Hartshorne Exercises

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Contents

I Varieties

Conventions. k is an algebraically closed field.

I.1 Affine Varieties

Conventions. $A = k[x_1, \dots, x_n]$.

I.1.1 INCOMPLETE

(a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be an irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof. (a) $A(Y) = k[x, y]/(y - x^2)$. Consider $k[x, y]/(y - x^2) \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$. This is an isomorphism.

(b) $A(Z) = k[x, y]/(xy - 1) \cong k[x, x^{-1}] \ncong k[t]$.

INCOMPLETE $f = ax^2 + bx + cxy + dy + ey^2 + f = (ax^2 + cxy + ey^2) + (bx + dy + f)$. By a change of variables we can diagonalize the homogeneous summand to a sum of squares. As k is algebraically closed it can be factored into linear polynomials, so after a change of variables we get $f = (Ax + By)(Cx + Dy) + (bx + dy + f)$.

something something nondegenerate conics something something matrix form

change variables something something homogeneous linear terms are lin dep/lin indep

See here for char not 2 and here for char 2. Alternatively, here (pdf) does this with projective geometry - pick 3 points on any conic and move them to $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$

□

I.1.2

The Twisted Cubic Curve. Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Proof. Take coordinates x, y, z and observe that Y satisfies the equations $z = x^3, y = x^2$. Then let $I = (z - x^3, y - x^2)$. We claim therefore that $Y = Z(I)$. Indeed, we certainly have $Y \subseteq Z(I)$. On the other hand, let $(a, b, c) \in Z(I)$. Then $c = a^3, b = a^2$ so $(a, b, c) = (a, a^2, a^3) \in Y$. Hence, $Y = Z(I)$ and is therefore an algebraic set.

Consider $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$. The kernel is the f such that $f(t, t^2, t^3) = 0$, which is precisely $I(Y)$ (algebraically closed fields are infinite and polynomials correspond to polynomial functions over an infinite field). Hence, $A(Y) \cong k[t]$. Hence, Y is an affine variety of dimension 1.

All that is left to do is to find generators for $I(Y)$. We claim therefore that $I = I(Y)$ so that the above generators work. Indeed, let $f \in I(Y)$. Then $f(t, t^2, t^3) = 0$. Write $f = \sum a_{ijk} x^i y^j z^k$. This condition therefore says that $\sum a_{ijk} t^{i+2j+3k} = 0$. Furthermore, in the quotient $k[x, y, z]/(z - x^3, y - x^2)$, $f = \sum a_{ijk} x^{i+2j+3k}$. This equals 0 by the above, so $f \in I$ as desired. $\square$

I.1.3

Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is the union of three irreducible components. Describe them and find their prime ideals.

Proof. By definition, $Y = Z(x^2 - yz) \cap Z(xz - x)$.

$$\begin{aligned}
Z(x^2 - yz) \cap Z(xz - x) &= Z(x^2 - yz) \cap Z((x)(z - 1)) \\
&= Z(x^2 - yz) \cap (Z(x) \cup Z(z - 1)) \\
&= Z(x^2 - yz) \cap Z(x) \cup Z(x^2 - yz) \cap Z(z - 1) \\
&= Z(x^2 - yz, x) \cup Z(x^2 - yz, z - 1) \\
&= Z(yz, x) \cup Z(x^2 - yz, z - 1) \\
&= Z(yz, x) \cup Z(x^2 - yz + y(z - 1), z - 1) \\
&= Z(yz, x) \cup Z(x^2 - y, z - 1) \\
&= Z(yz) \cap Z(x) \cup Z(x^2 - y, z - 1) \\
&= (Z(y) \cup Z(z)) \cap Z(x) \cup Z(x^2 - y, z - 1) \\
&= (Z(y) \cap Z(x) \cup Z(z) \cap Z(x)) \cup Z(x^2 - y, z - 1) \\
&= Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1)
\end{aligned}$$

We now seek to show that $(x^2 - y, z - 1)$ is prime. Indeed, define the map $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto 1$. This is an isomorphism and $k[t]$ is a domain. Thus, we have $Y = Z(x, y) \cup Z(x, z) \cup Z(x^2 - y, z - 1)$ a union of three irreducible components. Their prime ideals are as given and we now geometrically describe each one of these.

$Z(x, y) = \{(a, b, c) \in \mathbb{A}^3 \mid a = b = 0\}$ so this is just the z axis. Similarly, $Z(x, z)$ is the y axis. Finally, $Z(x^2 - y, z - 1) = \{(a, b, c) \in \mathbb{A}^3 \mid a^2 = b, c = 1\} = \{(a, a^2, 1)\}$. This is a parabola sitting in the plane $z = 1$ with vertex at $(0, 0, 1)$ opening in the y axis. Hence, Y is the union of two lines, the y and z axes, and the parabola just described. $\square$

I.1.4

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Proof. The diagonal of $\mathbb{A}^2$ is defined by $Z(x-y)$, which is closed. If these topologies agreed, then the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$ will be closed so $\mathbb{A}^1$ will be Hausdorff. However, the topology on $\mathbb{A}^1$ is the cofinite topology and is therefore not Hausdorff. $\square$

I.1.5

Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Proof. For $Y \subseteq \mathbb{A}^n$ closed, we know that $I(Y)$ is radical. Hence, $A(Y) = k[x_1, \ldots, x_n]/I(Y)$ is reduced finitely generated k-algebra.

On the other hand, let B be some reduced finitely generated k-algebra. Then we can write $B \cong k[x_1, \ldots, x_n]/I$. As B is reduced, I is radical, so by the Nullstellensatz we have $I(Z(I)) = \sqrt{I} = I$. Hence, $A(Z(I)) = k[x_1, \ldots, x_n]/I \cong B$. $\square$

I.1.6

Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof. Let $\emptyset \neq U, V \subseteq X$ be open, X irreducible. We claim that $U \cap V \neq \emptyset$. If they were disjoint, $X = X - (U \cap V) = (X - U) \cup (X - V)$. These are proper closed sets so this contradicts irreducibility of X.

Now, let $Y \subseteq X$ be irreducible in the subspace topology. Let $\overline{Y} = A \cup B$ be closed sets. We want to show that one of A or B are equal to $\overline{Y}$. Indeed, $(A \cap Y) \cup (B \cap Y) = Y$, which is irreducible, so WLOG say $A \cap Y = Y$. Hence, $Y \subseteq A$ so $A \cup B = \overline{Y} \subseteq A$ and $A = \overline{Y}$, proving irreducibility. $\square$

I.1.7

(a) Show that the following conditions are equivalent for a topological space X:

  (i) X is noetherian;

  (ii) every nonempty family of closed subsets has a minimal element;

  (iii) X satisfies the ascending chain condition for open subsets;

  (iv) every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Proof. (a) The equivalence of (i) and (ii) is the same argument from algebra. (i) is equivalent to (iii) and (ii) is equivalent to (iv) via taking complements, which reverses order.

(b) Let $X = \bigcup \mathcal{U}$ and let $S = \{\bigcup \mathcal{V} \mid \mathcal{V} \subseteq \mathcal{U} \text{ finite}\}$. Then by (a), S has a maximal element $U_1 \cup \cdots \cup U_n$. If there is an $x \in X - (U_1 \cup \cdots \cup U_n)$ then let $x \in U_{n+1}$ open. Then $U_1 \cup \cdots \cup U_n \subsetneq U_1 \cup \cdots \cup U_{n+1}$ a contradiction. Then $X = U_1 \cup \cdots \cup U_n$ and X is compact.

(c) Let $A \subseteq X$. Let $A_1 \supseteq A_2 \supseteq \dots$ be a decreasing chain of closed sets in A. Then $A_i = \overline{A_i} \cap A$ and $\overline{A_1} \supseteq \overline{A_2} \supseteq \dots$ is a decreasing chain in X, which is noetherian. The chain upstairs terminates so the chain downstairs does as well.

(d) Let X be noetherian and Hausdorff. Let A be an irreducible closed subset of X. Let $x, y \in A$ and $x \in U, y \in V$ open neighborhoods in A. $U \cap V \neq \emptyset$ by irreducibility, so $x = y$ by Hausdorffness. Thus, the irreducible components of X are singletons and X is finite. Finite Hausdorff spaces are discrete.

□

I.1.8

Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Proof. Let $A = k[x_1, \dots, x_n]$. As H is a hypersurface, $H = Z(f)$ for some f irreducible. Let $Y = Z(\mathfrak{p})$ with $\mathfrak{p}$ prime. Then $B = A/\mathfrak{p}$ is the coordinate ring of Y. Consider $\overline{f}$ in B. As $Y \nsubseteq H$, $(f) \nsubseteq \mathfrak{p}$. Thus, $f \notin \mathfrak{p}$ so $\overline{f}$ is nonzero in B.

Let $F \subseteq Y \cap H$ be an irreducible component. Then $F = Z(\mathfrak{q})$ for some prime $\mathfrak{q}$. Furthermore, every irreducible closed subset of $Y \cap H$ is contained in some irreducible component. In other words, irreducible components are maximal with respect to being an irreducible closed subset of $F \cap Y$. Thus, $\mathfrak{q}$ is minimal with respect to primes containing $\mathfrak{p} + (f)$ as $Y \cap H = Z(\mathfrak{p} + (f))$.

Observe that $A/(\mathfrak{p} + (f)) \cong B/(\overline{f})$. B is a domain and as discussed, $\overline{f} \neq 0$ in B. By [Eisenbud, 13.11], $\dim B/(\overline{f}) = \dim B - 1$ so by [Eisenbud, 13.4], $\operatorname{codim}(\overline{f}) = 1$. We are trying to compute $\dim A/\mathfrak{q} = \dim B/(\mathfrak{q}/\mathfrak{p})$. By [Eisenbud, 13.4], this is $\dim B - \operatorname{codim} \mathfrak{q}/\mathfrak{p}$. The claim is therefore that $\operatorname{codim} \mathfrak{q}/\mathfrak{p} = 1$. $f \in \mathfrak{q}$ so $\overline{f} \in \mathfrak{q}/\mathfrak{p}$. Furthermore, $\mathfrak{q}$ is minimal with respect to containing $\mathfrak{p}$ and f so $\mathfrak{q}/\mathfrak{p}$ is minimal with respect to containing $\overline{f}$. By Krull's principal ideal theorem, $\operatorname{codim} \mathfrak{q}/\mathfrak{p} \leq 1$. As $\operatorname{codim}(\overline{f}) = 1$ and $(\overline{f}) \subseteq \mathfrak{q}/\mathfrak{p}$, we have $\operatorname{codim} \mathfrak{q}/\mathfrak{p} = 1$. In summary,

$$\begin{aligned}
\dim F &= \dim A/\mathfrak{q} \\
&= \dim B/(\mathfrak{q}/\mathfrak{p}) \\
&= \dim B - \operatorname{codim} \mathfrak{q}/\mathfrak{p} \\
&= \dim B - 1 \\
&= \dim Y - 1
\end{aligned}$$

□

I.1.9

Let $\mathfrak{a} \subseteq A = k[x_1..., x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

Proof. Let F be an irreducible component of $Z(\mathfrak{a})$. Then $F = Z(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. As F is an irreducible component, it is maximal with respect to being an irreducible closed subset of $Z(\mathfrak{a})$ and $\mathfrak{p}$ is therefore minimal with respect to containing $\mathfrak{a}$. $\mathfrak{a}$ can be generated by r elements so by Krull's height theorem (a generalization of the principal ideal theorem), $\operatorname{codim} p \leq r$. Using [Eisenbud, 13.4], we arrive at the equation $\dim A/\mathfrak{p} + \operatorname{codim} p = \dim A$. Of course, $\dim A = n$ and $\dim A/\mathfrak{p} = \dim F$. Hence, $\dim F + \operatorname{codim} p = n$. As $\operatorname{codim} p \leq r$, $\dim F \geq n - r$. □

I.1.10

(a) If Y is any subset of a topological space X, then $\dim Y \leq \dim X$.

(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

Proof. (a) Take a chain of irreducible closed subsets $Z_0 < \cdots < Z_r \subseteq Y$. The closures $\overline{Z_i}$ are irreducible and $\overline{Z_i} < \overline{Z_{i+1}}$ as $Z_i = \overline{Z_i} \cap Y$. Thus, every chain of irreducible closed subsets of Y lifts to a chain of irreducible closed subsets of X with the same length. Hence, $\dim Y \leq \dim X$.

(b) By (a) $\dim U_i \leq \dim X$. Hence, $\sup \dim U_i \leq \dim X$. We claim that if $Z_0 < \cdots < Z_n$ is a chain of irreducible closed subsets of X then some U_i has a chain of the same length. This will show that $\dim X \leq \sup \dim U_i$, proving equality. Let $U_i \cap Z_0 \neq \emptyset$. We claim $Z_j \cap U_i < Z_{j+1} \cap U_i$, which will become our chain in U_i of the same length. Indeed, $Z_j \cap U_i \subseteq Z_j$ is a nonempty open subset and is therefore dense and irreducible. Hence, $\overline{Z_j \cap U_i} = Z_j$. Hence, if $Z_j \cap U_i = Z_{j+1} \cap U_i$ then their closures are equal and $Z_j = Z_{j+1}$ a contradiction.

(c) Consider the space $X = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. The closed sets are therefore $\{\{a, b, c\}, \{b, c\}, \{c\}, \emptyset\}$. Hence, $\{c\} < \{b, c\}$ is a chain of irreducible closed subsets so $\dim X \geq 1$. Furthermore, $\{a\}$ is an open generic point (it intersects all nonempty open subsets) so it is a dense open set with dimension $0 < 1$.

(d) Take a maximal chain $Z_0 < \cdots < Z_r$ in Y. Then these Z_i are closed and irreducible in X. Hence, we have the chain $Z_0 < \cdots < Z_r \subseteq X$. As $\dim X = \dim Y = r$, $X = Z_r \subseteq Y$.

(e) Let R be a noetherian ring of infinite dimension (see the example due to Nagata [Stacks, 02JC]). Then $\operatorname{Spec} R$ is a noetherian topological space of infinite dimension.

□

I.1.11

Let $Y \subseteq \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection*—cf. (Ex. I.2.17).

Proof. First of all, consider the map $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$. As discussed previously, we are justified in using the same name for the variables of the coordinate ring and the coordinates on affine space themselves because for an infinite field (such as the algebraically closed field k we work over) polynomials are identified with polynomials. This map has kernel $I(Y)$ by definition. Its image is $k[t^3, t^4, t^5]$, a domain, so $I(Y)$ is prime. Furthermore, $\dim k[x, y, z]/I(Y) = 3 - \operatorname{codim} I(Y)$. Hence, to compute the codimension of $I(Y)$, we need only compute the dimension of $k[x, y, z]/I(Y) \cong k[t^3, t^4, t^5]$. Observe that $k[t^3, t^4, t^5] \subseteq k[t]$ is integral. Indeed, t satisfies the monic polynomial $s^3 - t^3$. Hence, its dimension is $\dim k[t] = 1$. Then $\operatorname{codim} I(Y) = 2$ as desired.

Let $\mathfrak{m} = (x, y, z)$. We claim that $(S)/\mathfrak{m}(S) \cong \sum_{s\in S} k \cdot s$ for any $S \subseteq k[x, y, z]$. Indeed, any element of $(S)/\mathfrak{m}(S)$ is represented by some $\sum f_i s_i$. Let's analyze one such $fs = \left(\sum a_{ijk} x^i y^j z^k\right) s$. For all $(i, j, k) \neq (0, 0, 0)$, $a_{ijk} x^i y^j z^k \in \mathfrak{m}$. Hence, $fs \equiv a_{000} s \pmod{\mathfrak{m}(S)}$. Thus, $\sum f_i s_i \equiv \sum f_i(0) s_i \pmod{\mathfrak{m}(S)}$. In other words, every element of $(S)/\mathfrak{m}(S)$ is represented by some element of $\sum_{s\in S} k \cdot s$. On the other hand, if $\sum a_i s_i \equiv \sum b_j s_j \pmod{\mathfrak{m}(S)}$ for $a_i, b_j \in k$ then $\sum a_i s_i - \sum b_j s_j \in \mathfrak{m}(S)$. Hence, evaluating the a_i, b_j at 0 sends this to 0. These are constants, so the difference is 0 and $\sum a_i s_i = \sum b_j s_j$. This proves the isomorphism.

Observe that $x^3 - zy, y^2 - xz, z^2 - x^2 y \in I(Y) = I$ all have different x degrees. Hence, they are k linearly independent. Let $(S) = I$ with $x^3 - zy, y^2 - xz, z^2 - x^2 y \in S$. Then as above, $I/\mathfrak{m}I \cong \sum_{s\in S} k \cdot s$. As $\{x^3 - zy, y^2 - xz, z^2 - x^2 y\}$ is k linearly independent, $\sum_{s\in S} k \cdot s$ has dimension at least 3. Furthermore, the above paragraph shows that any generating set of I spans $I/\mathfrak{m}I$. Hence, any generating set of I must have at least $\dim I/\mathfrak{m}I \geq 3$ elements. □

I.1.12

Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $Z(f)$ in $\mathbb{A}^2$ is not irreducible.

Proof. Consider the irreducible polynomial $f = x^2 + y^2 + 1$. Its zero set is empty and therefore not irreducible.

⟨Let's find a less trivial example - such an f with $Z(f) \neq \emptyset$.⟩ □

I.2 Projective Varieties

Conventions. $S = k[x_0, \ldots, x_n]$.

I.2.1

Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [*Hint*: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, [Hartshorne, 1.3A].

Proof. $\mathbb{A}^{n+1}$ has coordinate ring S. For notational ease, I will denote $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ to be the affine variety defined by $\mathfrak{a}$, as opposed to $Z(\mathfrak{a})$ which refers heretofor only to the projective variety. The given condition on f is that for all $P \in Z(\mathfrak{a})$, $f(P) = 0$. By definition, $f(P) = 0$ means that $f(a_0, \ldots, a_n) = 0$ for any homogeneous coordinates $(a_0, \ldots, a_n)$ of P. Take now some $(a_0, \ldots, a_n) \in V(\mathfrak{a})$. We claim that $f(a_0, \ldots, a_n) = 0$. As f is homogeneous, $f(0) = 0$ so suppose $(a_0, \ldots, a_n)$ is nonzero and let $P \in \mathbb{P}^n$ be the point it represents. Let $g \in \mathfrak{a}$ be homogeneous. Then $g(a_0, \ldots, a_n) = 0$ so $g(P) = 0$ and $P \in Z(\mathfrak{a})$. Hence, $f(P) = 0$ so by definition, $f(a_0, \ldots, a_n) = 0$. Thus, $f \in I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ by the Nullstellensatz. □

I.2.2

For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \emptyset$;

(ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;

(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Proof. Note that $S - S_+ = k - 0$. Hence, statement (ii) is equivalent to $\sqrt{a} \supseteq S_+$.

(iii) $\implies$ *(i)*. If $\mathfrak{a} \supseteq S_d$ then $x_i^d \in \mathfrak{a}$ for all i. Hence, $Z(\mathfrak{a}) \subseteq \cap Z(x_i^d) = \emptyset$.

(i) $\implies$ *(ii)*. If $\deg f > 0$ then $f \in S_+$. Furthermore, it holds vacuously that for all $P \in Z(\mathfrak{a}) = \emptyset$, $f(P) = 0$. Hence, by the homogeneous Nullstellensatz, $f^n \in \mathfrak{a}$ for some n. Hence, $\sqrt{\mathfrak{a}} \supseteq S_+$. Thus, $\mathfrak{a} = S_+$ or $\mathfrak{a} = S$ as $S - S_+ = k$. Note that we applied the homogeneous Nullstellensatz on $Z(\mathfrak{a}) = \emptyset$. This is justified as $Z(\mathfrak{a}) = \emptyset$ implies that $V(\mathfrak{a}) - 0 = \emptyset$. Indeed, letting $\pi : \mathbb{A}^{n+1} \longrightarrow \mathbb{P}^n$, $\pi^{-1}[Z(\mathfrak{a})] = V(\mathfrak{a}) - 0$. With this formula in mind, we can see that the proof of I.2.1 is valid for the empty set.

(ii) $\implies$ *(iii)*. Let $\sqrt{\mathfrak{a}} \supseteq S_+$. Then $x_i \in \sqrt{\mathfrak{a}}$ for all i. Hence, $x_i^{d_i} \in \mathfrak{a}$ for some d_i. Let $d = \sum d_i$. Now, let $x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n}$ be a generic (modulo constants) degree d monomial. That is, $\sum k_i = d$. If all $k_i > d_i$ then $d = \sum k_i > \sum d_i = d$. Hence, some $k_i \leq d_i$ so the term $x_i^{d_i}$ appears in this expression. Hence, $x_0^{k)} x_1^{k_1} \cdots x_n^{d_n} \in \mathfrak{a}$. Hence, $S_d \subseteq \mathfrak{a}$. □

I.2.3

(a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$.

Proof. (a) Let $P \in Z(T_2)$ and $f \in T_1 \subseteq T_2$. Then $f(P) = 0$ so $P \in Z(T_1)$.

(b) Let $f \in I(Y_2)$ and $P \in Y_1 \subseteq Y_2$. Then $f(P) = 0$ so $f \in I(Y_1)$.

(c) Each $Y_i \subseteq Y_1 \cup Y_2$ so by part (b), $I(Y_i) \supseteq I(Y_1 \cup Y_2)$. Hence, $I(Y_1) \cap I(Y_2) \supseteq I(Y_1 \cup Y_2)$. On the other hand,let $f \in (I(Y_1) \cap I(Y_2))^h$. Then $f[Y_1], f[Y_2] \subseteq \{0\}$. Thus, $f[Y_1 \cup Y_2] \subseteq \{0\}$ so $f \in I(Y_1 \cup Y_2)^h$. As these ideals are homogeneous, the homogeneous elements generate so we have $I(Y_1) \cap I(Y_2) \subseteq I(Y_1 \cup Y_2)$.

(d) Let $f \in I(Z(\mathfrak{a}))^h$ with $\deg f > 0$. Then by definition, for all $P \in Z(\mathfrak{a})$, $f(P) = 0$. Then by the homogenous Nullstellensatz (**??**), $f \in \sqrt{\mathfrak{a}}$. Hence, 0 and all nonconstant homogeneous polynomials $f \in I(Z(\mathfrak{a}))$ are in $\sqrt{a}$. Furthermore, as $Z(\mathfrak{a}) \neq 0$, $k \cap I(Z(\mathfrak{a}))^h = \{0\}$. Of course, 0 is also in $I(Z(\mathfrak{a}))^h$ and $\sqrt{\mathfrak{a}}$. Hence, $I(Z(\mathfrak{a}))^h \subseteq \sqrt{\mathfrak{a}}$. As all these ideals are homogeneous, this proves $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$. Of course, if $f^n(P) = 0$ then $f(P) = 0$. Thus, $I(Z(\mathfrak{a}))$ is radical and $\mathfrak{a} \subseteq I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ so we achieve equality.

(e) We of course have $Y \subseteq Z(I(Y))$, so $\overline{Y} \subseteq Z(I(Y))$. On the other hand, let $Y \subseteq Z(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$. By definition, this is a generic closed set containing Y. Furthermore, this means that for all $P \in Y$ and $f \in \mathfrak{a}^h$, $f(P) = 0$. Hence, $\mathfrak{a}^h \subseteq I(Y)$. As $\mathfrak{a}$ is homogeneous, $\mathfrak{a} \subseteq I(Y)$ so $Z(\mathfrak{a}) \supseteq Z(I(Y))$. As $Z(\mathfrak{a})$ was arbitrary, $Z(I(Y)) = \overline{Y}$.

□

I.2.4

(a) There is a $1-1$ inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note*: Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

(b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

(c) Show that $\mathbb{P}^n$ itself is irreducible.

Proof. (a) $Z(\mathfrak{a})$ is, by definition, always algebraic. Furthermore, it's easy to see that $I(Y)$ is always radical and homogeneous. We must show then that $I(Y)$ can never equal S_+ for Y algebraic. Indeed, if it was the case that $I(Z(\mathfrak{a}))) = S_+$then $x_i \in I(Z(\mathfrak{a}))$ for all i. Thus, each x_i sends all of $Z(\mathfrak{a})$ to 0. But $\bigcap Z(x_i) = \emptyset$ so $Z(\mathfrak{a}) = \emptyset$. However, $I(\emptyset) = S \neq S_+$.

That I and Z are inverses on these restricted domains follows from parts (d) and (e) of problem I.2.3.

(b) Let $I(Y)$ be prime. Then let $Y \subseteq Z(I_1) \cup Z(I_2)$. Hence, $I(Y) \supseteq I(Z(I_1) \cup Z(I_2))$. By I.2.3 part (c), this is $I(Z(I_1)) \cap I(Z(I_2)) = \sqrt{I}_2 \cap \sqrt{I}_2 \supseteq I_1 I_2$. Hence, $I(Y) \supseteq I_1 I_2$. It is then a general fact of commutative algebra that some $I_j \subseteq I(Y)$. Indeed, if neither is contained in $I(Y)$ then let $a_j \in I_j - I(Y)$. $a_1 a_2 \in I_1 I_2$ but cannot be in $I(Y)$ as it is prime, a contradiction.

On the other hand, let Y be irreducible. Let f, g be homogeneous such that $fg \in I(Y)^h$. Then $Y \subseteq Z(fg) \subseteq Z(f) \cup Z(g)$. Then as Y is irreducible, it is contained in one of these, WLOG say $Y \subseteq Z(f)$. Hence, $f \in I(Y)$. As $I(Y)$ is homogeneous, this proves its primality.

(c) $\mathbb{P}^n = I(0)$.

□

I.2.5

1. $\mathbb{P}^n$ is a Noetherian topological space.

2. Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Proof. 1. Let $Y_1 > Y_2 > \dots$ be closed subsets of $\mathbb{P}^n$. Then we have $I(Y_1) < I(Y_2) < \dots$ in the coordinate ring $k[x_0, \dots, x_n]$. But this ring is Noetherian - contradiction.

2. First note that a closed subspace of a Noetherian space is itself Noetherian. Indeed, a chain in the subspace is a chain in the superspace. So we prove this result for a general Noetherian space X.

If X is irreducible then we are done. Else, we have $X = X_0 \cup X_1$ for some proper closed subsets X_i. We can proceed this process to $X_0 = X_{00} \cup X_{01}$ and $X_1 = X_{10} \cup X_{11}$. Continue this until everything in sight is irreducible. This will happen in finitely many steps, as if not there is some binary string $b_0 b_1 \dots$ such that $X_{b_0} > X_{b_0 b_1} > \dots$ - contradicting Noetheriannessitudity. Now, if $X = X_0 \cup \dots \cup X_n$ are irreducible, we can omit any containments so suppose no $X_i \subseteq X_j$. If in addition we have $X = Y_0 \cup \dots \cup Y_m$ an irreducible decomposition. Then each $X_i \subseteq Y_{\sigma(i))}$ for some $0 \leq \sigma(i) \leq m$. Similarly, each $Y_j \subseteq X_{\tau(j)}$ for some $0 \leq \tau(j) \leq n$. Then $X_i \subseteq X_{\tau(\sigma(i))}$, so by assumption $X_i = X_{\tau(\sigma(i))}$. Similarly, $Y_j = Y_{\sigma(\tau(i))}$. Thus, these are inverses so these decompositions are the same up to permutation.

□

I.2.6

If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ be the homeomorphism of [Hartshorne, 2.2], let Y_i be the affine variety $\phi_i[Y \cap U_i]$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use [Hartshorne, 1.7], [Hartshorne, 1.8A], and (Ex I.1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

Proof. We will follow this hint. We want to identity $A(Y_i)$ with $(S(Y)_{x_i})^0$, which is the set of all $\frac{f}{x_i^d}$ such that $f \in k[x_0, \ldots, x_n]$ is homogeneous of degree d. We will consider the coordinate ring $A(Y_i)$ to have coordinates $x_0, \ldots, \widehat{x_i}, \ldots, x_n$. We define the map

$$\begin{aligned} k[x_0, \ldots, \widehat{x_i}, \ldots, x_n] &\longrightarrow (S(Y)_{x_i})^0 \\ f &\mapsto f\left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) \end{aligned}$$

It's not hard to see that this lands in the degree 0 subring. We want this map to factor through $A(Y_i)$. What then is $J(Y_i)$? By definition, it is the $f \in k[a_0, \ldots, \widehat{a_i}, \ldots, a_n]$ such that $f(a) = 0$ for all $a \in Y_i$. Of course, $Y_i = \phi_i[Y \cap U_i]$ so any such a will look like $\left(\frac{a_0}{a_1}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$. Indeed, take some $f \in J(Y_i)$. Then f is mapped to $\frac{\beta_i(f)}{a_i^d}$ where $\beta_i(f)$ is the homogenization of f with respect to a_i, i.e. $\beta_i(f) = a_i^d f\left(\frac{a_0}{a_i}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$. Here we of course mean $d = \deg f$. Now, if $d = 0$ then f is constant and vanishes on Y_i, so it must be 0 (or Y_i must be empty but this is a case we ignore). If $d \neq 0$ then consider some $[a_0 : \cdots : a_n] \in Y$. If $a_i = 0$ then $\beta_i(f)([a_0 : \cdots : a_n]) = a_i^d f\left(\frac{a_0}{a_i}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right) = 0$ as $d > 0$. On the other hand, if $a_i \neq 0$ then the term $f\left(\frac{a_0}{a_i}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right) = 0$ as $f \in J(Y_i)$. In any case, we see therefore that $\beta_i(f) \in I(Y)$. Hence, $\frac{\beta_i(f)}{x_i^d} \in I(Y)_{x_i}$. By flatness of localization, $S(Y)_{x_i} = k[x_0, \ldots, x_n]_{x_i}/I(Y)_{x_i}$. Thus, we can quotient this map to yield $A(Y_i) \longrightarrow (S(Y)_{x_i})^0$.

We now seek to show that this map is an isomorphism, which we will do by exhibiting aninverse. Recall the "dehomogenization" of f with respect to x_i is $\alpha_i(f) = f(x_0, \ldots, 1, \ldots, x_n)$ with 1 in the i^{th} position. We consider the map $(S(Y)_{x_i})^0 \longrightarrow A(Y_i)$ via $\frac{f}{x_i^d} \mapsto \alpha_i(f)$, where of course f is taken to be homogeneous and degree d. In other words, this map evaluates a rational function at $(x_0, \ldots, 1, \ldots, x_n)$.

We first have to show that this map is actually well defined on these quotients. Indeed, let's begin with some $f \in I(Y)_{x_i}$ of homogeneous degree 0, so $f = \frac{g}{x_i^d}$ with $g \in I(Y)^h$ of degree d. Then this maps to $g(x_0, \ldots, 1, \ldots, x_n)$. Take some $\left(\frac{a_0}{a_1}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right) \in Y_i$. Then $g(x_0, \ldots, 1, \ldots, x_n)\left(\frac{a_0}{a_1}, \ldots, \frac{\widehat{a_i}}{a_i}, \ldots, \frac{a_n}{a_i}\right) = g\left(\frac{a_0}{a_i}, \ldots, 1, \ldots, \frac{a_n}{a_i}\right) = 0$ as $g \in I(Y)^h$. Thus, this does descend to a well defined map on the quotient $(S(Y)_{x_i})^0$.

It is now easy enough to verify that these maps are inverse. Let $f \in A(Y_i)$. Then $f \mapsto \frac{\beta_i(f)}{x_i^d} \mapsto \alpha_i(\beta_i(f))$. This is $f\left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right)(x_0, \ldots, 1, \ldots, x_n) = f$. On the other hand, let $\frac{f}{x_i^d} \in (S(Y)_{x_i})^0$. Then $\frac{f}{x_i^d} \mapsto \alpha_i(f) \mapsto \frac{\beta_i(\alpha_i(f))}{x_i^d}$. The numerator here is $\beta_i(f(x_0, \ldots, 1, \ldots, x_n)) = x_i^d f\left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}\right) = f$ as f is homogeneous of degree f.

In summary, we have defined the following maps:

$$\begin{array}{ccc} x_j & \longmapsto & \frac{x_j}{x_i} \\ A(Y_i) & \overset{\sim}{\longleftrightarrow} & (S(Y)_{x_i})^0 \\ \begin{cases} 1 & i = j \\ x_j & i \neq j \end{cases} & \longleftarrow & x_j \end{array}$$

Next, we will extend this to a map $A(Y_i)[x_i, x_i^{-1}] \longrightarrow S(Y)_{x_i}$ via sending $x_i \mapsto x_i$. The image of this map contains x_i, x_i^{-1} and $(S(Y)_{x_i})^0$. Given f homogeneous, $\frac{f}{x_i^d} \in (S(Y)_{x_i})^0$, so $f \in (S(Y)_{x_i})^0[x_i]$. Hence, this map is onto. By flatness, it suffices to show that the map $A(Y_i)[x_i] \longrightarrow S(Y)_{x_i}$ is injective.

Recall that each $\phi(f_k)$ has degree 0 in $S(Y)_{x_i}$. Then $\phi(f_k)x_i^k$ has degree k in this ring, so each term in $\sum \phi(f_k)x^k$ has different degree, so these are necessarily linearly independent. Hence, for this sum to vanish, each term $\phi(f_k)x_i^k$ must vanish as well. Since we are in a domain (Y is a variety), this means that each $\phi(f_k) = 0$. ϕ has already been shown to be an isomorphism $A(Y_i) \longrightarrow (S(Y)_{x_i})^0$, so each f_k is therefore 0. Thus, the extension of ϕ to $A(Y_i)[x_i, x_i^{-1}] \longrightarrow S(Y)_{x_i}$ is an isomorphism as well.

Now, to compute $\dim S(Y)$ we need to compute $\dim A(Y_i)[x_i, x_i^{-1}]$, which is equal to the transcendence degree of $qf(A(Y_i)[x_i]) = qf(A(Y_i))(x_i)$. x_i was a formal variable for $A(Y_i)$, as we took A to have coordinates $x_0, \ldots, \widehat{x_i}, \ldots, x_n$. Hence, $\operatorname{trdeg}_k qf(A(Y_i))(x_i) = \operatorname{trdeg}_k qf(A(Y_i)) + 1 = \dim A(Y_i) + 1$. By I.1.10(b), $\dim Y = \sup \dim Y_i$. Y_i is empty when $Y \cap U_i = \emptyset$, i.e. when $Y \subseteq Z(x_i)$. In that case, $x_i \in I(Y)$ so $x_i = 0$ in $S(Y)_{x_i}$. When this is not the case (and it must not be the case for some x_i), then $x_i \neq 0$ in $S(Y)$ so we have inclusions $S(Y) \subseteq S(Y)_{x_i} \subseteq qf(S(Y))$. We have computed that $qf(S(Y)_{x_i}) = qf(A(Y_i))(x_i)$, which has transcendence degree $\dim A(Y_i) + 1$, which is $\dim Y_i + 1$. Thus, for all i such that $Y_i \neq \emptyset$, we have $\dim Y_i = \dim S(Y) - 1$. Taking sups yields $\dim Y = \dim S(Y) - 1$, and that $\dim Y = \dim Y_i$ whenever $Y_i \neq \emptyset$. □

I.2.7

(a) $\dim \mathbb{P}^n = n$.

(b) If $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.
Hint: Use (Ex. I.2.6) to reduce to [Hartshorne, 1.10].

Proof. (a) $S(\mathbb{P}^n) = k[x_0, \ldots, x_n]$ has dimension $n + 1$ so this follows from I.2.6.

(b) By I.1.10(b), $\dim \overline{Y} = \sup \dim \overline{Y} \cap U_i$. By [Hartshorne, 1.10], $\dim Y \cap U_i = \dim \overline{Y \cap U_i}$, where the closure is computed in $U_i \cong \mathbb{A}^n$. Furthermore, $\overline{Y \cap U_i} = \overline{Y} \cap U_i$, where on the right hand side the closure is in $\mathbb{P}^n$. This is an elementary fact about the subspace topology. In other words, we have $\dim \overline{Y} \cap U_i = \dim Y \cap U_i$. Taking sup on both sides and using I.1.10(b) we get the equality $\dim \overline{Y} = \dim Y$.

□

I.2.8

A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbb{P}^n$.

Proof. ($\Longleftarrow$). The ideal (f) has height 1 so by I.2.6, $\dim Z(f) = \dim S/(f) - 1 = (n+1) - 1 - 1 = n-1$.

($\Longrightarrow$). Let $Y = Z(\mathfrak{p})$ have dimension $n-1$. Then by I.2.6, $\dim S(Y) = n$. As $\dim S(Y) = \dim S - \operatorname{codim} \mathfrak{p}$, this tells us that $\operatorname{codim} \mathfrak{p} = 1$. As S is a UFD, this means that $\mathfrak{p} = (f)$ for some $\deg f > 0$.

We must therefore show that f is homogeneous. As the ideal $(f) = \mathfrak{p}$ is homogeneous, we can decompose $(f) = \bigoplus (f) \cap S_d$. Consider therefore the homogeneous decomposition $f = \sum f_e$. As (f) is homogeneous, each $f_e \in (f)$. Furthermore, as (f) is not the unit ideal, $(f) \cap S_0 = 0$. As $(f) \neq 0$, we must have some $d > 0$ such that $(f) \cap S_d \neq 0$. Let d be minimal with respect to this property and let $g \in (f) \cap S_d - 0$. Then $g \in (f)$ so $f \mid g$. Write $fh = g$, so in the homogeneous decompositions we get $(\sum f_e)(\sum h_e) = g_d = g$. We can compute this sum as follows:

$$\begin{aligned}\left(\sum f_e\right)\left(\sum h_e\right) &= \sum_{e,e'} f_e h_{e'} \\ &= \sum_{k \geq 0} \sum_{k=e+e'} f_e h_{e'}\end{aligned}$$

So we have $\sum_{e+e'=d} f_e h_{e'} = g_d$ and all other terms are 0. We additionally have $f_0 = 0$ as $f_0 \in (f) \cap S_0 = 0$. Furthermore, by minimality of d, there can be no $0 < e < d$ such that $f_e \neq 0$. Hence, for all $0 \leq e < d$, $f_e = 0$. As $f \mid g$, f cannot have any terms of degree higher than d. Thus, $f = f_d$ is homogeneous. □

I.2.9 INCOMPLETE

Projective Closure of an Affine Variety. If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta[J(Y)]$, using the notation of the proof of [Hartshorne, 2.2].

(b) Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic of (Ex. I.1.2). Its projective closure $\overline{Y} \subseteq \mathbb{P}^3$ is called the twisted cubic curve in $\mathbb{P}^3$. Find generators for $J(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $J(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

Proof. (a) First and foremost, observe that $I(\overline{Y}) = I(Z(I(Y))) = \sqrt{(}I(Y)) = I(Y)$. Hence, we will focus our attention on $I(Y)$.

Recall the definition of $\beta(f) = \beta_0(f) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ where $d = \deg f$. Let $f \in J(Y)$. Then for all $[a_0 : \cdots : a_n] \in Y$ (by which we mean $\phi^{-1}[Y]$), $0 = f(\phi([a_0 : \cdots : a_n])) = f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)$. Then indeed, $\beta(f) \in I(Y)^h$. This yields the easier inclusion $(\beta[J(Y)]) \subseteq I(Y)$.

On the other hand, let $f \in I(Y)^h$. Then for all $\phi^{-1}(a_1, \ldots, a_n) = [1 : a_1 : \cdots : a_n]$, $f([1 : a_1 : \cdots : a_n])$. Letting $\alpha(f) = \alpha_0(f) = f(1, x_1, \ldots, x_n)$ this says that $\alpha(f) \in J(Y)$. Naively, we'd just apply β and go home, but tragically this fails. Indeed, consider $\beta(\alpha(x_0^d)) = \beta(1) = 1$. α has the potential to lose the data of the x_0, but all hope is not lost. Consider $\beta(\alpha(x_0 + x_1)) = \beta(1 + x_1) = x_0 + x_1$, so the issue seems to be with powers of x_0. Wih that in mind, let $x_0^e \mid\mid f$ and write $f = x_0^e g$. Then $\alpha(g) = \alpha(f)$, so we'll try to compute $\beta(\alpha(g))$.

Indeed, $\alpha(g) = g(1, x_1, \ldots, x_n)$ so $\beta(\alpha(g)) = g\left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)x_0^d$. Here, $d = \deg \alpha(g)$ As f is homogeneous and $g = \frac{f}{x_0^e}$, g is homogeneous as well. Thus, if it were the case that $\deg g = d$ then this would be precisely g. Indeed, as $x_0 \nmid g$ there must be some monomial summand of g which has no x_0 term. This term has the same degree as g, as g is homogeneous. Furthermore, applying α to such a term leaves it unchanged. As α cannot increase degree, this means that the degree of $\alpha(g)$ is indeed equal to the degree of g, so $g = \beta(\alpha(g))$.

As discussed, this means that $g = \beta(\alpha(g)) = \beta(\alpha(g)) \in \beta[J(Y)]$. Hence, $f = x_0^e g \in (\beta[J(Y)])$, so $I(Y)^h \subseteq (\beta[J(Y)]) \subseteq I(Y)$. As $I(Y)$ is homogeneous, we get the desired equality $I(Y) = (\beta[J(Y)])$.

(b) The twisted cubic was defined by $Y = \{(t, t^2, t^3) : t \in k\} \subseteq \mathbb{A}^3$. As shown in I.1.2, Z is a variety with ideal $I(Y) = (x_3^3 - x_1, x_2^2 - x_1)$. Applying β to these yields $x_3^3 - x_0^2 x_1$ and $x_2^2 - x_0 x_1$. We therefore seek to prove that $I(\overline{Y}) > (x_3^3 - x_0^2 x_1, x_2^2 - x_0 x_1)$.

It'd be nice to get some kind of picture here, so in figure 1 below we show a plot of the twisted cubic curve in affine space.

Now let's try to visualize this in $\mathbb{P}^3$. We'll appeal to the usual CW complex structure on $\mathbb{RP}^n$. Indeed, we consider $\mathbb{P}^3$ to be the unit ball in $\mathbb{R}^3$ where the boundary sphere has the usual antipoal gluing. If we scale the plot above into the open unit disk, then we get figure 2

This suggests that the closure in $\mathbb{P}^3$ should be the affine twisted cubic along with this blue line $[0 : 0 : 0 : 1]$, and that this line is approached by the affine twisted cubic as one "tends to infinity." How do we make rigorous this idea of tending to infinity? Well the twisted cubic is parametrized by $\mathbb{A}^1$, so it stands to reason that to parametrize its projective closure, and hence to "tend to infinity" that we would want a parametrization by $\mathbb{P}^1$. Indeed, the affine parametrization is given by $t \mapsto (t, t^2, t^3)$. We'll homogenenize this to get a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ via $[t_0 : t_1] \mapsto [t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3]$.

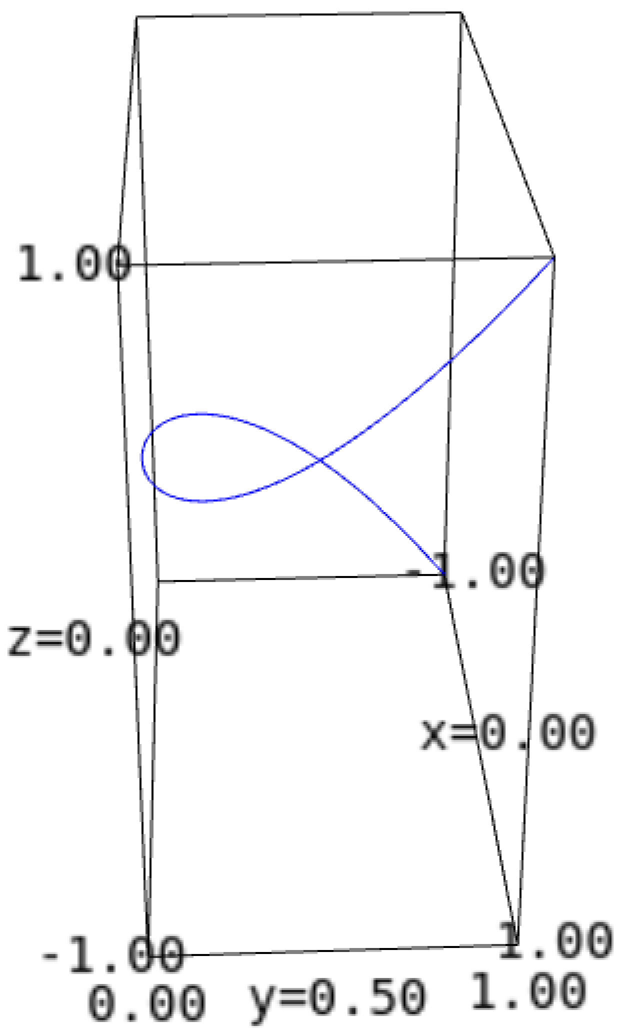

Figure 1: The twisted cubic in $\mathbb{A}^3$

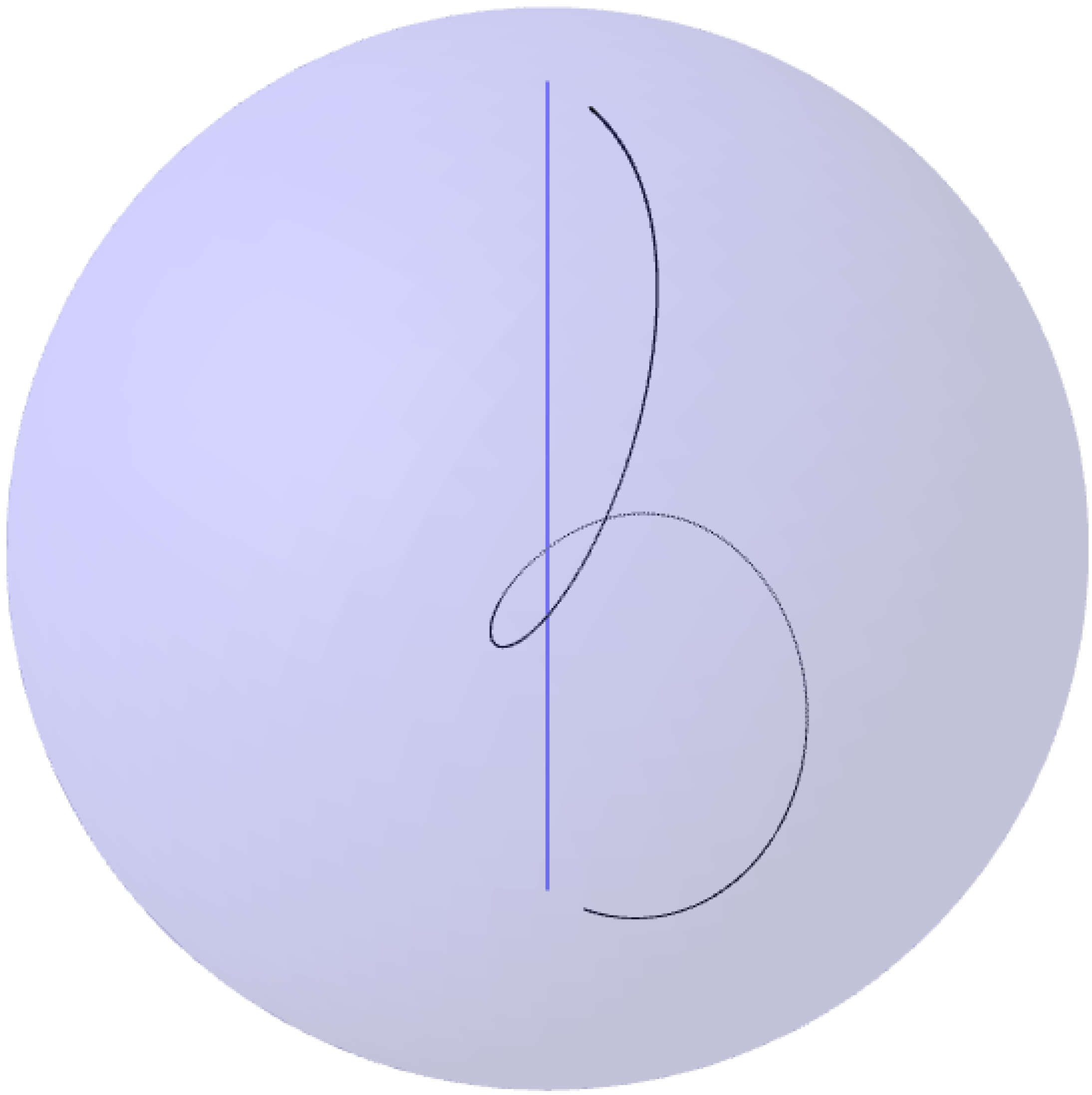

Figure 2: The twisted cubic in $\mathbb{P}^3$

Restricting to $t_0 = 1$ gives us the original affine parametrization, and the point at infinity $[0:1]$ maps to $[0:0:0:1]$ as we expected.

We claim therefore that the image of this map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ (see I.2.12 for the general case!) is the projective closure of Y. Observe that the image of this map is contained in $Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$. If we intersect this algebraic set with $x_0 = 1$ we get Y back. If we consider $x_0 = 0$ then these equations yield $x_1^3 = x_1^2 = x_1x_2 = 0$, which leaves only $[0:0:0:1]$. As our intuition suggested, we therefore expect that $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ has image $Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2) = Y \cup \{[0:0:0:1]\}$ and that this is precisely $\overline{Y}$. Indeed, we can do this same trick of considering $t_0 = 1$ and $t_0 = 0$ to prove that this is indeed the image, so it suffices to show that $\overline{Y} = Y \cup \{[0:0:0:1]\}$.

Certainly $Y \cup \{[0:0:0:1]\}$ is closed. Thus, if Y itself is not closed then certainly this must be the closure. If Y was closed, then $Y \cup \{[0:0:0:1]\} = Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$ seperates this closed set into two disjoint closed sets. However, it is not hard to see that our map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ is continuous, and $\mathbb{P}^1$ is connected, so the image must be connected as well. Thus, this decomposition would be a contradiction so Y is not closed. Hence, we have computed $\overline{Y} = \operatorname{im}(\mathbb{P}^1 \longrightarrow \mathbb{P}^3) = Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$.

We can finally show that $(x_3^3 - x_0^2x_1, x_2^2 - x_0x_1) < I(\overline{Y})$. Indeed, $[0:0:1:1] \in Z(x_3^3 - x_0^2x_1, x_2^2 - x_0x_1)$ but is not in $\overline{Y}$, so we have the strict containment.

However, we are still asked to find actual generators for $I(\overline{Y})$. Our best guess is of course $(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$, but we only know that the radical of this ideal is $I(\overline{Y})$. There's probably a way to do all this nonsense, but we'll defer it to I.2.12. Should that problem remain unsolved, mutter something about Gröbner bases to stop worrying about this computation.

□

I.2.10

The Cone Over a Projective Variety (3). Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{(0,\ldots,0)\} \longrightarrow \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0,\ldots,a_n)$ to the point with homogeneous coordinates $(a_0,\ldots,a_n)$· We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}[Y] \cup \{(0,\ldots,0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0,\ldots x_n]$.

(b) $C(Y)$ is irreducible if and only if Y is.

(c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1}$. This is called the *projective cone* over Y.

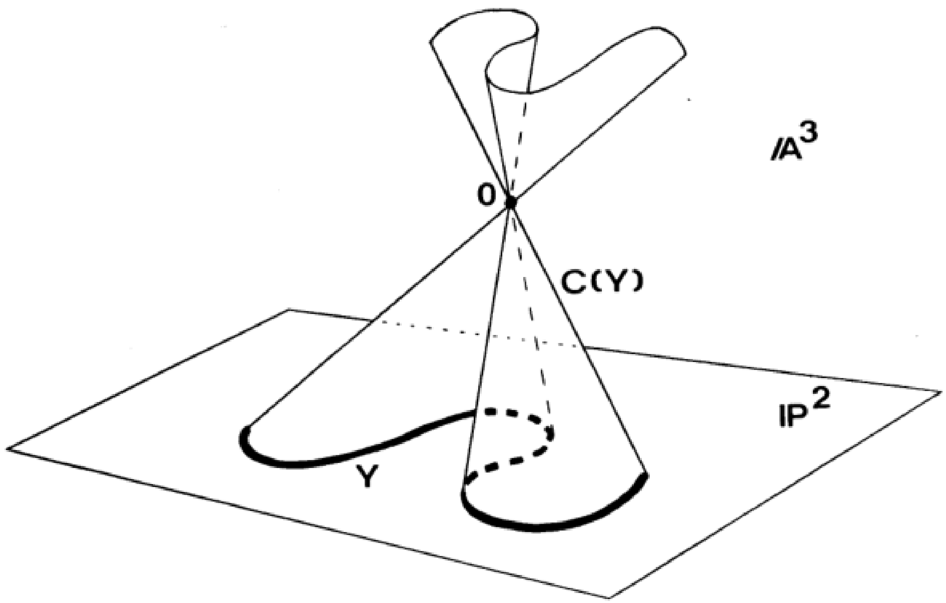

Figure 3: The cone over a curve in $\mathbb{P}^2$.

Proof. (a) Let $I = I(Y)$, so that $Y = Z(I)$. We seek to compute $\theta^{-1}[Z(I)]$. Indeed, we claim that this is precisely $V(I) - 0$. If $(a_0, \ldots, a_n) \in V(I) - 0$ then $\theta(a_0, \ldots, a_n) = [a_0 : \cdots : a_n] \in \mathbb{P}^n$. Let $f \in I^h$. Then $f(a_0, \ldots, a_n) = 0$ so indeed, $f([a_0 : \cdots : a_n]) = 0$ and $\theta(a_0, \ldots, a_n) \in Z(I)$. Hence, $V(I) - 0 \subseteq \theta^{-1}[Z(I)]$. Conversely, let $\theta(a_0, \ldots, a_n) \in Z(I)$. Then of course $(a_0), \ldots, a_n) \neq 0$. Let $f \in I$ and write $f = \sum f_e$ the homogeneous decomposition. As $I = I(Y)$ is homogeneous, each $f_e \in I^h$. Then as $[a_0 : \cdots : a_n] \in Z(I)$, we have that $f_e(a_0, \ldots, a_n) = 0$ for all e. Hence, $f(a_0, \ldots, a_n) = 0$ and we have computed $\theta^{-1}[Z(I)] = V(I) - 0$.

Additionally, every $f \in I^h$ vanishes on 0. Furthermore, $Y \neq \emptyset$ so $1 \notin I$. Hence, every nonzero element of I is the sum of homogeneous elements of I, all of which vanish on 0, so every element of I vanishes on 0. Thus, we conclude that $V(I) = \theta^{-1}[Z(I)] \cup 0$. Then $J(C(Y)) = I$ by the Nullstellensatz.

(b) $C(Y)$ is irreducible iff $I(C(Y)) = I(Y)$ irreducible iff Y is irreducible.

(c) $\dim C(Y) = \dim k[x_0, \ldots, x_n]/I(C(Y)) = \dim k[x_0, \ldots, x_n]/I(Y) = \dim Y + 1$.

□

I.2.11

Linear Varieties in $\mathbb{P}^n$. A hypersurface defined by a linear polynomial is called a *hyperplane*.

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbb{P}^n$:

(i) $I(Y)$ can be generated by linear polynomials.

(ii) Y can be written as an intersection of hyperplanes.

In this case we say that Y is a linear variety in $\mathbb{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbb{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials.

(c) Let Y, Z be linear varieties in $\mathbb{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r+s-n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r+s-n$. (Think of $\mathbb{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

Proof. (a) $(i \implies ii)$. Let $I(Y) = (f_1, \ldots, f_r)$ linear. Then $Y = Z(I(Y)) = \bigcap Z(f_i)$.

$(ii \implies i)$. Let $Y = \bigcap_{i \in I} Z(f_i)$. We therefore have $I(Y) = \sqrt{(f_i : i \in I)}$. As S is Nötherian we can find a finite set of generators among the f_i, i.e. $(f_1, \ldots, f_r) = (f_i : i \in I)$. Indeed, find the maximal such finitely generated sub-ideal. Thus, $Y = Z(f_1, \ldots, f_r)$. Write $f_i = \sum f_{ij} x_j$. Then $[a_0 : \cdots : a_n] \in Y$ iff $(f_{ij}) \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = 0$. The kernel of the matrix (f_{ij}) will have some basis $v_1, \ldots, v_s$ so take a matrix A which sends $e_i \mapsto v_i$ and the rest to whatever basis you extend it to. Then $(f_{ij})AP = 0$ precisely when $P = (*, \ldots, *, 0, \ldots, 0)$ with s many $*'s$. Thus A represents a linear change of coordinates on $\mathbb{P}^n$ which sends $Z(f_1, \ldots, f_r)$ to $Z(y_{s+1}, \ldots, y_n)$, whose ideal is of course just $(y_{s+1}, \ldots, y_n)$. This leads us to the general fact that a proper ideal generated by finitely many linear polynomials is necessarily prime. If you don't like this, take the transformation on the polynomials themselves.

(b) Suppose that $I(Y) = (f_1, \ldots, f_s)$ linear polynomials. Then by Krull's principal ideal theorem, $\operatorname{codim} I(Y) \leq s$. As $I(Y)$ is prime, (see (a) or the assumption that Y is a variety) we compute $\dim S(Y) = (n+1) - \operatorname{codim} I(Y) \geq n+1-s$. Of course, by I.2.6, $\dim S(Y) = \dim Y + 1 = r+1$. Hence, we must have $r+1 \geq n+1-s$ and therefore that $s \geq n-r$.

(c) Recall from I.2.10 the map $\theta : \mathbb{A}^{n+1} - 0 \longrightarrow \mathbb{P}^n$ sending $(a_0, \ldots, a_n) \mapsto [a_0 : \cdots : a_n]$, and the cone $C(Y) = \theta^{-1}[Y] \cup \{0\}$ for a closed subset $Y \subseteq \mathbb{P}^n$. Then as $\dim Y = r$ and $\dim Z = s$ we have $\dim C(Y) = r+1$ and $\dim C(Z) = s+1$. The condition that $r+s-n \geq 0$ is equivalent to $(r+1)+(s+1)-(n+1) \geq 1$, i.e. that $\dim C(Y) + \dim C(Z) - \dim A^{n+1} \geq 1$. The idea then is to use the fact that a line in A^{n+1} is precisely a point in $\mathbb{P}^n$.

Recall also that $C(Z(I)) = V(I) \subseteq \mathbb{A}^{n+1}$. As Y, Z are linear they are defined by the zero set of finitely many homogeneous linear polynomials. Hence, their affine cones are the affine zero sets of those same polynomials, and are therefore subspaces of the vector space $\mathbb{A}^{n+1} = k^{n+1}$. Hence, from the second isomorphism theorem, we observe that $\dim C(Y) + \dim C(Z) - \dim A^{n+1} \leq \dim(C(Y) \cap C(Z))$.

To sum the above up, $r+s-n \geq 0$ iff $\dim(C(Y) \cap C(Z)) \geq 1$. In other words, that there is some nonzero $v \in \dim(C(Y) \cap C(Z))$ Then the line $\theta(v) \in Y \cap Z$ witnesses the fact that this is nonempty. Long story short, $Y \cap Z \neq \emptyset$ iff their cones share a common line.

$\square$

I.2.12

The $d-Uple$ Embedding. For given $n, d > 0$, let $M_0, M_1, \dots, M_N$ be all the monomials of degree d in the $n+1$ variables $x_0, \dots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ by sending the point $P = (a_0, \dots, a_n)$ to the point $\rho_d(P) = (M_0(a), \dots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

(a) Let $\theta : k[y_0, \dots, y_N] \longrightarrow k[x_0, \dots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. I.2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

Proof. (a) Proving the homogeneity of $\mathfrak{a}$ comes from θ being defined as evaluation by homogeneous polynomials. Indeed, let $f \in \mathfrak{a}$, so that $\theta(f) = f(M_1, \dots, M_N) = 0$. Let $f = \sum f_e$ be the homogeneous decomposition of f. Then $\sum f_e(M_1, \dots, M_N) = 0$. Furthermore, as each M_i is homogeneous of degree d, the degree of $f_e(M_1, \dots, M_N)$ is de. As $d \geq 1$, each $f(M_1, \dots, M_N)$ has a distinct degree. Thus, by linear independence, each $f_e(M_1, \dots, M_N) = 0$. Hence, they are all in $\mathfrak{a}$ so $\mathfrak{a}$ is homogeneous. Primality is immediate from the fact that $k[x_0, \dots, x_n]$ is a domain.

(b) For the easier inclusion, take some $\rho_d([a_0 : \cdots : a_n]) = [M_0(a) : \cdots : M_N(a)]$ and let $f \in Z(\mathfrak{a})^h$. Then $f(M_0(a), \dots, M_N(a)) = \theta(f)(a) = 0$ so $\operatorname{im} \rho_d \subseteq Z(\mathfrak{a})$.

We will define an inverse map $\psi : Z(\mathfrak{a}) \longrightarrow \mathbb{P}^n$. For a multi-index I we will define $I_j = (i_0, \dots, i_j{+}1, \dots, i_n)$. If we take $\sum I = d{-}1$ then we will define a map $\psi_I : Z(\mathfrak{a}) \longrightarrow \mathbb{P}^n$ via $[\{a_J\}] \mapsto [a_{I_0} : \cdots : a_{I_n}]$. Essentially, we are extracting $n+1$ coordinates from $[\{a_J\}]$, starting from I_0 and proceeding in lexicographic order. Essentially, we seek to show that this, with all the relations of $\mathfrak{a}$, are all we need to recover $[\{a_I\}]$.

First, we will show that ψ_I is independent of the choice of I. Indeed, take another multi-index K with $\sum K = d - 1$. Then to compare $\psi_I([\{a_J\}])$ and $\psi_K([\{a_J\}])$ we need to compare the ratios $\frac{a_{I_j}}{a_{K_j}}$. They define the same point in projective space iff all these ratios are the same (ignoring division by 0). Observe that $I_j + K_0 = I_0 + K_j$, so $\mathfrak{a}$ contains the polynomial $y_{I_j} y_{K_0} - y_{I_0} y_{K_j}$. As we are defining our ψ on $Z(\mathfrak{a})$, we therefore have $a_{I_j} a_{K_0} = a_{I_0} a_{K_j}$, i.e. that $\frac{a_{I_j}}{a_{K_j}} = \frac{a_{I_0}}{a_{K_0}}$ for all J. Hence, $\psi_I = \psi_K$ so we can simply call it ψ.

Start with some $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$. Take some $a_i \neq 0$ and take $I = (0, \dots, 0, d-1, 0, \dots, 0)$ in the i^{th} position. $\rho_d(a) = [\{a^J\}]$, and applying $\psi = \psi_I$ to this yields

$[a^{I_0} : \cdots : a^{I_n}]$. $a^{I_j} = a^{(0,\ldots,0,d-1,0,\ldots,0)+(0,\ldots,0,1,0,\ldots,0)} = a_i^{d-1}a_j$. Hence, $\psi(\rho_d(a)) = [a_i^d : a_i^{d-1}a_1 : \cdots : a_i^{d-1}a_n] = a$.

On the other hand, take some $[\{a_J\}] \in Z(\mathfrak{a})$. There is some index I_0 for which $a_{I_0} \neq 0$ (ok fine maaayybbeee all of the $(0,*,\ldots,*)$ terms vanish but I'm sure this won't work if I bash some relations so whatever). Applying $\psi = \psi_I$ yields $[a_{I_0} : \cdots : a_{I_n}]$. If we apply ρ_d to this we end up with $\left[\prod_i a_{I_i}^{j_i}\right]$ ranging over all $J = (j_0, \ldots, j_n)$. We want this to equal our original $[\{a_J\}]$,so again we consider the ratios $\left(a_{I_i}^{j_i}/a_J\right)$ vs $\left(a_{I_i}^{k_i}/a_K\right)$. In other words, we seek to show that $a_K \prod a_{I_i}^{j_i} = a_J \prod a_{I_i}^{k_i}$. Indeed, this corresponds to the polynomial $y_K \prod y_{I_i}^{j_i} - y_J \prod y_{I_i}^{k_i}$. Note that on the left hand side, the "signature" is $K + \sum j_i I_i$ and on the right hand side it is $J + \sum k_i I_i$. We'll let $\hat{i} = (0,\ldots,0,1,0,\ldots,0)$. Then $J + \sum k_i I_i = J + \sum k_i(I + \hat{i})$. This is $J + \sum k_i I + \sum k_i \hat{i} = J + K + dI$. The same computation works for the other term, so the signatures agree and therefore $y_K \prod y_{I_i}^{j_i} - y_J \prod y_{I_i}^{k_i} \in \mathfrak{a}$. Thus, $[\{a_J\}]$ vanishes on this so we get the desired equality $a_K \prod a_{I_i}^{j_i} = a_J \prod a_{I_i}^{k_i}$ so $\rho_d(\psi([\{a_J\}])) = [\{a_J\}]$.

We have therefore proven that these two maps are inverse. In fact, we could have done this for just $Z(\{\prod y_{K_i} - \prod y_{J_i} : \sum K_i = \sum J_i\})$, so these generate $\mathfrak{a}$. If we try hard enough we may even show that we only need the degree two examples, but whatever.

(c) We'd expect that $\rho_d^{-1}[Z(I)] = Z(\theta[I])$ and that $\rho_d[Z(J)] = Z(\theta^{-1}[J^d])$, where by J^d I mean to take your favorite homogeneous generators $J = (S)$ and let S^d be raising everthing in S to the d^{th} power so that it is in the image of θ. It seems pretty believable that ψ and ρ_d are continuous though, as they're all just evaluation at a bunch of homogeneous polynomials of the same degree so they better be continuous. In fact, see I.3.4 to see that they are morphisms of varieties, hence continuous. No need to worry about circularity, we never used continuity in the proof of I.3.4, only that these were inverse.

(d) We did this is I.2.9 already.

□

I.2.13

Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbb{P}^5$ such that $V \cap Y = Z$.

Proof. Take indeed some closed curve $Z \subseteq Y$, i.e. a variety of dimension 1. By I.2.12 we have a homeomorphism $\rho_2 : \mathbb{P}^2 \longrightarrow Y = Z(\mathfrak{a})$ where $\mathfrak{a}$ is defined as in I.2.12. Let $\Gamma = \rho_2^{-1}[Z]$. This is a codimension 1 variety so it equals $Z(f)$ for some $f \in k[x_0, x_1, x_2]$ irreducible and homogeneous (I.2.8). To find a hypersurface $V \subseteq \mathbb{P}^5$ we therefore want some $g \in k[y_0, \ldots, y_5]$ homogeneous and irreducible such that $V = Z(g)$. The intersection $Y \cap V = Z(\mathfrak{a}) \cap Z(g) = Z(\mathfrak{a} + (g))$. We therefore want the image of ρ_2 to be $Z(\mathfrak{a} + (g))$. On the coordinate rings, this would mean we have $\theta : k[y_0, \ldots, y_5]/\sqrt{(\mathfrak{a} + (g))} \longrightarrow k[x_0, x_1, x_2]/(f)$ an isomorphism. Then we essentially need $\theta[\sqrt{(\mathfrak{a} + (g))}] = (f)$. As $\mathfrak{a}$ is the kernel we're looking for $\sqrt{(\theta(g))} = (f)$

Let's first consider the image of θ in this case. Now, $\theta(f) = f(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$ so certainly anything in the image has all homogeneous summands of even degree. On the other hand, consider an even degree monomial $x_0^i x_1^j x_2^k$, i.e. $i + j + k \equiv 0 \pmod 2$ If these are all 0 then we are done via $y_{200}^{i/2} y_{020}^{j/2} y_{002}^{k/2}$. Alternatively, we could have one of these 0 and the other two are 1. For instance, if $i, j \equiv 1 \pmod 2$ then we could take $y_{200}^{\frac{i-1}{2}} y_{020}^{\frac{j-1}{2}} y_{110} y_{002}^{k/2}$. Therefore we have that $\operatorname{im}\theta$ is the set of polynomials whose homogeneous components are all even degree. And in fact, for a homogeneous even degree guy, we can find a preimage which is also homogeneous. Thus, we can solve $\sqrt{(\theta(g))} = (f)$ by taking $\theta(g) = f^2$ and g homogeneous.

We now need to find an irreducible g satisfying the above. Suppose we were lucky and found a homogeneous g such that $\theta(g) = f$. If $g = ab$ then $f = \theta(a)\theta(b)$, so WLOG take $\theta(a) \in k^\times$, as f is irreducible. Then as θ sends each y to a homogeneous degree monomial, only constants can map to constants. Hence, $a \in k^\times$ so g is irreducible. On the other hand, if we only have $\theta(g) = f^2$ then again write $g = ab$ whence $f^2 = \theta(a)\theta(b)$. If g was already irreducible then we'd of course be done, so suppose not. Then $\theta(a) = \theta(b) = f$ as $k[x_0, x_1, x_2]$ is a UFD. As $ab = g$ homogeneous, a, b are homogeneous. Then by the previous case, they are irreducible.

In any case, we have shown that we can find an irreducible homogeneous $g \in k[\{y_I\}]$ such that $\theta(g) = f$ or f^2. As discussed, we therefore take $V = Z(g)$ as our candidate hypersurface whose "shadow" onto the d-uple embedded $\mathbb{P}^2$ is our fixed curve Z. That is, we want to show that $V \cap Y = Z(\mathfrak{a} + (g)) = Z = \rho_2[Z(f)]$. Of course, we already know $Z \subseteq Y$. Take some $P \in Z$, which we know is of the form $\rho_2(Q)$ for $Q \in Z(f)$. Then $g(P) = g(\rho_2(Q)) = \theta(g)(Q) = f(Q) = 0$ (or $f^2(Q)$ but kjwe;lkjsDLf). Hence, we also have $Z \subseteq Z(g)$ so $Z \subseteq Y \cap V$. On the other hand, take some $P \in Y \cap V = Z(\mathfrak{a} + (g))$. Then $P \in Y$ so write $P = \rho_2(Q)$. We want to show that $Q \in Z(f)$. Indeed, let $g(P) = g(\rho_2(Q)) = \theta(g)(Q) = 0$. Of course, $\theta(g) = f^{1.5\pm0.5}$ so $f(Q) = 0$ and $P \in \rho_2[Z(f)] = Z$ as desired. Thus, we have $V \cap Y \subseteq Z \subseteq V \cap Y$. □

I.2.14

The Segre Embedding. Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_ib_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a subvariety of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \mid i = 0, \ldots, r, j = 0, \ldots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[z_{ij}] \longrightarrow k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends z_{ij} to x_iy_j. Then show that $\operatorname{im}\psi = Z(\mathfrak{a})$.]

Proof. As suggested, we will show that the image of ψ is $Z(\mathfrak{a})$ as defined in the body of the question. Indeed, let $(P, Q) \in \mathbb{P}^r \times \mathbb{P}^s$ and $f \in \mathfrak{a}^h$. Then $f(\psi(P, Q)) = \theta(f)(a_0, \ldots, b_s) = 0$.

On the other hand take some $p \in Z(\mathfrak{a})$. Write $P = [c_{ij}] \in \mathbb{P}^N$. Then some $c_{kl} \neq 0$, so we work in the affine patch on which is it 1. We'll start defining our preimage to this via $a_k = b_l = 1$. We're trying to make it so that $c_{ij} = a_ib_j$ for all i, j. In particular, this would mean that $a_i = a_ib_l = c_{il}$ and $b_j = a_kb_j = c_{kj}$. So indeed, we take these as our definitions and claim that $([a_0 : \cdots : a_r], [b_0 : \cdots : b_s]) \mapsto P$. Indeed, that means we want to show that $c_{ij} = a_ib_j$. Consider the relations $(x_iy_j)(x_my_n) = (x_iy_n)(x_my_j)$. Hence, the polynomials

$z_{ij}z_{mn} - z_{in}z_{mj} \in \mathfrak{a}^h$. In particular, we have that $c_{ij}c_{kl} = c_{kj}c_{il}$. Using the definitions we already have, this means exactly that $c_{ij} = a_ib_j$. Thus, $P \in \operatorname{im}\psi$. □

I.2.15

The Quadric Surface in $\mathbb{P}^3$ See 4. Consider the surface Q (a surface is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}$, $\{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all t, u, $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

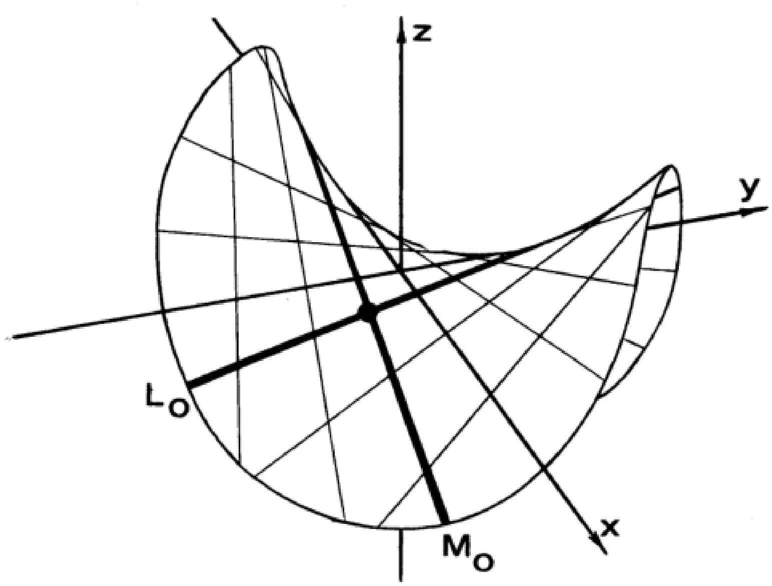

Figure 4: The quadric surface in $\mathbb{P}^3$.

Proof. (a) This is defined by $\psi([a_0 : a_1], [b_0 : b_1]) = [a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1]$. We had the map $k[z_{00}, z_{01}, z_{10}, z_{11}] \longrightarrow k[x_0, x_1, y_0, y_1]$ via $z_{ij} \mapsto x_iy_j$. By I.2.14 above, $\operatorname{im}\psi = Z(\ker\theta)$. Thus, we seek to show $\ker(\theta) = (z_{00}z_{11} - z_{01}z_{10})$.

We will approach this with dimension theory. Note $z_{00}z_{11} - z_{01}z_{10}$ is irreducible and is in $\ker(\theta)$. Hence, $(z_{00}z_{11} - z_{01}z_{10}) \subseteq \ker(\theta)$ is an inclusion of primes, so we can prove equality by proving equality of their codimensions. By Krull's principal ideal theorem, the codimension of $(z_{00}z_{11} - z_{01}z_{10})$ is 1. Furthermore, we have $\dim \operatorname{im} \theta = 4 - \operatorname{codim} \ker(\theta)$.

We are therefore left to compute $\dim \operatorname{im} \theta$. Indeed, $\operatorname{im} \theta = k[x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. To show that this has dimension 3, we claim that the first three generators are independent and that the last is algebraic over the first three (really we could choose any 3 of 4). Indeed, the easy part is that x_1y_1 satisfies $(x_0y_0)t - (x_0y_1)(x_1y_0)$. Hence, $k(x_0y_0, x_0y_1, x_1y_0, x_1y_1)/k(x_0y_0, x_0y_1, x_1y_0)$ is algebraic so their transcendence degrees agree.

Now we want algebraic independence of these first 3. Take indeed some $\sum a_{ijk}(x_0y_0)^i(x_0y_1)^j(x_1y_0)^k = 0$. Then $\sum a_{ijk}x_0^{i+j}x_1^k y_0^{i+k} y_1^j = 0$. Observe that the map $(i, jk) \mapsto (i + j, k, i + k, j)$ is injective. Thus, each coefficient a_{ijk} is attached to a monomial of a unique "signature" $(i+j, k, i+k, j)$. x_0, x_1, y_0, y_1 are algebraically independent so these distinct signature monomials are linearly independent. Thus, each $a_{ijk} = 0$ and we have our algebraic independence.

In conclusion, $\dim k[x_0y_0, x_0y_1, x_1y_0, x_1y_1] = 3$. Thus, $\ker(\theta)$ and $(z_{00}z_{11} - z_{10}z_{01})$ have the same codimension. Thus, they are equal and we conclude $\operatorname{im} \psi = Z(\ker \theta) = Z(z_{00}z_{11} - z_{01}z_{10})$.

(b) We have our map $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow Q$. Fix some $t = [a : b] \in \mathbb{P}^1$. We'll define $L_t = \operatorname{im}(\psi(t, -))$ and $M_t = \operatorname{im}(\psi(-, t))$. This essentially transfers the coordinate grid on the plane onto quadric surface. As all these set theoretic properties hold for the coordinate grid in $\mathbb{P}^2$, the bijection ψ proves that they hold in Q. We therefore need only show that these are in fact lines.

Let $t = [a_0 : a_1]$. Then $L_t = \{[a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1] : [b_0 : b_1] \in \mathbb{P}^1\}$. We can observe that these satisfy the equations $a_1z_{00} - a_0z_{10}$ and $a_0z_{11} - a_1z_{01}$. Thus we easily have $L_t \subseteq Z(a_1z_{00} - a_0z_{10}) \cap Z(a_0z_{11} - a_1z_{01})$. On the other hand, suppose $[c_{00} : c_{01} : c_{10} : c_{11}]$ is in this line. Then if $a_0 \neq 0$ take $b_0 = c_{00}$ and $b_1 = c_{01}$. If $a_0 = 0$ take $b_0 = c_{10}$ and $b_1 = c_{11}$. One can check that the image of this in the Segre embedding along t lies in L_t. We can do something analogous for M_t.

(c) Q contains the image of the 2-uple embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$, which has image $\{[a_0^2 : a_0a_1 : a_1a_0 : a_1^2]\}$. See I.2.12 for why this is a curve. Note that it is also the image of the diagonal $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ under ψ. However, the diagonal is not closed as $\mathbb{P}^1$ is not Hausdorff. In fact, it is irreducible! This immediately shows that ψ is not a homeomorphism, but for the sake of completeness, we will show that $\mathbb{P}^1 \times \mathbb{P}^1$ contains precisely the curves already described.

First of all, let $L'_t = \{t\} \times \mathbb{P}^1$ and $M'_t = \mathbb{P}^1 \times \{t\}$, so that ψ takes $L'_t \mapsto L_t$ and $M'_t \mapsto M_t$. We claim that these are the only curves in $\mathbb{P}^1 \times \mathbb{P}^1$. A curve is an irreducible closed subset of dimension 1, which is a purely topological notion. To do this, we will first explore irreducible closed subsets of a product of spaces.

Lemma I.1. *Let X_0, X_1 be topological spaces. Then the irreducible closed subsets of $X_0 \times X_1$ are all of the form $A_0 \times A_1$ where A_iX_i are irreducible closed subsets of their respective spaces.*

Proof. First, let's show that these proposed subsets are actually closed and irreducible. Closedness is immediate, so we focus on irreducibility. Take $\emptyset \neq U_0, U_1 \subseteq A_0 \times A_1$ open. Then we can find $U_{00} \times U_{01} \subseteq U_0$ and $U_{10} \times U_{11} \subseteq U_i$ open and nonempty. By irreducibilty of the A_i, $U_{i0} \cap U_{i1} \neq \emptyset$. Hence, $U_0 \cap U_1 \neq \emptyset$.

Now, take $F \subseteq X_0 \times X_1$ irreducible and closed. Then we have $F \subseteq \pi_0[F] \times \pi_1[F]$, but these certainly need not be closed, so instead take $F \subseteq \overline{\pi_0[F]} \times \overline{\pi_1[F]} = Z$. Then F is an irreducible closed subset of Z. As it is closed, $Z - F$ is open and disjoint from F. Assume that this is a proper containent and take therefore a basic open set $\emptyset \neq U_0 \times U_1 \subseteq Z - F$. Now, as $U_0 \times U_1 \subseteq Z - F$, $Z - (U_0 \times U_1) \supseteq F$. Furthermore, $Z - (U_0 \times U_1) \subseteq (X_0 - U_0) \times \overline{\pi_1[F]} \cup \overline{\pi_0[F]} \times (X_1 - U_1)$, which is closed. Then as F is irreducible, it is contained in one of these, say $F \subseteq (X_0 - U_0) \times \overline{\pi_1[F]}$. In particular, $\pi_0[F] \subseteq X_0 - U_0$, so $\overline{\pi_0[F]} \subseteq X_0 - U_0$. But we took $U_0 \subseteq \overline{\pi_0[F]}$, a contradiction. Hence, $F = \overline{\pi_0[F]} \times \overline{\pi_1[F]}$. Why are these irreducible? Uhhhhhhhh cuz □

Note that the irreducible closed subsets of $\mathbb{P}^1$ are $\emptyset$, points, and the whole of $\mathbb{P}^1$. This therefore shows that the only curves in $\mathbb{P}^1$ are L'_t and M'_t as describe.

□

I.2.16

(a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

Proof. (a) We have $Q_1 \cap Q_2 = Z(x^2 - yw, xy - zw)$. Recall that $Z(I(Y_1 \cup Y_2)) = Z(I(Y_1) \cap I(Y_2)) = Y_1 \cup Y_2$, when these are closed. This suggests that to realize something as the union of two closed subsets, we want to find a primary decomposition.

Indeed, observe that $(x^2 - yw, xy - zw) \subseteq (x, w) \cap (xz - y^2, yw - x^2, zw - xy)$. The former defines a line and the latter defines a twisted cubic (just believe, the proofs of I.2.9 and I.2.12 are insufficient but it shouldn't be tooooo hard to show that these degree two guys generate, in fact I think (?) this holds in general). We therefore seek to show equality. Take let's consider elements of $(x, w) \cap (xz - y^2, yw - x^2, zw - xy)$. First of all, since $f \in (x, w)$ iff each of its summands are (this can be seen directly or through the theory of monomial ideals ala [CLO07]. In any case, take some $f(xz - y^2) + g(yw - x^2) + h(zw - xy) \in (x, w) \cap (xz - y^2, yw - x^2, zw - xy)$ The last two terms are clearly in (x, w), but we additionally need $f(xz - y^2) \in (x, w)$. Indeed, this will occur iff $f \in (x, w)$. Observe that in the quotient by $(x^2 - yw, xy - zw)$, we have $xzw - wy^2 = x^2y - wy^2 = ywy - wy^2 = 0$ and $x^2z - xy^2 = ywz - xy^2 = ywz - zwy = 0$. Thus, whenever $f \in (x, w)$ the associated summand $f(xz - y^2) \in (x^2 - yw, xy - zw)$. Thus, we indeed that $(x, w) \cap (xz - y^2, yw - x^2, zw - xy) = (x^2 - yw, xy - zw)$. Hence,

$Q_1 \cap Q_2 = Z((x,w) \cap (xz - y^2, yw - x^2, zw - xy)) = Z(x,w) \cup Z((xz - y^2, yw - x^2, zw - xy))$, which is a union of a line and a twisted cubic.

(b) First, observe that $C \cap L = Z(x^2 - yz, y) = Z(x^2, y) = Z(x, y) = \{[0:0:1]\}$. On the other hand, $I(C) + I(L) = (x^2 - yz, y) = (x^2, y) \neq (x, y) = I(P)$. Note that the equations given for C an L are irreducible so there is no fuss.

□

I.2.17

Complete intersections. A variety Y of dimension r in $\mathbb{P}^n$ is a *(strict) complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

(a) Let Y be a variety in $\mathbb{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geq n - q$.

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbb{P}^3$ (Ex. I.2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

(d) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III, §5] for commentary.

⟨*My note*: These references are not to this textbook, but to the references in the textbook itself. For laziness, I do not include these in the bibliography.⟩

Proof. (a) We want to apply Krull's principal ideal theorem here, but there's the issue that that is meant to hold for minimal primes, whereas here we are looking for homogeneous minimal primes over our homogeneous $\mathfrak{a}$. So indeed, we first show the following lemma:

Lemma I.2. *Let S be an ω graded ring. Then a minimal prime $\mathfrak{p}$ of S is homogeneous.*

Proof. To do this, we want to show that any prime $\mathfrak{p}$ contains another prime $\mathfrak{q}$ that is homogeneous. The natural choice is to take $\mathfrak{q} = (\mathfrak{p}^h)$, the ideal generated by the homogeneous elements of $\mathfrak{p}$. This certainly tells us that $\mathfrak{q} \subseteq \mathfrak{p}$ and that $\mathfrak{q}$ is homogeneous. Furthermore, $\mathfrak{p}^h \subseteq \mathfrak{q}^h \subseteq \mathfrak{p}^h$ so we have $\mathfrak{q}^h = \mathfrak{p}^h$. Then to show that $\mathfrak{q}$ is prime, we need only check this on homogeneous elements. Indeed, take $a, b \in S^h$ such that $ab \in \mathfrak{q}^h = \mathfrak{p}^h$. Then by primality of $\mathfrak{p}$, $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. As these are homogeneous, one of these is therefore in $\mathfrak{p}^h = \mathfrak{q}^h \subseteq \mathfrak{q}$. Thus, $\mathfrak{q}$ is prime.

Now if $\mathfrak{p}$ was minimal then $\mathfrak{q} \subseteq \mathfrak{p}$ is an equality, and $\mathfrak{p} = \mathfrak{q}$ is homogeneous. □

Note that this proof also works for minimal primes over homogeneous ideals I, as we can apply the lemma to S/I, which is itself graded. Hence, any minimal prime over out given homogeneous ideal $\mathfrak{a}$ must itself be homogeneous.

Now, by Krull's principal ideal theorem, an minimal prime $\mathfrak{p} \supseteq \mathfrak{a}$ must have $\operatorname{codim} \mathfrak{p} \leq q$. As discussed, $\mathfrak{p}$ is homogeneous so we form the inclusion $Z(\mathfrak{p}) \subseteq Z(\mathfrak{a}) = Y$ of a maximal irreuducible susbet of Y. Then $\dim Y \geq \dim Z(\mathfrak{p})$. $\dim Z(\mathfrak{p}) = \dim S(\mathfrak{p}) - 1 = (n+1) - \operatorname{codim} \mathfrak{p} - 1 = n - \operatorname{codim} \mathfrak{p} \geq n - q$.

(b) Let $I(Y) = (a_1, \ldots, a_s)$ where $s = n - r$. Recall that Y was assumed to be a variety so $I(Y)$ must be prime. Then for each i there is some irreducible $f_i \mid a_i$ such that $f_i \in I(Y)$. Then $(f_1, \ldots, f_s) \subseteq I(Y)$. On the other hand, $f_i \mid a_i$ so $a_i \in (f_i)$. Thus, $I(Y) = (a_1, \ldots, a_s) \subseteq (f_1, \ldots, f_s) \subseteq I(Y)$ so we achieve equality. Hence, we have $Y = \bigcap Z(f_i)$, which are all hypersurfaces.

Of course, the problem insists that it is the intersection of $n - r$ hypersurfaces. The index of the intersection ranges from 1 to $s = n - r$, but a priori there could be repeats in this intersections. However, the fact that $\dim Y = r$ precludes this. Indeed, suppose there were some repeats in the intersection, i.e. that some $Z(f_i) = Z(f_j)$. That would mean that we could generate $I(Y)$ by fewer than s elements. But if $I(Y)$ is generated by l many elements, part (a) above tells us that $r = \dim Y \geq n - l$. Thus, $l \geq n - r = s$, so s is the minimal number of generators this ideal can have. Hence, it is the minimal number of hypersurfaces we need to intersect in Y.

(c) Take $Y = Z(xz - yz, yw - z^2, xw - yz)$. To write $Y = H_1 \cap H_2$ for hypersurfaces $H_i = Z(f_i)$ means that $\sqrt{(f_1, f_2)} = (xz - yz, yw - z^2, xw - yz)$. Wikipedia claims that we can take $f_1 = xz - y^2$ and $f_2 = z(yw - z^2) - w(xw - yz)$. One can then bash out this computation.

On the other hand, we want to show that $I(Y)$ cannot be generated by 2 elements. The naïve dimension approach using (a) won't work, as generating $I(Y)$ by two elements yields the inequality $\dim Y \geq 3 - 2$ which is correct. Hence, we'll approach this with Nakayama's lemma, noting that the given generators $a_1 = xz - yz$, $a_2 = yw - z^2$, $a_3 = xw - yz$ are linearly independent over k.

Let's let $\mathfrak{p} = I(Y)$. Now, we want a maximal ideal $\mathfrak{p} \subseteq \mathfrak{m}$ so that we can get a corresponding prime $\mathfrak{p}S_\mathfrak{m}$ in the local ring $S_\mathfrak{m}$ on which we can apply Nakayama's lemma. Specifically, we note that generators of $\mathfrak{p}S_\mathfrak{m}$ correspond to generators of $\mathfrak{p}S_\mathfrak{m}/\mathfrak{p}\mathfrak{m}S_\mathfrak{m}$ as a $S_\mathfrak{m}/\mathfrak{m}S_\mathfrak{m} = k$ vector space, and in fact that minimal generating sets correspond to bases. We therefore want to compute the dimension of this latter space. Note that by flatness of localization, and the fact that $k = S/\mathfrak{m}$ is a field, $\mathfrak{p}S_\mathfrak{m}/\mathfrak{p}\mathfrak{m}S_\mathfrak{m} = \mathfrak{p}/\mathfrak{p}\mathfrak{m}$.

We therefore want to prove that $\dim_k \mathfrak{p}/\mathfrak{p}\mathfrak{m} \geq 3$. Indeed, we will show that the $a_i \in \mathfrak{p}$ are linearly independent in the quotient. Of course, we have no actually picked a maximal ideal $\mathfrak{m}$ so we will take $\mathfrak{m} = (x, y, z, w)$. Note that $0 \in Y$ so $\mathfrak{p} \subseteq \mathfrak{m}$. Now, let's try to understand the quotient. Take some $\sum f_i a_i \in \mathfrak{p}$, where $f_i = \sum f_{iJ} X^J$ for $X = (x, y, z, w)$. We see that each term $X^J a_i \in \mathfrak{p}\mathfrak{m}$ for $J \neq (0,0,0,0)$. Then in the quotient $\mathfrak{p}/\mathfrak{p}\mathfrak{m}$, we have $\sum f_i a_i = \sum f_i(0) a_i$, so it is spanned as a k vector space by the a_i. In other words, we have a well defined isomorphism $\mathfrak{p}/m\mathfrak{p} \longrightarrow \sum k a_i$ via $\sum f_i a_i \mapsto \sum f_i(0) a_i$ due to the isomorphism $S/\mathfrak{m} \longrightarrow k$ via $f \mapsto f(0)$. Furthermore, the a_i are k-linearly independent. Thus, $\dim_k \mathfrak{p}/\mathfrak{p}\mathfrak{m} = 3$. If there was any smaller generating set for $\mathfrak{p}$ then we could run through the same procedure and find a strictly smaller basis for $\mathfrak{p}/\mathfrak{m}\mathfrak{p}$ - a contradiction.

$\square$

I.3 Morphisms

I.3.1

(a) Show that any conic in $\mathbb{A}^2$ is either isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 - 0$ (c.f **??**).

(b) Show that $\mathbb{A}^1$ is *not* isomorphic to any proper open subset of itself. (This result is generalized by **??** below).

(c) Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

(d) We will see later (I.4.8) that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.

(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

Proof. (a) A conic is the zero set of an irreducible polynomial f of degree 2. By the results of this section, it will be fruitful to consider the coordinate rings. Indeed, by **??**.c, $A(V(f)) \cong k[x,y]/(y-x^2)$ or $k[x,y]/(xy-1)$. The former is isomorphic to $k[t]$ via $x \mapsto t$, $y \mapsto t^2$, corresponding to the map $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$ via $t \mapsto (t, t^2)$. The algebra isomorphism yields an isomorphism of varieties.

In the latter case the obvious maps are $V(xy-1) \longrightarrow \mathbb{A}^1 - 0$ via $(x,y) \mapsto x$ and $\mathbb{A}^1 - 0 \longrightarrow V(xy-1)$ sending $t \mapsto (t, 1/t)$. The former is induced by the map $k[t] \longrightarrow k[x,y]/(xy-1)$ sending $t \mapsto x$, and it of course has image equal to exactly $\mathbb{A}^1 - 0$. For the latter, we consider the map $k[x,y]/(xy-1) \longrightarrow \mathcal{O}(\mathbb{A}^1 - 0)$ sending xt and $y \mapsto 1/t$. As [Hartshorne, I.3.5] was proven in this generality, this induces the map $\mathbb{A}^1 - 0 \longrightarrow V(xy-1)$ we just described. It is easy to check that these are inverse to each other, and we have additionally shown that they are both maps of varieties via their rings of regular functions.

(b) Take $U < \mathbb{A}^1$. Of course, if $\emptyset = U$ then we are done so suppose otherwise. Then $Y = \mathbb{A}^1 - U$ satisfies $0 < Y < \mathbb{A}^1$. We can therefore write $Y = V(f)$ for some nonzero, nonconstant polynomial $f \in k[x]$. We can also assume WLOG that f is squarefree. If indeed $\mathbb{A}^1 \cong U$, then $k[x] = \mathcal{O}(\mathbb{A}^1) \cong \mathcal{O}(U)$ as k-algebras. Consider then our hypothetical isomorphism $k[x] \cong \mathcal{O}(U)^\times$. As this is an isomorphism of k-algebras, we must have had the following commutative diagram.

$$\begin{tikzcd} k[x] \arrow[rr, "\sim"] & & \mathcal{O}(U) \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

This descends to the unit groups as follows.

$$\begin{tikzcd} k^\times \arrow[rr, "\sim"] & & \mathcal{O}(U)^\times \\ & k^\times \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Which would imply that $\mathcal{O}(U)^\times = k^\times$. But as $V(f) \cap U = \emptyset$, $f \in \mathcal{O}(U)^\times$. Furthermore, f was assumed to be nonconstant, so this is a contradiction.

(c) We're going to cheat a bit here and piggy back off of the part of **??** I didn't check, namely the change of coordinates. I gave references for the case of even and odd/zero characteristic there. Taking this for granted, the result is there is a change of coordinates $x \mapsto a_1X + a_2Y + a_3$ and $y \mapsto b_1X + b_2Y + b_3$ representing an isomorphism $k[x,y] \longrightarrow k[X,Y]$ such that an irreducible quadratic $f(x,y)$ becomes $XY-1$ or $Y-X^2$. This gives us an automorphism $\mathbb{A}^2 \longrightarrow \mathbb{A}^2$ which sends the conic $V(f)$ to either $V(XY-1)$ or $V(Y-X^2)$. These two homogenize to essentially the same polynomial, so the hope is that we can homogenize this to get our desired automorphism of $\mathbb{P}^2$.

Now take a conic $Z(f) \subseteq \mathbb{P}^2$ be a conic, so that $f(x,y,z)$ is an irreducible homogeneous quadratic. Assume without loss of generality that $U_z \cap Z(f) \neq \emptyset$, where $U_z = \mathbb{P}^2 - Z(z)$ is the usual affine patch. Hence, $z \nmid f$ so some term of f contains no z. This means that α_z doesn't collapse the information of f. More formally, it means that $\alpha_z(f)$ has degree two, as the term without a z is unchanged. Thus, $\beta_z(\alpha_z(f)) = f$. This tells us that we can faithfully attempt to work in U_z. Indeed, consider the above change of coordinates $k[x,y] \longrightarrow k[X,Y]$. Then this sens $\alpha_z(f)(x,y)$ to one of these special conics $XY-1$ or $Y-X^2$. We'll homogenize this coordinate change as follows.

$$\begin{aligned} x &\mapsto a_1X + a_2Y + a_3Z \\ y &\mapsto b_1X + b_2Y + b_3Z \\ z &\mapsto Z. \end{aligned}$$

This yields an isomorphism $k[x,y,z] \longrightarrow k[X,Y,Z]$ and hence a linear automorphism $\mathbb{P}^2 \longrightarrow \mathbb{P}^2$. This sends $Z(f(x,y,z))$ to $Z(f(X,Y,Z))$. Furthermore, we can see plainly by the definition of these coordinates that $\alpha_z(f)(X,Y) = \alpha_Z(f(X,Y,Z))$. Hence, $\alpha_Z(f(X,Y,Z)) = XY-1$ or $Y-X^2$. By the above discussion, we can apply β_Z to recover $f((X,Y,Z)$. Indeed, this tells us that $f(X,Y,Z) = XY - Z^2$ or $YZ - X^2$. These, of course, define the same variety. Hence, all conics in $\mathbb{P}^2$ are isomorphic.

Of course, we still need to show that these conics are isomorphic to $\mathbb{P}^1$ itself. We'll defer part of this proof until I.3.4, which shows that the d-uple embedding is an isomorphism onto its image. By I.2.12 of the last section, we know the ideal of the image is the kernel of the map $k[y_0,y_1,y_2] \longrightarrow k[x_0,x_1]$ sending $y_0 \mapsto x_0^2$, $y_1 \mapsto x_0x_1$, and $y_2 \mapsto x_1^2$. Letting $\mathfrak{a}$ be the kernel, we want to show that the inclusion $(xy - z^2) \subseteq \mathfrak{a}$ is an equality. This should proceed as in I.2.15, i.e. we'll show they have the same codimension. Equivalently, that $\operatorname{codim} \mathfrak{a} = 1$. Hence, we just want to show that the image $k[x_0^2, x_0x_1, x_1^2]$ is two dimensional. Indeed, it's obvious enough that x_0^2 and x_1^2 are algebraically independent. Furthermore, x_0x_1 satisfies $t^2 - x_0^2x_1^2$ and is hence integral over the other two. Thus, the dimension is indeed two, so $\operatorname{codim} \mathfrak{a} = 3 - 2 = 1$ and we have equality. Thus, the image of the 2-uple embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ is an isomorphism onto $Z(xy - z^2)$. Thus, all conics are indeed isomorphic to $\mathbb{P}^1$.

Sidenote: it'd be interesting to prove that all plane conics are isomorphic using methos from projective geometry itself, rather than this incidental change of coordinates.

Perhaps we can consider the Veronese surface, i.e. the image of $\mathbb{P}^2 \longrightarrow \mathbb{P}^5$ under the 2-uple embedding. A conic takes the form $ax^2+bxy+cxz+dy^2+eyz+fy^2$, which corresponds exactly to the point $[a:b:c:d:e:f] \in \mathbb{P}^5$. This idea of "straightening out" the variety to get a linear variety via the d-uple embedding is explored more in I.3.5.

(d) Recall that a curve in X is a one dimensional irreducible closed subset of X. This is a purely topological notion, so properties thereof are invariant under homeomorphism. We see here the topological distinction between the affine and projective plane. In $\mathbb{A}^2$ we can find disjoint curves, but in $\mathbb{P}^2$ we cannot. For $\mathbb{A}^2$ simply consider the lines $V(x)$ and $V(x-1)$, whose intersection is of course $V(x, x-1) = V(1) = \emptyset$. However, as we shall see in I.3.7.a, all curves in $\mathbb{P}^2$ intersect. Thus, these two spaces cannot be homeomorphic.

(e) Let X be an affine variety that is isomorphic to a projective variety Y. Then $\mathcal{O}(X) \cong \mathcal{O}(Y) \cong k$. Of course, know the regular functions on a point: $\mathcal{O}(\{*\}) = k$. Hence, $A(X) \cong A(\{*\})$ so $A \cong \{*\}$.

$\square$

I.3.2

A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that ϕ defines a bijective bicontinuous morphism on $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that ϕ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p > 0$ and define a map $\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ by $t \mapsto t^p$. Show that ϕ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism.*

Proof. (a) This map is induced by the algebra homomorphism $k[x,y] \longrightarrow k[t]$ via $x \mapsto t^2$, $y \mapsto t^3$. The image is certainly contained in $V(y^2 - x^3)$. Any $(a, b) \in V(y^2 - x^3)$ is in the image. Indeed, if $a \neq 0$ take $t = \frac{b}{a}$. Else, take $t = 0$. In fact, this defines an inverse to ϕ, so it is a bijective morphism.

Next we will show that it is a homeomorphism. Indeed, we will do this by showing hat the subspace topology on $V(y^2 - x^2)$ is the cofinite topology. As the topology on $\mathbb{A}^1$ is also the cofinite topology, this bijection will automatically be a homeomorphism. Indeed, any nonempty closed subset of $V(y^2 - x^3)$ is (uniquely) the finite union of irreducible closed subsets as it is a Noetherian space. Hence, we consider some $F \subseteq V(y^2 - x^3)$ closed and irreducible. If $F < V(y^2 - x^3)$ then we want to show that it is finite. Indeed, if strict inclusion holds then $\dim F < \dim V(y^2 - x^3)$. As the latter has dimension 1 – for instance because the integral closure of its coordinate ring is $k[t]$ – $\dim F = 0$. Hence, it is a single point. Thus, any proper closed subset of $V(y^2 - x^3)$

is finite, so its topology is the cofinite one. Hence, the bijection ϕ takes proper closed (finite) subsets precisely to proper closed (finite) subsets and it is a homeomorphism.

Finally, we will show that ϕ is not an isomorphism of varieties. These are affine varieties so it suffices to check their coordinate rings. Indeed, on coordinate rings, the map is $k[x,y]/(y^2 - x^3) \longrightarrow k[t]$ via $x \mapsto t^2$, $y \mapsto t^3$. This has image $k[t, t^2] < k[t]$ so it is not an isomorphism. In fact, we can do better. Not only is ϕ not an isomorphism, there is no isomorphism between these varieties. Again, it suffices to look at their coordinate rings. $k[t]$ is a UFD but $k[x,y]/(y^2 - x^3) \cong k[t^2, t^3]$ is not. Indeed, $y^2 = x^3$ (and $(t^2)^3 = (t^3)^2$) are distinct irreducible factorizations. Also just for fun, the $k[t^2, t^3]$ is not normal by $k[t]$ is.

As an aside, that we do not prove in any detail, the inverse map we defined is a rational function away from 0, so this suggests that the difference between these two varieties is at 0. Indeed, look at figure 5 below.

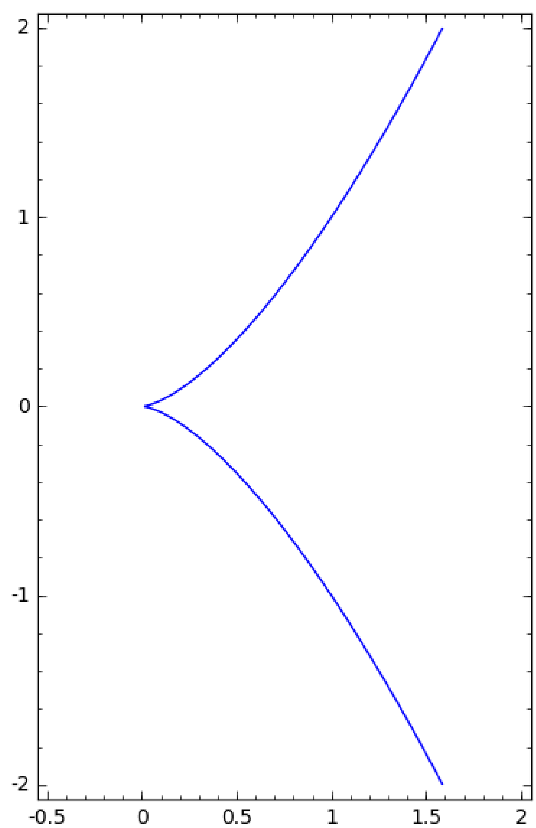

Figure 5: The cuspidal cubic in $\mathbb{A}^2$

Of course this was draw in the real plane, but the point is that there is a cusp at 0, suggesting geometrically what we said above. Also notable is that this variety is sort of "parametrized" by the affine line, but it lacks "smoothness" at 0 so this is not an isomorphism. I.3.17 covers this a bit more.

(b) This map is injective because it is a field homomorphism. It is onto because k is algebraically closed. Indeed, take some $a \in k$. Then the polynomial $t^p - a$ has a root in k. As described in part (a), because $\mathbb{A}^1$ has the cofinite topology, this is therefore a homeomorphism. It is given by the algebra homomorphism $k[t] \longrightarrow k[t]$ via $t \mapsto t^p$ so it is a morphism of varieties. Of course, this map has image $k[t^p] < k[t]$, so it is not an isomorphism.

□

I.3.3

(a) Let $\phi : X \longrightarrow Y$ be a morphism. Then for each $P \in X$, ϕ induces a homomorphism of local rings $\phi_P^* : \mathcal{O}_{\phi(P),Y} \longrightarrow \mathcal{O}_{P,X}$.

(b) Show that a morphism ϕ is an isomorphism if and only if ϕ is a homeomorphism, and the induced map ϕ_P^* on local rings is an isomorphism for all $P \in X$.

(c) Show that if $\phi[X]$ is dense in Y, then the map ϕ_P^* is injective for all $P \in X$.

Proof. (a) Take indeed some $(V, f) \in \mathcal{O}_{\phi(P),Y}$. Then $\phi(P) \in V$ and f is regular on V. Then $f \circ \phi$ is regular on $\phi^{-1}[V]$, which contains P. Hence, we map $(V, f) \mapsto (\phi^{-1}[V], f \circ \phi)$. I don't want to prove that this is well defined an a homomorphism, so I appeal to the general fact that ϕ induces a map of sheaves $\mathcal{O}_Y \longrightarrow \phi_*\mathcal{O}_X$. These are sheaves of rings (over Y), so functoriality of the colimit yields this exact map on stalks.

We furthermore want to show that this is a morphism of *local* rings. That is, $\mathfrak{m}_{\phi(P)} \mapsto \mathfrak{m}_P$. This is actually quite trivial: $\mathfrak{m}_{\phi(P)}$ is precisely those (V, f) such that $f(\phi(P)) = 0$. Then $f \circ \phi$ vanishes on P so $(\phi^{-1}[V], f \circ \phi) \in \mathfrak{m}_P$.

(b) The forward direction is obvious (see functoriality of the stalks) so suppose that ϕ is a morphism, a homeomorphism, and induces isomorphisms on the stalks. We just want to show that for all $f : U \longrightarrow k$ regular, $U \subseteq X$ open and nonempty, that $f \circ \phi^{-1}$ is regular. Take indeed $P \in U$. Then $(U, f) \in \mathcal{O}_{X,P}$. The map $\mathcal{O}_{\phi(P),Y} \longrightarrow \mathcal{O}_{P,X}$ is an isomorphism by assumption. In particular (U, f) is in the image, so there is some $(V, g) \mapsto (U, f)$. Then (suppressing restrictions), $g \circ \phi = f$ so $f \circ \phi^{-1} = g$. Thus, $f \circ \phi^{-1}$is regular on a neighborhood of P. Hence, $f \circ \phi^{-1}$ is regular and ϕ^{-1} is a morphism and ϕ is an isomorphsm.

(c) Note first that if we have a commutative diagram

$$X \xrightarrow{\phi} Y \underset{\alpha}{\overset{\beta}{\rightrightarrows}} k$$

then $\beta = \alpha$. Indeed, $\beta - \alpha : Y \longrightarrow k$ is regular and its zero set is closed and contains $\phi[X]$, which is dense. Thus, $\alpha - \beta = 0$. Warning - density is not enough, as maps with dense images are not in general epimorphisms in **Top**!

Now, take the map $\phi_P^* : \mathcal{O}_{Y,\phi(P)} \longrightarrow \mathcal{O}_{X,P}$ and suppose $(V, f) \mapsto 0$. By definition, $(V, f) \mapsto (\phi^{-1}[V], f \circ \phi)$. Then we must have some $\emptyset \neq W \subseteq \phi^{-1}[V]$ such that

$(f \circ \phi)|_W = 0$. Hence, we have the following commutative diagram.

$$\begin{tikzcd} \phi^{-1}[V] \arrow[r, "\phi"] & V \arrow[r, "f"] & k \\ W \arrow[u, hook] \arrow[urr, "0"'] & & \end{tikzcd}$$

We claim that the restriction $\phi : W \longrightarrow V$ also has dense image. Indeed, let $\emptyset \neq O \subseteq V$ be open. Then $\emptyset \neq \phi^{-1}[O] \subseteq \phi^{-1}[V]$ is open. It is therefore open in X, which is assumed to be irreducible, and is hence dense. Then $W \cap \phi^{-1}[O] \neq \emptyset$ an $\phi[W] \cap O \neq \emptyset$. We have the following commutative diagram.

$$\begin{tikzcd} W \arrow[r, "\phi"] & V \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & k \end{tikzcd}$$

so as $W \longrightarrow V$ is dominant, the explanation above tells us that $f = 0$ on V. Thus, the kernel of ϕ_P^* is trivial.

□

I.3.4

Show that the d-uple embedding of $\mathbb{P}^n$ (I.2.12) is an isomorphism onto its image.

Proof. In the proof of I.2.12.b, we defined an inverse ψ to the d-uple embedding ρ_d. This took some point $[\{a_J\}] \mapsto [a_{I_0} : \cdots : a_{I_n}]$, with the notation as described above. Point being, this map is defined by a bunch of homogeneous polynomials of the same degree, in this case $y_{I_0}, \ldots, y_{I_n}$. The $d-$uple embedding is the same sort of thing - homogeneous coordinate is defined by a homogeneous polynomial and each one has the same degree d. Since we already know these maps are inverse, we need only check that they are morphisms of varieties. To do so, we appeal to the following general lemma.

Lemma I.3. *Let $f_1, \ldots, f_m \in k[x_0, \ldots, x_n]$ be nonzero homogeneous polynomials of the same degree. Then if $Y \subseteq \mathbb{P}^n$ is a variety such that $Z(f_0, \ldots, f_m) \cap Y = \emptyset$, the map $\phi : Y \longrightarrow \mathbb{P}^m$ via $P \mapsto [f_0(P) : \cdots : f_m(P)]$ is a morphism of varieties.*

Proof. First of all, this map is well defined on Y because we chose it not to vanish there, and we insisted that all f_i are homogeneous of the same degree.

Now, we want to show that ϕ induces a map $\phi^* : \mathcal{O}_{\mathbb{P}^m} \longrightarrow \phi_* \mathcal{O}_Y$. Indeed, let $\emptyset \neq V \subseteq \mathbb{P}^m$ be open and let $f : V \longrightarrow k$ look like $\frac{g}{h}$ on some $\emptyset \neq W \subseteq V$ with $g, h \in k[x_0, \ldots, x_m]$ homogeneous polynomials of the same degree such that $Z(h) \cap V = \emptyset$. Then $\emptyset \neq \phi^{-1}[W] \subseteq \phi^{-1}[V]$ and on $\phi^{-1}[W]$ we have $f \circ \phi = \frac{g}{h} \circ \phi$. This evaluates as

$$P \mapsto [f_0(P) : \cdots : f_m(P)] \mapsto \frac{g([f_0(P) : \cdots : f_m(P)])}{h([f_0(P) : \cdots : f_m(P)])}.$$

So we have $\frac{g}{h} \circ \phi = \frac{g(f_0, \ldots, f_m)}{h(f_0, \ldots, f_m)}$. As each f_i has the same degree, this is a quotient of same degree homogeneous polynomials in $k[x_0, \ldots, x_n]$. Thus, $f \circ \phi$ is regular and ϕ induces a morphism on these sheaves, and is hence a morphism of varieties $Y \longrightarrow \mathbb{P}^m$. □

The only additional condition we have to check for the lemma to apply is that the coordinate functions of ρ_d and ψ vanish nowhere on $\mathbb{P}^n$ and $\operatorname{im}\rho_d$ respectively. Really this should have been checked in I.2.12 for these maps to even make sense, but it's clear enough. Indeed, for ρ_d, its components are the monomial of degree d, and $Z(x_0^d, \ldots, x_n^d) = \emptyset$. On the other hand, recall that the definition of ψ was independent of the choice of tuple I. As I ranges over all tuples with $\sum I = d-1$, $[a_{I_0} : \cdots a_{I_n}]$ will eventually contain each a_J, $\sum J = d$. As one of these components must be nonzero, the components of ψ cannot vanish on $\operatorname{im}\rho_d$. Hence, the lemma tells us that ρ_d and ψ are morphisms of varieties, and we know that they are inverse already. Hence, $\rho_d : \mathbb{P}^n \longrightarrow \operatorname{im}\rho_d$ is an isomorphism of varieties. □

I.3.5

By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbb{P}^n$ is any hypersurface, show that $\mathbb{P}^n - H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ and use the fact that $\mathbb{P}^N$ minus a hyperplane is affine.]

Proof. Let $H = Z(f)$ for some irreducible homogeneous polynomial $f \in k[x_0, dots, x_n]$. Let $d = \deg f$. Then by I.3.4, the map $\rho_d : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ is an isomorphism onto its image Y. We write $f = \sum a_I x^I$ ranging over multi-indices I with $\sum I = d$. Let $\overline{f} = \sum a_I y_I$, where y_I are the coordinates on $\mathbb{P}^N$, indexed by those same multi-indices. The d-uple embedding is defined then by $y_I \mapsto x^I$. We claim that, for $P \in \mathbb{P}^n$, that $f(P) = 0$ iff $\overline{f}(\rho_d(P)) = 0$. Indeed, $f(P) = 0$ means that $\sum a_I P^I = 0$ and $\rho_d(P) = [\{P^I\}]$, so $\sum a_I P^I = \overline{f}(\rho_d(P))$.

This tells us that ρ_d descends to a map $\mathbb{P}^n - H \longrightarrow Y - Z(\overline{f})$. As $\overline{f}$ is linear and irreducible, $Z(\overline{f})$ is a hypersurface in $\mathbb{P}^N$. We must, strictly speaking, show that this map is still a map of varieties, but we will defer this until I.3.10 which handles this in greater generality. Using this, we therefore conclude that $\mathbb{P}^n - H \cong Y - Z(\overline{f})$, which is $\mathbb{P}^n$ minus a hyperplane.

It remains to show that $\mathbb{P}^n$ minus a hyperplane is affine. Indeed, take $Z \subseteq \mathbb{P}^n$ a hyperplane. Let $T : \mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{n+1}$ be a linear isomorphism sending Z to $V(x_0)$. Then quotienting to $\mathbb{P}^n$ yields a linear automorphism $\mathbb{P}^n \longrightarrow \mathbb{P}^n$ sending $Z \mapsto Z(x_0)$. We know that $\mathbb{P}^n - Z(x_0) = U_0$ is affine by the usual map $\phi_0 : U_0 \longrightarrow \mathbb{A}^n$. □

I.3.6

There are quasi-affine varieties which are not affine. For example, show that $X = \mathbb{A}^2 - 0$ is not affine. [*Hint*: Show that $\mathcal{O}(X) \cong k[x, y]$ and use I.3.5. See **??** for another proof.]

Proof. Note: I used Gathmann's notes for the idea to look at the ideal of denominators of a regular functions (see: here).

Observe first that $X = D(x) \cup D(y)$. Then $\mathcal{O}(X) = \mathcal{O}(D(x)) \cap \mathcal{O}(D(y))$, where the intersection takes place in $k(X) = k(x, y)$. We seek then to compute the ring of regular functions of one of these basic open sets.

Lemma I.4. *Let X be an affine variety. Then $\mathcal{O}(D(f) \cap X) = \mathcal{O}(X)_f$.*

Proof. First of all, we have an inclusion $\mathcal{O}(X)_f \longrightarrow \mathcal{O}(D(f) \cap X)$, interpreted in the function field $k(X)$. Conversely, take some $\alpha \in \mathcal{O}(D(f) \cap X)$. As mentioned above, we will use the ideal of denominators of α. Indeed, let $I = \{g \in \mathcal{O}(X) : g\alpha \in \mathcal{O}(X)\}$. In other words $g \in I$ iff $\alpha = \frac{h}{g}$ for some $h \in \mathcal{O}(X)$. We want to show that some power $f^n \in I$. In other words, that $f \in \sqrt{I}$.

Now, if we have $X \subseteq \mathbb{A}^n$ defined as $V(\mathfrak{p})$ we'd like to pull back to $k[x_1, \ldots, x_n]$ to apply the Nullstellensatz. Indeed, I pulls back to $I + \mathfrak{p}$. We then want to show that (a representative of) f vanishes on $V(I + \mathfrak{p})$. Equivalently, that $D(f) \subseteq D(I + \mathfrak{p})$. Indeed, let $P \in D(f)$ and write $f = \frac{g}{h}$ on a neighborhood of P in $\mathbb{A}^n$. Then $h(P) \neq 0$ and any representative of h is in $\mathfrak{p} + I$. Thus, $p \in D(I + \mathfrak{p})$. We therefore have $J(V(f)) \subseteq J(V(I + \mathfrak{p})) = \sqrt{I + \mathfrak{p}}$. Thus, $f \in \sqrt{I}$, i.e. some power of f appears as a denominator of α. Hence, $\alpha \in \mathcal{O}(X)_f$ as desired. □

Hence, we have $\mathcal{O}(X) = k[x, y]_x \cap k[x, y]_y \subseteq k(x, y)$. Take some $\frac{f}{x^n} = \frac{g}{x^m}$ in the intersection. Then $y^m f = x^n g$, so $y^m \mid g$ and $x^n \mid d$. Hence, $\frac{f}{x^n} = \frac{g}{y^m} \in k[x, y]$ and indeed, $\mathcal{O}(X) = k[x, y]$. □

I.3.7

(a) Show that any two curves in $\mathbb{P}^2$ have a nonempty intersection.

(b) More generally, show that if $Y \subseteq \mathbb{P}^n$ is a projective variety of dimension ≥ 1, and if H is a hypersurface, then $Y \cap H \neq \emptyset$. [*Hint*: Use I.3.5 and I.3.1.e. See [Hartshorne, I.7.2] for a generalization.]

Proof. (a) Take indeed two curves $Z(f)$ and $Z(g)$, where f, g are irreducible homogeneous polynomials. Then their intersection is $Z(f, g)$. By I.2.17.a, the dimension of $Z(f, g) \geq 0$. Hence, it is nonempty. If this argument is a bit scary (which, to me, it is) we can unpack it. All we're really doing is saying that, by Krull's principal ideal theorem (which works well for homogeneous primes by I.2.17), $\operatorname{codim}(f, g) \leq 2$. Thus, the inclusion $(f, g) \subseteq (x, y, z)$ is strict. As (x, y, z) is the irrelevant ideal, the correspondence tells us that (f, g) defines a nonempty $Z(f, g)$, as only the unit ideal can define the empty variety.

(b) The hint really gives us the whole idea. Suppose indeed that $Y \cap H = \emptyset$ and consider then $\mathbb{P}^n - H$. By I.3.5, this is an affine variety. Furthermore, we have $Y \subseteq \mathbb{P}^n - H$. Then Y is an irreducible closed subset of an affine variety and is, therefore, itself an affine variety. By I.3.1.e, it must therefore be a point, i.e. $\dim Y = 0$. Thus, if $\dim Y \geq 1$ we must have $Y \cap H \neq \emptyset$.

□

I.3.8

Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$ with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of [Hartshorne, I.3.4a] in the case $Y = \mathbb{P}^n$.)

Proof. We can write $\mathbb{P}^n - (H_i \cap H_j) = (\mathbb{P}^n - H_i) \cup (\mathbb{P}^n - H_j)$. Hence, $\mathcal{O}(\mathbb{P}^n - (H_i \cap H_j)) = \mathcal{O}(\mathbb{P}^n - H_i) \cap \mathcal{O}(\mathbb{P}^n - H_j)$. Recall that we write $U_i = \mathbb{P}^n - H_i$ and that we had an isomorphism of varieties $\phi_i : U_i \longrightarrow \mathbb{A}^n$. Indeed, since we know $\mathcal{O}(\mathbb{A}^n)$ is the polynomial ring over k in n variables, we can compute U_i. More precisely, take $\mathcal{O}(\mathbb{A}^n) = k[x_0, \ldots, \widehat{x_i}, \ldots, x_n]$. Then the isomorphism $\mathcal{O}(\mathbb{A}^n) \longrightarrow \mathcal{O}(U_i)$ takes $f(x_0, \ldots, \widehat{x_i}, \ldots, x_n) \mapsto \frac{\beta_i(f)}{x_i^{\deg f}}$, where $\beta_i(f) = x_i^{\deg f} f\left(\frac{x_0}{x_i}, \ldots, \frac{\widehat{x_i}}{x_i}, \ldots, \frac{x_n}{x_i}\right)$. Now, one maybe tempted to cancel the $x_i^{\deg f}$ from the numerator and denominator, but one must please control themself. The numerator $\beta_i(f)$ is a degree $\deg f$ homogeneous polynomial that is not divisible by x_i, as f has no x_i terms. Hence, keeping it as this quotient makes it readily obvious exactly how this is interpreted as a regular function on $\mathcal{O}(U_i)$.

Now, take some $\alpha \in \mathcal{O}(U_i) \cap \mathcal{O}(U_j)$. Then we can write $\alpha = \frac{\beta_i(f)}{x_i^{\deg f}}$ and $\alpha = \frac{\beta_j(g)}{x_j^{\deg g}}$ for $f \in k[x_0, \ldots, \widehat{x_i}, \ldots, x_n]$ and $g \in k[x_0, \ldots, \widehat{x_j}, \ldots, x_n]$. Then, setting these two quotients equal, we get $x_j^{\deg g} \beta_i(f) = x_i^{\deg f} \beta_j(g)$. Now, as $i \neq j$, x_i and x_j are coprime. Hence, $x_i^{\deg f} | \beta_i(f)$. But as discussed above, no positive power of x_i can divide $\beta_i(f)$. Hence, $\deg f = 0$ so $\frac{\beta_i(f)}{x_i^{\deg f}}$ is a constant, i.e. α is a constant. Thus, $\mathcal{O}(\mathbb{P}^n - (H_i \cap H_j)) = \mathcal{O}(U_i) \cap \mathcal{O}(U_j) = k$. Less formally, the idea is that if there was a positive power of x_i in the denominator then this could not be defined on $U_j \cap H_i$, lest we divide by 0. $\square$

I.3.9

The homogeneous coordinate ring is not an invariant under isomorphism. For example, let $X = \mathbb{P}^1$ an let Y be the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y$ (I.3.4). But show that $S(X) \ncong S(Y)$.

Proof. As explained in I.3.1.c, $Y = Z(z^2 - xy)$. Then $S(Y) = k[x, y, z]/(z^2 - xy)$, which we claim is not isomorphic to $S(X) = k[x, y]$. First of all, the given map $S(Y) \longrightarrow S(X)$ via xx^2, $y \mapsto y^2$, $z \mapsto xy$ is not as isomorphism, as its image is $k[x^2, xy, y^2] < k[x, y]$. But the question was not that this specific map is not an isomorphism, it is that no isomorphism at all can exist. First, consider the plot of $V(z^2 - xy) \subseteq \mathbb{A}^3$ below (figure 6).

There seems to be an issue at the origin. Again, this is a plot in $\mathbb{R}^3$ but we're only using it for ideas. As such, let's consider the localization of $S(Y)$ at the maximal ideal $\mathfrak{m} = (x, y, z)$. First of all, note that every localization of $S(X) = k[x, y]$ by a maximal ideal is a regular local ring of dimension 2, so in particular the Zariski cotangent spaces $\mathfrak{n}/\mathfrak{n}^2$ are all 2 dimensional k vector spaces, for $\mathfrak{n}$ a maximal ideal of $k[x, y]$.

First of all, note that since $(z^2 - xy) \subseteq (x, y, z)$, $\mathfrak{m}$ does indeed correspond to a maximal ideal of $S(Y)$. Specifically, this is $\mathfrak{m}/(z^2 - xy)$. In fact, we have that $\mathfrak{m}^2 \supseteq (z^2 - xy)$, so $\mathfrak{m}^2$ corresponds to the ideal $\mathfrak{m}^2/(z^2 - xy)$. Now, these are ideals of $S(Y)$, so they are $S(Y)$ modules. Hence, they are also $k[x, y, z]$ modules. By the third isomorphism theorem, we have $\frac{\mathfrak{m}/(z^2 - xy)}{\mathfrak{m}^2/(z^2 - xy)} \cong \mathfrak{m}/\mathfrak{m}^2$ as $k[x, y, z]$ modules. We can appeal to the general fact that $k[x, y, z]$ is a three dimensional regular ring, or just see directly that $(x, y, z)/(x, y, z)^2$ has $\{x, y, z\}$ as a k-basis (any higher order things are killed in the quotient). Thus, $\mathfrak{m}/\mathfrak{m}^2$ is three dimensional over k, so in particular $S(Y)$ localized at $\mathfrak{m}$ is not a regular local ring. Thus, $S(Y) \ncong S(X)$.

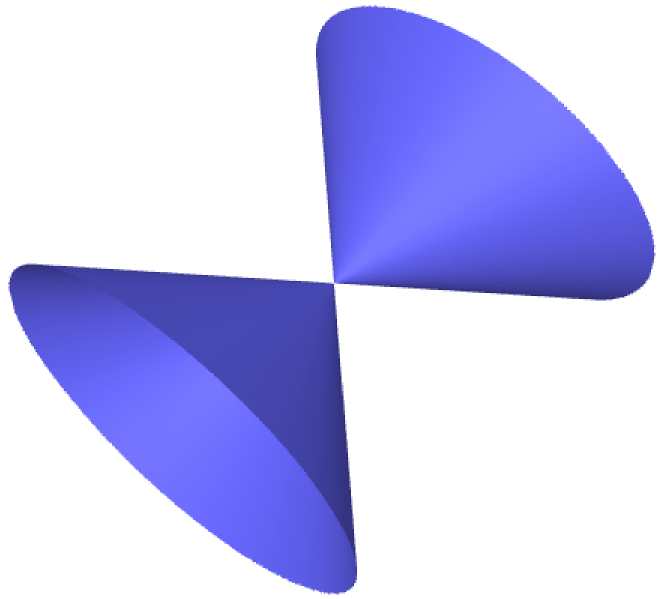

Figure 6: $V(z^2 - xy) \subseteq \mathbb{A}^3$

My grasp on this connection is a bit tenuous, but the fact that the Zariski cotangent space at the origin of this variety is three dimensional, as opposed to the desired two dimensions, reflects the three tangent directions at the origin. Everywhere else on this variety is nice and smooth, so we'd expect a two dimensional regular local ring everywhere else. $\square$

I.3.10

Subvarieties

A subset of a topological space is *locally closed* if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine of projective space. We call this the *induced structure* on Y, and we call Y a *subvariety* of X.

Now let $\phi : X \longrightarrow Y$ be a morphism and let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\phi[X'] \subseteq Y'$. Show that $\phi|_{X'} : X' \longrightarrow Y'$ is a morphism.

Proof. We have to show that $\phi|_{X'}$ induces a map on sheaves $\phi^* : \mathcal{O}_{Y'} \longrightarrow \phi_* \mathcal{O}_{X'}$. Indeed, take some (U, f), $\emptyset \neq U \subseteq Y'$ open and f regular on U. We want to show that $\phi^*(f) = f \circ \phi$ is regular. Now we can write $U = V \cap Y'$ for $V \subseteq Y$ open. It'd be pretty nice if we could extend f to V, as then we can use the fact that $\phi : X \longrightarrow Y$ is regular. This may be asking too much, but we still have the following lemma.

Lemma I.5. *Let $Y' \subseteq Y$ be a subvariety and $P \in Y'$. Then the restriction map $\mathcal{O}_{Y,P} \longrightarrow \mathcal{O}_{Y',P}$ on local rings is onto.*

Proof. Take indeed some $(U,f) \in \mathcal{O}_{Y',P}$. Some restriction to a neighborhood of P will be the quotient of polynomials, so by relabeling assume without loss of generality that $f = \frac{g}{h}$ for some polynomials g, h in the underlying polynomial ring of Y. The specifics will depend on if Y is projective or affine, but this distinction isn't very important right now. Indeed, we can consider the open set $D(h) \cap Y$, which contains P. This is an open subset of Y, so $\left(D(h) \cap Y, \frac{g}{h}\right) \in \mathcal{O}_{Y,P}$. The restriction of this to $\mathcal{O}_{Y',P}$ is $\left(D(h) \cap Y', \frac{g}{h}\right)$, which equals (U,f) in the local ring. Hence, we have found a preimage to (U,f) and the restriction map is onto. $\square$

Now let's return to our regular function (U,f) on Y'. Take some $P \in U$. Then we can view $(U,f) \in \mathcal{O}_{Y',P}$. By the lemma, we can take some $(V,g) \in \mathcal{O}_{Y,P}$ which restricts to (U,f). That is, $(V \cap Y', g|_{Y'}) = (U,f)$. As $\phi : X \longrightarrow Y$ is a map of varieties, we have $(\phi^{-1}[V], g \circ \phi)$ a regular function on anything in $\phi^{-1}[P]$. Thus, on some neighborhood of any of these points in $\phi^{-1}[P]$, $g \circ \phi$ is a quotient of polynomials on X. Then indeed, the restriction of any quotient of polynomials will remain regular so $g \circ \phi$ restricts to a regular function on Y' at any point in $\phi^{-1}[P]$. Of course, as g was a lift of f, $g \circ \phi$ is a lift of $f \circ \phi$. Thus, $f \circ \phi$ is regular on any point in $\phi^{-1}[P]$, hence it is regular on $\phi^{-1}[U]$. This therefore gives us our desired map of sheaves $\mathcal{O}_{Y'} \longrightarrow \phi_* \mathcal{O}_{X'}$. $\square$

I.3.11 INCOMPLETE

Let X be any variety and let $P \in X$. Show that there is a $1-1$ correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

Proof. Let's focus on the case that X is a (quasi) affine variety. We'll try to eliminate the quasi. Indeed, if we had $Y_1, Y_2 \subseteq X$ irreucible closed subsets then $\overline{Y_1}, \overline{Y_2} \subseteq \overline{X}$ are irreducible closed subsets. Furthermore, if $\overline{Y_1} = \overline{Y_2}$ then $\overline{Y_1} \cap X = \overline{Y_2} \cap X$. These should hopefully be Y_1, Y_2 again idk ugh. If so, irreducible closed subsets of X are in bijection to those in $\overline{X}$ so we can kill the quasi.

It's easy enough to do this in the affine variety case. Indeed, irreducible closed subsets of X are in correspondence to prime ideals of $A(X)$. $P \in Y$ iff $I(Y) \subseteq \mathfrak{m}_P$, and prime ideals in $A(X)_{\mathfrak{m}_P}$ correspond exactly to prime ideals in $A(X)$ contained in $\mathfrak{m}_P$. Hence, irreducible closed subsets containing P do indeed correspond to primes in $A(X)_{\mathfrak{m}_P} \cong \mathcal{O}_{X,P}$.

Now let's do the (quasi) projective part. We can hopefully eliminate the quasi as above. Then recall that closed subvarieties of X correspond to homogeneous primes in $S(X)$ not equal to the irrelevant ideal. Hopefully the localization correspondence still works. $\square$

I.3.12 INCOMPLETE

If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$. [*Hint*: Reduce to the affine case and use [Hartshorne, I.3.2c]].

Proof. idea: go to affine patch. can kill quasi part by closures, which preserves local ring. boom. $\square$

I.3.13 INCOMPLETE

The Local Ring of a Subvariety

Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes (U, f) where $U \subseteq X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function of U. We say that (U, f) is equivalent to (V, g) if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the *local ring* of Y on X. Note that if $Y = \{P\}$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

Proof. We claim that the unique maximal ideal of $\mathcal{O}_{Y,X}$ is $\mathfrak{m}_Y := \{(U, f) \in \mathcal{O}_X : f|_Y = 0\}$, in analogy to $\mathfrak{m}_P$. This is, of course, the kernel of the map $\mathcal{O}_{Y,X} \longrightarrow \mathcal{O}_Y \subseteq k(Y)$ by restriction.

idea: easy to see that you can invert away from Y so it's local. the lifting stuff from I.3.10 should give that the residue field is appropriate. idk about the dimension. we can embed the quotient field in $k(X)$, which has trdeg $\dim X$ over k and the residue field is $k(Y)$ which has trdeg $\dim Y$ over k. can we subtract these?

ok so $\mathcal{O}(X) \longrightarrow \mathcal{O}_{Y,X}$ via $f \mapsto (X, f)$. should be injective by density of opens, or because $(dom(f), f)$ is a representative of the equivalence class of (U, f), where $dom(f)$ is the largest open subset f is defined on. in any case, when $Y = \{P\}$ this is the inclusion $\mathcal{O}(X) \longrightarrow \mathcal{O}_P$, and when $Y = \{X\}$ this is the inclusion $\mathcal{O}(X) \longrightarrow k(X)$. so we'll analogously take (overloaded but if we're right it's ok) $\mathfrak{m}_Y = \{f \in \mathcal{O}(X) : f|_Y = 0\}$. Is it then the case that $\mathcal{O}(X)_{\mathfrak{m}_Y} = \mathcal{O}_{Y,X}$? we may have to be careful of affine vs projective, as for projective we really need a homogeneous localization.

concept: $\mathcal{O}_{Y,X} = \bigcup_{P \in Y} \mathcal{O}_P \subseteq k(X)$. also seems like $\mathfrak{m}_Y = \bigcap_{P \in Y} \mathfrak{m}_P$. can we use this to kill the projective and affine cases at the same time. □

I.3.14

Projection from a Point

Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Define a mapping $\phi : \mathbb{P}^{n+1} - \{P\} \longrightarrow \mathbb{P}^n$ by $\phi(Q) =$ the intersection of the unique line containing P and Q with $\mathbb{P}^n$.

(a) Show that ϕ is a morphism.

(b) Let $Y \subseteq \mathbb{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbb{P}^1$ (I.2.12). If t, u are homogeneous coordinates on $\mathbb{P}^1$, we say that Y is a curve given *parametrically* by $(x, y, z, w) = (t^3, t^2u, tu^2, y^3)$. Let $P = [0 : 0 : 1 : 0]$, and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

Proof. (a) We first do the particular case of $\mathbb{P}^n = Z(x_{n+1})$ and $P = [0 : \cdots : 0 : 1]$. Let $Q \in \mathbb{P}^{n+1} - \{P\}$ and let l be the line connecting the two. To understand this with legitimate linear algebra, we appeal to the cone of I.2.10. We then have lines $C(P), C(Q)$ and the plane $C(l)$ which contains both of them. All of this is, of course, in the vector space $\mathbb{A}^{n+2}$. $Q \neq P$ so this is indeed a plane. Furthermore, we have the hyperplane $C(Z(x_{n+1}))$. The intersection point of l and $Z(x_{n+1})$ lifts to the intersection line of $C(Z(x_n)) \cap C(l)$. Let's say Q has homogeneous coordinates $[a_0 : \cdots : a_{n+1}]$. Then $C(l)$ is the span of $(a_0, \ldots, a_{n+1})$ and $(0, \ldots, 0, 1)$. So for instance, it contains the point $(a_0, \ldots, a_n, 0)$. Note that one of the a_i, $i \le n$, must be nonzero as $Q \neq P = [0 : \cdots : 0 : 1]$. Thus, $(a_0, \ldots, a_n, 0)$ is a nonzero point in the one dimensional subspace $C(Z(x_{n+1})) \cap C(l)$. Hence, $[a_0 : \cdots : a_n : 0] \in l \cap Z(x_{n+1})$ so $\phi([a_0 : \cdots : a_{n+1}]) = [a_0 : \cdots : a_n : 0]$. A modified version of the lemma in I.3.4 shows that this is a morphism.

Now, we can reduce the general case to the above case. Consider a point $P \in \mathbb{P}^{n+1}$ and a hyperplane $H \subseteq \mathbb{P}^{n+1}$ not containing P. Then there is a linear isomorphism $T : \mathbb{A}^{n+2} \longrightarrow \mathbb{A}^{n+2}$ sending $C(P)$ to the span of $(0, \ldots, 0, 1)$ and $C(H)$ to the hyperplane $V(x_{n+1})$. This reduces to a linear isomorphism $\mathbb{P}^{n+1} \longrightarrow \mathbb{P}^{n+1}$, which we can use to appeal to the above special case.

(b) By the same logic as above, $\phi([t^3 : t^2u : tu^2 : u^3]) = [t^3 : t^2u : 0 : u^3]$. We of course need $[0 : 0 : 1 : 0] \notin Y$ for this to make sense. Indeed, if $[t^3 : t^2u : tu^2 : u^3] = [0 : 0 : 1 : 0]$ then $tu^2 \neq 0$ so $t, u \neq 0$ so t^3, t^2u, u^3 are all nonzero. Anyways, we want to say that $\phi[Y] \subseteq Z(x_2) = \mathbb{P}^2$ is a cuspidal cubic curve. The cuspidal cubic in $\mathbb{A}^2$ was given in I.3.2.a as the image of $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$, $t \mapsto (t^2, t^3)$. We modify this to $t \mapsto (t^3, t^2, 0, 1) \in D(x_3) \subseteq \mathbb{P}^2$. If we homogeneize this to a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^2$ we get $[t : u] \mapsto [t^3 : t^2u : 0 : u^3]$. The image of this is precisely $\phi[Y]$. We seek equations for this variety, and inspired by I.3.2.a we take the homogenization of $x_1^3 - x_0^2$ and get $x_1^2 - x_0^2x_3$.

Certainly $\phi[Y] \subseteq Z(x_1^2 - x_0^2x_3)$. On the other hand, take some $[a : b : 0 : d] \in Z(x_1^2 - x_0^2x_3)$. Then we have the relation $b^3 = a^2d$. We want to find some $[t : u]$ such that $[a : b : 0 : d] = [t^3 : t^2u : 0 : u^3]$. If $a = 0$ then by the above relation, $b = 0$ so $[a : b : 0 : d] = [0 : 0 : 0 : 1]$, in which case we take $t = 0, u = 1$. On the other hand, suppose $a \neq 0$. Then $[a : b : 0 : d] = [a^3 : ba^2 : 0 : da^2]$. As $b^3 = a^2d$, this becomes $[a^3 : ba^2 : 0 : b^3]$, so we take $[t : u] = [a : b]$. Thus, $Z(x_1^3 - x_0^2x_3) \subseteq \phi[Y] \subseteq Z(x_1^3 - x_0^2x_3)$. Hence, $\phi[Y]$ is indeed a cuspidal cubic.

We conclude with the plot (7 below) of $\phi[Y]$, which of course must be done over $\mathbb{R}$. We do this by taking the affine patch $D(x_3) \subseteq \mathbb{P}^2$ and shrinking it to an open disk in the plane $\mathbb{R}^2$. The boundary circle of this then represents the line at infinity $\mathbb{P}^1$. This is the usual CW complex structure of $\mathbb{RP}^2$.

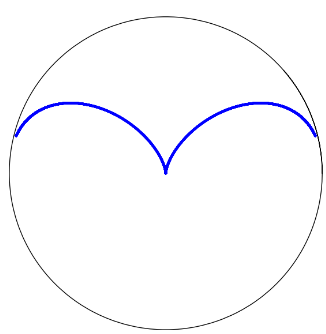

Figure 7: The cuspidal cubic in $\mathbb{P}^2$

□

I.3.15 INCOMPLETE

Products of Affine Varieties
Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X | x \times Y \subseteq Z_i\}$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology (I.1.4).

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties.

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

Proof. (a) kinda hard : /

(b) First, write $A(X) = k[x_1, \ldots, x_n]/\mathfrak{p}$ and $A(Y) = k[y_1, \ldots, y_m]/\mathfrak{q}$. Then by the Yoneda lemma, we can find natural isomorphisms

$$A(X) \otimes_k A(Y) \cong \frac{k[x_1, \ldots, x_n] \otimes_k k[y_1, \ldots, y_m]}{(\mathfrak{p} \otimes 1 + 1 \otimes \mathfrak{q})} \cong \frac{k[x_1, \ldots, x_n, y_1, \ldots, y_m]}{(\mathfrak{p} + \mathfrak{q})}.$$

We interpret $k[x_1, \ldots, y_m]$ as the coordinate ring of $\mathbb{A}^n \times \mathbb{A}^m$, so to show $A(X \times Y) \cong A(X) \otimes_k A(Y)$ we need to show that $X \times Y = V(\mathfrak{p} + \mathfrak{q})$. By the way, neither $\mathfrak{p}$ nor $\mathfrak{q}$ are ideals of $k[x_1, \ldots, y_m]$ but we treat them as subsets via the inclusions $k[x_1, \ldots, x_n], k[y_1, \ldots, y_m] \subseteq k[x_1, \ldots, y_m]$.

" $\subseteq$ " Let $(a, b) \in X \times Y$ and $f + g \in \mathfrak{p} + q$. Then $(f + g)(a, b) = f(a, b) + g(a, b) = f(a) + g(b) = 0$ as $X = V(\mathfrak{p})$ and $Y = V(\mathfrak{q})$. Thus, $X \times Y \subseteq V(\mathfrak{p} + \mathfrak{q})$.

" $\supseteq$ " Let $(a, b) \in V(\mathfrak{p} + \mathfrak{q})$. Then for any $f \in \mathfrak{p}$, $0 = f(a, b) = f(a)$ so $a \in V(\mathfrak{p}) = X$. Similarly, $b \in Y$ so $(a, b) \in X \times Y$. Thus, $V(\mathfrak{p} + \mathfrak{q}) \subseteq X \times Y$.

Hence, we have shown $X \times Y = V(\mathfrak{p} + \mathfrak{q})$ and as discussed, we therefore have $A(X \times Y) = A(X) \otimes_k A(Y)$. Additionally, by part (a) this is a domain. On the other hand, we can show part (a) without the given hint by showing the general fact of commutative algebra that the tensor prodict of finite type domains over an algebraically closed field is a domain.

(c) This comes from part (b) and the following lemma.

Lemma I.6. *Let A, B be finite type algebras over a field k (not necessarily algebraically closed). Then* $\dim A \otimes_k B = \dim A + \dim B$.

Proof. By Nöther's normalization lemma, we can find finite inclusions $k[x_1, \ldots, x_a] \longrightarrow A$ and $k[y_1, \ldots, y_b] \longrightarrow B$. Then $a = \dim A$ and $B = \dim B$. Furthermore, these together yield a finite map $k[x_1, \ldots, x_a, y_1, \ldots, y_b] \longrightarrow A \otimes_k B$, so $\dim A \otimes_k B = a + b = \dim A + \dim B$. □

(d) Let Var denote the category of varieties and dom the category of finite type domains over k, which is a full subcategory of Alg. We're suppressing the dependence of these categories on the underlying field k. We have the following natural isomorphisms via [Hartshorne, I.3.5]:

$$\begin{aligned}\mathsf{Var}(Z, X \times Y) &\cong \mathsf{dom}(A(X \times Y), \mathcal{O}(Z)) \\ &\cong \mathsf{dom}(A(X) \otimes_k A(Y), \mathcal{O}(Z)) \\ &\cong \mathsf{dom}(A(X), \mathcal{O}(Z)) \times \mathsf{dom}(A(Y), \mathcal{O}(Z)) \\ &\cong \mathsf{Var}(Z, X) \times \mathsf{Var}(Z, Y).\end{aligned}$$

Hence, $X \times Y$ is indeed a product in Var.

□

I.3.16 INCOMPLETE

Products of Quasi-Projective Varieties

Use the Segre embedding I.2.14 to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of a projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$

(a) Show that $X \times Y$ is a quasi-projective variety.

(b) If X, Y are both projective, show that $X \times Y$ is projective.

(c) Show that $X \times Y$ is a product in the category of varieties.

Proof. (a) In the course of proving (b), we will show that if $X, Y \subseteq \mathbb{P}^n, \mathbb{P}^m$ are closed subsets (not necessarily irreducible) then their product $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a closed subset, i.e. $\psi[X \times Y] \subseteq \psi[\mathbb{P}^n \times \mathbb{P}^m]$ is closed. This, combined with the actual statement of (b), will let us solve this problem.

Indeed, as X, Y are quasiprojective take some projective varieties $X', Y' \subseteq \mathbb{P}^n, \mathbb{P}^m$. Then by part (b), $X' \times Y' \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is a projective variety, so we need only show that $X \times Y \subseteq X' \times Y'$ is an open subset. Indeed, we can decompose $X' \times Y' - X \times Y = (X' - X) \times Y' \cup X' \times (Y' - Y)$. Both parts of these union are closed as explained in the above paragraph. Hence, $X \times Y \subseteq X' \times Y'$ in the topology induced from ψ. In conclusion, $X \times Y$ is a quasiprojective variety.

(b) We want to show that $\psi[X \times Y] \subseteq \psi[\mathbb{P}^n \times \mathbb{P}^m]$ is an irreducible closed subset. We'll first show that if X, Y are closed subsets of $\mathbb{P}^n, \mathbb{P}^m$ then $\psi[X \times Y] \subseteq \psi[\mathbb{P}^n \times \mathbb{P}^m]$ is closed. Indeed, recall that we had the map $\theta : k[z_{ij}] \longrightarrow k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ via

$z_{ij}|.x_iy_j$. Then the image of the Segre embedding $\psi : \mathbb{P}^n \times \mathbb{P}^m \longrightarrow \mathbb{P}^{(n+1)(m+1)-1}$ is given by $Z(\ker(\theta))$. This was proven in I.2.14.

So we need to find equations defining $\psi[X \times Y] \subseteq \mathbb{P}^{(n+1)(m+1)-1}$. As we're assuming X, Y are closed in their respective projective spaces, we can write $X = Z(I)$ and $Y = Z(J)$ for homogeneous ideals $I, J \subseteq k[x_0, \ldots, x_n], k[y_0, \ldots, y_m]$. From now on, I may use the abbreviations $k[x] := k[x_0, \ldots, x_n]$ and $k[y] = k[y_0, \ldots, y_m]$. We have the quotient map $^- : k[x, y] \longrightarrow k[x, y]/(I \cup J)$. We consider then the composition $\overline{\theta} = {}^- \circ \theta : k[z] \longrightarrow k[x, y]/(I \cup J)$. We claim then that $Z(\ker(\overline{\theta})) = X \times Y$.

Before we show any containment, we need this expression to make sense. That is, we need $\ker(\overline{\theta})$ to be homogeneous. This is true, as since I, J are homogeneous we can write $(I \cup J) = (I^h \cup J^h)$, which is a homogeneous ideal in $k[x, y]$. Hence, θ and $^-$ are both maps of graded rings which respect the grading (although θ doubles grading). So $\overline{\theta}$ preserves grading and its kernel is therefore homogeneous.

Now, we will show the equality $\psi[X \times Y] = Z(\ker \overline{\theta})$.

" $\subseteq$ " Let $(a, b) \in X \times Y$ and $f(z) \in \ker(\overline{\theta})^h$. Then $f(\psi(a, b)) = \theta(f)(a, b)$. As $f \in \ker \overline{\theta}$, $\theta(f) \in \ker{}^- = (I^h \cup J^h)$. If we had any $g \in I^h$ then $g(a, b) = g(a) = 0$ as $a \in X = Z(I)$. Similarly, if $g \in J^h$ we'd have $g(a, b) = 0$. We can write $\theta(f)$ as a $k[x, y]$ linear combination of terms in $I^h \cup J^h$, so $\theta(f)(a, b) = 0$. Thus, $f(\psi(a, b)) = 0$ so $\psi(a, b) \in Z(\ker \overline{\theta})$. In conclusion, $\psi[X \times Y] \subseteq Z(\ker \overline{\theta})$.

" $\supseteq$ " Let $P \in Z(\ker \overline{\theta}) \subseteq \ker \theta$. Then as ψ is a bijection $\mathbb{P}^n \times \mathbb{P}^m \longrightarrow Z(\ker \theta)$, we can find unique $(a, b) \in \mathbb{P}^n \times \mathbb{P}^m$ such that $P = \psi(a, b)$. We seek to show that $(a, b) \in X \times Y$. Indeed, let's take some $f \in I^h$. If we can show that $f(a) = 0$ then we will have shown $a \in Z(I) = X$.

All we know is that $P \in Z(\ker(\overline{\theta}))$, so we somehow want to pull f back to $k[z]$. We have that some component b_i of b is nonempty. Then the polynomial $f(x_0y_i, \ldots, x_ny_i)$ is still homogeneous. Furthermore, by homogeneity of f, this is equal to $y_i^d f(x)$, where $d = \deg f$. Thus, $f(x_0y_i, \ldots, x_ny_i) \in (I \cup J)^h$. Let us write $f = \sum a_K x^K$. Then $f(x_0y_i, \ldots, x_ny_i) = \sum a_{k_0,\ldots,k_n}(x_0y_i)^{k_0} \cdots (x_ny_i)^{k_n}$. Hence, we take $g = \sum a_{k_0,\ldots,k_n} z_{0i}^{k_0} \cdots z_{ni}^{k_n}$. This will be homogeneous (of degree d) and of course, $\theta(g) = f(x_0y_i, \ldots, x_ny_i) \in (I \cup J) = \ker{}^-$. Hence, $g \in (\ker \overline{\theta})^h$.

This therefore tells us that $g(P) = 0$. We have then that $0 = g(\psi(a, b)) = \theta(g)(a, b) = f(a_0b_i, \ldots, a_nb_i) = b_i^d f(a_0, \ldots, a_n)$. As we chose $b_i \neq 0$, we must have $f(a) = 0$. Thus, $a \in Z(I) = X$. The same reasoning applies to show that $b \in Z(J) = Y$. Hence, $P \in \psi[X \times Y]$ and we have shown $Z(\ker \overline{\theta}) \subseteq \psi[X \times Y]$

We have shown that $\psi[X \times Y] = Z(\ker \overline{\theta})$, which is a closed subset of $\mathbb{P}^{(n+1)(m+1)-1}$. Equivalently, $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$ is closed in the structure induced by ψ. To conclude this, we need to show that if X, Y are irreducible then their product $X \times Y$ is irreducible, which again really means that $\psi[X \times Y]$ is irreducible. As shown above, the equations defining $\psi[X \times Y]$ are given as $\ker(k[z] \longrightarrow k[x, y] \longrightarrow k[x, y]/(\mathfrak{p} \cup \mathfrak{q}))$, where $\mathfrak{p} = I(X)$ and $\mathfrak{q} = I(Y)$. As discussed in I.3.15.a, this is isomorphic to $k[x]/\mathfrak{p} \otimes_k k[y]/\mathfrak{q}$. That this is a domain follows from I.3.15.a. Alternatively, this can be shown directly using algebraic closure of k. This can be found in [Algebra II, Ch. 5 §17.5 Cor. 3].

(c) Glue the affine open cover???

Let π_X, π_Y be the set theoretic projections from $X \times Y$. Let $Z = \psi[X \times Y]$. We could hope that the maps $p_X = \pi_X \circ \psi^{-1}$ and $p_Y = \pi_Y \circ \psi^{-1}$ are projections witnessing Z as a product of the varieties X, Y. If we do indeed take this as our projections, then observe the isomorphisms

$$\begin{aligned}\mathsf{Set}(W, Z) &\cong \mathsf{Set}(W, X \times Y)\\ &\cong \mathsf{Set}(W, X) \times \mathsf{Set}(W, Y)\end{aligned}$$

given by $f \mapsto \psi^{-1} \circ f \mapsto (\pi_X \circ \psi^{-1} \circ f, \pi_Y \circ \psi^{-1} \circ f) = (p_X \circ f, p_Y \circ f)$.

So we need to show two things: that p_X, p_Y are morphisms of varieties and that this isomorphism $f \mapsto (p_X \circ f, p_Y \circ f)$ restricts to $\mathsf{Var}(W, Z) \longrightarrow \mathsf{Var}(W, X) \times \mathsf{Var}(W, Y)$.

I'll first show that p_X and p_Y are morphisms of varieties. Indeed, I'll explicitly describe a formula for it. Take some $P = [\{c_{ij}\}] \in Z$. Then $\psi^{-1}(P) = (a, b) \in X \times Y$ can be found as follows. Take some (k, l) such that $c_{kl} \neq 0$. Then define $a_i = c_{il}$ and $b_j = c_{kj}$. That this actually works was done in I.2.4. Hence, $p_X(P) = [c_{0l} : \cdots : c_{nl}]$ and $p_Y(P) = [c_{k0} : \cdots : c_{km}]$. This looks like a morphism, but (k, l) depend on P so we are not done. What saves us is that the choice of (k, l) can be taken locally constant in P, as if $c_{kl} \neq 0$ then we can take the same choice of (k, l) throughout the open neighborhood $P \in Z - Z(z_{kl})$. Being a morphism is a local property, so p_X and p_Y are indeed morphisms.

(actually also need to check well definedness, i.e. independence of choice of affine nbhd but this should follow from the relations)

By the above computation, we already know that p_X, p_Y give us a bijection $\mathsf{Set}(W, Z) \cong \mathsf{Set}(W, X) \times \mathsf{Set}(W, Y)$ for any variety W. We need this to restrict to Var. That is, we want to say that $(p_x \circ \phi, p_Y \circ \phi)$ are morphisms of varieties if and only if ϕ is. We just showed that p_X and p_Y are morphisms of varieties, so one direction is immediate. We are now left to show that if $\phi : W \longrightarrow Z$ is a set map such that its projections $p_X \circ \phi$ and $p_Y \circ \phi$ are morphisms of varieties, then ϕ is a morphism of varieties.

I'm not sure how this works honestly. The product diagram looks like this:

$$\begin{tikzcd} & W \arrow[dl, "\mathsf{Var}"'] \arrow[d, "\mathsf{Set}"] \arrow[dr, "\mathsf{Var}"] & \\ Y & Z \arrow[l, "\mathsf{Var}"] \arrow[r, "\mathsf{Var}"'] & X \end{tikzcd}$$

so we sorta want a map of sheaves like this:

$$\begin{tikzcd} & \mathcal{O}_W & \\ \mathcal{O}_Y \arrow[ur] \arrow[r] & \mathcal{O}_Z \arrow[u, dashed, "?"] & \mathcal{O}_X \arrow[ul] \arrow[l] \end{tikzcd}$$

which somehow makes me want to say there's a tensor product thing happening but I really don't know.

$\square$

I.3.17 INCOMPLETE

Normal Varieties

A variety Y is *normal at a point* $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is *normal* if it is normal at every point.

(a) Show that every conic in $\mathbb{P}^2$ is normal.

(b) Show that the quadric suraces Q_1, Q_2 in $\mathbb{P}^3$ given by the equations $Q_1 = Z(xy - zw)$, $Q_2 = Z(xy - z^2)$ are normal. (cd. **??** for the latter.)

(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbb{A}^3$ is not normal.

(d) If Y is affine, then Y is normal iff $A(Y)$ is integrally closed.

(e) Let Y be an affine variety. Show that there is a normal variety $\widetilde{Y}$, and a morphism $\pi : \widetilde{Y} \longrightarrow Y$, with the property that whenever Z is a normal variety and $\phi : Z \longrightarrow Y$ is a *dominant* morphism, (i.e. $\phi[Z]$ is dense in Y), then there is a unique morphism $\theta : Z \longrightarrow \widetilde{Y}$ such that $= \pi \circ \theta$. $\widetilde{Y}$ is called the *normalization* of Y. You will need [Hartshorne, I.3.9A] above.

Proof. (a) By I.3.1.c, all conics are isomorphic to $\mathbb{P}^1$. Furthermore, local rings are functorial (being colimits) so it suffices to show that $\mathbb{P}^1$ is normal. Take $P \in \mathbb{P}^1$. We can use the isomorphism $\mathcal{O}_P \cong S(Y)_{(\mathfrak{m}_P)}$, but the degree 0 stipulation makes it a bit more annoying. An easier thing to do would be to take $P \in \mathbb{A}^1 \subseteq \mathbb{P}^1$ for any open embedded copy of $\mathbb{A}^1$. Then $\mathcal{O}_P$ can be computed in $\mathbb{A}^1$ or $\mathbb{P}^1$. It's easy to do this for $\mathbb{A}^1$, as then $\mathcal{O}_P \cong k[t]_{\mathfrak{m}_P}$. $k[t]$ is a UFD therefore normal, and integral closure commutes with localization. Hence, the localization of a normal domain is normal, so $\mathcal{O}_P$ is normal.

(b) I haven't done these computations yet.

(c) This follows from part (d) of this problem, but I'll do it directly too. Above (5) we have a picture of the cuspidal cubic. The messed up part (the cusp!) is at the origin, so we suspect that the local ring at the origin will not be integrally closed. Indeed, $\mathcal{O}_0 \cong A(X)_{\mathfrak{m}_0}$, where X is the cuspidal cubic $V(y^2 - x^3)$. The maximal ideal $\mathfrak{m}_0$ is given by those polynomials which vanish at 0, so it is generated by (x, y). Note that we have the isomorphism $k[x, y]/(y^2 - x^3) \longrightarrow k[t^2, t^3]$ via $x \mapsto t^2$, $y \mapsto t^3$. Furthermore, this isomorphism sends $\mathfrak{m}_0 = (x, y) \mapsto (t^2, t^3)$. Thus, $\mathcal{O}_0 \cong k[t^2, t^3]_{(t^2,t^3)}$. The quotient field of this ring is $k(t)$. Note that $t \in k(t)$ is integral over $k[t^2, t^3]_{(t^2,t^3)}$ but is not in the ring. Hence, $\mathcal{O}_0$ is not normal and thus, X is not normal.

(d) ($\Longleftarrow$) Integral closure commutes with localization and $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$.

($\Longrightarrow$) Suppose that $\mathcal{O}_P$ is normal for all P. Then $A(Y)_{\mathfrak{m}_P}$ is normal for all $P \in Y$. By correspondence and the Nullstellensatz, this means that for all $\mathfrak{m} \in Max(A(Y))$, $A(Y)_{\mathfrak{m}}$ is normal. It is a general fact of commutative algebra that $A(Y)$ is therefore normal. Indeed, let $A(Y) \subseteq R \subseteq k(Y)$ be the integral closure. Then by flatness of localization , we have $A(Y)_{\mathfrak{m}} \subseteq R_{\mathfrak{m}} \subseteq k(Y)$. As integral closure commutes with localization, the inclusion $R_\mathfrak{m}$ is the integral closure of $A(Y)_\mathfrak{m}$ in $k(Y)$. We assumed that $A(Y)_\mathfrak{m}$ is integrally closed, so the inclusion $A(Y)_\mathfrak{m} \subseteq R_\mathfrak{m}$ is onto. Thus, the inclusion $A(Y) \subseteq R$ is onto at all localizations by maximal ideals. In other words, its cokernel is locally 0, therefore 0. Hence, it is onto and $A(Y) = R$ is normal.

(e) I haven't solved this yet. I suspect that the normalization will be defined as follows. Take $A(Y) \subseteq R \subseteq k(Y)$ the integral closure. Then by [Hartshorne, 3.9A], R is a finite type domain over k and therefore corresponds to an affine variety $\widetilde{Y}$. Furthermore, the inclusion map $i : A(Y) \subseteq R$ yields a map $\pi : \widetilde{Y} \longrightarrow Y$.

For example, take $Y = V(y^2 - x^3)$ the cuspidal cubic. Then its coordinate ring is $A(Y) = k[x,y]/(y^2 - x^3) \cong k[t^2, t^3]$, whose integral closure is $k[t]$. This corresponds to the map $\mathbb{A}^1 \longrightarrow Y$ via $t \mapsto (t^2, t^3)$.

Anyways, it is a fact of commutative algebra that an integral extension like $i : A(Y) \subseteq A(\widetilde{Y})$ yields a surjective map $\operatorname{Spec}(A(\widetilde{Y})) \longrightarrow \operatorname{Spec}(A(Y))$, which reduces to a surjective map $Max(A(\widetilde{Y})) \longrightarrow Max(A(Y))$. By the Nullstellensatz, this yields the map $\pi : \widetilde{Y} \longrightarrow Y$ which is therefore onto.

I'm a bit lost here, so here are some facts:

(i) A commutative diagram of varieties

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "\pi"] & Y \\ Z \arrow[u, dashed, "\exists!"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

is the same as a commutative diagram of k-algebras

$$\begin{tikzcd} A(\widetilde{Y}) \arrow[d, dashed, "\exists!"'] & A(Y) \arrow[l, "i"'] \arrow[dl, "\Phi"] \\ \mathcal{O}(Z) & \end{tikzcd}$$

(ii) What do dominant maps on varieties look like on coordinate rings? One fact we know is that they induce injections on local rings $\mathcal{O}_{Y,\phi(P))} \longrightarrow \mathcal{O}_{Z,P}$ and hence yield an extension of function fields $k(Y) \longrightarrow k(Z)$.

(iii) If we had a diagram of coordinate rings as in (i), then recall that every $a \in A(\widetilde{Y})$ is the root of some monic polynomial $f \in A(Y)[t]$. Thus, $\Phi(f)(\Phi(a)) = 0$. So we need $\Phi(a)$ to be integral over $A(Y)$ with respect to the algebra structure induced by Φ. That is, Φ must be an integral map. Is this the precise condition to get a dominant map of varieties? Does this give us the desired mapping property? The integral closure is the "smallest" integral extension right? Is there a UP of integral closure?

$\square$

I.3.18 INCOMPLETE

Projectively Normal Varieties

A projective variety $Y \subseteq \mathbb{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

(a) If Y is projectively normal then Y is normal.

(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbb{P}^3$ given parametrically by $(x, y, z, w) = (t^4, t^3u, t^2u^2, tu^3, u^4)$. Then Y is normal by not projectively normal. See **??** for more examples.

(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbb{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

Proof. (a) $\mathcal{O}_P \cong S(Y)_{(\mathfrak{m}_P)}$ which should be normal. I cheated out of proving this in I.3.17.a but maybe that same cheat works? Localization preserves integral closure but I'm not entirely sure what to do about the degree 0 part. Somehow I expect this to work.

(b) This seems to satisfy the equation $xw = yz$, so hopefully when I finish I.3.17.b that'll give me a proof. Also, I could just appeal to (c) to show that Y is normal but this is not really wholly satisfying.

(c) Can we find an inverse ala the approach we took for the d-uple embedding? This is very close to that.

□

I.3.19 INCOMPLETE

Automorphisms of $\mathbb{A}^n$

Let $\phi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \ldots, f_n$ of n variables $x_1, \ldots, x_n$. Let $J = \det \left| \frac{\partial f_i}{\partial x_j} \right|$ be the Jacobian polynomial of ϕ.

(a) If ϕ is an isomorphism (in which case we call ϕ an *automorphism* of $\mathbb{A}^n$) show that J is a nonzero constant polynomial.

(b) The converse of (a) is an unsolved problem, even for $n = 2$. See, for example, Vitushkin [1].

Proof. This would follow immediate from an "algebraic chain rule," which I was unable (or rather, unwilling...) to bash out at the moment. Indeed, if we let J_ϕ be the Jacobian matrix of ϕ then we hope to prove the identity $(J_\phi \circ \psi)(J_\psi) = J_{\phi \circ \psi}$ by analogy to the normal multivariable chain rule. If we had that result, then multiplicativity of the determinant would give us the result. □

I.3.20 INCOMPLETE

Let Y be a variety of dimension ≥ 2, an let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show that this would be false of $\dim Y = 1$. See **??** for a generalization.

Proof. (a) I'm sorta stumped here. For an example, let's consider the cone $Y = V(z^2 - xy) \subseteq \mathbb{A}^3$. I plotted this above (6). I expect that it isn't normal at the origin, which will probably come from the isomorphism $k[x,y,z]/(z^2 - xy) \longrightarrow k[s^2, st, t^2]$, which has normalization $k[s,t]$. We hope that the map $\mathcal{O}(Y) \longrightarrow (Y - 0)$ is not onto. For instance, maybe we'd have $\sqrt{x}$ or something (corresponding to s in $k[s^2, st, t^2] \subseteq k[s,t]$) defined on $Y - 0$ but not on Y?? In any case, how does this help me?

Concept: Does every normal point admit a neighborhood isomorphic to an open subset of affine space? I think the computation $\mathcal{O}(A^2 - 0) = k[x,y]$ in I.3.6 could generalize to higher affine spaces. Maybe that'll work?

(b) Take $Y = \mathbb{A}^1$ and $P = 0$. Then the map $\mathcal{O}(Y) \longrightarrow \mathcal{O}(Y - 0)$ is given by $k[x] \longrightarrow k[x]_x$ which is not onto.

□

I.3.21

Group Varieties

A group variety consists of a variety Y together with a morphism $Y \times Y \longrightarrow Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \longrightarrow Y$.

(a) The *additive group* $\mathbf{G}_a$ is given by the variety $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ defined by $\mu(a,b) = a + b$. Show it is a group variety.

(b) The *multiplicative group* $\mathbf{G}_m$ is given by the variety $\mathbb{A}^1 - 0$ and the morphism $\mu(a,b) = ab$. Show it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $\mathrm{Hom}(X, G)$ has a natural group structure.

(d) For any variety X show that $\mathrm{Hom}(X, \mathbf{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

(e) For any variety X show that $\mathrm{Hom}(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$ under multiplication.

Proof. (a) The map $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$ is induced from the coordinate rings via $k[t] \longrightarrow k[x,y]$ sending $t \mapsto x + y$ and is therefore a morphism of varieties. Furthermore, the inversion map is given by $k[t] \longrightarrow k[t]$ via $t \mapsto -t$, which is also therefore a map of varieties.

(b) This is the restriction of the map $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$ via $k[t] \longrightarrow k[x,y]$ sending $t \mapsto xy$. As for inversion, I'll denote $\phi : \mathbf{G}_m \longrightarrow \mathbf{G}_m$ via $\phi(a) = a^{-1}$. Now, let $f \in \mathcal{O}_{\mathbf{G}_m}(U)$ be given by $f = \frac{g}{h}$. Then $f \circ \phi$ is given by $\frac{f(x^{-1})}{g(x^{-1})}$. Let $d = \deg f$ and $e = \deg g$. Then $\frac{f(x^{-1})}{g(x^{-1})} = \frac{x^{d+e}f(x^{-1})}{x^{d+e}g(x^{-1})}$ is a quotient of polynomials. Furthermore, $\frac{1}{x}$ is well defined on U since $U \subseteq \mathbf{G}_m = \mathbb{A}^1 - 0$. Hence, $f \circ \phi$ is a regular function so ϕ is a morphism of varieties.

(c) This is really a general categorical fact. Indeed, we have the induced map $\operatorname{Hom}(X, G^2) \longrightarrow \operatorname{Hom}(X, G)$. By definition of a product, we have a natural isomorphism $\operatorname{Hom}(X, G^2) \cong \operatorname{Hom}(X, G)^2$. The group axioms can be defined as certain commutative diagrams, and this will give us corresponding diagrams for $\operatorname{Hom}(X, G)$ and its square, thus proving that it is a group.

(d) We have natural isomorphisms $\mathsf{Var}(X, \mathbf{G}_a) \cong \mathsf{Alg}(k[t], \mathcal{O}(X)) \cong \mathcal{O}(X)$. Does this respect the group structure? The first isomorphism certainly does, as it comes from the equivalence of categories $\mathsf{Var}^{op} \longrightarrow \mathsf{Dom}$, which sends the group object $\mathbf{G}_a$ to the cogroup object $k[t]$.

This cogroup structure is given by $k[t] \longrightarrow k[t] \otimes_k k[t]$ via $t \mapsto t \otimes 1 + 1 \otimes t$. This yields the map $\mathsf{Alg}(k[t] \otimes_k k[t], \mathcal{O}(X)) \longrightarrow \mathsf{Alg}(k[t], \mathcal{O}(X))$ via $\phi \mapsto (t \mapsto \phi(t \otimes 1 + 1 \otimes t))$. By universal property, any $\phi \in \mathsf{Alg}(k[t] \otimes_k k[t], \mathcal{O}(X))$ is given uniquely by two maps $f, g : k[t] \longrightarrow \mathcal{O}(X)$ such that $\phi(a \otimes b) = f(a)g(b)$. Furthermore, any map $k[t] \longrightarrow \mathcal{O}(X)$ is given uniquely by a choice of $\alpha \in \mathcal{O}(X)$ such that $t \mapsto \alpha$. Hence, ϕ is given by a pair $(\alpha, \beta) \in \mathcal{O}(X)$. The map $t \mapsto \phi(t \otimes 1 + 1 \otimes t)$ is therefore equal to $\phi(t \otimes 1) + \phi(1 \otimes t) = \alpha + \beta$. Thus, ϕ, given by (α, β), yields the map $k[t] \longrightarrow \mathcal{O}(X)$ given by $\alpha + \beta$. Finally, the last isomorphism $\mathsf{Alg}(k[t], \mathcal{O}(X)) \cong \mathcal{O}(X)$ is an isomorphism of groups.

(e) We have $\mathbf{G}_m \cong V(xy - 1) \subseteq \mathbb{A}^2$ so it is an affine variety. Thus, we can say that $\mathsf{Var}(X, \mathbf{G}_m) \cong \mathsf{Alg}(\mathcal{O}(\mathbf{G}_m), \mathcal{O}(X))$. We know that $\mathcal{O}(\mathbf{G}_m) = \mathcal{O}(\mathbb{A}^1 - 0) = k[t]_t$. This is a cogroup given by $k[t]_t \longrightarrow k[t]_t \otimes_k k[t]_t$ via $t \mapsto (t \otimes 1)(1 \otimes t)$. A similar argument to (d) above shows that the isomorphisms $\mathsf{Var}(X, \mathbf{G}_m) \cong \mathsf{Alg}(k[t]_t, \mathcal{O}(X)) \cong \mathcal{O}(X)^*$ are group isomorphisms.

$\square$

I.4 Rational Maps

I.4.1

If f, g are regular functions on open subsets U, V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a *rational* function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f is *defined* at the poonts of U.

Proof. The essence of the proof is that being a regular function is a local property. Let $h : U \cup V \longrightarrow k$ be the unique extension of f, g. Let $x \in U \cup V$. We want to show that h is regular at x. Suppose that $x \in U$. Then as f is regular at x, there is some $x \in U' \subseteq U$ such that f is a quotient of polynomials on U'. Then as $h|_{U'} = f|_{U'}$, h is regular at x. The same logic applies to the case $x \in V$. Thus, h is regular.

Now let $f \in k(X)$ a rational function. Then f is an equivalence class of regular functions on nonempty open subsets of X. Index the elements of this equivalence class as $f = \{(U_i, f_i) : i \in I\}$. Now let $U = \bigcup U_i$. There is a unique $g : U \longrightarrow k$ such that $g|_{U_i} = f_i$. By the same logic as the previous paragraph, i.e. locality of regularity, g is regular. By definition, $(U, g) \in f$ is the largest representative of f. $\square$

I.4.2

Same problem for rational maps. If ϕ is a rational map of X to Y, show there is a largest open set on which ϕ is represented by a morphism. We say the rational map is *defined* at the points of that open set.

Proof. This proof will be identical to I.4.1, given that we can show that being a morphism of varieties is a local property. Formally, we prove the following:

Lemma I.7. *Let X, Y be varieties and $X = \bigcup U_i$ an open cover. Suppose that $\phi : X \longrightarrow Y$ is a set map such that each restriction $\phi|_{U_i}$ is a morphism of varieties $U_i \longrightarrow Y$. Then ϕ is a morphism of varieties.*

Proof. Let $V \subseteq Y$ nonempty and let $f : V \longrightarrow k$ be regular. We want to show that $f \circ \phi : \phi^{-1}[V] \longrightarrow k$ is a regular function on X. Recall first that continuity is a local property, so $\phi^{-1}[V]$ is open. Indeed, we write it as $\bigcup \phi^{-1}[V] \cap U_i = \bigcup \phi|_{U_i}^{-1}[V]$. Now, let $\phi_i = \phi|_{U_i}$. Then $f \circ \phi_i : \phi_i^{-1}[V] \longrightarrow k$ is regular as ϕ_i was assumed to be a morphism of varieties. Thus, $f \circ \phi$ is locally regular and, as discussed in the previous problem, is therefore regular. This is precisely the definition of ϕ being a morphism of varieties. $\square$

Now, the exact same argument as in the previous problem will suffice. $\square$

I.4.3

(a) Let f be the rational function on $\mathbb{P}^2$ given by $f = \frac{x_1}{x_0}$. Find the set of points where f is defined and describe the corresponding regular function.

(b) Now think of this function as a rational map $\mathbb{P}^2 \longrightarrow \mathbb{A}^1$. Embed $\mathbb{A}^1$ in $\mathbb{P}^1$, and let $\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ be the resulting rational map. Find the set of points where ϕ is defined, and describe the corresponding morphism.

Proof. (a) As written, this function is defined on the open subset $\mathbb{P}^2 - Z(x_0)$. Suppose that there was an extension F of this function which was regular at some P. Then in some neighborhood V of P, F can be written as a quotient $F = \frac{f}{g}$ of homogeneous polynomials of the same degree. Thus, $\frac{f}{g}$ and $\frac{x_1}{x_0}$ must be equal as functions on the nonempty open set $V \cap (\mathbb{P}^2 - Z(x_0))$. Thus, they must be equal as rational polynomials, so $\frac{f}{g} = \frac{x_1}{x_0}$. In particular, $x_0 | g$. We must have $g(P) \neq 0$, so $P \notin Z(x_0)$. Thus, $\mathbb{P}^2 - Z(x_0)$ is the largest open set on which f is defined.

We can, of course, view $\mathbb{P}^2 - Z(x_0)$ as $\mathbb{A}^2$ via the isomorphism $[x_1, x_2] \mapsto [1 : x_1 : x_2]$. Pulling f back to $\mathbb{A}^2$, we get the regular function $\mathbb{A}^2 \longrightarrow k$ defined by $(x_1, x_2) \mapsto x_1$, which is induced by $k[t] \longrightarrow k[x_1, x_2]$ via $t \mapsto x_1$.

(b) We embed $\mathbb{A}^1 \longrightarrow \mathbb{P}^1$ via $t \mapsto [1 : t]$. The composition $\mathbb{P}^2 \dashrightarrow \mathbb{A}^1 \to \mathbb{P}^1$ is given by $[x_0 : x_1 : x_2] \mapsto [1 : x_1/x_0]$, which is defined on $\mathbb{P}^2 - Z(x_0)$. This has an extension $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$, which is defined on $\mathbb{P}^2 - \{[0 : 0 : 1]\}$. I'd like to remark, before continuing, that on $x_0 = 0$ this yields $[0 : 1]$. On the other hand, this "equals," via the previous formula, $[1 : 1/0] = [1 : \infty]$. All this is to say that this extension is possible because the inclusion $\mathbb{A}^1 \subseteq \mathbb{P}^1$ includes a point at infinity which allows this division by zero to become sensible.

Anyways, the issue now is to determine if this is the largest open set on which ϕ is defined. I took the idea of looking at the closure of fibers from here. Suppose we had some morphism $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^1$ such that $\Phi|_{\mathbb{P}^2 - \{[0:0:1]\}} = \phi$. Then if $\Phi([0 : 0 : 1]) = P$ then the fiber $\Phi^{-1}[P]$ will be closed by continuity. Furthermore, $[0 : 0 : 1]$ can only be in one fiber, so the question is which one? We will show that it wants to be in all of them.

Indeed, take some $[a_0 : a_1] \in \mathbb{P}^1$. Then the preimage $\phi^{-1}[[a_0 : a_1]] \subseteq \mathbb{P}^2 - \{[0 : 0 : 1]\}$ is closed in the subspace topology. Observe additionally that $\phi^{-1}[[a : b]] \subseteq Z(a_1x_0 - a_0x_1)$. In fact, $Z(a_1x_0 - a_0x_1) = \phi^{-1}[[a_0 : a_1]] \cup \{[0 : 0 : 1]\}$. Thus, $[0 : 0 : 1] \in \overline{\phi^{-1}[[a_0 : a_1]]}$ for any $[a_0 : a_1] \in \mathbb{P}^1$. Furthermore, $\Phi^{-1}[[a_0 : a_1]] \supseteq \overline{\phi^{-1}[[a_0 : a_1]]}$. This is actually an equality, as $\phi^{-1}[[a_0 : a_1]]$ is not closed by irreducibiltiy of $Z(a_1x_0 - a_0x_1)$. The point is, though, that $[0 : 0 : 1]$ is therefore an element of every single $\Phi^{-1}[P]$, $P \in \mathbb{P}^1$. This is what I mean by saying $[0 : 0 : 1]$ wants to be in all of the fibers, a contradiction.

In conclusion, the largest open subset on which $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ via $[x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$ is defined is $\mathbb{P}^2 - \{[0 : 0 : 1]\}$.

□

I.4.4

A variety Y is *rational* if it is birationally equivalent to $\mathbb{P}^n$ for some n (or, equivalently by [Hartshorne, I.4.5], if $k(Y)$ is a purely transcendental extension of k).

(a) Any conic in $\mathbb{P}^2$ is a rational curve.

(b) The cuspidal cubic $y^2 = x^3$ is a rational curve.

(c) Let Y be the nodal cubic curve $y^2z = x^2(x+z)$ in $\mathbb{P}^2$. Show that the projection ϕ from the point $P = [0 : 0 : 1]$ to the line $z = 0$ (Exc. I.3.14) induces a birational map from Y to $\mathbb{P}^1$. Thus, Y is a rational curve.

Proof. (a) By exercise I.3.1.c, all conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$, and are hence birational to it and therefore rational.

(b) Let $Y = V(y^2 - x^3) \subseteq \mathbb{A}^2$. Then consider the map $\mathbb{A}^1 - 0 \longrightarrow Y - 0$ via $t \mapsto (t^2, t^3)$. This is a morphism of varieties with inverse $Y - 0 \longrightarrow \mathbb{A}^1 - 0$ given by $(x, y) \mapsto \frac{y}{x}$. Thus, Y is birational to $\mathbb{A}^1$, which is isomorphic to an open subset of $\mathbb{P}^1$ and is hence rational.

(c) Recall from the proof of I.3.14 that this particular projection has the formula $\phi([x : y : z]) = [x : y]$. We remark that the equation describing Y here is the homogenization of the equation describing the affine nodal cubic $y^2 = x^3 + x^2$. Thus, we will homogenize the parametrization of the affine nodal cubic. Indeed, consider the map $\psi : \mathbb{P}^1 \longrightarrow Y$ given by $[t : u] \mapsto [u^2t - t^3 : u^3 - ut^2 : t^3]$. Restrict the domain to the affine open patch $\{u = 1\}$. Restrict further to the open subset on which $t \neq \pm 1$. Then $\phi(\psi([t : 1])) = [t - t^3 : 1 - t^2] = [t : 1]$. Thus, the composition $\mathbb{P}^1 \dashrightarrow Y \dashrightarrow \mathbb{P}^1$ is the identity.

On the other hand, $Y \dashrightarrow \mathbb{P}^1 \dashrightarrow Y$ is given by $[x : y : z] \mapsto [x : y] \mapsto [y^2x - x^3 : y^3 - yx^2 : x^3]$. Restrict this, of course, to the open set where $[x : y : z] \neq [0 : 0 : 1]$. Observe that the second component $y^3 - yx^2$ factors as $y(y^2 - x^2)$, so we claim that $(y^2 - x^2)[x : y : z] = [y^2x - x^3 : y^3 - yx^2 : x^3]$. For this to work, we have to restrict to the open subset on which $x \neq \pm y$. Anyways, we certainly have $x(y^2 - x^2) = xy^2 - x^3$. Furthermore, as $[x : y : z] \in Y$ we have $x^3 = y^2z - x^2z = (y^2 - x^2)z$. Hence, $[x : y : z] = [y^2x - x^3 : y^3 - yx^2 : x^3]$ when this makes sense. Thus, the composition $Y \dashrightarrow \mathbb{P}^1 \dashrightarrow Y$ is the identity. In conclusion, ϕ and ψ are birationally inverse, so Y is rational.

□

I.4.5

Show that the quadric surface Q defined by $xy = zw$ in $\mathbb{P}^3$ is birational to $\mathbb{P}^2$, but not isomorphic to $\mathbb{P}^2$ (cf. Exc. I.2.15).

Proof. Recall that $Q = \mathbb{P}^1 \times \mathbb{P}^1$. Consider then $(\mathbb{P}^1 - P) \times (\mathbb{P}^1 - P)$ for some point $P \in \mathbb{P}^1$. This is an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$. Furthermore, $\mathbb{P}^1 - P \cong \mathbb{A}^1$ so this open subset is isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$. Hence, an open subset of Q is isomorphic to $\mathbb{A}^2$, which is rational.

Also, recall by I.3.7.a that any two curves in $\mathbb{P}^2$ intersect. However, by I.2.15, there are curves in Q which do not intersect. For instance, let $P \neq P'$ be distinct points in $\mathbb{P}^1$. Then $\mathbb{P}^1 \times P$ and $\mathbb{P}^1 \times P'$ are disjoint curves in Q. Thus, Q and $\mathbb{P}^2$ are not even homeomorphic.

Remark: By the link in exc. I.4.3.b we have that every map $\mathbb{P}^2 \longrightarrow \mathbb{P}^1$ is constant, so every map $\mathbb{P}^2 \longrightarrow Q$ is constant as well, so there is absolutely no chance of an isomorphism. □

I.4.6 INCOMPLETE

Plane Cremona Transformations.

A birational map $\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is called a *plane Cremona transformation*. We give an example, called a *quadratic transformation*. It is the rational map $\phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ given by $[a_0 : a_1 : a_2] \mapsto [a_1 a_2 : a_0 a_2 : a_0 a_1]$ when no two of the a_0, a_1, a_2 are 0.

(a) Show that ϕ is birational, and is its own inverse.

(b) Find open sets $U, V \subseteq \mathbb{P}^2$ such that $\phi : U \longrightarrow V$ is an isomorphism.

(c) Find the open sets where ϕ and ϕ^{-1} are defined, and describe the corresponding morphisms.

Proof. (a) Note that when a_0, a_1, a_2 are all nonzero then $[a_1 a_2 : a_0 a_2 : a_0 a_1] = [1/a_0 : 1/a_1 : 1/a_2](a_0 a_1 a_2)$, and $[a_0 : a_1 : a_2] \mapsto [1/a_0 : 1/a_1 : 1/a_2]$ is indeed its own inverse.

(b) As just discussed in part (a), on the open subset $\mathbb{P}^2 - Z(x_0 x_1 x_2)$ this map is an involution.

(c) The definition in the problem statement holds on $\mathbb{P}^2 - Z(x_1 x_2, x_0 x_2, x_0 x_1)$. Is this open subset maximal? Observe that $P \in Z(x_1 x_2, x_0 x_2, x_0 x_1)$ iff exactly two of its entries are 0. That is, it consists of $\{[0 : 0 : 1], [0 : 1 : 0], [1 : 0 : 0]\}$. We therefore want to determine if we can extend ϕ to any of these three points. I'm not sure how to do this, but I'm pretty sure you can't.

□

I.4.7

Let X and Y be two varieties. Suppose there are points $P \in X$ and $Q \in Y$ such that the local rings $\mathcal{O}_{X,P}$ and $\mathcal{O}_{Y,Q}$ are isomorphic as k-algebras. Then show that there are open sets $P \in U \subseteq X$ and $Q \in V \subseteq Y$ and an isomorphism $U \longrightarrow V$ which sends P to Q.

Proof. We will prove this first for X, Y affine varieties. This will be sufficient, as we know that all varieties admit a basis of open affine neighborhoods. Indeed, let's say that $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$. Then we have $\mathcal{O}_{X,P} = A(X)_{\mathfrak{m}_P}$ and $\mathcal{O}_{Y,Q} = A(Y)_{\mathfrak{m}_Q}$. Suppose then that we had an isomorphism of k-algebras $\phi : A(Y)_{\mathfrak{m}_Q} \longrightarrow A(X)_{\mathfrak{m}_P}$. This is uniquely determined by the choice of $y_i \mapsto \frac{f_i}{g_i} \in A(X)_{\mathfrak{m}_P}$. Of course, this choice is not free. It must respect the relations $I(Y)$ and send the localized elements in $A(Y)_{\mathfrak{m}_Q}$ to units.

Now, recall that isomorphism $\mathsf{Alg}(A(Y), \mathcal{O}(U)) \cong \mathsf{Var}(U, Y)$. Then we want our choice of $\phi(y_i) = \frac{f_i}{g_i}$ to represent functions on some open subset $U \subseteq X$. They will, under this isomorphism, yield coordinates of a map $U \longrightarrow Y$. So what open set must this be? Well we certainly need each $g_i \neq 0$ on U for this to work, so take $U = D(\prod g_i)$. This makes sense, as each g_i was necessarily nonzero in $A(X)$, which is a domain. Then each $\frac{f_i}{g_i} \in \mathcal{O}(U)$ so we do indeed get a map $A(Y) \longrightarrow \mathcal{O}(U)$, and subsequently a map $\Phi : U \longrightarrow Y$.

We must first check two essential facts to ensure that we're on the right track. First, that $P \in U$ and second that $\Phi(P) = Q$. For the former, observe that $\prod g_i(P) \neq 0$ as each $g_i \notin \mathfrak{m}_P$ and hence, by maximality, their product $\prod g_i \notin \mathfrak{m}_P$. Now, we must show that $\Phi(P) = Q$. By the Nullstellensatz, it suffices to show that for every $f \in \mathfrak{m}_Q$ we have $f(\Phi(P)) = 0$. Indeed, take such an f. We want to show that $f(\phi(y_1)(P), \dots, \phi(y_m)(P)) = 0$. Now, observe that $\phi(f) = f(\phi(y_1), \dots, \phi(y_m))$, as ϕ is given by evaluation at the $\phi(y_i)$. We therefore want to show that $0 = f(\Phi(P)) = (\phi(f))(P)$. Since ϕ was a ring isomorphism, it takes the $\mathfrak{m}_Q A(Y)_{\mathfrak{m}_Q} \mapsto \mathfrak{m}_P A(X)_{\mathfrak{m}_P}$ isomorphically. Thus, $\phi(f) \in \mathfrak{m}_P A(X)_{\mathfrak{m}_P}$ so we may write $\phi(f) = \frac{g}{h}$ where $g \in \mathfrak{m}_P$ and $h \notin \mathfrak{m}_P$. Hence, $(\phi(f))(P) = \frac{g(P)}{h(P)} = 0$ as desired. Then $f(\Phi(P)) = 0$ for all $f \in \mathfrak{m}_Q$ so $\Phi(P) = Q$.

So far, we have taken our isomorphism ϕ and used it to find an open neighborhood $P \in U \subseteq X$ and an induced map $\Phi : U \longrightarrow Y$ which takes $P \mapsto Q$. We will now apply the same process to ϕ^{-1} to find an open neighborhood $Q \in V \subseteq Y$ and a map $\Psi : V \longrightarrow X$ which sends $Q \mapsto P$. Since Ψ arose from ϕ^{-1} via the same construction as $\phi \mapsto \Phi$, it better be the case that $\Psi = \Phi^{-1}$.

Indeed, take some $y \in Y$ on which the composition $\Phi \circ \Psi$ is defined. This works, for instance, on $V \cap \Psi^{-1}[U]$. This contains Q so it is a nonempty open subset. We want to show that $\Phi(\Psi(y)) = y$. As discussed before, it suffices to show that to vanishes on all $f \in \mathfrak{m}_y$. Recalling as before that $f(\Phi(P)) = (\phi(f))(P)$ and the analogous fact for Ψ and ϕ^{-1}, we have the following.

$$\begin{aligned} f(\Phi(\Psi(y))) &= (\phi(f))(\Psi(y)) \\ &= (\phi^{-1}(\phi(f)))(y) \\ &= f(y) \\ &= 0. \end{aligned}$$

We can proceed similarly to show that $\Psi \circ \Phi$ restricts to the identity on an open neighborhood of P. Hence, Φ yields an isomorphism $U' \longrightarrow V'$ sending $P \mapsto Q$. □

I.4.8

(a) Show that any variety of positive dimension over k has the same cardinality as k. [*Hints:* Do $\mathbb{A}^n$ and $\mathbb{P}^n$ first. Then for any X, use induction on the dimension of n. Use [Hartshorne, I.4.9] to make X birational to a hypersurface $H \subseteq \mathbb{P}^{n+1}$. Use Exc. I.3.7 to show that the projection of H to $\mathbb{P}^n$ from a point not on H is finite-to-one and surjective.]

(b) Deduce that any two *curves* over k are homeomorphic (cf. Exc. I.3.1).

Proof. (a) I don't actually use the hint, but it's pretty interesting so I'll include it as a lemma.

Lemma I.8. *Let $H \subseteq \mathbb{P}^{n+1}$ a hypersurface, $P \notin H$, and $P \notin \mathbb{P}^n$ a hyperplane. Then the projection $\phi : \mathbb{P}^{n+1} - P \longrightarrow \mathbb{P}^n$ restricts to a finite to one surjection $H \longrightarrow \mathbb{P}^n$.*

Proof. Let $Q \in \mathbb{P}^n$ and consider the preimage $\phi^{-1}[Q] \subseteq \mathbb{P}^{n+1} - P$. Its closure in $\mathbb{P}^{n+1}$ is the line L connecting P and Q. By exercise I.3.7.b, the intersection. Thus, $\phi|_H$ is onto.

We want to show that this intersection is, in fact, finite. The intersection $H \cap L$ is a nonempty closed subset of L. It is proper as $P \in L$ and $P \notin H$. Thus, its dimension must be strictly smaller than L, which is a line and therefore one dimensional. Hence, $\dim H \cap L = 0$. As L is a line in $\mathbb{P}^{n+1}$, it is isomorphic to $\mathbb{P}^1$. We claim therefore that a zero dimensional nonempty closed subset of $\mathbb{P}^1$ is finite. On each affine open patch, a dimension 0 closed subset is finite, so it is also true of $\mathbb{P}^1$. Thus, $H \cap L = \phi|_H^{-1}[Q]$ is finite and nonempty for all $Q \in \mathbb{P}^n$. □

Now, onto the actual question. As k is infinite, the absorption laws tell us that $|\mathbb{A}^n| = |k|^n = |k|$ for all $n \geq 1$. Furthermore, as $\mathbb{P}^n$ contains a copy of $\mathbb{A}^n$ and is surjected onto by $\mathbb{A}^{n+1} - 0$, we have $|\mathbb{P}^n| = |k|$ as well. Now, let X be any variety of positive dimension. Then it is a locally closed subset of some $\mathbb{P}^N$, so its cardinality is at most $|k|$.

On the other hand, X contains an open affine variety, as these in fact form a basis for the topology on X. Thus, as we need only show $|X| \geq |k|$ it suffices to take the case of X affine. Then by Nöther's normalization lemma, there is some finite inclusion $k[t_1, \ldots, t_r] \longrightarrow A(X)$. Here $r = \dim X \geq 1$. We then get a corresponding map $X \longrightarrow \mathbb{A}^r$ of affine varieties. It is a surjection as an integral extension of rings induces a surjection on the prime spectra, and integrality furthermore allows this to restrict to maximal ideals. By the Nullstellensatz, this corresponds to a surjection of points $X \longrightarrow \mathbb{A}^r$. Hence, $|X| \geq |\mathbb{A}^r| = |k|$.

(b) It suffices, by part (a) to show that all curves have the cofinite topology. Indeed, let C be a curve and let $F < C$ a nonempty closed subset. If C is a affine then the coordinate ring $A(F)$ will be zero dimensional Nötherian, hence Artinian. Thus, it will have only finitely many maximal ideals so F will be finite. In general, C has a basis of open affine curves, so it is in particular covere by these. In fact, as C is a Nötherian space, it is compact, so it has a finite cover by open affine curves. Then the intersection of F with any of these curves will be a proper closed subset of an affine variety and therefore finite. Hence, in general, F is finite. Furthermore, any bijection between cofinite spaces is a homeomorphism, so all curves are homeomorphic.

□

I.4.9 INCOMPLETE

Let X be a projective variety of dimension r in $\mathbb{P}^n$ with $n \geq r + 2$. Show that for a suitable choice of $P \notin X$, and a linear $\mathbb{P}^{n-1} \subseteq \mathbb{P}^n$, the projection from P to $\mathbb{P}^{n-1}$ (Exc. I.3.4) induces a *birational* morphism of X onto its image $X' \subseteq \mathbb{P}^{n-1}$. You will need to use [Hartshorne, I.4.6A, I.4.7A, I.4.8A]. This shows in particular that the birational map of [Hartshorne, I.4.9] can be obtained by a finite number of such projections.

Proof. Well let's think about how this would affect function fields. Though I still don't know why X' is even a variety. Let's imagine $[0 : \cdots : 0 : 1] \notin X$ and let $\mathbb{P}^{n-1} = Z(x_n)$. Then the projection $\phi : \mathbb{P}^n - P \longrightarrow \mathbb{P}^{n-1}$ is given by $[a_0 : \cdots : a_n] \mapsto [a_0 : \cdots : a_{n-1}]$. This is finite to one and onto (see I.4.8) so it is of course dominant. Thus it induces a map on function fields $k(\mathbb{P}^{n-1}) \longrightarrow k(\mathbb{P}^n)$ via $(U, f) \mapsto (\phi^{-1}[U], f \circ \phi)$. We can write elements of $k(\mathbb{P}^{n-1})$ as rational functions $\frac{f(x_0,\ldots,x_{n-1})}{g(x_0),\ldots,x_{n-1}}$ which are homogeneous of the same degree. Then similarly, $\frac{f}{g} \circ \phi$ can be written as a homogeneous element of $k(x_0, \ldots, x_n)$ of degree 0. . This then sends $[a_0 : \cdots : a_n]$ to $\frac{f(a_0,\ldots,a_{n-1})}{g(a_0,\ldots,a_{n-1})}$, so the induced map $k(\mathbb{P}^{n-1}) \longrightarrow k(\mathbb{P}^n)$ is just the inclusion $k(x_0, \ldots, x_{n-1})^0 \longrightarrow k(x_0, \ldots, x_n)^0$.

How about on $X \longrightarrow X'$? It's described in the same way as $(U, f) \mapsto (\phi^{-1}[U], f \circ \phi)$. Also, we can write $k(X) = S(X)_{((0))}$ the homogeneous localization, and analogously for X' (if it's even a variety??). For this to be birational, I'd hope $k(X') \longrightarrow k(X)$ is an isomorphism. But I am lost. □

I.4.10 INCOMPLETE

Let Y be the cuspial cubic curve $y^2 = x^3$ in $\mathbb{A}^2$. Blow up the point $O = (0, 0)$, let E be the exceptional curve, and let $\widetilde{Y}$ be the strict transform of Y. Show that E meets $\widetilde{Y}$ in one point, and that $\widetilde{Y} \cong \mathbb{A}^1$. In this case the morphism $\phi : \widetilde{Y} \longrightarrow Y$ is a homeomorphism, but it is not an isomorphism.

Proof. We can write $\phi^{-1}[Y] = E \cup \phi^{-1}[Y - 0]$, where $E = \phi^{-1}[0]$ is the exceptional curve. Taking closures, we get $\phi^{-1}[Y] = E \cup \widetilde{Y}$. This is the irreducible decomposition of $\phi^{-1}[Y]$, which is defined in $\mathbb{A}^2 \times \mathbb{P}^1$ by the equations $y^2 = x^3$ and $xu = yt$. If we intersect $\phi^{-1}[Y]$ with some nonempty open subset U we therefore get $(E \cap U) \cup (\widetilde{Y} \cap U)$, which are both irreducible and not contained in each other. We will therefore try to understand how E and $\widetilde{Y}$ intersect in affine open patches.

Indeed, let's start with $D(t) = \{t = 1\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$. This is isomorphic to $\mathbb{A}^3$, and $\phi^{-1}[Y] \cap D(t)$ is given by $V(y^2 - x^3, xu - y)$. It's a simple computation to see that this equals $V(x, y) \cup V(u^2 - x, xu - y)$. The former component is $E \cap D(t)$. The latter, which one can show is irreducible, is therefore $\widetilde{Y} \cap D(t)$. Thus, $\widetilde{Y} \cap E \cap D(t) = V(x, y) \cap V(u^2 - x, xu - y)$, which equals $V(x, y, u) = \{0\}$.

Below is an image of the intersection $\widetilde{Y} \cap D(t)$ in white, along with the exceptional curve $E \cap D(t)$ in yellow. (8 below)

On the other hand, if we intersect with $D(u) = \{u = 1\}$ we get $V(y^2 - x^3, x - yt) = V(x, y) \cup V(1 - yt^3, x - yt^3)$. The intersection of these is $V(x, y, 1 - yt^3, x - yt) = V(x, y, 1) = \emptyset$.

Thus, we have computed the intersection $\widetilde{Y} \cap E = (0, 0, [1 : 0])$. This, additionally, shows that $\widetilde{Y} = \phi^{-1}[Y - 0] \cup \{(0, 0, [1 : 0])\}$. The parametrization $\mathbb{A}^1 \longrightarrow Y$ via $t \mapsto (t^2, t^3)$ yields a parametrization $\mathbb{A}^1 - 0 \longrightarrow \phi^{-1}[Y - 0]$ via $t \mapsto (t^2, t^3, [t^2 : t^3])$. We can extend this to $\mathbb{A}^1$ via $0 \mapsto (0, 0, [1 : 0])$. I claim that this is an isomorphism of varieties $\mathbb{A}^1 \longrightarrow \widetilde{Y}$. It is given by maps $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$ via $t \mapsto (t^2, t^3)$ and $\mathbb{A}^1 \longrightarrow \mathbb{P}^1$ via $t \mapsto [1 : t]$, so it's at least a morphism of varieties. Its inverse is given by the projection $Y \longrightarrow \mathbb{P}^1$, whose image consists of those elements of the form $[1 : t]$, which maps isomorphically to $\mathbb{A}^1$ via $[1 : t] \mapsto t$.

The projection back down to Y returns us to the usual parametrization of Y, which is not an isomorphism but is a homeomorphism.

NOTE: I'm not completely happy with the rigor here. □

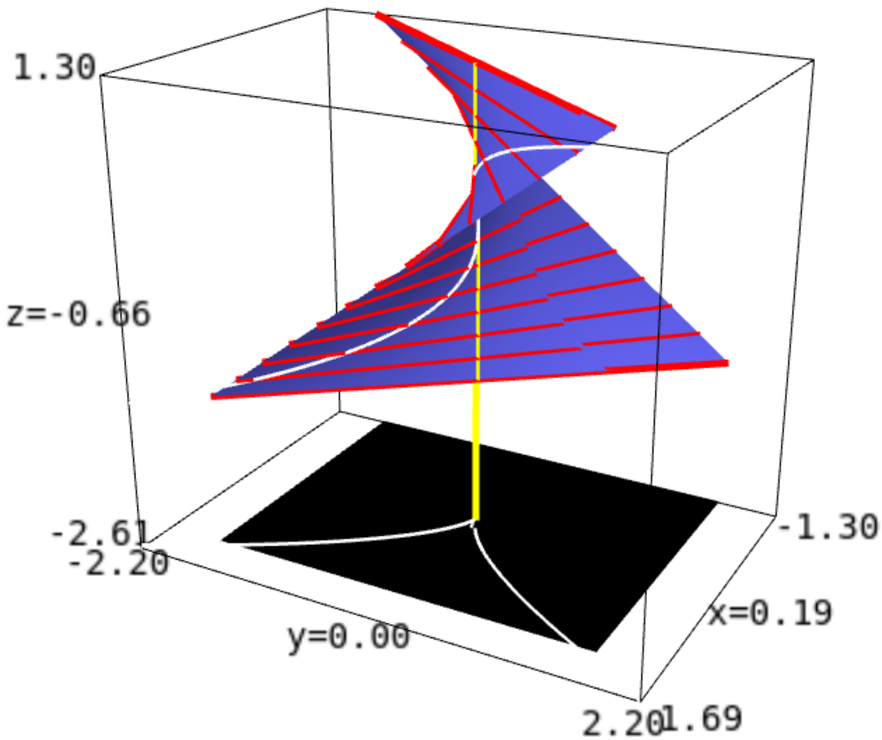

Figure 8: The blowup of the cuspidal cubic

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