Edge-maximal graphs of branchwidth k

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Abstract

In this extended abstract we state some definitions and results from our earlier paper [2] and use these to characterize the class of edge-maximal graphs of branchwidth k. Similarly to the maximal graphs of treewidth k (the k-trees) they turn out to be a subclass of chordal graphs where every minimal separator has size k.

Keywords: Graph, Tree-decomposition, Width parameters

1 Introduction

Branchwidth and treewidth are connectivity parameters of graphs introduced in the proof of the Graph Minors Theorem by Robertson and Seymour [3]. In a recent paper [2] we introduced some useful tools for branchwidth, like k-troikas, k-good chordal graphs and good subtree representations, that allow us to prove results for branchwidth that are analogous to similar results for treewidth. For example, we arrive at a succinct expression of the common

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basis of treewidth and branchwidth: For any $k \geq 2$ a graph G on vertices $v_1, v_2, ..., v_n$ has branchwidth at most k (treewidth at most $k-1$) if and only if there is a cubic tree T with subtrees $T_1, T_2, ..., T_n$ such that if v_i and v_j adjacent then subtrees T_i and T_j share at least one edge (node) of T, and each edge (node) of T is shared by at most k of the subtrees (replace underlined words by the words in parenthesis.) In this extended abstract we state some definitions and results from our earlier paper [2] and use these to characterize the class of edge-maximal graphs of branchwidth k. Similarly to the maximal graphs of treewidth k (the k-trees) they turn out to be a subclass of chordal graphs where every minimal separator has size k.

2 Definitions and earlier results

A *branch-decomposition* (T, μ) of a graph G is a tree T with nodes of degree one and three only, together with a bijection μ from the edge-set of G to the set of degree-one nodes (leaves) of T. For an edge e of T let T_1 and T_2 be the two subtrees resulting from $T \setminus \{e\}$, let G_1 and G_2 be the graphs induced by the edges of G mapped by μ to leaves of T_1 and T_2 respectively, and let $mid(e) = V(G_1) \cap V(G_2)$. The width of (T, μ) is the size of the largest $mid(e)$ thus defined. For a graph G its *branchwidth* $bw(G)$ is the smallest width of any branch-decomposition of G [3].

A tree-decomposition $(T, \mathcal{X})$ of a graph G is an arrangement of the vertex subsets $\mathcal{X}$ of G, called bags, as nodes of the tree T such that for any two adjacent vertices in G there is some bag containing them both, and for each vertex of G the bags containing it induce a connected subtree.

Definition 2.1 A *subtree-representation* $R = (T, \{T_1, T_2, ..., T_n\})$ is a pair where T is a tree with vertices of degree at most three and $T_1, T_2, ..., T_n$ are subtrees of T. Its *edge intersection graph* $EI(R)$ has vertex set $\{v_1, v_2, ..., v_n\}$ and edge set $\{v_i v_j : T_i \text{ and } T_j \text{ share an edge of } T\}$, while its *vertex intersection graph* $VI(R)$ has the same vertex set but edge set $\{v_i v_j : T_i \text{ and } T_j \text{ share a node of } T\}$. For a node u of T, we call the set of vertices $X_u = \{v_i : T_i \text{ contains } u\}$ the *bag* of u, and $\{X_u : u \in V(T)\}$ the bags of R.

With the above terminology we can easily move between the view of a subtree-representation R as a tree T with a set of subtrees $\{T_1, T_2, ..., T_n\}$ or as a tree T with a set of bags $\{X_u : u \in V(T)\}$. When manipulating the

[3] The graphs of branchwidth 1 are the stars, and constitute a somewhat pathological case. To simplify certain statements we therefore restrict attention to graphs having branchwidth $k \geq 2$.

latter we must simply ensure that for any vertex in $EI(R)$ the set of bags containing that vertex corresponds to a set of nodes of T inducing a subtree, i.e. a connected subgraph.

Definition 2.2 The *edge-weight* of subtree-representation $R = (T, \{T_1, ...T_n\})$ is the maximum, over all edges uv of T, of the number of subtrees in $\{T_1, ...T_n\}$ that contain edge uv. R is a *good* subtree-representation if $EI(R) = VI(R)$.

Lemma 2.3 *[2] A graph G has branchwidth at most $k \Leftrightarrow$ there is a good subtree-representation R of edge-weight at most k with G a spanning subgraph of $EI(R)$.*

Definition 2.4 A k-troika [4] (A, B, C) of a set X are 3 subsets of X such that $|A| \leq k$, $|B| \leq k$, $|C| \leq k$, and $A \cup B = A \cup C = C \cup B = X$. (A, B, C) *respects* $S_1, S_2, ..., S_q$ if any $S_i, 1 \leq i \leq q$ is contained in at least one of A, B or C.

Lemma 2.5 *[2] X has a k-troika respecting S_1, S_2 (assume $|S_1 \cup S_2| > k$) if and only if $|X| \leq \lfloor 3k/2 \rfloor$, $|S_1| \leq k$, $|S_2| \leq k$ and $|X| \leq 2k - |S_1 \cap S_2|$. For any $q \geq 0$ if X has a k-troika respecting $S_1, S_2, ..., S_q$ then $|S_i| \leq k$ for each $1 \leq i \leq q$ and $|X| \leq \lfloor 3k/2 \rfloor$.*

Definition 2.6 A *k-good chordal graph* is a chordal graph where every maximal clique X has a k-troika respecting the minimal separators contained in X.

Theorem 2.7 *[2] A graph G has branchwidth at most $k \Leftrightarrow G$ is a spanning subgraph of a k-good chordal graph*

3 Edge-maximal graphs of branchwidth k

Definition 3.1 A graph G of branchwidth k is called a k-branch if adding any edge to G will increase its branchwidth.

The edge-maximal graphs of treewidth k are the well-known k-trees, definable as chordal graphs where every minimal separator has size k and no clique is larger than $k + 1$ [4]. It is known that the 1-branches are exactly the stars, i.e. a subclass of 1-trees, that 2-branches are exactly the 2-trees, and it can be deduced from Lemmas 3 and 7 and Theorem 6 of [1] that the 3-branches are the 3-trees having no three-dimensional cube as a minor. For larger values

[4] A troika is a horse-cart drawn by three horses, and when the need arises any two of them should also be able to pull the cart

of k the connection between k-trees and k-branches is not so tight, allthough our first observation implies that k-branches are also chordal with minimal separators of size k.

Definition 3.2 Let G be a chordal graph with C_G its set of maximal cliques and S_G its set of minimal separators. A tree-decomposition $(T, \mathcal{X})$ of G is called *k-full* if the following conditions hold. The set of bags $\mathcal{X}$ is in 1-1 correspondence with $C_G \cup S_G$. We call the nodes with bags in C_G the *maxclique nodes* and the nodes with bags in S_G the *minsep nodes*. The minsep bags all have size k. There is an edge ij in the tree T iff $X_i \in S_G, X_j \in C_G$ and $X_i \subseteq X_j$. Every maxclique bag X_j has a k-troika respecting its neighbor minsep bags.

Note that the conditions that every minimal separator have size k and every maximal clique have a k-troika respecting the minimal separators contained in it implies that if G has a k-full tree-decomposition then it is unique. Note also that if G has a k-full tree-decomposition then it is a k-good chordal graph and thus has branchwidth at most k by Theorem 2.7

Lemma 3.3 *If G is a k-branch then it has a k-full tree-decomposition.*

Proof: By Theorem 2.7 we know that G must be a k-good chordal graph. We show that every minimal separator S has size k. Note that this suffices since by taking one node for each element of $C_G \cup S_G$ and adding edges according to the criteria in the lemma we must get a tree-decomposition $(T, \mathcal{X})$ since otherwise G would not be chordal. Moreover, since G is a k-good chordal graph each maxclique bag of $(T, \mathcal{X})$ would have a k-troika respecting its minsep neighbors.

Let S be a minimal (a, b)-separator of G and consider a good subtree-representation $R = (T, S)$ of edge-weight k with $G = EI(R) = VI(R)$, guaranteed to exist by Lemma 2.3. There is a unique path P in T between the subtrees corresponding to a and b. For every node i on this path its bag X_i contains S and there must exist two adjacent nodes i, j for whom $X_i \cap X_j = S$, otherwise S is not a minimal a, b-separator. But then we must have $|S| \leq k$ since otherwise the edge-weight of R would be more than k. We now show that if $|S| < k$ then we can add an edge to G without increasing its branchwidth. Assume that moving from left to right on path P we first hit i and then its neighbor j. Move left from node i and right from node j until encountering the first nodes l and r with bags not contained in S, say $c \in X_l \setminus S$ and $d \in X_r \setminus S$. We now add vertex c to every bag corresponding to a node on the path from l to i and vertex d to every bag on the path from r to j. Note that the intersection of any two bags corresponding to adjacent nodes on the l to

r path now has size $|S|+1$. Now subdivide the edge ij with the new node having bag $S \cup \{c,d\}$ and also having a leaf attached to it with bag $\{c,d\}$. If $|S| < k$ we would now have a good subtree-representation R' of edge-weight k. By Lemma 2.3 this would mean that the graph $EI(R')$ which is G with added edge cd has branchwidth k so G could not have been a k-branch. $\square$

Definition 3.4 Let $(T, \mathcal{X})$ be a k-full tree-decomposition of a graph G. Let T' be a subtree of T having at least one edge with all its leaves being maxclique nodes. Let $Contract(T, T')$ be the tree resulting from T by contracting all edges of T' and let $X_{T'} = \{v : v \in X \text{ and } X \text{ a maxclique node in } T'\}$ be all the vertices of G contained in some bag of T'. Let $(Contract(T, T'), \mathcal{X}')$ be the contraction induced on the tree-decomposition, with a node of $V(T) \setminus V(T')$ having the same bag in $\mathcal{X}'$ and $\mathcal{X}$, and the new contracted node having bag $X_{T'}$. We say that T' is a mergeable subtree of the k-full tree-decomposition $(T, \mathcal{X})$ if $(Contract(T, T'), \mathcal{X}')$ is a k-full tree-decomposition of the graph G' which we get from G by making a clique out of the vertices in $X_{T'}$.

Theorem 3.5 *G is a k-branch $\Leftrightarrow$ $|V(G)| \geq \lfloor 3(k-1)/2 \rfloor + 1$ and G has a k-full tree-decomposition having no mergeable subtree.*

Proof: $\Rightarrow$: If $|V(G)| \leq \lfloor 3(k-1)/2 \rfloor$ then by Theorem 2.7 G has branchwidth less than k since the clique on this many vertices has a $(k-1)$-troika. By Lemma 3.3 it follows that G has a k-full tree-decomposition and if this had a mergeable subtree then we could add edges to G and still have a k-good chordal graph of branchwidth at most k.

$\Leftarrow$: Since G has $|V(G)| \leq \lfloor 3(k-1)/2 \rfloor$ and a k-full tree-decomposition it has branchwidth k by Theorem 2.7 and Lemma 2.5. Assume for sake of contradiction that some strict supergraph H of G is a k-branch and that it has a k-full tree-decomposition T_H. Note first that since every minimal separator of both G and H is of size k then H cannot contain a minimal separator that is not also a minimal separator of G. Thus the minsep nodes of T_H are a subset of the minsep nodes of T_G. Consider the connected subtrees that result from removing the minsep nodes of T_H from T_G. It follows that the maximal cliques of H must be in 1-1 correspondence with these subtrees. As H is a strict supergraph of G, there is at least one such component, say T', containing at least two maxclique nodes, and any such T' would be a mergeable subtree of T_G. $\square$

To characterize k-branches all that remains is to characterize the k-full tree-decompositions having no mergeable subtrees.

Lemma 3.6 *A k-full tree-decomposition $(T, \mathcal{X})$ has a mergeable subtree $\Leftrightarrow$ T*

has a non-trivial subtree T' whose leaves are maxclique nodes and satisfying $|X_{T'}| \leq \lfloor 3k/2 \rfloor$. Moreover, either the node with bag $X_{T'}$ in $(Contract(T, T')$ has at most one neighbor or else T' is a path X, B, Y with X, B, Y and all their neighbors in T inducing a path A, X, B, Y, C satisfying $B \setminus (A \cup C) = \emptyset$.

Proof: $\Leftarrow$: We show that T' as described would be a mergeable subtree of $(T, \mathcal{X})$ since $X_{T'}$ would in $Contract(T, T')$ have a k-troika respecting its minsep neighbors. If it had at most one minsep neighbor this is obvious. Otherwise by the conditions in the Lemma we would have $|A \cap C| = 2k - |(X \cup Y)|$ satisfying Lemma 2.5.

$\Rightarrow$: Assume $(T, \mathcal{X})$ has a mergeable subtree T''. Certainly we must have $|X_{T''}| \leq \lfloor 3k/2 \rfloor$ since the clique on more vertices than this does not have branchwidth k. If $X_{T''}$ had more than two minsep neighbors in $Contract(T, T'')$ then some two of them would have union strictly smaller than $X_{T''}$ and so $X_{T''}$ could not have a k-troika respecting its minsep neighbors. Thus wlog we have that $X_{T''}$ has two minsep neighbors A, C in $Contract(T, T'')$ and $X_{T''}$ having a k-troika respecting A, C. We show that T'' must contain a subtree like T' described in the Lemma. If T'' has at least 3 maxclique nodes then all nodes of T'' must have degree 2 in T since otherwise $A \cup C \neq X_{T''}$ and $X_{T''}$ could not have a k-troika respecting A, C. Assume the path T'' is $X_1, S_1, X_2, ..., S_{i-1}, X_i$, with A a neighbor of X_1 and C a neighbor of X_i. We claim that X_1, S_1, X_2 would already be a mergeable subtree. Since $X_{T''}$ has a k-troika respecting A, C we must have $A \cup C = X_1 \cup X_2 \cup ... \cup X_i$. By the interval structure of these maximal cliques we therefore have $S_1 \setminus (A \cup S_2) = \emptyset$. Note that we have $A \cap S_2 \subseteq S_1$ and since $|A| = |S_2| = k$ we have $|A \cap S_2| = 2k - |(X \cup Y)|$ so that by Lemma 2.5 $X \cup Y$ has a k-troika respecting A, S_2. □

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