METIS

Lesson 5: Moments

Introduction

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Lecture Overview:

Goals of the lecture:

1. Understand the moments of random variables

Moments of Random Variables

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Moments of Random Variables

Definition:

We described Random Variables in terms of their distributions. Well, if we use distributions to characterize Random Variables, we should also be able to characterize distributions and this is what **moments** are – characteristics of a distribution.

There are many moments but the two most important are:

1. **Expected Value**
2. **Variance (or Standard Deviation)**

- Depending on if the Random Variable is Discrete or Continuous, the moments will be formulated differently
- There exist many important properties of these two moments, some of which we will cover in the subsequent slides

Expected Value

Definition:

Expected Value of a Random Variable refers to the mean of the Random Variable. In other words, what value does the Random Variable take on average throughout many iterations of the Random Experiment.

Example 1:

Motivating Example: Let's say I toss 2 coins in my Random Experiment and define the Random Variable to be the sum of the result (H – 1, T – 0) and this sum will be equal to my payout (i.e. get 2 Heads, make $2). I can then define my Probability Mass Function associated with the possible values that the Random Variable can take (0,1,2) and each possible value will be assigned a probability.

I might want to ask myself the following question: If I toss the coin 1000 times, on average, what is my payout after each toss?

Expected Value – Discrete Random Variables

Definition:

The **Expected Value** of a Discrete Random Variable is intuitive and what you would expect. It is simply the sum of the products of the possible values of the Random Variable and the probability that these values are obtained. Formally, we define the Expected Value as the following:

$$\mathbb{E}(X) = \sum_i x_i \mathbb{P}(X = x_i) = \mu_X$$

Example 2:

Following the motivating example from Example 14, the Probability Mass Function would be:

$$X = \begin{cases} 0, TT \text{ with probability } 0.25 \\ 1, HT, TH \text{ with probability } 0.5 \\ 2, HH \text{ with probability } 0.25 \end{cases}$$

Therefore, if I tossed the coins 1000 times, I would expect to get 250 TT, 500 HT/TH, and 250 HH

Expected Value – Discrete Random Variables

Example 2 (Continued):

If I were to calculate my expected payoff for any given toss after running the experiment 1000 times, I may intuitively calculate the formula as a sum of the recorded events of each possible values, weighted by their payoff, divided by the total number of events in the experiment:

$$\frac{0(250) + 1(500) + 2(250)}{1000} = \$1$$

Or, using the formula: $\mathbb{E}(X) = \sum_i x_i \mathbb{P}(X = x_i) = (0 * 0.25) + (1 * 0.50) + (2 * 0.25) = \$1$

Expected Value – Continuous Random Variables

Definition:

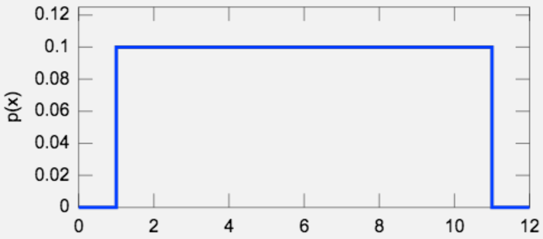
The **Expected Value** of a Continuous Random Variable is formulated differently than that of a Discrete Random Variable since we are now working with Probability Density Functions. The formula now involves an integral since we are interested in the area under the curve between two points:

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \mu_X$$

Example 3:

If the Probability Density Function of a Random Variable X is represented as $\frac{1}{b-a}$ and shown in the picture below, then what is the Expected Value of X?

NOTE: We will name this specific Random Variable later on in the lecture as a Uniform Random Variable



Expected Value – Continuous Random Variables

Example 3 (Continued):

We can use the formula to calculate the Expected Value of the Random Variable X

$$\mathbb{E}(X) = \int_{1}^{11} x \frac{1}{11 - 1} dx$$

$$\mathbb{E}(X) = \left[\frac{x^2}{2} \times \frac{1}{10}\right] = \left[\frac{11^2}{2} \times \frac{1}{10}\right] - \left[\frac{1^2}{2} \times \frac{1}{10}\right] = 6$$

NOTE: Since this is actually a “special” Random Variable, we simply memorize the formula for its Expected Value as: $\frac{1}{2}(a + b)$

Variance

Definition:

Variance of a Random Variable refers to the spread of the distribution. In other words, on average, how far is each value of the Random Variable away from the Expected Value.

Example 4:

Motivating Example: Following from the motivating previous example:

I might also want to ask myself the following question: If I toss the coin 1000 times, on average, how far away is the payout of each toss from the average payout?

Variance – Discrete Random Variables

Definition:

The **Variance** of a Discrete Random Variable is the sum of the products of the squared variance of the possible values of the Random Variable and the probabilities these values are obtained.
Formally, we define the Variance as the following:

$$V(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \sum_i (x_i - \mu)^2 \mathbb{P}(X = x_i) = \sigma^2$$

Example 4:

Following the motivating example from a previous example, the Variance of the Random Variable X is:

$$V(X) = (0 - 1)^2(0.25) + (1 - 1)^2(0.5) + (2 - 1)^2(0.25) = 0.5$$

Variance – Continuous Random Variables

Definition:

The **Variance** of a Continuous Random Variable is again formulated differently than that of a Discrete Random Variable since we are now working with Probability Density Functions which requires the use of integrals:

$$V(X) = \mathbb{E}\{[X - \mathbb{E}(X)]^2\} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \sigma^2$$

Example 5:

Following the example from a previous example, the Variance of the Random Variable X is:

$$V(X) = \int_1^{11} (x - 6)^2 \times \frac{1}{11-1} dx = [\frac{(x-6)^3}{3} \times \frac{1}{10}]_1^{11} = \left[\frac{(11-6)^3}{3} \times \frac{1}{10}\right] - \left[\frac{(1-6)^3}{3} \times \frac{1}{10}\right] = \frac{25}{3}$$

NOTE: Since this is a “special” Random Variable, we simply memorize the formula as $\frac{1}{12}(b - a)^2$

QUESTIONS?