GABARITO LISTA IV

1. Calcule $\varphi(x,t)$, a função de onda de partícula livre evoluída no tempo, dado que

$$\varphi(x,0) = \frac{\sqrt{\sigma}}{\pi^{1/4}} \exp\left\{ ikx - \frac{1}{2}\sigma^2 x^2 \right\}$$

onde $k, \sigma \in \mathbb{R}$ e são constantes. Obtenha $\Delta x(t)$ e $\Delta p(t)$

·) Usando o propagador livre, temos que

$$\varphi(x,t) = \int dx' \underbrace{\langle x | U(t,0) | x' \rangle}_{J(x,t;x',0)} \varphi(x',0),$$

onde

$$J(x,t;x',t') = \sqrt{\frac{m}{2\pi i \hbar (t-t')}}\, e^{i\frac{m(x-x')^2}{2\hbar(t-t')}} . \qquad t > t'.$$

Assim

$$\varphi(x,t) = \sqrt{\frac{m}{2\pi i \hbar t}} \int_{-\infty}^{\infty} dx'\, e^{i\frac{m(x-x')^2}{2\hbar t}} \frac{\sqrt{\sigma}}{\pi^{1/4}} e^{ikx' - \frac{1}{2}\sigma^2 x'^2}$$

A integral é feita usando que

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2 + bx + c} = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2}{4a} + c}$$

Obtemos que

$$\varphi(x,t) = \frac{1}{\pi^{1/4}} \sqrt{\frac{m\sigma}{m + i\hbar\sigma^2 t}} \exp\left\{ \frac{2mkx + im\sigma^2 x^2 - \hbar k^2 t}{2\hbar\sigma^2 t - 2im} \right\}.$$

Ora, para determinar $\Delta x(t)$ e $\Delta p(t)$, devemos calcular

$$\Delta x(t) = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

$$\Delta p(t) = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

•) $\Delta x(t)$. Calculamos

*) $\langle x \rangle(t) = \int_{-\infty}^{\infty} dx \; x \; |\varphi(x,t)|^2$,

resolvendo a integral, encontramos,

$$\langle x \rangle(t) = \frac{\hbar k t}{m}$$

*) $$\langle x^2 \rangle(t) = \int_{-\infty}^{\infty} dx \; x^2 \; |\varphi(x,t)|^2$$

$$= \frac{(\hbar\sigma t)^2 (2k^2 + \sigma^2) + m^2}{2 m^2 \sigma^2}.$$

Portanto,

$$\Delta x(t) = \sqrt{\frac{\hbar^2\sigma^2 t^2(2k^2+\sigma^2)+m^2}{2m^2\sigma^2} - \frac{\hbar^2 k^2 t^2}{m^2}}$$

$$= \sqrt{\frac{\hbar^2\sigma^4 t^2+m^2}{2m^2\sigma^2}}$$

$$\Delta x(t) = \frac{1}{\sqrt{2}\,\sigma}\sqrt{1+\frac{\hbar^2\sigma^4 t^2}{m^2}} \,. \rightarrow$$ A dispersão do pacote cresce com o tempo.

•) $\Delta p(t)$. Calculamos,

*] $$\langle p\rangle(t) = \int_{-\infty}^{\infty} dx \;\; \varphi(x,t)^* \frac{\hbar}{i}\frac{\partial}{\partial x}\varphi(x,t)$$

$$= \hbar k,$$

*] $$\langle p^2\rangle(t) = -\int_{-\infty}^{\infty} dx \;\; \varphi(x,t)^* \hbar^2 \frac{\partial^2}{\partial x^2}\varphi(x,t)$$

$$= \frac{\hbar^2}{2}(2k^2+\sigma^2),$$

Assim,

$$\Delta p(t) = \sqrt{\frac{\hbar^2}{2}(\cancel{2k^2}+\sigma^2) - \cancel{\hbar^2 k^2}}$$

$$= \frac{\hbar\sigma}{\sqrt{2}} \,.//$$ Independente do tempo.

2) Considere uma partícula se movendo em um potencial linear unidimensional, dado por

$$V(x) = ax$$

Determinar

a) $J(x',t';x,t) = \langle x't' | x,t \rangle$

b) $J(x't';pt) = \langle x't' | pt \rangle$

c) $J(p't';pt) = \langle p't' | pt \rangle$

a) O propagador $J(x't;xt)$ é

$$J(x't';x,t) = \langle x't' | x,t \rangle$$

$$= \langle x' | U(t',t) | x \rangle;$$

lembrando que a Hamiltoniana não depende do tempo, podemos escrever

$$J(x't';xt) = \langle x' | \underbrace{e^{-\frac{i}{\hbar}H(t'-t)}}_{U(t'-t)} | x \rangle .$$

Para determinar tal propagador consideraremos dois métodos.

1] Fórmula de Zassenhaus.

A formula de Zassenhaus é

$$\exp\{\lambda(A+B)\} = \exp\{\lambda A\} \exp\{\lambda B\} \exp\{-\tfrac{1}{2}\lambda^2 [A,B]\}$$
$$\exp\left\{\frac{\lambda^3}{6}[A+2B,[A,B]]\right\} \cdots$$

Para nosso caso,

$$\lambda = \frac{i\tau}{\hbar} \qquad \tau = t' - t$$

$$A = \frac{p^2}{2m}, \quad B = ax$$

Calculamos os commutadores,

◦) $$[A,B] = \left[\frac{p^2}{2m}, ax\right] = \frac{a}{2m}[p^2, x] = -\frac{i\hbar a}{2m}2p = -\frac{i\hbar a}{m}p.$$

◦) $$[A+2B,[A,B]] = \left[\frac{p^2}{2m} + 2ax, -\frac{i\hbar a}{m}p\right] = -\frac{2i\hbar a^2}{m}\underbrace{[x,p]}_{i\hbar} = \frac{2\hbar^2 a^2}{m}$$

◦) $$[A,[A,[A,B]]] = [B,[A,[A,B]]] = [A,[B,[A,B]]] = [B,[B,[A,B]]] = 0$$

Portanto,

$$\exp\left\{-\frac{i}{\hbar}\tau H\right\} = \exp\left\{\frac{-i}{\hbar}\tau\frac{p^2}{2m}\right\}\exp\left\{\frac{-i}{\hbar}\tau ax\right\}\exp\left\{-\frac{1}{2}\left(\frac{-i\tau}{\hbar}\right)^2\cdot\left(-\frac{i\hbar a}{m}p\right)\right\}$$

$$\cdot\exp\left\{\frac{1}{6}\left(\frac{-i\tau}{\hbar}\right)^3\frac{2\hbar^2a^2}{m}\right\}$$

$$= \exp\left\{-\frac{i}{\hbar}\tau\frac{p^2}{2m}\right\}\exp\left\{-\frac{i}{\hbar}\tau ax\right\}\exp\left\{+\frac{ia}{2m\hbar}\tau^2 p\right\}\exp\left\{+\frac{i\tau^3a^2}{3\hbar m}\right\}$$

Assim

$$J(x't';xt) = \langle x'| e^{-\frac{i\tau}{\hbar}\frac{p^2}{2m}} e^{-\frac{i\tau}{\hbar}ax} e^{+\frac{ia}{\hbar m}\tau^2 p} e^{+\frac{ia^2\tau^3}{3\hbar m}} |x\rangle$$

$$J(x't';xt) = e^{+\frac{i a^2 \tau^3}{3\hbar m}} \langle x'| e^{-\frac{i\tau}{\hbar}\frac{p^2}{2m}} e^{-\frac{i\tau}{\hbar} a x} e^{+\frac{i a \tau^2}{2\hbar m} p} \underset{\mathbb{1}}{\uparrow} |x\rangle$$

$$= e^{+\frac{i a^2 \tau^3}{3\hbar m}} \int dp' \langle x'| e^{-\frac{i\tau}{\hbar}\frac{p^2}{2m}} e^{-\frac{i\tau a}{\hbar} x} e^{+\frac{i a \tau^2}{2\hbar m} p} |p'\rangle\langle p'|x\rangle$$

$$= e^{+\frac{i a^2 \tau^3}{3\hbar m}} \int dp'\, e^{+\frac{i a \tau^2}{2\hbar m} p'} \langle x'| e^{-\frac{i\tau}{\hbar}\frac{p^2}{2m}} e^{-\frac{i\tau a}{\hbar} x} \underset{\mathbb{1}}{\uparrow} |p'\rangle\langle p'|x\rangle$$

$$= e^{+\frac{i a^2 \tau^3}{3\hbar m}} \int dp' \int dx''\, e^{+\frac{i a \tau^2}{2\hbar m} p'} \langle x'| e^{-\frac{i\tau}{2m\hbar} p^2} \underset{\mathbb{1}}{\uparrow} e^{-\frac{i a\tau}{\hbar} x} |x''\rangle\langle x''|p'\rangle \langle p'|x\rangle$$

$$= e^{+\frac{i a^2 \tau^3}{3\hbar m}} \int dp' \int dx''\, e^{+\frac{i a \tau^2}{2\hbar m} p'} e^{-\frac{i a\tau}{\hbar} x''} \int dp'' \langle x'| e^{-\frac{i\tau}{2m\hbar} p^2} |p''\rangle \cdot \langle p''|x''\rangle\langle x''|p'\rangle \langle p'|x\rangle$$

$$= e^{+\frac{i a^2 \tau^3}{3\hbar m}} \int dp' \int dx'' \int dp''\, \exp\left\{+\frac{i a \tau^2}{2\hbar m} p' - \frac{i a\tau}{\hbar} x'' - \frac{i\tau}{2m\hbar} p''^2\right\} \cdot \langle x'|p''\rangle\langle p''|x''\rangle\langle x''|p'\rangle\langle p'|x\rangle.$$

usando que

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{ipx}{\hbar}} \qquad \langle p|x\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{ipx}{\hbar}}$$

temos que

$$e^{\frac{-ia^2\tau^3}{3\hbar m}} J(x't';xt) = \int dp' \int dx'' \int dp''$$

$$\exp\left\{ + \frac{ia\tau^2}{2\hbar m} p' - \frac{ia\tau}{\hbar} x'' - \frac{i\tau}{2m\hbar} p''^2 + \frac{ix'p''}{\hbar} - \frac{ip''x''}{\hbar} + \frac{ix''p'}{\hbar} - \frac{ip'x}{\hbar} \right\} \frac{1}{(2\pi\hbar)^2}$$

$$= \int dp' \int dp'' \exp\left\{ + \frac{ia\tau^2}{2\hbar m} p' - \frac{i\tau}{2m\hbar} p''^2 + \frac{ix'p''}{\hbar} - \frac{ip'x}{\hbar} \right\} \frac{1}{2\pi\hbar}$$

$$\cdot \left\{ \int \frac{dx''}{2\pi\hbar} \exp\left\{ \frac{ix''}{\hbar}\left(-a\tau - p'' + p'\right)\right\}\right\}.$$

Usando que

$$\int \frac{dx}{2\pi\hbar} e^{\frac{ipx}{\hbar}} = \delta(p),$$

temos que

$$e^{\frac{ia^2\tau^3}{3\hbar m}} J(x't';xt) = \int dp' \int dp'' \frac{1}{2\pi\hbar} \exp\left\{ + \frac{ia\tau^2}{2\hbar m} p' - \frac{i\tau}{2m\hbar} p''^2 + \frac{ix'p''}{\hbar} - \frac{ip'x}{\hbar} \right\}$$

$$\cdot\, \delta\left(p' - (p'' + a\tau)\right)$$

$$= \int dp'' \frac{1}{2\pi\hbar} \exp\left\{ + \frac{ia\tau^2}{2\hbar m}(p'' + a\tau) - \frac{i\tau}{2m\hbar} p''^2 + \frac{ix'p''}{\hbar} - \frac{ix}{\hbar}(p'' + a\tau)\right\}$$

$$= \exp\left\{ - \frac{ia^2\tau^3}{2\hbar m} - \frac{iax\tau}{\hbar}\right\} \int \frac{dp''}{2\pi\hbar} \exp\left\{ -\frac{i\tau}{2m\hbar} p''^2 + \frac{ip''}{\hbar}\left(x' - x + \frac{a\tau^2}{2m}\right)\right\}$$

A integral é feita trivialmente. Obtemos

$$e^{-\frac{i a^2 \tau^3}{3\hbar m}} J(x't';xt) = \exp\left\{ \frac{-i a^2 \tau^3}{2\hbar m} - \frac{i a x \tau}{\hbar} \right\} \frac{1}{2\pi\hbar} \cdot \sqrt{\frac{m\hbar \pi 2}{i \tau}}$$

$$\exp\left\{ + \frac{i}{8m\hbar\tau} \left(a\tau^2 + 2m(x'-x) \right)^2 \right\}$$

$$J(x't';xt) = \sqrt{\frac{m}{2\pi i \hbar \tau}} \exp\left\{ + \frac{i a^2 \tau^3}{3\hbar m} - \frac{i a^2 \tau^3}{2\hbar m} - \frac{i a x \tau}{\hbar} \right.$$

$$\left. + \frac{i}{8m\hbar\tau} \left(a\tau^2 + 2m(x'-x) \right)^2 \right\}$$

Simplificando, obtemos,

$$-\frac{i a^2 \tau^3}{6\hbar m} - \frac{i a x \tau}{\hbar} + \frac{i}{8m\hbar\tau} \left(a\tau^2 + 2m(x'-x) \right)^2 =$$

$$= -\frac{i a^2 \tau^3}{24\hbar m} - \frac{i a \tau}{2\hbar}(x+x') + \frac{i m (x'-x)^2}{2\hbar\tau}$$

Portanto

$$J(x't';xt) = \sqrt{\frac{m}{2\pi i \hbar \tau}} \exp\left\{ + \frac{i m (x'-x)^2}{2\hbar\tau} - \frac{i a \tau}{2\hbar}(x'+x) - \frac{i a^2 \tau^3}{24\hbar m} \right\}$$

2] Equações Diferenciais.

Determinamos as equações de Heisenberg para os operadores $x(t)$ e $p(t)$,

$$i\hbar \frac{d}{dt'} x(t') = [x(t'), H] = \left[x(t), \frac{p^2}{2m} + ax\right] = i\hbar \frac{p(t)}{m},$$

$$i\hbar \frac{d}{dt'} p(t') = [p(t'), H] = \left[p(t), \frac{p^2}{2m} + ax\right] = -i\hbar a.$$

Assim,

$$\frac{dx}{dt'} = \frac{p}{m}, \qquad \frac{dp}{dt'} = -a.$$

A solução para o operador p é simplesmente

$$p(t') = p(t) - a(t'-t). \qquad t' > t$$

Substituindo em x, temos que

$$\frac{dx}{dt''} = \frac{p(t) - a(t''-t)}{m}$$

$$x(t') - x(t) = \int_t^{t'} dt'' \, \frac{p(t) - a(t''-t)}{m}$$

$$x(t') = x(t) + \frac{p(t)}{m}(t'-t) - \frac{a}{m}\left\{\frac{(t'-t)^2}{2} + t(t'-t)\right\}$$

$$X(t') = X(t) + \frac{P(t)}{m}(t'-t) - \frac{a}{2m}\left\{t'^2 - \cancel{2t't} + t^2 + \cancel{2tt'} - 2t^2\right\}$$

$$X(t') = X(t) + \frac{P(t)}{m}(t'-t) - \frac{a}{2m}\left(t'^2 - t^2\right).$$

Agora, usando a equação do operador de evolução temporal

$$i\hbar \frac{\partial}{\partial t'} U(t',t) = H(t')U(t',t),$$

podemos calcular

$$i\hbar \frac{\partial}{\partial t'} \langle x't' | xt\rangle = i\hbar \frac{\partial}{\partial t'} \langle x' | U(t',t) | x\rangle$$

$$= \langle x' | H(t') U(t',t) | x\rangle,$$

lembrando que a Hamiltoniana não depende do tempo,

$$H(t')U(t',t) = H\,U(t'-t)$$
$$= H e^{-\frac{iH}{\hbar}(t'-t)}$$
$$= e^{-\frac{iH}{\hbar}t'}\, H\, e^{\frac{iH}{\hbar}t},$$

pelo que obtemos,

$$i\hbar \frac{\partial}{\partial t'} \langle x't' | xt\rangle = \langle x't' | H | xt\rangle.$$

Por outro lado, podemos considerar

$$\langle x't'|p(t')|xt\rangle = \langle x't'|p(t)|xt\rangle - a\underbrace{(t'-t)}_{\tau}\langle x't'|xt\rangle,$$

que implica a equação

$$\frac{\hbar}{i}\frac{\partial}{\partial x'}\langle x't'|xt\rangle = -\frac{\hbar}{i}\frac{\partial}{\partial x}\langle x't'|xt\rangle - a\tau\langle x't'|xt\rangle,$$

ou seja

$$\frac{\hbar}{i}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)J(x',x;\tau) + a\tau\, J(x',x;\tau) = 0$$

Assim, temos duas equações que o propagador deve obedecer,

$$i\hbar\frac{\partial}{\partial t'}J(x',x;\tau) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2}J(x',x;\tau) + a x' J(x',x;\tau), \qquad (1)$$

$$\frac{\hbar}{i}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)J(x',x;\tau) + a\tau J(x',x;\tau) = 0. \qquad (2)$$

Notemos que estamos considerando explícitamente que o propagador depende da diferença $\tau = t'-t$. Isto é provado considerando

$$i\hbar\frac{\partial}{\partial t'}\langle x't'|xt\rangle = \langle x't'|H|xt\rangle$$

$$-i\hbar\frac{\partial}{\partial t}\langle x't'|xt\rangle = \langle x't'|H|xt\rangle$$

Portanto

$$i\hbar\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial t'}\right)\langle x't'|xt\rangle = 0,$$

assim

$$\langle x't'|xt\rangle = J(x',x;\underbrace{t'-t}_{\tau}).$$

Para resolver as equações, consideramos o Ansatz,

$$J(x',x;\tau) = \sqrt{\frac{m}{2\pi i\hbar\tau}}\,\exp\left\{ i\,\frac{m(x'-x)^2}{2\hbar\tau} + i\,\varphi(x',x;\tau)\right\}.$$

Substituindo na equação (2), usando que

- $$\frac{\partial J}{\partial x} = i\left\{-\frac{m(x'-x)}{2\hbar\tau} + \frac{\partial\varphi}{\partial x}\right\}J,$$

- $$\frac{\partial J}{\partial x'} = i\left\{\frac{m(x'-x)}{2\hbar\tau} + \frac{\partial\varphi}{\partial x'}\right\}J,$$

temos

$$\frac{\hbar}{i}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial x'}\right)J + a\tau J = 0$$

$$\frac{\hbar}{\cancel{i}}\cancel{i}\left\{-\cancel{\frac{m(x'-x)}{2\hbar\tau}} + \frac{\partial\varphi}{\partial x} + \cancel{\frac{m(x'-x)}{2\hbar\tau}} + \frac{\partial\varphi}{\partial x'}\right\}J$$

$$+\, a\tau J = 0$$

temos a equação

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial x'} + \frac{a\tau}{\hbar} = 0$$

A solução é

$$\varphi(x', x; \tau) = -\frac{a\tau}{2\hbar}(x+x') + f(\tau),$$

que pode ser verificada substituindo na equação. Para determinar a função $f(\tau)$, substituimos o Ansatz para o propagador na eq. de Schrödinger (1), levando em conta que

∘) $$\frac{\partial J}{\partial x'} = i\left\{\frac{m(x'-x)}{\hbar\tau} - \frac{a\tau}{2\hbar}\right\} J$$

∘) $$\frac{\partial^2 J}{\partial x'^2} = \frac{im}{\hbar\tau} J - \left\{\frac{m(x'-x)}{\hbar\tau} - \frac{a\tau}{2\hbar}\right\}^2 J$$

∘) $$\frac{\partial J}{\partial t'} = \left\{-\frac{im(x'-x)^2}{2\hbar\tau^2} - \frac{ia(x'+x)}{2\hbar} + i\frac{df}{d\tau}\right\} J - \frac{1}{2\tau} J$$

Substituindo,

$$i\hbar\frac{\partial}{\partial t'} J = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x'^2} J + ax' J$$

$$i\hbar\left\{-\frac{im(x'-x)^2}{2\hbar\tau^2} - \frac{ia(x'+x)}{2\hbar} - \frac{1}{2\tau} + i\frac{df}{d\tau}\right\} J = -\frac{\hbar^2}{2m}\left\{\frac{im}{\hbar\tau} - \frac{m^2(x'-x)^2}{\hbar^2\tau^2}\right.$$

$$\left. - \frac{2m(x'-x)}{\hbar\cancel{\tau}}\frac{a\cancel{\tau}}{2\hbar} + \frac{a^2\tau^2}{4\hbar^2}\right\} J + ax' J$$

$$\cancel{\frac{m(x'-x)^2}{2\tau^2}} + \frac{a}{2}(x'+x) - \cancel{\frac{i\hbar}{2\tau}} - \hbar\frac{df}{d\tau} = ax' - \cancel{\frac{i\hbar}{2\tau}} + \cancel{\frac{m(x'-x)^2}{2\tau^2}}$$
$$+ \frac{a^2\tau^2}{8m} - \frac{a(x'-x)}{2}$$

$$-\hbar\frac{df}{d\tau} + \cancel{\frac{a}{2}(x'+x)} = \frac{a^2\tau^2}{8m} + \cancel{\frac{a}{2}\underbrace{(2x'-x+x)}_{x'+x}}$$

$$-\hbar\frac{df}{d\tau} = \frac{a^2\tau^2}{8m} \quad \Rightarrow \quad f(\tau) = -\frac{a^2\tau^3}{24\hbar m} + c.$$

Portanto, a solução é

$$J(x't';xt) = \sqrt{\frac{m}{2\pi i\hbar\tau}}\, \mathrm{Exp}\left\{ i\frac{m(x'-x)^2}{2\hbar\tau} - \frac{ia\tau}{2\hbar}(x'+x) - \frac{i a^2\tau^3}{24\hbar m} \right\},$$

que é idêntico ao obtido no primeiro método.

b)

Para determinar $J(x't';pt)$, notemos que

$$\langle x't'|pt\rangle = \int dx\, \langle x't'|xt\rangle\langle xt|pt\rangle$$
$$= \int \frac{dx}{\sqrt{2\pi\hbar}}\, e^{\frac{ipx}{\hbar}}\, \langle x't|xt\rangle.$$

Calculamos

$$\langle x't'|pt\rangle = \int \frac{dx}{\sqrt{2\pi\hbar}} \sqrt{\frac{m}{2\pi\hbar i\tau}} \exp\left\{ \frac{ipx}{\hbar} + \frac{im(x'-x)^2}{2\hbar\tau} - \frac{ia\tau}{2\hbar}(x'+x) - \frac{ia^2\tau^3}{24\hbar m} \right\}$$

$$= \exp\left\{ \frac{imx'^2}{2\hbar\tau} - \frac{ia\tau x'}{2\hbar} - \frac{ia^2\tau^3}{24\hbar m} \right\} \sqrt{\frac{m}{(2\pi\hbar)^2 i\tau}}$$

$$\int_{-\infty}^{\infty} dx \exp\left\{ \frac{ipx}{\hbar} + \frac{im(x^2-2x'x)}{2\hbar\tau} - \frac{ia\tau x}{2\hbar} \right\},$$

resolvemos a integral de modo padrão, para obter

$$\langle x't'|pt\rangle = \sqrt{\frac{\cancel{m}}{(2\pi\hbar)^2 \cancel{i\tau}}} \exp\left\{ \frac{imx'^2}{2\hbar\tau} - \frac{ia\tau x'}{2\hbar} - \frac{ia^2\tau^3}{24\hbar m} \right\}$$

$$\cdot \sqrt{\frac{\cancel{2\pi i\hbar\tau}}{\cancel{m}}} \exp\left\{ - \frac{i(2mx' + \tau(a\tau-2p))^2}{8\hbar m\tau} \right\}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \exp\left\{ -\frac{ip^2\tau}{2m\hbar} + ip\left[\frac{x'}{\hbar} + \frac{a\tau^2}{2\hbar m}\right] - \frac{ia\tau x'}{\hbar} - \frac{ia^2\tau^3}{6\hbar m} \right\}.$$

c)

Analogamente,

$$\langle p't'|pt\rangle = \int dx' \langle p't'|x't'\rangle \langle x't'|pt\rangle$$

$$= \int \frac{dx'}{\sqrt{2\pi\hbar}} e^{-\frac{ip'x'}{\hbar}} \langle x't'|pt\rangle$$

$$\langle p't'|pt\rangle = \int \frac{dx'}{2\pi\hbar} \exp\left\{ -\frac{ip'x'}{\hbar} - \frac{ip^2\tau}{2m\hbar} + \frac{ipx'}{\hbar} + \frac{iap\tau^2}{2\hbar m} - \frac{ia\tau x'}{\hbar} - \frac{ia^2\tau^3}{6\hbar m} \right\}$$

$$= \exp\left\{ -\frac{ip^2\tau}{2m\hbar} + \frac{iap\tau^2}{2\hbar m} - \frac{ia^2\tau^3}{6\hbar m} \right\} \underbrace{\int \frac{dx'}{2\pi\hbar} \exp\left\{ \frac{ix'}{\hbar}(p-p'-a\tau) \right\}}_{\delta(p-p'-a\tau)}$$

$$\langle p't'|pt\rangle = \exp\left\{ -\frac{i\tau}{2m\hbar}\left[p^2 - ap\tau + \frac{a^2\tau^2}{3} \right] \right\} \delta(p-p'-a\tau)$$