PUFFIn: Theory

1 General formulation

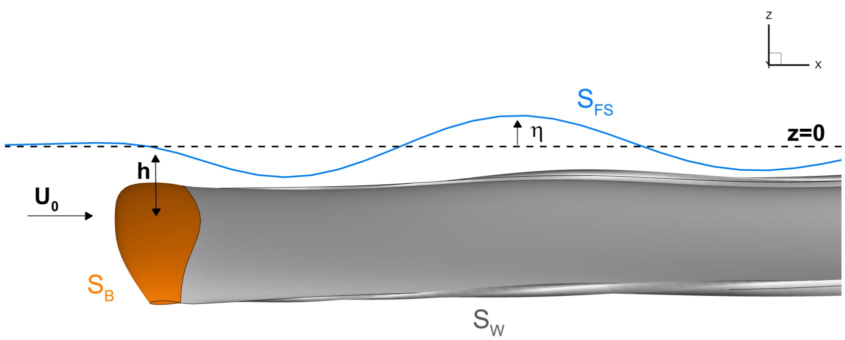

Figure 1: Typical configuration for foil computation.

PUFFIn is based on an incompressible potential approach, i.e. the viscous effects are neglected and the flow is supposed to be irrotational and incompressible. For typical configurations the domain boundary S is the union of the body surface S_B, the free surface boundary S_{FS} and the wake surface S_W (figure 1). The velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is obtained as the superposition of the undisturbed flow $\boldsymbol{U}_0$ and the gradient of the disturbance potential $\phi(\boldsymbol{x}, t)$:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{U}_0 + \boldsymbol{\nabla}\phi(\boldsymbol{x}, t) \tag{1}$$

With this assumptions, the mass conservation reduced to a Laplacian equation on the velocity potential:

$$\Delta\phi(\boldsymbol{x}, t) = 0 \tag{2}$$

Applying the second Green identity to the previous equation, the potential might be obtained through a domain Boundary Integral Equation (BIE) written on the domain boundaries S (Katz et Plotkin [2]):

$$\phi = -\frac{1}{4\pi}\int_S \left[\sigma\frac{1}{r} - \mu\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] dS \quad \text{where} \quad \frac{\partial}{\partial n} = \boldsymbol{n}\cdot\boldsymbol{\nabla} \tag{3}$$

with $\boldsymbol{n}$ the outward pointing normal vector. Equation 3 involves distributions of doublets μ and sources σ on the boundary S, related with the potential by the relations:

$$\begin{aligned} \mu &= -\phi \\ \sigma &= -\frac{\partial\phi}{\partial n} \end{aligned} \tag{4}$$

On the hydrofoil, the sources are given by the non-penetration condition:

$$\sigma = -\boldsymbol{U}_0 \cdot \boldsymbol{n}, \tag{5}$$

Under the assumption that the wake should not support hydrodynamic loads, the sources distribution on the wake surface must be null and equation 3 can be written:

$$\phi = -\frac{1}{4\pi} \int_{S_B+S_{FS}} \left[\sigma \frac{1}{r} - \mu \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] dS + \frac{1}{4\pi} \int_{S_W} \left[\mu \frac{\partial}{\partial n} \left(\frac{1}{r} \right) \right] dS \tag{6}$$

In addition, a kinematic condition is imposed on the free surface :

$$\frac{\partial \eta}{\partial t} + (\boldsymbol{U}_0 + \boldsymbol{\nabla}\phi) \cdot \boldsymbol{\nabla}\eta = \frac{\partial \phi}{\partial z} \quad \text{for} \quad z = \eta \tag{7}$$

A dynamic condition is also obtained on the free surface with the Bernoulli relation:

$$\frac{\partial \phi}{\partial t} + \boldsymbol{U}_0 \cdot \boldsymbol{\nabla}\phi + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + g\eta = 0 \quad \text{for} \quad z = \eta \tag{8}$$

These two conditions are non-linear since both relations contain quadratic terms and are written on the deformed free surface ($z = \eta$), which is unknown *a priori*. In order to reduce the computation time, linearized conditions are used, as discussed in section 3.

To construct a numerical solution, the boundaries are discretized using quadrilateral elements and the sources and doublets distributions are supposed to be constant on each element. Writting equation 6 at the geometrical center of each element, a general matricial form of the Boundary Element Method (BEM) can be obtained (Filippas and Belibassakis [1]):

$$\boldsymbol{\mathcal{A}}(\eta) \begin{bmatrix} \boldsymbol{\mu_b} \\ \boldsymbol{\sigma_f} \end{bmatrix} = \boldsymbol{\mathcal{B}}(\eta) \begin{bmatrix} \boldsymbol{\sigma_b} \\ \boldsymbol{\mu_f} \end{bmatrix} + \boldsymbol{\mathcal{W}}(\eta)\boldsymbol{\mu_w} \tag{9}$$

where $\boldsymbol{\sigma}_b$, $\boldsymbol{\mu}_b$ are vectors containing the sources and doublets strengths on the hydrofoil panels, $\boldsymbol{\sigma}_f$, $\boldsymbol{\mu}_f$ contains the sources and doublets on the free surface and $\boldsymbol{\mu}_w$ are the doublet strengths on the wake panels. The influence matrices $\boldsymbol{\mathcal{A}}(\eta)$, $\boldsymbol{\mathcal{B}}(\eta)$ and $\boldsymbol{\mathcal{W}}(\eta)$ can be computed exactly for constant strength singularity panels (Katz et Plotkin [2]). In equation 9, $\boldsymbol{\sigma_b}$ is directly given by relation 5, while $\boldsymbol{\mu_b}$, $\boldsymbol{\mu_w}$, $\boldsymbol{\mu_f}$ et $\boldsymbol{\sigma_f}$ are unknowns of the problem. A major drawback of this formulation is that the influence matrix $\boldsymbol{\mathcal{A}}(\eta)$ depends on the position of the free surface η, and thus should be built and inversed at each time step of the numerical procedure. However, this computational bottleneck disappears with linearized free surface conditions (section 3), and the matrix inversion is only done once at the beginning of the computation using a lower-upper decomposition.

Once the velocity potential is known, the pressure distribution on the hydrofoil is obtained with the Bernoulli relation. Integration of the pressure over the surface S_B gives the hydrodynamic forces and moments acting on the hydrofoil.

2 Wake and Kutta condition

A Lagrangian approach is used to construct the wake surface, i.e. the position of the wake panels $\boldsymbol{x}_W^n$ at the current time step t^n are obtained from the velocity value at the previous time step t^{n-1}:

$$\boldsymbol{x}_W^{n+1} = \boldsymbol{x}_W^n + \boldsymbol{U}(\boldsymbol{x}_W^n, t^n)\Delta t \tag{10}$$

While higher order schemes are necessary for true unsteady problems (Filippas et Belibassakis [1]), for the steady configuration studied in the current work, a first order scheme in time is sufficient to obtain an accurate solution at convergence. For inviscid flow, the doublet strength on a wake panel should remain constant between two time steps. Thus, the wake doublet distribution only depends on the strengths of the doublets on the panels located downstream the trailing edge. These strengths are obtained using a Kutta condition, imposing the equality of the extrados and intrados pressure at the trailing edge:

$$p_{\mathrm{TE}}^{\mathrm{int}} = p_{\mathrm{TE}}^{\mathrm{ext}} \tag{11}$$

where $p_{\mathrm{TE}}^{\mathrm{int}}$ is the pressure on the intrados et $p_{\mathrm{TE}}^{\mathrm{int}}$ the pressure on the extrados. With the Bernoulli relation, the previous equation might be written:

$$p_{\mathrm{TE}}^{\mathrm{int}} - p_{\mathrm{TE}}^{\mathrm{ext}} = \left[\rho \frac{\partial \phi}{\partial t} + \frac{1}{2}\rho(\boldsymbol{U}_0 + \nabla\phi)^2 + gz\right]_{\mathrm{int}}^{\mathrm{ext}} \tag{12}$$

where $[a]_{\mathrm{int}}^{\mathrm{ext}}$ is the difference between the intrados and extrados values of the quantity a. A parabolic extrapolation is used to impose equation 12 at the trailing edge. Since the Kutta condition is non-linear, a Picard iterative procedure is used at each time step.

3 Linearized free surface condition

For non-linear free surface condition, the kinematic and dynamic conditions should be imposed on the deformed free interface. To reduce the computational time, the non-linear free surface conditions might be linearized around the initial free surface position $z = 0$. Keeping only the first order terms, the linearized conditions are:

$$\begin{aligned} \frac{\partial \eta}{\partial t} + \boldsymbol{U}_0 \cdot \boldsymbol{\nabla}\eta &= \frac{\partial \phi}{\partial z} \quad \text{en} \quad z = 0 \\ \frac{\partial \phi}{\partial t} + \boldsymbol{U}_0 \cdot \boldsymbol{\nabla}\phi &= -g\eta \quad \text{en} \quad z = 0 \end{aligned} \tag{13}$$

Combining these two equations, the well known Neumann-Kelvin (NK) formulation is obtained:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U_0} \cdot \boldsymbol{\nabla}\right)^2 \phi + g\frac{\partial \phi}{\partial z} = 0 \tag{14}$$

The NK condition is solved using the finite difference method, with a third order upwind scheme for the spatial derivative and a second order backward scheme for the time derivative.

References

[1] ES Filippas and KA Belibassakis. Hydrodynamic analysis of flapping-foil thrusters operating beneath the free surface and in waves. *Engineering Analysis with Boundary Elements*, 41:47–59, 2014.

[2] J Katz and A Plotkin. *Low-speed aerodynamics*, volume 13. Cambridge university press, 2001.

[3] JN Newman. *Marine hydrodynamics*. The MIT press, 2018.