# Derivation for the minimum skewness of CMB maps

Jay A. Trevino

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## 1 Introduction

In this short document, I will explain the derivation of the skewness for CMB by finding a solution set of weights. This derivation is motivated by the fact that the foreground contaminants in the CMB maps are not completely "Gaussian" i.e. they do not reflect a reflect a symmetrical normal distribution. This asymmetry stems from the fact that the contaminants across the sky are not evenly distributed, in large part due to the dust in the Milky Way plane. The dust in this region is significantly greater than dust in other regions of the sky. By minimizing the skewness, we can minimize the effects of this asymmetry and achieve a normal distribution for analysis.

## 2 Derivation

Given the standard equation for the skewness of the ILC map

$$S^3_{\hat{y}} = \frac{1}{N_{pix}} \sum_p (\hat{y}(p) - \langle y \rangle)^3$$

using $\hat{y}(p)$ to define the ILC weights $w_i$ as follows

$$\hat{y}(p) = \sum_{i=1}^{N_{obs}} w_i \Delta T_i(p)$$

we can redefine the skewness equation into a different form that shows how we can factor out the weights, $w_i$ in order to more efficiently precede with the derivation:

$$S^3_{\hat{y}} = Skew(\sum_{i=1}^{N_{obs}} w_i \Delta T_i(p))$$

$$\sum_{i=1}^{N_{obs}} Skew(w_i \Delta T_i(p))$$

$$\sum_{i,j,k=1}^{N_{obs}} w_i w_j w_k Skew(\Delta T_i(p))$$

where $Skew(\Delta T_i(p)) = \hat{B}_{ijk}$

Thus, we now have

$$\sum_{i,j,k=1}^{N_{obs}} w_i w_j w_k \hat{B}_{ijk}$$

Using the method of Lagrange multipliers, with $\lambda$ representing an arbitrary constant, we can find the partial derivatives of $w_i$ and set this equal to zero. With our constraint for the Lagrange multiplier being $\sum_i w_i a_i = 1$. As we established in the previous derivation for the variance of ILC maps, the constraint ensures that the weighted sums not exceed 1 and the CMB be "properly preserved in the final ILC map". We now arrive at the partial derivative:

$$\frac{\partial}{\partial w_i}(\sum_{i,j,k=1}^{N_{obs}} w_i w_j w_k \hat{B}_{ijk} - \lambda(1 - \sum_i w_i a_i)) = 0$$

The partial derivative of the constraint is just unity and we are left with just $\lambda$. Taking $\sum_{ijk}^{N_{obs}} w_i w_j w_k$ as $\mathbf{w}^3$, we can write:

$$3w^2 \hat{B}_{ijk} - \lambda = 0$$

$$3\sum_{jk}(w_j w_k)^2 B_{ijk} = \lambda a_j a_k$$

$$(w_j w_k)^2 = \frac{1}{3}\sum_{jk}(\hat{B}^{-1})_{ijk}\lambda a_j a_k$$

Since $\sum_{ijk}^{N_{obs}} w_i = 1$, we can solve for $\lambda$:

$$(w_j w_k)^2 = \mathbf{w} = 1 = \frac{1}{3}\sum_{jk}(\hat{B}^{-1})_{ijk}\lambda a_j a_k$$

$$\lambda = \frac{3}{\sum_{jk}(\hat{B}^{-1})_{ijk}\lambda a_j a_k}$$

Thus, solving for the weights, we finally arrive at:

$$w_i = (w_j w_k)^2 = \frac{1}{3}\sum_{jk}(\hat{B}^{-1})_{ijk}\lambda a_j a_k$$

$$w_i = \frac{\sum_{jk}(\hat{B}^{-1})_{ijk} a_j a_k}{\sum_{lm}(\hat{B}^{-1})_{lmn} a_l a_m a_n}$$