Hartshorne Exercises

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Contents

I Varieties

Conventions. k is an algebraically closed field.

I.1 Affine Varieties

Conventions. $A = k[x_1, \ldots, x_n]$.

I.1.1 INCOMPLETE

(a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

(c) Let f be an irreducible quadratic polynomial in $k[x, y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

Proof. (a) $A(Y) = k[x, y]/(y - x^2)$. Consider $k[x, y]/(y - x^2) \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$. This is an isomorphism.

(b) $A(Z) = k[x, y]/(xy - 1) \cong k[x, x^{-1}] \ncong k[t]$.

INCOMPLETE $f = ax^2 + bx + cxy + dy + ey^2 + f = (ax^2 + cxy + ey^2) + (bx + dy + f)$. By a change of variables we can diagonalize the homogeneous summand to a sum of squares. As k is algebraically closed it can be factored into linear polynomials, so after a change of variables we get $f = (Ax + By)(Cx + Dy) + (bx + dy + f)$.

something something nondegenerate conics something something matrix form

change variables something something homogeneous linear terms are lin dep/lin indep

See here for char not 2 and here for char 2. Alternatively, here (pdf) does this with projective geometry - pick 3 points on any conic and move them to $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$

□

I.1.2

The Twisted Cubic Curve. Let $Y \subseteq \mathbb{A}^3$ be the set $Y = \{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Proof. Take coordinates x, y, z and observe that Y satisfies the equations $z = x^3, y = x^2$. Then let $I = (z - x^3, y - x^2)$. We claim therefore that $Y = Z(I)$. Indeed, we certainly have $Y \subseteq Z(I)$. On the other hand, let $(a, b, c) \in Z(I)$. Then $c = a^3, b = a^2$ so $(a, b, c) = (a, a^2, a^3) \in Y$. Hence, $Y = Z(I)$ and is therefore an algebraic set.

Consider $k[x,y,z] \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$. The kernel is the f such that $f(t,t^2,t^3) = 0$, which is precisely $I(Y)$ (algebraically closed fields are infinite and polynomials correspond to polynomial functions over an infinite field). Hence, $A(Y) \cong k[t]$. Hence, Y is an affine variety of dimension 1.

All that is left to do is to find generators for $I(Y)$. We claim therefore that $I = I(Y)$ so that the above generators work. Indeed, let $f \in I(Y)$. Then $f(t,t^2,t^3) = 0$. Write $f = \sum a_{ijk}x^iy^jz^k$. This condition therefore says that $\sum a_{ijk}t^{i+2j+3k} = 0$. Furthermore, in the quotient $k[x,y,z]/(z-x^3, y-x^2)$, $f = \sum a_{ijk}x^{i+2j+3k}$. This equals 0 by the above, so $f \in I$ as desired. $\square$

I.1.3

Let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is the union of three irreducible components. Describe them and find their prime ideals.

Proof. By definition, $Y = Z(x^2 - yz) \cap Z(xz - x)$.

$$\begin{aligned}
Z(x^2 - yz) \cap Z(xz - x) &= Z(x^2 - yz) \cap Z((x)(z-1)) \\
&= Z(x^2 - yz) \cap (Z(x) \cup Z(z-1)) \\
&= Z(x^2 - yz) \cap Z(x) \cup Z(x^2 - yz) \cap Z(z-1) \\
&= Z(x^2 - yz, x) \cup Z(x^2 - yz, z-1) \\
&= Z(yz, x) \cup Z(x^2 - yz, z-1) \\
&= Z(yz, x) \cup Z(x^2 - yz + y(z-1), z-1) \\
&= Z(yz, x) \cup Z(x^2 - y, z-1) \\
&= Z(yz) \cap Z(x) \cup Z(x^2 - y, z-1) \\
&= (Z(y) \cup Z(z)) \cap Z(x) \cup Z(x^2 - y, z-1) \\
&= (Z(y) \cap Z(x) \cup Z(z) \cap Z(x)) \cup Z(x^2 - y, z-1) \\
&= Z(x,y) \cup Z(x,z) \cup Z(x^2 - y, z-1)
\end{aligned}$$

We now seek to show that $(x^2-y, z-1)$ is prime. Indeed, define the map $k[x,y,z] \longrightarrow k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto 1$. This is an isomorphism and $k[t]$ is a domain. Thus, we have $Y = Z(x,y) \cup Z(x,z) \cup Z(x^2-y, z-1)$ a union of three irreducible components. Their prime ideals are as given and we now geometrically describe each one of these.

$Z(x,y) = \{(a,b,c) \in \mathbb{A}^3 \mid a = b = 0\}$ so this is just the z axis. Similarly, $Z(x,z)$ is the y axis. Finally, $Z(x^2 - y, z-1) = \{(a,b,c) \in \mathbb{A}^3 \mid a^2 = b, c = 1\} = \{(a,a^2,1)\}$. This is a parabola sitting in the plane $z = 1$ with vertex at $(0,0,1)$ opening in the y axis. Hence, Y is the union of two lines, the y and z axes, and the parabola just described. $\square$

I.1.4

If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, show that the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.

Proof. The diagonal of $\mathbb{A}^2$ is defined by $Z(x-y)$, which is closed. If these topologies agreed, then the diagonal of $\mathbb{A}^1 \times \mathbb{A}^1$ will be closed so $\mathbb{A}^1$ will be Hausdorff. However, the topology on $\mathbb{A}^1$ is the cofinite topology and is therefore not Hausdorff. $\square$

I.1.5

Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbb{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Proof. For $Y \subseteq \mathbb{A}^n$ closed, we know that $I(Y)$ is radical. Hence, $A(Y) = k[x_1, \ldots, x_n]/I(Y)$ is reduced finitely generated k-algebra.

On the other hand, let B be some reduced finitely generated k-algebra. Then we can write $B \cong k[x_1, \ldots, x_n]/I$. As B is reduced, I is radical, so by the Nullstellensatz we have $I(Z(I)) = \sqrt{I} = I$. Hence, $A(Z(I)) = k[x_1, \ldots, x_n]/I \cong B$. $\square$

I.1.6

Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

Proof. Let $\emptyset \neq U, V \subseteq X$ be open, X irreducible. We claim that $U \cap V \neq \emptyset$. If they were disjoint, $X = X - (U \cap V) = (X - U) \cup (X - V)$. These are proper closed sets so this contradicts irreducibility of X.

Now, let $Y \subseteq X$ be irreducible in the subspace topology. Let $\overline{Y} = A \cup B$ be closed sets. We want to show that one of A or B are equal to $\overline{Y}$. Indeed, $(A \cap Y) \cup (B \cap Y) = Y$, which is irreducible, so WLOG say $A \cap Y = Y$. Hence, $Y \subseteq A$ so $A \cup B = \overline{Y} \subseteq A$ and $A = \overline{Y}$, proving irreducibility. $\square$

I.1.7

(a) Show that the following conditions are equivalent for a topological space X:

  (i) X is noetherian;

  (ii) every nonempty family of closed subsets has a minimal element;

  (iii) X satisfies the ascending chain condition for open subsets;

  (iv) every nonempty family of open subsets has a maximal element.

(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Proof. (a) The equivalence of (i) and (ii) is the same argument from algebra. (i) is equivalent to (iii) and (ii) is equivalent to (iv) via taking complements, which reverses order.

(b) Let $X = \bigcup \mathcal{U}$ and let $S = \{\bigcup \mathcal{V} \mid \mathcal{V} \subseteq \mathcal{U} \text{ finite}\}$. Then by (a), S has a maximal element $U_1 \cup \cdots \cup U_n$. If there is an $x \in X - (U_1 \cup \cdots \cup U_n)$ then let $x \in U_{n+1}$ open. Then $U_1 \cup \cdots \cup U_n \subsetneq U_1 \cup \cdots \cup U_{n+1}$ a contradiction. Then $X = U_1 \cup \cdots \cup U_n$ and X is compact.

(c) Let $A \subseteq X$. Let $A_1 \supseteq A_2 \supseteq \ldots$ be a decreasing chain of closed sets in A. Then $A_i = \overline{A_i} \cap A$ and $\overline{A_1} \supseteq \overline{A_2} \supseteq \ldots$ is a decreasing chain in X, which is noetherian. The chain upstairs terminates so the chain downstairs does as well.

(d) Let X be noetherian and Hausdorff. Let A be an irreducible closed subset of X. Let $x, y \in A$ and $x \in U, y \in V$ open neighborhoods in A. $U \cap V \neq \emptyset$ by irreducibility, so $x = y$ by Hausdorffness. Thus, the irreducible components of X are singletons and X is finite. Finite Hausdorff spaces are discrete.

□

I.1.8

Let Y be an affine variety of dimension r in $\mathbb{A}^n$. Let H be a hypersurface in $\mathbb{A}^n$, and assume that $Y \not\subseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$.

Proof. Let $A = k[x_1, \ldots, x_n]$. As H is a hypersurface, $H = Z(f)$ for some f irreducible. Let $Y = Z(\mathfrak{p})$ with $\mathfrak{p}$ prime. Then $B = A/\mathfrak{p}$ is the coordinate ring of Y. Consider $\overline{f}$ in B. As $Y \not\subseteq H$, $(f) \not\subseteq \mathfrak{p}$. Thus, $f \notin \mathfrak{p}$ so $\overline{f}$ is nonzero in B.

Let $F \subseteq Y \cap H$ be an irreducible component. Then $F = Z(\mathfrak{q})$ for some prime $\mathfrak{q}$. Furthermore, every irreducible closed subset of $Y \cap H$ is contained in some irreducible component. In other words, irreducible components are maximal with respect to being an irreducible closed subset of $F \cap Y$. Thus, $\mathfrak{q}$ is minimal with respect to primes containing $\mathfrak{p} + (f)$ as $Y \cap H = Z(\mathfrak{p} + (f))$.

Observe that $A/(\mathfrak{p} + (f)) \cong B/(\overline{f})$. B is a domain and as discussed, $\overline{f} \neq 0$ in B. By [Eisenbud, 13.11], $\dim B/(\overline{f}) = \dim B - 1$ so by [Eisenbud, 13.4], $\operatorname{codim}(\overline{f}) = 1$. We are trying to compute $\dim A/\mathfrak{q} = \dim B/(\mathfrak{q}/\mathfrak{p})$. By [Eisenbud, 13.4], this is $\dim B - \operatorname{codim} \mathfrak{q}/\mathfrak{p}$. The claim is therefore that $\operatorname{codim} \mathfrak{q}/\mathfrak{p} = 1$. $f \in \mathfrak{q}$ so $\overline{f} \in \mathfrak{q}/\mathfrak{p}$. Furthermore, $\mathfrak{q}$ is minimal with respect to containing $\mathfrak{p}$ and f so $\mathfrak{q}/\mathfrak{p}$ is minimal with respect to containing $\overline{f}$. By Krull's principal ideal theorem, $\operatorname{codim} \mathfrak{q}/\mathfrak{p} \leq 1$. As $\operatorname{codim}(\overline{f}) = 1$ and $(\overline{f}) \subseteq \mathfrak{q}/\mathfrak{p}$, we have $\operatorname{codim} \mathfrak{q}/\mathfrak{p} = 1$. In summary,

$$\begin{aligned}
\dim F &= \dim A/\mathfrak{q} \\
&= \dim B/(\mathfrak{q}/\mathfrak{p}) \\
&= \dim B - \operatorname{codim} \mathfrak{q}/\mathfrak{p} \\
&= \dim B - 1 \\
&= \dim Y - 1
\end{aligned}$$

□

I.1.9

Let $\mathfrak{a} \subseteq A = k[x_1 ..., x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geq n - r$.

Proof. Let F be an irreducible component of $Z(\mathfrak{a})$. Then $F = Z(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. As F is an irreducible component, it is maximal with respect to being an irreducible closed subset of $Z(\mathfrak{a})$ and $\mathfrak{p}$ is therefore minimal with respect to containing $\mathfrak{a}$. $\mathfrak{a}$ can be generated by r elements so by Krull's height theorem (a generalization of the principal ideal theorem), $\operatorname{codim} p \leq r$. Using [Eisenbud, 13.4], we arrive at the equation $\dim A/\mathfrak{p} + \operatorname{codim} p = \dim A$. Of course, $\dim A = n$ and $\dim A/\mathfrak{p} = \dim F$. Hence, $\dim F + \operatorname{codim} p = n$. As $\operatorname{codim} p \leq r$, $\dim F \geq n - r$. □

I.1.10

(a) If Y is any subset of a topological space X, then $\dim Y \leq \dim X$.

(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

(e) Give an example of a noetherian topological space of infinite dimension.

Proof. (a) Take a chain of irreducible closed subsets $Z_0 < \cdots < Z_r \subseteq Y$. The closures $\overline{Z_i}$ are irreducible and $\overline{Z_i} < \overline{Z_{i+1}}$ as $Z_i = \overline{Z_i} \cap Y$. Thus, every chain of irreducible closed subsets of Y lifts to a chain of irreducible closed subsets of X with the same length. Hence, $\dim Y \leq \dim X$.

(b) By (a) $\dim U_i \leq \dim X$. Hence, $\sup \dim U_i \leq \dim X$. We claim that if $Z_0 < \cdots < Z_n$ is a chain of irreducible closed subsets of X then some U_i has a chain of the same length. This will show that $\dim X \leq \sup \dim U_i$, proving equality. Let $U_i \cap Z_0 \neq \emptyset$. We claim $Z_j \cap U_i < Z_{j+1} \cap U_i$, which will become our chain in U_i of the same length. Indeed, $Z_j \cap U_i \subseteq Z_j$ is a nonempty open subset and is therefore dense and irreducible. Hence, $\overline{Z_j \cap U_i} = Z_j$. Hence, if $Z_j \cap U_i = Z_{j+1} \cap U_i$ then their closures are equal and $Z_j = Z_{j+1}$ a contradiction.

(c) Consider the space $X = \{a, b, c\}$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\}$. The closed sets are therefore $\{\{a, b, c\}, \{b, c\}, \{c\}, \emptyset\}$. Hence, $\{c\} < \{b, c\}$ is a chain of irreducible closed subsets so $\dim X \geq 1$. Furthermore, $\{a\}$ is an open generic point (it intersects all nonempty open subsets) so it is a dense open set with dimension $0 < 1$.

(d) Take a maximal chain $Z_0 < \cdots < Z_r$ in Y. Then these Z_i are closed and irreducible in X. Hence, we have the chain $Z_0 < \cdots < Z_r \subseteq X$. As $\dim X = \dim Y = r$, $X = Z_r \subseteq Y$.

(e) Let R be a noetherian ring of infinite dimension (see the example due to Nagata [Stacks, 02JC]). Then $\operatorname{Spec} R$ is a noetherian topological space of infinite dimension.

□

I.1.11

Let $Y \subseteq \mathbb{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection*—cf. (Ex. I.2.17).

Proof. First of all, consider the map $k[x, y, z] \longrightarrow k[t]$ via $x \mapsto t^3$, $y \mapsto t^4$, $z \mapsto t^5$. As discussed previously, we are justified in using the same name for the variables of the coordinate ring and the coordinates on affine space themselves because for an infinite field (such as the algebraically closed field k we work over) polynomials are identified with polynomials. This map has kernel $I(Y)$ by definition. Its image is $k[t^3, t^4, t^5]$, a domain, so $I(Y)$ is prime. Furthermore, $\dim k[x, y, z]/I(Y) = 3 - \operatorname{codim} I(Y)$. Hence, to compute the codimension of $I(Y)$, we need only compute the dimension of $k[x, y, z]/I(Y) \cong k[t^3, t^4, t^5]$. Observe that $k[t^3, t^4, t^5] \subseteq k[t]$ is integral. Indeed, t satisfies the monic polynomial $s^3 - t^3$. Hence, its dimension is $\dim k[t] = 1$. Then $\operatorname{codim} I(Y) = 2$ as desired.

Let $\mathfrak{m} = (x, y, z)$. We claim that $(S)/\mathfrak{m}(S) \cong \sum_{s \in S} k \cdot s$ for any $S \subseteq k[x, y, z]$. Indeed, any element of $(S)/\mathfrak{m}(S)$ is represented by some $\sum f_i s_i$. Let's analyze one such $fs = \left(\sum a_{ijk} x^i y^j z^k\right) s$. For all $(i, j, k) \neq (0, 0, 0)$, $a_{ijk} x^i y^j z^k \in \mathfrak{m}$. Hence, $fs \equiv a_{000} s \pmod{\mathfrak{m}(S)}$. Thus, $\sum f_i s_i \equiv \sum f_i(0) s_i \pmod{\mathfrak{m}(S)}$. In other words, every element of $(S)/\mathfrak{m}(S)$ is represented by some element of $\sum_{s \in S} k \cdot s$. On the other hand, if $\sum a_i s_i \equiv \sum b_j s_j \pmod{\mathfrak{m}(S)}$ for $a_i, b_j \in k$ then $\sum a_i s_i - \sum b_j s_j \in \mathfrak{m}(S)$. Hence, evaluating the a_i, b_j at 0 sends this to 0. These are constants, so the difference is 0 and $\sum a_i s_i = \sum b_j s_j$. This proves the isomorphism.

Observe that $x^3 - zy, y^2 - xz, z^2 - x^2 y \in I(Y) = I$ all have different x degrees. Hence, they are k linearly independent. Let $(S) = I$ with $x^3 - zy, y^2 - xz, z^2 - x^2 y \in S$. Then as above, $I/\mathfrak{m}I \cong \sum_{s \in S} k \cdot s$. As $\{x^3 - zy, y^2 - xz, z^2 - x^2 y\}$ is k linearly independent, $\sum_{s \in S} k \cdot s$ has dimension at least 3. Furthermore, the above paragraph shows that any generating set of I spans $I/\mathfrak{m}I$. Hence, any generating set of I must have at least $\dim I/\mathfrak{m}I \geq 3$ elements. □

I.1.12

Give an example of an irreducible polynomial $f \in \mathbb{R}[x, y]$, whose zero set $Z(f)$ in $\mathbb{A}^2$ is not irreducible.

Proof. Consider the irreducible polynomial $f = x^2 + y^2 + 1$. Its zero set is empty and therefore not irreducible.

⟨Let's find a less trivial example - such an f with $Z(f) \neq \emptyset$.⟩ □

I.2 Projective Varieties

Conventions. $S = k[x_0, \dots, x_n]$.

I.2.1

Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with $\deg f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbb{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [*Hint*: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, [Hartshorne, 1.3A].

Proof. $\mathbb{A}^{n+1}$ has coordinate ring S. For notational ease, I will denote $V(\mathfrak{a}) \subseteq \mathbb{A}^{n+1}$ to be the affine variety defined by $\mathfrak{a}$, as opposed to $Z(\mathfrak{a})$ which refers heretofor only to the projective variety. The given condition on f is that for all $P \in Z(\mathfrak{a})$, $f(P) = 0$. By definition, $f(P) = 0$ means that $f(a_0, \dots, a_n) = 0$ for any homogeneous coordinates $(a_0, \dots, a_n)$ of P. Take now some $(a_0, \dots, a_n) \in V(\mathfrak{a})$. We claim that $f(a_0, \dots, a_n) = 0$. As f is homogeneous, $f(0) = 0$ so suppose $(a_0, \dots, a_n)$ is nonzero and let $P \in \mathbb{P}^n$ be the point it represents. Let $g \in \mathfrak{a}$ be homogeneous. Then $g(a_0, \dots, a_n) = 0$ so $g(P) = 0$ and $P \in Z(\mathfrak{a})$. Hence, $f(P) = 0$ so by definition, $f(a_0, \dots, a_n) = 0$. Thus, $f \in I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ by the Nullstellensatz. □

I.2.2

For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \emptyset$;

(ii) $\sqrt{\mathfrak{a}} =$ either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;

(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Proof. Note that $S - S_+ = k - 0$. Hence, statement (ii) is equivalent to $\sqrt{a} \supseteq S_+$.

(iii) $\implies$ *(i)*. If $\mathfrak{a} \supseteq S_d$ then $x_i^d \in \mathfrak{a}$ for all i. Hence, $Z(\mathfrak{a}) \subseteq \cap Z(x_i^d) = \emptyset$.

(i) $\implies$ *(ii)*. If $\deg f > 0$ then $f \in S_+$. Furthermore, it holds vacuously that for all $P \in Z(\mathfrak{a}) = \emptyset$, $f(P) = 0$. Hence, by the homogeneous Nullstellensatz, $f^n \in \mathfrak{a}$ for some n. Hence, $\sqrt{\mathfrak{a}} \supseteq S_+$. Thus, $\mathfrak{a} = S_+$ or $\mathfrak{a} = S$ as $S - S_+ = k$. Note that we applied the homogeneous Nullstellensatz on $Z(\mathfrak{a}) = \emptyset$. This is justified as $Z(\mathfrak{a}) = \emptyset$ implies that $V(\mathfrak{a}) - 0 = \emptyset$. Indeed, letting $\pi : \mathbb{A}^{n+1} \longrightarrow \mathbb{P}^n$, $\pi^{-1}[Z(\mathfrak{a})] = V(\mathfrak{a}) - 0$. With this formula in mind, we can see that the proof of I.2.1 is valid for the empty set.

(ii) $\implies$ *(iii)*. Let $\sqrt{\mathfrak{a}} \supseteq S_+$. Then $x_i \in \sqrt{\mathfrak{a}}$ for all i. Hence, $x_i^{d_i} \in \mathfrak{a}$ for some d_i. Let $d = \sum d_i$. Now, let $x_0^{k_0} x_1^{k_1} \cdots x_n^{k_n}$ be a generic (modulo constants) degree d monomial. That is, $\sum k_i = d$. If all $k_i > d_i$ then $d = \sum k_i > \sum d_i = d$. Hence, some $k_i \leq d_i$ so the term $x_i^{d_i}$ appears in this expression. Hence, $x_0^{k_)} x_1^{k_1} \cdots x_n^{d_n} \in \mathfrak{a}$. Hence, $S_d \subseteq \mathfrak{a}$. □

I.2.3

(a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

(c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \emptyset$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

(e) For any subset $Y \subseteq \mathbb{P}^n$, $Z(I(Y)) = \overline{Y}$.

Proof. (a) Let $P \in Z(T_2)$ and $f \in T_1 \subseteq T_2$. Then $f(P) = 0$ so $P \in Z(T_1)$.

(b) Let $f \in I(Y_2)$ and $P \in Y_1 \subseteq Y_2$. Then $f(P) = 0$ so $f \in I(Y_1)$.

(c) Each $Y_i \subseteq Y_1 \cup Y_2$ so by part (b), $I(Y_i) \supseteq I(Y_1 \cup Y_2)$. Hence, $I(Y_1) \cap I(Y_2) \supseteq I(Y_1 \cup Y_2)$. On the other hand,let $f \in (I(Y_1) \cap I(Y_2))^h$. Then $f[Y_1], f[Y_2] \subseteq \{0\}$. Thus, $f[Y_1 \cup Y_2] \subseteq \{0\}$ so $f \in I(Y_1 \cup Y_2)^h$. As these ideals are homogeneous, the homogeneous elements generate so we have $I(Y_1) \cap I(Y_2) \subseteq I(Y_1 \cup Y_2)$.

(d) Let $f \in I(Z(\mathfrak{a}))^h$ with $\deg f > 0$. Then by definition, for all $P \in Z(\mathfrak{a})$, $f(P) = 0$. Then by the homogenous Nullstellensatz (I.1.1), $f \in \sqrt{\mathfrak{a}}$. Hence, 0 and all nonconstant homogeneous polynomials $f \in I(Z(\mathfrak{a}))$ are in $\sqrt{a}$. Furthermore, as $Z(\mathfrak{a}) \neq 0$, $k \cap I(Z(\mathfrak{a}))^h = \{0\}$. Of course, 0 is also in $I(Z(\mathfrak{a}))^h$ and $\sqrt{\mathfrak{a}}$. Hence, $I(Z(\mathfrak{a}))^h \subseteq \sqrt{\mathfrak{a}}$. As all these ideals are homogeneous, this proves $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$. Of course, if $f^n(P) = 0$ then $f(P) = 0$. Thus, $I(Z(\mathfrak{a}))$ is radical and $\mathfrak{a} \subseteq I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ so we achieve equality.

(e) We of course have $Y \subseteq Z(I(Y))$, so $\overline{Y} \subseteq Z(I(Y))$. On the other hand, let $Y \subseteq Z(\mathfrak{a})$ for some homogeneous ideal $\mathfrak{a}$. By definition, this is a generic closed set containing Y. Furthermore, this means that for all $P \in Y$ and $f \in \mathfrak{a}^h$, $f(P) = 0$. Hence, $\mathfrak{a}^h \subseteq I(Y)$. As $\mathfrak{a}$ is homogeneous, $\mathfrak{a} \subseteq I(Y)$ so $Z(\mathfrak{a}) \supseteq Z(I(Y))$. As $Z(\mathfrak{a})$ was arbitrary, $Z(I(Y)) = \overline{Y}$.

□

I.2.4

(a) There is a $1-1$ inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note*: Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

(b) An algebraic set $Y \subseteq \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

(c) Show that $\mathbb{P}^n$ itself is irreducible.

Proof. (a) $Z(\mathfrak{a})$ is, by definition, always algebraic. Furthermore, it's easy to see that $I(Y)$ is always radical and homogeneous. We must show then that $I(Y)$ can never equal S_+ for Y algebraic. Indeed, if it was the case that $I(Z(\mathfrak{a}))) = S_+$then $x_i \in I(Z(\mathfrak{a}))$ for all i. Thus, each x_i sends all of $Z(\mathfrak{a})$ to 0. But $\bigcap Z(x_i) = \emptyset$ so $Z(\mathfrak{a}) = \emptyset$. However, $I(\emptyset) = S \neq S_+$.

That I and Z are inverses on these restricted domains follows from parts (d) and (e) of problem I.2.3.

(b) Let $I(Y)$ be prime. Then let $Y \subseteq Z(I_1) \cup Z(I_2)$. Hence, $I(Y) \supseteq I(Z(I_1) \cup Z(I_2))$. By I.2.3 part (c), this is $I(Z(I_1)) \cap I(Z(I_2)) = \sqrt{I}_2 \cap \sqrt{I}_2 \supseteq I_1 I_2$. Hence, $I(Y) \supseteq I_1 I_2$. It is then a general fact of commutative algebra that some $I_j \subseteq I(Y)$. Indeed, if neither is contained in $I(Y)$ then let $a_j \in I_j - I(Y)$. $a_1 a_2 \in I_1 I_2$ but cannot be in $I(Y)$ as it is prime, a contradiction.

On the other hand, let Y be irreducible. Let f, g be homogeneous such that $fg \in I(Y)^h$. Then $Y \subseteq Z(fg) \subseteq Z(f) \cup Z(g)$. Then as Y is irreducible, it is contained in one of these, WLOG say $Y \subseteq Z(f)$. Hence, $f \in I(Y)$. As $I(Y)$ is homogeneous, this proves its primality.

(c) $\mathbb{P}^n = I(0)$.

□

I.2.5

1. $\mathbb{P}^n$ is a Noetherian topological space.

2. Every algebraic set in $\mathbb{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

Proof. 1. Let $Y_1 > Y_2 > \dots$ be closed subsets of $\mathbb{P}^n$. Then we have $I(Y_1) < I(Y_2) < \dots$ in the coordinate ring $k[x_0, \dots, x_n]$. But this ring is Noetherian - contradiction.

2. First note that a closed subspace of a Noetherian space is itself Noetherian. Indeed, a chain in the subspace is a chain in the superspace. So we prove this result for a general Noetherian space X.

If X is irreducible then we are done. Else, we have $X = X_0 \cup X_1$ for some proper closed subsets X_i. We can proceed this process to $X_0 = X_{00} \cup X_{01}$ and $X_1 = X_{10} \cup X_{11}$. Continue this until everything in sight is irreducible. This will happen in finitely many steps, as if not there is some binary string $b_0 b_1 \dots$ such that $X_{b_0} > X_{b_0 b_1} > \dots$ - contradicting Noetheriannessitudity. Now, if $X = X_0 \cup \dots \cup X_n$ are irreducible, we can omit any containments so suppose no $X_i \subseteq X_j$. If in addition we have $X = Y_0 \cup \dots \cup Y_m$ an irreducible decomposition. Then each $X_i \subseteq Y_{\sigma(i))}$ for some $0 \leq \sigma(i) \leq m$. Similarly, each $Y_j \subseteq X_{\tau(j)}$ for some $0 \leq \tau(j) \leq n$. Then $X_i \subseteq X_{\tau(\sigma(i))}$, so by assumption $X_i = X_{\tau(\sigma(i))}$. Similarly, $Y_j = Y_{\sigma(\tau(i))}$. Thus, these are inverses so these decompositions are the same up to permutation.

□

I.2.6

If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\varphi_i : U_i \longrightarrow \mathbb{A}^n$ be the homeomorphism of [Hartshorne, 2.2], let Y_i be the affine variety $\phi_i[Y \cap U_i]$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use [Hartshorne, 1.7], [Hartshorne, 1.8A], and (Ex I.1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

Proof. We will follow this hint. We want to identity $A(Y_i)$ with $(S(Y)_{x_i})^0$, which is the set of all $\frac{f}{x_i^d}$ such that $f \in k[x_0, \dots, x_n]$ is homogeneous of degree d. We will consider the coordinate ring $A(Y_i)$ to have coordinates $x_0, \dots, \widehat{x_i}, \dots, x_n$. We define the map

$$\begin{aligned} k[x_0, \dots, \widehat{x_i}, \dots, x_n] &\longrightarrow (S(Y)_{x_i})^0 \\ f &\mapsto f\left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i}\right) \end{aligned}$$

It's not hard to see that this lands in the degree 0 subring. We want this map to factor through $A(Y_i)$. What then is $J(Y_i)$? By definition, it is the $f \in k[a_0, \dots, \widehat{a_i}, \dots, a_n]$ such that $f(a) = 0$ for all $a \in Y_i$. Of course, $Y_i = \phi_i[Y \cap U_i]$ so any such a will look like $\left(\frac{a_0}{a_1}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right)$. Indeed, take some $f \in J(Y_i)$. Then f is mapped to $\frac{\beta_i(f)}{a_i^d}$ where $\beta_i(f)$ is the homogenization of f with respect to a_i, i.e. $\beta_i(f) = a_i^d f\left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right)$. Here we of course mean $d = \deg f$. Now, if $d = 0$ then f is constant and vanishes on Y_i, so it must be 0 (or Y_i must be empty but this is a case we ignore). If $d \neq 0$ then consider some $[a_0 : \dots : a_n] \in Y$. If $a_i = 0$ then $\beta_i(f)([a_0 : \dots : a_n]) = a_i^d f\left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right) = 0$ as $d > 0$. On the other hand, if $a_i \neq 0$ then the term $f\left(\frac{a_0}{a_i}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right) = 0$ as $f \in J(Y_i)$. In any case, we see therefore that $\beta_i(f) \in I(Y)$. Hence, $\frac{\beta_i(f)}{x_i^d} \in I(Y)_{x_i}$. By flatness of localization, $S(Y)_{x_i} = k[x_0, \dots, x_n]_{x_i} / I(Y)_{x_i}$. Thus, we can quotient this map to yield $A(Y_i) \longrightarrow (S(Y)_{x_i})^0$.

We now seek to show that this map is an isomorphism, which we will do by exhibiting aninverse. Recall the "dehomogenization" of f with respect to x_i is $\alpha_i(f) = f(x_0, \dots, 1, \dots, x_n)$ with 1 in the i^{th} position. We consider the map $(S(Y)_{x_i})^0 \longrightarrow A(Y_i)$ via $\frac{f}{x_i^d} \mapsto \alpha_i(f)$, where of course f is taken to be homogeneous and degree d. In other words, this map evaluates a rational function at $(x_0, \dots, 1, \dots, x_n)$.

We first have to show that this map is actually well defined on these quotients. Indeed, let's begin with some $f \in I(Y)_{x_i}$ of homogeneous degree 0, so $f = \frac{g}{x_i^d}$ with $g \in I(Y)^h$ of degree d. Then this maps to $g(x_0, \dots, 1, \dots, x_n)$. Take some $\left(\frac{a_0}{a_1}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right) \in Y_i$. Then $g(x_0, \dots, 1, \dots, x_n)\left(\frac{a_0}{a_1}, \dots, \frac{\widehat{a_i}}{a_i}, \dots, \frac{a_n}{a_i}\right) = g\left(\frac{a_0}{a_i}, \dots, 1, \dots, \frac{a_n}{a_i}\right) = 0$ as $g \in I(Y)^h$. Thus, this does descend to a well defined map on the quotient $(S(Y)_{x_i})^0$.

It is now easy enough to verify that these maps are inverse. Let $f \in A(Y_i)$. Then $f \mapsto \frac{\beta_i(f)}{x_i^d} \mapsto \alpha_i(\beta_i(f))$. This is $f\left(\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right)(x_0, \ldots, 1, \ldots, x_n) = f$. On the other hand, let $\frac{f}{x_i^d} \in (S(Y)_{x_i})^0$. Then $\frac{f}{x_i^d} \mapsto \alpha_i(f) \mapsto \frac{\beta_i(\alpha_i(f))}{x_i^d}$. The numerator here is $\beta_i(f(x_0, \ldots, 1, \ldots, x_n)) = x_i^d f\left(\frac{x_0}{x_i}, \ldots, \frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}\right) = f$ as f is homogeneous of degree f.

In summary, we have defined the following maps:

$$\begin{array}{ccc} x_j & \longmapsto & \frac{x_j}{x_i} \\ A(Y_i) & \xleftrightarrow{\sim} & (S(Y)_{x_i})^0 \\ \begin{cases} 1 & i = j \\ x_j & i \neq j \end{cases} & \longleftarrow & x_j \end{array}$$

Next, we will extend this to a map $A(Y_i)[x_i, x_i^{-1}] \longrightarrow S(Y)_{x_i}$ via sending $x_i \mapsto x_i$. The image of this map contains x_i, x_i^{-1} and $(S(Y)_{x_i})^0$. Given f homogeneous, $\frac{f}{x_i^d} \in (S(Y)_{x_i})^0$, so $f \in (S(Y)_{x_i})^0[x_i]$. Hence, this map is onto. By flatness, it suffices to show that the map $A(Y_i)[x_i] \longrightarrow S(Y)_{x_i}$ is injective.

Recall that each $\phi(f_k)$ has degree 0 in $S(Y)_{x_i}$. Then $\phi(f_k)x_i^k$ has degree k in this ring, so each term in $\sum \phi(f_k)x^k$ has different degree, so these are necessarily linearly independent. Hence, for this sum to vanish, each term $\phi(f_k)x_i^k$ must vanish as well. Since we are in a domain (Y is a variety), this means that each $\phi(f_k) = 0$. ϕ has already been shown to be an isomorphism $A(Y_i) \longrightarrow (S(Y)_{x_i})^0$, so each f_k is therefore 0. Thus, the extension of ϕ to $A(Y_i)[x_i, x_i^{-1}] \longrightarrow S(Y)_{x_i}$ is an isomorphism as well.

Now, to compute $\dim S(Y)$ we need to compute $\dim A(Y_i)[x_i, x_i^{-1}]$, which is equal to the transcendence degree of $qf(A(Y_i)[x_i]) = qf(A(Y_i))(x_i)$. x_i was a formal variable for $A(Y_i)$, as we took A to have coordinates $x_0, \ldots, \widehat{x_i}, \ldots, x_n$. Hence, $\operatorname{trdeg}_k qf(A(Y_i))(x_i) = \operatorname{trdeg}_k qf(A(Y_i)) + 1 = \dim A(Y_i) + 1$. By I.1.10(b), $\dim Y = \sup \dim Y_i$. Y_i is empty when $Y \cap U_i = \emptyset$, i.e. when $Y \subseteq Z(x_i)$. In that case, $x_i \in I(Y)$ so $x_i = 0$ in $S(Y)_{x_i}$. When this is not the case (and it must not be the case for some x_i), then $x_i \neq 0$ in $S(Y)$ so we have inclusions $S(Y) \subseteq S(Y)_{x_i} \subseteq qf(S(Y))$. We have computed that $qf(S(Y)_{x_i}) = qf(A(Y_i))(x_i)$, which has transcendence degree $\dim A(Y_i) + 1$, which is $\dim Y_i + 1$. Thus, for all i such that $Y_i \neq \emptyset$, we have $\dim Y_i = \dim S(Y) - 1$. Taking sups yields $\dim Y = \dim S(Y) - 1$, and that $\dim Y = \dim Y_i$ whenever $Y_i \neq \emptyset$. □

I.2.7

(a) $\dim \mathbb{P}^n = n$.

(b) If $Y \subseteq \mathbb{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \overline{Y}$.
Hint: Use (Ex. I.2.6) to reduce to [Hartshorne, 1.10].

Proof. (a) $S(\mathbb{P}^n) = k[x_0, \ldots, x_n]$ has dimension $n + 1$ so this follows from I.2.6.

(b) By I.1.10(b), $\dim \overline{Y} = \sup \dim \overline{Y} \cap U_i$. By [Hartshorne, 1.10], $\dim Y \cap U_i = \dim \overline{Y \cap U_i}$, where the closure is computed in $U_i \cong \mathbb{A}^n$. Furthermore, $\overline{Y \cap U_i} = \overline{Y} \cap U_i$, where on the right hand side the closure is in $\mathbb{P}^n$. This is an elementary fact about the subspace topology. In other words, we have $\dim \overline{Y} \cap U_i = \dim Y \cap U_i$. Taking sup on both sides and using I.1.10(b) we get the equality $\dim \overline{Y} = \dim Y$.

□

I.2.8

A projective variety $Y \subseteq \mathbb{P}^n$ has dimension $n-1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbb{P}^n$.

Proof. ($\Longleftarrow$). The ideal (f) has height 1 so by I.2.6, $\dim Z(f) = \dim S/(f) - 1 = (n+1) - 1 - 1 = n-1$.

($\Longrightarrow$). Let $Y = Z(\mathfrak{p})$ have dimension $n-1$. Then by I.2.6, $\dim S(Y) = n$. As $\dim S(Y) = \dim S - \operatorname{codim} \mathfrak{p}$, this tells us that $\operatorname{codim} \mathfrak{p} = 1$. As S is a UFD, this means that $\mathfrak{p} = (f)$ for some $\deg f > 0$.

We must therefore show that f is homogeneous. As the ideal $(f) = \mathfrak{p}$ is homogeneous, we can decompose $(f) = \bigoplus (f) \cap S_d$. Consider therefore the homogeneous decomposition $f = \sum f_e$. As (f) is homogeneous, each $f_e \in (f)$. Furthermore, as (f) is not the unit ideal, $(f) \cap S_0 = 0$. As $(f) \neq 0$, we must have some $d > 0$ such that $(f) \cap S_d \neq 0$. Let d be minimal with respect to this property and let $g \in (f) \cap S_d - 0$. Then $g \in (f)$ so $f \mid g$. Write $fh = g$, so in the homogeneous decompositions we get $(\sum f_e)(\sum h_e) = g_d = g$. We can compute this sum as follows:

$$\begin{aligned}\left(\sum f_e\right)\left(\sum h_e\right) &= \sum_{e,e'} f_e h_{e'} \\ &= \sum_{k \geq 0} \sum_{k=e+e'} f_e h_{e'}\end{aligned}$$

So we have $\sum_{e+e'=d} f_e h_{e'} = g_d$ and all other terms are 0. We additionally have $f_0 = 0$ as $f_0 \in (f) \cap S_0 = 0$. Furthermore, by minimality of d, there can be no $0 < e < d$ such that $f_e \neq 0$. Hence, for all $0 \leq e < d$, $f_e = 0$. As $f \mid g$, f cannot have any terms of degree higher than d. Thus, $f = f_d$ is homogeneous. □

I.2.9 INCOMPLETE

Projective Closure of an Affine Variety. If $Y \subseteq \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbb{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(Y)$ is the ideal generated by $\beta[J(Y)]$, using the notation of the proof of [Hartshorne, 2.2].

(b) Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic of (Ex. I.1.2). Its projective closure $\overline{Y} \subseteq \mathbb{P}^3$ is called the twisted cubic curve in $\mathbb{P}^3$. Find generators for $J(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $J(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

Proof. (a) First and foremost, observe that $I(\overline{Y}) = I(Z(I(Y))) = \sqrt{(}I(Y)) = I(Y)$. Hence, we will focus our attention on $I(Y)$.

Recall the definition of $\beta(f) = \beta_0(f) = x_0^d f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$ where $d = \deg f$. Let $f \in J(Y)$. Then for all $[a_0 : \cdots : a_n] \in Y$ (by which we mean $\phi^{-1}[Y]$), $0 = f(\phi([a_0 : \cdots : a_n])) = f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)$. Then indeed, $\beta(f) \in I(Y)^h$. This yields the easier inclusion $(\beta[J(Y)]) \subseteq I(Y)$.

On the other hand, let $f \in I(Y)^h$. Then for all $\phi^{-1}(a_1, \ldots, a_n) = [1 : a_1 : \cdots : a_n]$, $f([1 : a_1 : \cdots : a_n])$. Letting $\alpha(f) = \alpha_0(f) = f(1, x_1, \ldots, x_n)$ this says that $\alpha(f) \in J(Y)$. Naively, we'd just apply β and go home, but tragically this fails. Indeed, consider $\beta(\alpha(x_0^d)) = \beta(1) = 1$. α has the potential to lose the data of the x_0, but all hope is not lost. Consider $\beta(\alpha(x_0 + x_1)) = \beta(1 + x_1) = x_0 + x_1$, so the issue seems to be with powers of x_0. Wih that in mind, let $x_0^e \mid\mid f$ and write $f = x_0^e g$. Then $\alpha(g) = \alpha(f)$, so we'll try to compute $\beta(\alpha(g))$.

Indeed, $\alpha(g) = g(1, x_1, \ldots, x_n)$ so $\beta(\alpha(g)) = g\left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)x_0^d$. Here, $d = \deg \alpha(g)$ As f is homogeneous and $g = \frac{f}{x_0^e}$, g is homogeneous as well. Thus, if it were the case that $\deg g = d$ then this would be precisely g. Indeed, as $x_0 \nmid g$ there must be some monomial summand of g which has no x_0 term. This term has the same degree as g, as g is homogeneous. Furthermore, applying α to such a term leaves it unchanged. As α cannot increase degree, this means that the degree of $\alpha(g)$ is indeed equal to the degree of g, so $g = \beta(\alpha(g))$.

As discussed, this means that $g = \beta(\alpha(g)) = \beta(\alpha(g)) \in \beta[J(Y)]$. Hence, $f = x_0^e g \in (\beta[J(Y)])$, so $I(Y)^h \subseteq (\beta[J(Y)]) \subseteq I(Y)$. As $I(Y)$ is homogeneous, we get the desired equality $I(Y) = (\beta[J(Y)])$.

(b) The twisted cubic was defined by $Y = \{(t, t^2, t^3) : t \in k\} \subseteq \mathbb{A}^3$. As shown in I.1.2, Z is a variety with ideal $I(Y) = (x_3^3 - x_1, x_2^2 - x_1)$. Applying β to these yields $x_3^3 - x_0^2 x_1$ and $x_2^2 - x_0 x_1$. We therefore seek to prove that $I(\overline{Y}) > (x_3^3 - x_0^2 x_1, x_2^2 - x_0 x_1)$.

It'd be nice to get some kind of picture here, so in 1 below we show a plot of the twisted cubic curve in affine space.

Now let's try to visualize this in $\mathbb{P}^3$. We'll appeal to the usual CW complex structure on $\mathbb{RP}^n$. Indeed, we consider $\mathbb{P}^3$ to be the unit ball in $\mathbb{R}^3$ where the boundary sphere has the usual antipoal gluing. If we scale the plot above into the open unit disk, then we get 2

This suggests that the closure in $\mathbb{P}^3$ should be the affine twisted cubic along with this blue line $[0 : 0 : 0 : 1]$, and that this line is approached by the affine twisted cubic as one "tends to infinity." How do we make rigorous this idea of tending to infinity? Well the twisted cubic is parametrized by $\mathbb{A}^1$, so it stands to reason that to parametrize its projective closure, and hence to "tend to infinity" that we would want a parametrization by $\mathbb{P}^1$. Indeed, the affine parametrization is given by $t \mapsto (t, t^2, t^3)$. We'll homogenenize this to get a map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ via $[t_0 : t_1] \mapsto [t_0^3 : t_0^2 t_1 : t_0 t_1^2 : t_1^3]$.

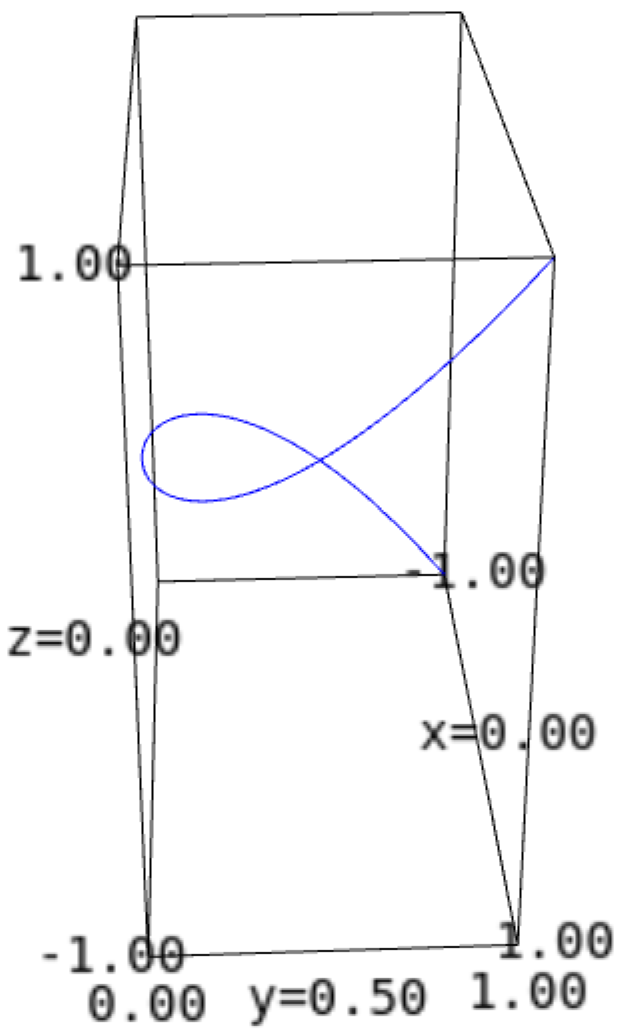

Figure 1: The twisted cubic in $\mathbb{A}^3$

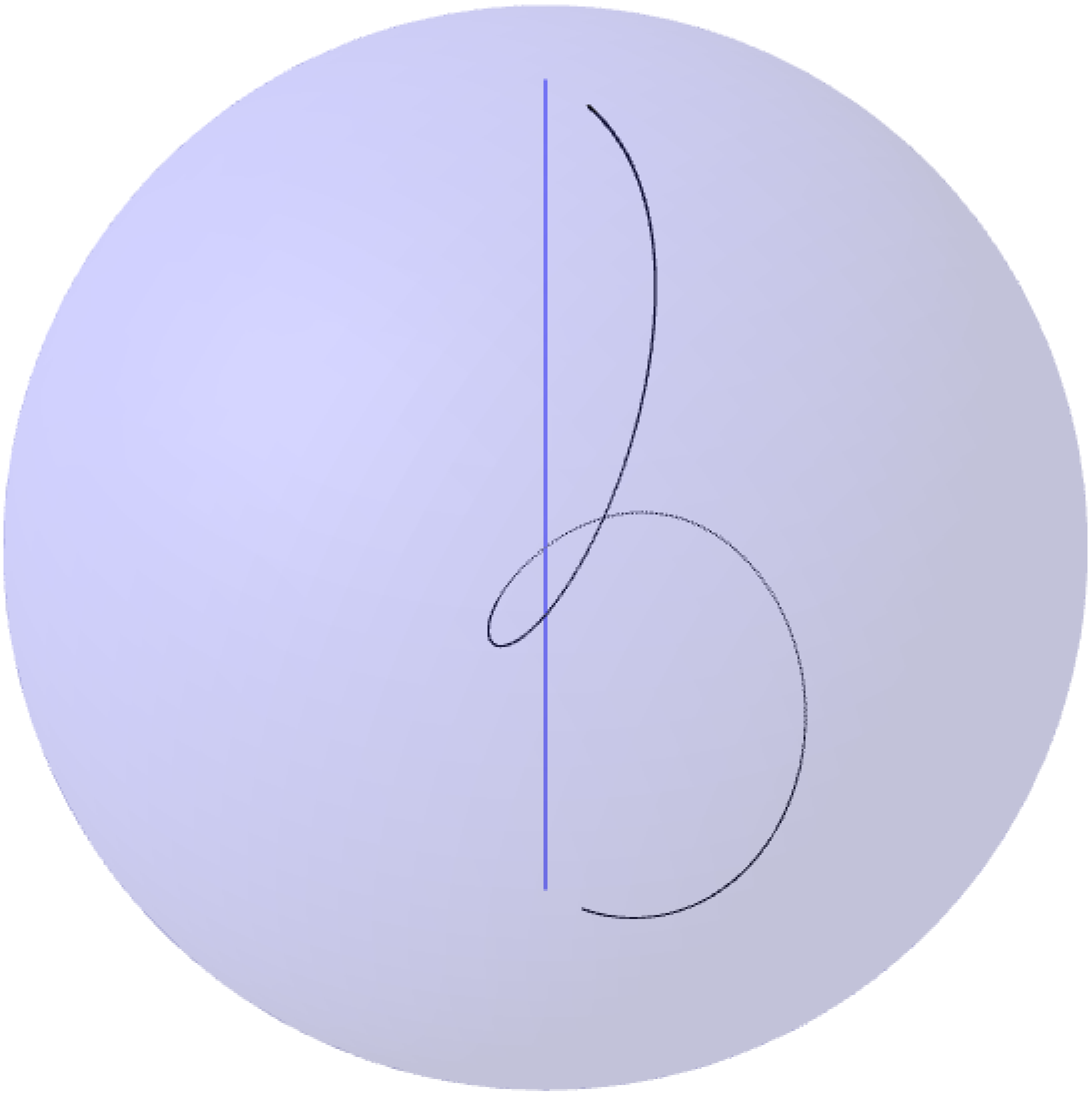

Figure 2: The twisted cubic in $\mathbb{P}^3$

Restricting to $t_0 = 1$ gives us the original affine parametrization, and the point at infinity $[0:1]$ maps to $[0:0:0:1]$ as we expected.

We claim therefore that the image of this map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ (see I.2.12 for the general case!) is the projective closure of Y. Observe that the image of this map is contained in $Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$. If we intersect this algebraic set with $x_0 = 1$ we get Y back. If we consider $x_0 = 0$ then these equations yield $x_1^3 = x_1^2 = x_1x_2 = 0$, which leaves only $[0:0:0:1]$. As our intuition suggested, we therefore expect that $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ has image $Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2) = Y \cup \{[0:0:0:1]\}$ and that this is precisely $\overline{Y}$. Indeed, we can do this same trick of considering $t_0 = 1$ and $t_0 = 0$ to prove that this is indeed the image, so it suffices to show that $\overline{Y} = Y \cup \{[0:0:0:1]\}$.

Certainly $Y \cup \{[0:0:0:1]\}$ is closed. Thus, if Y itself is not closed then certainly this must be the closure. If Y was closed, then $Y \cup \{[0:0:0:1]\} = Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$ seperates this closed set into two disjoint closed sets. However, it is not hard to see that our map $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$ is continuous, and $\mathbb{P}^1$ is connected, so the image must be connected as well. Thus, this decomposition would be a contradiction so Y is not closed. Hence, we have computed $\overline{Y} = \mathrm{im}(\mathbb{P}^1 \longrightarrow \mathbb{P}^3) = Z(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$.

We can finally show that $(x_3^3 - x_0^2x_1, x_2^2 - x_0x_1) < I(\overline{Y})$. Indeed, $[0:0:1:1] \in Z(x_3^3 - x_0^2x_1, x_2^2 - x_0x_1)$ but is not in $\overline{Y}$, so we have the strict containment.

However, we are still asked to find actual generators for $I(\overline{Y})$. Our best guess is of course $(x_0^2x_3 - x_1^3, x_0x_2 - x_1^2, x_0x_3 - x_1x_2)$, but we only know that the radical of this ideal is $I(\overline{Y})$. There's probably a way to do all this nonsense, but we'll defer it to I.2.12. Should that problem remain unsolved, mutter something about Gröbner bases to stop worrying about this computation.

□

I.2.10

The Cone Over a Projective Variety (3). Let $Y \subseteq \mathbb{P}^n$ be a nonempty algebraic set, and let $\theta : \mathbb{A}^{n+1} - \{(0, \ldots, 0)\} \longrightarrow \mathbb{P}^n$ be the map which sends the point with affine coordinates $(a_0, \ldots, a_n)$ to the point with homogeneous coordinates $(a_0, \ldots, a_n)\cdot$ We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}[Y] \cup \{(0, \ldots, 0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbb{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0, \ldots x_n]$.

(b) $C(Y)$ is irreducible if and only if Y is.

(c) $\dim C(Y) = \dim Y + 1$.

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbb{P}^{n+1}$. This is called the *projective cone* over Y.

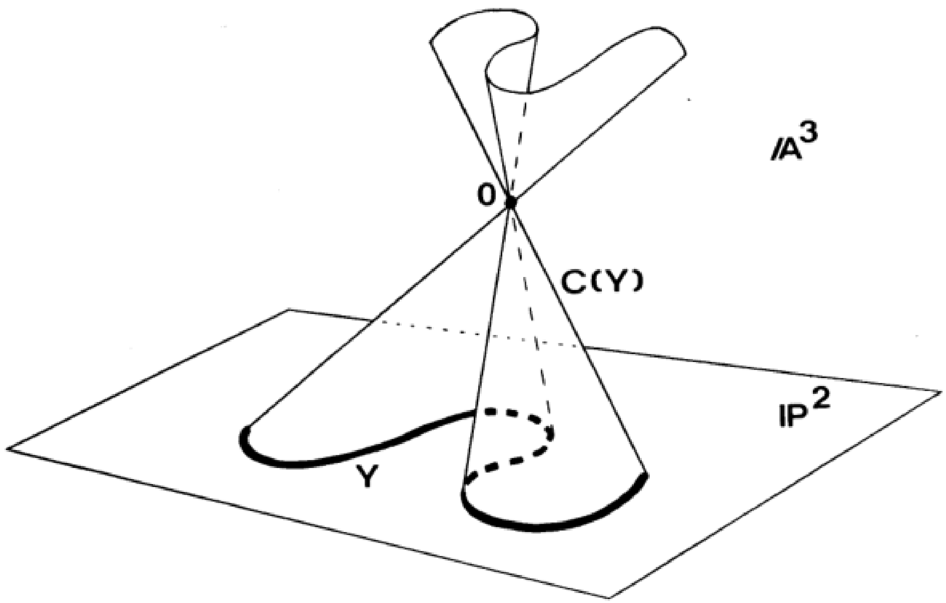

Figure 3: The cone over a curve in $\mathbb{P}^2$.

Proof. (a) Let $I = I(Y)$, so that $Y = Z(I)$. We seek to compute $\theta^{-1}[Z(I)]$. Indeed, we claim that this is precisely $V(I) - 0$. If $(a_0, \dots, a_n) \in V(I) - 0$ then $\theta(a_0, \dots, a_n) = [a_0 : \cdots : a_n] \in \mathbb{P}^n$. Let $f \in I^h$. Then $f(a_0, \dots, a_n) = 0$ so indeed, $f([a_0 : \cdots : a_n]) = 0$ and $\theta(a_0, \dots, a_n) \in Z(I)$. Hence, $V(I) - 0 \subseteq \theta^{-1}[Z(I)]$. Conversely, let $\theta(a_0, \dots, a_n) \in Z(I)$. Then of course $(a_0), \dots, a_n) \neq 0$. Let $f \in I$ and write $f = \sum f_e$ the homogeneous decomposition. As $I = I(Y)$ is homogeneous, each $f_e \in I^h$. Then as $[a_0 : \cdots : a_n] \in Z(I)$, we have that $f_e(a_0, \dots, a_n) = 0$ for all e. Hence, $f(a_0, \dots, a_n) = 0$ and we have computed $\theta^{-1}[Z(I)] = V(I) - 0$.

Additionally, every $f \in I^h$ vanishes on 0. Furthermore, $Y \neq \emptyset$ so $1 \notin I$. Hence, every nonzero element of I is the sum of homogeneous elements of I, all of which vanish on 0, so every element of I vanishes on 0. Thus, we conclude that $V(I) = \theta^{-1}[Z(I)] \cup 0$. Then $J(C(Y)) = I$ by the Nullstellensatz.

(b) $C(Y)$ is irreducible iff $I(C(Y)) = I(Y)$ irreducible iff Y is irreducible.

(c) $\dim C(Y) = \dim k[x_0, \dots, x_n]/I(C(Y)) = \dim k[x_0, \dots, x_n]/I(Y) = \dim Y + 1$.

□

I.2.11

Linear Varieties in $\mathbb{P}^n$. A hypersurface defined by a linear polynomial is called a *hyperplane.*

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbb{P}^n$:

(i) $I(Y)$ can be generated by linear polynomials.

(ii) Y can be written as an intersection of hyperplanes.

In this case we say that Y is a linear variety in $\mathbb{P}^n$.

(b) If Y is a linear variety of dimension r in $\mathbb{P}^n$, show that $I(Y)$ is minimally generated by $n-r$ linear polynomials.

(c) Let Y, Z be linear varieties in $\mathbb{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r+s-n \geq 0$, then $Y \cap Z \neq \emptyset$. Furthermore, if $Y \cap Z \neq \emptyset$, then $Y \cap Z$ is a linear variety of dimension $\geq r+s-n$. (Think of $\mathbb{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

Proof. (a) ($i \implies ii$). Let $I(Y) = (f_1, \dots, f_r)$ linear. Then $Y = Z(I(Y)) = \bigcap Z(f_i)$.

($ii \implies i$). Let $Y = \bigcap_{i \in I} Z(f_i)$. We therefore have $I(Y) = \sqrt{(f_i : i \in I)}$. As S is Nötherian we can find a finite set of generators among the f_i, i.e. $(f_1, \dots, f_r) = (f_i : i \in I)$. Indeed, find the maximal such finitely generated sub-ideal. Thus, $Y = Z(f_1, \dots, f_r)$. Write $f_i = \sum f_{ij} x_j$. Then $[a_0 : \cdots : a_n] \in Y$ iff $(f_{ij}) \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = 0$. The kernel of the matrix (f_{ij}) will have some basis $v_1, \dots, v_s$ so take a matrix A which sends $e_i \mapsto v_i$ and the rest to whatever basis you extend it to. Then $(f_{ij})AP = 0$ precisely when $P = (*, \dots, *, 0, \dots, 0)$ with s many $*'s$. Thus A represents a linear change of coordinates on $\mathbb{P}^n$ which sends $Z(f_1, \dots, f_r)$ to $Z(y_{s+1}, \dots, y_n)$, whose ideal is of course just $(y_{s+1}, \dots, y_n)$. This leads us to the general fact that a proper ideal generated by finitely many linear polynomials is necessarily prime. If you don't like this, take the transformation on the polynomials themselves.

(b) Suppose that $I(Y) = (f_1, \dots, f_s)$ linear polynomials. Then by Krull's principal ideal theorem, $\operatorname{codim} I(Y) \leq s$. As $I(Y)$ is prime, (see (a) or the assumption that Y is a variety) we compute $\dim S(Y) = (n+1) - \operatorname{codim} I(Y) \geq n+1-s$. Of course, by I.2.6, $\dim S(Y) = \dim Y + 1 = r+1$. Hence, we must have $r+1 \geq n+1-s$ and therefore that $s \geq n-r$.

(c) Recall from I.2.10 the map $\theta : \mathbb{A}^{n+1} - 0 \longrightarrow \mathbb{P}^n$ sending $(a_0, \dots, a_n) \mapsto [a_0 : \cdots : a_n]$, and the cone $C(Y) = \theta^{-1}[Y] \cup \{0\}$ for a closed subset $Y \subseteq \mathbb{P}^n$. Then as $\dim Y = r$ and $\dim Z = s$ we have $\dim C(Y) = r+1$ and $\dim C(Z) = s+1$. The condition that $r+s-n \geq 0$ is equivalent to $(r+1)+(s+1)-(n+1) \geq 1$, i.e. that $\dim C(Y) + \dim C(Z) - \dim A^{n+1} \geq 1$. The idea then is to use the fact that a line in A^{n+1} is precisely a point in $\mathbb{P}^n$.

Recall also that $C(Z(I)) = V(I) \subseteq \mathbb{A}^{n+1}$. As Y, Z are linear they are defined by the zero set of finitely many homogeneous linear polynomials. Hence, their affine cones are the affine zero sets of those same polynomials, and are therefore subspaces of the vector space $\mathbb{A}^{n+1} = k^{n+1}$. Hence, from the second isomorphism theorem, we observe that $\dim C(Y) + \dim C(Z) - \dim A^{n+1} \leq \dim(C(Y) \cap C(Z))$.

To sum the above up, $r+s-n \geq 0$ iff $\dim(C(Y) \cap C(Z)) \geq 1$. In other words, that there is some nonzero $v \in \dim(C(Y) \cap C(Z))$ Then the line $\theta(v) \in Y \cap Z$ witnesses the fact that this is nonempty. Long story short, $Y \cap Z \neq \emptyset$ iff their cones share a common line.

$\square$

I.2.12

The $d-Uple$ Embedding. For given $n,d>0$, let $M_0,M_1,\ldots,M_N$ be all the monomials of degree d in the $n+1$ variables $x_0,\ldots,x_n$, where $N=\binom{n+d}{n}-1$. We define a mapping $\rho_d:\mathbb{P}^n\longrightarrow\mathbb{P}^N$ by sending the point $P=(a_0,\ldots,a_n)$ to the point $\rho_d(P)=(M_0(a),\ldots,M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbb{P}^n$ in $\mathbb{P}^N$. For example, if $n=1$, $d=2$, then $N=2$, and the image Y of the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ is a conic.

(a) Let $\theta:k[y_0,\ldots,y_N]\longrightarrow k[x_0,\ldots,x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbb{P}^N$.

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

(c) Now show that ρ_d is a homeomorphism of $\mathbb{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

(d) Show that the twisted cubic curve in $\mathbb{P}^3$ (Ex. I.2.9) is equal to the 3-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

Proof. (a) Proving the homogeneity of $\mathfrak{a}$ comes from θ being defined as evaluation by homogeneous polynomials. Indeed, let $f\in\mathfrak{a}$, so that $\theta(f)=f(M_1,\ldots,M_N)=0$. Let $f=\sum f_e$ be the homogeneous decomposition of f. Then $\sum f_e(M_1,\ldots,M_N)=0$. Furthermore, as each M_i is homogeneous of degree d, the degree of $f_e(M_1,\ldots,M_N)$ is de. As $d\geq 1$, each $f(M_1,\ldots,M_N)$ has a distinct degree. Thus, by linear independence, each $f_e(M_1,\ldots,M_N)=0$. Hence, they are all in $\mathfrak{a}$ so $\mathfrak{a}$ is homogeneous. Primality is immediate from the fact that $k[x_0,\ldots,x_n]$ is a domain.

(b) For the easier inclusion, take some $\rho_d([a_0:\cdots:a_n])=[M_0(a):\cdots:M_N(a)]$ and let $f\in Z(\mathfrak{a})^h$. Then $f(M_0(a),\ldots,M_N(a))=\theta(f)(a)=0$ so $\operatorname{im}\rho_d\subseteq Z(\mathfrak{a})$.

For the converse JustDoIt or possibly seek this reference if I resign.

We will define an inverse map $\psi:Z(\mathfrak{a})\longrightarrow\mathbb{P}^n$. For a multi-index I we will define $I_j=(i_0,\ldots,i_j+1,\ldots,i_n)$. If we take $\sum I=d-1$ then we will define a map $\psi_I:Z(\mathfrak{a})\longrightarrow\mathbb{P}^n$ via $[\{a_J\}]\mapsto[a_{I_0}:\cdots:a_{I_n}]$. Essentially, we are extracting $n+1$ coordinates from $[\{a_J\}]$, starting from I_0 and proceeding in lexicographic order. Essentially, we seek to show that this, with all the relations of $\mathfrak{a}$, are all we need to recover $[\{a_I\}]$.

First, we will show that ψ_I is independent of the choice of I. Indeed, take another multi-index K with $\sum K=d-1$. Then to compare $\psi_I([\{a_J\}])$ and $\psi_K([\{a_J\}])$ we need to compare the ratios $\frac{a_{I_j}}{a_{K_j}}$. They define the same point in projective space iff all these ratios are the same (ignoring division by 0). Observe that $I_j+K_0=I_0+K_j$, so $\mathfrak{a}$ contains the polynomial $y_{I_j}y_{K_0}-y_{I_0}y_{K_j}$. As we are defining our ψ on $Z(\mathfrak{a})$, we therefore have $a_{I_j}a_{K_0}=a_{I_0}a_{K_j}$, i.e. that $\frac{a_{I_j}}{a_{K_j}}=\frac{a_{I_0}}{a_{K_0}}$ for all J. Hence, $\psi_I=\psi_K$ so we can simply call it ψ.

Start with some $a = [a_0 : \cdots : a_n] \in \mathbb{P}^n$. Take some $a_i \neq 0$ and take $I = (0, \ldots, 0, d-1, 0, \ldots, 0)$ in the i^{th} position. $\rho_d(a) = [\{a^J\}]$, and applying $\psi = \psi_I$ to this yields $[a^{I_0} : \cdots : a^{I_n}]$. $a^{I_j} = a^{(0,\ldots,0,d-1,0,\ldots,0)+(0,\ldots,0,1,0,\ldots,0)} = a_i^{d-1}a_j$. Hence, $\psi(\rho_d(a)) = [a_i^d : a_i^{d-1}a_1 : \cdots : a_i^{d-1}a_n] = a$.

On the other hand, take some $[\{a_J\}] \in Z(\mathfrak{a})$. There is some index I_0 for which $a_{I_0} \neq 0$ (ok fine maaayybbeee all of the $(0, *, \ldots, *)$ terms vanish but I'm sure this won't work if I bash some relations so whatever). Applying $\psi = \psi_I$ yields $[a_{I_0} : \cdots : a_{I_n}]$. If we apply ρ_d to this we end up with $\left[\prod_i a_{I_i}^{j_i}\right]$ ranging over all $J = (j_0, \ldots, j_n)$. We want this to equal our original $[\{a_J\}]$,so again we consider the ratios $\left(a_{I_i}^{j_i}/a_J\right)$ vs $\left(a_{I_i}^{k_i}/a_K\right)$. In other words, we seek to show that $a_K \prod a_{I_i}^{j_i} = a_J \prod a_{I_i}^{k_i}$. Indeed, this corresponds to the polynomial $y_K \prod y_{I_i}^{j_i} - y_J \prod y_{I_i}^{k_i}$. Note that on the left hand side, the "signature" is $K + \sum j_i I_i$ and on the right hand side it is $J + \sum k_i I_i$. We'll let $\hat{i} = (0, \ldots, 0, 1, 0, \ldots, 0)$. Then $J + \sum k_i I_i = J + \sum k_i (I + \hat{i})$. This is $J + \sum k_i I + \sum k_i \hat{i} = J + K + dI$. The same computation works for the other term, so the signatures agree and therefore $y_K \prod y_{I_i}^{j_i} - y_J \prod y_{I_i}^{k_i} \in \mathfrak{a}$. Thus, $[\{a_J\}]$ vanishes on this so we get the desired equality $a_K \prod a_{I_i}^{j_i} = a_J \prod a_{I_i}^{k_i}$ so $\rho_d(\psi([\{a_J\}])) = [\{a_J\}]$.

We have therefore proven that these two maps are inverse. In fact, we could have done this for just $Z(\{\prod y_{K_i} - \prod y_{J_i} : \sum K_i = \sum J_i\})$, so these generate $\mathfrak{a}$. If we try hard enough we may even show that we only need the degree two examples, but whatever.

(c) We'd expect that $\rho_d^{-1}[Z(I)] = Z(\theta[I])$ and that $\rho_d[Z(J)] = Z(\theta^{-1}[J^d])$, where by J^d I mean to take your favorite homogeneous generators $J = (S)$ and let S^d be raising everthing in S to the d^{th} power so that it is in the image of θ. It seems pretty believable that ψ and ρ_d are continuous though, as they're all just evaluation at a bunch of homogeneous polynomials of the same degree so they better be continuous.

(d) We did this is I.2.9 already.

□

I.2.13

Let Y be the image of the 2-uple embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbb{P}^5$ such that $V \cap Y = Z$.

Proof. Take indeed some closed curve $Z \subsetneq Y$, i.e. a variety of dimension 1. By I.2.12 we have a homeomorphism $\rho_2 : \mathbb{P}^2 \longrightarrow Y = Z(\mathfrak{a})$ where $\mathfrak{a}$ is defined as in I.2.12. Let $\Gamma = \rho_2^{-1}[Z]$. This is a codimension 1 variety so it equals $Z(f)$ for some $f \in k[x_0, x_1, x_2]$ irreducible and homogeneous (I.2.8). To find a hypersurface $V \subsetneq \mathbb{P}^5$ we therefore want some $g \in k[y_0, \ldots, y_5]$ homogeneous and irreducible such that $V = Z(g)$. The intersection $Y \cap V = Z(\mathfrak{a}) \cap Z(g) = Z(\mathfrak{a} + (g))$. We therefore want the image of ρ_2 to be $Z(\mathfrak{a} + (g))$. On the coordinate rings, this would mean we have $\theta : k[y_0, \ldots, y_5]/\sqrt{(\mathfrak{a} + (g))} \longrightarrow k[x_0, x_1, x_2]/(f)$ an isomorphism. Then we essentially need $\theta[\sqrt{(\mathfrak{a} + (g))}] = (f)$. As $\mathfrak{a}$ is the kernel we're looking for $\sqrt{(\theta(g))} = (f)$

Let's first consider the image of θ in this case. Now, $\theta(f) = f(x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2)$ so certainly anything in the image has all homogeneous summands of even degree. On the other hand, consider an even degree monomial $x_0^i x_1^j x_2^k$, i.e. $i + j + k \equiv 0 \pmod 2$ If these are all 0 then we are done via $y_{200}^{i/2} y_{020}^{j/2} y_{002}^{k/2}$. Alternatively, we could have one of these 0 and the other two are 1. For instance, if $i, j \equiv 1 \pmod 2$ then we could take $y_{200}^{\frac{i-1}{2}} y_{020}^{\frac{j-1}{2}} y_{110} y_{002}^{k/2}$. Therefore we have that $\operatorname{im}\theta$ is the set of polynomials whose homogeneous components are all even degree. And in fact, for a homogeneous even degree guy, we can find a preimage which is also homogeneous. Thus, we can solve $\sqrt{(\theta(g))} = (f)$ by taking $\theta(g) = f^2$ and g homogeneous.

We now need to find an irreducible g satisfying the above. Suppose we were lucky and found a homogeneous g such that $\theta(g) = f$. If $g = ab$ then $f = \theta(a)\theta(b)$, so WLOG take $\theta(a) \in k^\times$, as f is irreducible. Then as θ sends each y to a homogeneous degree monomial, only constants can map to constants. Hence, $a \in k^\times$ so g is irreducible. On the other hand, if we only have $\theta(g) = f^2$ then again write $g = ab$ whence $f^2 = \theta(a)\theta(b)$. If g was already irreducible then we'd of course be done, so suppose not. Then $\theta(a) = \theta(b) = f$ as $k[x_0, x_1, x_2]$ is a UFD. As $ab = g$ homogeneous, a, b are homogeneous. Then by the previous case, they are irreducible.

In any case, we have shown that we can find an irreducible homogeneous $g \in k[\{y_I\}]$ such that $\theta(g) = f$ or f^2. As discussed, we therefore take $V = Z(g)$ as our candidate hypersurface whose "shadow" onto the d-uple embedded $\mathbb{P}^2$ is our fixed curve Z. That is, we want to show that $V \cap Y = Z(\mathfrak{a} + (g)) = Z = \rho_2[Z(f)]$. Of course, we already know $Z \subseteq Y$. Take some $P \in Z$, which we know is of the form $\rho_2(Q)$ for $Q \in Z(f)$. Then $g(P) = g(\rho_2(Q)) = \theta(g)(Q) = f(Q) = 0$ (or $f^2(Q)$ but kjwe;lkjsDLf). Hence, we also have $Z \subseteq Z(g)$ so $Z \subseteq Y \cap V$. On the other hand, take some $P \in Y \cap V = Z(\mathfrak{a} + (g))$. Then $P \in Y$ so write $P = \rho_2(Q)$. We want to show that $Q \in Z(f)$. Indeed, let $g(P) = g(\rho_2(Q)) = \theta(g)(Q) = 0$. Of course, $\theta(g) = f^{1.5\pm0.5}$ so $f(Q) = 0$ and $P \in \rho_2[Z(f)] = Z$ as desired. Thus, we have $V \cap Y \subseteq Z \subseteq V \cap Y$. $\square$

I.2.14

The Segre Embedding. Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \longrightarrow \mathbb{P}^N$ be the map defined by sending the ordered pair $(a_0, \dots, a_r) \times (b_0, \dots, b_s)$ to $(\dots, a_ib_j, \dots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a subvariety of $\mathbb{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbb{P}^N$ be $\{z_{ij} \mid i = 0, \dots, r, j = 0, \dots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[z_{ij}] \longrightarrow k[x_0, \dots, x_r, y_0, \dots, y_s]$ which sends z_{ij} to x_iy_j. Then show that $\operatorname{im}\psi = Z(\mathfrak{a})$.]

Proof. As suggested, we will show that the image of ψ is $Z(\mathfrak{a})$ as defined in the body of the question. Indeed, let $(P, Q) \in \mathbb{P}^r \times \mathbb{P}^s$ and $f \in \mathfrak{a}^h$. Then $f(\psi(P, Q)) = \theta(f)(a_0, \dots, b_s) = 0$.

On the other hand take some $p \in Z(\mathfrak{a})$. Write $P = [c_{ij}] \in \mathbb{P}^N$. Then some $c_{kl} \neq 0$, so we work in the affine patch on which is it 1. We'll start defining our preimage to this via $a_k = b_l = 1$. We're trying to make it so that $c_{ij} = a_ib_j$ for all i, j. In particular, this would mean that $a_i = a_ib_l = c_{il}$ and $b_j = a_kb_j = c_{kj}$. So indeed, we take these as our definitions and claim that $([a_0 : \cdots : a_r], [b_0 : \cdots : b_s]) \mapsto P$. Indeed, that means we want to show that $c_{ij} = a_ib_j$. Consider the relations $(x_iy_j)(x_my_n) = (x_iy_n)(x_my_j)$. Hence, the polynomials

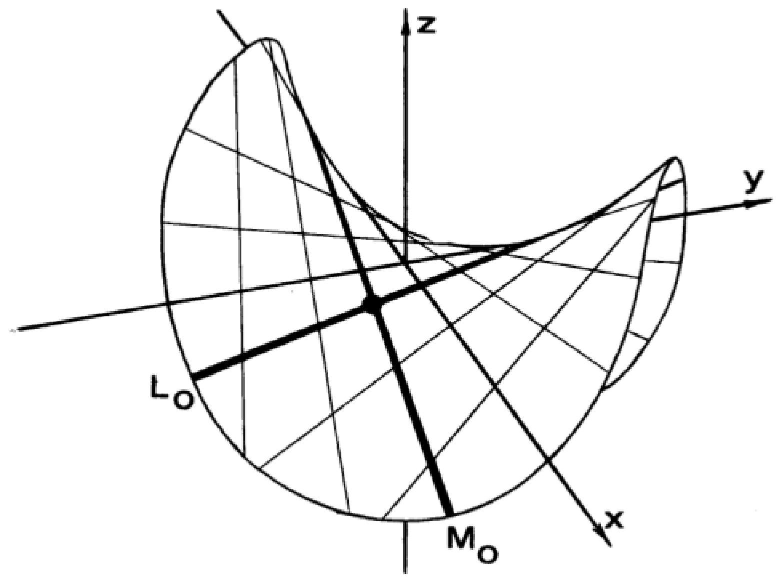

Figure 4: The quadric surface in $\mathbb{P}^3$.

$z_{ij}z_{mn} - z_{in}z_{mj} \in \mathfrak{a}^h$. In particular, we have that $c_{ij}c_{kl} = c_{kj}c_{il}$. Using the definitions we already have, this means exactly that $c_{ij} = a_i b_j$. Thus, $P \in \operatorname{im}\psi$. □

I.2.15

The Quadric Surface in $\mathbb{P}^3$ See 4. Consider the surface Q (a surface is a variety of dimension 2) in $\mathbb{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, for suitable choice of coordinates.

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}$, $\{M_t\}$, each parametrized by $t \in \mathbb{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \emptyset$; if $M_t \neq M_u$, $M_t \cap M_u = \emptyset$, and for all t, u, $L_t \cap M_u =$ one point.

(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbb{P}^1 \times \mathbb{P}^1$ (where each $\mathbb{P}^1$ has its Zariski topology).

Proof. (a) This is defined by $\psi([a_0 : a_1], [b_0 : b_1]) = [a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1]$. We had the map $k[z_{00}, z_{01}, z_{10}, z_{11}] \longrightarrow k[x_0, x_1, y_0, y_1]$ via $z_{ij} \mapsto x_i y_j$. By I.2.14 above, $\operatorname{im}\psi = Z(\ker\theta)$. Thus, we seek to show $\ker(\theta) = (z_{00}z_{11} - z_{01}z_{10})$.

We will approach this with dimension theory. Note $z_{00}z_{11} - z_{01}z_{10}$ is irreducible and is in $\ker(\theta)$. Hence, $(z_{00}z_{11} - z_{01}z_{10}) \subseteq \ker(\theta)$ is an inclusion of primes, so we can prove equality by proving equality of their codimensions. By Krull's principal ideal theorem, the codimension of $(z_{00}z_{11} - z_{01}z_{10})$ is 1. Furthermore, we have $\dim \operatorname{im} \theta = 4 - \operatorname{codim} \ker(\theta)$.

We are therefore left to compute $\dim \operatorname{im} \theta$. Indeed, $\operatorname{im} \theta = k[x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. To show that this has dimension 3, we claim that the first three generators are independent and that the last is algebraic over the first three (really we could choose any 3 of 4). Indeed, the easy part is that x_1y_1 satisfies $(x_0y_0)t - (x_0y_1)(x_1y_0)$. Hence, $k(x_0y_0, x_0y_1, x_1y_0, x_1y_1)/k(x_0y_0, x_0y_1, x_1y_0)$ is algebraic so their transcendence degrees agree.

Now we want algebraic independence of these first 3. Take indeed some $\sum a_{ijk}(x_0y_0)^i(x_0y_1)^j(x_1y_0)^k = 0$. Then $\sum a_{ijk}x_0^{i+j}x_1^k y_0^{i+k} y_1^j = 0$. Observe that the map $(i, jk) \mapsto (i + j, k, i + k, j)$ is injective. Thus, each coefficient a_{ijk} is attached to a monomial of a unique "signature" $(i + j, k, i + k, j)$. x_0, x_1, y_0, y_1 are algebraically independent so these distinct signature monomials are linearly independent. Thus, each $a_{ijk} = 0$ and we have our algebraic independence.

In conclusion, $\dim k[x_0y_0, x_0y_1, x_1y_0, x_1y_1] = 3$. Thus, $\ker(\theta)$ and $(z_{00}z_{11} - z_{10}z_{01})$ have the same codimension. Thus, they are equal and we conclude $\operatorname{im} \psi = Z(\ker \theta) = Z(z_{00}z_{11} - z_{01}z_{10})$.

(b) We have our map $\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow Q$. Fix some $t = [a : b] \in \mathbb{P}^1$. We'll define $L_t = \operatorname{im}(\psi(t, -))$ and $M_t = \operatorname{im}(\psi(-, t))$. This essentially transfers the coordinate grid on the plane onto quadric surface. As all these set theoretic properties hold for the coordinate grid in $\mathbb{P}^2$, the bijection ψ proves that they hold in Q. We therefore need only show that these are in fact lines.

Let $t = [a_0 : a_1]$. Then $L_t = \{[a_0b_0 : a_0b_1 : a_1b_0 : a_1b_1] : [b_0 : b_1] \in \mathbb{P}^1\}$. We can observe that these satisfy the equations $a_1z_{00} - a_0z_{10}$ and $a_0z_{11} - a_1z_{01}$. Thus we easily have $L_t \subseteq Z(a_1z_{00} - a_0z_{10}) \cap Z(a_0z_{11} - a_1z_{01})$. On the other hand, suppose $[c_{00} : c_{01} : c_{10} : c_{11}]$ is in this line. Then if $a_0 \neq 0$ take $b_0 = c_{00}$ and $b_1 = c_{01}$. If $a_0 = 0$ take $b_0 = c_{10}$ and $b_1 = c_{11}$. One can check that the image of this in the Segre embedding along t lies in L_t. We can do something analogous for M_t.

(c) Q contains the image of the 2-uple embedding $\mathbb{P}^1 \longrightarrow \mathbb{P}^3$, which has image $\{[a_0^2 : a_0a_1 : a_1a_0 : a_1^2]\}$. See I.2.12 for why this is a curve. Note that it is also the image of the diagonal $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ under ψ. However, the diagonal is not closed as $\mathbb{P}^1$ is not Hausdorff. In fact, it is irreducible! This immediately shows that ψ is not a homeomorphism, but for the sake of completeness, we will show that $\mathbb{P}^1 \times \mathbb{P}^1$ contains precisely the curves already described.

First of all, let $L'_t = \{t\} \times \mathbb{P}^1$ and $M'_t = \mathbb{P}^1 \times \{t\}$, so that ψ takes $L'_t \mapsto L_t$ and $M'_t \mapsto M_t$. We claim that these are the only curves in $\mathbb{P}^1 \times \mathbb{P}^1$. A curve is an irreducible closed subset of dimension 1, which is a purely topological notion. To do this, we will first explore irreducible closed subsets of a product of spaces.

Lemma I.1. *Let X_0, X_1 be topological spaces. Then the irreducible closed subsets of $X_0 \times X_1$ are all of the form $A_0 \times A_1$ where A_iX_i are irreducible closed subsets of their respective spaces.*

Proof. First, let's show that these proposed subsets are actually closed and irreducible. Closedness is immediate, so we focus on irreducibility. Take $\emptyset \neq U_0, U_1 \subseteq A_0 \times A_1$ open. Then we can find $U_{00} \times U_{01} \subseteq U_0$ and $U_{10} \times U_{11} \subseteq U_i$ open and nonempty. By irreducibilty of the A_i, $U_{i0} \cap U_{i1} \neq \emptyset$. Hence, $U_0 \cap U_1 \neq \emptyset$.

Now, take $F \subseteq X_0 \times X_1$ irreducible and closed. Then we have $F \subseteq \pi_0[F] \times \pi_1[F]$, but these certainly need not be closed, so instead take $F \subseteq \overline{\pi_0[F]} \times \overline{\pi_1[F]} = Z$. Then F is an irreducible closed subset of Z. As it is closed, $Z - F$ is open and disjoint from F. Assume that this is a proper containent and take therefore a basic open set $\emptyset \neq U_0 \times U_1 \subseteq Z - F$. Now, as $U_0 \times U_1 \subseteq Z - F$, $Z - (U_0 \times U_1) \supseteq F$. Furthermore, $Z - (U_0 \times U_1) \subseteq (X_0 - U_0) \times \overline{\pi_1[F]} \cup \overline{\pi_0[F]} \times (X_1 - U_1)$, which is closed. Then as F is irreducible, it is contained in one of these, say $F \subseteq (X_0 - U_0) \times \overline{\pi_1[F]}$. In particular, $\pi_0[F] \subseteq X_0 - U_0$, so $\overline{\pi_0[F]} \subseteq X_0 - U_0$. But we took $U_0 \subseteq \overline{\pi_0[F]}$, a contradiction. Hence, $F = \overline{\pi_0[F]} \times \overline{\pi_1[F]}$. Why are these irreducible? Uhhhhhhhh cuz □

Note that the irreducible closed subsets of $\mathbb{P}^1$ are $\emptyset$, points, and the whole of $\mathbb{P}^1$. This therefore shows that the only curves in $\mathbb{P}^1$ are L'_t and M'_t as describe.

□

I.2.16

(a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbb{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbb{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

Proof. (a) We have $Q_1 \cap Q_2 = Z(x^2 - yw, xy - zw)$. Recall that $Z(I(Y_1 \cup Y_2)) = Z(I(Y_1) \cap I(Y_2)) = Y_1 \cup Y_2$, when these are closed. This suggests that to realize something as the union of two closed subsets, we want to find a primary decomposition.

Indeed, observe that $(x^2 - yw, xy - zw) \subseteq (x, w) \cap (xz - y^2, yw - x^2, zw - xy)$. The former defines a line and the latter defines a twisted cubic (just believe, the proofs of I.2.9 and I.2.12 are insufficient but it shouldn't be tooooo hard to show that these degree two guys generate, in fact I think (?) this holds in general). We therefore seek to show equality. Take let's consider elements of $(x, w) \cap (xz - y^2, yw - x^2, zw - xy)$. First of all, since $f \in (x, w)$ iff each of its summands are (this can be seen directly or through the theory of monomial ideals ala [CLO07]. In any case, take some $f(xz - y^2) + g(yw - x^2) + h(zw - xy) \in (x, w) \cap (xz - y^2, yw - x^2, zw - xy)$ The last two terms are clearly in (x, w), but we additionally need $f(xz - y^2) \in (x, w)$. Indeed, this will occur iff $f \in (x, w)$. Observe that in the quotient by $(x^2 - yw, xy - zw)$, we have $xzw - wy^2 = x^2y - wy^2 = ywy - wy^2 = 0$ and $x^2z - xy^2 = ywz - xy^2 = ywz - zwy = 0$. Thus, whenever $f \in (x, w)$ the associated summand $f(xz - y^2) \in (x^2 - yw, xy - zw)$. Thus, we indeed that $(x, w) \cap (xz - y^2, yw - x^2, zw - xy) = (x^2 - yw, xy - zw)$. Hence,

$Q_1 \cap Q_2 = Z((x,w) \cap (xz - y^2, yw - x^2, zw - xy)) = Z(x,w) \cup Z((xz - y^2, yw - x^2, zw - xy))$, which is a union of a line and a twisted cubic.

(b) First, observe that $C \cap L = Z(x^2 - yz, y) = Z(x^2, y) = Z(x, y) = \{[0:0:1]\}$. On the other hand, $I(C) + I(L) = (x^2 - yz, y) = (x^2, y) \neq (x, y) = I(P)$. Note that the equations given for C an L are irreducible so there is no fuss.

□

I.2.17

Complete intersections. A variety Y of dimension r in $\mathbb{P}^n$ is a *(strict) complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.

(a) Let Y be a variety in $\mathbb{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geq n - q$.

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbb{P}^3$ (Ex. I.2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees $2, 3$ respectively, such that $Y = H_1 \cap H_2$.

(d) It is an unsolved problem whether every closed irreducible curve in $\mathbb{P}^3$ is a set-theoretic intersection of two surfaces. See Hartshorne [1] and Hartshorne [5, III, §5] for commentary.

⟨*My note*: These references are not to this textbook, but to the references in the textbook itself. For laziness, I do not include these in the bibliography.⟩

Proof. (a) We want to apply Krull's principal ideal theorem here, but there's the issue that that is meant to hold for minimal primes, whereas here we are looking for homogeneous minimal primes over our homogeneous $\mathfrak{a}$. So indeed, we first show the following lemma:

Lemma I.2. *Let S be an ω graded ring. Then a minimal prime $\mathfrak{p}$ of S is homogeneous.*

Proof. To do this, we want to show that any prime $\mathfrak{p}$ contains another prime $\mathfrak{q}$ that is homogeneous. The natural choice is to take $\mathfrak{q} = (\mathfrak{p}^h)$, the ideal generated by the homogeneous elements of $\mathfrak{p}$. This certainly tells us that $\mathfrak{q} \subseteq \mathfrak{p}$ and that $\mathfrak{q}$ is homogeneous. Furthermore, $\mathfrak{p}^h \subseteq \mathfrak{q}^h \subseteq \mathfrak{p}^h$ so we have $\mathfrak{q}^h = \mathfrak{p}^h$. Then to show that $\mathfrak{q}$ is prime, we need only check this on homogeneous elements. Indeed, take $a, b \in S^h$ such that $ab \in \mathfrak{q}^h = \mathfrak{p}^h$. Then by primality of $\mathfrak{p}$, $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. As these are homogeneous, one of these is therefore in $\mathfrak{p}^h = \mathfrak{q}^h \subseteq \mathfrak{q}$. Thus, $\mathfrak{q}$ is prime.

Now if $\mathfrak{p}$ was minimal then $\mathfrak{q} \subseteq \mathfrak{p}$ is an equality, and $\mathfrak{p} = \mathfrak{q}$ is homogeneous. □

Note that this proof also works for minimal primes over homogeneous ideals I, as we can apply the lemma to S/I, which is itself graded. Hence, any minimal prime over out given homogeneous ideal $\mathfrak{a}$ must itself be homogeneous.

Now, by Krull's principal ideal theorem, an minimal prime $\mathfrak{p} \supseteq \mathfrak{a}$ must have $\operatorname{codim} \mathfrak{p} \leq q$. As discussed, $\mathfrak{p}$ is homogeneous so we form the inclusion $Z(\mathfrak{p}) \subseteq Z(\mathfrak{a}) = Y$ of a maximal irreuducible susbet of Y. Then $\dim Y \geq \dim Z(\mathfrak{p})$. $\dim Z(\mathfrak{p}) = \dim S(\mathfrak{p}) - 1 = (n+1) - \operatorname{codim} \mathfrak{p} - 1 = n - \operatorname{codim} \mathfrak{p} \geq n - q$.

(b) Let $I(Y) = (a_1, \ldots, a_s)$ where $s = n - r$. Recall that Y was assumed to be a variety so $I(Y)$ must be prime. Then for each i there is some irreducible $f_i \mid a_i$ such that $f_i \in I(Y)$. Then $(f_1, \ldots, f_s) \subseteq I(Y)$. On the other hand, $f_i \mid a_i$ so $a_i \in (f_i)$. Thus, $I(Y) = (a_1, \ldots, a_s) \subseteq (f_1, \ldots, f_s) \subseteq I(Y)$ so we achieve equality. Hence, we have $Y = \bigcap Z(f_i)$, which are all hypersurfaces.

Of course, the problem insists that it is the intersection of $n - r$ hypersurfaces. The index of the intersection ranges from 1 to $s = n - r$, but a priori there could be repeats in this intersections. However, the fact that $\dim Y = r$ precludes this. Indeed, suppose there were some repeats in the intersection, i.e. that some $Z(f_i) = Z(f_j)$. That would mean that we could generate $I(Y)$ by fewer than s elements. But if $I(Y)$ is generated by l many elements, part (a) above tells us that $r = \dim Y \geq n - l$. Thus, $l \geq n - r = s$, so s is the minimal number of generators this ideal can have. Hence, it is the minimal number of hypersurfaces we need to intersect in Y.

(c) Take $Y = Z(xz - yz, yw - z^2, xw - yz)$. To write $Y = H_1 \cap H_2$ for hypersurfaces $H_i = Z(f_i)$ means that $\sqrt{(f_1, f_2)} = (xz - yz, yw - z^2, xw - yz)$. Wikipedia claims that we can take $f_1 = xz - y^2$ and $f_2 = z(yw - z^2) - w(xw - yz)$. One can then bash out this computation.

On the other hand, we want to show that $I(Y)$ cannot be generated by 2 elements. The naïve dimension approach using (a) won't work, as generating $I(Y)$ by two elements yields the inequality $\dim Y \geq 3 - 2$ which is correct. Hence, we'll approach this with Nakayama's lemma, noting that the given generators $a_1 = xz - yz$, $a_2 = yw - z^2$, $a_3 = xw - yz$ are linearly independent over k.

Let's let $\mathfrak{p} = I(Y)$. Now, we want a maximal ideal $\mathfrak{p} \subseteq \mathfrak{m}$ so that we can get a corresponding prime $\mathfrak{p}S_{\mathfrak{m}}$ in the local ring $S_{\mathfrak{m}}$ on which we can apply Nakayama's lemma. Specifically, we note that generators of $\mathfrak{p}S_{\mathfrak{m}}$ correspond to generators of $\mathfrak{p}S_{\mathfrak{m}}/\mathfrak{p}\mathfrak{m}S_{\mathfrak{m}}$ as a $S_{\mathfrak{m}}/\mathfrak{m}S_{\mathfrak{m}} = k$ vector space, and in fact that minimal generating sets correspond to bases. We therefore want to compute the dimension of this latter space. Note that by flatness of localization, and the fact that $k = S/\mathfrak{m}$ is a field, $\mathfrak{p}S_{\mathfrak{m}}/\mathfrak{p}\mathfrak{m}S_{\mathfrak{m}} = \mathfrak{p}/\mathfrak{p}\mathfrak{m}$.

We therefore want to prove that $\dim_k \mathfrak{p}/\mathfrak{p}\mathfrak{m} \geq 3$. Indeed, we will show that the $a_i \in \mathfrak{p}$ are linearly independent in the quotient. Of course, we have no actually picked a maximal ideal $\mathfrak{m}$ so we will take $\mathfrak{m} = (x, y, z, w)$. Note that $0 \in Y$ so $\mathfrak{p} \subseteq \mathfrak{m}$. Now, let's try to understand the quotient. Take some $\sum f_i a_i \in \mathfrak{p}$, where $f_i = \sum f_{iJ} X^J$ for $X = (x, y, z, w)$. We see that each term $X^J a_i \in \mathfrak{p}\mathfrak{m}$ for $J \neq (0,0,0,0)$. Then in the quotient $\mathfrak{p}/\mathfrak{p}\mathfrak{m}$, we have $\sum f_i a_i = \sum f_i(0) a_i$, so it is spanned as a k vector space by the a_i. In other words, we have a well defined isomorphism $\mathfrak{p}/m\mathfrak{p} \longrightarrow \sum k a_i$ via $\sum f_i a_i \mapsto \sum f_i(0) a_i$ due to the isomorphism $S/\mathfrak{m} \longrightarrow k$ via $f \mapsto f(0)$. Furthermore, the a_i are k-linearly independent. Thus, $\dim_k \mathfrak{p}/\mathfrak{p}\mathfrak{m} = 3$. If there was any smaller generating set for $\mathfrak{p}$ then we could run through the same procedure and find a strictly smaller basis for $\mathfrak{p}/\mathfrak{m}\mathfrak{p}$ - a contradiction.

$\square$

I.3 Morphisms

I.3.1 INCOMPLETE

(a) Show that any conic in $\mathbb{A}^2$ is either isomorphic to $\mathbb{A}^1$ or $\mathbb{A}^1 - 0$ (c.f I.1.1).

(b) Show that $\mathbb{A}^1$ is *not* isomorphic to any proper open subset of itself. (This result is generalized by **??** below).

(c) Any conic in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^1$.

(d) We will see later (**??**) that any two curves are homeomorphic. But show now that $\mathbb{A}^2$ is not even homeomorphic to $\mathbb{P}^2$.

(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

Proof. (a) A conic is the zero set of an irreducible polynomial f of degree 2. By the results of this section, it will be fruitful to consider the coordinate rings. Indeed, by I.1.1.c, $A(V(f)) \cong k[x,y]/(y-x^2)$ or $k[x,y]/(xy-1)$. The former is isomorphic to $k[t]$ via $x \mapsto t$, $y \mapsto t^2$, corresponding to the map $\mathbb{A}^1 \longrightarrow \mathbb{A}^2$ via $t \mapsto (t,t^2)$. The algebra isomorphism yields an isomorphism of varieties.

In the latter case the obvious maps are $V(xy-1) \longrightarrow \mathbb{A}^1 - 0$ via $(x,y) \mapsto x$ and $\mathbb{A}^1 - 0 \longrightarrow V(xy-1)$ sending $t \mapsto (t, 1/t)$. The former is induced by the map $k[t] \longrightarrow k[x,y]/(xy-1)$ sending $t \mapsto x$, and it of course has image equal to exactly $\mathbb{A}^1 - 0$. For the latter, we consider the map $k[x,y]/(xy-1) \longrightarrow \mathcal{O}(\mathbb{A}^1 - 0)$ sending xt and $y \mapsto 1/t$. As [Hartshorne, I.3.5] was proven in this generality, this induces the map $\mathbb{A}^1 - 0 \longrightarrow V(xy-1)$ we just described. It is easy to check that these are inverse to each other, and we have additionally shown that they are both maps of varieties via their rings of regular functions.

(b) Take $U < \mathbb{A}^1$. Of course, if $\emptyset = U$ then we are done so suppose otherwise. Then $Y = \mathbb{A}^1 - U$ satisfies $0 < Y < \mathbb{A}^1$. We can therefore write $Y = V(f)$ for some nonzero, nonconstant polynomial $f \in k[x]$. We can also assume WLOG that f is squarefree. If indeed $\mathbb{A}^1 \cong U$, then $k[x] = \mathcal{O}(\mathbb{A}^1) \cong \mathcal{O}(U)$ as k-algebras. Consider then our hypothetical isomorphism $k[x] \cong \mathcal{O}(U)^\times$. As this is an isomorphism of k-algebras, we must have had the following commutative diagram.

$$\begin{tikzcd} k[x] \arrow[rr, "\sim"] & & \mathcal{O}(U) \\ & k \arrow[ul] \arrow[ur] & \end{tikzcd}$$

This descends to the unit groups as follows.

$$\begin{tikzcd} k^\times \arrow[rr, "\sim"] & & \mathcal{O}(U)^\times \\ & k^\times \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Which would imply that $\mathcal{O}(U)^\times = k^\times$. But as $V(f) \cap U = \emptyset$, $f \in \mathcal{O}(U)^\times$. Furthermore, f was assumed to be nonconstant, so this is a contradiction.

□

I.3.2

A morphism whose unerlying map on the topological spaces is a homeomorphism need not be an isomorphism.

(a) For example, let $\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that ϕ defines a bijective bicontinuous morphism on $\mathbb{A}^1$ onto the curve $y^2 = x^3$, but that ϕ is not an isomorphism.

(b) For another example, let the characteristic of the base field k be $p >$) and define a map $\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ by $t \mapsto t^p$. Show that ϕ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism*.

Proof. □

I.3.3 INCOMPLETE

(a) Let $\phi : X \longrightarrow Y$ be a morphism. Then for each $P \in X$, ϕ induces a homomorphism of local rings $\phi_P^* : \mathcal{O}_{\phi(P),Y} \longrightarrow \mathcal{O}_{P,X}$.

(b) Show that a morphism ϕ is an isomorphism if and only if ϕ is a homeomorphism, and the induced map ϕ_P^* on local rings is an isomorphism for all $P \in X$.

(c) Show that if $\phi[X]$ is dense in Y, then the map ϕ_P^* is injective for all $P \in X$.

Proof. (a) Take indeed some $(V, f) \in \mathcal{O}_{\phi(P),Y}$. Then $\phi(P) \in V$ and f is regular on V. Then $f \circ \phi$ is regular on $\phi^{-1}[V]$, which contains P. Hence, we map $(V, f) \mapsto (\phi^{-1}[V], f \circ \phi)$. I don't want to prove that this is well defined an a homomorphism, so I appeal to the general fact that ϕ induces a map of sheaves $\mathcal{O}_Y \longrightarrow \phi_* \mathcal{O}_X$. These are sheaves of rings (over Y), so functoriality of the colimit yields this exact map on stalks.

We furthermore want to show that this is a morphism of *local* rings. That is, $\mathfrak{m}_{\phi(P)} \mapsto \mathfrak{m}_P$. This is actually quite trivial: $\mathfrak{m}_{\phi(P)}$ is precisely those (V, f) such that $f(\phi(P)) = 0$. Then $f \circ \phi$ vanishes on P so $(\phi^{-1}[V], f \circ \phi) \in \mathfrak{m}_P$.

(b) The forward direction is obvious (see functoriality of the stalks) so suppose that ϕ is a morphism, a homeomorphism, and induces isomorphisms on the stalks. We just want to show that for all $f : U \longrightarrow k$ regular, $U \subseteq X$ open and nonempty, that $f \circ \phi^{-1}$ is regular. Take indeed $P \in U$. Then $(U, f) \in \mathcal{O}_{X,P}$. The map $\mathcal{O}_{\phi(P),Y} \longrightarrow \mathcal{O}_{P,X}$ is an isomorphism by assumption. In particular (U, f) is in the image, so there is some $(V, g) \mapsto (U, f)$. Then (suppressing restrictions), $g \circ \phi = f$ so $f \circ \phi^{-1} = g$. Thus, $f \circ \phi^{-1}$is regular on a neighborhood of P. Hence, $f \circ \phi^{-1}$ is regular and ϕ^{-1} is a morphism and ϕ is an isomorphsm.

□

I.3.4

Show that the d-uple embedding of $\mathbb{P}^n$ (I.2.12) is an isomorphism onto its image.

Proof. □

I.3.5

By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbb{P}^n$ is any hypersurface, show that $\mathbb{P}^n - H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$ and use the fact that $\mathbb{P}^N$ minus a hyperplace is affine.]

Proof. □

I.3.6

There are quasi-affine varieties which are not affine. For example, show that $X = \mathbb{A}^2 - 0$ is not affine. [*Hint*: Show that $\mathcal{O}(X) \cong k[x, y]$ and use I.3.5. See **??** for another proof.]

Proof. □

I.3.7

(a) Show that any two curves in $\mathbb{P}^2$ have a nonempty intersection.

(b) More generally, show that if $Y \subseteq \mathbb{P}^n$ is a projective variety of dimension ≥ 1, and if H is a hypersurface, then $Y \cap H \neq \emptyset$. [*Hint*: Use I.3.5 and I.3.1.e. See [Hartshorne, I.7.2] for a generalization.]

Proof. □

I.3.8

Let H_i and H_j be the hyperplanes in $\mathbb{P}^n$ defined by $x_i = 0$ and $x_j = 0$ with $i \neq j$. Show that any regular function on $\mathbb{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of [Hartshorne, I.3.4a] in the case $Y = \mathbb{P}^n$.)

Proof. □

I.3.9

The homogeneous coordinate ring is not an invariant under isomorphism. For example, let $X = \mathbb{P}^1$ an let Y be the 2-uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$. Then $X \cong Y$ (I.3.4). But show that $S(X) \ncong S(Y)$.

Proof. □

I.3.10

Subvarieties

S subset of a topological space is *locally closed* if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine of projective space. We call this the *induced structure* on Y, and we call Y a *subvariety* of X.

Now let $\phi : X \longrightarrow Y$ be a morphism and let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\phi[X'] = Y'$. Show that $\phi|_{X'} : X' \longrightarrow Y'$ is a morphism.

Proof. □

I.3.11

Let X be any variety and let $P \in X$. Show that there is a $1-1$ correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

Proof. □

I.3.12

If P is a point on a variety X, then $\dim \mathcal{O}_p = \dim X$. [*Hint*: Reduce to the affine case and use [Hartshorne, I.3.2c]].

Proof. □

I.3.13

The Local Ring of a Subvariety

Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes (U, f) where $U \subseteq X$ is open, $U \cap Y \neq \emptyset$, and f is a regular function of U. We say that (U, f) is equivalent to (V, g) if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the *local ring* of Y on X. Note that if $Y = \{P\}$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

Proof. □

I.3.14

Projection from a Point

Let $\mathbb{P}^n$ be a hyperplane in $\mathbb{P}^{n+1}$ and let $P \in \mathbb{P}^{n+1} - \mathbb{P}^n$. Define a mapping $\phi : \mathbb{P}^{n+1} = \{P\} \longrightarrow \mathbb{P}^n$ by $\phi(Q) =$ the itnersection of the unique line containing P and Q with $\mathbb{P}^n$.

(a) Show that ϕ is a morphism.

(b) Let $Y\mathbb{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbb{P}^1$ (I.2.12). If t, u are homogeneous coordinates on $\mathbb{P}^1$, we say that Y is a curve given *parametrically* by $(x, y, z, w) = (t^3, t^2u, tu^2, y^3)$. Let $P = [0 : 0 : 1 : 0]$, and let $\mathbb{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

Proof. □

I.3.15

Products of Affine Varieties

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbb{A}^{n+m}$ with its induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X | x \times Y \subseteq Z_i\}$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology (I.1.4).

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

(c) Show that $X \times Y$ is a product in the category of varieties.

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

Proof. □

I.3.16

Products of Quasi-Projective Varieties

Use the Segre embedding I.2.14 to identify $\mathbb{P}^n \times \mathbb{P}^m$ with its image and hence give it a structure of a projective variety. Now for any teo quasi-projective varieties $X \subseteq \mathbb{P}^n$ and $Y \subseteq \mathbb{P}^m$, consider $X \times Y \subseteq \mathbb{P}^n \times \mathbb{P}^m$

(a) Show that $X \times Y$ is a quasi-projective variety.

(b) If X, Y are both projective, show that $X \times Y$ is projective.

(c) Show that $X \times Y$ is a product in the category of varieties.

Proof. □

I.3.17

Normal Varieties

A variety Y is *normal at a point* $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is *normal* if it is normal at every point.

(a) Show that every conic in $\mathbb{P}^2$ is normal.

(b) Show that the quadric suraces Q_1, Q_2 in $\mathbb{P}^3$ given by the equations $Q_1 = Z(xy - zw)$, $Q_2 = Z(xy - z^2)$ are normal. (cd. **??** for the latter.)

(c) Show that the cuspidal cubic $y^2 = x^2$ in $\mathbb{A}^3$ is not normal.

(d) If Y is affine, then Y is normal iff $A(Y)$ is integrally closed.

(e) Let Y be an affine variety. Show that there is a normal variety $\widetilde{Y}$, and a morphism $\pi : \widetilde{Y} \longrightarrow Y$, with the property that whenever Z is a normal variety and $\phi : Z \longrightarrow Y$ is a *dominant* morphism, (i.e. $\phi[Z]$ is dense in Y), then there is a unique morphism $\theta : Z \longrightarrow \widetilde{Y}$ such that $= \pi \circ \theta$. $\widetilde{Y}$ is called the *normalization* of Y. You will need [Hartshorne, I.3.9A] above.

Proof. □

I.3.18

Projectively Normal Varieties

A projective variety $Y \subseteq \mathbb{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.

(a) If Y is projectively normal then Y is normal.

(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbb{P}^3$ given parametrically by $(x, y, z, w) = (t^4, t^3u, t^2u^2, tu^3, u^4)$. Then Y is normal by not projectively normal. See **??** for more examples.

(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbb{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

Proof. □

I.3.19

Automorphisms of $\mathbb{A}^n$

Let $\phi : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ be a morphism of $\mathbb{A}^n$ to $\mathbb{A}^n$ given by n polynomials $f_1, \dots, f_n$ of n variables $x_1, \dots, x_n$. Let $J = \det \left| \frac{\partial f_i}{\partial x_j} \right|$ be the Jacobian polynomial of ϕ.

(a) If ϕ is an isomorphism (in which case we call ϕ an *automotphism* of $\mathbb{A}^n$) show that J is a nonzero constant polynomial.

(b) The converse of (a) is an unsolved problem, even for $n = 2$. See, for example, Vitushkin [1].

Proof. $\square$

I.3.20

Let Y be a variety of dimension ≥ 2, an let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.

(a) Show that f extends to a regular function on Y.

(b) Show that this would be false of $\dim Y = 1$. See **??** for a generalization.

Proof. $\square$

I.3.21

Group Varieties

A group variety consists of a variety Y together with a morphism $Y \times Y \longrightarrow Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \mapsto y^{-1}$ is also a morphism of $Y \longrightarrow Y$.

(a) The *additive group* $\mathbf{G}_a$ is given by the variety $\mathbb{A}^1$ and the morphism $\mu : \mathbb{A}^2 \longrightarrow \mathbb{A}^1$ defined by $\mu(a, b) = a + b$. Show it is a group variety.

(b) The *multiplicative group* $\mathbf{G}_m$ is given by the variety $\mathbb{A}^1 - 9$ and the morphism $\mu(a, b) = ab$. Show it is a group variety.

(c) If G is a group variety, and X is any variety, show that the set $\mathrm{Hom}(X, G)$ has a natural group structure.

(d) For any variety X show that $\mathrm{Hom}(X, \mathbf{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

(e) For any variety X show that $\mathrm{Hom}(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$ under multiplication.

Proof. $\square$

References

[Eisenbud] D. Eisenbud. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995. ISBN: 9780387942698. URL: https://books.google.com/books?id=Fm%5C_yPgZBucMC.

[CLO07] David A. Cox, John B. Little, and Donal O'Shea. *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*. en. 3rd ed. Undergraduate texts in mathematics. New York: Springer, 2007. ISBN: 978-0-387-35650-1 978-0-387-35651-8.

[Hartshorne] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN: 9781475738490. URL: https://books.google.com/books?id=7z4mBQAAQBAJ.

[Stacks] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu. 2020.