Math 225B Lecture Notes
Prof. Ko Honda
Winter 2020

Contents

1 Lecture 1 - 1/6

1.1 Sard's Theorem

Sard's theorem is the main technical lemma to make differential topology work.

Theorem 1.1.1. *Let $f : M \longrightarrow N$. Then the set of critical values of f has measure 0 in N.*

Of course, this is meaningless until we defined what "measure 0" means on a manifold. This is much easier than defining a measure on a manifold, as the standard approach of working by patches will not be invariant under coordinate change. In fact, a change of coordinates results in multiplying the measure by a positive function. However, for the specific case of measure 0 sets, this issue does not occur.

Definition 1.1.1. A subset $S \subseteq N$ has *measure zero* if there exists some countable atlas $\{(\phi_i, U_i)\}$ such that for all $\varepsilon > 0$ we have that $\phi_i[U \cap S]$ can be covered by a countable union of rectangles with total volume less than ε. In other words, each $\phi_i[U_i \cap S]$ has measure 0 under the Lebesgue measure.

Note that via laziness, we will begin to identify patches with their images in Euclidean space. Under this regime, we can for instance say that $S \subseteq N$ has measure 0 if S can be covered by a countable union of rectangles. Precisely, a rectangle means the preimage of a rectangle (contained in $\phi[U]$) under some chart ϕ.

Remarks.

1. $S \subseteq N$ has measure 0 iff it can be covered by a countable union of rectangles with total volume arbitrarily small. Indeed, cover each U_i by rectangles of total volume less than $\varepsilon/2^i$. Then these collect to a countable cover of S by rectangles with total volume less than $\sum \varepsilon/2^i = \varepsilon$.

2. Nonempty open subsets of $\mathbb{R}^n$ have positive measure. Hence, Sard's theorem implies that the set of regular values is dense in N.

Proof. It suffices to prove Sard's theorem on Euclidean space via the existence of a countable atlas and the standard $\varepsilon/2^i$ trick. Indeed, let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$. We will prove the case $n = 1$. See *Milnor* for a full proof.

Define now $C_k = \{p \in \mathbb{R}^n : \text{all partials of order up to } k \text{ vanish at } p\}$. Note that C_1 consists of all points on which all *df* is 0. As we are assuming $f : \mathbb{R}^m \longrightarrow \mathbb{R}$, this tells us that $f[C_1]$ is the set of critical values of f. Furthermore, observe that $C_1 \supseteq C_2 \ldots$.

We proceed by induction on m. This result is clear for $m = 0$. Now suppose that it holds for all smooth maps $\mathbb{R}^{m-1} \longrightarrow \mathbb{R}$. We proceed in three steps.

1. $f[C_1 - C_2]$ has measure 0.

2. $f[C_k - C_{k+1}]$ has measure 0.

3. $f[C_k]$ has measure 0 for $k \geq m$.

These together will indeed imply that $f[C_1]$ has measure 0.

1. Let $p \in C_1 - C_2$. Then we claim that there exists some neighborhood $p \in V$ such that $f[(C_1 - C_2) \cap V]$ has measure 0. By countable addivity, this is sufficient. Note that by assumption, we have $\frac{\partial f}{\partial x_i}(p) = 0$ for all i and some $\frac{\partial^2 f}{\partial x_i \partial x_j}(p) \neq 0$. WLOG say $i = 1$. Then consider the map $h : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ via $(x_1, \ldots, x_m) \mapsto \left(\frac{\partial^2 f}{\partial x_1 \partial x_j}, x_2, \ldots, x_m\right)$. Consider these to be new coordinates $(\widetilde{x_1}, \widetilde{x_2}, \ldots, \widetilde{x_m}) = \left(\frac{\partial^2 f}{\partial x_1 \partial x_j}, x_2, \ldots, x_m\right)$. Furthermore, we have that $dh(p)$ is invertible. Then by the inverse function theorem, h restricts to a diffeomorphism $h : V \longrightarrow V'$, $p \in V$. The set of critical values of $f|_V$ is therefore equal to the set of critical values of $f|_V \circ h^{-1}$ as this is a diffeomorphism. If $y \in \mathbb{R}$ is a critival value of $f \circ h^{-1}$, it is also a critical value of $(f \circ h^{-1})_{\{\widetilde{x_1}=0\}}$. The set of critical values of this last function has measure 0 by induction and we are done.

2. Same as 1.

3. Let $p \in C_k$ for $k \geq m$. In local coordinates, view $f : \left[-\frac{1}{2}, \frac{1}{2}\right]^m \longrightarrow \mathbb{R}$. Apply Taylor's theorem to get $f(x+h) = f(x) + R(x,h)$ with $|R(x,h)| \leq c|h|^{k+1}$, $c > 0$. This holds for $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]^m$ and $|h| < \delta$ small. Fix some δ small and pick a cube Q of width δ containing p. Then the volume of Q is δ^m. Hence, $\mu(f[Q]) \leq c\delta^{k+1}$ by Taylor's theorem. Hence, the total measure is $\leq c\delta^{k+1}/\delta^m$. As $k \geq m$, this tends to zero as $\delta \to 0$.

$\square$

2 Lecture 2 - 1/8

2.1 Transversality day 1

Definition 2.1.1. Let $f_i : M_i \longrightarrow N$, $i = 1, 2$. We say that f_1 and f_2 are transverse (and write $f_1 \pitchfork f_2$) if for all $(x_1, x_2) \in M_1 \times M_2$ such that $f_1(x_1) = f_2(x_2)$ we have $T_{f_i(x_i)} = (f_1)_*[T_{x_1}M_1] + (f_2)_*[T_{x_2}M_2]$.
Being transverse to a submanifold means being transverse to the inclusion map.

Examples.

1. Let $Y = \{x = 0\}, X = \{y = 0\} \subseteq \mathbb{R}^2$. Then $X \cap Y = \{(0, 0)\}$ and $X \pitchfork Y$.
2. Let $X = \{y = 0\}, Y = \{y = x^2\} \subseteq \mathbb{R}^2$. Then $X \cap Y = \{(0, 0)\}$ but X and Y do not intersect transversely.

Theorem 2.1.1. *Let $f : M^m \longrightarrow N^n$ be transverse to some $Z \subseteq N$. Then $f^{-1}[Z]$ is a submanifold of M. Furthermore, the codimension of Z in Y equals the codimension of $f^{-1}[Z]$ in M.*

Proof. Given $p \in Z$ we can take local coordinates $x_1, \ldots, x_n$ on $p \in U \subseteq N$ open satisfying $Z \cap U = \{x_{k+1} = \cdots = x_n = 0\}$ by the implicit function theorem. Here, $k = \dim Z$. Let $g : U \longrightarrow \mathbb{R}^{n-k}$ have coordinates $(x_{k+1}, \ldots, x_n)$. By transversality of f and Z, this map is a submersion. Hence, $f^{-1}[Z \cap U] = (g \circ f)^{-1}[0]$ is a submanifold. This locally equips $f^{-1}[Z]$ with a smooth manifold structure of codimension $n - k = \operatorname{codim} Z$. Being a manifold is a local property, so we are done. $\square$

Here's the plan from here. Take $f : M \longrightarrow N$, $Z \subseteq N$ a submanifold. We seek to "perturb" f into f' such that $f' \pitchfork Z$. If we have complementary dimensions $\dim M + \dim Z = \dim Z$ and if M is compact, we can try to count the total intersection number of f' and Z, which will hopefully be an invariant of f and Z. Of course, doing this literally is hopeless. Take example 2 above of the parabola intersecting the x-axis. Moving the parabola slightly up yields no intersection, but doing it downwards slightly yields 2 intersections. However, using orientations we will see that the leftmost intersection can be given a positive sign, and the rightmost one can be given a negative sign. Then the signed count becomes $1 - 1 = 0$. This also suggests that we can view this as a mod 2 invariant. We begin this program with the following.

Theorem 2.1.2. *Suppose $F : M \times S \longrightarrow N$ is transverse to some $Z \subseteq N$. Here, we are interpreting S as a parameter space. Then for almost all $s \in S$, $f_s = F(-, s)$ is transverse to Z.*

Proof. We have $F^{-1}[Z] \subseteq M \times S$ a submanifold. Consider the projection $\pi : M \times S \longrightarrow S$. Let $\widetilde{\pi}$ be the restriction of π to $F^{-1}[Z]$. By Sard's theorem, almost every $s \in S$ is a regular value of $\widetilde{\pi}$. Take such an $s \in S$. Let $(x, s) \in \widetilde{\pi}^{-1}[s]$. Then $T_{(x,s)}M \times S = T_xM + T_{(x,s)}F^{-1}[Z]$, as $\widetilde{\pi}_*[T_{(x,s)}F^{-1}[Z]] = T_sS$. Hence, $F_*[T_{(x,s)}M \times S] = (f_s)_*[T_xM] + F_*[T_{(x,s)}F^{-1}[Z]]$, so $F_*[T_{(x,s)}M \times S] + T_{F(x,s)}Z \subseteq (f_s)_*[T_xM] + T_{F(x,s)}Z$. Of course, $T_{F(x,s)}Z = T_{f_s(x)}Z$ and $F_*[T_{(x,s)}M \times S] + T_{F(x,s)}Z = T_{F(x,s)}N$ by assumption. Hence, $f_s \pitchfork Z$ for any such s. $\square$

3 Lecture 3 - 1/10

3.1 Transversality day 2

Definition 3.1.1. Two maps $f_0, f_1 : M \longrightarrow N$ are (smoothly) homotopic if there exists some $F : M \times [0,1] \longrightarrow N$ such that $f_i = F(-,i), i = 0, 1$. We often write $f_0 \sim f_1$. This is easily checked to be an equivalence relation. This formalizes the intuitive notion of a smooth deformation of a map.

3.1.1 Transverse Homotopy Theorem

Theorem 3.1.1 (Transverse Homotopy)**.** *Let $f : M \longrightarrow N$, $Z \subseteq N$. Then there exists a $g : M \longrightarrow N$ such that $f \sim g$ and $g \pitchfork Z$.*

The idea is that except for stupid things (which are measure 0), any way we can push f will force transversality. Consider the nontransverse examples such as the parabola and the axis for a visual of this.
This follows from the following two theorems.

Theorem 3.1.2. *Given $f : M \longrightarrow N$, $Z \subseteq N$, there exists a thiccening $F : M \times S \longrightarrow N$ with $F \pitchfork Z$ with S an open ball in $\mathbb{R}^m$ and some $f_{s_0} = f$.*

Theorem 3.1.3. *If $F : M \times S \longrightarrow N$ is transverse to some $Z \subseteq N$ then for almost every $s \in S$, $f_s \pitchfork Z$.*

The latter was proven last lecture, so it suffices to prove the former.

Proof. For an easy case, take $N = \mathbb{R}^n$. Now let $F : M \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ via $(x,y) \mapsto f(x) + y$. Then $F_0 = f$. This is how to formalize "pushing" f around. Clearly, this is a submersion at all points, so $F \pitchfork Z$. Furthermore, by the above theorem, this formalizes the intuition that almsot every direction we can push f works.

We now prove the result in greater generality. Suppose for simplicity that N^n is compact. For $p \in N$, let $x_1, \ldots, x_n$ be local coordinates on $p \in U_p \subseteq N$. View $U_p \subseteq \mathbb{R}^n$ via these coordinates. Take concentric closed balls $B_p \subseteq B'_p \subseteq U_p$ centered at p. Then there exists a vector field X_i supported on B'_p which equals $\frac{\partial}{\partial x_i}$ on B_p. By compactness, cover N by finitely many interiors of B_p. Let $Y_1, \ldots, Y_k$ be the vector fields corresponding to these finitely many points we just defined.

Now let $\phi^i_t(x)$ be the time t flow of Y_i starting at x. Let $F : M \longrightarrow \mathbb{R}^k \longrightarrow N$ via $(x, t_1, \ldots, t_k) \mapsto \phi^k_{t_k} \circ \ldots \phi^1_{t_1}(x)$. This starts at x and flows for time t_1 along Y_1, t_2 along $Y_2, \ldots, t_k$ along Y_k. This is (clearly?) a submersion (set all but t_1 to 0 and differentiate to get Y_1). $\square$

We also have the following enhancements to the transverse homotopy theorem, which we state without proof.

1. The relative version. Let $f : M \longrightarrow N$, $Z \subseteq N$, and some $C \subseteq M$ closed. Suppose that $f \pitchfork Z$ on C (transversality is an infinitesimal condition, so this means it holds for

points on C). Then there is a $g \pitchfork Z$ that is homotopic to f *relative to* C, i.e. there is a homotopy constant on C.

The idea for the proof is to extend the infinitesimal to the local and show transversality in a neighborhood of C. Then apply the above to the complement of this neighborhood.

2. Manifolds with boundary. Let $f : M \longrightarrow N$, $Z \subseteq N$ with M having boundary and N, Z boundaryless. Then there is a $g \sim f$ with $g|_{\partial M} \pitchfork Z$ and $g \pitchfork M$. The definition of the tangent space on the boundary is obvious in local coordinates where we view it as a half plane.

   The idea for this proof is to first show transversality on ∂M and use the relative version on $C = \partial M$.

4 Lecture 4 - 1/13

4.1 Morse Functions

Definition 4.1.1. A critical point of a map $f : M \longrightarrow \mathbb{R}$ is a point $p \in M$ such that df_p is not onto (cf. regular points vs. values). This is called nondegenerate if there are local coordinates $x_1, \ldots, x_n$ about p such that the Hessian $H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is nonsingular at p.

Independence of coordinates in this definition is a trivial application of the homework functor HW : **UnsolvedProblems** $\longrightarrow$ **SolvedProblems**.

Examples. There are 3 prototypes for critical points of some $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

1. $f(x, y) = \frac{1}{2}(x^2 + y^2)$. At the critical point at the origin, $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

2. $f(x, y) = \frac{1}{2}(x^2 - y^2)$. At the critical point at the origin, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

3. $f(x, y) = \frac{1}{2}(-x^2 - y^2)$. At the critical point at the origin, $H = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

An important result which we state without proof (cf. Milnor's *Morse Theory*) is the Morse Lemma.

Theorem 4.1.1 (Morse Lemma)**.** *Given $p \in M$ a nondegenerate critical point of $f : M \longrightarrow \mathbb{R}$, there exist local coordinates $x_1, \ldots, x_n$ about $p = 0$ such that $f(x_1, \ldots, x_n) = f(p) + \sum \lambda_i x_i^2$, $\lambda_i = \pm 1$.*

Definition 4.1.2. We view the Hessian at p as a quadratic form on T_pM. For p a nondegenerate critical point (sometimes called a Morse critical point), the index is defined to the the rank of the maximum negative definite subspace of $H(p)$. This equals the number of λ_i which are negative in the Morse Lemma. In the above 3 examples, the indices at the origin are 0, 1, 2 respectively.

Definition 4.1.3. $f : M \longrightarrow \mathbb{R}$ is Morse if all of its critical points are Morse.

Remark. For technical applications, we often want f to be proper.

Example. Sit the torus standing up on the xy-plane and take the "height function" $T^2 \longrightarrow \mathbb{R}$. The critical points are at the cusps on the torus. Starting from the bottom most point, the indices of these critical points are 0, 1, 1, 2.

Remark. A Morse function on a compact manifold can be used to glue a cellular decomposition.

Theorem 4.1.2. *A generic function is Morse.*

Here, "generic" means that a "random" function $f : M \longrightarrow \mathbb{R}$ is Morse. This is equally vague, so what we really mean is something akin to the transverse homotopy theorem, which we intuit as saying that a "generic" function is transverse to a fixed submanifold.

For this proof, we appeal first to the homework functor, which yields f Morse iff $df : M \longrightarrow T^*M$ is transverse to the 0 section.

Proof. Our goal is to construct a thicc $F : M_x \times S_s \longrightarrow \mathbb{R}$ such that $\Phi : M \times S \longrightarrow T^*M$ via $(x, s) \mapsto df_s(x)$ is transverse to the 0 section. We proceed similarly to the proof of the transverse homotopy theorem.

Assuming M to be compact, for $p \in M$ we take local coordinates $x_1, \dots, x_n$ and balls $p \in B_p \subsetneq B'_p$. Take now functions $\widetilde{x_i}$ supported on B'_p such that which agree with x_i on B_p. By compactness, take finitely many p such that $intB_p$ cover M. Now let $f_1, \dots, f_k$ be the list of all $\widetilde{x_i}$ constructed as above. Now let $F(x, s_1, \dots, s_k) = f(x) + \sum s_i f_i(x)$. Then $\Phi(x, s_1, \dots, s_k) = df(x) + \sum s_i df_i(s)$ (we are only differentiating in the x direction). Then Φ is transverse to the 0 section, as the $d\widetilde{x_i}$ span the whole tangent space of any point in their associated B_p . Hence, as in the proof of the transverse homotopy theorem, $df_s = \Phi(-, s) \pitchfork 0$ for almost all $s \in S$. $\square$

5 Lecture 5 - 1/15

5.1 Whitney Embedding Theorem

Theorem 5.1.1 (Whitney Embedding)**.** *Given an n-dimensional manifold M, there is an embedding $M \longrightarrow \mathbb{R}^{2n+1}$.*

Proof. We do the case where M is compact and leave the rest to homework. We do this in two steps.

Lemma 5.1.1.1 (Step 1)**.** *There is an embedding $M \hookrightarrow \mathbb{R}^N$, $N >> 0$.*

Lemma 5.1.1.2 (Step 2)**.** *Given an embedding $M \hookrightarrow \mathbb{R}^N$ with $N \geq 2n+2$, there exists a projection $\mathbb{R}^N \longrightarrow \mathbb{R}^{2n+1}$ such that $M \hookrightarrow \mathbb{R}^N \longrightarrow \mathbb{R}^{2n+1}$ is an embedding.*

Proof of step 1. "A generic function into $\mathbb{R}^N, N >> 0$ is an embedding".

Cover M by finitely many rectangles $R_{p_1}, \ldots, R_{p_l}$, each contained in some coordinate patch, with $p \in R_p = [-1,1]_{x_1} \times \cdots \times [-1,1]_{x_n}$ and $p = 0$. Choose a bump function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ supported on $[-1,1]$, $\phi'(x) \neq 0$ on $(-1,0) \cup (0,1)$ and $\phi(a) = \phi(b)$ iff $a = -b$ on $[-1,1]$. Let $\phi_\varepsilon(x) = \phi(x - \varepsilon)$.

Case $n = 1$. Let $g(x) = (\phi_{-\varepsilon}(x), \phi_\varepsilon(x))$. Observe that $g(x) = 0$ iff $x \notin (-1-\varepsilon, 1+\varepsilon)$, g is an immersion on $(-1-\varepsilon, 1+\varepsilon)$, and g is injective on $(-1-\varepsilon, 1+\varepsilon)$.

Case $n = 2$. Let $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ be a bump function symmetric about the origin which is identically 1 on $[-1-\varepsilon, 1+\varepsilon]$ and supported on $(-1-2\varepsilon, 1+2\varepsilon)$. Now, let $g(x_1, x_2) = (\phi_{-\varepsilon}(x_1)\psi(x_2), \phi_\varepsilon(x_1)\psi(x_2), \phi_{-\varepsilon}(x_2)\psi(x_1), \phi_\varepsilon(x_2)\psi(x_1))$. The ψ ensure compact support. It's not hard to show that this is an embedding on $(-1-\varepsilon, 1+\varepsilon)^2$.

Case n. Define analogously to $n = 2$.

Now, let $\psi_1, \ldots, \psi_{2nl}$ be the components of the associated g defined on each rectangles $R_{p_1}, \ldots, R_{p_l}$ (each g has $2n$ components and there are l many rectangles, so l many g). Let $f = (\psi_1, \ldots, \psi_{2nl})$. This is our proposed embedding $M \longrightarrow \mathbb{R}^{2nl}$.

Immersion. This is a local property and can therefore be done on the rectangles, where it is clear.

Proper. M is compact.

Injectivity. Let $x, y \in M$ with $f(x) = f(y)$. If x, y lie in the same rectangle, then we can simply use the properties of g discussed above to get $x = y$. Else, by how the distinct g for each rectangle are constructed, x and y are nonzero on distinct functions, so $f(x) \neq f(y)$.

For the noncompact case, the rough idea is to replace the rectangles with a countable disjoint union of rectangles which finitely cover. ☐

Proof of step 2. Take such an embedding $M \overset{f}{\hookrightarrow} \mathbb{R}^N$,$N \geq 2n+2$. We define the secant and tangent maps

$$\sigma : M \times M \times \mathbb{R} \longrightarrow \mathbb{R}^N \qquad (x, y, t) \mapsto t(f(x) - f(y))$$
$$\tau : TM \longrightarrow \mathbb{R}^N \qquad (x, v) \mapsto df_x(v)$$

respectively. The names are suggestive: σ paramemtrizes the line defined by $f(x) - f(y)$, which is viewed as a secant line through $M = f[M]$, and τ of course relates to the tangent space viewed in $\mathbb{R}^N$. Indeed, we will use σ to prove injectivity and τ to prove immersiveness.

$\dim M \times M \times \mathbb{R} = 2n+1$ and $\dim TM = 2n$, so "im(σ) is at most $2n+1$ dimensional" and "im(τ) is at most $2n$ dimensional". Phrasing this more rigorously, as $N \geq 2n+2$, Sard's theorem implies that both im(σ) and im(τ) have measure 0 in $\mathbb{R}^N$. Hence, choose some $v \in \mathbb{R}^N$ which is not in the image of σ or τ.

Now let $\pi : \mathbb{R}^N \longrightarrow (\mathbb{R}v)^\perp$, the orthogonal complement of the line spanned by v. Note that $0 \in \operatorname{im}(\sigma)$, so $v \neq 0$, so $\dim(\mathbb{R}v)^\perp = N-1$. We claim that $M \overset{f}{\hookrightarrow} \mathbb{R}^N \xrightarrow{\pi} (\mathbb{R}v)^\perp$ is an embedding.

Indeed, for injectivity, suppose $\pi(f(x)) = \pi(f(y))$. Then $f(x) - f(y) \in \ker(\pi) = \mathbb{R}v$. Hence, $f(x) - f(y) = tv$. If $t \neq 0$ then $\frac{1}{t}(f(x) - f(y)) = v$, but $v \notin \operatorname{im}(\sigma)$. Hence, $t = 0$ so $f(x) = f(y)$. As f is an embedding, $x = y$.

The same idea works to so immersiveness using τ. Hence, by induction, we can orthogonally project away dimensions until $\mathbb{R}^{2n+1}$ and remain an embedding. □

□

6 Lecture 6 - 1/17

6.1 Orientations day 1

6.1.1 Oriented Vector Spaces

Definition 6.1.1. Let V be a finite dimensional vector space over $\mathbb{R}$. Then $F(V)$ is the set of all ordered bases of V. An element of $F(V)$ is called a frame of V.

Remark. We view $F(V) \subseteq V^n$. Linear independence is an open condition, so this is an open subset and inherits the topology/smooth structure from V^n. Furthermore, there is a smooth action of $GL(V)$ on $F(V)$ via termwise evaluation, which is more or less just matrix multiplication. Restricting this action to $GL^+(V)$ yields two orbits: $GL^+(V) * (v_1, \ldots, v_n)$ and $GL^+(V) * (-v_1, \ldots, v_n)$. We denote the equivalence class of a frame $v = (v_1, \ldots, v_n)$ by $[v] = [v_1, \ldots, v_n]$ and let $-[v]$ be the other class.

Definition 6.1.2. An orientation of V is an equivalence class of $F(V)$. Denote $\mathcal{O}(V) = F(V)/\sim$. The standard orientation on $\mathbb{R}^n$ is $[e_1, \ldots, e_n]$.

Note that for any *nonzero* vector space, there are precisely two orientations. Of course, the 0 vector space has $F(0) = \{()\}$, the "empty tuple". Therefore, for consistency, we say that an orientation of the 0 vector space is simply a choice of $\{+, -\}$.

We wish to consider when two oriented vector spaces (a vector space with an orientation) are equivalent. The natural definition is, of course, that a linear isomorphism $\phi : V \longrightarrow V'$ of oriented vector spaces $(V, \mathcal{O}), (V', \mathcal{O}')$ is orientation preserving if $\phi(\mathcal{O}) = \mathcal{O}'$.

To extend this to manifolds, the natural thing to do is to talk about frames and orientations of each tangent space. Furthermore, this should these frames and orientations should somehow vary smoothly as parameterized by M. Indeed, we define

Definition 6.1.3. For a manifold M, let the frame bundle be a "bundle" $F(M) \longrightarrow M$ whose fiber over x is $F(T_xM)$. Let its orientation bundle be $\mathcal{O}(M) \longrightarrow M$ whose fiber over x is $\mathcal{O}(T_xM)$. Defining the smooth structure on these is homework. Note that $\mathcal{O}(M)$ is a 2 to 1 map. In fact, it's a double cover (and a 2 to 1 proper submersion).

6.1.2 Oriented Manifolds

This new framework allows us to formalize the notion of a smoothly varying orientation on each tangent space of M. Indeed,

Definition 6.1.4. An orientation of M is a global section of the orientation bundle $\mathcal{O}(M) \longrightarrow M$.

We previously defined M to be orientable iff the transition functions of TM had positive determinant. Indeed, these notions are compatible.

Remarks.

1. M is orientable iff $\mathcal{O}(M) \longrightarrow M$ admits a global section.

2. If M is connected, M is orientable iff $\mathcal{O}(M)$ has two connected components. These are the two orientations of M.

Example. **INSERT PICTURE FROM NOTES REGARDING MÖBIUS BAND**

Conventions. Product Orientation. Let $(M, \mathcal{O}_M), (N, \mathcal{O}_N)$ be oriented manifolds. We orient the product as $(M \times N, \mathcal{O}_M \times \mathcal{O}_N)$ via $(\mathcal{O}_M \times \mathcal{O}_N)(p, q) = [\mathcal{O}_M(p), \mathcal{O}_N(q)]$. By this, we mean that if $\mathcal{O}_M(p)$ (the orientation of T_pM) is $[v_1, \ldots, v_m]$ and $\mathcal{O}_N(q) = [w_1, \ldots, w_n]$, then we take $(\mathcal{O}_M \times \mathcal{O}_N)(p, q) = [v_1, \ldots, v_m, w_1, \ldots, w_n]$.

Note that the order of M and N is extremely important. $M \times N \cong N \times M$ as manifolds, but this need not hold as oriented manifolds. Indeed, it takes $(\dim M)(\dim N)$ swaps to get from $[v_1, \ldots, w_n]$ from $[w_1, \ldots, v_m]$. Hence, $\mathcal{O}_M \times \mathcal{O}_N = (-1)^{(\dim M)(\dim N)} \mathcal{O}_N \times \mathcal{O}_M$.

Boundary Orientation. Let M be an oriented manifold with boundary. Let n be the outward pointing normal (this definition is clear in local coordinates when you're looking at a half plane, and is (?????) invariant under positively oriented coordinate changes). $\mathcal{O}_{\partial M}$ is defined to satisfy $\mathcal{O}_M = [n, \mathcal{O}_{\partial M}]$. The convention of the normal vector being first is important.

Examples. 1. $[0, 1] \subseteq \mathbb{R}$ is a submanifold with boundary. Its interior is $(0, 1)$, which inherits the standard orientation $[e_1]$ on $\mathbb{R}$. We have $\partial[0, 1] = \{0, 1\}$. Let $\varepsilon_i = \pm 1, i = 0, 1$, be the orientation of $i \in \partial[0, 1]$. The boundary orientation insists that $[e_1] = [n_i, \varepsilon_i]$ for n_i the normal vector at i. n_1 points in the e_1 direction whereas n_0 points in the $-e_1$ direction. Hence, $\varepsilon_1 = +$ and $\varepsilon_0 = -$.

2. We consider the orientation of $[0, 1]_t \times M$ with M oriented (boundaryless??), given the product orientation. Indeed, $\partial([0, 1] \times M) = \{0\} \times M \cup \{1\} \times M$. By example 1 and the above conventions, the boundary orientation on $\{0\} \times M$ is $-\mathcal{O}_M$ and the boundary orientation on $\{1\} \times M$ is $\mathcal{O}_M$.

7 Lecture 7 - 1/22

7.1 Oriented Intersection Numbers

7.1.1 More on orientation

Let $V = V_1 \oplus V_2$ finite dimensional vector spaces over $\mathbb{R}$. Orientations on any two of these yield orientations on the third satisfying $\mathcal{O}_V = \mathcal{O}_{V_1} \times \mathcal{O}_{V_2}$. This extends in a natural way to any short exact sequence.

We will use the following setup a lot, so we define condition $(*)$ to be $f : M \longrightarrow N$, $Z \subseteq N$, $f \pitchfork Z$.

Preimage Orientation. If we are in condition $(*)$ with all manifolds oriented, then there is an induced orientation of $f^{-1}[Z]$ via the following procedure.

Take a complement V of $T_x f^{-1}[Z]$ in M, i.e. $V \oplus T_x f^{-1}[Z] = T_x M$. We claim that $f_* : V \longrightarrow f_*[V]$ is an isomorphism. Indeed, if $f_*(v) = 0$ then $v \in T_x f^{-1}[Z]$. Hence, by transversality, we have $f_*[V] \oplus T_{f(x)} Z = T_{f(x)} N$. Now,

1. $T_{f(x)} Z$ and $T_{f(x)} N$ are oriented, so we get an induced orientation on $f_*[V]$.
2. The isomorphism $f_* : V \longrightarrow f_*[V]$ and the above orientation on $f_*[V]$ induces an orientation on V.
3. Now, $T_x M$ and V are oriented, so $T_x f^{-1}[Z]$ is oriented.

One can check that this procedure yields a smoothly varying orientation on the $T_x f^{-1}[Z]$ which is independent of the choice of complement V. We shall denote this orientation by $f^* \mathcal{O}_Z$ or $f^{-1} \mathcal{O}_Z$.

Suppose now that we are in condition $(*)$ with all manifolds oriented and that only M has boundary. The conventions on boundary and preimage orientations yield two orientations on $\partial f^{-1}[Z]$, as this can be written as $\partial(f^{-1}[Z])$ or $(\partial f)^{-1}[Z]$, where ∂f is f restricted to the boundary. (c.f. the boundary version of the transverse homotopy theorem). Indeed, we have

1. The boundary orientation on $(f^{-1}[Z], f^* \mathcal{O}_Z)$.
2. The preimage orientation $((\partial f)^{-1}[Z], (\partial f)^* \mathcal{O}_Z)$.

To distinguish between the two, we call the first orientation $\partial(f^{-1}[Z])$ and the second $(\partial f)^{-1}[Z]$. Remember that these are the same as submanifolds of $f^{-1}[Z]$. However, their orientations differ via

Theorem 7.1.1. $\partial(f^{-1}[Z]) = (-1)^{\operatorname{codim} Z} (\partial f)^{-1}[Z]$.

Proof. $\mathcal{O}_M = [n, \mathcal{O}_{\partial M}] = [n, \mathcal{O}_V, (\partial f)^{-1}[Z]]$. Here, n is the outward pointing normal and V is the complement as in the construction of the preimage orientation. On the other hand, $\mathcal{O}_M = [\mathcal{O}_V, \mathcal{O}_{f^{-1}[Z]}] = [\mathcal{O}_V, n, \mathcal{O}_{\partial(f^{-1}[Z])}]$. These are equal, and swapping $\mathcal{O}_V$ and n requires $\dim V = \operatorname{codim} Z$ swaps. $\square$

7.1.2 Oriented Intersection Numbers

Conventions. We denote the following by condition $(**)$. $f : M \longrightarrow N$, $Z \subseteq N$ a submanifold, all manifolds oriented and boundaryless, M compact, and $\dim M + \dim N = \dim Z$. Guillemin and Pollack denote this as being appropriate for intersection theory.

We are finally at the point where we can try to rigorously study the intersection number of f and Z, which was our plan since we defined transversality. Indeed, we define the intersection number $I(f, Z)$ as follows.

Definition 7.1.1. In condition $(**)$, we define $I(f, Z)$ as follows.

1. Take a smooth homotopy $f \sim g$ with $g \pitchfork Z$.
2. $g^{-1}[Z]$ is a submanifold by transversality. By the complemantary dimension assumption, $\dim g^{-1}[Z] = 0$. As M is compact, $g^{-1}[Z]$ is finite.
3. $g^{-1}[Z]$ can be given the preimage orientation. As it consists of isolated points, this consists of a choice of ± 1 per point in $g^{-1}[Z]$. We now define $I(f, Z)$ as the sum of these signs, which makes sense by finiteness of $g^{-1}[Z]$.

Of course, we must show that this definition is independent of the choice of homotopic replacement map g. Indeed, for $g_0 \sim g_1 : M \longrightarrow N$ with both transverse to $Z \subseteq N$, we claim that $I(g_0, Z) = I(g_1, Z)$. Indeed, let $G : [0, 1] \times M \longrightarrow N$ witness this homotopy. By the relative version of the transverse homotopy theorem, we may assume that $G \pitchfork Z$ without changing $G(i, -) = g_i$ for $i = 0, 1$. Consider therefore $G^{-1}[Z]$. This is a one dimensional manifold with boundary. Also, assuming Z to be closed (which we usually mean), compactness of $I \times M$ implies that this is a compact one dimensional manifold with boundary. By the classification of compact one manifolds, which we do not prove, the only connected compact one manifolds are S^1 and I. Thus, $\partial(G^{-1}[Z])$ comes in pairs of points with opposite orientations, so the signed intersection count $\#\partial(G^{-1}[Z]) = 0$. As shown before, this implies that $\#(\partial G)^{-1}[Z] = 0$, and this is just $\#g_1^{-1}[Z] - \#g_0^{-1}[Z] = I(g_1, Z) - I(g_0, Z)$.

8 Lecture 8 - 1/24

Recall that we defined the oriented intersection number $I(f, Z)$ by homotoping f to some $g \pitchfork Z$ and computing the signed count $\#g^{-1}[Z]$. This required all manifolds involved to be oriented. Looking back at the proof of well definition of this fact, note that the boundary of a compact connected 1 manifold always has an even number of points. Hence, we can similarly define an invariant $I_2(f, Z) = |g^{-1}[Z]| \bmod 2$.

8.1 Degree

Definition 8.1.1. Suppose that N is connected and $\dim M = \dim N$. Let $f : M \longrightarrow N$. We define the degree of f to be $\deg(f) = I(f, pt)$ for some $pt \in N$ which we say has an orientation $+$. We are therefore also assuming that M is compact and all manifolds are oriented and boundaryless.
As a brief note, $f \pitchfork pt$ iff pt is a regular value of f. Also, we have not shown that this is independent of the point chosen, but in due time we shall.

The degree of a map formalizes the notion of being n to 1. Indeed, we are rigorously analyzing the "size" of the fiber of a regular value, keeping in mind orientation. Furthermore, for this notion to not be horrible, it must be independent of the point chosen, so this suggests some kind of uniformity in the size of the fibers.

Example. $S^1 \longrightarrow S^1$ via $\theta \mapsto n\theta$, or $z \mapsto z^n$. The degree of this map is n.

Regarding the well definition with respect to the point, suppose $p, q \in N$. Take a 1 parameter family of diffeomorphisms connecting p and q (ϕ_t orientation preserving diffeos such that $\phi_0 = id$ and $\phi_1(p) = q$). Pulling this back yields a homotopy of f, showing invariance.

We had a definition using cohomology last quarter, and this turns out to be the same. Indeed, this was more or less proven in the notes last quarter, but just not using the sophisticated jargon of intersection theory.

Theorem 8.1.1. *Let $f : M \longrightarrow N$ between compact, oriented n-manifolds with N connected. Suppose that f can be extended to a thicc map $F : W \longrightarrow N$ such that $\partial W = M$, W compact and oriented, and $\partial F = f$. Then* $\deg(f) = 0$.

Proof. Take some regular value y of f and let $F : W \longrightarrow N$ be such an extension. Using the transverse homotopy theorem relative to ∂W, we can say WLOG that $F \pitchfork y$. We have $\#(\partial F)^{-1}[y] = \pm\#\partial(F^{-1}[y])$. As discussed before, $\partial(F^{-1}[y])$ is the boundary of a compact 1 manifold and therefore comes in oppositely oriented pairs. Hence, this is 0, so $0 = \#(\partial F)^{-1}[y] = \#f^{-1}[y] = \deg(f)$. □

Example. We now show an application of degree theory by proving the fundamental theorem of algebra. Indeed, let $p \in \mathbb{C}[x] - \mathbb{C}$ have no roots and without loss of generality let it be monic. Now let $R >> 0$ such that on $|z| = R$, $p(z)/|p(z)| \approx z^n/R^n$. Then $f(z) = p(z)/|p(z)| : \{|z| = R\} \longrightarrow S^1$ has degree $n > 0$. However, as p has no roots, f may be extended to a map on $\{|z| \leq R\}$ and therefore has degree 0 by the lemma, a contradiction. In fact, we can enhance the result to the following.

Theorem 8.1.2. *Let W be a compact 2 submanifold of $\mathbb{C}$ with boundary. If $p : W \longrightarrow \mathbb{C}$ is a polynomial (or more generally, a holomorphic function) with no zeroes on ∂W, then the number of zeroes (counted with multiplicity) is* $\deg\left(\frac{p}{|p|} : \partial W \longrightarrow S^1\right)$.

Proof. We recall the following facts from complex analysis. z_0 is a zero of p iff $z - z_0 | p$ and p has only finitely many roots in W.

Now, let $z_1, \ldots, z_k$ be the distinct roots of p, with respective multiplicity a_i. Now remove a small neighborhood $D_\varepsilon(z_i)$ around each z_i and consider $\frac{p}{|p|} : W - \bigcup D_\varepsilon(z_i)$. Compute that on $\partial D_\varepsilon(z_i)$, $\deg\left(\frac{p}{|p|} : \partial D_\varepsilon(z_i) \longrightarrow S^1\right) = a_i$. By the above theorem,

$$\deg\left(\frac{p}{|p|} : \partial(W - \bigcup D_\varepsilon(z_i)\right) = 0,$$

as we have removed all of the roots by removing these disks. Furthermore,

$$\deg(\partial(W - \bigcup D_\varepsilon(z_i)) = \deg(\partial W) - \sum \deg(\partial D_\varepsilon(z_i)),$$

so $\deg(\partial W) = \sum a_i$, the number of roots of p counted with multiplicity. Here, we were lazy with notation by calling things $\deg(\partial W)$, but it's clear what we mean. $\square$

9 Lecture 9 - 1/27

9.1 Winding Numbers

This is a further application of degree theory. Recall from complex analysis that the winding number of a complex curve at a point is a rigorous definition of the number of times, counted with sign, that the curve wraps around the point. For instance, the winding number of a circle oriented counterclockwise about its center is 1, and the winding number of a circle wrapped n times around its center is n. A common definition of the winding number is to take rays starting at the point in question and computing the signed intersection count of the ray with the curve. With all this in mind, we are led to the following definition.

Definition 9.1.1. Let $f : M^n \longrightarrow \mathbb{R}^{n+1}$ with M compact, boundaryless, and oriented (if we don't want to assume oriented, work mod 2). For $z \in \mathbb{R}^{n+1} - f[M]$ (such points are generic by Sard's theorem), the winding number $wind(f, z)$ is the degree of the map $M^n \longrightarrow S^n$ via $x \mapsto \frac{f(x)-z}{|f(x)-z|}$. $\frac{f(x)-z}{|f(x)-z|}$ can be viewed as the unit vector pointing from z to $f(x)$, so this is counting (with sign) how often f points in a direction.

Theorem 9.1.1. *Given $f : M \longrightarrow \mathbb{R}^{n+1}$ as in the definition, suppose $M = \partial W$ with W^{n+1} compact and oriented. Let $F : W \longrightarrow \mathbb{R}^{n+1}$ be a thiccening of f, i.e. $\partial F = f$. If $z \in \mathbb{R}^{n-1} - f[M]$ is a regular value of F, then $F^{-1}[z]$ is finite and $wind(f, z) = \#F^{-1}[z]$.*

We present two proofs of the fact.

Proof 1. This is similar to the enhancement to the fundamental theorem of algebra proven before.

Finiteness of $F^{-1}[z]$ is standard, so let it be $\{y_1, \ldots, y_k\}$. Take small neighborhoods $D_\varepsilon(y_i) \subseteq W$ disjoint. Let $W' = W - \bigcup D_\varepsilon(y_i)$ and define $\Phi : W' \longrightarrow S^n$ via $x \mapsto \frac{F(x)-z}{|F(x)-z|}$. Thus, $\deg \partial\Phi = 0$, so $wind(F|_{\partial W'}) = 0$. Hence, $wind(f, z) = \sum wind(F|_{\partial D_\varepsilon(y_i)}, z)$. For ε sufficiently small, $F|_{\partial D_\varepsilon}$ maps diffeomorphically onto its target. The winding number is therefore ± 1 depending on the orientation, so $wind(f, z) = \#F^{-1}[z]$. □

Proof 2. Take a generic ray γ from $z \to z'$ with $|z'| >> 0$. WLOG say that $F, f \pitchfork \gamma$. The winding number changes by ± 1 when γ crosses $f[M]$ generically. A more rigorous treatment of this is in Guillemin and Pollack. □

9.2 Jordan-Brouwer Separation Theorem

Theorem 9.2.1 (Jordan-Brouwer)**.** *The complement of a compact, connected hypersurface $M \subseteq \mathbb{R}^{n+1}$ is has two connected components.*

Proof. There are two steps to this proof: showing that there are at most 2 connected components and that there are exactly two connected components.

Lemma 9.2.1.1 (Step 1)**.** *There are at most 2 connected components in $\mathbb{R}^{n+1} - M$*

Proof of step 1. Take some $x_0 \in M$. By the implicit function theorem, there is a neighborhood $U \subseteq \mathbb{R}^{n+1}$ on x_0 with some coordinates such that the inclusion $M \cap U \longrightarrow U$ looks like the inclusion of a hyperplane. Let U^+ and U^- be the connected components of $U - M \cap U$.

Now, let $z \in \mathbb{R}^{n+1} - M$. Take a ray $z \to x_0$. If this ray does not intersect M, then z is connected to U^+ or U^-. If not, let z_1 be the first intersection point of this ray with M. As M is connected, there is a path $\gamma : z_1 \to x_0$ in M. This path can be deformed (???) to a path $\gamma' : z_1 \to x_0$ which only intersects M at the endpoints. Hence, z is connected to U^+ or U^- (??concatenating isn't enough as at intersects at z_1, so have to push it off more??). Thus, $\mathbb{R}^{n+1} - M$ has at most two path components, associated to U^+ and U^-. $\square$

Lemma 9.2.1.2 (Step 2)**.** *The two components can be distinguished by $wind(i, z)$ with $i : M \longrightarrow \mathbb{R}^{n+1}$ the inclusion.*

Proof of step 2. Indeed, take some arbitrarily short arc $\gamma : z_0 \to z_1$, which intersects M once transversely with z_0 connected to U^- and z_1 connected to U^+. Then letting $wind_2$ be the mod 2 winding number, $wind_2(i, z_0) - wind_2(i, z_1) = 1$. As the mod 2 winding number is locally constant on $\mathbb{R}^{n+1} - M$ (paths induce homotopies of the unit vector thing), this shows that U^+ and U^- necessarily define different path components of $\mathbb{R}^{n+1} - M$. $\square$

$\square$

10 Lecture 10 - 1/29

We proceed with another application of degree theory.

10.1 Borsuk - Ulam Theorem

Theorem 10.1.1 (Borsuk - Ulam). *Let $f : S^n \longrightarrow \mathbb{R}^{n+1}$ with $0 \notin f[S^n]$ and $f(x) = -f(-x)$. Then $wind(f, 0)$ is odd.*

Proof. (?????) Let $\overline{f} = \frac{f}{|f|} : S^n \longrightarrow S^n$. Then $wind(f, 0) = \deg(\overline{f})$, so we seek to compute the mod 2 degree of $\overline{f}$. We proceed by induction.

For $n = 1$, use polar coordinates and believe it. Suppose now that the result holds for some $n \geq 1$. Take our $\overline{f} : S^{n+1} \longrightarrow S^{n+1}$. Let H^+ and H^- be the upper and lower hemispheres of S^{n+1}. Take now some $a \in S^{n+1} - \overline{f}[H^+]$. Equivalently, $-a \notin \overline{f}[H^-]$. This induces a partition $\overline{f}^{-1}[a] = \overline{f}|_{H^+}^{-1}[a] \coprod \overline{f}|_{H^-}^{-1}[a]$. Hence, $\#\overline{f}^{-1}[a] = \#\overline{f}|_{H^+}^{-1}[a] + \#\overline{f}|_{H^-}^{-1}[a]$.

Now, compose $\overline{f}$ with an orthogonal projection $\pi : \mathbb{R}^{n+2} \longrightarrow \mathbb{R}^{n+1}$ with kernel $\mathbb{R}a$. Then $(\pi \circ \overline{f}|_{H^+})^{-1}[0] = \overline{f}|_{H^+}^{-1}[\{-a, a\}]$, which has cardinality $|\overline{f}^{-1}[a]|$. Hence,

$$\begin{aligned} wind(f, 0) &= \#\overline{f}^{-1}[a] \\ &= wind(\pi \circ \overline{f}, 0) \end{aligned}$$

which is odd by induction. $\square$

10.2 More on Intersection Theory

Convention. Throughout this section, we will be working with $f : L \longrightarrow N$, $g : M \longrightarrow N$ with L, M compact, all manifolds oriented, and $\dim L + \dim M = \dim N$.

Definition 10.2.1. The intersection number was quite asymmetric before, so we define more generally the oriented intersection number of f and g to be $I(f, g)$ as follows. As usual, if we don't want things to be oriented we can take $I_2(f, g)$ to be the mod 2 version.

1. Homotope f, g so that $f \pitchfork g$.
2. For each pair (x, y) with $f(x) = g(y)$, assign a sign according to how the direct sum orientation aligns with the decomposition $df_x[T_xL] \oplus dg_y[T_yM] = T_{f(x)=g(y)}N$ (by transversality).
3. Sum these orientation numbers.

For this to make sense, realize that $(f \times g)^{-1}[\Delta]$, $\Delta \subseteq N$ the diagonal, is the set of points (x, y) with $f(x) = g(y)$. Furthermore, $f \pitchfork g$ iff $f \times g \pitchfork \Delta$. From here, everything is standard.

The oriented intersection number of smooth maps has the following three properties, which we state without proof.

1. $I(f, g) = (-1)^{\dim M} I(f \times g, \Delta)$.

2. $I(f,g) = (-1)^{\dim M \dim L} I(g,f)$.

3. If $f_0 \sim f_1, g_0 \sim g_1$ then $I(f_0,g_0) = I(f_1,g_1)$.

This provides an alternative proof for why the Möbius band N is not orientable. Indeed, let γ be the central band of N and let γ' be something homotopic to γ which intersects γ once (PICTURE). If N were orientable then we could do oriented intersection theory. Then we would have $I(\gamma,\gamma) = -I(\gamma,\gamma)$ by the above facts. However, as $|\gamma \cap \gamma'| = 1$, $I(\gamma,\gamma) = I(\gamma,\gamma') = \pm 1$. Hence, $1 = -1$, a contradiction. Note that this also shows how mod 2 intersection theory cannot "see" orientation.

11 Lecture 11 - 2/3

Convention. M is a compact, oriented manifold.

11.1 Euler Characteristic

Definition 11.1.1. Let $\Delta \subseteq M$ be the diagonal. Then the Euler characteristic of M is

$$\chi(M) = I(\Delta, \Delta). \tag{1}$$

We proceed to state, without proof, various alternate definitions of the Euler characteristic. For the following discussion, we will frequently refer to the torus T^2.

1. Suppose that M has a triangulation, i.e. a decomposition into simplices Δ^n. For example, taking $T^2 = [0,1]^2/\sim$, drawing the diagonal line from $(0,0) \to (1,1)$ induces a triangulation. Then

   $$\chi(M) = \sum (-1)^i (\text{number of } i \text{ simplices}). \tag{2}$$

   Using the above triangulation of T^2, we count that there is one 0-simplex (the point $(0,0) \sim (1,1)$), three 1-simplices (the diagonal and the two boundary), and two 2-simplices (the triangles cut out by the diagonal). Hence, $\chi(T^2) = 0$.

2. Given a Morse function $f : M \longrightarrow \mathbb{R}$, we have

   $$\chi(M) = \sum (-1)^i (\text{number of critical points of index } i). \tag{3}$$

   View now $T^2 \subseteq \mathbb{R}^3$ sitting tangent to the xy-plane. Take the height function $f : T^2 \longrightarrow \mathbb{R}$, which is Morse. As discussed in the section on Morse functions, f has one index 0 critical point, two index 1 critical points, and one index 2 critical point. Hence, $\chi(T^2) = 0$.

   This discussion about T^2 generalizes to an arbitrary genus g surface Σ_g. Indeed, sit Σ_g on the xy-plane in the same way and take a Morse function. Once again, the bottom cusp has index 0, the top cusp has index 2, and all intermediate cusps have index 1. Then $\chi(\Sigma_g) = 2 - 2g$.

11.2 Lefschetz Fixed Point Theory

Vaguely speaking, the goal here is to study the fixed points of maps $f : M \longrightarrow M$. Of course, these can range wildly and be horribly infinite (consider the identity). We would furthermore, like some homotopy invariant study of this, so we seek to rephrase this in terms of intersection theory.

Definition 11.2.1. Let $f : M \longrightarrow M$. Let $\Gamma(f) = \{(x, f(x))\} \subseteq M \times M$ the graph of f and $\Delta \subseteq M \times M$ the diagonal (which can be viewed as the graph of the identity). Note that $\Gamma(f) \cap \Delta$ is the fixed points of f, so we define the Lefschetz number of f to be $L(f) = I(\Delta, \Gamma(f))$.

Remarks. 1. This is a homotopy invariant. Indeed, wiggling f wiggles its graph.

2. If $f : M \longrightarrow M$ with $\Gamma(f) \pitchfork \Delta$, then $|L(f)|$ is a lower bound for the number of fixed points of f. We call such a map Lefschetz.

Theorem 11.2.1. *If $L(f) \neq 0$ then f has a fixed point.*

Proof. If f had no fixed points, $\Delta \cap \Gamma(f) = \emptyset$. Hence, these trivially intersect transversely so the above applies and $L(f) = 0$. ☐

We make an obvious point, which is that $L(id) = \chi(M)$. Hence, if $f \sim id$, $L(f) = \chi(M)$. Thus, if there is a map $f : M \longrightarrow M$ homotopic to the identity with no fixed points, then $\chi(M) = 0$. Indeed, consider $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and take $(x, y) \mapsto (x+\varepsilon, y+\varepsilon)$ for $\varepsilon > 0$ sufficiently small. This is homotopic to the identity (shrink ε) and has no fixed points, so we again see $\chi(T^2) = 0$.

Theorem 11.2.2. *Every map $f : M \longrightarrow M$ is homotopic to a Lefschetz map.*

Proof. This is a typical transversality argument, similar to $df \pitchfork 0$ iff f is Morse. ☐

11.3 Local Calculations

Definition 11.3.1. Suppose $x \in \Delta \pitchfork \Gamma(f)$, i.e. transversality holds at this point. This is called a Lefschetz fixed point of f.

Note that $T_{(x,x)}\Delta = \{(v, v) : v \in T_xM\}$ and $T_{(x,x)}\Gamma(f) = \{(v, df_xv) : v \in T_xM\}$. This leads us to the following.

Theorem 11.3.1. *TFAE*

1. *x is a Lefschetz fixed point of f.*

2. *$df_x - id$ is invertible.*

3. *1 is not an eigenvalue of df_x.*

Proof. (2) $\iff$ (3) is linear algebra. Furthermore,

$$\begin{aligned} x \text{ is a Lefschetz fixed point of } f &\iff T_{(x,x)}\Delta \cap T_{(x,x)}\Gamma(f) = 0 \\ &\iff \nexists v \neq 0 \text{ such that } df_xv = v \\ &\iff df_x - id \text{ is invertible.} \end{aligned}$$

☐

This leads us to the following HW/lemma: the local contribution of a Lefschetz fixed point x to $L(f)$ is just $L_x(f) = sgn(\det(df_x - id))$.

Examples. Consider a linear map $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. Then $f(0) = 0$ and $df_0 = f$. It often suffices to work infinitesimally where we can assume linearity (via the derivative), so this is important to look at.

1. $f = \lambda id$ with $\lambda > 1$. Then $L_0(f) = 1$. This arises, for example, when looking at a finite time flow of a vector field with a source at 0, such as $X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$.

2. $f = \lambda id$, $0 < \lambda < 1$. Then $L_0(f) = 1$. This arises, for example, when looking at a finite time flow of a vector field with a sink at 0, such as $X = -x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$.

3. $f = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for $\lambda_1 > 0$, $0 < \lambda_2 < 1$. Then $L_0(f) = -1$. This arises, for example, when looking at a finite time flow of a vector field with a more complicated zero, such as $X = x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}$.

The moral of these examples is that a uniform expansion/contraction correspond to $L_x(f) = 1$.

12 Lecture 12 - 2/5

12.1 Lefschetz Fixed Point Theory day 2

Conventions. M^n is compact, oriented. We are interested in fixed points of $f : M \longrightarrow M$

Recall from last time that we had a formula for $\chi(M)$ in terms of the critical points of a Morse function $g : M \longrightarrow \mathbb{R}$. We'll start with a bit more exposition on this. Take a downward gradient vector field X on M, i.e. it points in the direction of steepest descent. Flow for a short time ε to get a diffeomorphism $f : M \longrightarrow M$.

We now seek to compute $L(f) = \chi(M)$. Note that the fixed points of f are exactly the critical points of g. Near an index 0 critical point x, X looks like uniform contraction onto the critical point. Hence, $L_x(f) = 1$. Near an index 1 critical point, the contraction is not uniform, so $L_x(f) = -1$. Similarly, index 2 critical points look like uniform expansion, and this easily generalizes to index k, proving the formula.

12.1.1 Isolated fixed points not of Lefschetz type

Let x be an isolated fixed point of $f : M \longrightarrow M$. Let U be an open neighborhood of x such that it is the only fixed point of f in U.

Lemma 12.1.1. *There exists a homotopy of $f = f_0$ which is constant on $M - U$ and such that all of the fixed points of f_1 are Lefschetz inside of U.*

Proof. This is a typical application of the relative transversality homotopy theorem. □

Now, if we assume $U = B^n$ some ball, then we can compute the local contribution of x to $L(f)$ as

$$L_x(f) = \deg \begin{pmatrix} \phi : \partial B^n \longrightarrow S^{n-1} \\ \qquad z \mapsto \dfrac{f(z) - z}{|f(z) - z|} \end{pmatrix}$$

Indeed, when $x = 0$ is Lefschetz and $f(z) = Az$ for some A linear (some infinitessimal calculation using Lefschetzness should allow reduction to this case), then $df_x - id = A - id$ and $\phi : \partial B^n \longrightarrow S^{n-1}$ is given by $z \mapsto \frac{(A-id)z}{|(A-id)z|}$, which has degree $\pm 1 = sgn(\det(A - id))$.

Hence, after perturbing f to f_1, we have $\deg(\phi|_{\partial B^n}) = \sum \deg(\phi_{\partial B_i})$, where the B_i are disjoint balls encapsulating all of the fixed points. These $\deg(\phi_{\partial B_i})$ are the local Lefschetz numbers of f_1.

12.1.2 The Lefschetz fixed point theorem

Theorem 12.1.1 (Lefschetz Fixed Point Theorem)**.** *Let M be compact, oriented and let $f : M \longrightarrow M$. Then*

$$L(f) = (-1)^{\dim M} \sum_{i=0}^{\dim M} (-1)^i Tr(f_* : H^i(M) \longrightarrow H^i(M)).$$

Note that in this theorem, it is unclear that the right hand side is even an integer. Also, the $(-1)^{\dim M}$ term appears because we defined $L(f) = I(\Delta, \Gamma_f) = (-1)^{\dim M} I(\Gamma_f, \Delta)$. We omit the proof for now (c.f. Peterson's notes), as the most natural proof uses Poincaré duality.

Example. Consider S^n for $n \geq 1$. Then

$$H^i(S^n) = \begin{cases} \mathbb{R} & i = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

Applying the Lefschetz Fixed Point Theorem, we have $\chi(S^n) = L(id) = 1 + (-1)^n$.

If we instead take some $f : S^n \longrightarrow S^n$ with $\deg(f) = k$, then the induced maps on cohomology are given by

$$f_* : H^0(S^n) \xrightarrow{\cdot 1} H^0(S^n)$$

$$f_* : H^n(S^n) \xrightarrow{\cdot k} H^n(S^n)$$

so $L(f) = 1 + (-1)^n k$.

13 Lecture 13 - 2/12

13.1 Poincaré-Hopf Theorem

Conventions. M^n is compact, oriented and X is a vector field on M with isolated (hence finitely many) zeroes. Let $Z(X)$ denote the zero set of X.

Definition 13.1.1. For $p \in Z(X)$, pick a small ball $p \in B^n$ with coordinates x such that $x(p) = 0$. Then the $ind(X, p)$, index of X at p, is the degree of the map $\partial B^n \longrightarrow \partial S^{n-1}$ via $x \mapsto \frac{X(x)}{|X(x)|}$. Note that the length here is defined from $B^n \subseteq \mathbb{R}^n$.

This is independent of the choice of coordinates via an application of the homework functor.

Theorem 13.1.1 (Poincaré-Hopf). $\chi(M) = \sum_{p \in Z(X)} ind(X, p)$

Corollary 13.1.1.1. *As* $\chi(S^2) = 2 \neq 0$*, we have the Hairy ball theorem, i.e. that there are no nonvanishing vector fields on* S^2.

We shall present two proofs of this theorem.

Proof 1. Our definition is $\chi(M) = L(id) = L(f_\varepsilon)$ for f_ε a time ε flow of X, $\varepsilon > 0$ small. Note that the fixed points of f_ε are the zeroes of X.

From last time, we had local Lefschetz number $L_p(f_\varepsilon)$ for $p \in Z(X)$. By assumption on X, these fixed points are isolated so this applies. We therefore have $L(f_\varepsilon) = \sum L_p(f_\varepsilon)$ where

$$L_p(f_\varepsilon) = \deg\left(\frac{f_\varepsilon(x) - x}{|f_\varepsilon(x) - x|} : S^{n-1} \longrightarrow S^{n-1}\right).$$

By Taylor's theorem, $f_\varepsilon(x) - x \approx \varepsilon X(x)$. Hence, $\frac{f_\varepsilon(x)-x}{|f_\varepsilon(x)-x|} \approx \frac{X(x)}{|X(x)|}$. As $\varepsilon \to 0$, this approximation gets better and better. Hence, for sufficiently small ε, these maps have the same degree, so $L_p(f_\varepsilon) = ind(X, p)$. □

Proof 2. We begin with a slight reformulation of $ind(X, p)$. We first view $X : M \longrightarrow TM$. We claim that $ind(X, p)$ is the local intersection of $I(B^n, X[B^n])$. Here B^n is a sufficiently small closed ball centered at p such that p is the only zero of X in B^n. We can shrink B^n enough that the boundaries of B^n and $X[B^n]$ don't intersect. Of course, we identify $M \subseteq TM$ via the 0 section.

Indeed, perturb X so that it's transverse to the 0 section. In fact, do this perturbation relative to ∂B^n. Now, the fixed points are Lefschetz (similar proof to Morse function in terms of 0 section of cotangent bundle). Also, by the same argument as the local Lefschetz number calculation in terms of degree, assume WLOG $X[B^n] \pitchfork B^n$ at $p = 0$. Then, calculate the contributions from each transverse intersection point, which will be the same as the index $ind(X, p)$.

Globalizing, this means that $\sum ind(X, p) = I(0, X)$. As X is homotopic to 0, this is just $I(0, 0)$. It therefore remains to embed $TM \subseteq M \times M$. in such a way that the 0 section maps to the diagonal, i.e. $0 \longrightarrow \Delta$ via $(x, 0) \mapsto (x, x)$.

The key point here is to identify a neighborhood of T_xM with a neighborhood of $x \in M$, which varies smoothly in x. The usual way to do this is to use a Riemannian metric g (which always exists). We now use the exponential map $exp : TM \longrightarrow M$, $(x, v) \mapsto exp_x(v)$. This is given locally as a geodesic starting at x in the direction (and magnitude!) of v.

Alternatively, we can embed $M \subseteq \mathbb{R}^N$ and identify $T_xM \subseteq T_x\mathbb{R}^N = \mathbb{R}^N$. Let N_x be the orthogonal complement of T_xM. Let $\varepsilon > 0$ small and consider $D_x = \{|v| < \varepsilon\} \subseteq T_xM$. M can be written locally as a graph of a function $\phi_x : D_x \longrightarrow N_x$. This assignment $\phi_x : D_x \longrightarrow M \subseteq D_x \times N_x = \mathbb{R}^N$ is smooth in x. This yields $\coprod_{x\in M} D_x \longrightarrow M \times M$ via $(x, v) \mapsto (x, \phi_x(v))$.

Using either of these two methods, we get

$$\begin{aligned} \chi(M) &= I(\Delta, \Delta) \\ &= I(0, 0) \\ &= \sum ind(X, p). \end{aligned}$$

$\square$

14 Lecture 14 - 2/13

14.1 Framed Cobordisms and the Pontryagin Construction

(c.f. Milnor's *Topology from a Differentiable Viewpoint*)
The goal is the study maps $M^m \longrightarrow S^n$ where m, n need not be the same.

14.1.1 Smooth Approximations

Theorem 14.1.1 (Smooth Approximation)**.** *Let M be a compact manifold and N an arbitrary manifold.*

1. *Every continuous map $f : M \longrightarrow N$ admits a C^0-small smooth approximation $g : M \longrightarrow N$, i.e. for all $\varepsilon > 0$ such a g exists such that $|f(x) - g(x)| < \varepsilon$ for all $x \in M$. For this to make sense, assume $N \subseteq \mathbb{R}^N$ or use a Riemannian metric. Compactness of M implies that any two such metrics are equivalent.*

2. *Let $f_0, f_1 : M \longrightarrow N$ be smooth and continuously homotopic. Then they are smoothly homotopic.*

Proof. *HW* : **UnsolvedProblems** $\longrightarrow$ **SolvedProblems**. □

Definition 14.1.1. $[M, N] = Top$/homotopy
$[M, N]_{sm} = SmMan(M, N)$/smooth homotopy

The smooth approximation theorem basically says that $[M, N]_{sm} \longrightarrow [M, N]$ (forgetful) is a natural isomorphism.

14.1.2 Framed Cobordisms

Definition 14.1.2. Let $N, N' \subseteq M$ have the same dimension, with all manifolds boundaryless. N and N' are said to be cobordant in M if $N \times [0, \varepsilon], N' \times [1 - \varepsilon, 1]$ can be extended to some compact submanifold $X \subseteq M \times [0, 1]$ with $\partial X = X \cap (M \times \{0, 1\}) = N \times 0 \cup N' \times 1$. Note that forcing the "caps" for $[0, \varepsilon], [1 - \varepsilon, 1]$ allows this to be an equivalence relation.

We shall assume for a while that $M \subseteq \mathbb{R}^N$. Now, given $N^n \subseteq M^m$, the normal bundle $n(N) = \{(x, v) \in N \times \mathbb{R}^N : v \perp T_xN, v \in T_xM\}$, a rank $m - n$ vector bundle over N. More canonically, we have $0 \longrightarrow TN \longrightarrow TM|_N \longrightarrow n(N) \longrightarrow 0$ exact.

Definition 14.1.3. A framing of $N^n \subseteq M^m$ is a smooth assignment $v : x \in n \longrightarrow v(x) = (v_1(x), \ldots, v_{m-n}(x))$ with $v_i(x) \in n_x(N)$. This is basically the choice of $m - n$ sections of $n(N)$, which witnesses a global trivialization $n(N) \cong N \times \mathbb{R}^{m-n}$. The pair (N, v) is called a framed submanifold of M.

Definition 14.1.4. $(N, v), (N', v')$ are framed cobordant if there exists a cobordism $X \subseteq M \times [0, 1]$ between N and N' and a framing u of X in $M \times [0, 1]$ which agrees with v, v' on $N \times [0, \varepsilon]$, $N' \times [1 - \varepsilon, 1]$ respectively.

14.1.3 Main Results

Given a smooth map $f : M^m \longrightarrow S^n$, let $y \in S^n$ be a regular value. Then $f^{-1}[y]$ is a submanifold of M of codimension n. Pick an oriented basis $v = (v_1, \ldots, v_n)$ of $T_y S^n$ and let $n(f^{-1}[y])$ be the normal bundle. We then have an isomorphism $f_* : n_x f^{-1}[y] \longrightarrow T_y S^n$ for all $x \in f^{-1}[y]$. This is clear by the exact sequence above. Hence, we can lift $(v_1, \ldots, v_n)$ to $(w_1, \ldots, w_n) \in n_x f^{-1}[y]$. We denote this by $f^{-1}[v] = \{(w_1, \ldots, w_n)\}$.

Definition 14.1.5. $(f^{-1}[y], f^{-1}[v])$ is a Pontryagin submanifold of M.

Theorems.

(A) Any two Pontrygain submnaifolds, defined with the same map f, are framed cobordant.

(B) Two maps $f, g : M \longrightarrow S^n$ are smoothly homotopic if and onlt if their Pontryagin submanifolds are framed cobordant.

(C) Any compact, framed $(m - n)$ manifold $(N, w) \subseteq M$ can be realized as a Pontryagin submanifold.

Together, these theorems imply that $[M^m, S^n]$ is nothing more than the set of framed $(m-n)$ dimensional cobordism classes in M. For example, we compute $[S^m, S^n]$ for $m < n$. Indeed, $f^{-1}[y] = \emptyset$ for any regular value y. Hence, there is only one framed cobordism class, so $[S^m, S^n] = \{*\}$. The equidimensional case of $[M^n, S^n]$ will be returned to (Hopf Degree Theorem). Also, studying $[S^3, S^2]$ is therefore reduced to studying "framed links" up to framed cobordism.

15 Lecture 15 - 2/14

We seek to prove the main theorems stated above. These can be wrapped up into the following. First of all, let $FC_n(M)$ be the set of framed codimension n cobordism classes in M

Theorem 15.0.1. *The map $\Phi : [M, S^n] \longrightarrow FC_n(M)$ via $f \mapsto (f^{-1}[y], f^{-1}[v])$ is a well defined bijection. Here, M is boundaryless, y is some regular value of f and v is some frame of T_yS^n.*

Proof. There are three steps to show, which are more or less the three theorems stated last time.

(A) Φ is well defined.

(B) Φ is injective.

(C) Φ is surjective.

We begin with showing (A). Indeed, if v, v' are positively oriented bases at y then $(f^{-1}[y], f^{-1}[v]), (f^{-1}[y], f^{-1}[v'])$ are framed cobordant. Indeed, there is a path $\gamma : v \to v'$ in $Fr(T_yS^n)$. Take therefore $X = f^{-1}[y] \times [0, 1]$ as our cobordism. The framing on the slice X_t is, of course, given by $\gamma(t)$.

Furthermore, if f, g are smoothly homotopic and $y \in S^n$ is a regular value of both, then $f^{-1}[y]$ and $g^{-1}[y]$ are framed cobordant. We can suppress the framing as the above already showed independence of this choice. Indeed, let $F : M \times [0, 1] \longrightarrow S^n$ witness this homotopy. WLOG assume that $F_t = f$ and $F_{1-t} = g$ for all $t < \varepsilon$. As y is a regular value for both f and g, we have that $F \pitchfork y$ near $\partial(M \times [0, 1])$ (ε close). Then by the relative version of the transverse homotopy theorem, there exists some $G : M \times [0, 1] \longrightarrow S^n$ which is transverse to y and $G_t = F_t$ for all $t < \varepsilon$. Then $G^{-1}[y]$ is our desired framed cobordism.

Now, if y, y' are regular values of f, we claim that $f^{-1}[y] \sim f^{-1}[y']$. Indeed, let R be a rotation of S^n such that $R(y') = y$. Then $(R \circ f)^{-1}[y] = f^{-1}[y']$. Of course, $R \circ f \sim f$, so by the above, we have $f^{-1}[y'] = (R \circ f)^{-1}[y] \sim f^{-1}[y]$.

We now show (C). This is an important technique, due to Thom. Let $N \subseteq M$ be a compact, framed submanifold with no boundary and codimension n.

Lemma 15.0.1.1. *There exists a tubular neighborhood $U \supseteq N$ diffeomorphic to $N \times \mathbb{R}^n$.*

Proof. Take a framing $(v_1, \ldots, v_n)$ along n (the v_i are vector fields on N), which can be extended to $v_1, \ldots, v_n$ on M via some cutoff function/partition of unity nonsense. Let ϕ_t^X denote the time t flow of a vector field X on M. We therefore have a smooth map $N \times B_\varepsilon^n \longrightarrow M$ via $(x, t_1, \ldots, t_n) \mapsto \phi_{t_n}^{v_n} \circ \cdots \circ \phi_{t_1}^{v_1}(x)$. For small ε, this is an embedding. $\square$

Lemma 15.0.1.2. *There exists a smooth map $\phi : B_1^n \longrightarrow S^n$ such that*

$$\begin{aligned} \phi[\{1 - \delta \leq |x| \leq 1\}] &= S \\ \phi : \{|x| < 1 - \delta\} &\longrightarrow S^n - \{S\} \text{ a diffeomorphism} \\ \phi(0) &= N \end{aligned}$$

where S and N are the south and north poles respectively.

Let ϕ_ε be the corresponding map $B^n_\varepsilon \longrightarrow S^n$ in the second lemma. Define $f : M \longrightarrow S^n$ such that on $N \times B^n_\varepsilon$ it takes $(y, x) \mapsto \phi_\varepsilon(x)$ and on $M - (N \times B^n_\varepsilon)$ it collapses onto the south pole. Then $N = f^{-1}[n]$, n the north pole.

We conclude with (B). Let $f^{-1}[y] \sim g^{-1}[y]$. We claim $f \sim g$. Assume that f, g are of the type constructed in (B). Let $X \subseteq M \times [0, 1]$ witness the framed cobordism. We construct our homotopy $F : M \times [0, 1] \longrightarrow S^n$ via a "parametrized version of the previous proof".

Indeed, take a tubular neighborhood $X \times B^n_\varepsilon \longrightarrow M \times [0, 1]$ extending the inclusion. Send the complement of $X \times B^n_\varepsilon$ to the south pole and send $(y, x) \in X \times B^n_\varepsilon$ to $\phi_\varepsilon(x)$. $\square$

16 Lecture 16 - 2/19

16.1 The Hopf Degree Theorem

Theorem 16.1.1 (Hopf Degree Theorem)**.** *If M^m is compact, oriented, connected, and boundaryless, then two maps $f, g : M \longrightarrow S^m$ are homotopic if and only if* $\deg(f) = \deg(g)$.

Proof. By the Pontryagin-Thom, $[M, S^m] = FC_0(M)$, the framed 0 dimensional cobordism classes of M. 0 dimensional framed cobordism classes in M are, by compactness, just finite sets $\{(x^1, v^1), \ldots, (x^n, v^n)\}$ with $x^i \in M$ and v^i a frame of $T_{x^i}M$. Let $sgn(x^i, v^i)$ be 1 if the orientation of v^i and M agree at x and -1 otherwise. We have

$$\begin{tikzcd} SmMan(M, S^m) \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \\ {[M, S^m]} \arrow[ur] & \end{tikzcd}$$

The degree map takes $f \mapsto \#f^{-1}[y]$ for some regular value y. We have that $f^{-1}[y] = \{(x^1, v^1), \ldots, (x^n, v^n)\}$. Then $\deg(f) = \sum sgn(x^i, v^i)$.

For surjectivity, take the framed cobordism class determined by $\{(x^1, v^1), \ldots, (x^n, v^n)\}$. The corresponding $f \in [M, S^m]$ has degree $\sum sgn(x^i, v^i)$, which can be any integer.

We now proceed to show injectivity. Indeed, if $sgn(x, v) = -sgn(x', v')$ then $\{(x, v), (x', v')\}$ is framed cobordant to $\emptyset$. Furthermore, if $sgn(x, v) = sgn(x', v')$ then $\{(x, v)\} \sim \{(x', v')\}$. Here, we use connectedness of M and take a path, which drags along a vector field. $\square$

As a variant, we have the mod 2 Hopf degree theorem. Here, if M is compact, connected, boundaryless, and not orientable, then maps $M \longrightarrow S^n$ are homotopic iff they have the same mod 2 degree.

17 Lecture 17 - 2/20

17.1 Classification of Vector Bundles

(c.f. Milnor-Stasheff *Characteristic Classes*)

17.1.1 Definitions

Let $E \longrightarrow M$, $F \longrightarrow N$ be vector bundles. A map between these is a commutative square

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d] & F \arrow[d] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

such that $\overline{f}_p : E(p) \longrightarrow F(f(p))$ is linear.
If we have

$$\begin{tikzcd} & F \arrow[d, "v.b."'] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

we can construct the pullback vector bundle $f^{-1}F = \{(x, v) \in M \times F : f(x) = \pi(v)\}$.

17.1.2 The Universal Bundle

Recall the Grassmannian of k-planes in $\mathbb{R}^n$, which we denote $Gr(k, n)$. This naturally comes equipped with a rank k vector bundle $\gamma(k, n) \longrightarrow Gr(k, n)$ called the universal bundle. This satisfies $\gamma(k, n)_W = W$.

Now, if we fix k and vary n, we have

$$\begin{tikzcd} \dots \arrow[r] & \gamma(k,n) \arrow[r] \arrow[d] & \gamma(k,n+1) \arrow[r] \arrow[d] & \gamma(k,n+2) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & Gr(k,n) \arrow[r] & Gr(k,n+1) \arrow[r] & ylGr(k,n+2) \arrow[r] & \dots \end{tikzcd}$$

so we can take the colimit $\gamma(k, \infty) \longrightarrow Gr(k, \infty) = BO(k)$. Note that these are not in general manifolds. Also, the inclusion $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ is always via 0 in the last coordinate.

17.1.3 Classification

Theorem 17.1.1. *There is a bijection* $\Phi : \{\textit{rank } k \textit{ vector bundles}/M\}/\cong \longrightarrow [M, BO(k)]$.

Proof sketch. 1. We begin by defining Φ. Let $E \xrightarrow{\pi} M$ be a vector bundle of rank k. By the Whitney Embedding Theorem, there is an embedding $f : E \longrightarrow \mathbb{R}^N$. We have a map

$$M \longrightarrow Gr(k, N) \hookrightarrow Gr(k, \infty)$$

$$p \longmapsto T_p f[E_p]$$

The homotopy class of this map is our definition of $\Phi(E \xrightarrow{\pi} M)$.

2. We note now that there is a serious issue of well definedness in the definition of Φ. This better be independent of the choice of embedding $E \longrightarrow \mathbb{R}^N$. That this works is a feature of the construction of $Gr(k, \infty)$ as a colimit.

   Indeed, take two embeddings $f_i : E \longrightarrow \mathbb{R}^{N_i}$, $i = 0, 1$. We claim that there is a one parameter family of embeddings $f_t : E \longrightarrow \mathbb{R}^N$, $N >> N_0, N_1$ connecting f_0 and f_1. Here, we always view inclusions $\mathbb{R}^n \subseteq \mathbb{R}^m$ via 0s in the trailing coordinates. The idea is to start with the straight line homotopy $f_t = (1-t)f_0 + tf_1$. By the same proof as the Whitney Embedding Theorem, these are all embeddings for $N >> N_0, N_1$.

   Given such a one parameter family of embeddings, we get a smooth family of maps Φ_t, defined as above using f_t. Note that after composing $f : E \longrightarrow \mathbb{R}^{N_i}$ with the inclusion $\mathbb{R}^{N_i} \subseteq \mathbb{R}^N$, the maps Φ_i are the same as before doing this, as we are mapping into the colimit $Gr(k, \infty)$. Hence, this induces a homotopy $\Phi_0 \sim \Phi_1$, so the homotopy class $\Phi(E \xrightarrow{\pi} M)$ is independent of the choice of embedding of E into Euclidean space.

3. Now, let $M \xrightarrow{g} BO(k)$ with M compact. This can be homotoped so that $\mathrm{im}(g) \subseteq Gr(k, N)$ some N. Essentially, we want to project $W \in Gr(k, n+1)$ to $Gr(k, n)$, which is possible when W is disjoint from the orthogonal complement of $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$. Compactness says that this will eventually be possible.

   Now, by smooth approximation, there is a homotopic and arbitrarily C^0 close smooth map $M \xrightarrow{h} Gr(k, N)$. Let $\Psi(g) = h^{-1}\gamma(k, N)$ the pullback. This is well defined as the homotopy can be taken to be in the finite portion, so any two homotopic maps induce isomorphic pullbacks. Call this vector bundle $\Psi(g)$.

4. We claim therefore that Φ and Ψ are inverses. Indeed, we show $\Psi \circ \Phi = id$. The other direction is a bit harder.

   Take some vector bundle $E \longrightarrow M$. $\Phi(E \longrightarrow M)$ is the map $p \mapsto T_p f[E_p]$ for some embedding $f : E \longrightarrow \mathbb{R}^N$. This map has image in $Gr(k, N)$. Ψ is therefore the pullback of $\gamma(k, N)$ along this map. This is nothing more than $\coprod f_p[T_p E_p] \longrightarrow M$. Each fiber is canonically identified with E_p, so this is isomorphic to $E \longrightarrow M$.

$\square$

18 Lecture 18 - 2/21

18.1 Cobordisms and Thom's work

We seek to classify the cobordism classes of n-manifolds.

Convention. Manifolds are compact.

Definition 18.1.1. M^n, $(M')^n$ are cobordant if there exists some X^{n+1} such that $\partial X = M \coprod M'$ (c.f. framed cobordisms).

Theorem 18.1.1 (Thom)**.** *There exists a sequence of spaces*

$$\dots \longrightarrow MO(k) \longrightarrow MO(k+1) \longrightarrow \dots$$

with colimit MO such that

$$\eta_n = \{\textit{cobordisms classes of } n \textit{ manifolds}\} \xleftrightarrow{\sim} [S^n, MO]$$

Proof sketch. We first begin with a definition.

Definition 18.1.2. The Thom space $\tau(E)$ of a vector bundle $E \longrightarrow M$ is the one point compactification of E. For example, the one point compactification of the trivial vector bundle $\mathbb{R}^n \times M \longrightarrow M$ is $S^n \times M/$glue all north poles(=point at infinity) $= S^n \times M/\{\infty\} \times M$.

Recall

$$\begin{tikzcd}
\dots \arrow[r] & \gamma(k,n) \arrow[r] \arrow[d] & \gamma(k,n+1) \arrow[r] \arrow[d] & \gamma(k,n+2) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & Gr(k,n) \arrow[r] & Gr(k,n+1) \arrow[r] & Gr(k,n+2) \arrow[r] & \dots
\end{tikzcd}$$

from the previous lecture on the universal bundle. We denote $MO(k) = \tau(\gamma(k,\infty))$.

Now, given M^n, take $M \longrightarrow \mathbb{R}^{k+n}$ an embedding. Consider the (rank k) normal bundle $N(M) \longrightarrow M$ viewed in $\mathbb{R}^{k+n}$ via this embedding. As discussed last time, there exists a pullback diagram of vector bundles

$$\begin{tikzcd}
N(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \gamma(k,k+n) \arrow[d] \\
M \arrow[r] & Gr(k,k+n)
\end{tikzcd}$$

where the map $M \longrightarrow Gr(k,k+n)$ is given by $p \mapsto T_{f(p)}f[M]$. Similarly, we have

$$\begin{tikzcd}
N(M) \arrow[r] \arrow[d] & \gamma(k,\infty) \arrow[d] \\
M \arrow[r] & Gr(k,\infty)
\end{tikzcd}$$

We therefore define a map $S^{k+n} \xrightarrow{\phi_M} MO(k) = \tau(\gamma(k,\infty))$ as follows. On $N(M) \subseteq \mathbb{R}^{k+n} \subseteq S^{k+n}$, this is the composition $N(M) \longrightarrow \tau(N(M)) \longrightarrow \tau(\gamma(k,\infty))$. On the complement, send everything to the ∞.

Now, given a cobordism X^{n+1} between M, M', there exists an embedding $X \longrightarrow \mathbb{R}^{k+n} \times [0,1]$ such that $X \cap \mathbb{R}^{k+n} \times 0 = M$ and $X \cap \mathbb{R}^{k+n} \times 1 = M'$. We use some kind of relative embedding theorem. Take the normal bundle $N(X) \longrightarrow X$, which restricts to $N(M)$, $M(M')$ on the slices at 0, 1 respectively. Define now a map $S^{k+n} \times [0,1] \longrightarrow \tau(\gamma(k,\infty))$ as before. Indeed, on $N(X)$, this is the composition $N(X) \longrightarrow \tau(N(X)) \longrightarrow \tau(\gamma(k,\infty))$. On the complement, send everything to ∞. This is a homotopy between $\phi_M \sim \phi_{M'}$. Hence, the map $\Phi(M) = \phi_M$ is well defined, as cobordant manifolds induce homotopic maps.

To go the other direction, given $f : S^{k+n} \longrightarrow MO(k)$, how do we get a manifold? Similarly to last time, we can homotope f so that its image lies in $\tau(\gamma(k, k+n))$. The codomain of f need not be a manifold, but it will be one away from ∞. We therefore replace f with a "smooth approximation away from ∞" that is transverse to the 0 section of $\gamma(k, k+n)$. We therefore take this manifold to be $\Phi^{-1}(M) = f^{-1}[0]$.

[PICTURES] $\square$

To summarize the work we've done recently, we have the following bijections.

$$\{\text{codimension } n \text{ framed submanifolds of } M\}/\text{framed cobordism} \cong [M, S^n] \qquad (*)$$
$$\{\text{rank } k \text{ vector bundles over } M\}/\text{isomorphism} \cong [M, BO(k)] \qquad (*)$$
$$\{n \text{ manifolds}\}/\text{cobordism} \cong [S^n, MO] \qquad (*)$$

19 Lecture 19 - 2/24

19.1 More de Rham Theory

(c.f. Bott and Tu ch. 1)

Recall the de Rham complex Ω^* of a manifold M, $\Omega^i M = \Gamma(\bigwedge^i T^*M, M)$. The cohomology of this complex was $H^*(M)$, which was a contravariant functor $\mathbf{SmMan}^{op} \longrightarrow \mathbb{R}\text{-}\mathbf{Mod}$. Today, we do a variant of this called compactly supported cohomology.

Recall that the support of $\omega \in \Omega^i(M)$ is $Supp(\omega) = \overline{\{x \in M : \omega(x) \neq 0\}}$. We therefore let $\Omega^i_c(M) = \{\omega \in \Omega^i(M) : Supp(\omega) \text{ compact}\}$. Observe that the differential of Ω^* restricts to $d : \Omega^i_c(M) \longrightarrow \Omega^{i+1}_c M$. This new cochain complex is denoted Ω^*_c, and its cohomology is denoted H^*_c (compactly supported cohomology).

Remarks. 1. If M is compact then $\Omega^*_c(M) = \Omega^*(M)$.

2. H^*_c is not functorial, as the pullback of a compactly supported form need not be compactly supported. Indeed, take $M \longrightarrow *$ for M not compact, and let f be the 0-form (smooth function) on $*$ which is constant at 1. Then the pullback of f to M is the constant function at 1, which is not compactly supported

We can remedy (2) in the following ways.

(A) Restrict the morphisms in **SmMan** to only include proper maps. The preimage of a compactly supported form under a proper map is, of course, compactly supported.

(B) (The one we use) Observe that H^*_c is a covariant functor from the poset of open sets of M.

To expound on (B) a bit, let $U \xrightarrow{f} V$ be an embedding with $\dim U = \dim V$. Then $f_* : \Omega^i_c(U) \longrightarrow \Omega^i_c(V)$ can be defined by taking ω to its extension via 0 outside of U. This will be smooth as ω has compact support inside of U. Of course, we really mean the extension of $(f^{-1})^*\omega \in \Omega^i_c(f[U])$, but who cares. This is easily seen to be a map of cochain complexes and therefore induces a map on cohomology.

As we would hope, there is a covariant analog of Mayer-Vietoris for compactly supported cohomology. Indeed, let $M = U \cup V$ be an open cover. Then we have

$$U \cap V \underset{i_V}{\overset{i_U}{\rightrightarrows}} U \coprod V \xrightarrow{j} M$$

Lemma 19.1.1. *There exists a short exact sequence of cochain complexes going the wrong way.*

$$0 \longrightarrow \Omega^i_c(U \cap V) \xrightarrow{((i_U)_*, -(i_V)_*)} \Omega^i_c(U) \oplus \Omega^i_c(V) \xrightarrow{j_*} \Omega^i_c(M) \longrightarrow 0$$

Here, j_* really means $(j_U)_* \oplus (j_V)_*$, where $j = j_U \coprod j_V$.

Proof. We check exactness at $\Omega^i_c(M)$ and the rest is homework. Indeed, let $\eta \in \Omega^i_c(M)$. Take a partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$. Then let $\alpha = \rho_U \eta \in \Omega^i_c(U)$ and $\beta = \rho_V \eta \in \Omega^i_c(V)$. Then $\eta = \alpha + \beta$. $\square$

19.1.1 Poincaré Lemma

Consider the projection map $\pi : M \times \mathbb{R} \longrightarrow M$. We will define a map $\pi_* : \Omega_c^i(M \times \mathbb{R}) \longrightarrow \Omega_c^{i-1}(M)$ called the vertical integration map. This notation is overloaded with the pushforward, but fuck it.

Apply the homework functor to prove that every form on $M \times \mathbb{R}$ is a linear combination of forms of type

(A) $(\pi^*\phi)f(x,t)$, $\phi \in \Omega^i(M)$, $f(x,t) \in \Omega^0(M \times \mathbb{R})$

(B) $\pi^*\phi \wedge f(x,t)dt$

We called π_* the vertical integration map, so let's put our money where our mouth is. Indeed, to define π_*, we send forms of type (A) to 0, as there is nothing vertical ($\mathbb{R}$-direction) to integrate. Forms of type (B) are sent to $\phi \int_{\mathbb{R}} f(x,t)dt$. This makes sense as the domain of π_* was assumed to be the compactly supported forms. Also note that a form has many representations as a linear combination of forms of type (A) and (B), but the map is independent of this. We refer to π_* as "integrating out the fibers" or "integrating out the dt direction".

By homework, π_* is a chain map. Hence, we get an induced map on cohomology $H_c^i(M \times \mathbb{R}) \xrightarrow{\pi_*} H_c^{i-1}M$. Furthermore, for any $e = e(t)dt$, a compactly supported function on $\mathbb{R}$ with $\int_{\mathbb{R}} e = 1$, we can defined a map $e_* : \Omega_c^{i-1}(M) \longrightarrow \Omega_c^i(M \times \mathbb{R})$ via $\phi \mapsto \pi^*\phi \wedge e$.

Theorem 19.1.1 (Poincaré Lemma)**.**

$$H_c^i(M \times \mathbb{R}) \underset{\pi_*}{\overset{e_*}{\rightleftarrows}} H_c^{i-1}(M)$$

are inverse.

Proof. 1. $\pi_* \circ e_*$ takes $\phi \mapsto \pi^*\phi \wedge e \mapsto \phi \int_{\mathbb{R}} e = \phi$ by definition.

2. We seek a chain homotopy operator $K : \Omega_c^i(M \times \mathbb{R}) \longrightarrow \Omega_c^{i-1}(M \times \mathbb{R})$ between id and $e_* \circ \pi_*$, i.e. $id - e_* \circ \pi_* = (-1)^?(dK \pm Kd)$. We're lazy about the signs, but in the end it doesn't even matter, as we're only using K to show that $e_* \circ \pi_*$ descends to the identity on cohomology. Indeed, we define K as follows.

$$\begin{aligned}
(\pi^*\phi)f(x,t) &\mapsto 0 \\
\pi^*\phi \wedge f(x,t)dt &\mapsto \phi\left(\int_{\mathbb{R}} e(\tau)d\tau \int_{-\infty}^t f(x,\tau)d\tau - \int_{\mathbb{R}} f(x,\tau)d\tau \int_{-\infty}^t e(\tau)d\tau\right)
\end{aligned}$$

This definition deserves some explanation. If we take the trivial case of $\pi : \mathbb{R} \longrightarrow *$, then π_* takes $\Omega_c^1(\mathbb{R}) \longrightarrow \Omega_c^0(*) = \mathbb{R}$ via $f(t)dt \mapsto \int f(t)dt$, while e_* takes $1 \mapsto e$. Then we have

$$\begin{aligned}
(id - e_* \circ \pi_*)f(t)dt &= f(t)dt - \left(\int_{\mathbb{R}} f(\tau)d\tau\right)e(t)dt \\
&= \left(\int_{\mathbb{R}} e(\tau)d\tau\right)f(t)dt - \left(\int_{\mathbb{R}} f(\tau)d\tau\right)e(t)dt
\end{aligned}$$

and K is therefore a generalization of this.

[The proof will be continued next time.] ∎

20 Lecture 20 - 2/26

20.1 Poincaré Duality

20.1.1 Poincaré Lemma cont.

We begin by continuing our proof of the Poincaré Lemma from the previous lecture.

Proof. Note that from here on, we shall abuse notation and write $\pi^*\phi = \phi$. This is not too heinous, as $\pi^*\phi$ is basically just ϕ extended constantly in the vertical direction.

Recall that we were left to show that the K we defined previously was indeed a chain homotopy operator between id and $e_* \circ \pi_*$. Indeed, we see that $(id - e_*\pi_*)\phi f = \phi f$. On the other hand,

$$\begin{aligned}
(dK - Kd)\phi f &= -Kd(\phi f) \\
&= -K\left(d(\phi \circ f) + (-1)^i \phi \frac{\partial f}{\partial x} dx + (-1)^i \phi \frac{\partial f}{\partial t} dt\right) \\
&= (-1)^{i+1}\phi\left(\int_{-\infty}^{t} \frac{\partial f}{\partial \tau} d\tau - \int_{\mathbb{R}} \frac{\partial f}{\partial \tau} d\tau \int_{-\infty}^{t} e(\tau) d\tau\right) \\
&= (-1)^{i+1}\phi f
\end{aligned}$$

so $id - e_* \circ \pi_* - (-1)^{i+1}(dK - Kd)$ and we have the desired isomorphism on cohomology. $\square$

We now state a key example of the Poincaré Lemma. Indeed, this allows us to compute

$$H_c^i(\mathbb{R}^n) = H_c^{i-1}(\mathbb{R}^{n-1}) = \cdots = H_c^0(\mathbb{R}^{n-i}) = \begin{cases} 0 & n > i \\ \mathbb{R} & n = i \end{cases}$$

Of course, for $i > n$ we of course have $H_c^i(\mathbb{R}^n) = 0$. This reveals as interesting pattern

	$H_c^i(\mathbb{R}^n)$	$H^i(\mathbb{R}^n)$
$i = n$	$\mathbb{R}$	0
$i = n-1$	0	0
$\vdots$	$\vdots$	$\vdots$
$i = 0$	0	$\mathbb{R}$

so we'd suspect to have $H^i(M) = H_c^{n-i}(M)$ for $n = \dim M$.

20.1.2 Poincaré Duality

Definition 20.1.1. An open cover of a manifold is said to be good if all nonempty finite intersections are diffeomorphic to $\mathbb{R}^n$. It is a fact, which we state without proof, that all manifolds admit a good cover.

Lemma 20.1.1. *If M has a finite good cover then $H^i(M)$ and $H_c^i(M)$ are finite dimensional.*

Proof. Homework. $\square$

Theorem 20.1.1 (Poincaré Duality). *If M^n is oriented and has a finite good cover, then the pairing*

$$\int : H^i(M) \otimes H_c^{n-i}(M) \longrightarrow \mathbb{R}$$
$$\omega \otimes \eta \mapsto \int_M \omega \wedge \eta$$

is nondegenerate. Equivalently, the associated map $H^i(M) \longrightarrow (H_c^{n-i}(M))^*$ is an isomorphism.

Corollary 20.1.1.1. *If M^n is oriented and compact, $H^i(M) \cong (H^{n-i}(M))*$*

Proof. Poincaré duality is a local statement which has been globalized.

We first show that result for $M = \mathbb{R}^n$. Indeed, for the map $H^i(\mathbb{R}^n) \longrightarrow (H_c^{n-i}(\mathbb{R}^n))^*$, the only nontrivial case is $i = 0$. Indeed, for $i = 0$, the map takes $1 \mapsto \left(\omega \mapsto \int_{\mathbb{R}^n} \omega\right)$. Obviously, forms with $\int_{\mathbb{R}^n} \omega \neq 0$ exist, so by dimension counting this is an isomorphism.

Now, suppose we are in the general case and have a finite good cover $\{U_1, \ldots, U_k\}$. Then by the above argument, Poincaré duality holds for each $U_i \cong \mathbb{R}^n$ and all finite nonempty intersections $U_{i_1} \cap \cdots \cap U_{i_l}$.

[Proof continued next time.] □

21 Lecture 21 - 2/28

We begin with a continuation of the proof of Poincaré duality from last time. Indeed, we now need only show

Lemma 21.0.1. *There is a commutative (up to signs) diagram*

$$\begin{tikzcd}
\dots \arrow[r] & H^i(U\cup V) \arrow[r] \arrow[d] & H^i(U)\oplus H^i(V) \arrow[r] \arrow[d] & H^i(U\cap V) \arrow[r] \arrow[d] & H^{i+1}(U\cup V) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_c^{n-i}(U\cup V)^* \arrow[r] & H_c^{n-i}(U)^*\oplus H_c^{n-i}(V)^* \arrow[r] & H_c^{n-i}(U\cap V)^* \arrow[r] & H^{i+1}(U\cup V)^* \arrow[r] & \dots
\end{tikzcd}$$

where the vertical maps are the usual integration maps.

By induction, many of these maps are isomorphisms, so this plus the 5 lemma yields the result.

Proof. We first prove commutativity of the first square. The second is similar.

Let $\omega \in H^i(U\cup V)$. We have

$$\begin{tikzcd}
\omega \arrow[d, mapsto] & \\
\left(\eta \mapsto \int_{U\cup V} \omega\wedge\eta\right) \arrow[r, mapsto] & \left(\zeta \mapsto \int_U \omega\wedge\zeta\right)
\end{tikzcd}$$

and the V coordinate is similar. For the other path, we have

$$\begin{tikzcd}
\omega \arrow[r, mapsto] & j_U^*\omega \arrow[d, mapsto] \\
& \left(\zeta \mapsto \int_U j_U^*\omega\wedge\zeta\right)
\end{tikzcd}$$

in the U coordinate. Hence, the first square commutes.

For the third square, we have to understand the connecting homomorphisms. Recall

$$0 \longrightarrow \Omega^i(U\cup V) \longrightarrow \Omega^i(U)\oplus\Omega^i(V) \longrightarrow \Omega^i(U\cap V) \longrightarrow 0$$

Take $\{\rho_U,\rho_V\}$ a partition of unity subordinate to $\{U,V\}$. Then as $\rho_U+\rho_V=1$, $d\rho_U+d\rho_V=0$. Now, let $\omega\in\Omega^i(U\cap V)$ closed. Then the connecting homomorphism is defined by

$$\begin{tikzcd}
& (\rho_V\omega, -\rho_U\omega) \arrow[d, mapsto] & \omega \arrow[l, mapsto] \\
d^*\omega = d\rho_U\wedge\omega|_{U\cap V} & (d\rho_V\wedge\omega, -d\rho_U\wedge\omega) \arrow[l, mapsto] &
\end{tikzcd}$$

In the other direction, we had

$$0 \longleftarrow \Omega_c^i(U\cup V) \longleftarrow \Omega_c^i(U)\oplus\Omega_c^i(V) \longleftarrow \Omega_c^i(U\cap V) \longleftarrow 0$$

Take $\zeta \in \Omega^i_c(U \cup V)$ closed. The connecting homomorphism is given by

$$\begin{array}{ccc} \zeta & \longmapsto & (\rho_U\zeta, \rho_V\zeta) \\ & & \downarrow \\ & & (d\rho_U \wedge \zeta, d\rho_V \wedge \zeta) \longmapsto d_*\omega = d\rho_U \wedge \zeta|_{U\cap V} \end{array}$$

Hence, to prove commutativity of the third square, take some $\omega \in H^i(U\cap V)$ and compute

$$\begin{array}{ccc} \omega & & \\ \downarrow & & \\ \left(\eta \mapsto \int_{U\cap V} \omega \wedge \eta\right) & \longmapsto & \left(\zeta \mapsto d_*\zeta \mapsto \int_{U\cap V} \omega \wedge d\rho_U \wedge \zeta\right) \end{array}$$

$$\begin{array}{ccc} \omega & \longmapsto & d^*\omega = d\rho_V \wedge \omega \\ & & \downarrow \\ & & \left(\zeta \mapsto \int_{U\cap V} d\rho_V \wedge \omega \wedge \zeta\right) \end{array}$$

As $d\rho_U + d\rho_V = 0$, these two paths are off by a sign. □

21.1 Compact Vertical Cohomology

Let $E \xrightarrow{\pi} M$ be a vector bundle of rank n. Define $\Omega^i_{cv}(E) = \{\omega \in \Omega^i(E) : \forall K \subseteq M$ compact, $\pi^{-1}[K] \cap Supp(\omega)$ is compact$\}$. These forms are called compact vertical, and are said to have compact support in the vertical direction. Note that in particular, taking $K = \{p\}$, $Supp(\omega) \cap E_p$ is compact.

Remark. We can view a compact vertical form $\omega \in \Omega^i_{cv}(E)$ as a form on the Thom space $\tau(E)$. This isn't really rigorous since the Thom space need not be a manifold.

Take now a local trivialization $U \times \mathbb{R}^n$ with coordinates $(x, t_1, \ldots, t_n)$. Elements of $\Omega^i_{cv}(U \times \mathbb{R}^n)$ can be written as sums of $\pi^*\phi f(x,t)dt_I$ with $f(x,-)$ having compact support. Here, $\pi : U \times \mathbb{R}^n \longrightarrow U$ and $\phi \in \Omega^*(U)$. Note that $I = \emptyset$ is allowed here. As before, we have a map

$$\pi_* : \pi^*\phi f(x,t)dt_I \mapsto \begin{cases} 0 & |I| < n \\ \phi \int_{\mathbb{R}^n} f(x,t)dt_I & |I| = n \end{cases}$$

HW: This is well defined.

Fact. π_* commutes with d, so it induces a map $\pi_* : H^i_{cv}(E) \longrightarrow H^{i-n}(M)$. Also, by the Poincaré lemma, $\pi_* : H^i_{cv}(U \times \mathbb{R}^n) \xrightarrow{\sim} H^{i-n}(U)$.

Theorem 21.1.1 (Thom isomorphism). *If M admits a finite good cover $\{U_\alpha\}$ such that each $\pi^{-1}[U_\alpha]$ is trivial, then $\pi_* : H^i_{cv}(E) \xrightarrow{\sim} H^{i-n}(M)$.*

Proof. This is similar to before (PD is a local statement). □

Definition 21.1.1. The Thom class of a vector bundle $E \longrightarrow M$ is the class $\Phi \in H^n_{cv}(E)$ such that $\pi_*\Phi = 1 \in H^0(M)$. The inverse of π_* is called the Thom isomorphism $\tau : H^j(M) \longrightarrow H^{j+n}_{cv}(E)$. Observe that $\pi_*(\pi^*\omega \wedge \Phi) = \omega \wedge 1$, so $\tau(\omega) = \pi^*\omega \wedge \Phi$.

Facts.

1. Φ is the unique cohomology class in H^n_{cv} which restricts to a generator of $H^n_c(F)$ for each fiber F. A family of volume forms on E_p with integral 1 "sweeps out" to Φ.

2. There exists a representative of Φ such that on $\pi^{-1}[U] \cong U \times \mathbb{R}^n$, $\Phi = f(|t|)dt_1 \wedge \cdots \wedge dt_n$, with $f(|t|)$ a radially symmetric bump function.

22 Lecture 22 - 3/2

22.1 Poincaré Dual of a Submanifold

Recall that we had the cohomology ring $H^*(M) = \bigoplus H^i(M)$ with multiplication $[\alpha]\otimes[\beta] \mapsto [\alpha\wedge\beta]$. Now, let M^n be a compact manifold without boundary and Y^k a compact submanifold of M without boundary.

Definition 22.1.1. The Poincaré dual of Y is the cohomology class $PD(Y) = \eta_Y \in H^{n-k}(M)$ such that $\int_Y i^*\omega = \int_M \omega \wedge \eta_Y$. Here, $i : Y \longrightarrow M$ is the inclusion, so $i^*\omega = \omega|_Y$. In other words, the Poincaré dual is characterized by $\int_Y \omega|_Y = \int_M \omega \wedge \eta_Y$.

Note that such a class exists and is unique by Poincaré duality. Indeed, $H^k(M) \longrightarrow \mathbb{R}$ via $\omega \mapsto \int_Y i^*\omega$ is a linear functional, so corresponds to a $\eta_Y \in H^{n-k}(M)$ via Poincaré duality.

Theorem 22.1.1. *Let $N(Y) \longrightarrow Y$ be the normal bundle of Y, which is diffeomorphic to some neighborhood of Y in M. Then the Poincaré dual of Y and the Thom class of $N(Y) \longrightarrow Y$ can be represented by the same element, i.e. are cohomologous. In particular, there exists a representative of η_Y which is supported on any small tubular neighborhood of Y.*

Examples.

1. Consider first $Y = *$ a point in M^n. Then η_Y is a bump n-form on M, which is supported on a small ball about $*$ with $\int \eta_Y = 1$. Indeed, the map $H^0(M) \longrightarrow \mathbb{R}$ taking $f = c \mapsto \int_M \eta\cdot c = c$. (Here we assumed connectedness of M).On the other hand, $f = c \mapsto \int_* c = c$, so this is indeed the Poincaré dual.

2. Take $M = T^2$ and Y some circle around the meat. Then $N(y) = Y \times \mathbb{R}$ and $PD(Y)$ is a one form $f(t)dt$ which is compactly supported and has $\int_{\mathbb{R}} f(t)dt = 1$.

Proof. Let $j : N(Y) \longrightarrow M$ be the inclusion and Φ the Thom class of $N(y) \longrightarrow Y$. Now, for ω a closed k form on M, we claim that $\int_M \omega \wedge j_*\Phi = \int_Y i^*\omega$. Here, j_* is the extension by 0. This will indeed show that $j_*\Phi$ is the Poincaré dual of Y.

We certainly have $\int_M \omega \wedge j_*\Phi = \int_{N(Y)} \omega \wedge \Phi$. Furthermore, $N(Y) \xrightarrow{\pi} Y \xrightarrow{i} N(Y)$ are homotopy inverses (here, i is the inclusion/0 section). Hence, $\omega = \pi^*i^*\omega + d\lambda$, i.e. these forms are cohomologous. Thus,

$$\begin{aligned}
\int_{N(Y)} \omega \wedge \Phi &= \int_{N(Y)} (\pi^*i^*\omega + d\lambda) \wedge \Phi \\
&= \int_{N(Y)} \pi^*i^*\omega \wedge \Phi + \int_{N(Y)} d\lambda \wedge \Phi \\
&= \int_{N(Y)} \pi^*i^*\omega \wedge \Phi \\
&= \int_Y o^*\omega \wedge \pi_*\Phi \\
&= \int_Y i^*\omega
\end{aligned}$$

$\square$

We now return to discuss transversality some more. Indeed, let $Y, Z \subseteq M^n$ with $Y \pitchfork Z$ and all three manifolds compact, oriented, boundaryless. Then one can see that $N(Y \cap Z) = N(Y)|_{Y \cap Z} \oplus N(Z)|_{Y \cap Z}$. Hence,

$$\begin{aligned}\Phi(N(Y \cap Z)) &= \Phi(N(Y)|_{Y \cap Z} \oplus N(Z)|_{Y \cap Z}) \\ &= \Phi(N(Y)|_{Y \cap Z}) \wedge \Phi(N(Z)|_{Y \cap Z})\end{aligned}$$

We know, by the above theorem, that the Thom class of the normal bundle represents the Poincaré dual of the submanifold. Hence, we have the formula $\eta_{Y \cap Z} = \eta_Y \wedge \eta_Z$. In the context of the cohomology ring, this tells us that taking the product is the same as intersecting. Unfortunately, this is imperfect as not all cohomology classes come from submanifolds.