{m\alpha}{p_1(\alpha+\beta)}, \frac{m\beta}{p_2(\alpha+\beta)})$$

the comsumer's utility is maximalized.