A width parameter useful for chordal and co-comparability graphs

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Abstract. In 2013 Belmonte and Vatshelle used mim-width, a graph parameter bounded on interval graphs and permutation graphs that strictly generalizes clique-width, to explain existing algorithms for many domination-type problems, also known as (σ, ρ)-problems or LC-VSVP problems, on many special graph classes. In this paper, we focus on chordal graphs and co-comparability graphs, that strictly contain interval graphs and permutation graphs respectively. First, we show that mim-width is unbounded on these classes, thereby settling an open problem from 2012. Then, we introduce two graphs $K_t \boxminus K_t$ and $K_t \boxminus S_t$ to restrict these graph classes, obtained from the disjoint union of two cliques of size t, and one clique of size t and one independent set of size t respectively, by adding a perfect matching. We prove that $(K_t \boxminus S_t)$-free chordal graphs have mim-width at most $t-1$, and $(K_t \boxminus K_t)$-free co-comparability graphs have mim-width at most $t-1$. From this, we obtain several algorithmic consequences, for instance, while Dominating Set is NP-complete on chordal graphs, like all LC-VSVP problems it can be solved in time $\mathcal{O}(n^t)$ on chordal graphs where t is the maximum among induced subgraphs $K_t \boxminus S_t$ in the given graph. We also show that classes restricted in this way have unbounded rank-width which validates our approach.
In the second part, we generalize these results to bigger classes. We introduce a new width parameter sim-width, special induced matching-width, by making only a small change in the definition of mim-width. We prove that chordal and co-comparability graphs have sim-width at most 1. Since Dominating Set is NP-complete on chordal graphs, an XP algorithm parameterized only by sim-width would imply P=NP. Therefore, to apply the algorithms for domination-type problems mentioned above, we parameterize by both sim-width w and a further parameter t, which is the smallest value such that the input has no induced minor isomorphic to $K_t \boxminus S_t$ or $K_t \boxminus S_t$. We show that such graphs have mim-width at most $8(w+1)t^3$ and that the resulting algorithms for domination-type problems have runtime $n^{\mathcal{O}(wt^3)}$, when the decomposition tree is given.

1 Introduction

Graph width parameters like tree-width and clique-width have been studied for many years, and their algorithmic use has been steadily increasing. In 2012 Vatshelle introduced mim-width[4]

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[4] Introduced formally by Vatshelle in [21], but implicitly used by Belmonte and Vatshelle in [3]

Fig. 1. $K_5 \boxminus S_5$ and $K_5 \boxminus K_5$.

which is a parameter with even stronger modelling power than clique-width. This parameter is defined using branch decompositions over the vertex set with the cut function computing the maximum induced matching of the bipartite graph obtained by removing the edges in both parts. Well-known graph classes have bounded mim-width, e.g. interval graphs and permutation graphs have mim-width 1 while their clique-width can be quadratic in the number of vertices [21]. Thus for mim-width k an XP algorithm - runtime $n^{f(k)}$ - is already very interesting. XP algorithms based on mim-width were used by Belmonte et al [3] and Bui-Xuan et al [5] to give a common explanation for the existence of polynomial-time algorithms on many well-known graph classes like interval graphs and permutation graphs, for LC-VSVP problems - Locally Checkable Vertex Subset and Vertex Partitioning problems. We define the class of LC-VSVP problems formally in Section 4. In this paper, we extend these algorithms for LC-VSVP problems to subclasses of chordal graphs and co-comparability graphs, that strictly contain interval graphs and permutation graphs respectively. We also show that mim-width is unbounded on general chordal and co-comparability graphs.

The LC-VSVP problems include the class of domination-type problems known as (σ, ρ)-problems, whose intractability on chordal graphs is well known. For two subsets of non-negative numbers σ and ρ, a set S of vertices is called a (σ, ρ)-*dominating set* if for every vertex $v \in S$, $|S \cap N(v)| \in \sigma$, and for every $v \notin S$, $|S \cap N(v)| \in \rho$. Golovach et al [10] showed that for chordal graphs, the problem of deciding if a graph has a (σ, ρ)-dominating set is NP-complete if σ and ρ are such that there exists at least one chordal graph containing more than one such set. Golovach, Kratochvíl and Suchý [11] extended these results to the parameterized setting, showing that the existence of a (σ, ρ)-dominating set of size k, and at most k, are W[1]-complete problems when parameterized by k for any pair of finite sets σ and ρ.

In this paper we apply a different parametrization to solve these problems efficiently on chordal graphs, and on co-comparability graphs. We introduce two graphs $K_t \boxminus K_t$ and $K_t \boxminus S_t$, which are obtained from the disjoint union of two cliques of size t, and one clique of size t and one independent set of size t respectively, by adding a perfect matching. See Figure 1. We prove that $(K_t \boxminus S_t)$-free chordal graphs have mim-width at most $t-1$, and $(K_t \boxminus S_t)$-free co-comparability graphs have linear mim-width at most $t-1$. These results comprise newly discovered graph classes of bounded mim-width, as already $(K_3 \boxminus S_3)$-free chordal graphs contain all interval graphs, and $(K_3 \boxminus K_3)$-free co-comparability graphs contain all permutation graphs. In particular, previous known classes including interval and permutation graphs have bounded linear mim-width, while $(K_t \boxminus S_t)$-free chordal graphs in general have unbounded linear mim-width. By applying algorithms of Bui-Xuan et al [5] we obtain the following.

Theorem 1.1. *Given an n-vertex $(K_t \boxminus S_t)$-free chordal graph or an n-vertex $(K_t \boxminus K_t)$-free co-comparability graph, we can solve any LC-VSVP problem in time $n^{O(t)}$, including all (σ, ρ)-domination problems for finite or co-finite σ and ρ. For example,* Minimum Dominating Set *is solved in time $O(n^{3t+4})$ and q-*Coloring *in time $O(qn^{3qt+4})$.*

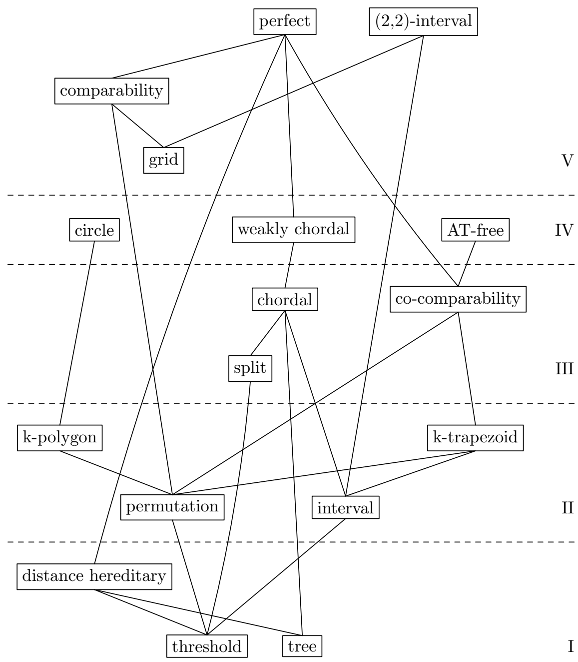

Fig. 2. Inclusion diagram of some well-known graph classes. (I) Classes where clique-width and rank-width are constant. (II) Classes where mim-width is constant. (III) Classes where sim-width is constant. (IV) Classes where it is unknown if sim-width is constant. (V) Classes where sim-width is unbounded.

In the second part of this paper we generalize these results to larger classes. We also ask the question - what algorithmic use can we make of a width parameter with even stronger modelling power than mim-width? We define the parameter *sim-width* (special induced matching-width) by making only a small change in the definition of mim-width, simply requiring that a special induced matching across a cut of the graph cannot contain edges between two vertices on the same side of the cut. See Section 2 for the precise definitions. The linear variant of sim-width will be called *linear sim-width.* We show that graphs of bounded sim-width are closed under taking induced minors, and the modelling power of sim-width is strictly stronger than mim-width.

Theorem 1.2. *Chordal graphs have sim-width at most* 1 *while split graphs have unbounded mim-width, and a branch-decomposition of sim-width at most* 1 *can be found in polynomial time.*

Co-comparability graphs have linear sim-width at most 1 *but unbounded mim-width, and a linear branch-decomposition of sim-width at most* 1 *can be found in polynomial time.*

Note that this also confirms a conjecture of Vatshelle and Belmonte from 2012 [21,3] that chordal graphs and co-comparability graphs have unbounded mim-width[5]. See Figure 2 for an inclusion diagram of some well-known graph classes. We conjecture that circle graphs and weakly chordal graphs, also have constant sim-width.

[5] Our result appeared on arxiv June 2016. In August 2016 a similar result by Mengel, developed independently, also appeared on arxiv [15]

The sim-width parameter thus meets our goal of having stronger modelling power than mim-width, but stronger modelling power automatically implies weaker analytic power, i.e. fewer problems will have FPT or XP algorithms when parameterized by sim-width. As an example, for problems like Minimum Dominating Set which are NP-complete on chordal graphs [4], we cannot expect an XP algorithm parameterized by sim-width, i.e. with runtime $|V(G)|^{f(\mathrm{simw}(G))}$, even if we are given a branch-decomposition, unless P=NP. Thus, for the algorithmic use of sim-width we must either strongly restrict the problems we consider, or we must put a further restriction on the input graphs. In this paper we take the latter approach, and apply the same graphs $K_t \boxminus K_t$ and $K_t \boxminus S_t$ as we did earlier for chordal and co-comparability graphs, but now we disallow them as induced minors, which is natural as the resulting classes are then closed under induced minors.

Theorem 1.3. *Every graph with sim-width w and no induced minor isomorphic to $K_t \boxminus K_t$ and $K_t \boxminus S_t$ has mim-width at most $8(w+1)t^3 - 1$.*

In a further result we show that we can also exclude these graphs as an induced subgraph to bound the mim-width, but in that case, we need to use Ramsey's theorem, and the bound will in general become worse. Combining again with the algorithms of [5] we get the following.

Theorem 1.4. *Given an n-vertex graph G having no induced minor isomorphic to $K_t \boxminus K_t$ and $K_t \boxminus S_t$, with a branch-decomposition of sim-width w, we can solve any LC-VSVP problem in time $O(n^{O(wt^3)})$*

The paper is organized as follows. Section 2 contains the main definitions. In Section 3 we prove that chordal graphs have sim-width at most 1, unbounded mim-width, but $(K_t \boxminus S_t)$-free chordal graphs have mim-width at most $t-1$. Similarly we show that co-comparability graphs have linear sim-width at most 1, unbounded mim-width, but $(K_t \boxminus K_t)$-free co-comparability graphs have linear mim-width at most $t-1$. We provide polynomial-time algorithms to find such decompositions. In Section 4 we introduce LC-VSVP problems and obtain the algorithmic results (Corollary 4.2) for chordal and co-comparability graphs. Then in Section 5 we generalize these algorithmic results to graphs of bounded sim-width, by bounding mim-width as a function of sim-width for graphs excluding $K_t \boxminus K_t$ and $K_t \boxminus S_t$, both as induced subgraphs and as induced minors.

In Section 6, we prove that chordal graphs having no induced minor isomorphic to $K_3 \boxminus K_3$ and $K_3 \boxminus S_3$ have unbounded rank-width, and in Section 7, we prove that sim-width is closed under induced minors. We list some questions on sim-width in Section 8.

2 Preliminaries

We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. We denote by $N_G(v)$ the set of neighbors of a vertex v in G, and let $N_G[v] := N_G(v) \cup \{v\}$. For two graphs G_1 and G_2 on disjoint vertex sets, the *union* of G_1 and G_2 is the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. For $v \in V(G)$ and $X \subseteq V(G)$, we denote by $G - v$ the graph obtained from G by removing v, and denote by $G - X$ the graph obtained from G by removing all vertices in X. For $e \in E(G)$, we denote by $G - e$ the graph obtained from G by removing e, and denote by G/e the graph obtained from G by contracting e. For a vertex v of G with exactly two neighbors v_1 and v_2 that are non-adjacent, the operation of removing v and adding the edge between its neighbors is called *smoothing* a vertex v. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X. A *clique* is a set of vertices that are pairwise adjacent, and an *independent set* is a set of vertices that are pairwise non-adjacent. A set of edges $\{v_1w_1, v_2w_2, \ldots, v_mw_m\}$ of G is called an *induced matching* in G if there are no other edges in $G[\{v_1, \ldots, v_m, w_1, \ldots, w_m\}]$. For a vertex partition (A, B) of G and an induced matching $\{v_1w_1, v_2w_2, \ldots, v_mw_m\}$ in G where $v_1, \ldots, v_m \in A$ and $w_1, \ldots, w_m \in B$, we say that it is an induced matching in G between A and B.

For two graphs H and G, H is a *subgraph* of G if H can be obtained from G by removing some vertices and edges, and H is an *induced subgraph* of G if $H = G[X]$ for some $X \subseteq V(G)$, and H is an *induced minor* of G if H can be obtained from G by a sequence of removing vertices and contracting edges, and H is a *minor* of G if H can be obtained from G by a sequence of removing vertices, removing edges, and contracting edges. We note that it is not allowed to remove an edge in the induced minor relation; for instance, the complete graph on 4 vertices cannot contain the cycle of length 4 as an induced minor. For a graph H, we say a graph is *H-free* if it contains no induced subgraph isomorphic to H.

A pair of vertex subsets (A, B) of a graph G is called a *vertex partition* if $A \cap B = \emptyset$ and $A \cup B = V(G)$. For a vertex partition (A, B) of a graph G, we denote by $G[A, B]$ the bipartite graph on the bipartition (A, B) where for $a \in A, b \in B$, a and b are adjacent in $G[A, B]$ if and only if they are adjacent in G. For a bipartite graph G with a bipartition (A, B), we say that a matrix M is a *bipartitie-adjacency* matrix of G, if the rows of M are indexed by A, the columns of M are indexed by B, and for $a \in A, b \in B$, $M_{a,b} = 1$ if a is adjacent to b in G, and $M_{a,b} = 0$ otherwise.

A tree is called *subcubic* if every internal node has exactly 3 neighbors. A tree T is called a *caterpillar* if contains a path P where for every vertex in T either it is in P or has a neighbor on P. A graph is called *chordal* if it contains no induced subgraph isomorphic to a cycle of length 4 or more. For a graph G, an ordering $v_1, \ldots, v_n$ of the vertex set of G is called a *co-comparability ordering* if for every triple i, j, k with $i < j < k$, v_j has a neighbor in each path from v_i to v_k avoiding v_j. A graph is called a *co-comparability graph* if it admits a co-comparability ordering. The complete graph on n vertices will be denoted by K_n.

2.1 Width parameters

For a graph G, we define functions $\mathrm{cutrk}_G, \mathrm{mimval}_G, \mathrm{simval}_G$ from $2^{V(G)}$ to $\mathbb{N}$ such that

- $\mathrm{cutrk}_G(A)$ is the rank of the bipartite-adjacency matrix of $G[A, V(G) \setminus A]$ where the rank is computed over the binary field,
- $\mathrm{mimval}_G(A)$ is the maximum size of an induced matching of $G[A, V(G) \setminus A]$,
- $\mathrm{simval}_G(A)$ is the maximum size of an induced matching $\{a_1b_1, a_2b_2, \ldots, a_mb_m\}$ in G where $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in V(G) \setminus A$.

For a graph G, a pair (T, L) of a subcubic tree T and a function L from $V(G)$ to the set of leaves of T is called a *branch-decomposition*. For each edge e of T, let (A_1^e, A_2^e) be the vertex partition of G where T_1^e, T_2^e are the two connected components of $T - e$, and for each $i \in \{1, 2\}$, A_i^e is the set of all vertices in G mapped to leaves contained in T_i^e. We call it the vertex partition of G associated with e. For a branch-decomposition (T, L) of a graph G and an edge e in T and a function $f : 2^{V(G)} \to \mathbb{N}$, the *width* of e with respect to f, denote by $f_{(T,L)}(e)$, is define as $f(A_1^e)$ where (A_1^e, A_2^e) is the vertex partition associated with e. The *width* of (T, L) with respect to f is the maximum width over all edges in T.

The *rank-width*, *mim-width*, and *sim-width* of a graph G are the minimum widths over all their branch-decompositions with respect to cutrk_G, mimval_G, and simval_G and denote by $\mathrm{rw}(G)$, $\mathrm{mimw}(G)$, and $\mathrm{simw}(G)$, respectively. For convenience, the width of a branch-decomposition will be called a rank-width, mim-width, or sim-width depending on the function f. If T is a subcubic caterpillar tree, then (T, L) is called a *linear branch-decomposition*. The *linear mim-width* and *linear sim-width* of a graph G are the minimum widths over all their linear branch-decompositions with respect to mimval_G and simval_G, and denote by $\mathrm{lmimw}(G)$ and $\mathrm{lsimw}(G)$, respectively. By definitions we have the following.

Lemma 2.1. *For a graph G, we have* $\mathrm{simw}(G) \le \mathrm{mimw}(G) \le \mathrm{rw}(G)$.

We also use tree-decompositions in Section 3. A *tree-decomposition* of a graph G is a pair $(T, \mathcal{B} = \{B_t\}_{t \in V(T)})$ such that (1) $\bigcup_{t \in V(T)} B_t = V(G)$, (2) for every edge in G, there exists B_t containing both end vertices, and (3) for $t_1, t_2, t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever t_2 is on the path from t_1 to t_3. Each vertex subset B_t is called a *bag* of the tree-decomposition. The *width* of a tree-decomposition is $w - 1$ where w is the maximum size of bags in the decomposition, and the *tree-width* of a graph is the minimum width over all tree-decompositions of the graph.

3 Mim-width of chordal and co-comparability graphs

In this section, we first show that *chordal graphs* and *co-comparability graphs* have sim-width at most 1, but have unbounded mim-width. Belmonte and Vatshelle [3] showed that chordal graphs either do not have constant mim-width or it is NP-complete to find such a decomposition. We strengthen their result. We further show that $(K_t \boxminus S_t)$-free chordal graphs have mim-width at most $t-1$, and similarly, $(K_t \boxminus S_t)$-free co-comparabililty graphshave mim-width at most $t-1$.

3.1 Chordal graphs

We use the fact that chordal graphs admit a tree-decomposition whose bags are maximal cliques.

Proposition 3.1. *Given a chordal graph, one can output a branch-decomposition of sim-width at most 1 in polynomial time. Moreover for every positive integer t, given a $(K_t \boxminus S_t)$-free chordal graph, one can output a mim-decomposition of width at most $t-1$ in polynomial time.*

Proof. To prove both statements, we construct a certain branch-decomposition explicitly. Let G be a chordal graph. We may assume that G is connected. We compute a tree-decomposition $(F, \mathcal{B} = \{B_t\}_{t \in V(F)})$ of G where every bag induces a maximal clique of G. It is known that such a decomposition can be computed in polynomial time; for instance, see [16]. Let us choose a root node r of F.

We construct a tree (T, L) from F as follows.

1. We attach a leaf r' to the root node r and regard it as the parent of r and let $B_r := \emptyset$.
2. For every $t \in V(F)$ with its parent t', we subdivide the edge tt' into a path $tv_1^t \cdots v_{|B_t \setminus B_{t'}|}^t t'$ with $|B_t \setminus B_{t'}|$ internal nodes, and for each internal node q, we attach a leaf q' and assign those leaves to the vertices of $B_t \setminus B_{t'}$ in any order as the images of L.
3. For every $t \in V(F)$, we do the following. Let $t_1, \ldots, t_m$ be the children of t in F. We remove t and introduce a path $w_1^t w_2^t \cdots w_m^t$. If t is a leaf, then we just remove it. We add an edge $w_1^t v_1^t$, and for each $i \in \{1, \ldots, m\}$, add an edge $w_i^t v_{|B_{t_i} \setminus B_t|}^{t_i}$.

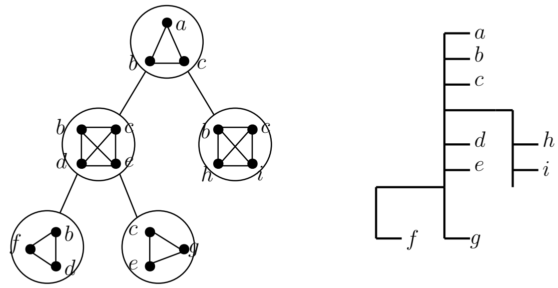

Fig. 3. Constructing a branch-decomposition (T, L) of a chordal graph G of sim-width at most 1 from its tree-decomposition.

See Figure 3 for an illustration of the construction. Let T' be the resulting tree, and we obtain a tree T from T' by smoothing all nodes of degree 2. Let (T, L) be the resulting branch-decomposition. Remark that F has $\mathcal{O}(|V(G)|)$ many nodes, and thus we can construct (T, L) in linear time. We claim that (T, L) has sim-width at most 1. We prove a stronger result that for every edge e of T with a partition (A, B) associated with e, either $N_G(A) \cap B$ or $N_G(B) \cap A$ is a clique.

Claim 1. Let e be an edge of T and let (A, B) be a partition of $V(G)$ associated with e. Then either $N_G(A) \cap B$ or $N_G(B) \cap A$ is a clique.

Proof. For convenience, we prove for T', which is the tree before smoothing. We may assume that both end nodes of e are internal nodes of T'. There are 4 types of e.

1. $e = v_i^t v_{i+1}^t$ for some t and i.
2. $e = w_1^t v_1^t$ for some t.
3. $e = v_{|B_{t_i} \setminus B_t|}^t w_i^t$ where t_i is a child of t.
4. $e = w_i^t w_{i+1}^t$ for some t and i.

Suppose $e = v_i^t v_{i+1}^t$ for some t and i, and let t' be the parent of t. We may assume that A corresponds to the part consisting of descendants of v_i^t. It is not difficult to check that for every $v \in A \setminus B_t$, $N_G(v) \cap B \subseteq B_t$. Furthermore, for $v \in A \cap B_t$, we have $N_G(v) \cap B \subseteq B_t$, because v is contained in $B_t \setminus B_{t'}$ by construction. Thus, $N_G(A) \cap B$ is a subset of B_t which is a clique. We can similarly prove for Cases 2 and 3.

We assume $e = w_i^t w_{i+1}^t$ for some t and i. Without loss of generality, we assume that $B_t \subseteq B$. We can observe that for every $v \in A$, $N_G(v) \cap B \subseteq B_t$. Thus $N_G(A) \cap B$ is a clique, as required. $\square$

Now, we prove that when G is $(K_t \boxminus S_t)$-free, (T, L) has mim-width at most $t - 1$.

Claim 2. If G is $(K_t \boxminus S_t)$-free, then (T, L) has mim-width at most $t - 1$.

Proof. We may assume $t \geq 2$. We show that for every edge e or T, $\text{mimval}_{(T,L)}(e) \leq t - 1$. Suppose for contradiction that there is an edge e of T and a partition (A, B) associated with e, where $\text{mimval}_G(A) \geq t$. We may assume that both end nodes of e are internal nodes of T.

By **Claim 1**, one of $N_G(A) \cap B$ and $N_G(B) \cap A$ is a clique. Without loss of generality we assume $N_G(B) \cap A$ is a clique C. If there is an induced matching $\{a_1 b_1, \ldots, a_t b_t\}$ in $G[A, B]$ where $a_1, \ldots, a_t \in A$, then we have $a_1, \ldots, a_t \in V(C)$. Furtheremore there are no edges between vertices in $\{b_1, \ldots, b_t\}$, otherwise, it creates an induced C_4. Thus, we have an induced subgraph isomorphic to $K_t \boxminus S_t$, which contradicts to our assumption. We conclude that (T, L) has mim-width at most $t-1$. $\square$

$\square$

We now prove the lower bound on the mim-width of general chordal graphs. We in fact show this for the class of split graphs that is a subclass of chordal graphs. A *split* graph is a graph that can be partitioned into two vertex sets C and I where C is a clique and I is an independent set. The Sauer-Shelah lemma [18] [19] is essential in the proof.

Theorem 3.2 (Sauer-Shelah lemma [18] [19]). *Let t be a positive integer and let M be an $X \times Y$ $(0,1)$-matrix such that $|Y| \geq 2$ and any two row vectors of M are distinct. If $|X| \geq |Y|^t$, then there are $X' \subseteq X$, $Y' \subseteq Y$ such that $|X'| = 2^t$, $|Y'| = t$, and all possible row vectors of length t appear in $M[X', Y']$.*

Proposition 3.3. *For every large enough n, there is a split graph on n vertices having mim-width at least $\sqrt{\log_2 \frac{n}{2}}$.*

Proof. Let $m \geq 10000$ be an integer and let $n := m + (2^m - 1)$. Let G be a split graph on the vertex partition (C, I) where C is a clique of size m, I is an independent set of size $2^m - 1$, and all vertices in I have pairwise distinct non-empty neighborhoods on C. We claim that every branch-decomposition of G has width at least $\sqrt{\log_2 \frac{n}{2}}$ with respect to the mimval_G function.

Let (T, L) be a branch-decomposition of G. It is well known that there is an edge of T inducing a balanced vertex partition, but we add a short proof for it. We subdivide an edge of T, and regard the new vertex as a root node. For each node $t \in V(T)$, let $\mu(t)$ be the number of leaves of T that are descendants of t. Now, we choose a node t that is farthest from the root node such that $\mu(t) > \frac{n}{3}$. By the choice of t, for each child t' of t, $\mu(t') \leq \frac{n}{3}$. Therefore, $\frac{n}{3} < \mu(t) \leq \frac{2n}{3}$. Let e be the edge connecting the node t and its parent. Then clearly, the vertex partition (A_1, A_2) of G induced by the edge e satisfies that for each $i \in \{1, 2\}$, $\frac{n}{3} < |A_i| \leq \frac{2n}{3}$. Without loss of generality, we may assume that $|A_1 \cap C| \geq |A_2 \cap C|$, and thus we have $\frac{m}{2} \leq |A_1 \cap C| \leq m$.

Note that $|A_2 \cap I| > \frac{n}{3} - m \geq \frac{2^m - 2m - 1}{3} \geq 2^{m-3}$. Since $|A_2 \cap C| < \frac{m}{2}$ and $m \geq 8$, there are at least

$$\frac{2^{m-3}}{2^{\frac{m}{2}}} \geq 2^{\frac{m}{2} - 3}$$

vertices in $A_2 \cap I$ that have pairwise distinct neighbors on $A_1 \cap C$. Let $I' \subseteq A_2 \cap I$ be the set of such vertices.

Now, by the Sauer-Shelah lemma, if $|I'| \geq |A_1 \cap C|^k$ for some positive integer k, then there will be an induced matching of size k between $A_1 \cap C$ and I' in $G[A_1, A_2]$. We choose $k := \sqrt{m}$. As $m \geq 10000$, we can deduce that $\frac{m}{2} - 3 \geq \sqrt{m} \log_2 m$. Therefore, we have

$$|I'| \geq 2^{\frac{m}{2} - 3} \geq m^{\sqrt{m}} \geq |A_1 \cap C|^{\sqrt{m}},$$

and there is an induced matching of size $\sqrt{m}$ between $A_1 \cap C$ and I' in $G[A_1, A_2]$. It implies that $\mathrm{mimval}_{(T,L)}(e) \geq \sqrt{m}$. As (T, L) was chosen arbitrary, the mim-width of G is at least $\sqrt{m} \geq \sqrt{\log_2 \frac{n}{2}}$. $\square$

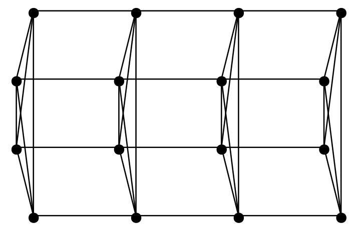

Fig. 4. The (4×4) column-clique grid.

3.2 Co-comparability graphs

We observe similar properties for co-comparability graphs. We recall that co-comparability graphs are exactly graphs that admit a co-comparability ordering.

Theorem 3.4 (McConnell and Spinrad [14]). *Given a co-comparability graph G, one can output a co-comparability ordering in polynomial time.*

Proposition 3.5. *Given a co-comparability graph, one can output a linear branch-decomposition of linear sim-width at most 1 in polynomial time. Moreover for every positive integer t, given a $(K_t \boxminus K_t)$-free co-comparability graph, one can output a linear mim-decomposition of width at most $t-1$ in polynomial time.*

Proof. Let G be a co-comparability graph. Using Theorem 3.4, we can obtain its co-comparability ordering $v_1, \ldots, v_n$. From this, we take a linear branch-decomposition (T, L) following the sequence. We claim that for each $i \in \{2, \ldots, n-1\}$, there is no induced matching of size 2 between $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$. Suppose there are $i_1, i_2 \in \{1, \ldots, i\}$ and $j_1, j_2 \in \{i+1, \ldots, n\}$ such that $\{v_{i_1}v_{j_1}, v_{i_2}v_{j_2}\}$ is an induced matching of G. Without loss of generality we may assume that $i_1 < i_2$. Then we have $i_1 < i_2 < j_1$, and thus by the definition of the co-comparability ordering, v_{i_2} should be adjacent to one of v_{i_1} and v_{j_1}, which contradicts to our assumption. Therefore, there is no induced matching of size 2. It implies that (T, L) has width at most 1.

Now, suppose that G is $(K_t \boxminus K_t)$-free. We claim that for every $i \in \{1, \ldots, n\}$, there are no induced matchings of size t in $G[\{v_1, \ldots, v_i\}, \{v_{i+1}, \ldots, v_n\}]$. For contradiction, suppose there is an induced matching $\{a_1b_1, \ldots, a_tb_t\}$ in $G[\{v_1, \ldots, v_i\}, \{v_{i+1}, \ldots, v_n\}]$ where $a_1, \ldots, a_t \in \{v_1, \ldots, v_i\}$. Without loss of generality, we may assume that for $1 \le x < y \le t$, a_x appears before a_y in the given ordering $v_1, \ldots, v_i$. So, one can observe that for $1 \le x < y \le t$, a_y is adjacent to a_x because a_y is in between a_x and b_x, and a_y is not adjacent to b_x. By the same reason, no matter how b_x and b_y lie, b_x and b_y are adjacent. It implies that $\{a_1, \ldots, a_t\}$ and $\{b_1, \ldots, b_t\}$ are cliques, and we have $K_t \boxminus K_t$, contradiction. We conclude that the same ordering gives a linear mim-decomposition of width at most $t-1$. □

To show that co-comparability graphs have unbounded mim-width, we provide a grid-like structure. For positive integers p, q, the $(p \times q)$ *column-clique grid* is the graph on the vertex set $\{v_{i,j} : 1 \le i \le p, 1 \le j \le q\}$ where

- for every $i \in \{1, \ldots, q\}$, $\{v_{1,j}, \ldots, v_{p,j}\}$ is a clique,
- for every $i \in \{1, \ldots, p\}$ and $j_1, j_2 \in \{1, \ldots, q\}$, v_{i,j_1} is adjacent to v_{i,j_2} if and only if $|j_2 - j_1| = 1$,

– for $i_1, i_2 \in \{1, \ldots, p\}$ and $j_1, j_2 \in \{1, \ldots, q\}$, v_{i_1,j_1} is not adjacent to v_{i_2,j_2} if neither $i_1 \neq i_2$ nor $j_1 \neq j_2$.

We depict an example in Figure 4. For each $1 \leq i \leq p$, we call $\{v_{i,1}, \ldots, v_{i,h}\}$ *the i-th row* of G, and define its columns similarly.

Lemma 3.6. *For integers $p, q \geq 12$, the $(p \times q)$ column-clique grid has mim-width at least $\min(\frac{p}{4}, \frac{q}{3})$.*

Proof. Let G be the $(p \times q)$ column-clique grid. Suppose that G has a branch-decomposition of width at most d with respect to the mimval_G function, for some positive integer d. It is enough to show that $d \geq \min(\frac{p}{4}, \frac{q}{3})$.

Firstly, assume that for each row R of G, $R \cap A \neq \emptyset$ and $R \cap B \neq \emptyset$. Then there is an edge between $R \cap A$ and $R \cap B$, as $G[R]$ is connected. For each i-th row R_i, we choose a pair of vertices $v_{i,a_i} \in R \cap A$ and $v_{i,b_i} \in R \cap B$ that are adjacent. We know that there is an index subset $X \subseteq \{1, \ldots, p\}$ such that $|X| \geq \frac{p}{2}$ and every pair (v_{i,a_i}, v_{i,b_i}) in $\{(v_{i,a_i}, v_{i,b_i}) : i \in X\}$ satisfies that $a_i + 1 = b_i$. By taking the same parity of a_i's, we know there is an index subset $Y \subseteq \{1, \ldots, p\}$ such that $|Y| \geq \frac{p}{4}$, all integers in $\{a_i : i \in Y\}$ have the same parity, and every pair (v_{i,a_i}, v_{i,b_i}) in $\{(v_{i,a_i}, v_{i,b_i}) : i \in Y\}$ satisfies that $a_i + 1 = b_i$.

We observe that $\{v_{i,a_i} v_{i,b_i} : i \in Y\}$ is an induced matching in $G[A, B]$. If it is not, then there are distinct integers $y, z \in Y$ such that either v_{y,a_y} is adjacent to v_{z,b_z}, or v_{y,b_y} is adjacent to v_{z,a_z}. But this is not possible; for instance, if v_{y,a_y} is adjacent to v_{z,b_z}, then $a_y = b_z$, and we have $a_y = a_z + 1$ as $z \in Y$. However, it contradicts to our assumption that all integers in $\{a_i : i \in Y\}$ have the same parity. Therefore, we conclude that $G[A, B]$ contains an induced matching of size at least $\frac{p}{4}$.

Now, we assume that there exists a row R such that R is fully contained in one of A and B. Without loss of generality, we may assume that R is contained in A. Since $|B| > \frac{|V(G)|}{3}$, we can choose an index set $X \subseteq \{1, \ldots, q\}$ such that $|X| > \frac{q}{3}$ and for each $i \in X$, the i-th column contains a vertex of B. For each i-th column where $i \in X$, we choose a vertex $v_{a_i,i}$ in B. It is not hard to verify that the edges between $\{v_{a_i,i} : i \in X\}$ and the rows in R form an induced matching of size $\frac{q}{3}$ in $G[A, B]$.

Therefore, we have $d \geq \min(\frac{p}{4}, \frac{q}{3})$. □

Corollary 3.7. *For every large enough n, there is a co-comparability graph on n vertices having mim-width at least $\sqrt{\frac{n}{12}}$.*

Proof. Let $p \geq 4$ be an integer, and let $n := 12p^2$. Let G be the $(4p \times 3p)$ column-clique grid. It is not hard to see that

$$v_{1,1}, v_{2,1}, \ldots, v_{4p,1}, v_{1,2}, v_{2,2}, \ldots, v_{4p-1,3p}, v_{4p,3p}$$

is a co-comparability ordering. Thus, G is a co-comparability graph. By Lemma 3.6, $\text{mimw}(G) \geq p = \sqrt{\frac{n}{12}}$. □

4 Algorithms for LC-VSVP problems

In this section, we describe algorithmic applications for restricted subclasses of chordal and co-comparability graphs described in Section 3. Telle and Proskurowski [20] classified a class of problems called *Locally Checkable Vertex Subset and Vertex Partitioning problems*, which is a subclass of MSO_1 problems. These problems generalize problems like Maximum Independent Set, Minimum Dominating Set, q-Coloring etc.

Let σ, ρ be finite or co-finite subsets of natural numbers. For a graph G and $S \subseteq V(G)$, we call S a (σ, ρ)-*dominating set* of G if

- for every $v \in S$, $|N_G(v) \cap S| \in \sigma$, and
- for every $v \in V(G) \setminus S$, $|N_G(v) \cap S| \in \rho$.

For instance, a $(0, \mathbb{N})$-set is an independent set as there are no edges inside of the set, and we do not care about the adjacency between S and $V(G) \setminus S$. Another example is that a $(\mathbb{N}, \mathbb{N}^+)$-set is a dominating set as we require that for each vertex in $V(G) \setminus S$, it has at least one neighbor in S. The class of *locally checkable vertex subset problems* consist of finding a minimum or maximum (σ, ρ)-dominating set in an input graph G, and possibly on vertex-weighted graphs.

For a positive integer q, a $(q \times q)$-matrix D_q is called a *degree constraint* matrix if each element is either a finite or co-finite subset of natural numbers. A partition $\{V_1, V_2, \ldots, V_q\}$ of the vertex set of a graph G is called a D_q-*partition* if

- for every $i, j \in \{1, \ldots, q\}$ and $v \in V_i$, $|N_G(v) \cap V_j| \in D_q[i, j]$.

For instance, if we take a matrix D_q where all diagonal entries are 0, and all other entries are $\mathbb{N}$, then a D_q-partition is a partition into q independent sets, which corresponds to a q-coloring of the graph. The class of *locally checkable vertex partitioning problems* consist of deciding if G admits a D_q-partition.

All these problems will be called *Locally Checkable Vertex Subset and Vertex Partitioning problems*, shortly LC-VSVP problems. As shown in [5] the runtime solving an LC-VSVP problem by dynamic programming relates to the finite or co-finite subsets of natural numbers used in its definition. The following function d is central.

1. Let $d(\mathbb{N}) = 0$.
2. For every finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max\{x \in \mathbb{N} : x \in \mu\}, \max\{x \in \mathbb{N} : x \notin \mu\})$.

For example, for Minimum Dominating Set and q-Coloring problems we plug in $d = 1$ because $\max(d(\mathbb{N}), d(\mathbb{N}^+)) = 1$ and $\max(d(0), d(\mathbb{N})) = 1$.

Theorem 4.1 (Belmonte and Vatshelle [3] and Bui-Xuan, Telle, and Vatshelle [5]). *Given an n-vertex graph and its branch-decomposition (T, L) of mim-width w we solve*

- *any (σ, ρ)-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $\mathcal{O}(n^{3dw+4})$,*
- *any D_q-vertex partitioning problem with $d = \max_{i,j} d(D_q[i, j])$ in time $\mathcal{O}(qn^{3dwq+4})$.*

Combining Theorem 4.1 with Propositions 3.1 and 3.5 we get the following.

Corollary 4.2. *Given an n-vertex chordal graph having no induced subgraph isomorphic to $K_t \boxminus S_t$, or co-comparability graph with no induced subgraph isomorphic to $K_t \boxminus K_t$, we solve*

- *any (σ, ρ)-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $\mathcal{O}(n^{3dt+4})$,*
- *any D_q-vertex partitioning problem with $d = \max_{i,j} d(D_q[i, j])$ in time $\mathcal{O}(qn^{3dtq+4})$.*

Dominating Set is NP-complete on chordal graphs [4], but for fixed t, it can be solved in polynomial on $(K_t \boxminus S_t)$-free chordal graphs by Corollary 4.2. For co-comparability graphs, Weighted Dominating Set is NP-complete on co-comparability graphs [13], but for every fixed t, it can be solved in polynomial time on $(K_t \boxminus K_t)$-free co-comparability graphs by Corollary 4.2. Remark that chordal graphs are $(K_2 \boxminus K_2)$-free, and co-comparability graphs are $(K_3 \boxminus S_3)$-free. In Section 5, we extend these two separate results into graphs of bounded sim-width having neither $K_t \boxminus S_t$ nor $K_t \boxminus S_t$ as an induced subgraph or an induced minor.

5 Extending to graphs of bounded sim-width

In Section 3, we proved that graphs of sim-width at most 1 contain all chordal and co-comparability graphs. A classical result on chordal graphs is that the problem of finding a minimum dominating set in a chordal graph is NP-complete [4]. So, even for this kind of locally-checkable problem, we cannot expect efficient algorithms on graphs of sim-width at most w. Therefore, to obtain a meta-algorithm for graphs of bounded sim-width encompassing many locally-checkable problems, we must impose some restrictions. We approach this problem in a way analogous to what has previously been done in the realm of rank-width [8].

It is well known that complete graphs have rank-width at most 1, but they have unbounded tree-width. Fomin, Oum, and Thilikos [8] showed that if G is K_r-minor free, then the tree-width of a graph is bounded by $c \cdot \mathrm{rw}(G)$ where c is a constant depending on r. This can be utilized algorithmically, to get a result for graphs of bounded rank-width when excluding a fixed minor, as the class of problems solvable in FPT time is strictly larger when parameterized by tree-width than rank-width [12].

We will do something similar by focusing on the distinction between mim-width and sim-width. However, K_r-minor free graphs are too strong, as one can show that on K_r-minor free graphs, the tree-width of a graph is also bounded by some constant factor of its sim-width. To see this, one can use Lemma 7.1 and the result on contraction obstructions for graphs of bounded tree-width [7].

Instead of using minors, we exclude certain graphs as induced subgraphs or induced minors. The induced minor operation is rather natural because sim-width does not increase when taking induced minors; we will observe in Section 7. We observe that indeed $K_t \boxminus K_t$ and $K_t \boxminus S_t$ are essential graphs to bound mim-width from graphs of bounded sim-width.

Proposition 5.1. *Every graph with sim-width w and no induced minor isomorphic to $K_t \boxminus K_t$ and $K_t \boxminus S_t$ has mim-width at most $8(w+1)t^3 - 1$.*

Note that we can also exclude these graphs as an induced subgraph to bound mim-width, but in that case, we need to use the Ramsey's theorem, and the bound on the exponent becomes again an exponential in w and t. Sometimes this Ramsey's bound can go down to a polynomial function depending on the underlying graphs, and thus it may also worth discussing about it. We denote by $R(k, \ell)$ the Ramsey number, which is the minimum integer satisfying that every graph with at least $R(k, \ell)$ vertices contains either a clique of size k or an independent set of size ℓ. By Ramsey's Theorem [17], $R(k, \ell)$ exists for every pair of positive integers k and ℓ.

Proposition 5.2. *Every graph with sim-width w and no induced subgraph isomorphic to $K_t \boxminus K_t$ and $K_t \boxminus S_t$ has mim-width at most $R(R(w+1,t), R(t,t))$.*

Propositions 3.1 and 3.5 can be seen as special cases of Proposition 5.2 because chordal graphs do not have any $K_t \boxminus K_t$ for $t \geq 2$, and co-comparability graphs do not have any $K_t \boxminus S_t$ for $t \geq 3$. Remark that Belmonte et al [2, Corollary 1] discussed that the Ramsey number can be a polynomial function of k and ℓ if the underlying graphs are some special classes such as chordal graphs, interval graphs, proper interval graphs, comparability graphs, co-comparability graphs, and permutation graphs. While comparability graphs have unbounded sim-width, all other graphs have either bounded mim-width, or sim-width 1. To have more applications of Proposition 5.2, it is interesting to see whether some graphs with constant sim-width admit a polynomial function for the Ramsey number.

We first prove Proposition 5.1. We use the following result. Notice that the optimal bound of Theorem 5.3 has been slightly improved by Fox [9], and then by Balogh and Kostochka [1].

Theorem 5.3 (Duchet and Meyniel [6]). *For positive integers k and n, every n-vertex graph contains either an independent set of size k or a K_t-minor where $t \geq \frac{n}{2k-1}$.*

Proof (Proof of Proposition 5.1). Let G be a graph with sim-width w and no induced minor isomorphic to $K_t \boxminus K_t$ and $K_t \boxminus S_t$. Let (T, L) be a branch-decomposition of G of width w with respect to the simval_G function. We claim that for each edge e of T, $\mathrm{mimval}_{(T,L)}(e) \leq 8(w+1)t^3 - 1$. It implies that G has mim-width at most $8(w+1)t^3 - 1$.

Let $e \in E(T)$, and (A, B) be the vertex partition of G associated with e. Suppose for contradiction that there is an induced matching $\{v_1w_1, \ldots, v_mw_m\}$ in $G[A, B]$ where $v_1, \ldots, v_m \in A$, $w_1, \ldots, w_m \in B$, and $m \geq 8(w+1)t^3$. Let f be the function from $\{v_1, \ldots, v_m\}$ to $\{w_1, \ldots, w_m\}$ such that $f(v_i) = w_i$ for each $i \in \{1, \ldots, m\}$. As $m \geq 8(w+1)t^3$, by Theorem 5.3, the subgraph $G[\{v_1, \ldots, v_m\}]$ contains either an independent set of size $2(w+1)t$, or a K_{2t^2}-minor.

If $G[\{v_1, \ldots, v_m\}]$ contains a K_{2t^2}-minor, then there exist pairwise disjoint subsets $S_1, \ldots, S_{2t^2}$ of $\{v_1, \ldots, v_m\}$ such that for each $i \in \{1, \ldots, 2t^2\}$, $G[S_i]$ is connected, and for two distinct integers $i, j \in \{1, \ldots, 2t^2\}$, there is an edge between S_i and S_j. In this case, for each $i \in \{1, \ldots, 2t^2\}$, we choose a representative d_i in each $f(S_i)$ and contract S_i to a vertex c_i. Let G' be the resulting graph. Then $G'[\{c_1, \ldots, c_{2t^2}\}, \{d_1, \ldots, d_{2t^2}\}]$ is an induced matching of size t, and $\{c_1, \ldots, c_{2t^2}\}$ is a clique in G'. We can do the same procedure for the set $\{d_1, \ldots, d_{2t^2}\}$, and by Theorem 5.3, the subgraph $G'[\{d_1, \ldots, d_{2t^2}\}]$ contains either an independent set of size t, or a K_t-minor. In both cases, one can observe that G' contains an induced minor isomorphic to $K_t \boxminus K_t$ or $K_t \boxminus S_t$, contradiction.

Now assume that $G[\{v_1, \ldots, v_m\}]$ contains an independent set $\{c_1, \ldots, c_{2(w+1)t}\}$, and for each $i \in \{1, \ldots, 2(w+1)t\}$, let $d_i := f(c_i)$. By Theorem 5.3, $G[\{d_1, \ldots, d_{2(w+1)t}\}]$ contains either an independent set of size $w+1$ or a K_t-minor. In the former case, we obtain an induced matching of size $w+1$, contradicting to the assumption that $\mathrm{simval}_{(T,L)}(e) \leq w$. In the latter case, we obtain an induced minor isomorphic to $K_t \boxminus S_t$, contradiction.

We conclude that $\mathrm{mimval}_{(T,L)}(e) \leq 8(w+1)t^3$. □

In a similar manner we can prove Proposition 5.2.

Proof (Proof of Proposition 5.2). One can easily modify from the proof of Proposition 5.1 by replacing the application of Theorem 5.3 with the Ramsey's Theorem to find an exact clique or an independent set. □

We extend Corollary 4.2 for general classes of graphs. In general, we do not have an algorithm to find a decomposition, and so we assume that the decomposition is given as an input.

Corollary 5.4. *Given an n-vertex graph G with a branch-decomposition of sim-width w,*

1. *if G has induced minor isomorphic to neither $K_t \boxminus K_t$ nor $K_t \boxminus S_t$, and $t' := 8(w+1)t^3$, then we can solve*
   - *any (σ, ρ)-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $\mathcal{O}(n^{3dt'+4})$,*
   - *any D_q-vertex partitioning problem with $d = \max_{i,j} d(D_q[i,j])$ in time $\mathcal{O}(qn^{3dt'q+4})$,*
2. *if G has no induced subgraph isomorphic to $K_t \boxminus K_t$ or $K_t \boxminus S_t$, and $t' := R(R(w+1,t), R(t,t))$, then we can solve*
   - *any (σ, ρ)-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $\mathcal{O}(n^{3dt'+4})$,*
   - *any D_q-vertex partitioning problem with $d = \max_{i,j} d(D_q[i,j])$ in time $\mathcal{O}(qn^{3dt'q+4})$.*

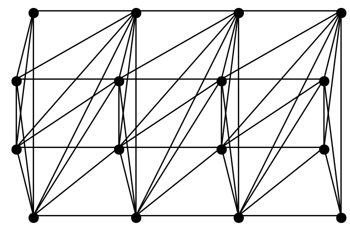

Fig. 5. The (4×4) Hsu-clique chain graph.

6 Unbounded rankwidth

One might wonder whether the class of graphs of sim-width w and having no induced minor isomorphic to $K_t \boxminus K_t$ or $K_t \boxminus S_t$ falls into a class of graphs of bounded rank-width. We confirm that this is not true, by showing that Hsu-clique chain graphs in Figure 5 are chordal and co-comparability, but do not contain $K_3 \boxminus K_3$ or $K_3 \boxminus S_3$ as an induced minor. Belmonte and Vatshelle showed that a $(p \times q)$ Hsu-clique chain graph has rank-width at least $\frac{p}{3}$ [3, Lemma 16] when $q = 3p + 1$. So, our algorithmic applications based on Proposition 5.1 are beyond algorithmic applications of graphs of bounded tree-width or rank-width. Since chordal graphs are closed under taking induced minors and $K_3 \boxminus K_3$ is not chordal, it is sufficient to prove that Hsu-clique chain graphs have no induced minor isomorphic to $K_3 \boxminus S_3$.

We formally define Hsu-clique chain graphs. For positive integers p, q, the $(p \times q)$ *Hsu-clique chain grid* is the graph on the vertex set $\{v_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ where

- for every $i \in \{1, \dots, q\}$, $\{v_{1,j}, \dots, v_{p,j}\}$ is a clique
- for every $i_1, i_2 \in \{1, \dots, p\}$ and $j \in \{1, \dots, q-1\}$, $v_{i_1,j}$ is adjacent to $v_{i_2,j+1}$ if and only if $i_1 \leq i_2$,
- for $i_1, i_2 \in \{1, \dots, p\}$ and $j_1, j_2 \in \{1, \dots, q\}$, v_{i_1,j_1} is not adjacent to v_{i_2,j_2} if $|j_1 - j_2| > 1$.

Proposition 6.1. *The class of chordal and co-comparability graphs having no induced minor isomorphic to $K_3 \boxminus S_3$ has unbounded rank-width.*

Proof. Let p be a positive integer and $q := 3p + 1$. Let G be a $(p \times q)$ Hsu-clique chain graph. Belmonte and Vatshelle showed that a $(p \times q)$ Hsu-clique chain graph has rank-width at least $\frac{p}{3}$ [3, Lemma 16]. Now, we claim that G has no induced minor isomorphic to $K_3 \boxminus S_3$. Let H be the graph on $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ where where $\{v_1, v_2, v_3\}$ is a clique, $\{w_1, w_2, w_3\}$ is an independent set, and $\{v_1w_1, v_2w_2, v_3w_3\}$ is the induced matching in $H[\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\}]$. For contradiction, suppose that G contains an induced minor isomorphic to H.

Since G contains H as an induced minor, there is a mapping μ from $V(H)$ to $2^{V(G)}$ where

- $\{\mu(v) : v \in V(H)\}$ are pairwise disjoint vertex subsets of G, and each set in $\{\mu(v) : v \in V(H)\}$ induces a connected subgraph of G,
- for two distinct vertices $v, w \in V(H)$, $vw \in E(H)$ if and only if there is an edge between $\mu(v)$ and $\mu(w)$.

For each $v \in V(H)$, let $I_v = \{i : v_{i,j} \in \mu(v)\}$. For convenience, we say that a finite set I of consecutive integers is an interval. Let $I := I_{v_1} \cup I_{v_2} \cup I_{v_3}$, and let ℓ, r be the least and greatest integers in I, respectively. Let $x, y \in \{v_1, v_2, v_3\}$ such that

1. $\mu(x)$ contains a vertex in the ℓ-th column, but for $z \in \{v_1, v_2, v_3\} \setminus \{x\}$, $\mu(z)$ has no vertex whose row index is higher than all vertices in $\mu(x)$,
2. similarly, $\mu(y)$ contains a vertex in the r-th column, but for $z \in \{v_1, v_2, v_3\} \setminus \{x\}$, $\mu(z)$ has no vertex whose row index is lower than all vertices in $\mu(y)$.

As $\{v_1, v_2, v_3\}$ is a clique, it is easy to observe that $I_x \cup I_y = I$. In other words, $\mu(x) \cup \mu(y)$ contains a vertex in each column from the ℓ-th column to the r-th column. Now, let $z \in \{v_1, v_2, v_3\} \setminus \{x, y\}$. By the choice of x and y, every vertex in G having a neighbor in $\mu(z)$ should have a neighbor in $\mu(x) \cup \mu(y)$. Thus, it contradicts to the assumption that G contains H as an induced minor. □

As a corollary of Proposition 6.2, we obtain the following.

Corollary 6.2. *The class of* $(K_3 \boxminus S_3)$*-free chordal graphs has unbounded rank-width, and the class of* $(K_3 \boxminus K_3)$*-free co-comparability graphs has unbounded rank-width.*

7 Sim-width and contraction

Let us start by showing that the sim-width of a graph does not increase when taking an induced minor. This is one of the main motivations to consider this parameter.

Lemma 7.1. *The sim-width of a graph does not increase when taking an induced minor.*

Proof. Clearly, the sim-width of a graph does not increase when removing a vertex. We prove for contractions.

Let G be a graph, $v_1v_2 \in E(G)$, and let (T, L) be a branch-decomposition of G of sim-width w. For convenience, let the contracted vertex in G/v_1v_2 be called v_1. We claim that G/v_1v_2 admits a branch-decomposition of G of sim-width at most w. We may assume that G has at least 3 vertices. For G/v_1v_2, we obtain a branch-decomposition (T', L') as follows:

– Let T' be the tree obtained from T by removing $L(v_2)$, and smoothing its neighbor. This neighbor of $L(v_2)$ has degree 3 in T because T is a subcubic tree and G has at least 3 vertices.
– Let L' be the function from $V(G/v_1v_2)$ to the set of leaves of T' such that $L'(w) = L(w)$ for $w \in V(G/v_1v_2) \setminus \{v_1\}$ and $L'(v_1) = L(v_1)$.

Let e_1 and e_2 be the two edges of T incident with the neighbor of $L(v_2)$, but not incident with $L(v_2)$. Let e_{cont} be the edge of T' obtained by smoothing.

Claim 1 *For each* $e \in E(T')$, $\mathrm{simval}_{(T',L')}(e) \le \mathrm{simval}_{(T,L)}(e)$ *if* $e \in E(T) \setminus \{e_1, e_2\}$*, and* $\mathrm{simval}_{(T',L')}(e_{cont}) \le \min(\mathrm{simval}_{(T,L)}(e_1), \mathrm{simval}_{(T,L)}(e_2))$.

Proof. Let $e \in E(T')$, and first assume that $e \in E(T) \setminus \{e_1, e_2\}$. Let (A, B) be the vertex partition of G/v_1v_2 associated with e. Without loss of generality, we may assume that $v_1 \in A$. Suppose there exists an induced matching $\{a_1b_1, \ldots, a_mb_m\}$ in G/v_1v_2 with $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$. Let (A', B') be the vertex partition of G associated with e. We will show that there is also an induced matching in G of same size between A' and B'.

We have either $A \cup \{v_2\} = A'$ and $B = B'$, or $A = A'$ and $B \cup \{v_2\} = B'$. If $v_1 \notin \{a_1, \ldots, a_m\}$, then $\{a_1b_1, \ldots, a_mb_m\}$ is also an induced matching between A' and B' in G. Without loss of generality, we may assume that $v_1 = a_1$.

Case 1. $A \cup \{v_2\} = A'$ **and** $B = B'$**.**

Proof. Note that in G, one of v_1 and v_2, say v', is adjacent to b_2. And also, v_1 and v_2 are not adjacent to any of $\{a_2, \ldots, a_m, b_1, \ldots, b_m\}$. Therefore, $\{v'b_1, a_2b_2, \ldots, a_mb_m\}$ is an induced matching in G between A' and B', as required. □

Case 2. $A = A'$ **and** $B \cup \{v_2\} = B'$**.**

Proof. If v_1 is adjacent to b_1 in G, then we have the same induced matching in G between A' and B'. Howver, v_1 is not necessary adjacent to b_1 in G. In this case, v_2 should be adjacent to b_1 in G. We now assume that v_1 is not adjacent to b_1 in G. In this case, $\{v_1v_2, a_2b_2, \ldots, a_mb_m\}$ is an induced matching between B_1 and B_2, because v_1 and v_2 are not adjacent to any of $\{a_2, \ldots, a_m, b_2, \ldots, b_m\}$. □

It shows that $\text{simval}_{(T',L')}(e) \le \text{simval}_{(T,L)}(e)$ if $e \in E(T) \setminus \{e_1, e_2\}$. We can follow the same procedure to show that the same holds for e_{cont} as well. □

Claim 1 implies that the width of (T', L') is at most the width of (T, L). We conclude that $\text{simw}(G/v_1v_2) \le \text{simw}(G)$. □

8 Concluding remarks

In this paper, we showed that every LC-VSVP problem can be solved in XP time parameterized by t on $(K_t \boxminus S_t)$-free chordal graphs and $(K_t \boxminus K_t)$-free co-comparability graphs. We further generalized this to every graph with sim-width at most w and having no induced minor isomorphic to $K_t \boxminus S_t$ or $K_t \boxminus K_t$ has mim-width at most $8(w+1)t^3$, by showing that every LC-VSVP problem can be solved in time $n^{\mathcal{O}(wt^3)}$ on such n-vertex graphs, when its branch-decomposition is given.

It would be interesting to find more classes having constant sim-width, but unbounded mim-width. We propose some possible classes, that are also presented in Figure 2.

Question 1 *Do weakly chordal graphs, AT-free graphs, or circle graphs have constant sim-width?*

We showed that Dominating Set can be solved in time $n^{\mathcal{O}(t)}$ on $(K_t \boxminus S_t)$-free chordal graphs, but we could not obtain an FPT algorithm. We ask whether it is W[1]-hard or not. This may be a right direction to show that Dominating Set is W[1]-hard parameterized by mim-width.

Question 2 *Is* Dominating Set *on chordal graphs W[1]-hard parameterized by the maximum t such that it has an $K_t \boxminus S_t$ induced subgraph?*

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