Gabarito: Lista 7

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Problema 1

Partindo da expressão para o potencial vetor

$$\vec{A}(\vec{x},t) = \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \vec{e}_{\alpha} + h.c. \right]$$

onde $kx = \vec{k} \cdot \vec{x} - \omega_k t$. Obtenha em termos dos operadores de criação e destruição

a) $\vec{E}$;

b) $\vec{B}$;

c) o momento linear
$$\vec{P} = \frac{1}{2c} \int d^3x \left[\vec{E} \times \vec{B} - \vec{B} \times \vec{E} \right].$$

Solução:

Primeiramente, vamos relembrar alguns fatos importantes. As equações de Maxwell, na ausência de cargas e correntes, têm a forma

$$\nabla \cdot \vec{E} = 0, \tag{1}$$
$$\nabla \cdot \vec{B} = 0, \tag{2}$$
$$\nabla \times \vec{E} + \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0, \tag{3}$$
$$\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 0. \tag{4}$$

Seguindo o procedimento padrão, escreveremos o campo $\vec{B}$ a partor do potencial vetor $\vec{A}(\vec{x},t)$, tal que

$$\vec{B} = \nabla \times \vec{A}. \tag{5}$$

Vamos também fazer uma "escolha de gauge" que consiste em considerar

$$\nabla \cdot \vec{A} = 0. \tag{6}$$

Essa escolha é chamada de Gauge de Coulomb. Uma consequência direta de (6) é que

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t}, \tag{7}$$

satisfazendo imediatamente que $\nabla \cdot \vec{E} = 0$. Temos então que determinar $\vec{A}$ é equivalente a determinar $\vec{E}$ e $\vec{B}$. Dessa forma, podemos ver que a combinação de (5) e (7) com (4) resulta em

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0. \tag{8}$$

Podemos então escrever o potencial vetor como

$$\vec{A}\left(\vec{x}, t\right) = \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \vec{e}_\alpha + e^{-ikx} a^\dagger_{\vec{k}\alpha} \vec{e}^*_\alpha \right], \tag{9}$$

onde

$$\left[a_{\vec{k}\alpha}, a^\dagger_{\vec{k}'\alpha'} \right] = \delta_{\vec{k}\vec{k}'} \delta_{\alpha\alpha'} \tag{10}$$

e

$$\vec{e}_\alpha \cdot \vec{e}_{\alpha'} = \vec{e}_\alpha \cdot \vec{e}^*_{\alpha'} = \vec{e}^*_\alpha \cdot \vec{e}^*_{\alpha'} = \delta_{\alpha\alpha'}. \tag{11}$$

Como consequência direta de (6) e (9)

$$\vec{k} \cdot \vec{A} = 0 \Longrightarrow \vec{k} \cdot \vec{e}_\alpha = \vec{k} \cdot \vec{e}^*_\alpha = 0. \tag{12}$$

a) Utilizando a expressão (9) para potencial vetor e (7) para o campo elétrico, temos que

$$\begin{aligned}
\vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} = -\frac{1}{c}\frac{\partial}{\partial t} \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \vec{e}_\alpha + e^{-ikx} a^\dagger_{\vec{k}\alpha} \vec{e}^*_\alpha \right] \\
&= -\frac{1}{c} \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[-i\omega_k e^{ikx} a_{\vec{k}\alpha} \vec{e}_\alpha + i\omega_k e^{-ikx} a^\dagger_{\vec{k}\alpha} \vec{e}^*_\alpha \right] \\
\vec{E} &= \frac{i}{c} \sum_{\vec{k}} \sum_{\alpha} \omega_k \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \vec{e}_\alpha - e^{-ikx} a^\dagger_{\vec{k}\alpha} \vec{e}^*_\alpha \right].
\end{aligned} \tag{13}$$

b) Utilizando (5), temos que

$$\begin{aligned}
B_j &= \epsilon_{jlm}\partial_l A_m = \epsilon_{jlm}\partial_l \left\{ \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \left(\vec{e}_\alpha\right)_m + e^{-ikx} a^\dagger_{\vec{k}\alpha} \left(\vec{e}^*_\alpha\right)_m \right] \right\} \\
&= \epsilon_{jlm} \left\{ \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[ik_l e^{ikx} a_{\vec{k}\alpha} \left(\vec{e}_\alpha\right)_m - ik_l e^{-ikx} a^\dagger_{\vec{k}\alpha} \left(\vec{e}^*_\alpha\right)_m \right] \right\} \\
&= i \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \epsilon_{jlm} k_l \left(\vec{e}_\alpha\right)_m - e^{-ikx} a^\dagger_{\vec{k}\alpha} \epsilon_{jlm} k_l \left(\vec{e}^*_\alpha\right)_m \right] \\
B_j &= i \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \left(\vec{k} \times \vec{e}_\alpha\right)_j - e^{-ikx} a^\dagger_{\vec{k}\alpha} \left(\vec{k} \times \vec{e}^*_\alpha\right)_j \right],
\end{aligned}$$

ou na forma vetorial

$$\vec{B} = i \sum_{\vec{k}} \sum_{\alpha} \sqrt{\frac{\hbar c}{2kV}} \left[e^{ikx} a_{\vec{k}\alpha} \vec{k} \times \vec{e}_\alpha - e^{-ikx} a^\dagger_{\vec{k}\alpha} \vec{k} \times \vec{e}^*_\alpha \right]. \tag{14}$$

c) Uma vez que conhecemos $\vec{E}$ e $\vec{B}$, podemos calcular o momento linear

$$P_j = \frac{1}{2c}\int d^3x\epsilon_{jlm}\left[E_lB_m - B_lE_m\right].$$

Temos então que

$$\begin{aligned}
E_lB_m &= -\frac{1}{c}\sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\omega_k\sqrt{\frac{\hbar c}{2kV}}\sqrt{\frac{\hbar c}{2k'V}}\left[e^{ikx}a_{\vec{k}\alpha}\left(\vec{e}_\alpha\right)_l - e^{-ikx}a^\dagger_{\vec{k}\alpha}\left(\vec{e}^*_\alpha\right)_l\right]\left[e^{ik'x}a_{\vec{k}'\alpha'}\left(\vec{k}'\times\vec{e}_{\alpha'}\right)_m - e^{-ik'x}a^\dagger_{\vec{k}'\alpha'}\left(\vec{k}'\times\vec{e}^*_{\alpha'}\right)_m\right]\\
&= -\frac{1}{c}\sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar c}{2\sqrt{kk'}V}\left[e^{i(k+k')x}a_{\vec{k}\alpha}a_{\vec{k}'\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{k}'\times\vec{e}_{\alpha'}\right)_m - e^{i(k-k')x}a_{\vec{k}\alpha}a^\dagger_{\vec{k}'\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{k}'\times\vec{e}^*_{\alpha'}\right)_m\right.\\
&\quad\left.-e^{-i(k-k')x}a^\dagger_{\vec{k}\alpha}a_{\vec{k}'\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{k}'\times\vec{e}_{\alpha'}\right)_m + e^{-i(k+k')x}a^\dagger_{\vec{k}\alpha}a^\dagger_{\vec{k}'\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{k}'\times\vec{e}^*_{\alpha'}\right)_m\right]\\
E_lB_m &= -\frac{1}{c}\sum_{\vec{k}}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar c}{2k}\left[-a_{\vec{k}\alpha}a_{-\vec{k}\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{k}\times\vec{e}_{\alpha'}\right)_m - a_{\vec{k}\alpha}a^\dagger_{\vec{k}\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{k}\times\vec{e}_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha}a^\dagger_{-\vec{k}\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)_m\right]
\end{aligned}$$

enquanto que

$$\begin{aligned}
B_lE_m &= -\frac{1}{c}\sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\omega_{k'}\sqrt{\frac{\hbar c}{2kV}}\sqrt{\frac{\hbar c}{2k'V}}\left[e^{ikx}a_{\vec{k}\alpha}\left(\vec{k}\times\vec{e}_\alpha\right)_l - e^{-ikx}a^\dagger_{\vec{k}\alpha}\left(\vec{k}\times\vec{e}^*_\alpha\right)_l\right]\left[e^{ik'x}a_{\vec{k}'\alpha'}\left(\vec{e}_{\alpha'}\right)_m - e^{-ik'x}a^\dagger_{\vec{k}'\alpha'}\left(\vec{e}^*_{\alpha'}\right)_m\right]\\
&= -\frac{1}{c}\sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\omega_{k'}\frac{\hbar c}{2\sqrt{kk'}V}\left[e^{i(k+k')x}a_{\vec{k}\alpha}a_{\vec{k}'\alpha'}\left(\vec{k}\times\vec{e}_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m - e^{i(k-k')x}a_{\vec{k}\alpha}a^\dagger_{\vec{k}'\alpha'}\left(\vec{k}\times\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right.\\
&\quad\left.-e^{-i(k-k')x}a^\dagger_{\vec{k}\alpha}a_{\vec{k}'\alpha'}\left(\vec{k}\times\vec{e}^*_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m + e^{-i(k+k')x}a^\dagger_{\vec{k}\alpha}a^\dagger_{\vec{k}'\alpha'}\left(\vec{k}\times\vec{e}^*_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right]\\
B_lE_m &= -\frac{1}{c}\sum_{\vec{k},}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar c}{2k}\left[a_{\vec{k}\alpha}a_{-\vec{k}\alpha'}\left(\vec{k}\times\vec{e}_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m - a_{\vec{k}\alpha}a^\dagger_{\vec{k}\alpha'}\left(\vec{k}\times\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha'}\left(\vec{k}\times\vec{e}^*_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m + a^\dagger_{\vec{k}\alpha}a^\dagger_{-\vec{k}\alpha'}\left(\vec{k}\times\vec{e}^*_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right],
\end{aligned}$$

onde utilizamos que

$$\frac{1}{V}\int d^3xe^{\pm i\left(\vec{k}\pm\vec{k}'\right)} = \delta_{\vec{k},\mp\vec{k}'}. \tag{15}$$

Temos então que

$$\begin{aligned}
P_j = -\frac{1}{2c^2}\epsilon_{jlm}\sum_{\vec{k}}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar c}{2k}&\left[-a_{\vec{k}\alpha}a_{-\vec{k}\alpha'}\left(\left(\vec{e}_\alpha\right)_l\left(\vec{k}\times\vec{e}_{\alpha'}\right)_m + \left(\vec{k}\times\vec{e}_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m\right)\right.\\
&- a_{\vec{k}\alpha}a^\dagger_{\vec{k}\alpha'}\left(\left(\vec{e}_\alpha\right)_l\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)_m - \left(\vec{k}\times\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right)\\
&- a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha'}\left(\left(\vec{e}^*_\alpha\right)_l\left(\vec{k}\times\vec{e}_{\alpha'}\right)_m - \left(\vec{k}\times\vec{e}^*_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m\right)\\
&\left.-a^\dagger_{\vec{k}\alpha}a^\dagger_{-\vec{k}\alpha'}\left(\left(\vec{e}^*_\alpha\right)_l\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)_m + \left(\vec{k}\times\vec{e}^*_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right)\right].
\end{aligned}$$

Podemos utilizar o símbolo anti-simétrico ϵ_{jlm} para simplificar a expressão logo acima:

$$\begin{aligned}
\vec{e}_\alpha\times\left(\vec{k}\times\vec{e}^*_{\alpha'}\right) &= \epsilon_{jlm}\left(\vec{e}_\alpha\right)_l\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)_m = \epsilon_{jlm}\left(\vec{e}_\alpha\right)_l\epsilon_{mpq}k_p\left(\vec{e}^*_{\alpha'}\right)_q = \epsilon_{mjl}\epsilon_{mpq}k_p\left(\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_q\\
&= \left(\delta_{jp}\delta_{lq} - \delta_{jq}\delta_{lp}\right)k_p\left(\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_q\\
&= k_j\left(\vec{e}_\alpha\right)_l\delta_{lq}\left(\vec{e}^*_{\alpha'}\right)_q - k_p\delta_{pl}\left(\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_j\\
&= \left(\vec{e}_\alpha\cdot\vec{e}^*_{\alpha'}\right)\vec{k} - \left(\vec{k}\cdot\vec{e}_\alpha\right)\vec{e}^*_{\alpha'}\\
\vec{e}_\alpha\times\left(\vec{k}\times\vec{e}^*_{\alpha'}\right) &= \delta_{\alpha\alpha'}\vec{k},
\end{aligned}$$

onde utilizamos que $\vec{k}\cdot\vec{e}_\alpha = 0$, como consequência do gauge de Coulomb (6), e $\vec{e}_\alpha\cdot\vec{e}^*_{\alpha'} = \delta_{\alpha\alpha'}$. Temos então que

$$\epsilon_{jlm}\left((\vec{e}_\alpha)_l\left(\vec{k}\times\vec{e}_{\alpha'}\right)_m+\left(\vec{k}\times\vec{e}_\alpha\right)_l(\vec{e}_{\alpha'})_m\right)=\left(\vec{e}_\alpha\times\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)-\vec{e}_{\alpha'}\times\left(\vec{k}\times\vec{e}^*_\alpha\right)\right)=0,$$
$$\epsilon_{jlm}\left((\vec{e}^*_\alpha)_l\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)_m+\left(\vec{k}\times\vec{e}^*_\alpha\right)_l(\vec{e}^*_{\alpha'})_m\right)=\left(\vec{e}^*_\alpha\times\left(\vec{k}\times\vec{e}^*_{\alpha'}\right)-\vec{e}^*_{\alpha'}\times\left(\vec{k}\times\vec{e}^*_\alpha\right)\right)=0,$$

permitindo então reescrever a expressão para $\vec{P}$ como

$$\begin{aligned}
\vec{P} &= -\frac{1}{2c^2}\sum_{\vec{k}}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar c}{2k}\left[-2a_{\vec{k}\alpha}a^\dagger_{\vec{k}\alpha'}\delta_{\alpha\alpha'}\vec{k}-2a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha'}\delta_{\alpha\alpha'}\vec{k}\right]\\
&= \frac{1}{c}\sum_{\vec{k}}\sum_{\alpha}ck\frac{\hbar}{2k}\left[a_{\vec{k}\alpha}a^\dagger_{\vec{k}\alpha}+a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha}\right]\vec{k}\\
&= \sum_{\vec{k}}\sum_{\alpha}\frac{\hbar}{2}\left[2a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha}+1\right]\vec{k}\\
&= \hbar\sum_{\vec{k}}\sum_{\alpha}\left[a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha}+\frac{1}{2}\right]\vec{k}.\\
\vec{P} &= \hbar\sum_{\vec{k}}\sum_{\alpha}a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha}, \qquad (16)
\end{aligned}$$

onde o fator de 1/2 foi elininado por conta da simetria do somatório em $\vec{k}$.

Problema 2

Mostre que

$$\left[\vec{x}\times\left(\vec{E}\times\vec{B}\right)\right]_i = E_l\left(\vec{x}\times\nabla\right)_i A_l - \nabla_l\left(E_l\epsilon_{ijk}x_jA_k\right)+\epsilon_{ilk}E_lA_k.$$

O último termo desta expressão independe da origem do sistema de coordenadas e representa o spin do fóton. Escreva o operador associado a este termo como função dos operadores de criação e destruição.

Solução:

Sabendo que o campo magnético $\vec{B}$ é dado a partir do potencial vetor $\vec{A}$ por meio da expressão (5), temos que

$$\vec{x}\times\left(\vec{E}\times\vec{B}\right) = \vec{x}\times\left(\vec{E}\times\left(\nabla\times\vec{A}\right)\right).$$

Vamos primeiro calcular

$$\begin{aligned}
\left(\vec{E}\times\left(\nabla\times\vec{A}\right)\right)_i &= \epsilon_{ijk}E_j\left(\nabla\times\vec{A}\right)_k = \epsilon_{ijk}\epsilon_{klm}E_j\nabla_lA_m \\
&= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)E_j\nabla_lA_m \\
\left(\vec{E}\times\left(\nabla\times\vec{A}\right)\right)_i &= E_m\nabla_iA_m - E_l\nabla_lA_i
\end{aligned}$$

nos permitindo então demonstrar que

$$\begin{aligned}
\left[\vec{x}\times\left(\vec{E}\times\vec{B}\right)\right]_i &= \epsilon_{ijk}x_j\left(E_l\nabla_kA_l - E_l\nabla_lA_k\right) \\
&= \epsilon_{ijk}x_jE_l\nabla_kA_l - \epsilon_{ijk}x_jE_l\nabla_lA_k \\
&= E_l\left(\epsilon_{ijk}x_j\nabla_k\right)A_l - \epsilon_{ijk}\nabla_l\left(x_jE_lA_k\right)+\epsilon_{ijk}\nabla_l\left(x_jE_l\right)A_k \\
&= E_l\left(\vec{x}\times\nabla\right)_iA_l - \nabla_l\left(E_l\epsilon_{ijk}x_jA_k\right)+\epsilon_{ijk}\left[\left(\nabla_lx_j\right)E_l + x_j\cancelto{0}{\left(\nabla_lE_l\right)}\right]A_k \\
\left[\vec{x}\times\left(\vec{E}\times\vec{B}\right)\right]_i &= E_l\left(\vec{x}\times\nabla\right)_iA_l - \nabla_l\left(E_l\epsilon_{ijk}x_jA_k\right)+\epsilon_{ilk}E_lA_k.
\end{aligned} \tag{17}$$

O último termo da expressão acima é o operador que mede o spin do fótion

$$\vec{J} = \frac{1}{2c}\int d^3x\left[\vec{E}\times\vec{A}-\vec{A}\times\vec{E}\right].$$

Temos então que

$$\begin{aligned}
\epsilon_{ilm}E_lA_m &= \epsilon_{ilm}\frac{i}{c}\sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar c}{2\sqrt{kk'}V}\left[e^{ikx}a_{\vec{k}\alpha}\left(\vec{e}_\alpha\right)_l - e^{-ikx}a^\dagger_{\vec{k}\alpha}\left(\vec{e}^*_\alpha\right)_l\right]\left[e^{ik'x}a_{\vec{k}'\alpha'}\left(\vec{e}_{\alpha'}\right)_m + e^{-ik'x}a^\dagger_{\vec{k}'\alpha'}\left(\vec{e}^*_{\alpha'}\right)_m\right] \\
&= i\epsilon_{ilm}\sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar}{2\sqrt{kk'}V}\left[e^{i\left(k+k'\right)x}a_{\vec{k}\alpha}a_{\vec{k}'\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m + e^{i\left(k-k'\right)x}a_{\vec{k}\alpha}a^\dagger_{\vec{k}'\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right. \\
&\quad\left. -e^{-i\left(k-k'\right)x}a^\dagger_{\vec{k}\alpha}a_{\vec{k}'\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m - e^{-i\left(k+k'\right)x}a^\dagger_{\vec{k}\alpha}a^\dagger_{\vec{k}'\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right] \\
&= i\epsilon_{ilm}\sum_{\vec{k}}\sum_{\alpha,\alpha'}\omega_k\frac{\hbar}{2k}\left[a_{\vec{k}\alpha}a_{-\vec{k}\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m + a_{\vec{k}\alpha}a^\dagger_{\vec{k}\alpha'}\left(\vec{e}_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha}a_{\vec{k}\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{e}_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha}a^\dagger_{-\vec{k}\alpha'}\left(\vec{e}^*_\alpha\right)_l\left(\vec{e}^*_{\alpha'}\right)_m\right].
\end{aligned}$$

Analogamente

$$\epsilon_{ilm} A_l E_m = \epsilon_{ilm} \left(E_m A_l\right)^\dagger = -\epsilon_{ilm} \left(E_l A_m\right)^\dagger$$

$$\epsilon_{ilm} A_l E_m = i\epsilon_{ilm} \sum_{\vec{k},} \sum_{\alpha,\alpha'} \omega_k \frac{\hbar}{2k} \left[a^\dagger_{\vec{k}\alpha} a^\dagger_{-\vec{k}\alpha'} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}^*_{\alpha'}\right)_m + a_{\vec{k}\alpha'} a^\dagger_{\vec{k}\alpha} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha'} a_{\vec{k}\alpha} \left(\vec{e}_\alpha\right)_l \left(\vec{e}^*_{\alpha'}\right)_m - a_{\vec{k}\alpha} a_{-\vec{k}\alpha'} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}^*_{\alpha'}\right)_m\right],$$

de forma que

$$J_i = \frac{i}{2c} \epsilon_{ilm} \sum_{\vec{k},} \sum_{\alpha,\alpha'} \omega_k \frac{\hbar}{2k} \left[a_{\vec{k}\alpha} a^\dagger_{\vec{k}\alpha'} \left(\vec{e}_\alpha\right)_l \left(\vec{e}^*_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha'} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m - a_{\vec{k}\alpha'} a^\dagger_{\vec{k}\alpha} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m + a^\dagger_{\vec{k}\alpha'} a_{\vec{k}\alpha} \left(\vec{e}_\alpha\right)_l \left(\vec{e}^*_{\alpha'}\right)_m\right].$$

Vamos então utilizar que

$$\vec{e}^*_\alpha \times \vec{e}_{\alpha'} = i\alpha\delta_{\alpha,\alpha'}\hat{k},$$

nos permitindo então a eliminar o somatório em α', de modo que

$$\begin{aligned}
J_i &= \frac{i}{2c} \epsilon_{ilm} \sum_{\vec{k},} \sum_{\alpha,\alpha'} \omega_k \frac{\hbar}{2k} \left[a_{\vec{k}\alpha} a^\dagger_{\vec{k}\alpha'} \left(\vec{e}^*_{\alpha'}\right)_m \left(\vec{e}_\alpha\right)_l - a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha'} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m - a_{\vec{k}\alpha'} a^\dagger_{\vec{k}\alpha} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m + a^\dagger_{\vec{k}\alpha'} a_{\vec{k}\alpha} \left(\vec{e}^*_{\alpha'}\right)_m \left(\vec{e}_\alpha\right)_l\right] \\
&= \frac{i}{2c} \epsilon_{ilm} \sum_{\vec{k},} \sum_{\alpha,\alpha'} \omega_k \frac{\hbar}{2k} \left[-a_{\vec{k}\alpha} a^\dagger_{\vec{k}\alpha'} \left(\vec{e}^*_{\alpha'}\right)_l \left(\vec{e}_\alpha\right)_m - a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha'} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m - a_{\vec{k}\alpha'} a^\dagger_{\vec{k}\alpha} \left(\vec{e}^*_\alpha\right)_l \left(\vec{e}_{\alpha'}\right)_m - a^\dagger_{\vec{k}\alpha'} a_{\vec{k}\alpha} \left(\vec{e}^*_{\alpha'}\right)_l \left(\vec{e}_\alpha\right)_m\right] \\
&= -\frac{i}{2c} \sum_{\vec{k},} \sum_{\alpha,\alpha'} i\alpha\delta_{\alpha,\alpha'} \omega_k \frac{\hbar}{2k} \left[a_{\vec{k}\alpha} a^\dagger_{\vec{k}\alpha'} + a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha'} + a_{\vec{k}\alpha'} a^\dagger_{\vec{k}\alpha} + a^\dagger_{\vec{k}\alpha'} a_{\vec{k}\alpha}\right] \hat{k}_i \\
&= \frac{1}{2c} \sum_{\vec{k}} \sum_{\alpha} \alpha\omega_k \frac{\hbar}{2k} 2 \left[a_{\vec{k}\alpha} a^\dagger_{\vec{k}\alpha} + a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha}\right] \hat{k}_i \\
&= \frac{1}{2c} \sum_{\vec{k}} \sum_{\alpha} \alpha k c \frac{\hbar}{k} \left[2a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} + 1\right] \hat{k}_i \\
J_i &= \hbar \sum_{\vec{k}} \sum_{\alpha} \alpha \left[a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} + \frac{1}{2}\right] \hat{k}_i.
\end{aligned}$$

Por fim, temos que

$$\vec{J} = \hbar \sum_{\vec{k}} \sum_{\alpha} \alpha a^\dagger_{\vec{k}\alpha} a_{\vec{k}\alpha} \hat{k}, \tag{18}$$

onde novamente o fator de 1/2 foi elininado por conta da simetria do somatório em $\vec{k}$.

Problema 3

Utilizando a base de helicidade para as polarizações, obtenha que

$$\vec{J} \cdot \frac{\vec{k}}{k} \left| \vec{k}, \lambda \right\rangle = \hbar\lambda \left| \vec{k}, \lambda \right\rangle .$$

Solução:

O operador $\vec{J}$ é dado pela expressão (18). Temos então que

$$\vec{J} \cdot \frac{\vec{k}}{k} = \vec{J} \cdot \hat{k} = \hbar \sum_{\vec{k}'} \sum_{\alpha} \alpha a^{\dagger}_{\vec{k}'\alpha} a_{\vec{k}'\alpha} \hat{k}' \cdot \hat{k}.$$

Dessa forma

$$\begin{aligned}
\vec{J} \cdot \hat{k} \left| \vec{k}, \alpha \right\rangle &= \hbar \sum_{\vec{k}'} \sum_{\alpha'} \alpha' a^{\dagger}_{\vec{k}'\alpha'} a_{\vec{k}'\alpha'} \hat{k}' \cdot \hat{k} a^{\dagger}_{\vec{k}\alpha} \left|0\right\rangle \\
&= \hbar \sum_{\vec{k}'} \sum_{\alpha'} \alpha' a^{\dagger}_{\vec{k}'\alpha'} \hat{k}' \cdot \hat{k} \left(\delta_{\vec{k}'\vec{k}} \delta_{\alpha\alpha'} + a^{\dagger}_{\vec{k}\alpha} a_{\vec{k}'\alpha'} \right) \left|0\right\rangle \\
&= \hbar\alpha a^{\dagger}_{\vec{k}\alpha} \hat{k} \cdot \hat{k} \left|0\right\rangle \\
\vec{J} \cdot \hat{k} \left| \vec{k}, \alpha \right\rangle &= \hbar\alpha \left| \vec{k}, \alpha \right\rangle . \qquad (19)
\end{aligned}$$

Problema 4

Mostre que o operador operador linear é o gerador de translações, i.e.,

$$T\left(\vec{s}\right)=e^{-i\vec{s}\cdot\vec{P}/\hbar}$$

é tal que

$$T^{\dagger}\left(\vec{s}\right)\vec{E}\left(\vec{r},t\right)T\left(\vec{s}\right)=\vec{E}\left(\vec{r}-\vec{s},t\right).$$

Solução:

Sabendo que o operador de momento linear (15) é dado por

$$\vec{P}=\hbar\sum_{\vec{k}}\sum_{\alpha}N_{\vec{k}\alpha}\vec{k}, \tag{20}$$

onde introduzimos o operador número

$$N_{\vec{k}\alpha}=a_{\vec{k}\alpha}^{\dagger}a_{\vec{k}\alpha},$$

temos que

$$T\left(\vec{s}\right)=\exp\left[-\frac{i}{\hbar}\hbar\sum_{\vec{k}}\sum_{\alpha}N_{\vec{k}\alpha}\vec{s}\cdot\vec{k}\right]=\prod_{\vec{k},\alpha}\exp\left[-iN_{\vec{k}\alpha}\vec{s}\cdot\vec{k}\right]. \tag{21}$$

Temos então que

$$T^{\dagger}\left(\vec{s}\right)\vec{E}\left(\vec{r},t\right)T\left(\vec{s}\right)=\frac{i}{c}\sum_{\vec{k}}\sum_{\alpha}\omega_k\sqrt{\frac{\hbar c}{2kV}}\left[e^{ikx}\left(T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}T\left(\vec{s}\right)\right)\vec{e}_{\alpha}-e^{-ikx}\left(T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}^{\dagger}T\left(\vec{s}\right)\right)\vec{e}_{\alpha}^{*}\right].$$

O problema agora se resume a calcular a seguinte quantidade:

$$\begin{aligned}
T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}T\left(\vec{s}\right)&=\prod_{\vec{k}',\alpha'}\prod_{\vec{k}'',\alpha''}\exp\left[iN_{\vec{k}'\alpha'}\vec{s}\cdot\vec{k}'\right]a_{\vec{k}\alpha}\exp\left[-iN_{\vec{k}''\alpha''}\vec{s}\cdot\vec{k}''\right]\\
T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}T\left(\vec{s}\right)&=\exp\left[iN_{\vec{k}\alpha}\vec{s}\cdot\vec{k}\right]a_{\vec{k}\alpha}\exp\left[-iN_{\vec{k}\alpha}\vec{s}\cdot\vec{k}\right],
\end{aligned}$$

onde lançamos mão do fato de que a relação de comutação entre $a_{\vec{k}\alpha}$ e $N_{\vec{k}''\alpha''}$ só é não-trivial quanto $\vec{k}=\vec{k}''$. Feito isso, utilizando o lema de Baker-Hausdorff

$$\exp\left(iG\lambda\right)A\exp\left(-iG\lambda\right)=A+i\lambda\left[G,A\right]+\left(\frac{i^2\lambda^2}{2!}\right)\left[G,\left[G,A\right]\right]+\left(\frac{i^3\lambda^3}{3!}\right)\left[G,\left[G,\left[G,A\right]\right]\right]+\cdots, \tag{22}$$

onde G é um operador hermitiano e λ é um parâmtro real, e a relação de comutação

$$\begin{aligned}
\left[a_{\vec{k}\alpha},N_{\vec{k}'\alpha'}\right]&=\left[a_{\vec{k}\alpha},a_{\vec{k}'\alpha'}^{\dagger}a_{\vec{k}'\alpha'}\right]\\
&=\left[a_{\vec{k}\alpha},a_{\vec{k}'\alpha'}^{\dagger}\right]a_{\vec{k}'\alpha'}+a_{\vec{k}'\alpha'}^{\dagger}\left[a_{\vec{k}\alpha},a_{\vec{k}'\alpha'}\right]\\
\left[a_{\vec{k}\alpha},N_{\vec{k}'\alpha'}\right]&=\delta_{\vec{k}\vec{k}'}\delta_{\alpha\alpha'}a_{\vec{k}'\alpha'},
\end{aligned} \tag{23}$$

é possível verificar que

$$
\begin{aligned}
T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}T\left(\vec{s}\right) &= a_{\vec{k}\alpha} - i\vec{s}\cdot\vec{k}a_{\vec{k}\alpha} + \left(\frac{i^{2}\left(\vec{s}\cdot\vec{k}\right)^{2}}{2!}\right)a_{\vec{k}\alpha} - \left(\frac{i^{3}\left(\vec{s}\cdot\vec{k}\right)^{3}}{3!}\right)a_{\vec{k}\alpha} + \cdots \\
&= a_{\vec{k}\alpha}\left[1 - i\vec{s}\cdot\vec{k} + \left(\frac{i^{2}\left(\vec{s}\cdot\vec{k}\right)^{2}}{2!}\right)a_{\vec{k}\alpha} - \left(\frac{i^{3}\left(\vec{s}\cdot\vec{k}\right)^{3}}{3!}\right)a_{\vec{k}\alpha} + \cdots\right] \\
T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}T\left(\vec{s}\right) &= a_{\vec{k}\alpha}\exp\left(-i\vec{s}\cdot\vec{k}\right). \qquad (24)
\end{aligned}
$$

Analogamente

$$
T^{\dagger}\left(\vec{s}\right)a^{\dagger}_{\vec{k}\alpha}T\left(\vec{s}\right) = a^{\dagger}_{\vec{k}\alpha}\exp\left(i\vec{s}\cdot\vec{k}\right). \qquad (25)
$$

Por fim, temos que

$$
\begin{aligned}
T^{\dagger}\left(\vec{s}\right)\vec{E}\left(\vec{r},t\right)T\left(\vec{s}\right) &= \frac{i}{c}\sum_{\vec{k}}\sum_{\alpha}\omega_{k}\sqrt{\frac{\hbar c}{2kV}}\left[e^{ikx}\left(T^{\dagger}\left(\vec{s}\right)a_{\vec{k}\alpha}T\left(\vec{s}\right)\right)\vec{e}_{\alpha} - e^{-ikx}\left(T^{\dagger}\left(\vec{s}\right)a^{\dagger}_{\vec{k}\alpha}T\left(\vec{s}\right)\right)\vec{e}^{*}_{\alpha}\right] \\
&= \frac{i}{c}\sum_{\vec{k}}\sum_{\alpha}\omega_{k}\sqrt{\frac{\hbar c}{2kV}}\left[e^{ikx}a_{\vec{k}\alpha}e^{-i\vec{s}\cdot\vec{k}}\vec{e}_{\alpha} - e^{-ikx}a^{\dagger}_{\vec{k}\alpha}e^{i\vec{s}\cdot\vec{k}}\vec{e}^{*}_{\alpha}\right] \\
&= \frac{i}{c}\sum_{\vec{k}}\sum_{\alpha}\omega_{k}\sqrt{\frac{\hbar c}{2kV}}\left[e^{i\vec{k}\cdot\left(\vec{r}-\vec{s}\right)}e^{-i\omega_{k}t}a_{\vec{k}\alpha}\vec{e}_{\alpha} - e^{-i\vec{k}\cdot\left(\vec{r}-\vec{s}\right)}e^{i\omega_{k}t}a^{\dagger}_{\vec{k}\alpha}\vec{e}^{*}_{\alpha}\right] \\
T^{\dagger}\left(\vec{s}\right)\vec{E}\left(\vec{r},t\right)T\left(\vec{s}\right) &= \vec{E}\left(\vec{r}-\vec{s},t\right). \qquad (26)
\end{aligned}
$$

Problema 5

Calcule os seguintes comutadores:

a) $[B_j(\vec{r}_1,t_1), B_l(\vec{r}_2,t_2)]$;

b) $[E_j(\vec{r}_1,t_1), B_l(\vec{r}_2,t_2)]$;

Solução:

Vamos começar calculando o seguinte comutador:

$$\begin{aligned}
[A_j(\vec{r}_1,t_1), A_l(\vec{r}_2,t_2)] &= \sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\frac{\hbar c}{2\sqrt{kk'}V}\left[e^{ikx_1}a_{\vec{k}\alpha}(\vec{e}_\alpha)_j + e^{-ikx_1}a^\dagger_{\vec{k}\alpha}(\vec{e}^*_\alpha)_j, e^{ik'x_2}a_{\vec{k}'\alpha'}(\vec{e}_{\alpha'})_l + e^{-ik'x_2}a^\dagger_{\vec{k}'\alpha'}(\vec{e}^*_{\alpha'})_l\right] \\
&= \sum_{\vec{k},\vec{k}'}\sum_{\alpha,\alpha'}\frac{\hbar c}{2\sqrt{kk'}V}\left(e^{i\left(kx_1-k'x_2\right)}\left[a_{\vec{k}\alpha}, a^\dagger_{\vec{k}'\alpha'}\right](\vec{e}_\alpha)_j(\vec{e}^*_{\alpha'})_l + e^{-i\left(kx_1-k'x_2\right)}\left[a^\dagger_{\vec{k}\alpha}, a_{\vec{k}'\alpha'}\right](\vec{e}^*_\alpha)_j(\vec{e}_{\alpha'})_l\right) \\
&= \sum_{\vec{k}}\sum_{\alpha}\frac{\hbar c}{2kV}\left(e^{ik(x_1-x_2)}(\vec{e}_\alpha)_j(\vec{e}^*_\alpha)_l - e^{-ik(x_1-x_2)}(\vec{e}^*_\alpha)_j(\vec{e}_\alpha)_l\right) \\
[A_j(\vec{r}_1,t_1), A_l(\vec{r}_2,t_2)] &= \sum_{\alpha}\int\frac{d^3k}{(2\pi)^3}\frac{\hbar c}{2k}\left(e^{ik(x_1-x_2)}(\vec{e}_\alpha)_j(\vec{e}^*_\alpha)_l - e^{-ik(x_1-x_2)}(\vec{e}^*_\alpha)_j(\vec{e}_\alpha)_l\right).
\end{aligned}$$

No resultado logo acima, utilizamos o limite para continuo

$$\frac{1}{V}\sum_{\vec{k}} \to \int\frac{d^3x}{(2\pi)^3}, \qquad \frac{V}{(2\pi)^3}\delta_{\vec{k},\vec{k}'} \to \delta\left(\vec{k}-\vec{k}'\right).$$

Considerando que $\vec{e}^*_\alpha = \vec{e}_\alpha$, podemos escrever que

$$t_{jl} = \sum_\alpha (\vec{e}_\alpha)_j(\vec{e}_\alpha)_l = \delta_{jl} - \hat{k}_j\hat{k}_l,$$

possibilitando então que

$$\begin{aligned}
[A_j(\vec{r}_1,t_1), A_l(\vec{r}_2,t_2)] &= \int\frac{d^3k}{(2\pi)^3}t_{jl}\frac{\hbar c}{2k}\left(e^{ik(x_1-x_2)} - e^{-ik(x_1-x_2)}\right) \\
&= \int\frac{d^3k}{(2\pi)^3}t_{jl}\frac{\hbar c}{2k}\left(e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{-i\omega_k(t_1-t_2)} - e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}e^{i\omega_k(t_1-t_2)}\right) \\
&= \int\frac{d^3k}{(2\pi)^3}t_{jl}\frac{\hbar c}{2k}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\left(e^{-i\omega_k(t_1-t_2)} - e^{i\omega_k(t_1-t_2)}\right) \\
[A_j(\vec{r}_1,t_1), A_l(\vec{r}_2,t_2)] &= -2i\hbar c\int\frac{d^3k}{2k(2\pi)^3}t_{jl}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\sin\left(\omega_k(t_1-t_2)\right).
\end{aligned} \tag{27}$$

Estamos agora em condições de calcular os comutadores pedidos pelo problema!

Sabendo que $\vec{B} = \nabla\times\vec{A}$, temos que

$$\begin{aligned}
[B_j(\vec{r}_1,t_1), B_l(\vec{r}_2,t_2)] &= \epsilon_{jab}\epsilon_{lmn}\nabla^{(1)}_a\nabla^{(2)}_m[A_b(\vec{r}_1,t_1), A_n(\vec{r}_2,t_2)] \\
&= -2i\hbar c\epsilon_{jab}\epsilon_{lmn}\nabla^{(1)}_a\nabla^{(2)}_m\int\frac{d^3k}{2k(2\pi)^3}t_{bn}e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\sin\left(\omega_k(t_1-t_2)\right) \\
[B_j(\vec{r}_1,t_1), B_l(\vec{r}_2,t_2)] &= -2i\hbar c\epsilon_{jab}\epsilon_{lmn}\int\frac{d^3k}{2k(2\pi)^3}t_{bn}(ik_a)(-ik_m)e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}\sin\left(\omega_k(t_1-t_2)\right).
\end{aligned}$$

Em order de simplificar a expressão acima, podemos utlizar que

$$\begin{aligned}
\epsilon_{jab}\epsilon_{lmn}t_{bn}k_ak_m &= \epsilon_{jab}\epsilon_{lmn}\left(\delta_{bn}-\hat{k}_b\hat{k}_n\right)k_ak_m \\
&= \epsilon_{jab}\epsilon_{lmb}k_ak_m - \cancelto{0}{\epsilon_{jab}\epsilon_{lmn}k_a\hat{k}_b\hat{k}_nk_m} \\
&= \left(\delta_{jl}\delta_{am}-\delta_{jm}\delta_{al}\right)k_ak_m \\
\epsilon_{jab}\epsilon_{lmn}t_{bn}k_ak_m &= \delta_{jl}k^2-k_jk_l,
\end{aligned}$$

resultando em

$$\begin{aligned}
\left[B_j\left(\vec{r}_1,t_1\right),B_l\left(\vec{r}_2,t_2\right)\right] &= -2i\hbar c\int\frac{d^3k}{2k\left(2\pi\right)^3}\left(\delta_{jl}k^2-k_jk_l\right)e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right) \\
&= -2i\hbar c\int\frac{d^3k}{2k\left(2\pi\right)^3}\left(\delta_{jl}\frac{1}{c^2}\frac{\partial^2}{\partial t_1\partial t_2}-\nabla_j^{(1)}\nabla_l^{(2)}\right)e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right) \\
&= -2i\hbar c\left(\delta_{jl}\frac{1}{c^2}\frac{\partial^2}{\partial t_1\partial t_2}-\nabla_j^{(1)}\nabla_l^{(2)}\right)\int\frac{d^3k}{2k\left(2\pi\right)^3}e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right) \\
\left[B_j\left(\vec{r}_1,t_1\right),B_l\left(\vec{r}_2,t_2\right)\right] &= 2i\hbar c\left(\delta_{jl}\frac{1}{c^2}\frac{\partial^2}{\partial t_1\partial t_2}-\nabla_j^{(1)}\nabla_l^{(2)}\right)D\left(r,t\right), \qquad (28)
\end{aligned}$$

onde

$$D\left(r,t\right) = -\int\frac{d^3k}{2k\left(2\pi\right)^3}e^{i\vec{k}\cdot\vec{x}}\sin\left(\omega_kt\right), \qquad (29)$$

com $\vec{x}=\vec{x}_1-\vec{x}_2$ e $t=t_1-t_2$.

Utilizando a expressão (27), podemos também calcular o comutador entre diferentes componentes do campo elétrico, ou seja,

$$\begin{aligned}
\left[E_j\left(\vec{r}_1,t_1\right),B_l\left(\vec{r}_2,t_2\right)\right] &= -\frac{1}{c}\frac{\partial}{\partial t_1}\epsilon_{lmn}\nabla_m^{(2)}\left[A_j\left(\vec{r}_1,t_1\right),A_n\left(\vec{r}_2,t_2\right)\right] \\
&= 2i\hbar\epsilon_{lmn}\frac{\partial}{\partial t_1}\nabla_m^{(2)}\int\frac{d^3k}{2k\left(2\pi\right)^3}t_{jn}e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right) \\
\left[E_j\left(\vec{r}_1,t_1\right),B_l\left(\vec{r}_2,t_2\right)\right] &= 2i\hbar\epsilon_{lmn}\frac{\partial}{\partial t_1}\int\frac{d^3k}{2k\left(2\pi\right)^3}\left(-ik_m\right)t_{jn}e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right).
\end{aligned}$$

A fim de simplificarmos a expressão logo acima, podemos utilizar que

$$\epsilon_{lmn}k_mt_{jn} = \epsilon_{lmn}k_m\left(\delta_{jn}-\hat{k}_j\hat{k}_n\right) = \epsilon_{lmj}k_m - \cancelto{0}{\epsilon_{lmn}k_m\hat{k}_j\hat{k}_n} = \epsilon_{lmj}k_m,$$

implicando que

$$\begin{aligned}
\left[E_j\left(\vec{r}_1,t_1\right),B_l\left(\vec{r}_2,t_2\right)\right] &= 2i\hbar\frac{\partial}{\partial t_1}\int\frac{d^3k}{2k\left(2\pi\right)^3}\left(-i\epsilon_{lmj}k_m\right)e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right) \\
&= 2i\hbar\epsilon_{lmj}\frac{\partial}{\partial t_1}\nabla_m^{(2)}\int\frac{d^3k}{2k\left(2\pi\right)^3}e^{i\vec{k}\cdot\left(\vec{x}_1-\vec{x}_2\right)}\sin\left(\omega_k\left(t_1-t_2\right)\right) \\
&= -2i\hbar\epsilon_{lmj}\frac{\partial}{\partial t_1}\nabla_m^{(2)}D\left(r,t\right) \\
\left[E_j\left(\vec{r}_1,t_1\right),B_l\left(\vec{r}_2,t_2\right)\right] &= 2i\hbar\epsilon_{ljm}\frac{\partial}{\partial t_1}\nabla_m^{(2)}D\left(r,t\right), \qquad (30)
\end{aligned}$$

onde é direto que comutador (30) é zero para os casos $j=l$. Para os casos $j\neq l$ uma analise mais cuidadosa deve ser feita.

O próximo passo é calcular efetivamente $D(r,t)$:

$$
\begin{aligned}
D(r,t) &= -\int \frac{d^3k}{2k(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \sin(\omega_k t) = \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} \frac{k^2 dk}{2k} e^{ikr\cos\theta} \sin(kct) \\
&= -\frac{2\pi}{2(2\pi)^3}\int_0^{\infty} kdk \sin(kct) \int_{-1}^{1} d\mu e^{ikr\mu} \\
&= -\frac{1}{(2\pi)^2}\int_0^{\infty} kdk \frac{\sin(kr)}{kr} \sin(kct) \\
&= -\frac{1}{(2\pi)^2}\int_0^{\infty} kdk \frac{\sin(kr)}{kr} \sin(kct) \\
&= -\frac{1}{4\pi^2 r}\int_0^{\infty} dk \sin(kr) \sin(kct) \\
&= -\frac{1}{8\pi^2 r}\int_{-\infty}^{\infty} dk \sin(kr) \sin(kct) \\
&= -\frac{1}{8\pi^2 r}\int_{-\infty}^{\infty} dk \left(\frac{e^{ikr} - e^{-ikr}}{2i}\right)\left(\frac{e^{ikct} - e^{-ikct}}{2i}\right) \\
&= \frac{1}{8\pi^2 r}\int_{-\infty}^{\infty} dk \left(\frac{e^{ik(r+ct)} - e^{ik(r-ct)} - e^{-ik(r-ct)} + e^{-ik(r+ct)}}{4}\right) \\
&= \frac{1}{16\pi r}\left[2\delta(r+ct) - 2\delta(r-ct)\right] \\
D(r,t) &= \frac{1}{8\pi r}\left[\delta(r+ct) - \delta(r-ct)\right].
\end{aligned} \tag{31}
$$

Por fim, obtemos que

$$
\begin{aligned}
\left[B_j(\vec{r}_1,t_1), B_l(\vec{r}_2,t_2)\right] &= \frac{i\hbar}{4\pi} c \left(\delta_{jl}\frac{1}{c^2}\frac{\partial^2}{\partial t_1 \partial t_2} - \nabla_j^1 \nabla_l^2\right)\left(\frac{\delta(r+ct) - \delta(r-ct)}{r}\right), \\
\left[E_j(\vec{r}_1,t_1), B_l(\vec{r}_2,t_2)\right] &= \frac{i\hbar}{4\pi} \epsilon_{ljm} \frac{\partial}{\partial t_1} \nabla_m^{(2)} \left(\frac{\delta(r+ct) - \delta(r-ct)}{r}\right).
\end{aligned} \tag{32}
$$

Os comutadores entre os campos eletromagnéticos calculados nos pontos do espaço-tempo x_1 e x_2 desaparecem a não ser que esses pontos possam ser conectados por um sinal de luz, ou em outras palavras, que a distância entre ele seja do tipo luz. Esse seria o principio de causalidade para os campos eletromagnéticos livres. O que é feito no processo de quantização canônica no contexto de teoria quântica de campos é impor relações de comutação a tempos iguais ($t_1 = t_2 = \tau$). No nosso caso, tal imposição seria equivalente à $t \to 0$, implicando diretamente que

$$D(r,t) = 0.$$

Consequentemente, também temos que

$$
\begin{aligned}
\lim_{t\to 0} \frac{\partial^2}{\partial t_1 \partial t_2} D(r,t) &= -\lim_{t\to 0} \frac{\partial^2}{\partial t^2} D(r,t) = 0, \\
\lim_{t\to 0} \frac{\partial}{\partial t_1} D(r,t) &= \lim_{t\to 0} \frac{\partial}{\partial t} D(r,t) \neq 0,
\end{aligned}
$$

resultando em

$$\left[B_j(\vec{r}_1,\tau), B_l(\vec{r}_2,\tau)\right] = \left[E_j(\vec{r}_1,\tau), E_l(\vec{r}_2,\tau)\right] = 0 \tag{33}$$

e

$$\left[E_j(\vec{r}_1,\tau), B_l(\vec{r}_2,\tau)\right] \neq 0, \qquad j \neq l. \tag{34}$$

Para qualquer instante τ é possível especificar o campo elétrico $\vec{E}$ ou com campo magnéitico $\vec{B}$, mas não os dois simultaneamente.