CS60010: Deep Learning

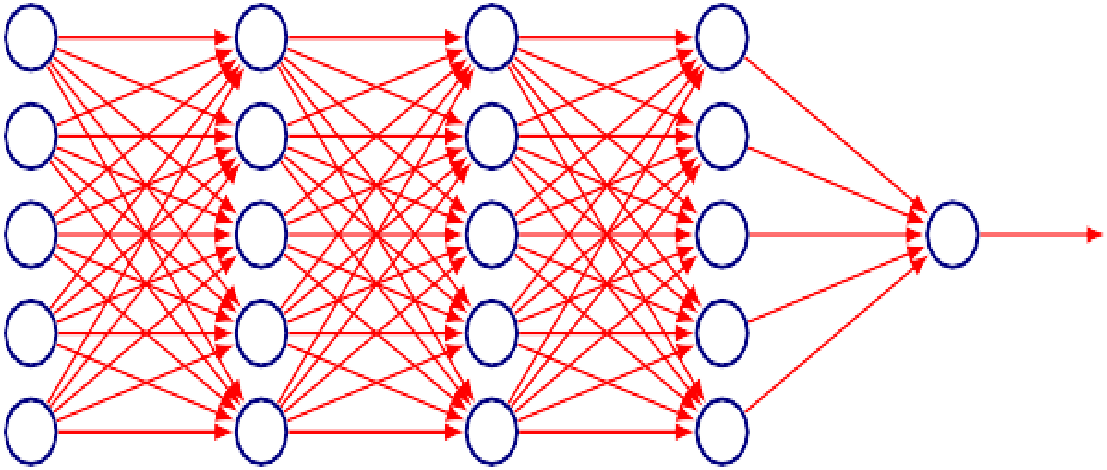
Sudeshna Sarkar

Spring 2019

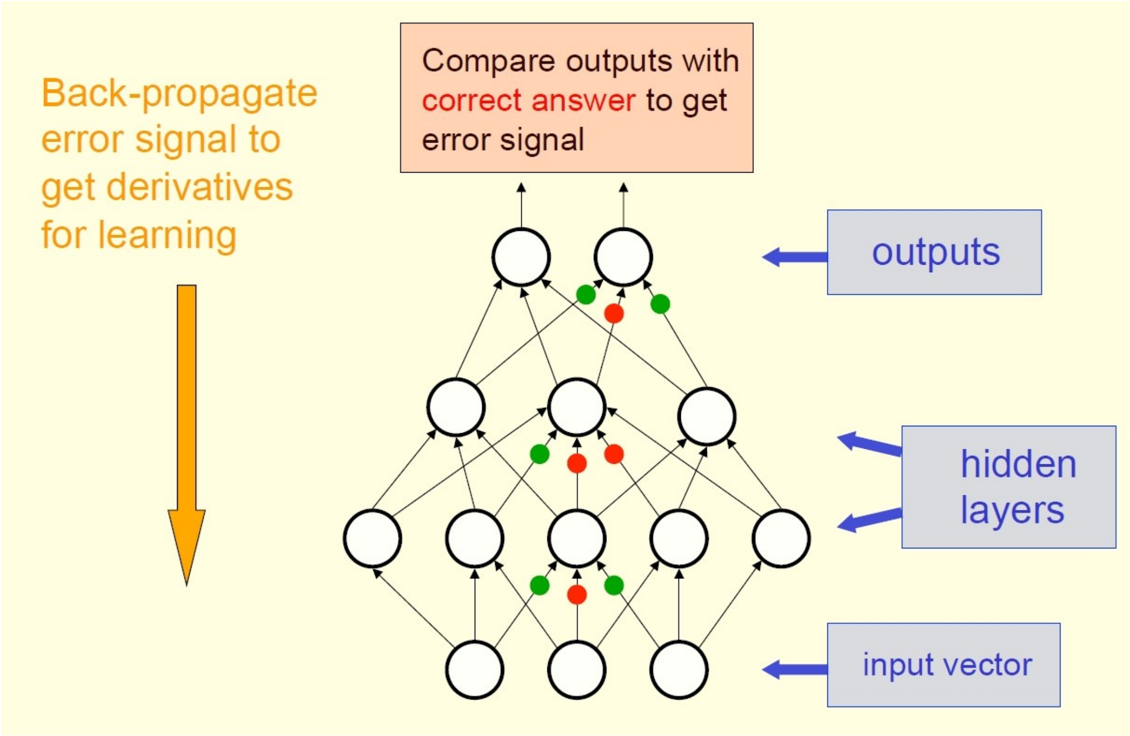
23 Jan 2019

BACKPROPAGATION: INTRODUCTION

How do we learn weights?



Backpropagation



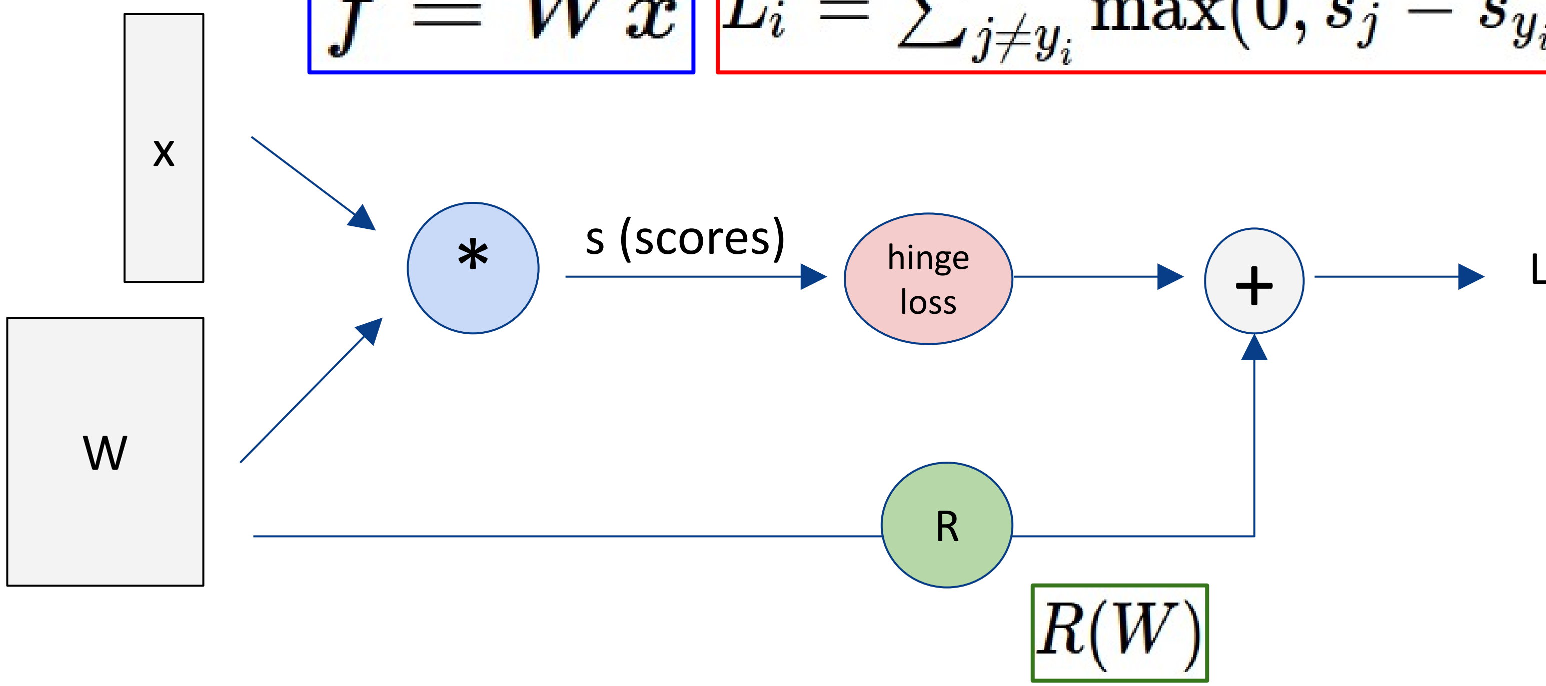
- **Feedforward Propagation**: Accept input x, pass through intermediate stages and obtain output $\hat{y}$
- **During Training**: Use $\hat{y}$ to compute a scalar cost $J(\theta)$
- Backpropagation allows information to flow backwards from cost to compute the gradient

Backpropagation

- We don't know what the hidden units should do
- But, we can compute how fast the error changes as we change a hidden activity
- Use error derivatives w.r.t hidden activities
- Each hidden unit can affect many output units and have separate effects on error – combine these effects
- Can compute error derivatives for hidden units efficiently (and once we have error derivatives for hidden activities, easy to get error derivatives for weights going in)

Computational Graph

$$\boxed{f = Wx} \quad \boxed{L_i = \sum_{j \neq y_i} \max(0, s_j - s_{y_i} + 1)}$$

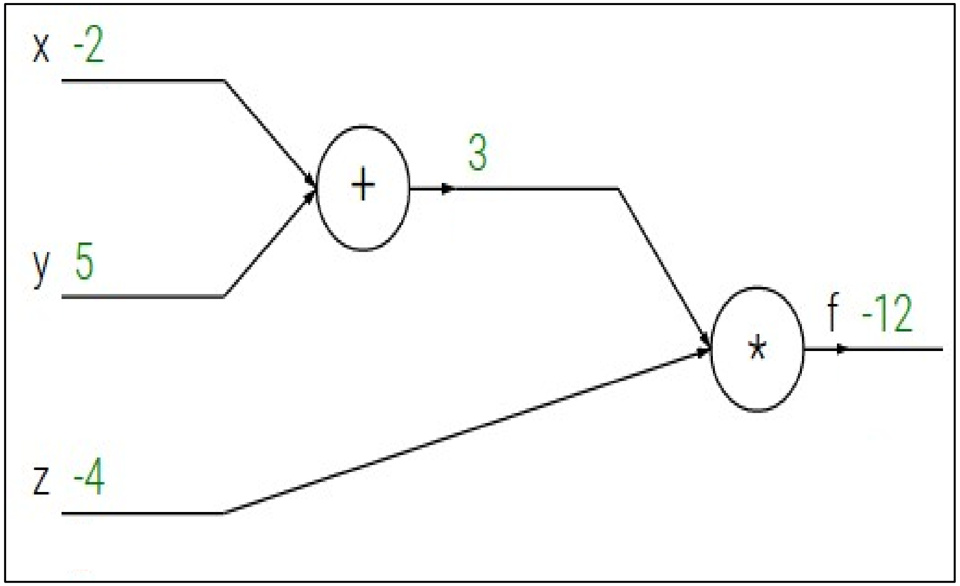


- Formalize computation as graphs
- Nodes indicate variables (scalar, vector, tensor or another variable)
- Operations are simple functions of one or more variables

Differentiating a Computation Graph

$$f(x, y, z) = (x + y)z$$

e.g. x = −2, y = 5, z = −4



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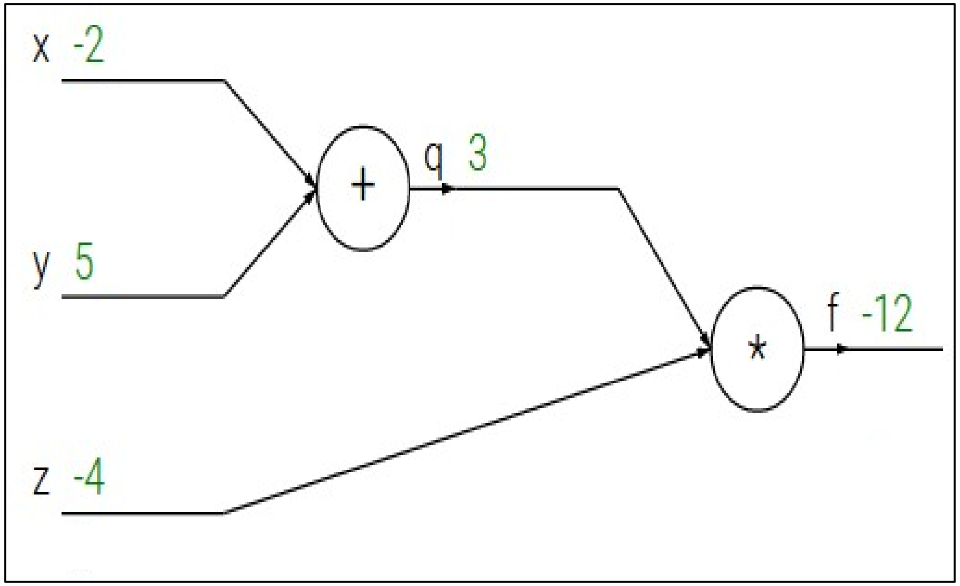
Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$



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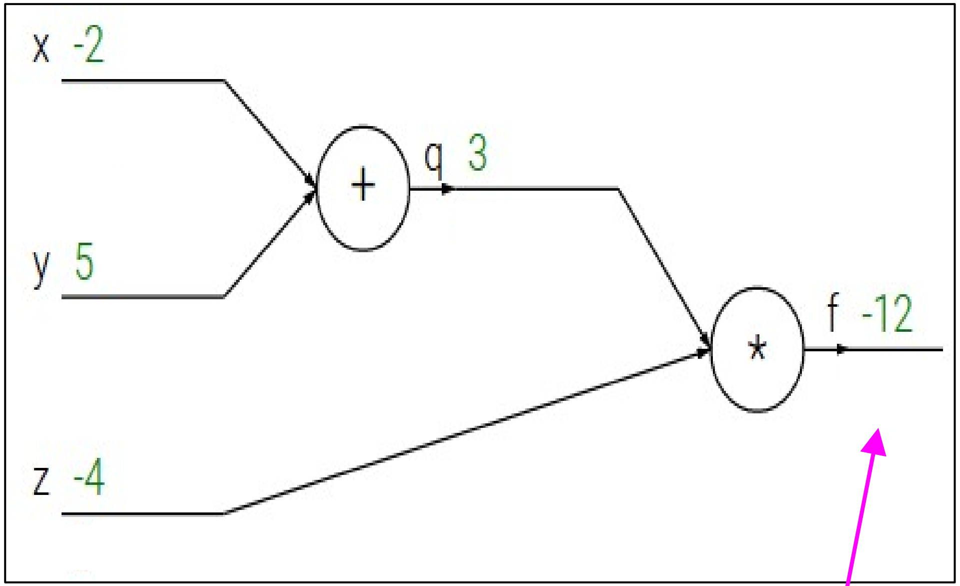

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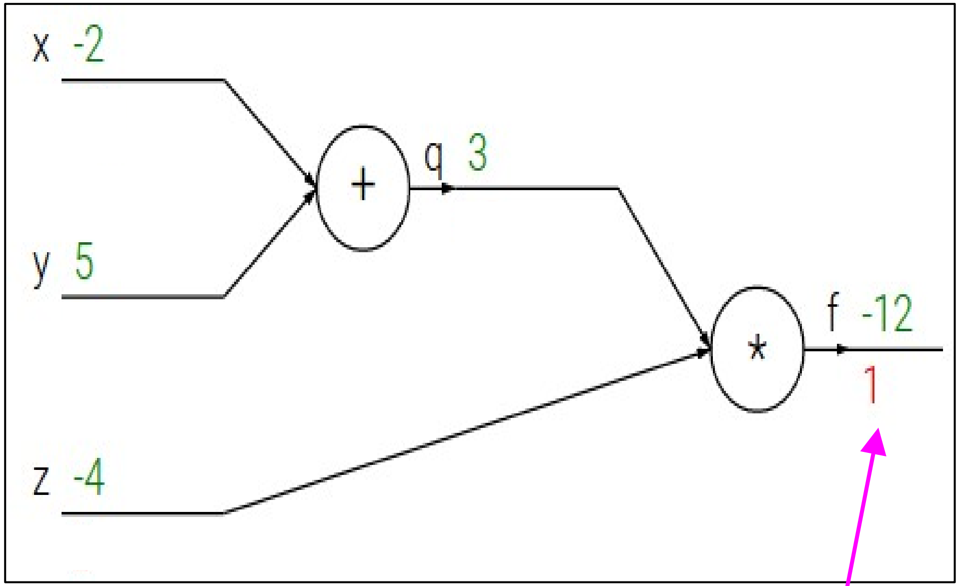

$$\frac{\partial f}{\partial f}$$

Based on cs231n by Fei-Fei Li & Andrej Karpathy & Justin Johnson

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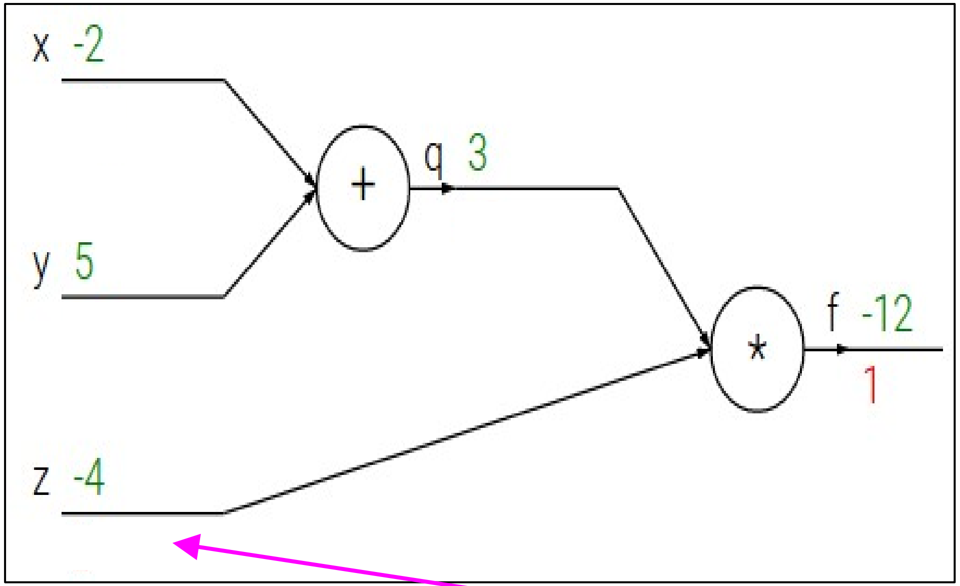

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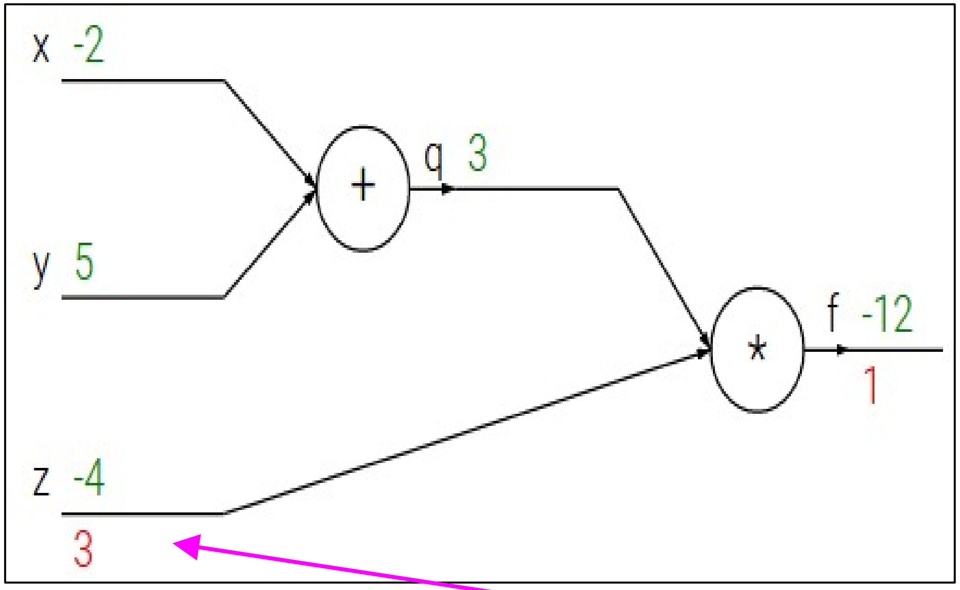

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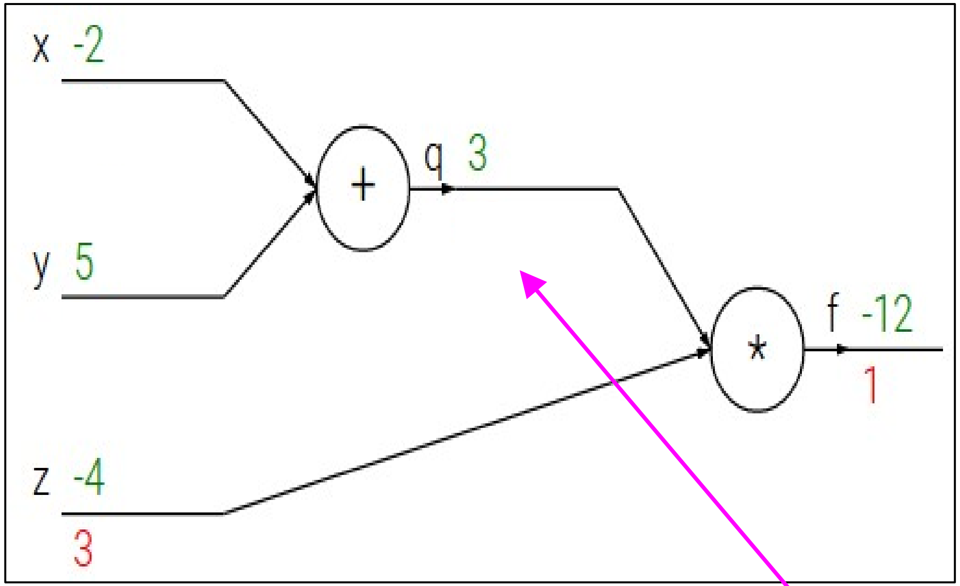

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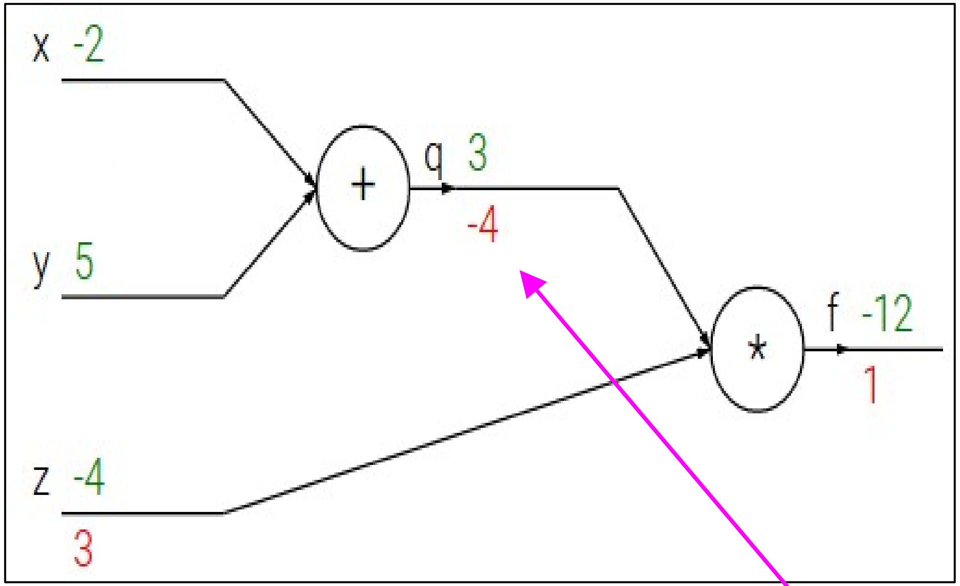

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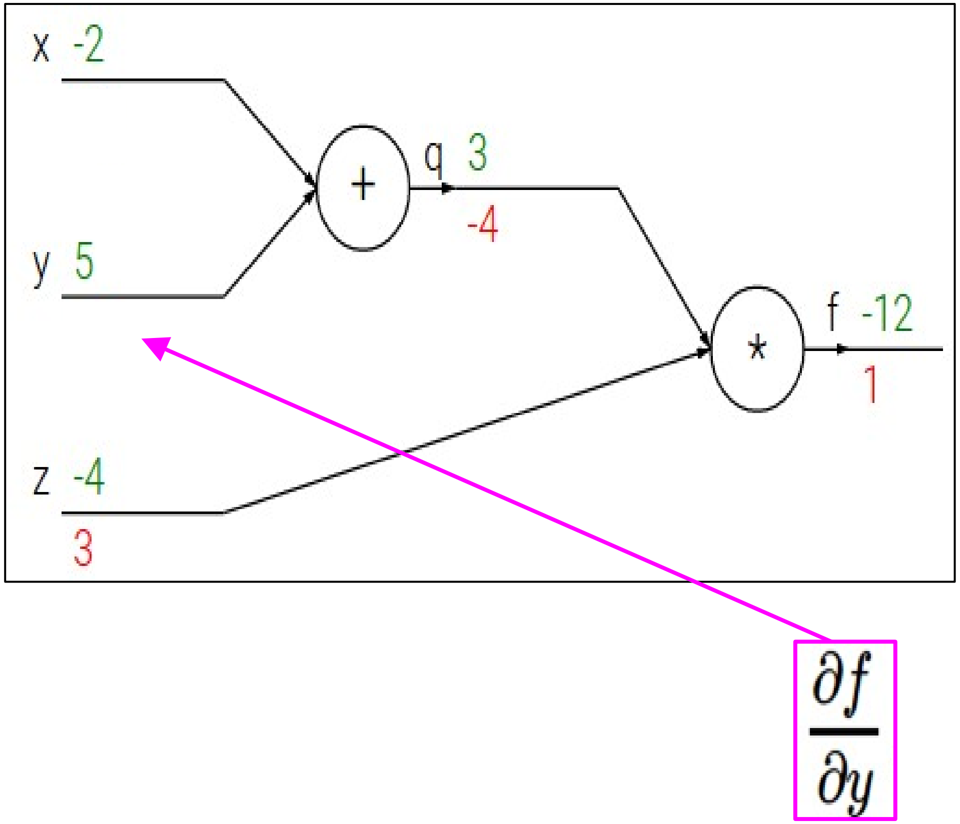

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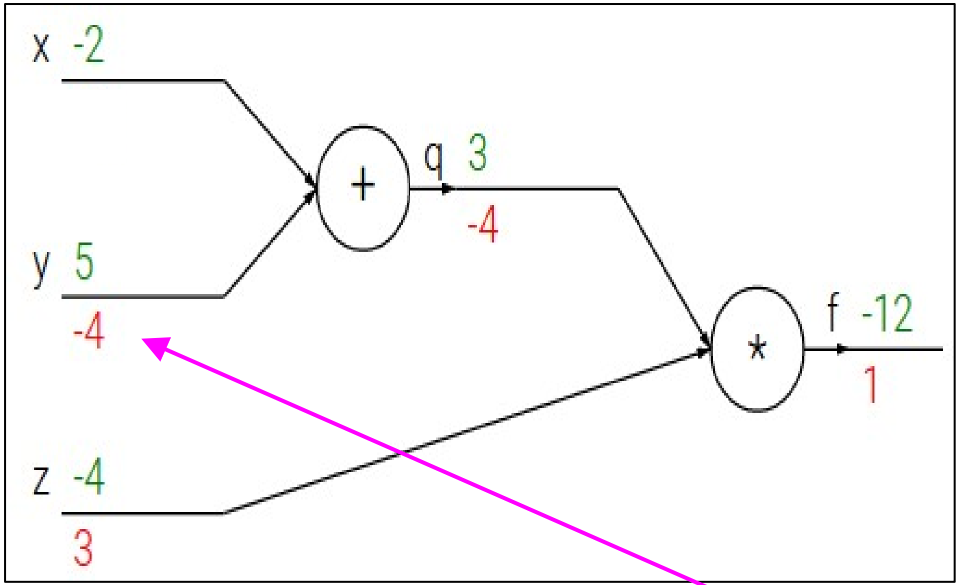

$$\frac{\partial f}{\partial y}$$

Chain rule:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q} \frac{\partial q}{\partial y}$$

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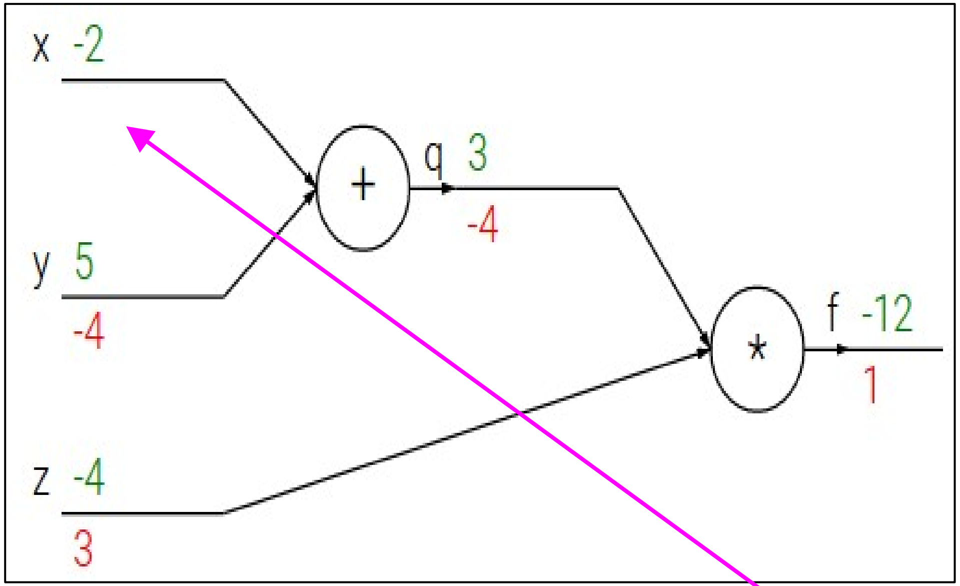

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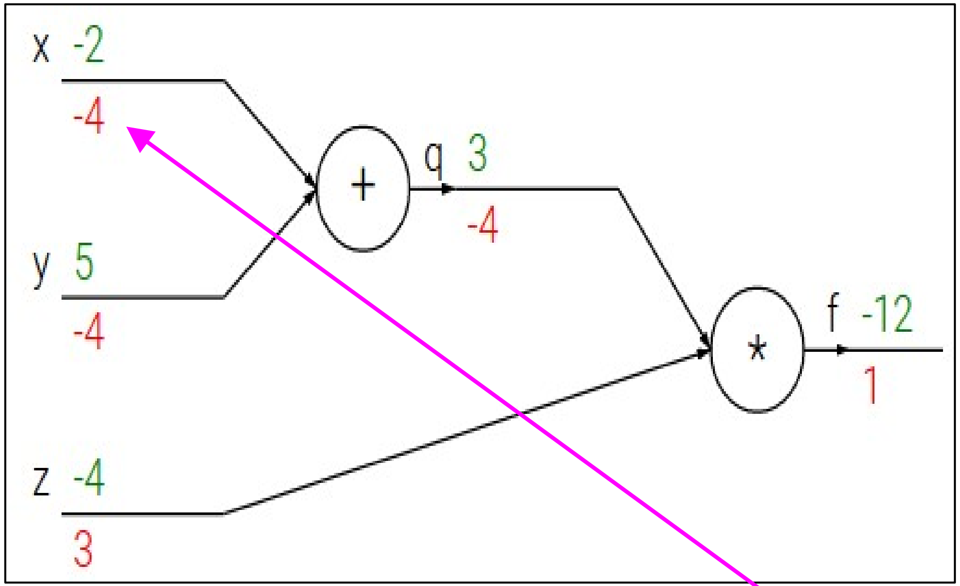

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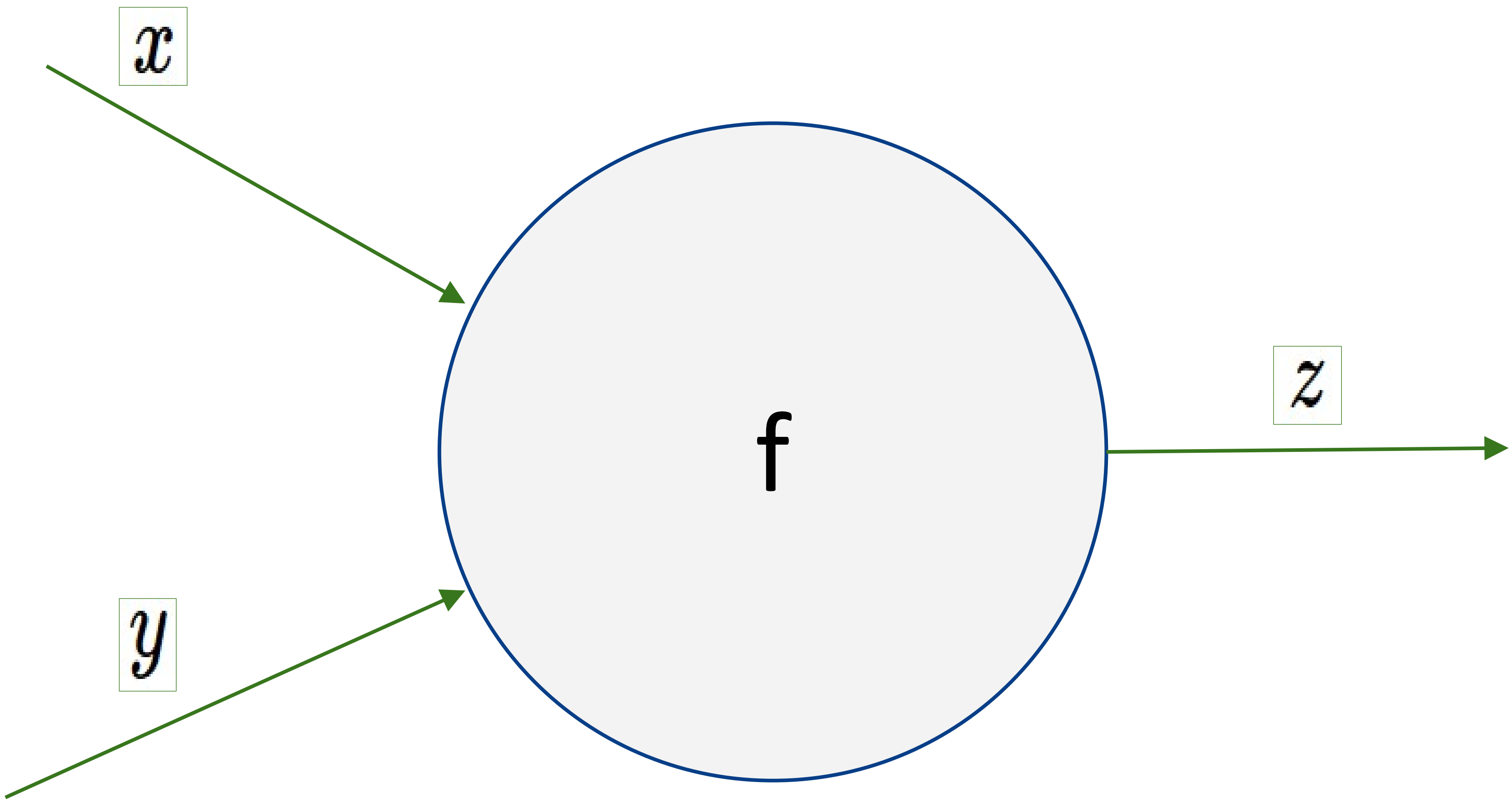
activations



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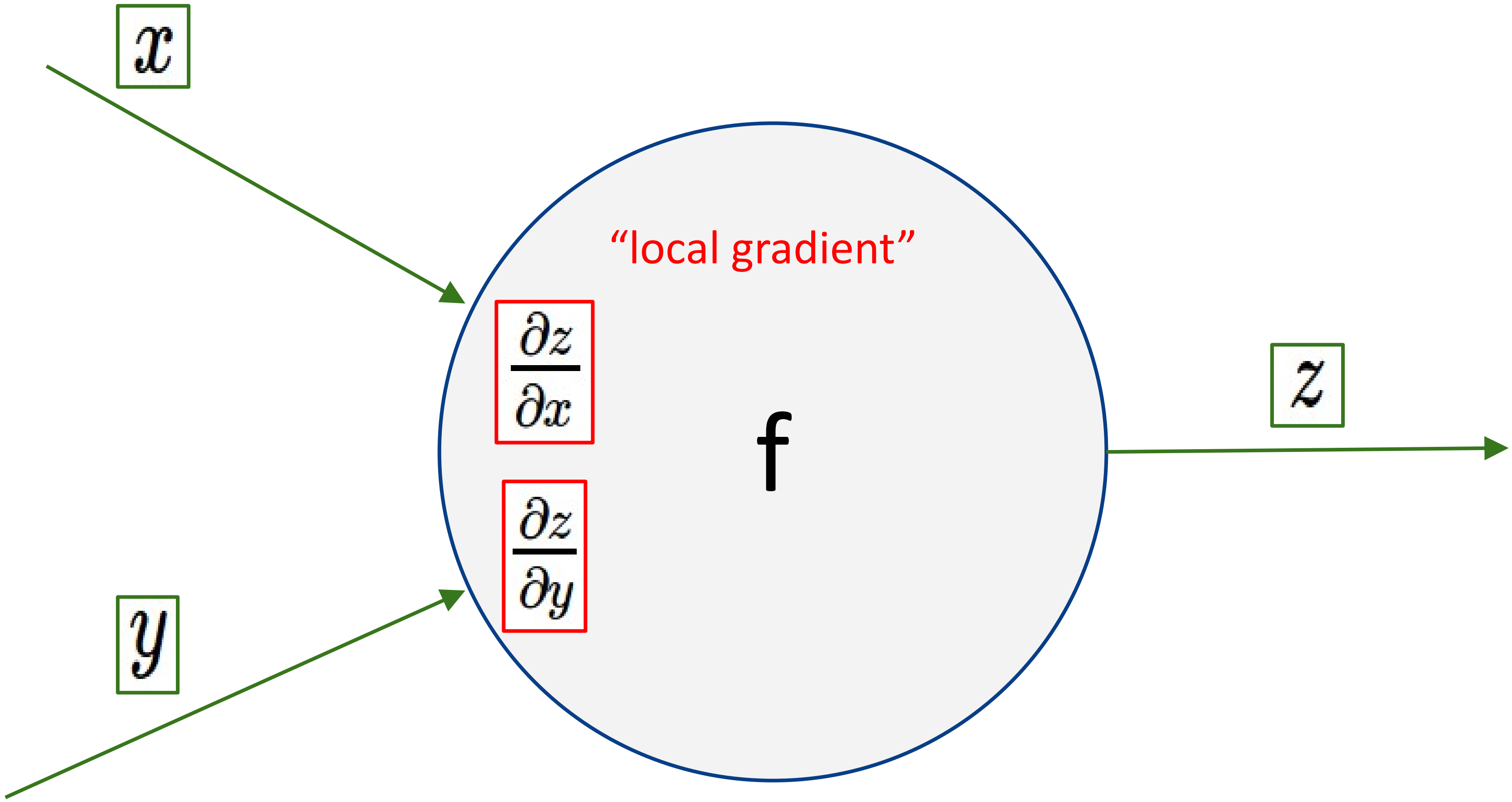
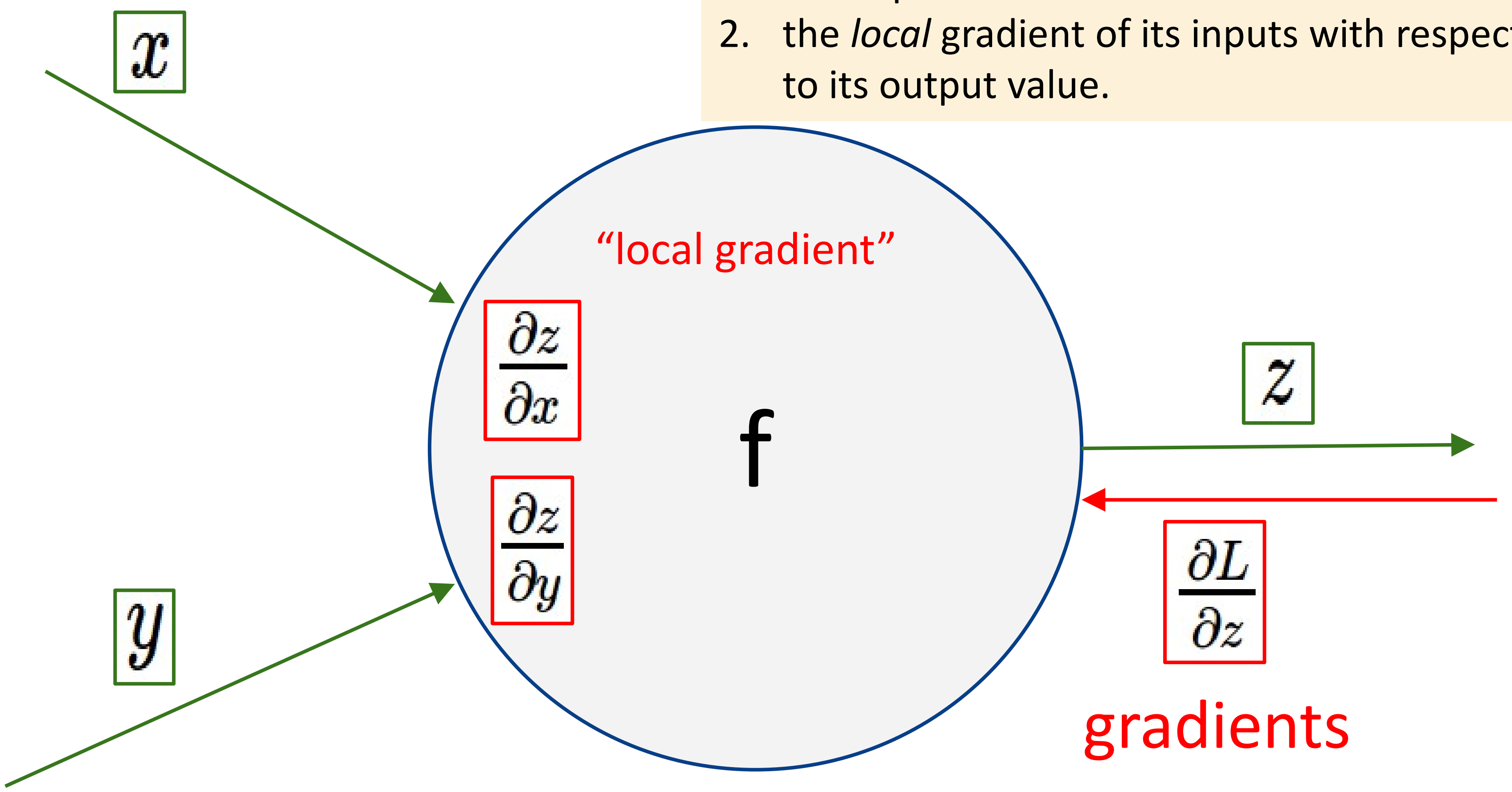


activations

Backpropagation: local process.
Every gate gets some inputs and can compute two things:

1. its output value
2. the *local* gradient of its inputs with respect to its output value.



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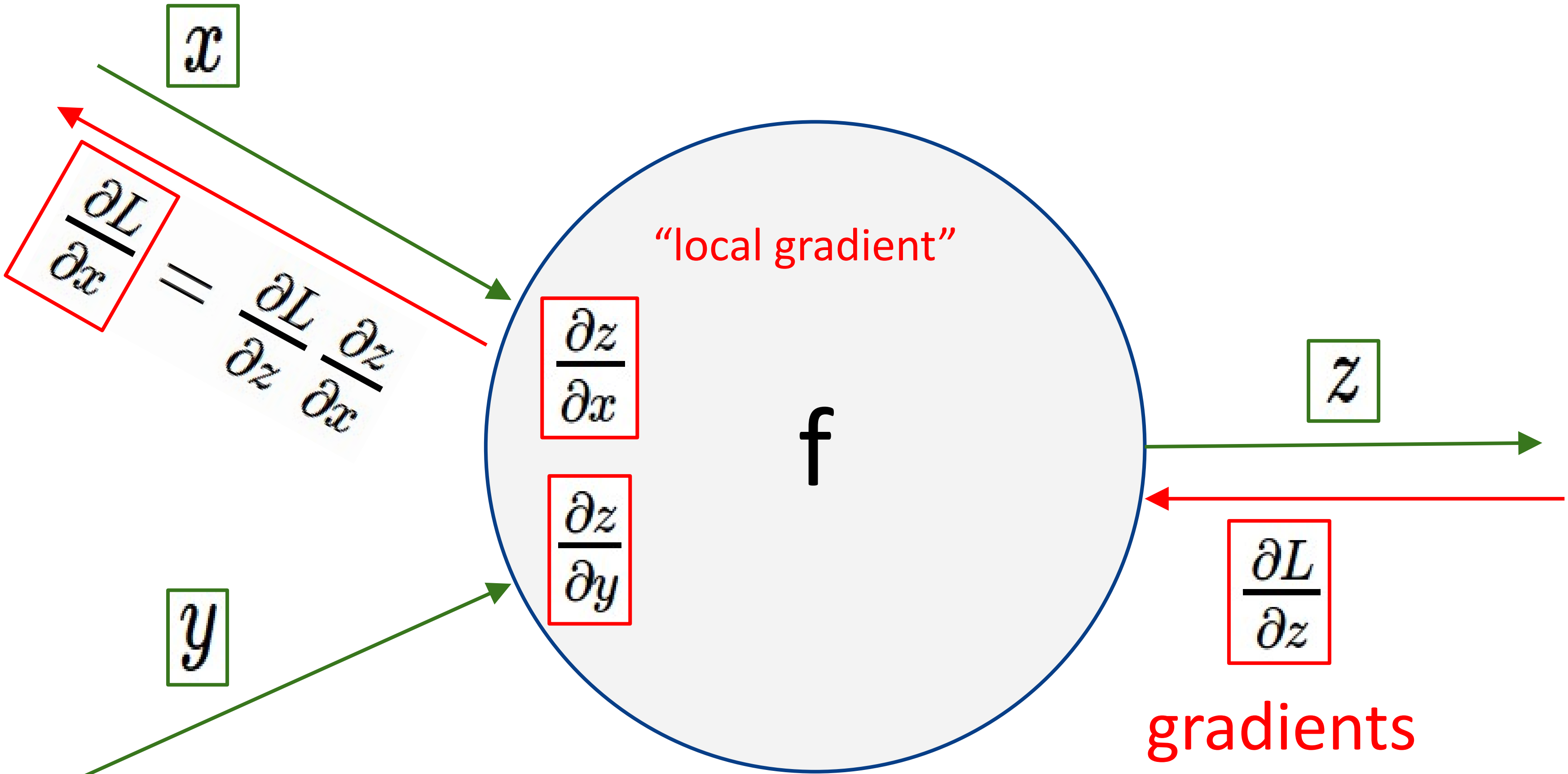

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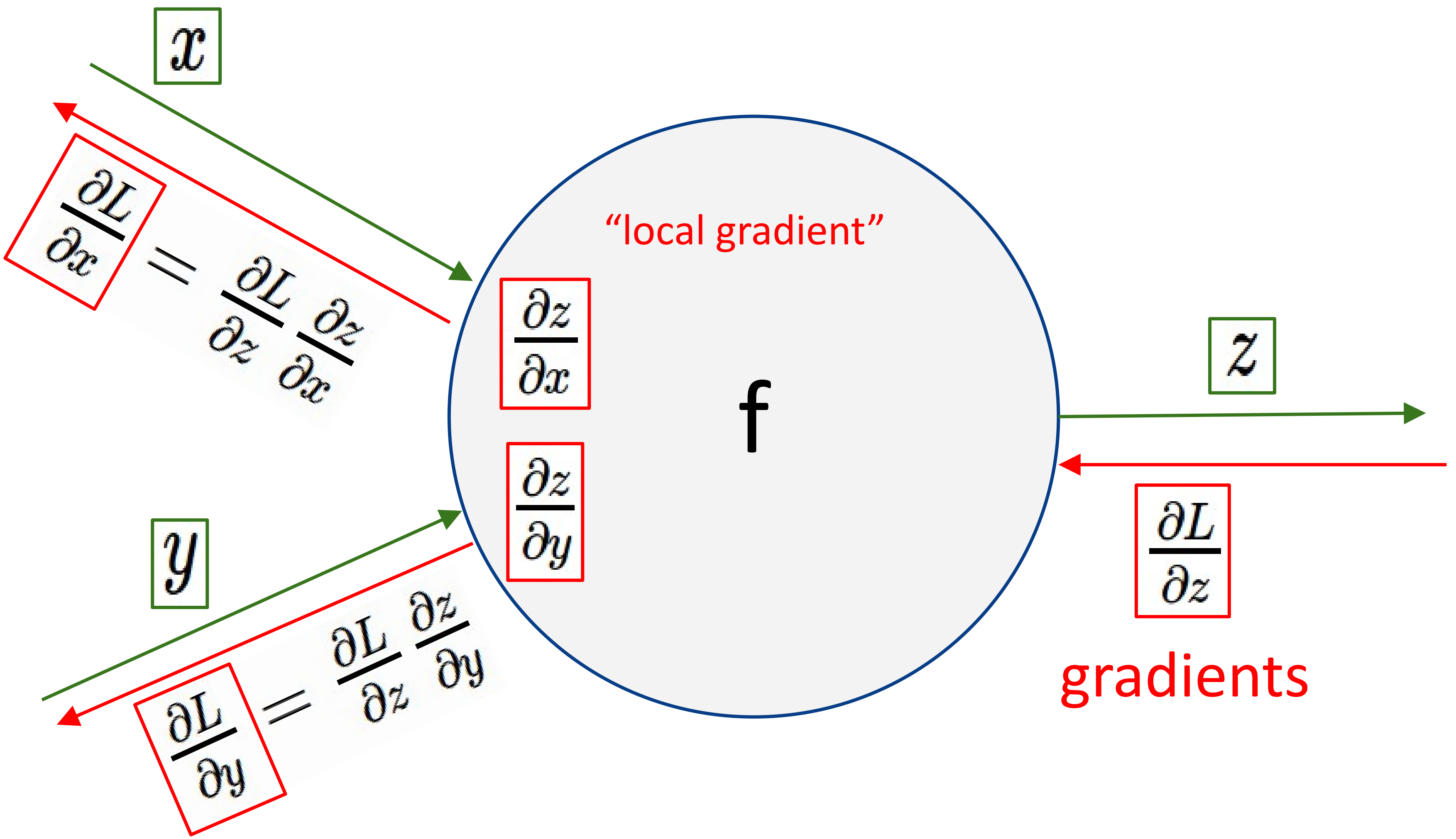

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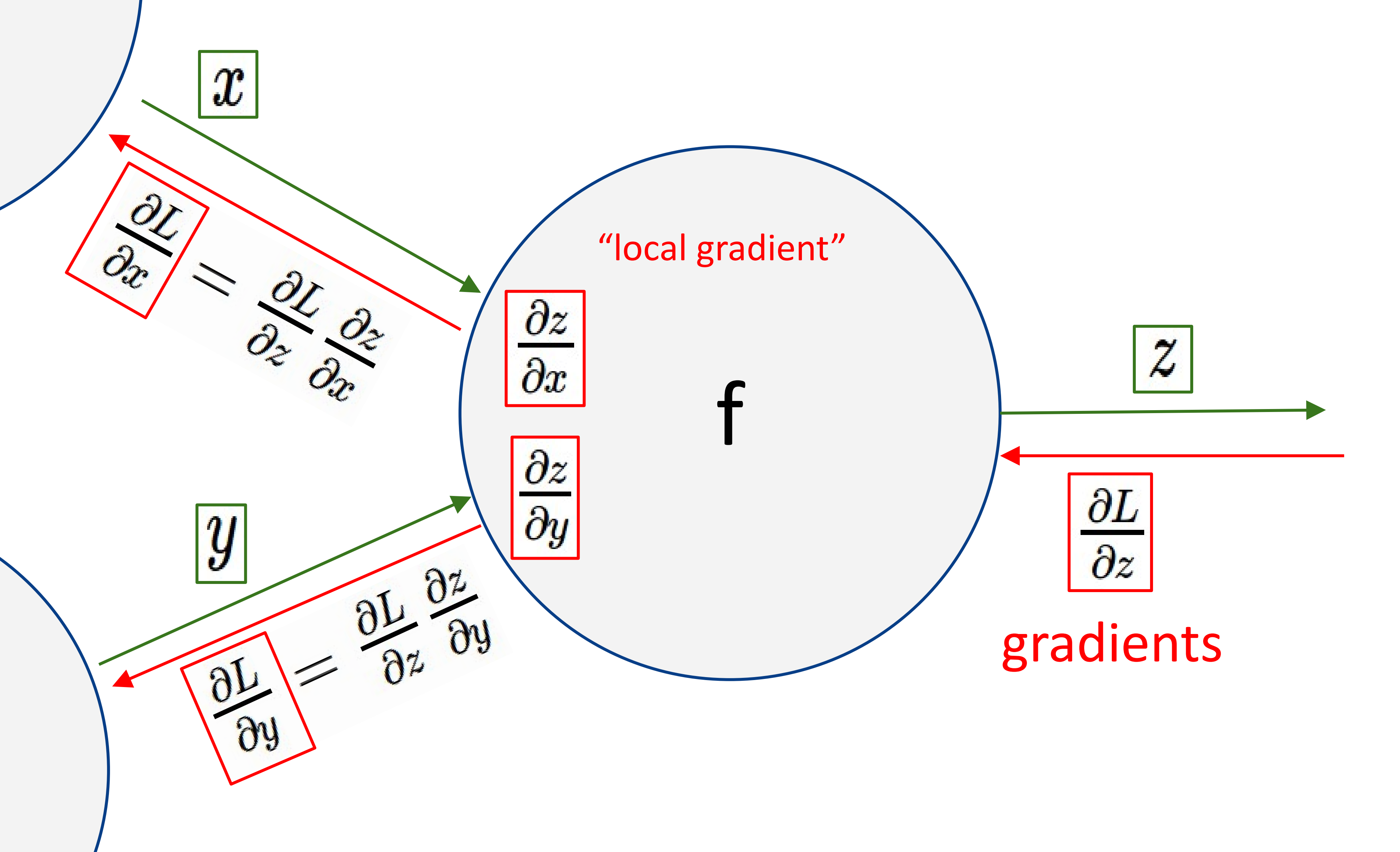
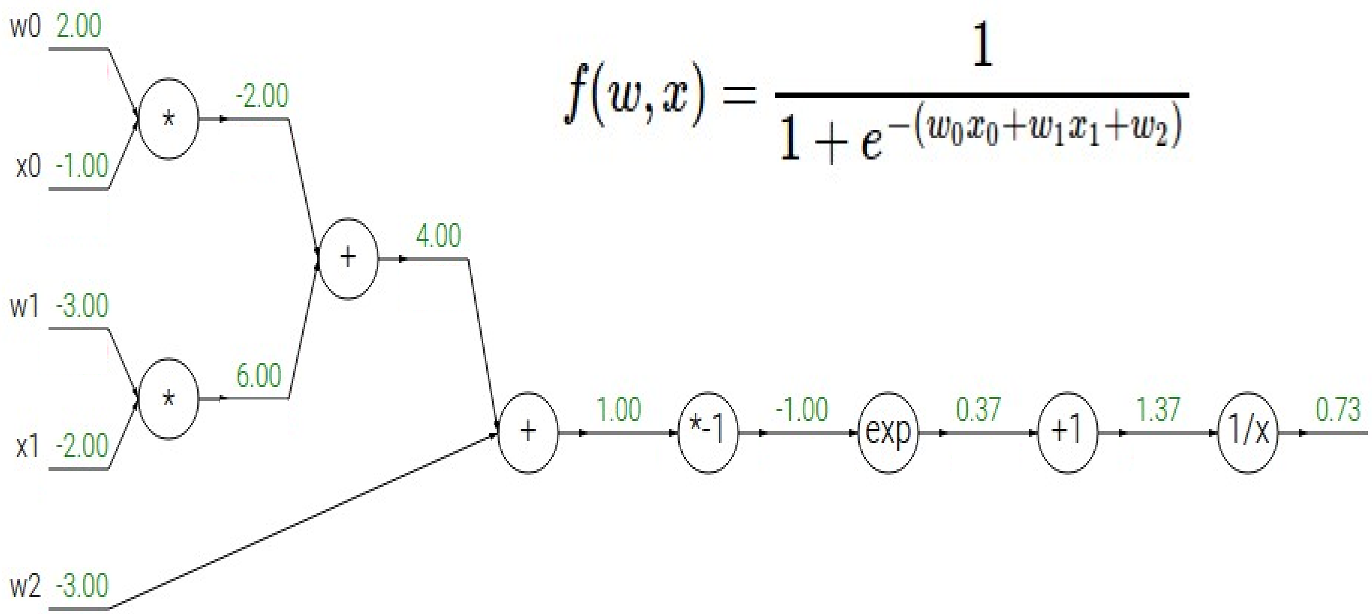


Another backprop example:

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$



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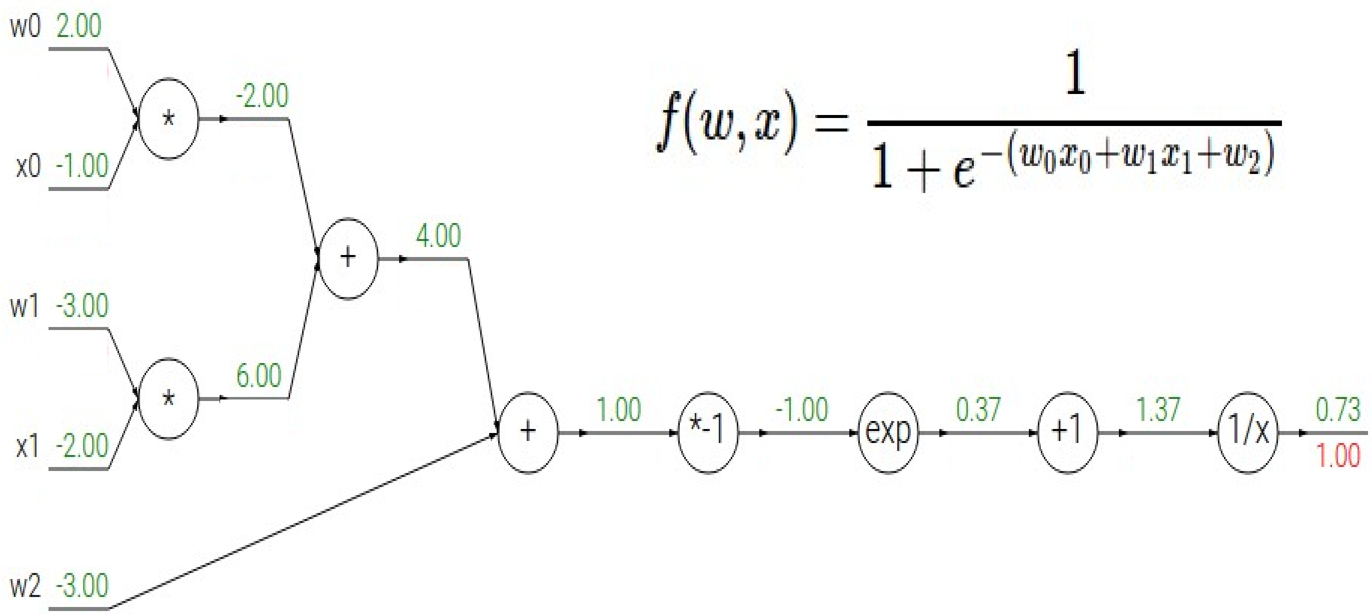

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x \qquad\qquad f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a \qquad\qquad f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

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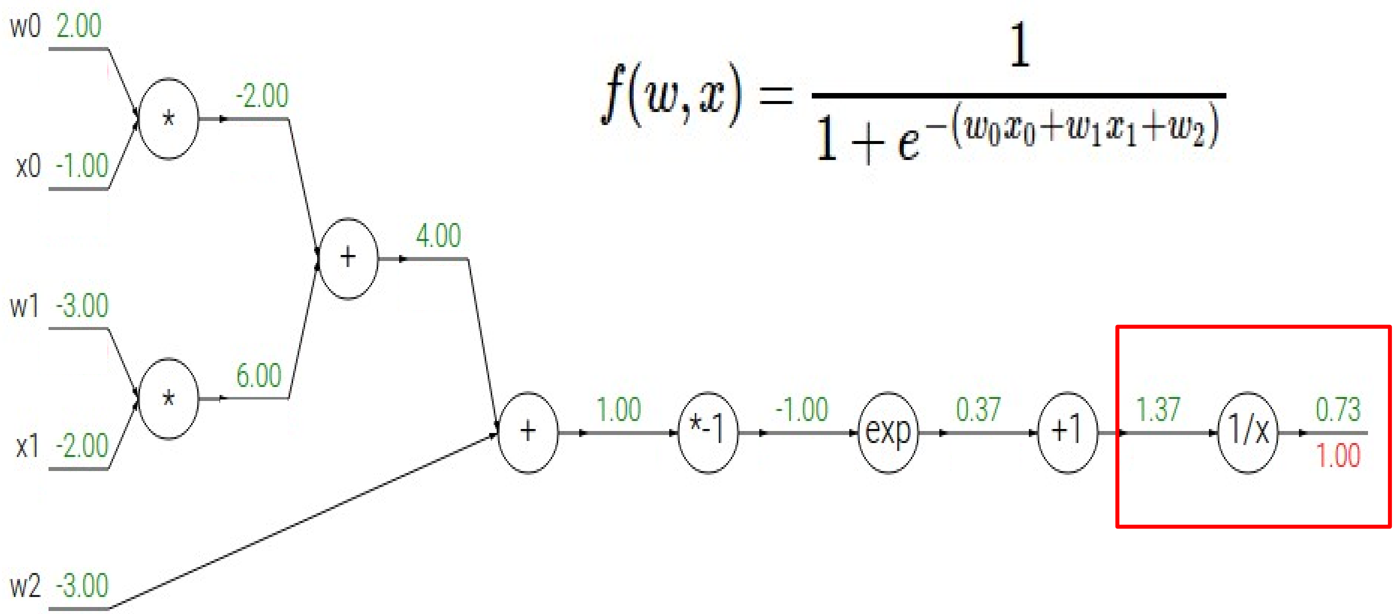

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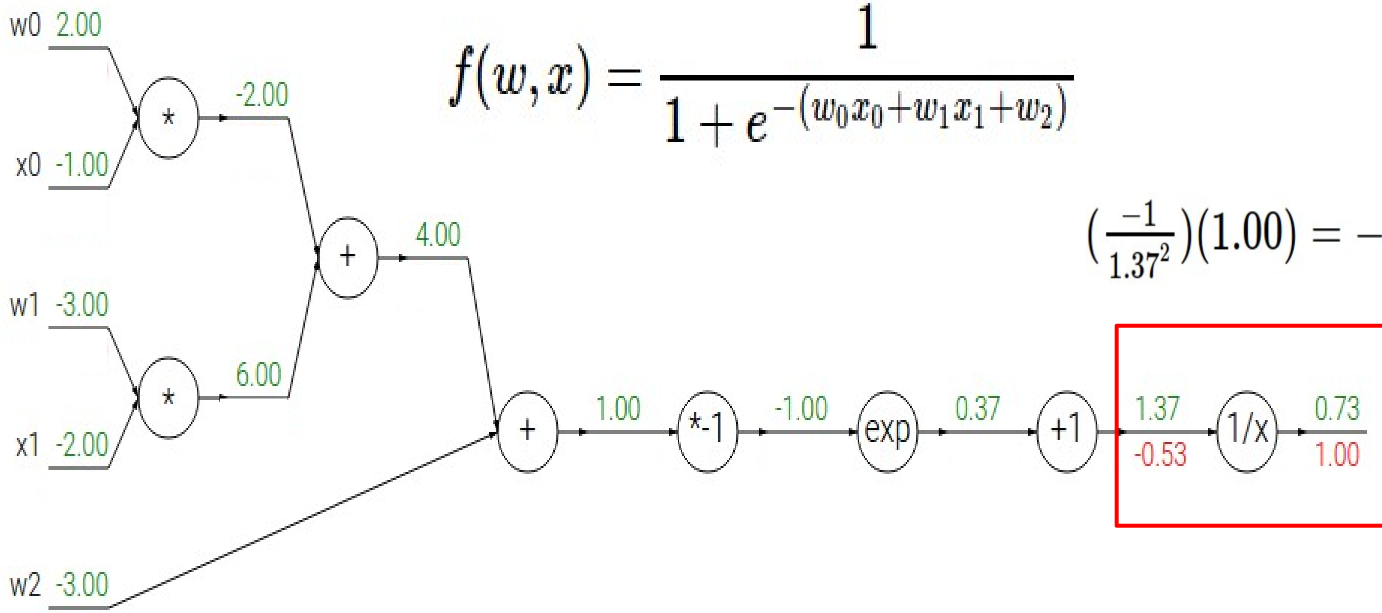

$$\left(\frac{-1}{1.37^2}\right)(1.00) = -0.53$$

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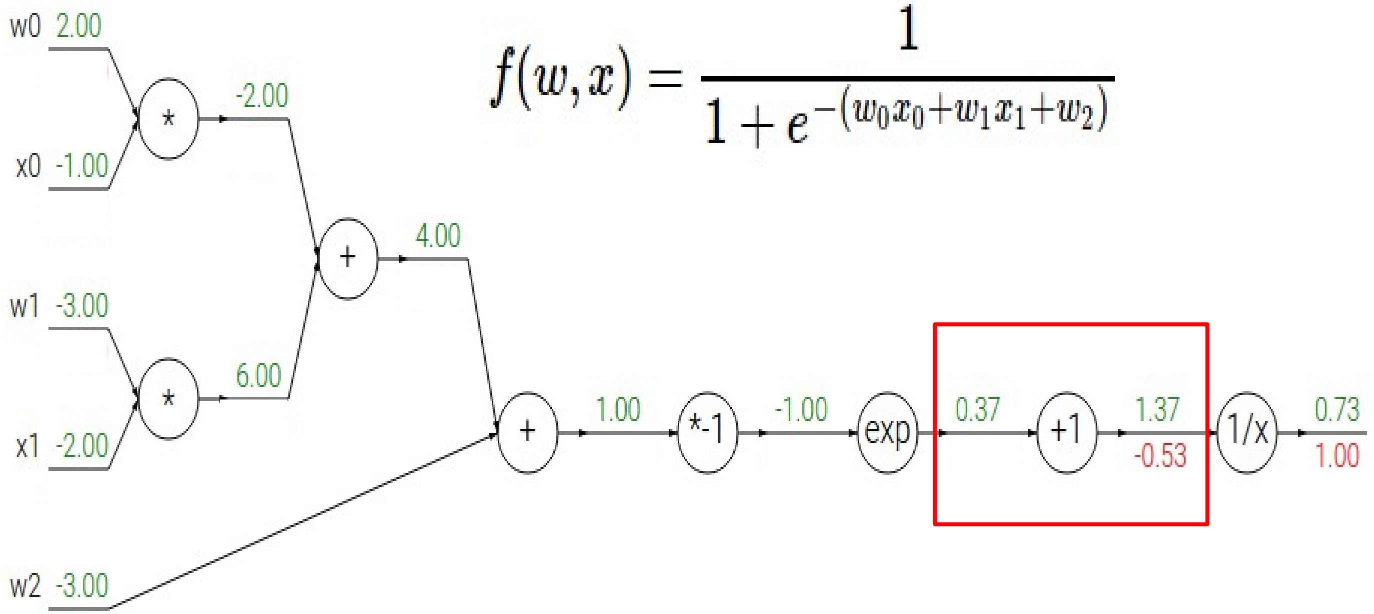

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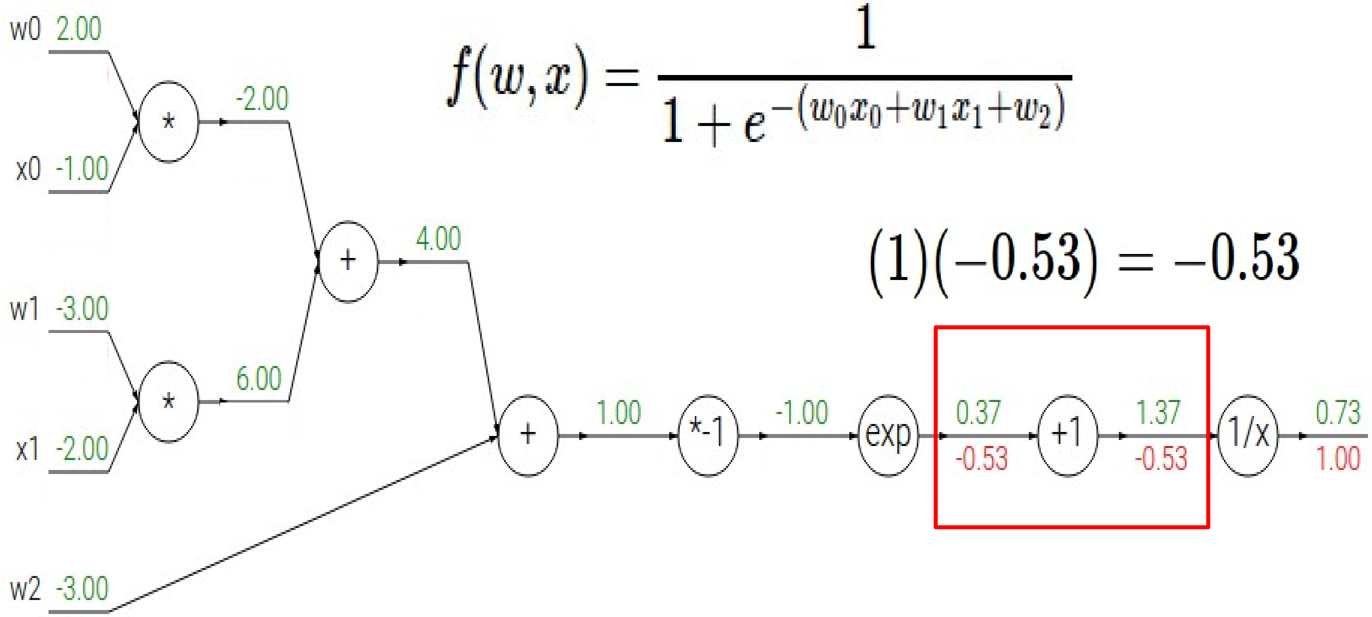

$$(1)(-0.53) = -0.53$$

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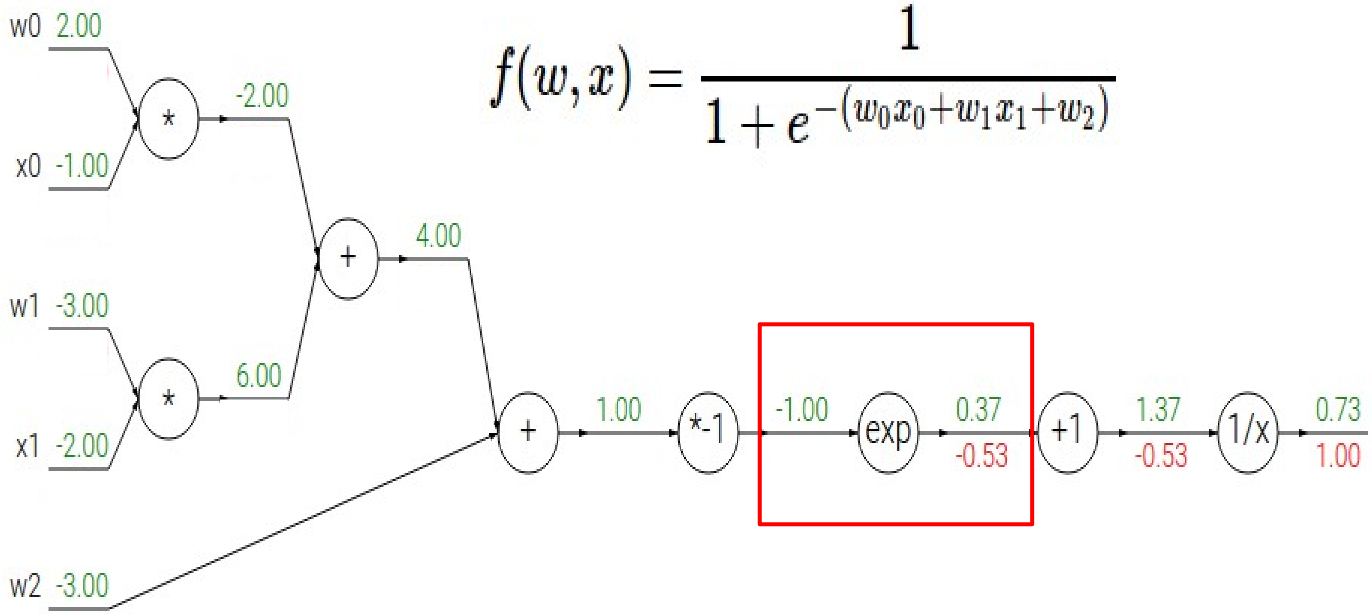

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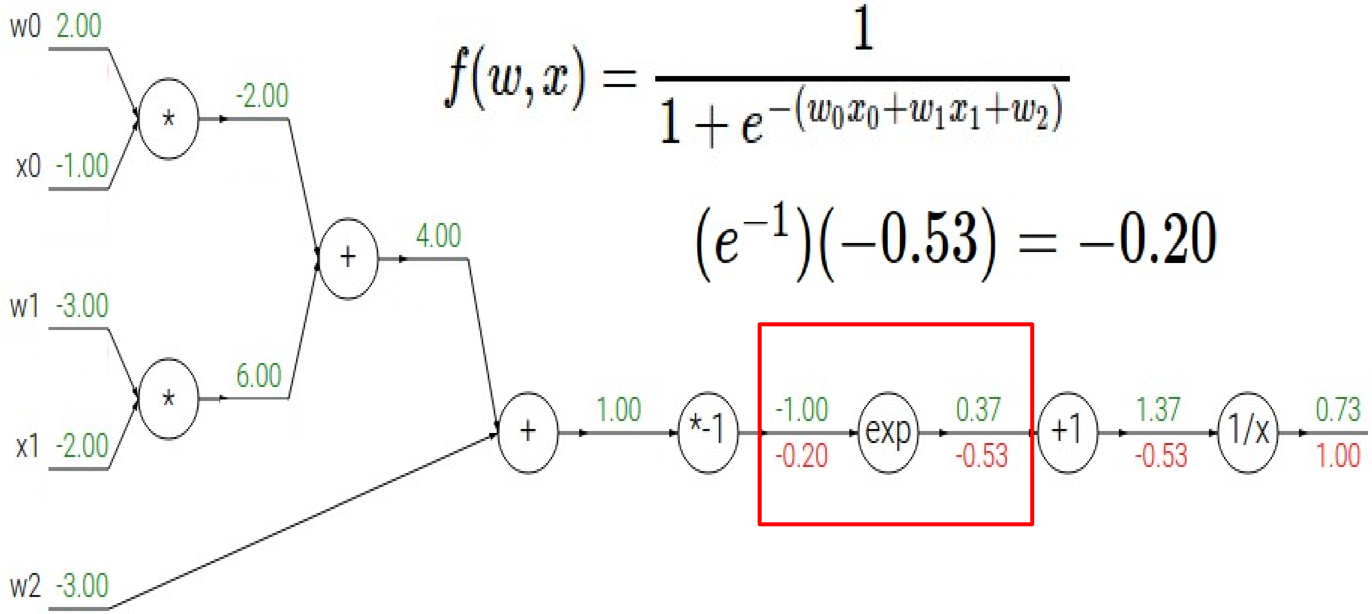

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$(e^{-1})(-0.53) = -0.20$$

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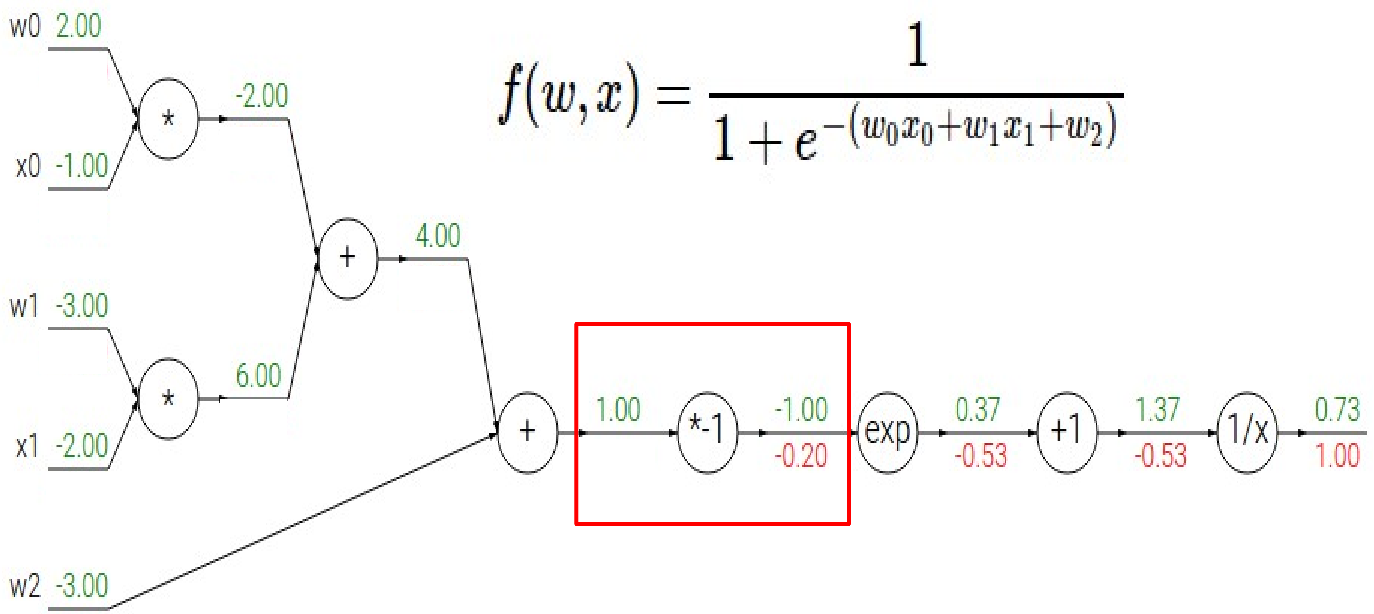

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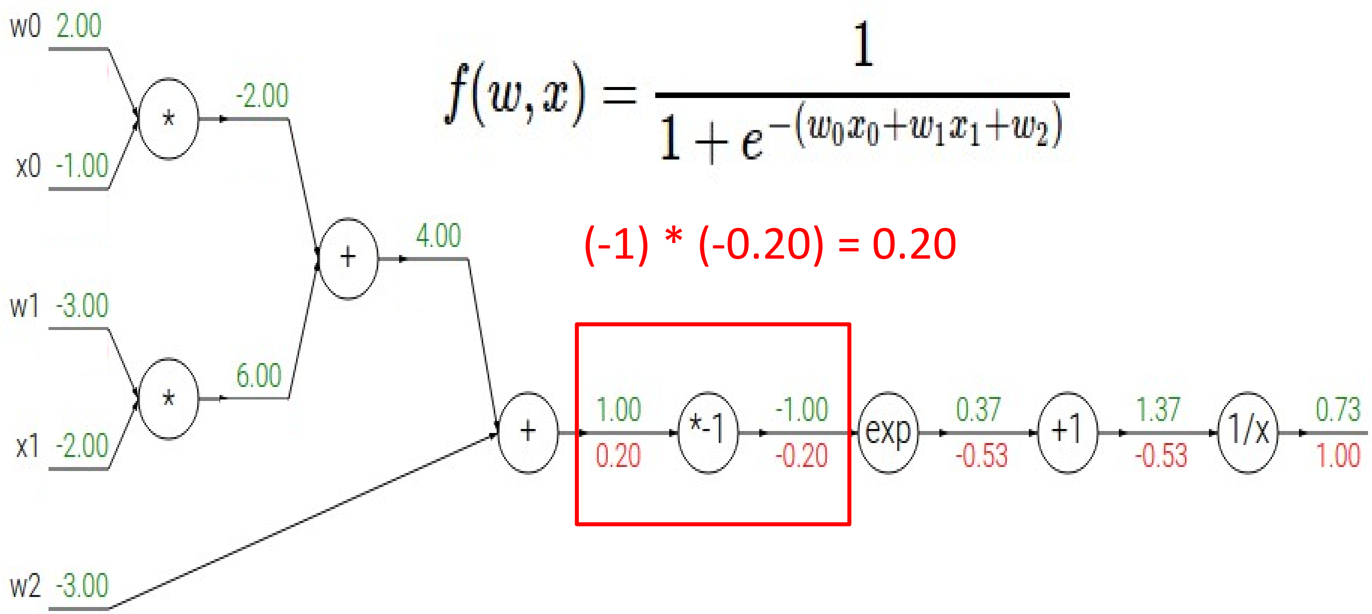

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(-1) * (-0.20) = 0.20

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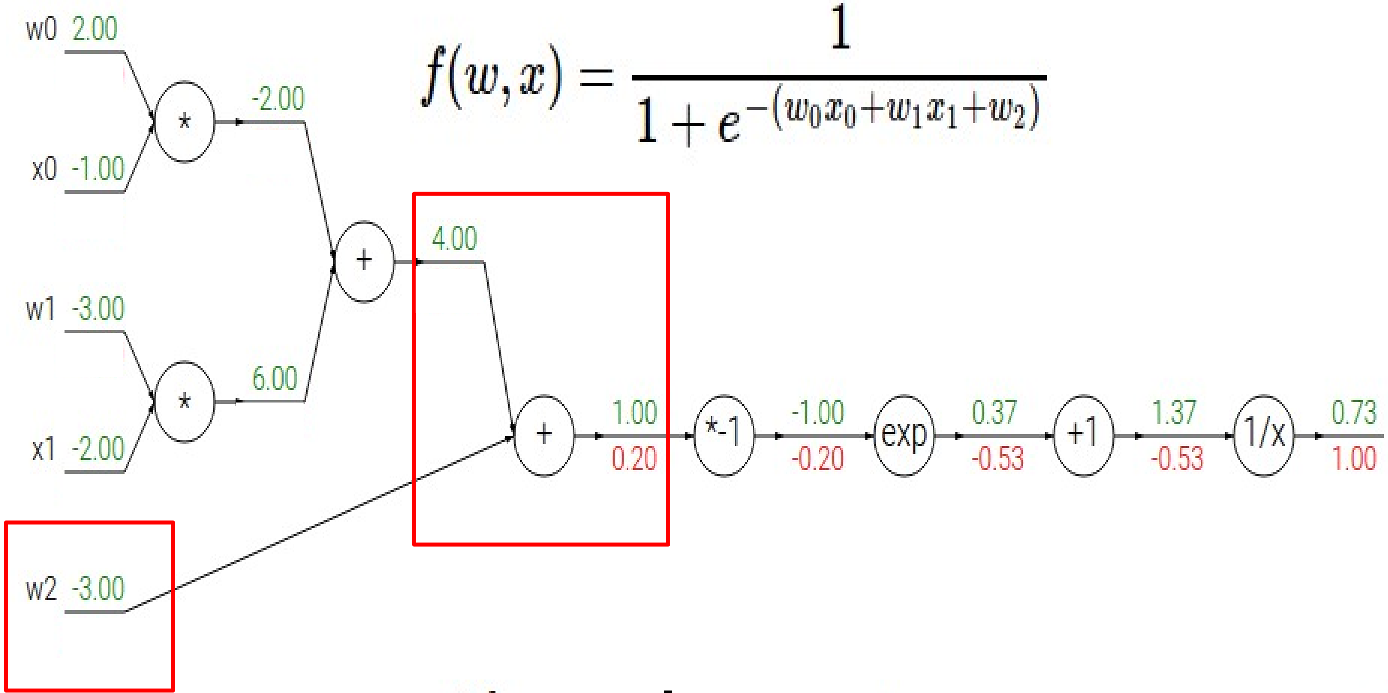

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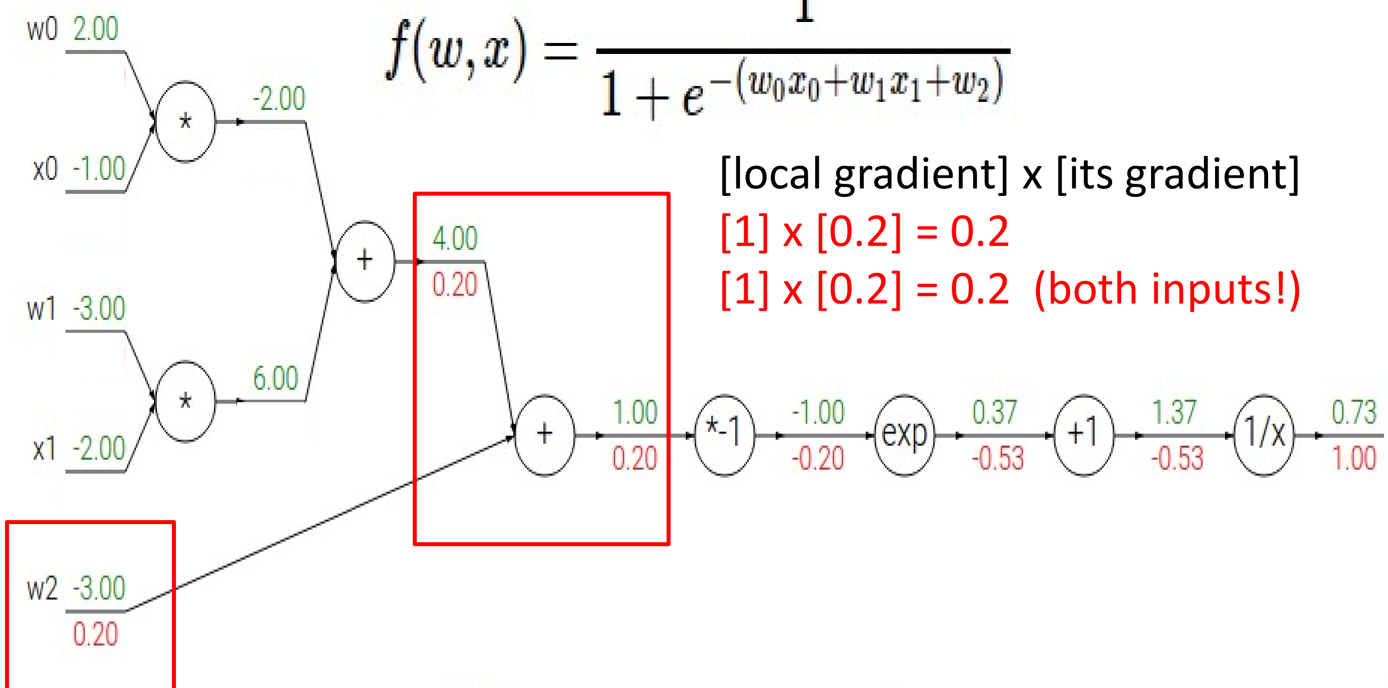
Another example:

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

[local gradient] x [its gradient]

[1] x [0.2] = 0.2

[1] x [0.2] = 0.2 (both inputs!)



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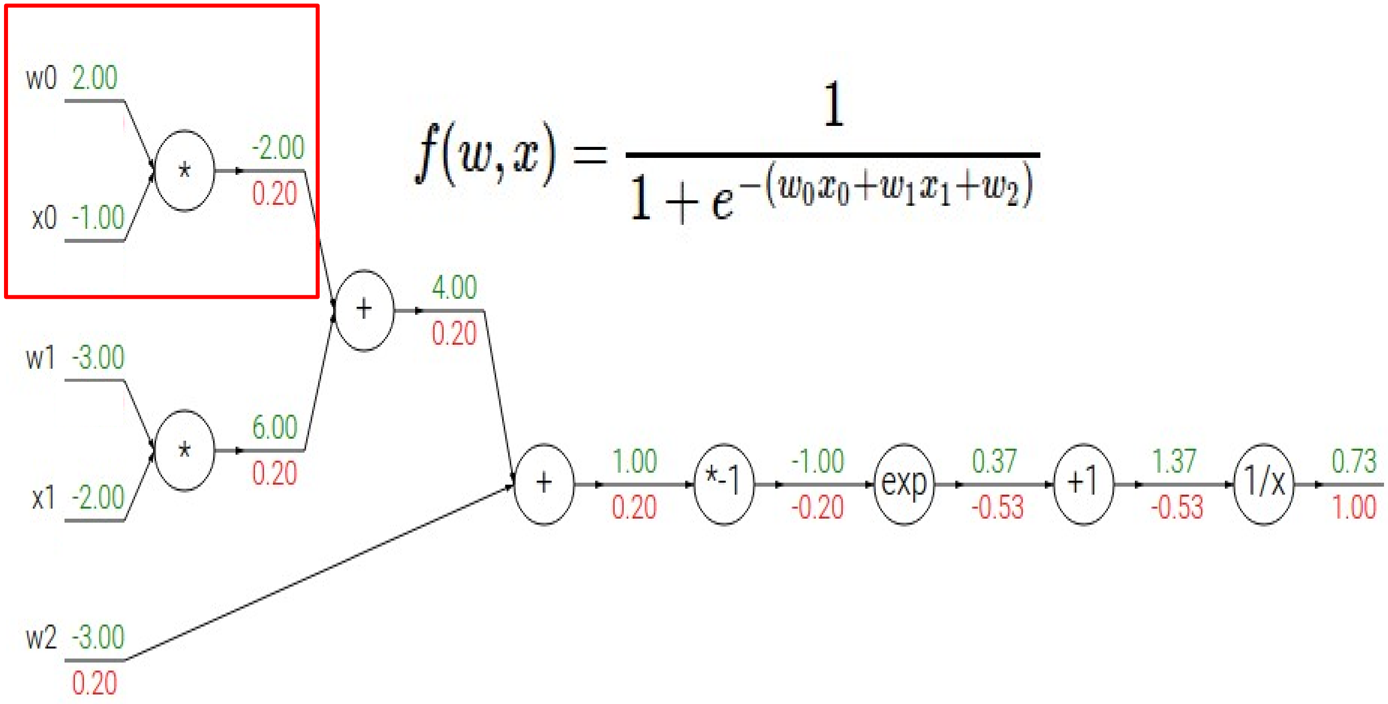

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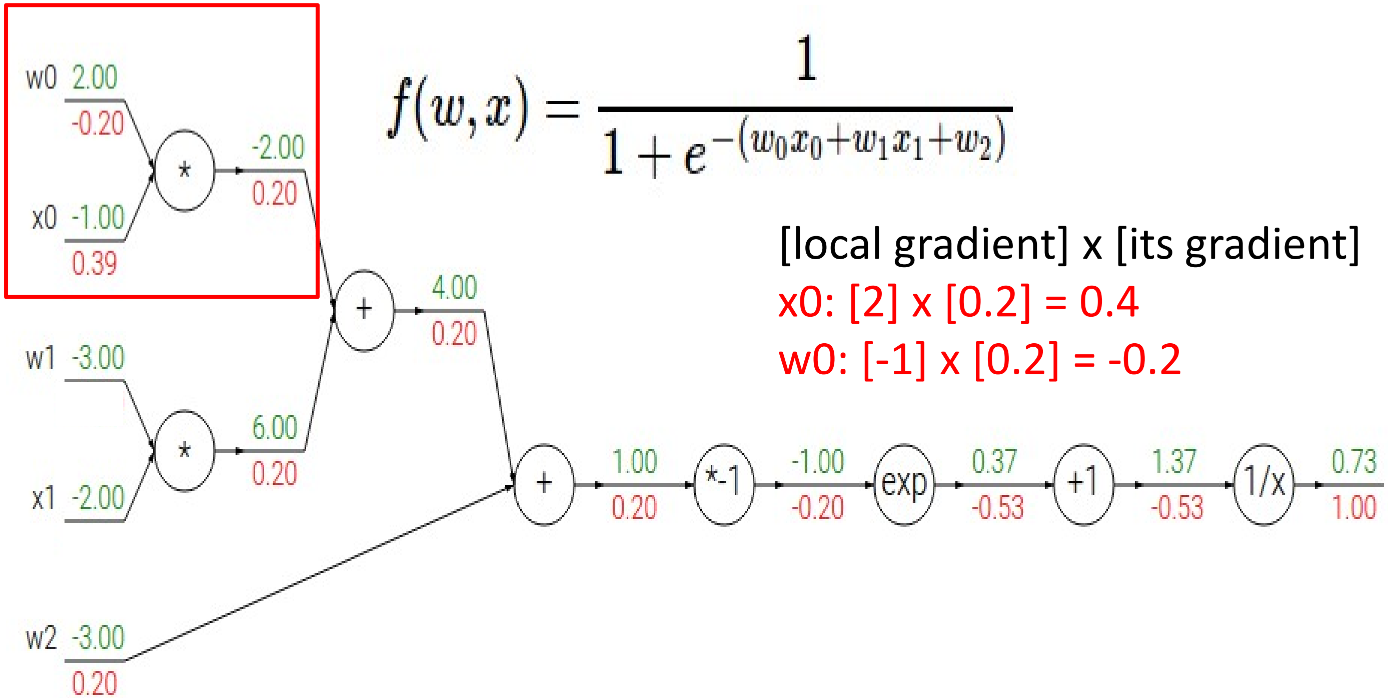


$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

[local gradient] x [its gradient]

x0: [2] x [0.2] = 0.4

w0: [-1] x [0.2] = -0.2

$$f(x) = e^x \quad \rightarrow \quad \frac{df}{dx} = e^x$$

$$f_a(x) = ax \quad \rightarrow \quad \frac{df}{dx} = a$$

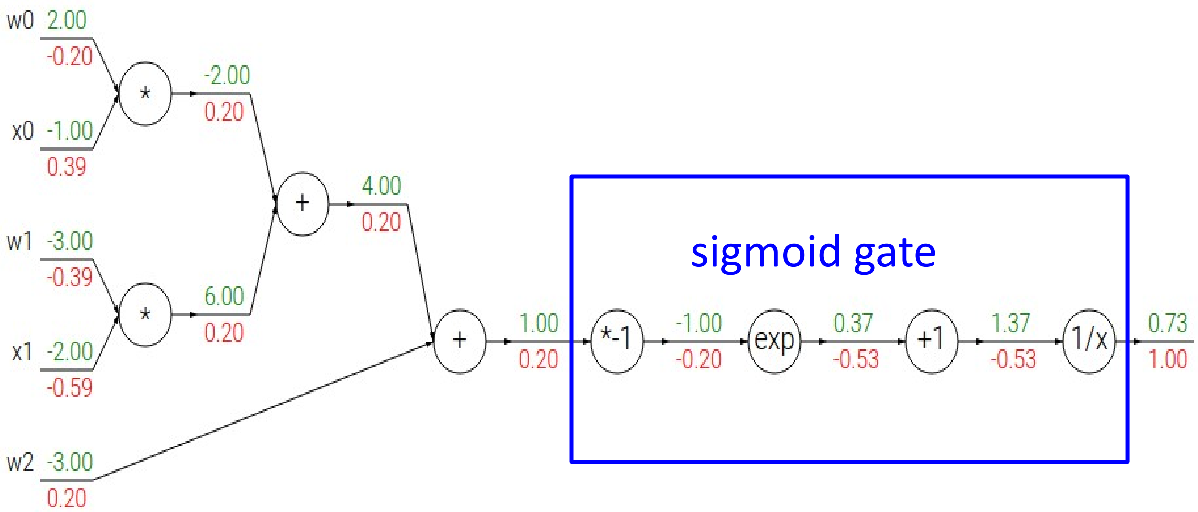
$$f(x) = \frac{1}{x} \quad \rightarrow \quad \frac{df}{dx} = -1/x^2$$

$$f_c(x) = c + x \quad \rightarrow \quad \frac{df}{dx} = 1$$

$$f(w,x) = \frac{1}{1+e^{-(w_0x_0+w_1x_1+w_2)}}$$

$$\boxed{\sigma(x) = \frac{1}{1+e^{-x}}}$$ sigmoid function

$$\frac{d\sigma(x)}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} = \left(\frac{1+e^{-x}-1}{1+e^{-x}}\right)\left(\frac{1}{1+e^{-x}}\right) = (1-\sigma(x))\sigma(x)$$



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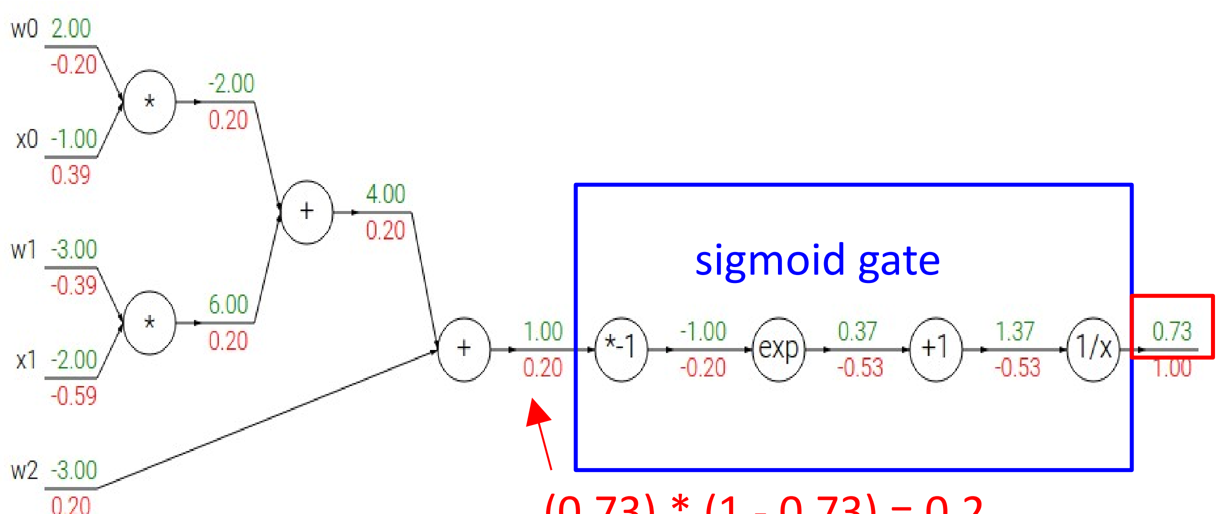
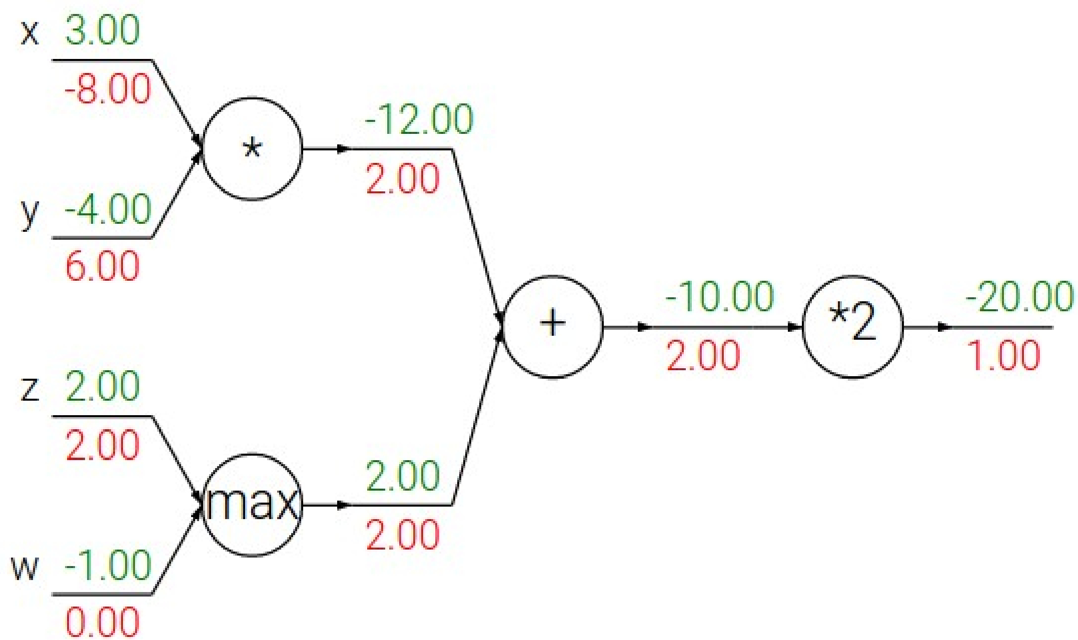


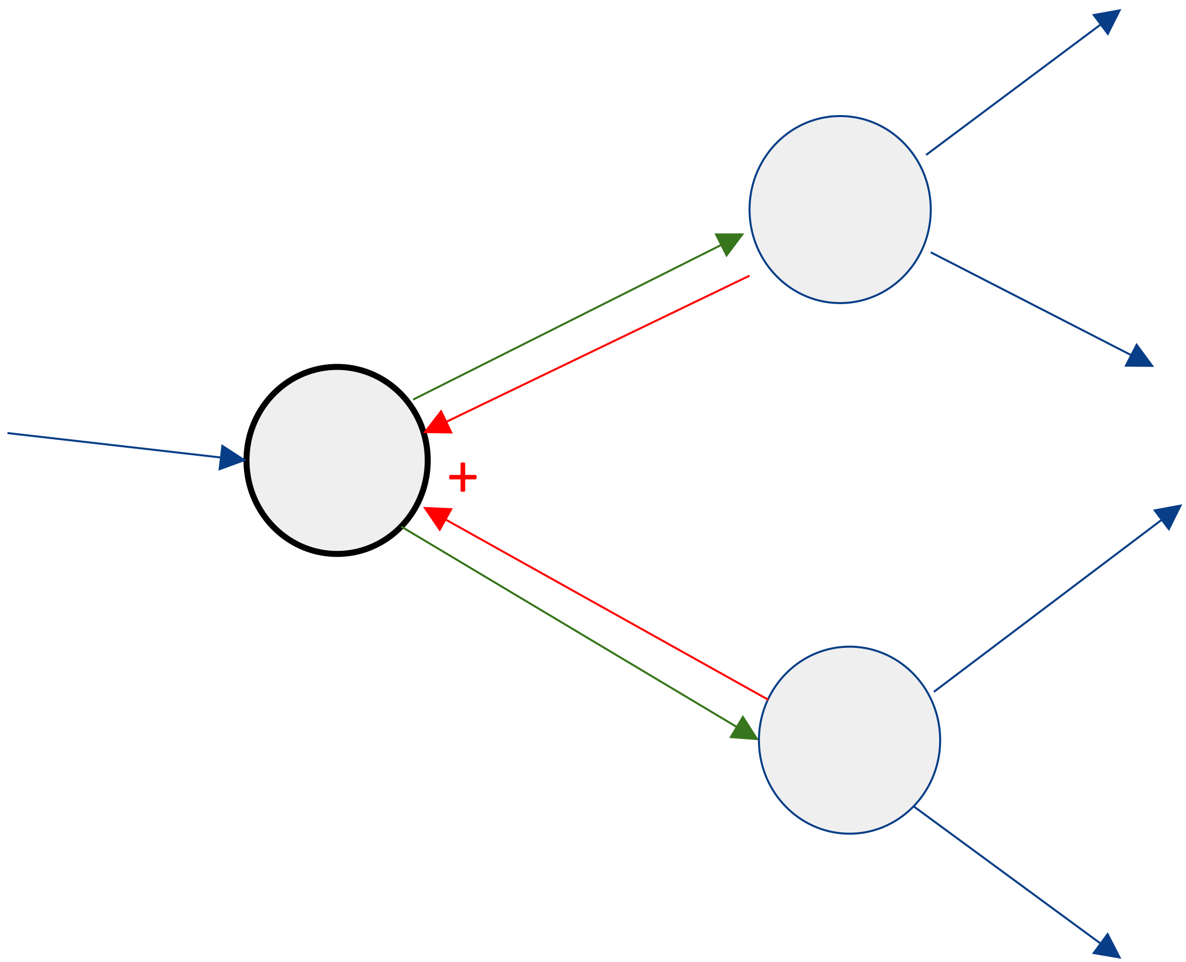
(0.73) * (1 - 0.73) = 0.2

Patterns in backward flow

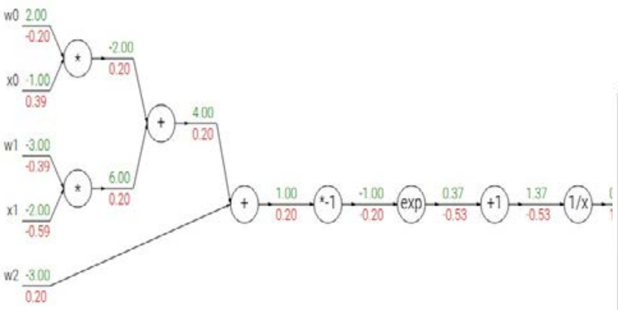
add gate: gradient distributor
max gate: gradient router
mul gate: gradient… “switcher”?



Gradients add at branches



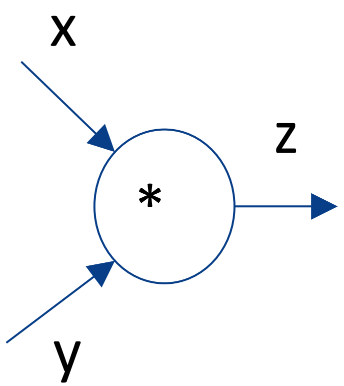
Implementation: forward/backward API



Graph (or Net) object. *(Rough psuedo code)*

```
class ComputationalGraph(object):
    #...
    def forward(inputs):
        # 1. [pass inputs to input gates...]
        # 2. forward the computational graph:
        for gate in self.graph.nodes_topologically_sorted():
            gate.forward()
        return loss # the final gate in the graph outputs the loss
    def backward():
        for gate in reversed(self.graph.nodes_topologically_sorted()):
            gate.backward() # little piece of backprop (chain rule applied)
        return inputs_gradients
```

Implementation: forward/backward API



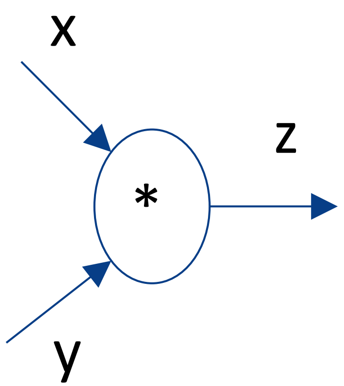
(x,y,z are scalars)

```
class MultiplyGate(object):
    def forward(x,y):
        z = x*y
        return z
    def backward(dz):
        # dx = ... #todo
        # dy = ... #todo
        return [dx, dy]
```

$$\frac{\partial L}{\partial z}$$

$$\frac{\partial L}{\partial x}$$

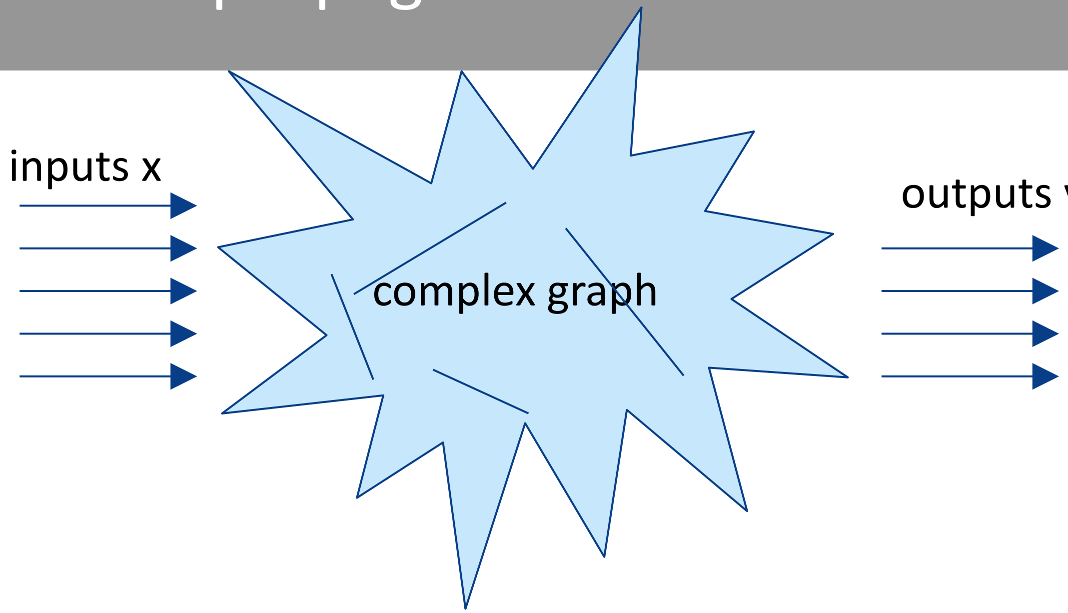
Implementation: forward/backward API



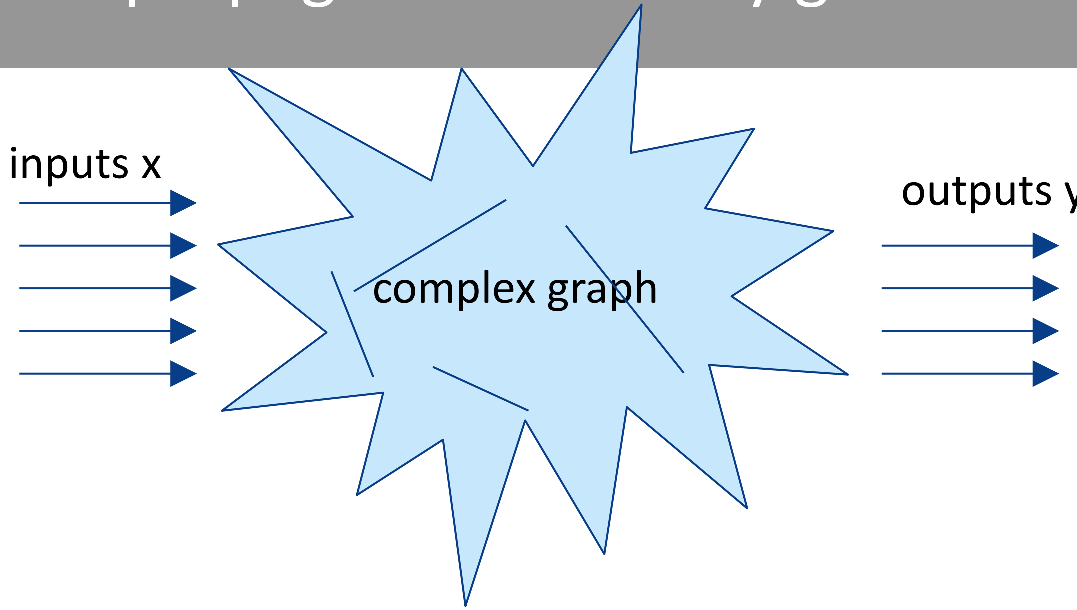
(x,y,z are scalars)

```
class MultiplyGate(object):
    def forward(x,y):
        z = x*y
        self.x = x # must keep these around!
        self.y = y
        return z
    def backward(dz):
        dx = self.y * dz # [dz/dx * dL/dz]
        dy = self.x * dz # [dz/dy * dL/dz]
        return [dx, dy]
```

Q: Why is it back-propagation?



Why is it back-propagation? i.e. why go backwards?



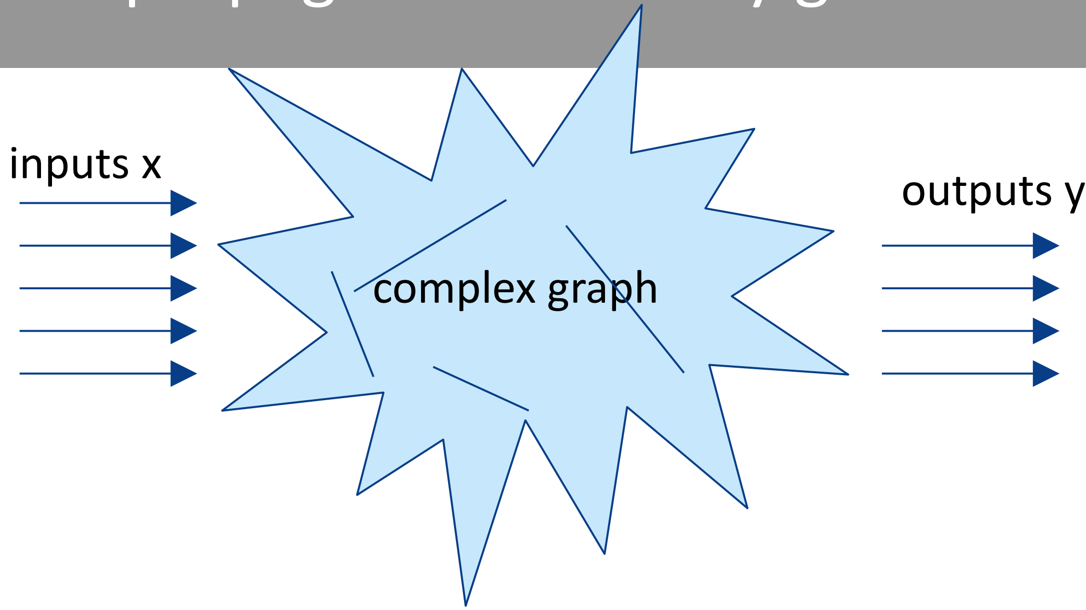
reverse-mode differentiation (if you want effect of many things on one thing)

$\frac{\partial y}{\partial x}$ for many different x

forward-mode differentiation (if you want effect of one thing on many things)

$\frac{\partial y}{\partial x}$ for many different y

Why is it back-propagation? i.e. why go backwards?

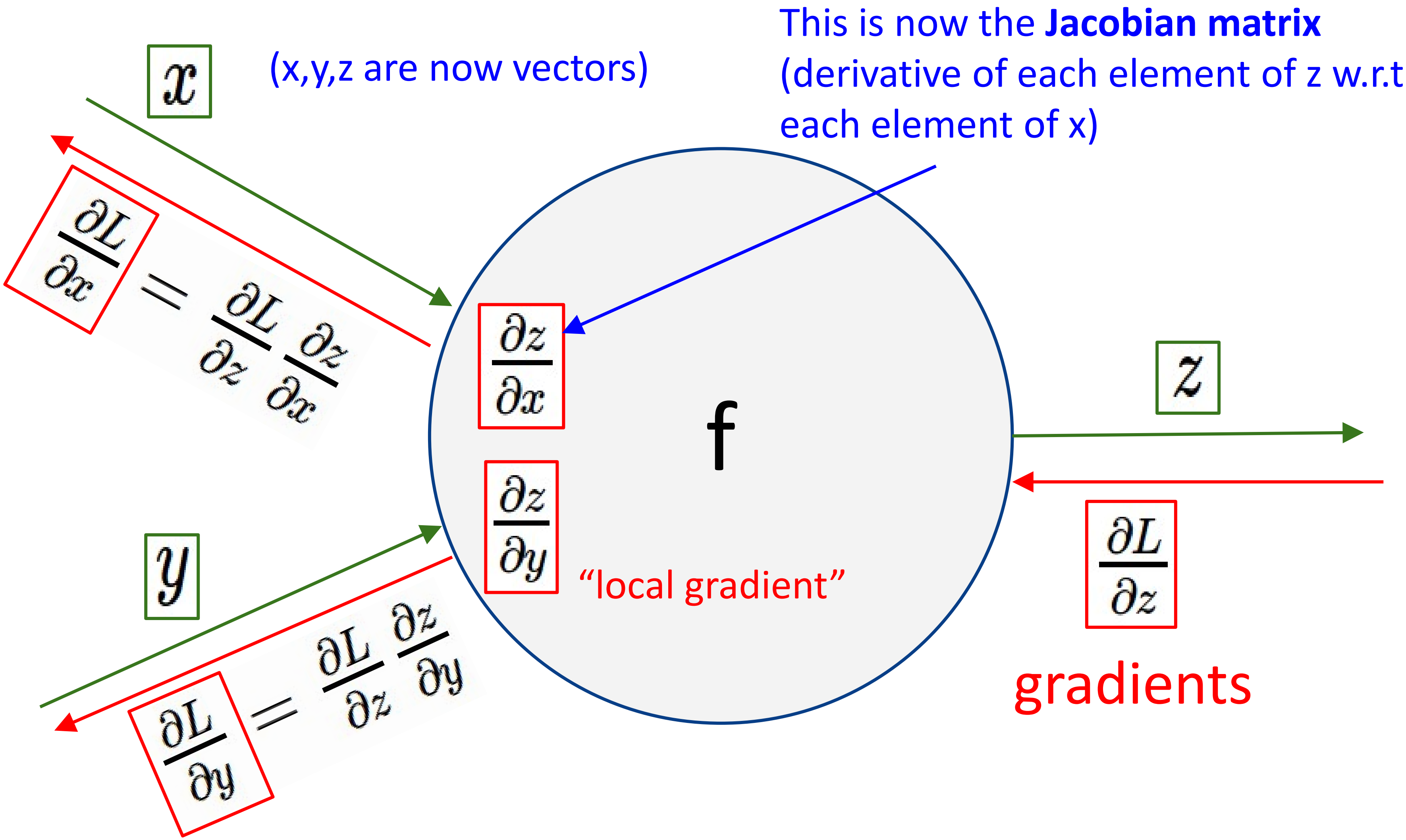


reverse-mode differentiation (if you want effect of many things on one thing)

$\frac{\partial y}{\partial x}$ for many different x

More common simply because many nets have a scalar loss function as output.

Gradients for vector data



Based on cs231n by Fei-Fei Li & Andrej Karpathy & Justin Johnson

Chain Rule

- Consider $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$
- Let $g : \mathbb{R}^m \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$
- Suppose $\mathbf{y} = g(\mathbf{x})$ and $z = f(\mathbf{y})$, then

$$\frac{\partial z}{\partial x_i} = \sum_j \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_i}$$

- In vector notation:

$$\begin{pmatrix} \frac{\partial z}{\partial x_1} \\ \vdots \\ \frac{\partial z}{\partial x_m} \end{pmatrix} = \begin{pmatrix} \sum_j \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_1} \\ \vdots \\ \sum_j \frac{\partial z}{\partial y_j} \frac{\partial y_j}{\partial x_m} \end{pmatrix} = \nabla_{\mathbf{x}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \right)^T \nabla_{\mathbf{y}} z$$

Chain Rule

$$\nabla_{\mathbf{x}} z = \left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)^T \nabla_{\mathbf{y}} z$$

- $\left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)$ is the $n \times m$ Jacobian matrix of g
- **Gradient** of $\mathbf{x}$ is a multiplication of a Jacobian matrix $\left(\frac{\partial \mathbf{y}}{\partial \mathbf{x}}\right)$ with a vector i.e. the gradient $\nabla_{\mathbf{y}} z$
- Backpropagation consists of applying such Jacobian-gradient products to each operation in the computational graph
- In general this need not only apply to vectors, but can apply to tensors w.l.o.g