Derivation of the minimum variance of CMB maps

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1 Introduction

In this document, I will explain the derivation for the minimum variance of CMB maps by finding the solution set of weights.

2 Derivation

Defining the ILC weight w_i as:

$$\hat{y}(p) = \sum_{i=1}^{N_{obs}} w_i \Delta T_i(p)$$

where $\hat{y}(p)$ is the weighted sum, $\Delta T_i(p)$ is the observed temperature fluctuation of the CMB, i is the index of the measurements, and N_{obs} is the total number of observations at different frequencies (Note: the final derivation is frequency independent).

We must also abide by the constraint $\sum_i w_i a_i = 1$. This constraint signifies that the sum of all the adjusted weights will be equal to 1, as each weight varies by the amount of noise. Weights with less noise have a greater value than those that have more noise in the measurement.

To find the variance $\sigma^2_{\hat{y}}$, we find the difference from the mean of $\hat{y}(p)$, square this result, and finally divide by the number of pixels N_{pix}.

$$\sigma^2_{\hat{y}} = \frac{1}{N_{pix}} \sum_p (\hat{y}(p) - \sum_p \frac{\hat{y}(p)}{N_{pix}})^2$$

Further simplified,

$$\sigma^2_{\hat{y}} = \frac{1}{N_{pix}} \sum_p (\hat{y}(p) - \langle \hat{y} \rangle)^2$$

In order to minimize the variance, we must equate the gradient with respect to the weights of the variance to zero. From the above equation:

$$\sigma_{\hat{y}}^2 = w_i w_j \hat{R_{ij}}$$

where $w_i w_j$ is factored out and $\hat{R_{ij}}$ is the sum of the differences between the observed temperature fluctuations at each measurement and the average of all the measurements, which is all then squared. Given this, we can now find the solutions for the weights w_i using Lagrange multipliers. Using the function $f(w) = \sigma_{\hat{y}}^2 = w_i w_j \hat{R_{ij}}$ and the constraint $g(w) = \sum_{i=1}^{k} w_i = 1$, we can find the partial derivatives of each of these. a_i is promptly ignored as a_i is assumed to equal 1 for the CMB. I have left out the derivations for R_{ij} from the variance equation.

$$\frac{\partial}{\partial w_i} f(w) = 2 \sum_{j=1}^{k} \hat{R_{ij}} w_j$$

$$\frac{\partial}{\partial w_i} g(w) = 1$$

Given the partial derivatives, take $\sum_{j=1}^{N_{obs}} \hat{R_{ij}}$ as R and $\sum_{j=1}^{N_{obs}} w_j$ as w. Thus,

$$2Rw - \lambda = 0$$

$$2Rw = \lambda$$

Since $w = 1$, then we get the equation $R = \frac{2}{\lambda}$. Using this we can solve for the weights:

$$w = \frac{\lambda}{2R} = (\frac{\lambda}{2})(R^{-1})$$

$$w = (R^{-1})^{-1})(R^{-1})$$

Finally, we arrive at:

$$w_j = \frac{\sum_i R_{ij}^{-1}}{\sum_{kl} R_{kl}^{-1}}$$