A note on the complexity of Feedback Vertex Set parameterized by mim-width

Lars Jaffke[*1], O-joung Kwon[†2], and Jan Arne Telle[1]

[1]Department of Informatics, University of Bergen, Norway.
{lars.jaffke, jan.arne.telle}@uib.no
[2]Logic and Semantics, Technische Universität Berlin, Berlin, Germany.
ojoungkwon@gmail.com

November 10, 2017

Abstract

We complement the recent algorithmic result that FEEDBACK VERTEX SET is XP-time solvable parameterized by the mim-width of a given branch decomposition of the input graph [3] by showing that the problem is W[1]-hard in this parameterization. The hardness holds even for linear mim-width, as well as for H-graphs, where the parameter is the number of edges in H. To obtain this result, we adapt a reduction due to Fomin, Golovach and Raymond [2], following the same line of reasoning but adding a new gadget.

1 Preliminaries

In this note (which will later be merged with the companion paper [3]), unless stated otherwise, a graph G with vertex set $V(G)$ and edge set $E(G)$ is finite, undirected, simple and connected. We let $|G| := |V(G)|$ and $||G|| := |E(G)|$. For an integer $k > 0$, we let $[k] := \{1, \ldots, k\}$.

For two (disjoint) vertex sets $X, Y \subseteq V(G)$, we denote by $G[X, Y]$ the bipartite subgraph of G with bipartition (X, Y) such that for $x \in X, y \in Y$, x and y are adjacent in G if and only if they are adjacent in $G[X, Y]$. A *cut* of G is a bipartition (A, B) of its vertex set. A set M of edges is a *matching* if no two edges in M share an endpoint, and a matching $\{a_1b_1, \ldots, a_kb_k\}$ is *induced* if there are no other edges in the subgraph induced by $\{a_1, b_1, \ldots, a_k, b_k\}$.

Let $uv \in E(G)$. We call the operation of adding a new vertex x to $V(G)$ and replacing uv by the path uxv the *edge subdivision* of uv. We call a graph G' a *subdivision of G* if it can be obtained from G by a series of edge subdivisions.

Mim-width. For a graph G and a vertex subset A of G, we define $\operatorname{mim}_G(A)$ to be the maximum size of an induced matching in $G[A, V(G) \setminus A]$.

A pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L}$ from $V(G)$ to the set of leaves of T is called a *branch decomposition*. If T is a caterpillar, then $(T, \mathcal{L})$ is called a *linear branch decomposition*.

For each edge e of T, let T_1^e and T_2^e be the two connected components of $T - e$, and let (A_1^e, A_2^e) be the vertex bipartition of G such that for each $i \in \{1, 2\}$, A_i^e is the set of all vertices in G mapped to leaves contained in T_i^e by $\mathcal{L}$. The *mim-width of* $(T, \mathcal{L})$, denoted by $\mathrm{mimw}(T, \mathcal{L})$, is defined as $\max_{e \in E(T)} \mathrm{mim}_G(A_1^e)$. The minimum mim-width over all branch decompositions of G is called the *mim-width of* G. We define the *linear mim-width* accordingly, additionally requiring the corresponding branch decomposition to be linear. If $|V(G)| \leq 1$, then G does not admit a branch decomposition, and the mim-width of G is defined to be 0.

H-Graphs. Let X be a set and $\mathcal{S}$ a family of subsets of X. The *intersection graph* of $\mathcal{S}$ is a graph with vertex set $\mathcal{S}$ such that $S, T \in \mathcal{S}$ are adjacent if and only if $S \cap T \neq \emptyset$. Let H be a (multi-) graph. We say that G is an H-*graph* if there is a subdivision H' of H and a family of subsets $\mathcal{M} := \{M_v\}_{v \in V(G)}$ (called an H-*representation*) of $V(H')$ where $H'[M_v]$ is connected for all $v \in V(G)$, such that G is isomorphic to the intersection graph of $\mathcal{M}$.

2 The Proof

Very recently, Fomin et al. [2] showed that H-graphs have linear mim-width at most $2 \cdot ||H||$ (Theorem 1) and that INDEPENDENT SET is W[1]-hard parameterized by $k + ||H||$, where k denotes the solution size (Theorem 6). This implies that INDEPENDENT SET is W[1]-hard for the combined parameter solution size plus linear mim-width. We will modify their reduction to show that MAXIMUM INDUCED FOREST parameterized by the mim-width of a given linear branch decomposition plus the solution size remains W[1]-hard. We formally define this parameterized problem below.

> MAXIMUM INDUCED FOREST/LINEAR MIM-WIDTH+k
> **Input:** A graph G, a linear branch decomposition $(T, \mathcal{L})$ of G and an integer k.
> **Parameter:** $w + k$, where $w := \mathrm{mimw}(T, \mathcal{L})$.
> **Question:** Does G contain an induced forest on k vertices?

The reduction is from MULTICOLORED CLIQUE where given a graph G and a partition $V_1, \ldots, V_k$ of $V(G)$, the question is whether G contains a clique of size k using precisely one vertex from each V_i ($i \in \{1, \ldots, k\}$). This problem is known to be W[1]-complete [1, 4].

Theorem 1. MAXIMUM INDUCED FOREST *is* W[1]-*hard when parameterized by* $w + k$ *and the hardness holds even when a branch decomposition of linear mim-width* w *is given.*

Proof. Let $(G, V_1, \ldots, V_k)$ be an instance of MULTICOLORED CLIQUE. We can assume that $k \geq 2$ and that $|V_i| = p$ for $i \in [k]$. If the second assumption does not hold, let $p := \max_{i \in [k]} |V_i|$ and add $p - |V_i|$ isolated vertices to V_i, for each V_i. We denote by $v_1^i, \ldots, v_p^i$ the vertices of V_i, for each $i \in [k]$.

We first obtain an H-graph G'' from an adapted version of the construction due to Fomin et al. [2, Proof of Theorem 6] as follows. The graph H remains the same and is constructed as follows.

1. Construct k nodes $u_1, \ldots, u_k$.

2. For every $1 \leq i < j \leq k$, construct a node $w_{i,j}$ and two pairs of parallel edges $u_i w_{i,j}$ and $u_j w_{i,j}$.

Note that $|H| = k + \binom{k}{2} = k(k+1)/2$ and $||H|| = 4 \cdot \binom{k}{2} = 2k(k-1)$. We then construct the subdivision H' of H by first subdividing each edge p times. We denote the subdivision nodes for 4 edges of H constructed for each pair $1 \leq i < j \leq k$ in Step 2 by $x_1^{(i,j)}, \ldots, x_p^{(i,j)}, y_1^{(i,j)}, \ldots, y_p^{(i,j)},$
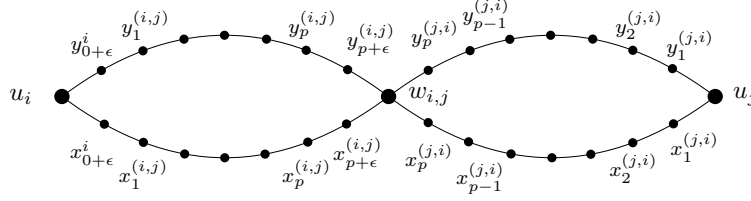
Figure 1: Illustration of the subdivision for a pair $1 \leq i < j \leq k$, assuming $j = i + 1 < k$. For $j \neq i + 1$, the vertices $x^i_{0+\epsilon}$ and $y^i_{0+\epsilon}$ do not exist.

$x^{(j,i)}_1, \ldots, x^{(j,i)}_p$, and $y^{(j,i)}_1, \ldots, y^{(j,i)}_p$. To simplify notation, we assume that $u_i = x^{(i,j)}_0 = y^{(i,j)}_0$, $u_j = x^{(j,i)}_0 = y^{(j,i)}_0$ and $w_{i,j} = x^{(i,j)}_{p+1} = y^{(i,j)}_{p+1} = x^{(j,i)}_{p+1} = y^{(j,i)}_{p+1}$.

Furthermore,[1] for $i \in [k-1]$, we subdivide the edges $x^{(i,i+1)}_0 x^{(i,i+1)}_1$ and $y^{(i,i+1)}_0 y^{(i,i+1)}_1$; we also subdivide $x^{(k,k-1)}_0 x^{(k,k-1)}_1$ and $y^{(k,k-1)}_0 y^{(k,k-1)}_1$. We call the new subdivision nodes (in either case) $x^i_{0+\epsilon}$ and $y^i_{0+\epsilon}$, for $i \in [k]$, respectively.

For each $1 \leq i < j \leq k$, we subdivide the edges $x^{(i,j)}_p x^{(i,j)}_{p+1}$ and $y^{(i,j)}_p y^{(i,j)}_{p+1}$ and denote the new subdivision vertices by $x^{(i,j)}_{p+\epsilon}$ and $y^{(i,j)}_{p+\epsilon}$, respectively. We illustrate this subdivision in Figure 1.

We now construct the H-graph G'' by defining its H-representation $\mathcal{M} = \{M_v\}_{v \in V(G')}$ where each M_v is a connected subset of $V(H')$. (Recall that G denotes the graph of the MULTICOLORED CLIQUE instance.)

1. For each $i \in [k]$, construct vertices α^i_x with model $M_{\alpha^i_x} := \{x^i_{0+\epsilon}\}$ and α^i_y with model $M_{\alpha^i_y} := \{y^i_{0+\epsilon}\}$.

2. For each $i \in [k]$ and $s \in [p]$, construct a vertex z^i_s with model
$$M_{z^i_s} := \{x^i_{0+\epsilon}, y^i_{0+\epsilon}\} \cup \bigcup_{j \in [k], j \neq i} \left(\left\{ x^{(i,j)}_0, \ldots, x^{(i,j)}_{s-1} \right\} \cup \left\{ y^{(i,j)}_0, \ldots, y^{(i,j)}_{p-s} \right\} \right).$$

3. For each $1 \leq i < j \leq k$, construct a vertex $\alpha^{(i,j)}_x$ with model $M_{\alpha^{(i,j)}_x} := \left\{ x^{(i,j)}_{p+\epsilon} \right\}$ and a vertex $\alpha^{(i,j)}_y$ with model $M_{\alpha^{(i,j)}_y} := \left\{ y^{(i,j)}_{p+\epsilon} \right\}$.

4. For each edge $v^i_s v^j_t \in E(G)$ for $s, t \in [p]$ and $1 \leq i < j \leq k$, construct a vertex $r^{(i,j)}_{s,t}$ with model
$$M_{r^{(i,j)}_{s,t}} := \left\{ x^{(i,j)}_{p+\epsilon}, y^{(i,j)}_{p+\epsilon} \right\} \cup \left\{ x^{(i,j)}_s, \ldots, x^{(i,j)}_{p+1} \right\} \cup \left\{ y^{(i,j)}_{p-s+1}, \ldots, y^{(i,j)}_{p+1} \right\}$$
$$\cup \left\{ x^{(j,i)}_t, \ldots, x^{(j,i)}_{p+1} \right\} \cup \left\{ y^{(j,i)}_{p-t+1}, \ldots, y^{(j,i)}_{p+1} \right\}.$$

Throughout the following, for $i \in [k]$ and $1 \leq i < j \leq k$, respectively, we use the notation
$$Z(i) := \bigcup_{s \in [p]} \{z^i_s\} \quad \text{and} \quad R(i,j) := \bigcup_{\substack{v^i_s v^j_t \in E(G), \\ s, t \in [p]}} \left\{ r^{(i,j)}_{s,t} \right\}$$

---

[1]To clarify, we would like to remark that this step (and everything revolving around the resulting vertices) did not appear in the reduction of Fomin et al. [2] and is vital to make it work for MAXIMUM INDUCED FOREST.
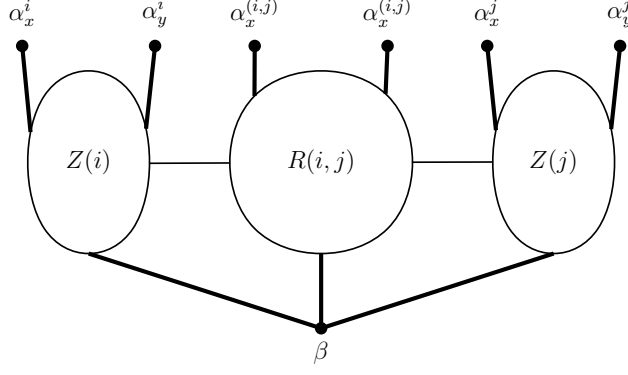
Figure 2: Illustration of a part of G', where $1 \leq i < j \leq k$. Bold edges imply that all possible edges between the corresponding (sets of) vertices are present.

and we let $Z_{+\alpha}(i) := Z(i) \cup \{\alpha_x^i, \alpha_y^i\}$ and $R_{+\alpha}(i,j) := R(i,j) \cup \{\alpha_x^{(i,j)}, \alpha_y^{(i,j)}\}$. We furthermore define

$$A := \bigcup_{i \in [k]} \{\alpha_x^i, \alpha_y^i\} \cup \bigcup_{1 \leq i < j \leq k} \left\{\alpha_x^{(i,j)}, \alpha_y^{(i,j)}\right\}.$$

We obtain the graph G' of the MAXIMUM INDUCED FOREST instance by taking the graph G'' and adding to it a vertex β which is adjacent to all vertices in $V(G'') \setminus A$. We illustrate this construction in Figure 2.

We now show that the linear mim-width of G' remains bounded by a function of k.[2]

Claim 2. G' has linear mim-width at most $4k(k-1) + 1$ and a linear branch decomposition of said width can be computed in polynomial time.

Proof. By [2, Theorem 1], G'' has linear mim-width at most $2||H|| = 4k(k-1) =: k'$. Given a linear branch decomposition of G'' of width k' we can add a new node to the branch decomposition in any place such that it stays linear and letting the new node be mapped to β. The resulting branch decomposition is a linear branch decomposition of G' with the mim-value in each cut increased by at most 1. The fact that this decomposition can be computed in polynomial time follows as well from [2, Theorem 1] and the fact that we constructed the H-representation of G'' above. ⌟

We now observe some crucial properties of the above construction.

Observation 3 (Claim 7 in [2]). For every $1 \leq i < j \leq k$, a vertex $z_h^i \in V(G')$ (a vertex $z_h^j \in V(G')$) is *not* adjacent to a vertex $r_{s,t}^{(i,j)}$ corresponding to the edge $v_s^i v_t^j \in E(G)$ if and only if $h = s$ ($h = t$, respectively).

Observation 4.

  (i) For every $i \in [k]$, $N(\alpha_x^i) = Z(i) = N(\alpha_y^i)$.

 (ii) For every $1 \leq i < j \leq k$, $N(\alpha_x^{(i,j)}) = R(i,j) = N(\alpha_y^{(i,j)})$.

(iii) A is an independent set in G' of size $2k + 2 \cdot \binom{k}{2}$.

(iv) For $i \in [k]$, $Z(i)$ induces a clique in G' and for $1 \leq i < j \leq k$, $R(i,j)$ induces a clique in G'.

---

[2]In fact, we will later show that G' is a K-graph for some $K \supseteq H$.

4

We are now ready to prove the correctness of the reduction. In particular we will show that G has a multicolored clique if and only if G' has an induced forest of size $k' := 3k + 3\binom{k}{2} + 1$.

Claim 5. If G has a multicolored clique on vertex set $\left\{v^1_{h_1}, \ldots, v^k_{h_k}\right\}$, then G' has an induced forest of size $k' = 3k + 3 \cdot \binom{k}{2} + 1$.

Proof. Using Observation 3, one can easily verify that the set

$$I := \left\{z^1_{h_1}, \ldots, z^k_{h_k}\right\} \cup \left\{r^{(i,j)}_{h_i, h_j} \mid 1 \leq i < j \leq k\right\} \tag{1}$$

is an independent set in G'. By Observation 4(iii) and the construction given above, we can conclude that $F := I \cup A \cup \{\beta\}$ induces a forest in G': I and A are both independent sets and $A \cup I$ induces a disjoint union of paths on three vertices, the middle vertices of which are contained in I. The only additional edges that are introduced are between β and vertices in I, so F induces a tree. Clearly, $|F| = |I| + |A| + |\{\beta\}| = k + \binom{k}{2} + 2k + 2 \cdot \binom{k}{2} + 1 = k'$, proving the claim. ⌟

We now prove the backward direction of the correctness of the reduction. This will be done by a series of claims and observations narrowing down the shape of any induced forest on k' vertices in G'. Eventually, we will be able conclude that any such induced forest contains an independent set of size $k + \binom{k}{2}$ of the shape (1). We can then conclude that G contains a multicolored clique by Observation 3.

The following is a direct consequence of Observation 4(iv).

Observation 6. Let F be an induced forest in G'. Then, $V(F)$ contains

(i) at most 2 vertices from $Z(i)$, where $i \in [k]$ and

(ii) at most 2 vertices from $R(i, j)$, where $1 \leq i < j \leq k$.

Next, we investigate the interaction of any induced forest with the sets $Z_{+\alpha}(i)$ and $R_{+\alpha}(i, j)$.

Claim 7. Let F be an induced forest in G'. If $V(F)$ contains two vertices from $Z(i)$, where $i \in [k]$ (from $R(i, j)$, where $1 \leq i < j \leq k$), then $V(F)$ cannot contain a vertex from $\{\alpha^i_x, \alpha^i_y\}$ (from $\{\alpha^{(i,j)}_x, \alpha^{(i,j)}_y\}$, respectively).

Proof. Suppose $V(F)$ contains two vertices $a, b \in Z(i)$. We show the claim for α^i_x and note that the same holds for α^i_y. By Observation 4(iv), a and b are adjacent and α^i_x is adjacent to both a and b by Observation 4(i). Hence, $\{\alpha^i_x, a, b\}$ induces a 3-cycle in G'.

An analogous argument can be given for the second statement. ⌟

In the light of Observation 6 and Claim 7, we make

Observation 8. Let F be an induced forest in G'. If $V(F)$ contains three vertices from $Z_{+\alpha}(i)$ for some $i \in [k]$ (three vertices from $R_{+\alpha}(i, j)$, respectively), then this set of three vertices must include α^i_x and α^i_y (resp., $\alpha^{(i,j)}_x$ and $\alpha^{(i,j)}_y$).

The previous observation implies that in G', any induced forest on $k' = 3k + 3 \cdot \binom{k}{2} + 1$ has the following form.

(I) For each $i \in [k]$, $V(F) \cap Z_{+\alpha}(i) = \{\alpha^i_x, \alpha^i_y, z^i_s\}$, for some $s \in [p]$.

5

(II) For each $1 \leq i < j \leq k$, $V(F) \cap R_{+\alpha}(i,j) = \{\alpha_x^{(i,j)}, \alpha_y^{(i,j)}, r_{t,t'}^{(i,j)}\}$, for some $t, t' \in [p]$.

(III) $\beta \in V(F)$.

To conclude the proof, we argue that any such induced forest F includes an independent set of size $k + \binom{k}{2}$ of the form (1). In particular, we use the following claim to establish the correctness of the reduction.

Claim 9. Let F be an induced forest in G' on k' vertices, $1 \leq i < j \leq k$ and $s_i, s_j, t_i, t_j \in [p]$. If $z_{s_i}^i, r_{t_i,t_j}^{(i,j)}, z_{s_j}^j \in V(F)$, then $s_i = t_i$ and $s_j = t_j$.

Proof. Suppose not and assume wlog. that $s_i \neq t_i$. Then, $\left\{\beta, z_{s_i}^i, r_{s_t,t_i}^{(i,j)}\right\}$ induces a 3-cycle in G': We have that $\beta \in V(F)$ by (III), and by construction β is adjacent to all vertices in $Z(i)$ and $R(i,j)$. By Observation 3 and the assumption that $s_i \neq t_i$, we have that $z_{s_i}^i r_{t_i,t_j}^{(i,j)} \in E(G')$. ⌟

Since by (I) and (II), any induced forest on k' vertices contains precisely one vertex from each $Z(i)$ (for $i \in [k]$) and $R(i,j)$ (for $1 \leq i < j \leq k$), we can conclude together with Claim 9 that $V(F)$ contains an independent set

$$\left\{z_{s_1}^1, \ldots, z_{s_k}^k\right\} \cup \left\{r_{s_i,s_j}^{(i,j)} \mid 1 \leq i < j \leq k\right\}$$

which by Observation 3 implies that G has a clique on vertex set $\left\{v_{s_1}^1, \ldots, v_{s_p}^p\right\}$. □

Since a graph on n vertices has an induced forest of size k if and only if it has a feedback vertex set of size $n - k$, we have the following consequence of Theorem 1.

Corollary 10. FEEDBACK VERTEX SET *is* W[1]-*hard parameterized by linear mim-width, even if a linear branch decomposition of bounded mim-width is given.*

We now show additionally that the above reduction can easily be modified to prove W[1]-hardness for MAXIMUM INDUCED FOREST and FEEDBACK VERTEX SET on H-graphs when the parameter includes $||H||$. In particular, we show the following (using the notation from the proof of Theorem 1.)

Proposition 11. *The graph G' is a K-graph for some $K \supseteq H$ with $|K| = 3 \cdot |H|$ and $||K|| = ||H|| + 2 \cdot |H|$.*

Proof. The graph K is obtained from H in the following way and is shown in Figure 3.

1. For each $i \in [k]$, add to H two neighbors π_i^x and π_i^y of u_i.

2. For each $1 \leq i < j \leq k$, add to H two neighbors $\pi_{(i,j)}^x$ and $\pi_{(i,j)}^y$ of $w_{(i,j)}$.

We let $\Pi := \bigcup_{i \in [k]}\{\pi_i^x, \pi_i^y\} \cup \bigcup_{1 \leq i < j \leq k}\{\pi_{(i,j)}^x, \pi_{(i,j)}^y\}$. The subdivision K' of K is obtained from subdividing each each edge of $K[V(K) \setminus \Pi]$ p times. (Note that this is the same subdivision done by Fomin et al. [2].) The graph G' is now constructed similarly to the construction given in the previous proof, except that we do not have the vertices $x_{0+\epsilon}, y_{0+\epsilon}, x_{p+\epsilon}$ and $y_{p+\epsilon}$ in K and hence in the models of the K-representation. For $i \in [k]$, the model of vertex α_x^i becomes $\{\pi_i^x\}$ and the model of α_y^i becomes $\{\pi_i^y\}$. For $1 \leq i < j \leq k$, the model of $\alpha_x^{(i,j)}$ becomes $\{\pi_{(i,j)}^x\}$ and the model for $\alpha_y^{(i,j)}$ becomes $\{\pi_{(i,j)}^y\}$. Furthermore, the model of each vertex z_s^i includes $\{\pi_i^x, \pi_i^y\}$ and the model of each $r_{s,t}^{(i,j)}$ includes the nodes $\pi_{(i,j)}^x$ and $\pi_{(i,j)}^y$. We can now represent the vertex β with model $V(K) \setminus \Pi$. It is straightforward to verify that this procedure gives a K-representation of G'. □
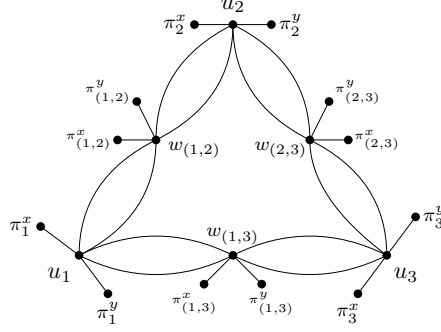
Figure 3: Illustration of the graph K for $k = 3$.

By Proposition 11 we have the following consequence of the proof of Theorem 1.

Corollary 12. MAXIMUM INDUCED FOREST *on H-graphs is* W[1]*-hard when parameterized by* $k + ||H||$ *and* FEEDBACK VERTEX SET *on H-graphs is* W[1]*-hard when parameterized by* $||H||$*. In both cases, the hardness even holds when an H-representation of the input graph is given.*

References

[1] Michael R. Fellows, Danny Hermelin, Frances Rosamond, and Stéphane Vialette. On the parameterized complexity of multiple-interval graph problems. *Theoretical Computer Science*, 410(1):53–61, 2009.

[2] Fedor V. Fomin, Petr A. Golovach, and Jean-Florent Raymond. On the tractability of optimization problems on H-graphs. *arXiv preprint*, 2017. arXiv:1709.09737.

[3] Lars Jaffke, O-joung Kwon, and Jan Arne Telle. A unified polynomial-time algorithm for feedback vertex set on graphs of bounded mim-width. *arXiv preprint*, 2017. arXiv:1710.07148.

[4] Krzysztof Pietrzak. On the parameterized complexity of the fixed alphabet shortest common supersequence and longest common subsequence problems. *Journal of Computer and System Sciences*, 67(4):757–771, 2003.