GABARITO LISTA VII

1) Obtenha os coeficientes de Clebsh-Gordan para a soma de dois momentos angulares iguais a 1. Utilize os dois métodos descritos em aula.

Método 1:

Notemos que para $j_1 = 1$, $j_2 = 1$ temos três possibilidades,

$$j = 2, 1, 0.$$

Consideramos cada caso.

- $j = 2$, $m = \pm 2, \pm 1, 0$.

O estado $|j, m\rangle = |2, 2\rangle$ é obtido únicamente dos estados $|j_1 = 1, m_1 = 1\rangle \otimes |j_2 = 1, m_2 = 1\rangle$, portanto,

$$|2, 2\rangle = |1, 1; 1, 1\rangle,$$

onde usamos a notação

$$|j_1, m_1; j_2, m_2\rangle = |j_1 m_1\rangle \otimes |j_2, m_2\rangle.$$

Aplicações sucessivas do operador

$$J_- = J_{1-} + J_{2-}$$

no estado $|2,2\rangle$ nos permitirá obter os outros estados.

$$J_- |2,2\rangle = (J_{1-} + J_{2-}) |1,1;1,1\rangle .$$

Usando que

$$J_- |j,m\rangle = \sqrt{j(j+1) - m(m-1)}\, |j,m-1\rangle ,$$

temos que,

$$\sqrt{2(2+1) - 2(2-1)}\, |2,1\rangle = \sqrt{1(1+1) - (1-1)}\, |1,0;1,1\rangle$$
$$+ \sqrt{2}\, |1,1;1,0\rangle$$

$$\sqrt{4}\, |2,1\rangle = \sqrt{2}\, |1,0;1,1\rangle + \sqrt{2}\, |1,1;1,0\rangle$$

$$|2,1\rangle = \frac{1}{\sqrt{2}} |1,0;1,1\rangle + \frac{1}{\sqrt{2}} |1,1;1,0\rangle .$$

Aplicando J_- novamente,

$$J_- |2,1\rangle = \frac{1}{\sqrt{2}} (J_{1-} + J_{2-}) |1,0;1,1\rangle + \frac{1}{\sqrt{2}} (J_{1-} + J_{2-}) |1,1;1,0\rangle$$

$$\sqrt{6}\, |2,0\rangle = \frac{1}{\sqrt{2}} \Big(\sqrt{2}\, |1,-1;1,1\rangle + \sqrt{2}\, |1,0;1,0\rangle$$
$$+ \sqrt{2}\, |1,0;1,1\rangle + \sqrt{2}\, |1,0;1,-1\rangle$$

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left\{ |1,-1;1,1\rangle + 2\,|1,0;1,0\rangle + |1,1;1,-1\rangle \right\}.$$

•) Novamente

$$J_-\,|2,0\rangle = \frac{1}{\sqrt{6}}\left(J_{1-} + J_{2-}\right)\left\{ |1,-1;1,1\rangle + 2\,|1,0;1,0\rangle + |1,1;1,-1\rangle \right\}$$

$$\sqrt{6}\,|2,-1\rangle = \frac{1}{\sqrt{6}}\left\{ \sqrt{2}\,|1,-1;1,0\rangle + 2\sqrt{2}\,|1,-1;1,0\rangle + 2\sqrt{2}\,|1,0;1,-1\rangle + \sqrt{2}\,|1,0;1,-1\rangle \right\}$$

$$|2,-1\rangle = \frac{1}{\sqrt{2}}\left\{ |1,-1;1,0\rangle + |1,0;1,-1\rangle \right\}.$$

Finalmente

$$J_-\,|2,-1\rangle = \frac{1}{\sqrt{2}}\left(J_{1-} + J_{2-}\right)\left\{ |1,-1;1,0\rangle + |1,0;1,-1\rangle \right\}$$

$$\sqrt{4}\,|2,-1\rangle = \frac{\sqrt{2}}{\sqrt{2}}\left\{ |1,-1;1,-1\rangle + |1,-1;1,-1\rangle \right\}$$

$$|2,-1\rangle = |1,-1;1,-1\rangle.$$

•) $j=1$

Neste caso, sabemos que uma combinação dos estados $|1,1;1,0\rangle$ e $|1,0;1,1\rangle$ vai dar um estado com $j=1$, $m=1$,

$$|1,1\rangle = A\,|1,1;1,0\rangle + B\,|1,0;1,1\rangle .$$

Tal estado deve ser ortogonal ao $|2,1\rangle$, portanto,

$$\frac{1}{\sqrt{2}}(A+B)=0 \quad \Rightarrow \quad B=-A,$$

e o estado deve ser normalizado

$$|A|^2 + |B|^2 = 1,$$

Portanto

$$A = \frac{1}{\sqrt{2}}, \qquad B = -\frac{1}{\sqrt{2}} .$$

$$|1,1\rangle = \frac{1}{\sqrt{2}}\left\{|1,1;1,0\rangle - |1,0;1,1\rangle\right\} .$$

Obtemos os estados $|1,0\rangle$, $|1,-1\rangle$ aplicando J_- :

a)

$$J_-|1,1\rangle = \frac{1}{\sqrt{2}}\left(J_{1-}+J_{2-}\right)\left\{|1,1;1,0\rangle - |1,0;1,1\rangle\right\}$$

$$\sqrt{2}\,|1,0\rangle = \frac{1}{\sqrt{2}}\left\{\sqrt{2}\,|1,0;1,0\rangle + \sqrt{2}\,|1,-1;1,1\rangle - \sqrt{2}\,|1,0;1,0\rangle - \sqrt{2}\,|1,0;1,-1\rangle\right\}$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left\{|1,+1;1,-1\rangle - |1,-1;1,1\rangle\right\}$$

$\circ)$

$$J_-|1,0\rangle = \frac{1}{\sqrt{2}}(J_{1-}+J_{2-})\left\{|1,1;1,-1\rangle - |1,-1;1,1\rangle\right\}$$

$$\sqrt{2}|1,-1\rangle = \frac{1}{\sqrt{2}}\left\{\sqrt{2}|1,0;1,-1\rangle - \sqrt{2}|1,-1;1,0\rangle\right\}$$

$$|1,-1\rangle = \frac{1}{\sqrt{2}}\left\{|1,0;1,-1\rangle - |1,-1;1,0\rangle\right\}$$

$\circ]$ $j=0$.

Notemos que a única combinação possível é

$$|0\ 0\rangle = \alpha|1,1;1,-1\rangle + \beta|1,0;1,0\rangle + \gamma|1,-1;1,1\rangle,$$

onde

$$|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1,$$

e

$$\langle 2\,0|0\,0\rangle = 0,$$

$$\langle 1\,0|0\,0\rangle = 0.$$

Obtemos que

$$\frac{1}{\sqrt{6}}(\alpha+2\beta+\gamma)=0,$$

$$\frac{1}{\sqrt{2}}(\alpha-\gamma)=0,$$

junto com a condição de normalização. Obtemos que

$$\alpha=\frac{1}{\sqrt{3}}\,;\ \beta=-\frac{1}{\sqrt{3}}\,;\ \gamma=\frac{1}{\sqrt{3}},$$

ou seja,

$$|0\,0\rangle=\frac{1}{\sqrt{3}}\{|1,1;1,-1\rangle-|1,0;1,0\rangle+|1,-1;1,1\rangle.$$

Método 2.

A relação de recurrência é

$$a_{\mp}(jm)\langle j_1,m_1;j_2m_2|j\,m\mp1\rangle=a_{\pm}(j_1m_1)\langle j_1,m_1\pm1;j_2,m_2|jm\rangle$$
$$+\,a_{\pm}(j_2m_2)\langle j_1,m_1;j_2,m_2\pm1|jm\rangle,$$

sendo

$$a_{\mp}(jm)=\sqrt{j(j+1)-m(m\pm1)}.$$

No caso em que $j=2$, $m=2$, de novo, temos que

$$|2,2\rangle = |1,1;1,1\rangle,$$

ou seja

$$\langle 1,1;1,1|2,2\rangle = 1.$$

Para o caso em que $j=m$, a relação de recorrência fica,

$$a_-(jj)\langle j_1,m_1;j_2,m_2|j\ j-1\rangle = a_+(j_1 m_1)\langle j_1,m_1+1;j_2 m_2|jm\rangle$$
$$+\ a_+(j_2 m_2)\langle j_1,m_1;j_2 m_2+1|jm\rangle.$$

onde

$$m_1+m_2 = m-1.$$

No caso de $j=2=m$, temos que,

•) $m_1=0$, $m_2=1$,

$$a_-(22)\langle 1,0;1,1|2\ 1\rangle = \underbrace{a_+(1\ 0)}_{\sqrt{2}}\underbrace{\langle 1,1;1,1|2\ 1\rangle}_{1}$$
$$+\ \cancel{a_+(1,1)\langle 1,0;1,2|2,1\rangle} \rightarrow 0$$

$$2\langle 1,0;1,1|2\ 1\rangle = \sqrt{2}$$

$$\langle 1,0;1,1|2\ 1\rangle = \frac{1}{\sqrt{2}}.$$

•) $m_1 = 1, \quad m_2 = 0,$

$$\underbrace{a_-(2\,2)}_{2} \langle 1,1;1,0|2\,1\rangle = a_+(1\,1) \underbrace{\cancel{\langle 1,2;1,0|2\,1\rangle}}_{\to 0} + \underbrace{a_+(1\,0)}_{\sqrt{2}} \underbrace{\langle 1,1;1,1|2\,1\rangle}_{2}$$

$$\langle 1,1;1,0|2\,1\rangle = \frac{1}{\sqrt{2}},$$

ou seja,

$$|2\,1\rangle = \frac{1}{\sqrt{2}}\left\{|1,1;1,0\rangle + |1,0;1,1\rangle\right\}.$$

•] $j=2, \; m=1.$

$$m_1 + m_2 = 0.$$

•) $m_1 = m_2 = 0$

$$\underbrace{a_-(2\,1)}_{\sqrt{6}} \langle 1,0;1,0|2\,0\rangle = \underbrace{a_+(1\,0)}_{\sqrt{2}} \underbrace{\langle 1,1;1,0|2\,1\rangle}_{\frac{1}{\sqrt{2}}}$$

$$+ \underbrace{a_+(1\,0)}_{\sqrt{2}} \underbrace{\langle 1,0;1,1|2\,1\rangle}_{\frac{1}{\sqrt{2}}}$$

$$\langle 1,0;1,0|2\,0\rangle = \frac{2}{\sqrt{6}}$$

∘) $m_1 = -1$, $m_2 = 1$

$$\underbrace{a_-(2\,1)}_{\sqrt{6}} \langle 1,-1;1,1|2\,0\rangle = \underbrace{a_+(1\,-1)}_{\sqrt{2}} \underbrace{\langle 1,0;1,1|2\,1\rangle}_{\frac{1}{\sqrt{2}}}$$

$$+\ \cancel{a_+(1+1)\langle 1,+1;1,2|2\,1\rangle}^{\,0}$$

$$\langle 1,-1;1,1|2\,0\rangle = \frac{1}{\sqrt{2}}$$

∘) $m_1 = 1$, $m_2 = -1$

$$\underbrace{a_-(2\,1)}_{\sqrt{6}} \langle 1,1;1,-1|2\,0\rangle = \cancel{a_+(1\,1)\langle 1,2;1,-1|2\,1\rangle}^{\,0}$$

$$+\ \underbrace{a_+(1\,-1)}_{\sqrt{2}} \underbrace{\langle 1,+1;1,0|2\,1\rangle}_{\frac{1}{\sqrt{2}}}$$

$$\langle 1,1;1,-1|2\,0\rangle = \frac{1}{\sqrt{6}}$$

Assim

$$|2\,0\rangle = \frac{1}{\sqrt{6}}\left\{|1,1;1,-1\rangle + 2\,|1,0;1,0\rangle + |1,-1;1,1\rangle\right\}.$$

∘) $j = 2$, $m = -1$

$$m_1 + m_2 = -1$$

•) $m_1 = 0, \quad m_2 = -1$

$$\underbrace{a_-(2\ 0)}_{\sqrt{6}} \langle 1,0;1,-1|2\ -1\rangle = \underbrace{a_+(1\ 0)}_{\sqrt{2}} \underbrace{\langle 1,1;1,-1|2\ 0\rangle}_{\frac{1}{\sqrt{6}}}$$

$$+ \underbrace{a_+(1\ -1)}_{\sqrt{2}} \underbrace{\langle 1,0;1,0|2\ 0\rangle}_{\frac{2}{\sqrt{6}}}$$

$$\langle 1,0;1,-1|2\ -1\rangle = \frac{\cancel{3}\sqrt{2}}{\cancel{6}_2} = \frac{1}{\sqrt{2}}$$

•) $m_1 = -1, \quad m_2 = 0$

$$\underbrace{a_-(2\ 0)}_{\sqrt{6}} \langle 1,-1;1,0|2\ -1\rangle = \underbrace{a_+(1\ -1)}_{\sqrt{2}} \underbrace{\langle 1,0;1,0|2\ 0\rangle}_{\frac{2}{\sqrt{6}}}$$

$$+ \underbrace{a_+(1\ 0)}_{\sqrt{2}} \underbrace{\langle 1,1;1,-1|2\ 0\rangle}_{\frac{1}{\sqrt{6}}}$$

$$\langle 1,-1;1,0|2\ -1\rangle = \frac{1}{\sqrt{2}}$$

Assim

$$|2\ -1\rangle = \frac{1}{\sqrt{2}} \left\{ |1,0;1,-1\rangle + |1,-1;1,0\rangle \right\}$$

•) $j=1\,,\; m=1$

Consideramos agora o caso $j=1,\, m=1$. Usamos que,

$$a_+(jj)\,\langle j_1, m_1; j_2, m_2 | j\, j+1\rangle = a_-(j_1\, m_1)\,\langle j_1, m_1 - 1; j_2, j-m_1+1 | jj\rangle$$

$$+\, a_-(j_2\; j+1-m_1)\,\langle j_1, m_1; j_2, j-m_1 | jj\rangle$$

Para nosso caso, $m_1 = 1 \quad m_2 = 1$

$$\cancelto{0}{a_+(1\,1)}\,\langle 1,0; 1,1 | 1\,2\rangle = \underbrace{a_-(1\,1)}_{\sqrt{2}}\,\langle 1,0; 1,1 | 1\,1\rangle$$

$$+\, \underbrace{a_-(1\,1)}_{\sqrt{2}}\,\langle 1,1; 1,0 | 1\,1\rangle,$$

Portanto

$$\langle 1,0; 1,1 | 1\,1\rangle = -\,\langle 1,1; 1,0 | 1\,1\rangle,$$

Normalizando, obtemos

$$\langle 1,0; 1,1 | 1\,1\rangle = \frac{1}{\sqrt{2}},$$

assim

$$|1\,1\rangle = \frac{1}{\sqrt{2}}\left\{ |1,0; 1,1\rangle - |1,1; 1,0\rangle \right\}.$$

Podemos obter agora os outros coeficientes de modo analogo ao feito anteriormente.

∘) $m_1 = m_2 = 0$

$$\underbrace{a_-(1\,1)}_{\sqrt{2}} \langle 1,0;1,0|1\,0\rangle = \underbrace{a_+(1\,0)}_{\sqrt{2}} \underbrace{\langle 1,1;1,0|1\,1\rangle}_{\frac{1}{\sqrt{2}}}$$

$$+ \underbrace{a_+(1\,0)}_{\sqrt{2}} \underbrace{\langle 1,0;1,1|1\,1\rangle}_{-\frac{1}{\sqrt{2}}}$$

$$\langle 1,0;1,0|1\,0\rangle = 0$$

∘) $m_1 = 1$, $m_2 = -1$

$$\langle 1,-1;1,1|1\,0\rangle = -\frac{1}{\sqrt{2}},$$

∘) $m_1 = -1$, $m_2 = 1$

$$\langle 1,1;1,-1|1\,0\rangle = \frac{1}{\sqrt{2}}.$$

$$|1\,0\rangle = \frac{1}{\sqrt{2}}\left\{|1,1;1,-1\rangle - |1,-1;1,1\rangle\right\}.$$

∘) $m_1 = 0, \quad m_2 = -1$

$$\langle 1,0;1,-1|1,-1\rangle = \frac{1}{\sqrt{2}}$$

∘) $m_1 = -1; \quad m_2 = 0$

$$\langle 1,-1;1,0|1,-1\rangle = -\frac{1}{\sqrt{2}}$$

Assim

$$|1,-1\rangle = \frac{1}{\sqrt{2}}\Big[|1,0;1,-1\rangle - |1,-1;1,0\rangle\Big].$$

Para o caso $j=0$, $m=0$, devemos aplicar a recorrência duas vezes:

∘) $m_1 = +1, \quad m_2 = j - m_1 + 1 = 0$

$$a_+(0\,0)\cancelto{0}{\langle 1,+1;1,0|0\,0\rangle} = \underbrace{a_-(1,+1)}_{\sqrt{2}}\langle 1,0;1,0|0\,0\rangle$$
$$+ \underbrace{a_-(1,0)}_{\sqrt{2}}\langle 1,1;1,-1|0\,0\rangle$$

$$\langle 1,0;1,0|0\,0\rangle = -\langle 1,1;1,-1|0\,0\rangle.$$

∘) $m_1 = 0, \quad m_2 = j - m_1 + 1 = 1$

$$\cancelto{0}{a_+(0\,0)}\langle 1,0;1,1|0\,0\rangle = \underbrace{a_-(1\,0)}_{\sqrt{2}}\langle 1,-1;1,0|0\,0\rangle$$
$$+ \underbrace{a_-(1\,1)}\langle 1,0;1,0|0\,0\rangle$$

$$\langle 1,0;1,0|0\ 0\rangle = -\langle 1,-1;1,-1|0\ 0\rangle .$$

Normalizando, obtemos

$$\langle 1,1;1,-1|0\ 0\rangle = \langle 1,-1;1,1|0\ 0\rangle = \frac{1}{\sqrt{3}}.$$

$$\langle 1,0;1,0|0\ 0\rangle = -\frac{1}{\sqrt{3}}.$$

Assim

$$|0\ 0\rangle = \frac{1}{\sqrt{3}}\left\{ |1,1;1,-1\rangle + |1,-1;1,1\rangle - |1,0;1,0\rangle \right\}.$$

2) Considere um problema de dois corpos, sujeito a uma interação central $V(r)$. Seja $\Psi_n(r)$ um autoestado ligado de onda s qualquer do sistema. Mostre que

$$|\Psi_n(0)|^2 = \frac{\mu}{2\pi}\int d^3r\, |\Psi_n(r)|^2 \frac{\partial V}{\partial r}.$$

A equação radial de Schrödinger, para $\ell = 0$ (onda s) é

$$-\frac{\hbar^2}{2\mu}\frac{d^2 u}{dr^2} + V(r)\, u = E u,$$

sendo $u(r) = r R(r)$. Multiplicando por $\dfrac{du}{dr}$

$$-\frac{\hbar^2}{2\mu}\frac{du}{dr}\frac{d^2u}{dr^2} + \left(V(r)-E\right)\frac{du}{dr}\,u = 0.$$

Usando que

∘) $$\frac{du}{dr}\frac{d}{dr}\left(\frac{du}{dr}\right) = \frac{1}{2}\frac{d}{dr}\left(\left(\frac{du}{dr}\right)^2\right)$$

∘) $$V(r)\frac{du}{dr}\,u = \frac{1}{2}\frac{d}{dr}\left(V\cdot u^2\right) - \frac{1}{2}\frac{dV}{dr}u^2$$

∘) $$E\frac{du}{dr}\,u = \frac{1}{2}E\frac{d}{dr}\left(u^2\right).$$

Temos que,

$$-\frac{\hbar^2}{2\mu}\frac{1}{2}\frac{d}{dr}\left(\left(\frac{du}{dr}\right)^2\right) + \frac{1}{2}\frac{d}{dr}\left(V\cdot u^2\right) - \frac{dV}{dr}u^2 - \frac{1}{2}E\frac{d}{dr}\left(u^2\right) = 0.$$

Integrando em r de $0 \to \infty$,

$$-\frac{\hbar^2}{2m}\int_0^\infty dr\,\frac{1}{2}\frac{d}{dr}\left(\frac{du}{dr}\right)^2 + \int_0^\infty dr\,\frac{1}{2}\frac{d}{dr}\left(V\cdot u^2\right) - \int_0^\infty dr\,\frac{dV}{dr}u^2 = \frac{1}{2}E\int_0^\infty dr\,\frac{d}{dr}u^2$$

$$-\frac{\hbar^2}{2m}\frac{1}{2}\left(\frac{du}{dr}\right)^2\Bigg|_0^\infty + \frac{1}{2}\cancelto{0}{\left(Vu^2\right)}\Bigg|_0^\infty - \frac{1}{2}\int_0^\infty dr\,\frac{dV}{dr}u^2 = \frac{1}{2}\cancelto{0}{Eu^2}\Bigg|_0^\infty$$

onde usamos que,

$$\lim_{r\to 0} u(r) = 0 \qquad \lim_{r\to\infty} u(r) = 0$$

Temos que

$$\frac{1}{2}\left(\frac{du}{dr}\right)^2\Bigg|_0^\infty = -\frac{2\mu}{\hbar^2}\frac{1}{2}\int_0^\infty dr\,\frac{dV}{dr}\,u^2$$

Lembrando que

$$u_n = r\,R_n(r),$$

$$\frac{1}{\sqrt{4\pi}}\,u_n = r\,\underbrace{\frac{1}{4\pi}\,R_n(r)}_{\Psi_n(r)}$$

$$u_n = \sqrt{4\pi}\,r\,\Psi_n(r)$$

$$\frac{du}{dr} = \sqrt{4\pi}\left[\Psi_n(r) + r\,\frac{d\Psi_n}{dr}\right],$$

isto é

$$\left(\frac{du}{dr}\right)^2\Bigg|_0^\infty = 4\pi\left\{|\Psi_n(r)|^2\Bigg|_0^\infty + \cancelto{0}{2r\,\Psi_n\,\frac{d\Psi_n}{dr}}\Bigg|_0^\infty + \cancelto{0}{\left(r\,\frac{d\Psi}{dr}\right)}\Bigg|_0^\infty\right.$$

$$= -4\pi\,|\Psi_n(0)|^2$$

Portanto,

$$-\frac{1}{\cancel{4}}\frac{\cancel{4}\pi}{2}|\Psi_n(0)|^2 = -\frac{\cancel{2}\mu}{\hbar^2}\frac{1}{\cancel{2}}\int_0^\infty dr\,\frac{dV}{dr}\cdot 4\pi\,|\Psi_n(r)|^2 r^2$$

$$|\Psi_n(0)|^2 = \frac{\mu}{\hbar^2 2\pi}\int_0^\infty r^2 dr\,\underbrace{4\pi}_{\int d\Omega}\,|\Psi_n(r)|^2\frac{dV}{dr}$$

$$|\Psi_n(0)|^2 = \frac{\mu}{2\pi\hbar^2}\int d^3r\,|\Psi_n(r)|^2\frac{dV}{dr}. \qquad \text{Q.E.D.}$$

3) Obtenha os autovalores e autofunções para a onda s $(\ell=0)$ no caso do potencial,

$$V(r) = -\frac{a^2}{8}e^{-\frac{r}{r_0}}.$$

Devemos resolver a equação para $u(r) = r\,R(r)$, no caso de $\ell=0$,

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + V(r)u = E\,u,$$

ou, equivalentemente

$$\frac{d^2u}{dr^2} - \left(U(r)+\beta^2\right)u = 0$$

Sendo,

$$U(r) = -\frac{2m}{\hbar^2}\frac{a^2}{8}e^{-\frac{r}{r_0}}$$

$$\beta^2 = -\frac{2mE}{\hbar^2}.$$

Para resolver a equação, fazemos a mudança de variáveis,

$$y = e^{-\frac{r}{2r_0}}$$

Temos que

$$\cdot)\ \frac{d}{dr} = \frac{dy}{dr}\frac{d}{dy} = -\frac{1}{2r_0}y\frac{d}{dy}$$

$$\cdot)\ \frac{d^2}{dr^2} = \frac{dy}{dr}\frac{d}{dy}\left(-\frac{1}{2r_0}y\frac{d}{dy}\right)$$

$$= +\frac{1}{4r_0^2}y\frac{d}{dy}\left(y\frac{d}{dy}\right) = \frac{1}{4r_0^2}y\left\{\frac{d}{dy}+y\frac{d^2}{dy^2}\right\}$$

A equação fica,

$$\frac{1}{4r_0^2}y\left\{\frac{d}{dy}+y\frac{d^2}{dy^2}\right\}u + \left(\frac{2m}{\hbar^2}\frac{a^2}{8}y^2 - \beta^2\right)u = 0$$

$$y^2\frac{d^2u}{dy^2} + y\frac{du}{dy} + \left(Ay^2 - \underbrace{4r_0^2\beta^2}_{\tilde{\beta}^2}\right)u = 0$$

onde

$$A = \frac{r_0^2 a^2 2m}{\hbar^2}$$

Fazemos uma segunda transformação para eliminar o A,

$$\tilde{y} = \sqrt{A}\, y$$

$$\frac{d}{d\tilde{y}} = \frac{1}{\sqrt{A}} \frac{d}{dy}, \qquad \frac{d^2}{d\tilde{y}^2} = \frac{1}{A} \frac{d^2}{dy^2}$$

Assim,

$$\tilde{y}^2 \frac{d^2}{d\tilde{y}^2} u + \tilde{y} \frac{d}{dy} u + (\tilde{y}^2 - \tilde{\beta}^2) u = 0,$$

que corresponde a uma equação de Bessel. A solução é

$$u(\tilde{y}) = C_1 J_{\tilde{\beta}}(\tilde{y}) + C_2 Y_{\tilde{\beta}}(\tilde{y}),$$

Notemos que a solução deve ser bem comportada para $r \to 0$ e $r \to \infty$. Para o segundo caso

$$r \to \infty \quad \to \quad \tilde{y} \to 0,$$

implica que

$$C_2 = 0$$

pois as funções de Neumann divergem em $\tilde{y} \to 0$. Além disso, os valores de $\tilde{\beta}$ devem ser numeros inteiros, para termos $J_{\tilde{\beta}}(y=0)=0$, então

$$\tilde{\beta}^2 = n,$$

ou seja,

$$4 r_0^2 \frac{2mE}{\hbar^2} = n$$

$$E = \frac{n\hbar^2}{8mr_0^2}.$$

Por outro lado, para $r \to 0$

$$r \to 0 \quad \Rightarrow \quad y \to \text{constante}.$$

queremos então que

$$J_{\tilde{\beta}}(\sqrt{A'}) = 0,$$

assim, $\sqrt{A'} = \sqrt{\frac{2m\alpha^2 r_0^2}{\hbar^2}}$ corresponderão aos zeros da funções de Bessel de ordem $\tilde{\beta}$, que são discretos.