# Hartshorne Exercises

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## Contents

# I Varieties

**Conventions.** $k$ is an algebraically closed field.

## I.1 Affine Varieties

**Conventions.** $A = k[x_1, \ldots, x_n]$.

### I.1.1 INCOMPLETE

(a) Let $Y$ be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$.

(b) Let $Z$ be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over $k$.

(c) Let $f$ be an irreducible quadratic polynomial in $k[x, y]$, and let $W$ be the conic defined by $f$. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

*Proof.* (a) $A(Y) = k[x, y]/(y - x^2)$. Consider $k[x, y]/(y - x^2) \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$. This is an isomorphism.

(b) $A(Z) = k[x, y]/(xy - 1) \cong k[x, x^{-1}] \ncong k[t]$.

INCOMPLETE $f = ax^2 + bx + cxy + dy + ey^2 + f = (ax^2 + cxy + ey^2) + (bx + dy + f)$. By a change of variables we can diagonalize the homogeneous summand to a sum of squares. As $k$ is algebraically closed it can be factored into linear polynomials, so after a change of variables we get $f = (Ax + By)(Cx + Dy) + (bx + dy + f)$.

something something nondegenerate conics something something matrix form

**change variables something something homogeneous linear terms are lin dep/lin indep**

□

### I.1.2

*The Twisted Cubic Curve.* Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that $Y$ is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over $k$. We say that $Y$ is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

*Proof.* Take coordinates $x, y, z$ observe that $Y$ satisfies the equations $z = x^3, y = x^2$. We claim therefore that $Y = Z(I)$. Indeed, we certainly have $Y \subseteq Z(I)$. On the other hand, let $(a, b, c) \in Z(I)$. Then $c = a^3, b = a^2$ so $(a, b, c) = (a, a^2, a^3) \in Y$. Hence, $Y = Z(I)$ and is therefore an algebraic set.

Consider $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$. The kernel is the $f$ such that $f(t, t^2, t^3) = 0$, which is precisely $I(Y)$ (algebraically closed fields are infinite and polynomials correspond to polynomial functions over an infinite field). Hence, $A(Y) \cong k[t]$. Hence, $Y$ is an affine variety of dimension 1.

All that is left to do is to find generators for $I(Y)$. We claim therefore that $I = I(Y)$ so that the above generators work. Indeed, let $f \in I(Y)$. Then $f(t, t^2, t^3) = 0$. Write $f = \sum a_{ijk}x^iy^jz^k$. This condition therefore says that $\sum a_{ijk}t^{i+2j+3k} = 0$. Furthermore, in the quotient $k[x, y, z]/(z - x^3, y - x^2)$, $f = \sum a_{ijk}x^{i+2j+3k}$. This equals 0 by the above, so $f \in I$ as desired. $\square$

### I.1.3

Let $Y$ be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that $Y$ is the union of three irreducible components. Describe them and find their prime ideals.

*Proof.* By definition, $Y = Z(x^2 - yz) \cap Z(xz - x)$.

$$\begin{aligned}
Z(x^2 - yz) \cap Z(xz - x) &= Z(x^2 - yz) \cap Z((x)(z - 1)) \\
&= Z(x^2 - yz) \cap (Z(x) \cup Z(z - 1)) \\
&= Z(x^2 - yz) \cap Z(x) \cup Z(x^2 - yz) \cap Z(z - 1) \\
&= Z(x^2 - yz, x) \cup Z(x^2 - yz, z - 1) \\
&= Z(yz, x) \cup Z(x^2 - yz, z - 1) \\
&= Z(yz, x) \cup Z(x^2 - yz + y(z - 1), z - 1) \\
&= Z(yz, x) \cup Z(x^2 - y, z - 1) \\
&= Z(yz) \cap Z(x) \cup Z(x^2 - y, z - 1) \\
&= (Z(y) \cup Z(z)) \cap Z(x) \cup Z(x^2 - y, z - 1) \\
&= (Z(y) \cap Z(x) \cup Z(z) \cap Z(x)) \cup Z(x^2 - y, z - 1) \\
&= Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1)
\end{aligned}$$

We now seek to show that $(x^2-y, z-1)$ is prime. Indeed, define the map $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto 1$. This is an isomorphism and $k[t]$ is a domain. Thus, we have $Y = Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1)$ a union of three irreducible components. Their prime ideals are as given and we now geometrically describe each one of these.

$Z(x, y) = \{(a, b, c) \in \mathbb{A}^3 \mid a = b = 0\}$ so this is just the $z$ axis. Similarly, $Z(x, z)$ is the $y$ axis. Finally, $Z(x^2 - y, z - 1) = \{(a, b, c) \in \mathbb{A}^3 \mid a^2 = b, c = 1\} = \{(a, a^2, 1)\}$. This is a parabola sitting in the plane $z = 1$ with vertex at $(0, 0, 1)$ opening in the $y$ axis. Hence, $Y$ is the union of two lines, the $y$ and $z$ axes, and the parabola just described. $\square$

### I.1.4

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

*Proof.* The diagonal of $\mathbb{A}^2$ is defined by $Z(x - y)$, which is closed. If these topologies agreed, then the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$ will be closed so $\mathbb{A}^1$ will be Hausdorff. However, the topology on $\mathbb{A}^1$ is the cofinite topology and is therefore not Hausdorff. $\square$

### I.1.5

Show that a $k$-algebra $B$ is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some $n$, if and only if $B$ is a finitely generated $k$-algebra with no nilpotent elements.

*Proof.* For $Y \subseteq \mathbb{A}^n$ closed, we know that $I(Y)$ is radical. Hence, $A(Y) = k[x_1, \dots, x_n]/I(Y)$ is reduced finitely generated $k$-algebra.

On the other hand, let $B$ be some reduced finitely generated $k$-algebra. Then we can write $B \cong k[x_1, \dots, x_n]/I$. As $B$ is reduced, $I$ is radical, so by the Nullstellensatz we have $I(Z(I)) = \sqrt{I} = I$. Hence, $A(Z(I)) = k[x_1, \dots, x_n]/I \cong B$. □

### I.1.6

Any nonempty open subset of an irreducible topological space is dense and irreducible. If $Y$ is a subset of a topological space $X$, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

*Proof.* Let $\emptyset \neq U, V \subseteq X$ be open, $X$ irreducible. We claim that $U \cap V \neq \emptyset$. If they were disjoint, $X = X - (U \cap V) = (X - U) \cup (X - V)$. These are proper closed sets so this contradicts irreducibility of $X$.

Now, let $Y \subseteq X$ be irreducible in the subspace topology. Let $\overline{Y} = A \cup B$ be closed sets. We want to show that one of $A$ or $B$ are equal to $\overline{Y}$. Indeed, $(A \cap Y) \cup (B \cap Y) = Y$, which is irreducible, so WLOG say $A \cap Y = Y$. Hence, $Y \subseteq A$ so $A \cup B = \overline{Y} \subseteq A$ and $A = \overline{Y}$, proving irreducibility. □

### I.1.7

(a) Show that the following conditions are equivalent for a topological space X:

   (i) X is noetherian;

   (ii) every nonempty family of closed subsets has a minimal element;

   (iii) X satisfies the ascending chain condition for open subsets;

   (iv) every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

*Proof.* (a) The equivalence of (i) and (ii) is the same argument from algebra. (i) is equivalent to (iii) and (ii) is equivalent to (iv) via taking complements, which reverses order.

(b) Let $X = \bigcup \mathcal{U}$ and let $S = \{\bigcup \mathcal{V} \mid \mathcal{V} \subseteq \mathcal{U} \text{ finite}\}$. Then by (a), $S$ has a maximal element $U_1 \cup \cdots \cup U_n$. If there is an $x \in X - (U_1 \cup \cdots \cup U_n)$ then let $x \in U_{n+1}$ open. Then $U_1 \cup \cdots \cup U_n \subsetneq U_1 \cup \cdots \cup U_{n+1}$ a contradiction. Then $X = U_1 \cup \cdots \cup U_n$ and $X$ is compact.

(c) Let $A \subseteq X$. Let $A_1 \supseteq A_2 \supseteq \ldots$ be a decreasing chain of closed sets in $A$. Then $A_i = \overline{A_i} \cap A$ and $\overline{A_1} \supseteq \overline{A_2} \supseteq \ldots$ is a decreasing chain in $X$, which is noetherian. The chain upstairs terminates so the chain downstairs does as well.

(d) Let $X$ be noetherian and Hausdorff. Let $A$ be an irreducible closed subset of $X$. Let $x, y \in A$ and $x \in U, y \in V$ open neighborhoods in $A$. $U \cap V \neq \emptyset$ by irreducibility, so $x = y$ by Hausdorffness. Thus, the irreducible components of $X$ are singletons and $X$ is finite. Finite Hausdorff spaces are discrete.

□

### I.1.8

Let $Y$ be an affine variety of dimension $r$ in $\mathbb{A}^n$. Let $H$ be a hypersurface in $\mathbb{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

*Proof.* Let $A = k[x_1, \ldots, x_n]$. As $H$ is a hypersurface, $H = Z(f)$ for some $f$ irreducible. Let $Y = Z(\mathfrak{p})$ with $\mathfrak{p}$ prime. Then $B = A/\mathfrak{p}$ is the coordinate ring of $Y$. Consider $\overline{f}$ in $B$. As $Y \nsubseteq H$, $(f) \nsubseteq \mathfrak{p}$. Thus, $f \notin \mathfrak{p}$ so $\overline{f}$ is nonzero in $B$.

Let $F \subseteq Y \cap H$ be an irreducible component. Then $F = Z(\mathfrak{q})$ for some prime $\mathfrak{q}$. Furthermore, every irreducible closed subset of $Y \cap H$ is contained in some irreducible component. In other words, irreducible components are maximal with respect to being an irreducible closed subset of $F \cap Y$. Thus, $\mathfrak{q}$ is minimal with respect to primes containing $\mathfrak{p} + (f)$ as $Y \cap H = Z(\mathfrak{p} + (f))$.

Observe that $A/(\mathfrak{p} + (f)) \cong B/\left(\overline{f}\right)$. $B$ is a domain and as discussed, $\overline{f} \neq 0$ in $B$. By [Eisenbud, 13.11], $\dim B/\left(\overline{f}\right) = \dim B - 1$ so by [Eisenbud, 13.4], $\operatorname{codim}\left(\overline{f}\right) = 1$. We are trying to compute $\dim A/\mathfrak{q} = \dim B/(\mathfrak{q}/\mathfrak{p})$. By [Eisenbud, 13.4], this is $\dim B - \operatorname{codim} \mathfrak{q}/\mathfrak{p}$. The claim is therefore that $\operatorname{codim} \mathfrak{q}/\mathfrak{p} = 1$. $f \in \mathfrak{q}$ so $\overline{f} \in \mathfrak{q}/\mathfrak{p}$. Furthermore, $\mathfrak{q}$ is minimal with respect to containing $\mathfrak{p}$ and $f$ so $\mathfrak{q}/\mathfrak{p}$ is minimal with respect to containing $\overline{f}$. By Krull's principal ideal theorem, $\operatorname{codim} \mathfrak{q}/\mathfrak{p} \leq 1$. As $\operatorname{codim}\left(\overline{f}\right) = 1$ and $\left(\overline{f}\right) \subseteq \mathfrak{q}/\mathfrak{p}$, we have $\operatorname{codim} \mathfrak{q}/\mathfrak{p} = 1$. In summary,

$$\begin{aligned}
\dim F &= \dim A/\mathfrak{q} \\
&= \dim B/(\mathfrak{q}/\mathfrak{p}) \\
&= \dim B - \operatorname{codim} \mathfrak{q}/\mathfrak{p} \\
&= \dim B - 1 \\
&= \dim Y - 1
\end{aligned}$$

□

**I.1.9**

Let $\mathfrak{a} \subseteq A = k[x_1 ..., x_n]$ be an ideal which can be generated by $r$ elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

*Proof.* Let $F$ be an irreducible component of $Z(\mathfrak{a})$. Then $F = Z(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. As $F$ is an irreducible component, it is maximal with respect to being an irreducible closed subset of $Z(\mathfrak{a})$ and $\mathfrak{p}$ is therefore minimal with respect to containing $\mathfrak{a}$. $\mathfrak{a}$ can be generated by $r$ elements so by Krull's height theorem (a generalization of the principal ideal theorem), $\operatorname{codim} p \leq r$. Using [Eisenbud, 13.4], we arrive at the equation $\dim A/\mathfrak{p} + \operatorname{codim} p = \dim A$. Of course, $\dim A = n$ and $\dim A/\mathfrak{p} = \dim F$. Hence, $\dim F + \operatorname{codim} p = n$. As $\operatorname{codim} p \leq r$, $\dim F \geq n - r$. □

**I.1.10**

(a) If $Y$ is any subset of a topological space $X$, then $\dim Y \leq \dim X$.

(b) If $X$ is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

(c) Give an example of a topological space $X$ and a dense open subset $U$ with $\dim U < \dim X$.

(d) If $Y$ is a closed subset of an irreducible finite-dimensional topological space $X$, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

*Proof.* (a) Take a chain of irreducible closed subsets $Z_0 < \cdots < Z_r \subseteq Y$. The closures $\overline{Z_i}$ are irreducible and $\overline{Z_i} < \overline{Z_{i+1}}$ as $Z_i = \overline{Z_i} \cap Y$. Thus, every chain of irreducible closed subsets of $Y$ lifts to a chain of irreducible closed subsets of $X$ with the same length. Hence, $\dim Y \leq \dim X$.

(b) By (a) $\dim U_i \leq \dim X$. Hence, $\sup \dim U_i \leq \dim X$. We claim that if $Z_0 < \cdots < Z_n$ is a chain of irreducible closed subsets of $X$ then some $U_i$ has a chain of the same length. This will show that $\dim X \leq \sup \dim U_i$, proving equality. Let $U_i \cap Z_0 \neq \emptyset$. We claim $Z_j \cap U_i < Z_{j+1} \cap U_i$, which will become our chain in $U_i$ of the same length. Indeed, $Z_j \cap U_i \subseteq Z_j$ is a nonempty open subset and is therefore dense and irreducible. Hence, $\overline{Z_j \cap U_i} = Z_j$. Hence, if $Z_j \cap U_i = Z_{j+1} \cap U_i$ then their closures are equal and $Z_j = Z_{j+1}$ a contradiction.

(c) Consider the space $X = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. The closed sets are therefore $\{\{a, b, c\}, \{b, c\}, \{c\}, \emptyset\}$. Hence, $\{c\} < \{b, c\}$ is a chain of irreducible closed subsets so $\dim X \geq 1$. Furthermore, $\{a\}$ is an open generic point (it intersects all nonempty open subsets) so it is a dense open set with dimension $0 < 1$.

(d) Take a maximal chain $Z_0 < \cdots < Z_r$ in $Y$. Then these $Z_i$ are closed and irreducible in $X$. Hence, we have the chain $Z_0 < \cdots < Z_r \subseteq X$. As $\dim X = \dim Y = r$, $X = Z_r \subseteq Y$.

(e) Let $R$ be a noetherian ring of infinite dimension (see the example due to Nagata [Stacks, 02JC]). Then $\operatorname{Spec} R$ is a noetherian topological space of infinite dimension.

$\square$

### I.1.11

Let $Y \subseteq \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say $Y$ is *not a local complete intersection*—cf. (Ex. I.2.17).

*Proof.* First of all, consider the map $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$. As discussed previously, we are justified in using the same name for the variables of the coordinate ring and the coordinates on affine space themselves because for an infinite field (such as the algebraically closed field $k$ we work over) polynomials are identified with polynomials. This map has kernel $I(Y)$ by definition. Its image is $k[t^3, t^4, t^5]$, a domain, so $I(Y)$ is prime. Furthermore, $\dim k[x, y, z]/I(Y) = 3 - \operatorname{codim} I(Y)$. Hence, to compute the codimension of $I(Y)$, we need only compute the dimension of $k[x, y, z]/I(Y) \cong k[t^3, t^4, t^5]$. Observe that $k[t^3, t^4, t^5] \subseteq k[t]$ is integral. Indeed, $t$ satisfies the monic polynomial $s^3 - t^3$. Hence, its dimension is $\dim k[t] = 1$. Then $\operatorname{codim} I(Y) = 2$ as desired.

Let $\mathfrak{m} = (x, y, z)$. We claim that $(S)/\mathfrak{m}(S) \cong \sum_{s \in S} k \cdot s$ for any $S \subseteq k[x, y, z]$. Indeed, any element of $(S)/\mathfrak{m}(S)$ is represented by some $\sum f_i s_i$. Let's analyze one such $fs = \left(\sum a_{ijk} x^i y^j z^k\right) s$. For all $(i, j, k) \neq (0, 0, 0)$, $a_{ijk} x^i y^j z^k \in \mathfrak{m}$. Hence, $fs \equiv a_{000} s \pmod{\mathfrak{m}(S)}$. Thus, $\sum f_i s_i \equiv \sum f_i(0) s_i \pmod{\mathfrak{m}(S)}$. In other words, every element of $(S)/\mathfrak{m}(S)$ is represented by some element of $\sum_{s \in S} k \cdot s$. On the other hand, if $\sum a_i s_i \equiv \sum b_j s_j \pmod{\mathfrak{m}(S)}$ for $a_i, b_j \in k$ then $\sum a_i s_i - \sum b_j s_j \in \mathfrak{m}(S)$. Hence, evaluating the $a_i, b_j$ at 0 sends this to 0. These are constants, so the difference is 0 and $\sum a_i s_i = \sum b_j s_j$. This proves the isomorphism.

Observe that $x^3 - zy, y^2 - xz, z^2 - x^2 y \in I(Y) = I$ all have different $x$ degrees. Hence, they are $k$ linearly independent. Let $(S) = I$ with $x^3 - zy, y^2 - xz, z^2 - x^2 y \in S$. Then as above, $I/\mathfrak{m}I \cong \sum_{s \in S} k \cdot s$. As $\{x^3 - zy, y^2 - xz, z^2 - x^2 y\}$ is $k$ linearly independent, $\sum_{s \in S} k \cdot s$ has dimension at least 3. Furthermore, the above paragraph shows that any generating set of $I$ spans $I/\mathfrak{m}I$. Hence, any generating set of $I$ must have at least $\dim I/\mathfrak{m}I \geq 3$ elements. $\square$

### I.1.12

Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $Z(f)$ in $\mathbb{A}^2$ is not irreducible.

*Proof.* Consider the irreducible polynomial $f = x^2 + y^2 + 1$. Its zero set is empty and therefore not irreducible.

⟨Let's find a less trivial example - such an $f$ with $Z(f) \neq \emptyset$.⟩ $\square$

## I.2 Projective Varieties

**Conventions.** $S = k[x_0, \ldots, x_n]$.

### I.2.1

Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [*Hint*: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is $S$, and use the usual Nullstellensatz, [Hartshorne, 1.3A].

*Proof.* $\mathbb{A}^{n+1}$ has coordinate ring $S$. For notational ease, I will denote $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ to be the affine variety defined by $\mathfrak{a}$, as opposed to $Z(\mathfrak{a})$ which refers heretofor only to the projective variety. The given condition on $f$ is that for all $P \in Z(\mathfrak{a})$, $f(P) = 0$. By definition, $f(P) = 0$ means that $f(a_0, \ldots, a_n) = 0$ for any homogeneous coordinates $(a_0, \ldots, a_n)$ of $P$. Take now some $(a_0, \ldots, a_n) \in V(\mathfrak{a})$. We claim that $f(a_0, \ldots, a_n) = 0$. As $f$ is homogeneous, $f(0) = 0$ so suppose $(a_0, \ldots, a_n)$ is nonzero and let $P \in \mathbb{P}^n$ be the point it represents. Let $g \in \mathfrak{a}$ be homogeneous. Then $g(a_0, \ldots, a_n) = 0$ so $g(P) = 0$ and $P \in Z(\mathfrak{a})$. Hence, $f(P) = 0$ so by definition, $f(a_0, \ldots, a_n) = 0$. Thus, $f \in I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ by the Nullstellensatz. $\square$

### I.2.2

For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \emptyset$;

(ii) $\sqrt{\mathfrak{a}} =$ either $S$ or the ideal $S_+ = \bigoplus_{d>0} S_d$;

(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

*Proof.* Note that $S - S_+ = k - 0$. Hence, statement (ii) is equivalent to $\sqrt{a} \supseteq S_+$.

*(iii)* $\implies$ *(i)*. If $\mathfrak{a} \supseteq S_d$ then $x_i^d \in \mathfrak{a}$ for all $i$. Hence, $Z(\mathfrak{a}) \subseteq \cap Z(x_i^d) = \emptyset$.

*(i)* $\implies$ *(ii)*. If $\deg f > 0$ then $f \in S_+$. Furthermore, it holds vacuously that for all $P \in Z(\mathfrak{a}) = \emptyset$, $f(P) = 0$. Hence, by the homogeneous Nullstellensatz, $f^n \in \mathfrak{a}$ for some $n$. Hence, $\sqrt{\mathfrak{a}} \supseteq S_+$. Thus, $\mathfrak{a} = S_+$ or $\mathfrak{a} = S$ as $S - S_+ = k$. Note that we applied the homogeneous Nullstellensatz on $Z(\mathfrak{a}) = \emptyset$. This is justified as $Z(\mathfrak{a}) = \emptyset$ implies that $V(\mathfrak{a}) - 0 = \emptyset$. Indeed, letting $\pi : \mathbb{A}^{n+1} \longrightarrow \mathbb{P}^n$, $\pi^{-1}[Z(\mathfrak{a})] = V(\mathfrak{a}) - 0$. With this formula in mind, we can see that the proof of I.2.1 is valid for the empty set.

*(ii)* $\implies$ *(iii)*. Let $\sqrt{\mathfrak{a}} \supseteq S_+$. Then $x_i \in \sqrt{\mathfrak{a}}$ for all $i$. Hence, $x_i^{d_i} \in \mathfrak{a}$ for some $d_i$. Let $d = \sum d_i$. Now, let $x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n}$ be a generic (modulo constants) degree $d$ monomial. That is, $\sum k_i = d$. If all $k_i > d_i$ then $d = \sum k_i > \sum d_i = d$. Hence, some $k_i \leq d_i$ so the term $x_i^{d_i}$ appears in this expression. Hence, $x_0^{k)} x_1^{k_1} \cdots x_n^{d_n} \in \mathfrak{a}$. Hence, $S_d \subseteq \mathfrak{a}$. $\square$

**I.2.3**

(a) If $T_1 \subseteq T_2$ are subsets of $S^h$, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(c) For any two subsets $Y_1, Y_2$ of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$.

*Proof.* (a) Let $P \in Z(T_2)$ and $f \in T_1 \subseteq T_2$. Then $f(P) = 0$ so $P \in Z(T_1)$.

(b) Let $f \in I(Y_2)$ and $P \in Y_1 \subseteq Y_2$. Then $f(P) = 0$ so $f \in I(Y_1)$.

(c) Each $Y_i \subseteq Y_1 \cup Y_2$ so by part (b), $I(Y_i) \supseteq I(Y_1 \cup Y_2)$. Hence, $I(Y_1) \cap I(Y_2) \supseteq I(Y_1 \cup Y_2)$. On the other hand,let $f \in (I(Y_1) \cap I(Y_2))^h$. Then $f[Y_1], f[Y_2] \subseteq \{0\}$. Thus, $f[Y_1 \cup Y_2] \subseteq \{0\}$ so $f \in I(Y_1 \cup Y_2)^h$. As these ideals are homogeneous, the homogeneous elements generate so we have $I(Y_1) \cap I(Y_2) \subseteq I(Y_1 \cup Y_2)$.

(d) Let $f \in I(Z(\mathfrak{a}))^h$ with $\deg f > 0$. Then by definition, for all $P \in Z(\mathfrak{a})$, $f(P) = 0$. Then by the homogenous Nullstellensatz (I.1.1), $f \in \sqrt{\mathfrak{a}}$. Hence, 0 and all nonconstant homogeneous polynomials $f \in I(Z(\mathfrak{a}))$ are in $\sqrt{a}$. Furthermore, as $Z(\mathfrak{a}) \neq 0$, $k \cap I(Z(\mathfrak{a}))^h = \{0\}$. Of course, 0 is also in $I(Z(\mathfrak{a}))^h$ and $\sqrt{\mathfrak{a}}$. Hence, $I(Z(\mathfrak{a}))^h \subseteq \sqrt{\mathfrak{a}}$. As all these ideals are homogeneous, this proves $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$. Of course, if $f^n(P) = 0$ then $f(P) = 0$. Thus, $I(Z(\mathfrak{a}))$ is radical and $\mathfrak{a} \subseteq I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ so we achieve equality.

(e) We of course have $Y \subseteq Z(I(Y))$, so $\overline{Y} \subseteq Z(I(Y))$. On the other hand, let $Y \subseteq Z(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$. By definition, this is a generic closed set containing $Y$. Furthermore, this means that for all $P \in Y$ and $f \in \mathfrak{a}^h$, $f(P) = 0$. Hence, $\mathfrak{a}^h \subseteq I(Y)$. As $\mathfrak{a}$ is homogeneous, $\mathfrak{a} \subseteq I(Y)$ so $Z(\mathfrak{a}) \supseteq Z(I(Y))$. As $Z(\mathfrak{a})$ was arbitrary, $Z(I(Y)) = \overline{Y}$.

□

**I.2.4**

(a) There is a $1-1$ inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of $S$ not equal to $S_+$, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note*: Since $S_+$ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of $S$.

(b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

(c) Show that $\mathbb{P}^n$ itself is irreducible.

*Proof.* (a) $Z(\mathfrak{a})$ is, by definition, always algebraic. Furthermore, it's easy to see that $I(Y)$ is always radical and homogeneous. We must show then that $I(Y)$ can never equal $S_+$ for $Y$ algebraic. Indeed, if it was the case that $I(Z(\mathfrak{a}))) = S_+$then $x_i \in I(Z(\mathfrak{a}))$ for all $i$. Thus, each $x_i$ sends all of $Z(\mathfrak{a})$ to 0. But $\bigcap Z(x_i) = \emptyset$ so $Z(\mathfrak{a}) = \emptyset$. However, $I(\emptyset) = S \neq S_+$.

That $I$ and $Z$ are inverses on these restricted domains follows from parts (d) and (e) of problem I.2.3.

(b) Let $I(Y)$ be prime. Then let $Y \subseteq Z(I_1) \cup Z(I_2)$. Hence, $I(Y) \supseteq I(Z(I_1) \cup Z(I_2))$. By I.2.3 part (c), this is $I(Z(I_1)) \cap I(Z(I_2)) = \sqrt{I}_2 \cap \sqrt{I}_2 \supseteq I_1 I_2$. Hence, $I(Y) \supseteq I_1 I_2$. It is then a general fact of commutative algebra that some $I_j \subseteq I(Y)$. Indeed, if neither is contained in $I(Y)$ then let $a_j \in I_j - I(Y)$. $a_1 a_2 \in I_1 I_2$ but cannot be in $I(Y)$ as it is prime, a contradiction.

On the other hand, let $Y$ be irreducible. Let $f, g$ be homogeneous such that $fg \in I(Y)^h$. Then $Y \subseteq Z(fg) \subseteq Z(f) \cup Z(g)$. Then as $Y$ is irreducible, it is contained in one of these, WLOG say $Y \subseteq Z(f)$. Hence, $f \in I(Y)$. As $I(Y)$ is homogeneous, this proves its primality.

(c) $\mathbb{P}^n = I(0)$.

□

**I.2.5**

1. $\mathbb{P}^n$ is a Noetherian topological space.

2. Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

*Proof.* 1. Let $Y_1 > Y_2 > \dots$ be closed subsets of $\mathbb{P}^n$. Then we have $I(Y_1) < I(Y_2) < \dots$ in the coordinate ring $k[x_0, \dots, x_n]$. But this ring is Noetherian - contradiction.

2. First note that a closed subspace of a Noetherian space is itself Noetherian. Indeed, a chain in the subspace is a chain in the superspace. So we prove this result for a general Noetherian space $X$.

If $X$ is irreducible then we are done. Else, we have $X = X_0 \cup X_1$ for some proper closed subsets $X_i$. We can proceed this process to $X_0 = X_{00} \cup X_{01}$ and $X_1 = X_{10} \cup X_{11}$. Continue this until everything in sight is irreducible. This will happen in finitely many steps, as if not there is some binary string $b_0 b_1 \dots$ such that $X_{b_0} > X_{b_0 b_1} > \dots$ - contradicting Noetheriannessitudity. Now, if $X = X_0 \cup \dots \cup X_n$ are irreducible, we can omit any containments so suppose no $X_i \subseteq X_j$. If in addition we have $X = Y_0 \cup \dots \cup Y_m$ an irreducible decomposition. Then each $X_i \subseteq Y_{\sigma(i))}$ for some $0 \leq \sigma(i) \leq m$. Similarly, each $Y_j \subseteq X_{\tau(j)}$ for some $0 \leq \tau(j) \leq n$. Then $X_i \subseteq X_{\tau(\sigma(i))}$, so by assumption $X_i = X_{\tau(\sigma(i))}$. Similarly, $Y_j = Y_{\sigma(\tau(i))}$. Thus, these are inverses so these decompositions are the same up to permutation.

□

**I.2.6**

If $Y$ is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ be the homeomorphism of [Hartshorne, 2.2], let $Y_i$ be the affine variety $\phi_i[Y \cap U_i]$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x, x^{-1}]$. Now use [Hartshorne, 1.7]), [Hartshorne, 1.8A], and (Ex I.1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever $Y_i$ is nonempty.]

*Proof.* We will follow this hint. We want to identity $A(Y_i)$ with $(S(Y)_{x_i})^0$, which is the set of all $\frac{f}{x_i^d}$ such that $f \in k[x_0, \dots, x_n]$ is homogeneous of degree $d$. We will consider the coordinate ring $A(Y_i)$ to have coordinates $x_0, \dots, \widehat{x_i}, \dots, x_n$. We define the map

$$\begin{aligned} k[x_0, \dots, \widehat{x_i}, \dots, x_n] &\longrightarrow (S(Y)_{x_i}) \\ f &\mapsto f\left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right) \end{aligned}$$

We want this map to factor through $A(Y_i)$. What then is $J(Y_i)$? By definition, it is the $f \in k[x_0, \dots, \widehat{x_i}, \dots, x_n]$ such that $f(a) = 0$ for all $a \in Y_i$. Of course, $Y_i = \phi_i[Y \cap U_i]$ so any such $a$ will look like $\left(\frac{a_0}{a_1}, \dots, \widehat{\frac{a_i}{a_i}}, , \dots, \frac{a_n}{a_i}\right)$. Indeed, we we take some $f \in J(Y_i)$ then for any $[a_0 : \dots : a_n] \in Y \cap U_i$, $f\left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right)$ vanishes on $[a_0 : \dots : a_n] = \left[\frac{a_0}{a_i} : \dots : 1 : \dots : \frac{a_n}{a_i}\right]$. Hence, this map sends $J(Y_i) \mapsto I(Y \cap U_i) \supseteq I(Y)$. Of course, we need to map into $I(Y)$. testtestteslskdjf;lskdjf;laksjdfl;asdkjf $\square$

**I.2.7**

(a) $\dim \mathbb{P}^n = n$.

(b) If $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.

*Hint*: Use (Ex. I.2.6) to reduce to
hartshorne.]

*Proof.* $\square$

**I.2.8**

A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial $f$ of positive degree. $Y$ is called a *hypersurface* in $\mathbb{P}^n$.

*Proof.* $\square$

### I.2.9

*Projective Closure of an Affine Variety.* If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism $\varphi_0$. Then we can speak of $\overline{Y}$, the closure of $Y$ in $\mathbb{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(Y)$ is the ideal generated by $\beta[I(Y)]$, using the notation of the proof of [Hartshorne, 2.2].

(b) Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic of (Ex. I.1.2). Its projective closure $\overline{Y} \subseteq \mathbb{P}^3$ is called the twisted cubic curve in $\mathbb{P}^3$. Find generators for $I(Y)$ and $I(\overline{})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

*Proof.* □

### I.2.10

*The Cone Over a Projective Variety* (2). Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{(0, \ldots, 0)\} \longrightarrow \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)\cdot$ We define the *affine cone* over $Y$ to be

$$C(Y) = \theta^{-1}[Y] \cup \{(0, \ldots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots x_n]$.

(b) $C(Y)$ is irreducible if and only if $Y$ is.

(c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1}$ . This is called the *projective cone* over $Y$.

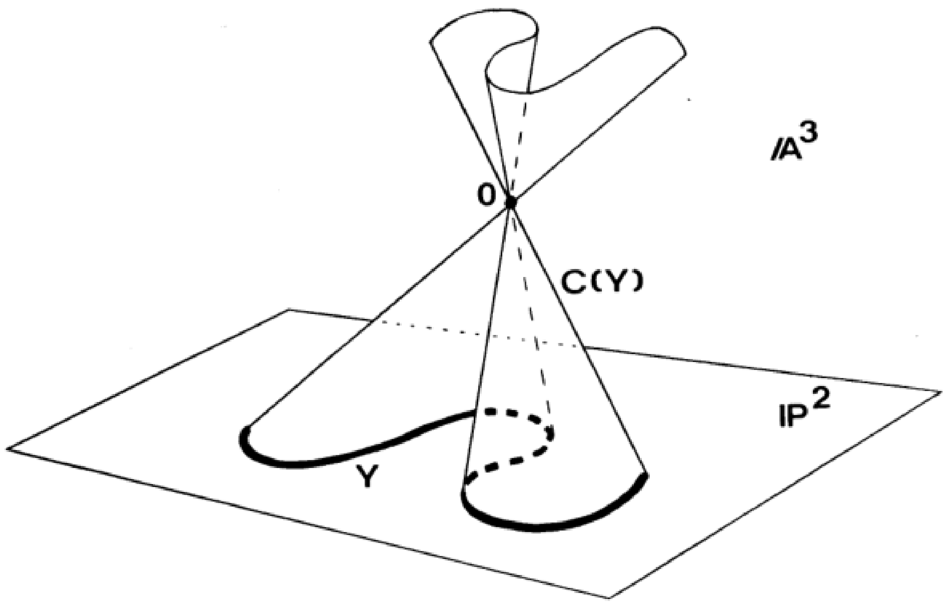


Figure 1: The cone over a curve in $\mathbb{P}^2$.

*Proof.* □

**I.2.11**

*Linear Varieties in* $\mathbb{P}^n$. A hypersurface defined by a linear polynomial is called a *hyperplane.*

(a) Show that the following two conditions are equivalent for a variety $Y$ in $\mathbb{P}^n$:

  (i) $I(Y)$ can be generated by linear polynomials.

  (ii) $Y$ can be written as an intersection of hyperplanes.

  In this case we say that $Y$ is a linear variety in $\mathbb{P}^n$.

(b) If $Y$ is a linear variety of dimension $r$ in $\mathbb{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials.

(c) Let $Y, Z$ be linear varieties in $\mathbb{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r+s-n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r+s-n$. (Think of $\mathbb{A}^{n+1}$ as a vector space over $k$, and work with its subspaces.)

*Proof.* □

**I.2.12**

*The $d-$Uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree $d$ in the $n+1$ variables $x_0, \ldots, x^n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$

obtained by substituting the $a_i$ in the monomials $M_j$. This is called the $d$-uple *embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image $Y$ of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

(a) Let $\theta : k[y_0, \dots, y_N] \longrightarrow k[x_0, \dots, x_n]$ be the homomorphism defined by sending $y_i$ to $M_i$, and let $\mathfrak{a}$ be the kernel of $\theta$. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of $\mathbb{P}^d$ is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that $\rho_d$ is a homeomorphism of $P^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. I.2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

*Proof.* □

**I.2.13**

Let $Y$ be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbb{P}^5$ such that $V \cap Y = Z$.

*Proof.* □

**I.2.14**

*The Segre Embedding.* Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_i b_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that $\psi$ is well-defined and injective. It is called the *Segre embedding*. Show that the image of $\psi$ is a subvariety of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \mid i = 0, \dots, r, j = 0, \dots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[z_{ij}] \longrightarrow k[x_0, \dots, x_r, y_0, \dots, y_s]$ which sends $z_{ij}$ to $x_i y_j$. Then show that $\Im\psi = Z(\mathfrak{a})$.]

*Proof.* □

**I.2.15**

*The Quadric Surface in* $\mathbb{P}^3$ (Fig. 2). Consider the surface $Q$ (a surface is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that $Q$ is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

(b) Show that $Q$ contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}$, $\{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all $t, u$, $L_t \cap M_u =$ one point.

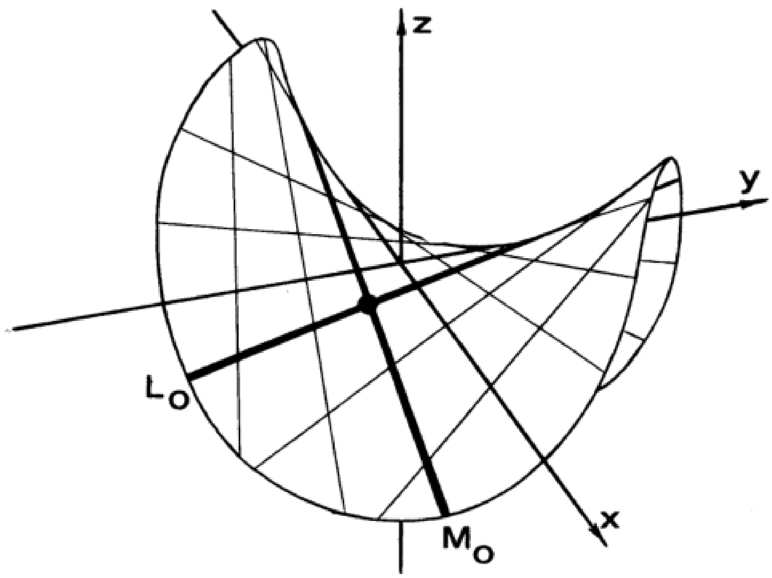


Figure 2: The quadric surface in $\mathbb{P}^3$.

(c) Show that $Q$ contains other curves besides these lines, and deduce that the Zariski topology on $Q$ is not homeomorphic via $\psi$ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

*Proof.* □

**I.2.16**

(a) The intersection of two varieties need not be a variety. For example, let $Q_1$ and $Q_2$ be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let $C$ be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let $L$ be the line given by $y = 0$. Show that $C \cap L$ consists of one point $P$, but that $I(C) + I(L) \neq I(P)$.

*Proof.* □

**I.2.17**

*Complete intersections.* A variety $Y$ of dimension $r$ in $\mathbb{P}^n$ is a *(strict) complete intersection* if $I(Y)$ can be generated by $n - r$ elements. $Y$ is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

(a) Let $Y$ be a variety in $\mathbb{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by $q$ elements. Then show that $\dim Y \geq n - q$.

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

(c) The converse of (b) is false. For example let $Y$ be the twisted cubic curve in $\mathbb{P}^3$ (Ex. I.2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces $H_1, H_2$ of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

(d) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III, §5] for commentary.

⟨*My note*: These references are not to this textbook, but to the references in the textbook itself. For laziness, I do not include these in the bibliography.⟩

*Proof.* □

# References

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