Computational Complexity of Covering Three-Vertex Multigraphs

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Abstract. A covering projection from a graph G onto a graph H is a mapping of the vertices of G onto the vertices of H such that, for every vertex v of G, the neighborhood of v is mapped bijectively onto the neighborhood of its image. Moreover, if G and H are multigraphs, then this local bijection has to preserve multiplicities of the neighbors as well. The notion of covering projection stems from topology, but has found applications in areas such as the theory of local computation and construction of highly symmetric graphs. It provides a restrictive variant of the constraint satisfaction problem with additional symmetry constraints on the behavior of the homomorphisms of the structures involved.
We investigate the computational complexity of the problem of deciding the existence of a covering projection from an input graph G to a fixed target graph H. Among other partial results this problem has been shown NP-hard for simple regular graphs H of valency greater than 2, and a full characterization of computational complexity has been shown for target multigraphs with 2 vertices. We extend the previously known results to the ternary case, i.e., we give a full characterization of the computational complexity in the case of multigraphs with 3 vertices. We show that even in this case a P/NP-completeness dichotomy holds.

Keywords: Computational Complexity, Graph Homomorphism, Covering Projection

1 Introduction

The concept of covering spaces or covering projections stems from topology, but has attracted a lot of attention in algebra, combinatorics, and also the theory of computation. For instance, it is used in algebraic graph theory as a very

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useful tool for construction of highly symmetric graphs. The applications in computability include the theory of local computations (cf. [2] and [7]). A lot of interest has been paid to graphs that allow finite planar covers. This class of graphs is closed in the minor order and hence recognizable in polynomial time, yet despite a lot of efforts no concrete recognition algorithm is known, since the obstruction set has not been determined yet. The class has been conjectured equal to projective planar graphs by Negami [20] (for the most recent results cf. [12, 13]).

Deciding the existence of a covering projection between input graphs G and H was shown NP-complete by Bodlaender in 1989 [4]. In [1], Abello et al. asked to characterize the computational complexity of deciding the existence of a covering projection from an input graph G onto a fixed graph H (hoping for a P/NP-completeness dichotomy depending on H). Such a characterization seems to be hard to obtain and only very partial results are known. The most general NP-completeness result states that for every simple regular graph H of valency at least 3, the problem is NP-complete [18]. No plausible conjecture on the borderline between polynomially solvable and NP-complete instances has been published so far, yet it is believed that a P/NP-completeness dichotomy will hold similarly as in the case of the Constraint Satisfaction Problem (CSP). See [10] for a nice overview of results up to 2008. More recent results include a focus on algebraically restricted coverings by planar graphs [9].

The relation to CSP is worth mentioning in more detail. As shown in [10], for every fixed graph H, the H-Cover problem can be reduced to CSP, but mostly to NP-complete cases of CSP, so this reduction does not help. In a sense a covering projection is itself a variant of CSP, but with further constraints of local symmetry. Thus the dichotomy conjecture for H-Cover does not follow from the well known Feder-Vardi dichotomy conjecture for CSP (cf. [8]).

In [17] the authors showed that in order to fully understand the H-Cover problem for simple graphs, one has to understand its generalization for colored mixed multigraphs. For this reason we are dealing with multigraphs (undirected) in this paper. Kratochvil et al. [17] completely characterized the computational complexity of the H-Cover problem for colored mixed multigraphs on two vertices. The aim of this paper is to extend this characterization to 3-vertex multigraphs (in the undirected and monochromatic case). The characterization is described in the next section. It is more involved than the case of 2-vertex multigraphs, but this should not be surprising. Ternary structures tend to be substantially more difficult than their binary counterparts. A parallel in CSP is the dichotomy of binary CSP proved by Schaefer in the 70's [21] followed by the characterization of CSP into ternary structures by Bulatov almost 30 years later [5].

2 Preliminaries and statement of our results

For the sake of brevity we reserve the term "graph" for a multigraph. We denote the set of vertices of a graph G by $V(G)$ and the set of edges by $E(G)$. For two vertices u, v of G we denote the number of distinct edges between u and v by $m_G(u, v)$ and we say that uv is an $m_G(u, v)$-edge. Degree of a vertex v of G is denoted by $\deg_G(v)$ (recall that in multigraphs, degree of a vertex v is defined as the number of edges going to other vertices plus twice the number of loops at v, i.e. $2m_G(v, v) + \sum_{u \neq v} m_G(u, v)$). By $N_G(v)$ we denote the multiset of neighbors of vertex v in G where the multiplicity of v in $N_G(v)$ is $2m_G(v, v)$ and for every $u \neq v$, the multiplicity of u is $m_G(u, v)$. We omit G in the subscript if G is clear from the context.

Suppose A and B are two multisets. Let A', resp. B' be the set of different elements from A, resp. B. We say that a mapping $g\colon A' \to B'$ is a bijection from A to B if for every $b' \in B'$ the sum of multiplicities of all elements from $g^{-1}(b')$ in A equals the multiplicity of b' in B. If C' is a set then by $A \cap C'$ we mean a multiset that contains only elements from $A' \cap C'$ with multiplicities corresponding to multiplicities in A. We denote the sum of multiplicities of all elements in A by $|A|$.

Let G and H be graphs. A *homomorphism* $f : V(G) \to V(H)$ is an edge preserving mapping from $V(G)$ to $V(H)$. A homomorphism f is a *covering projection* if $N_G(v)$ is mapped to $N_H(f(v))$ bijectively for every $v \in V(G)$. See Figure 1 for an example. A covering projection is also known as a *locally bijective homomorphism* or simply a *cover*. In this paper we denote a covering projection f from G to H by $f\colon G \to H$.

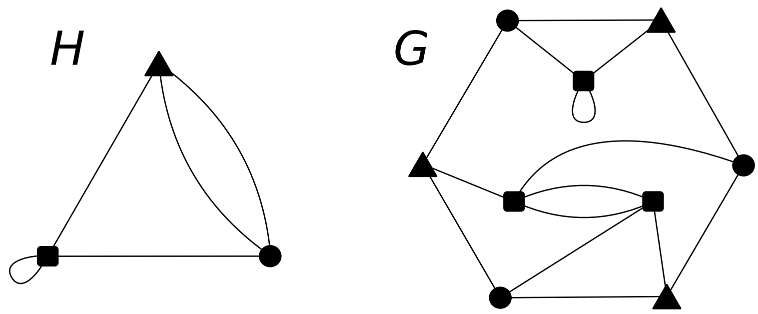

Fig. 1. Example of graph covering.

Strictly speaking, as the notion of a covering projection stems from topology it should be defined by a pair of mappings – one on the vertices and one on the edges of the graphs involved. However, it was shown in [17] (using König's theorem and 2-factorization of $2k$-regular multigraphs) that every cover (as defined here) can be extended to a topological covering projection $f : V(G) \cup E(G) \to V(H) \cup E(H)$.

In this paper we consider the following decision problem.

Problem: H-Cover
Parameter: Fixed graph H.
Input: Graph G.
Task: Does there exist a covering projection $f : G \to H$?

Note that the problem H-Cover belongs to class NP as we can guess a mapping $f\colon V(G) \to V(H)$ and verify if f is a covering projection in polynomial time. It means that in our NP-completeness results we will only be proving the NP-hardness part.

An equitable partition of a graph G is a partition of its vertex set into blocks $B_1, \ldots, B_d$ (ordered in some canonical way; see Corneil and Gotlieb [6]) such that for every $i, j = 1, \ldots, d$ and every vertex v in B_i it holds that $|N_G(v) \cap B_j| = r_{i,j}$ (recall that $N_G(v)$ is generally a multiset). We call the matrix $M = (r_{i,j})$ corresponding to the coarsest equitable partition $B_1, \ldots, B_d$ of G the degree refinement matrix of G, denoted by $drm(G)$, and we say that G is a d-block graph. Note that 1-block graphs are exactly regular graphs (despite the fact that vertices can contain different numbers of loops).

It is also known that if G covers H via a covering f, then $drm(G) = drm(H)$. In particular, f preserves the coarsest equitable partition of G, i.e., if $B'_1, \ldots, B'_d$, resp. $B_1, \ldots, B_d$ are the blocks in partition of G, resp. H then $f(B'_i) = B_i$ for every $i = 1, \ldots, d$. Since the matrix $drm(G)$ can be computed in time polynomial in the size of G, throughout this paper we will assume that $drm(G) = drm(H)$.

For every non-negative integers k, l, x, y we define a graph $S(k, l, x, y)$ on vertex set $\{a, b, c\}$ such that (see Figure 2):

- $m(a, c) = m(b, c) = k$
- $m(c, c) = l$
- $m(a, a) = m(b, b) = x$
- $m(a, b) = y$

In this paper we focus on graphs H on exactly three vertices. Note that such a graph will have either 1, 2 or 3 blocks. For such graphs we show an P/NP-completeness dichotomy of H-Cover by Observation 1, Theorem 1 and Theorem 2.

Observation 1 *Let H be a 3-block graph on three vertices. Then H-Cover is polynomially solvable.*

This follows from the fact that there exists only one mapping $f\colon V(G) \to V(H)$ that preserves blocks from the coarsest equitable partitions and it is easy to check if this mapping is a covering projection.

Theorem 1. *Let H be a 2-block graph on three vertices. If H is isomorphic to $S(k', l, x, 0), S(k', l, 0, y)$ or $S(2, l, 0, 0)$, where $k' \in \{0, 1\}$ and $l, x, y \geq 0$, then H-Cover is polynomially solvable. Otherwise H-Cover is NP-complete.*

Theorem 2. *Let H be a t-regular graph on three vertices (equivalently, a 1-block graph on three vertices). If H is disconnected or $t \leq 2$, then H-Cover is polynomially solvable. Otherwise H-Cover is NP-complete.*

In Section 3 we state the necessary lemmata for the proof of Theorem 1. Section 4 is devoted to the proof of Theorem 2, with the polynomial cases covered by Lemma 5. We introduce a new decision problem - H-Cover* and prove that this problem is NP-complete for all connected t-regular graphs H with $t \geq 4$. The proof is based on mathematical induction where we are able to use a stronger induction hypothesis than with simple H-Cover. The NP-hardness of H-Cover then follows from the fact that H-Cover* is reducible to H-Cover in polynomial time.

Let us give a few more technical definitions and notations. Throughout the rest of the paper we reserve the letter H for a graph on 3 vertices a, b, and c.

Let m, n, z be integers such that $m \geq n > 0$ and $z \geq 0$. We define a graph $H(m, n, z)$ to be the graph on vertex set $\{a, b, c\}$ with edges (see Figure 2):

- $m(a, a) = m$
- $m(a, b) = z$
- $m(a, c) = z + 2n$
- $m(b, b) = n$
- $m(b, c) = z + 2m$
- $m(c, c) = 0$

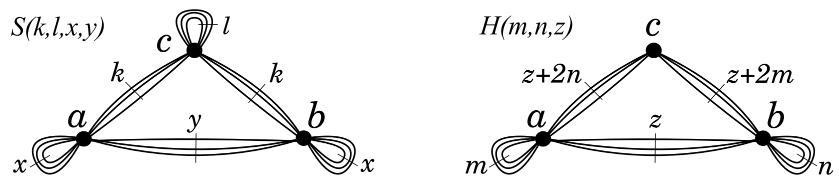

Fig. 2. Graphs $S(k, l, x, y)$ and $H(m, n, z)$.

Let G, F and H be graphs. From the definition of covering projection it is easy to show that if $f : G \to F$ and $g : F \to H$ are covering projections then their composition $g \circ f : G \to H$ is also a covering projection. Since every graph isomorphism is a covering projection, every time we investigate the complexity of H-Cover where H is isomorphic to $S(k, l, x, y)$ or $H(m, n, z)$, we can and we will assume that $H = S(k, l, x, y)$ or $H = H(m, n, z)$.

By a boundary $\delta_G(F)$ of an induced subgraph F of a graph G we mean the set of vertices of F that are adjacent to a vertex outside F.

Let A, B be sets and $f : A \to B$ be a mapping. Then we define $f(A) = \bigcup_{a \in A}\{f(a)\}$. If $A = \{a_1, \ldots, a_n\}$ then we will write $f(a_1, \ldots, a_n)$ instead of $f(\{a_1, \ldots, a_n\})$. If $f(A)$ contains only one element, say x, then we simply write $f(A) = x$ instead of $f(A) = \{x\}$.

3 Complexity for 2-block graphs on three vertices

In this section we provide the proof of Theorem 1. We will assume that H is a 2-block graph with blocks $\{a, b\}$ and $\{c\}$. By definition of equitable partition we have $deg_H(a) = deg_H(b) \neq deg_H(c)$. The next proposition shows the connection between graphs $S(k, l, x, y)$ and 2-block graphs:

Proposition 1. *Every 2-block graph H on three vertices is isomorphic to some $S(k, l, x, y)$, where $2x + y \neq 2l + k$.*

Proof. Since we can not distinguish vertices a and b in block $\{a, b\}$ we have $m(a, a) = m(b, b) = x$ and $m(a, c) = m(b, c) = k$. It means that H is isomorphic to $S(k, l, x, y)$, where $l = m(c, c)$ and $y = m(a, b)$. The inequality $2x + y \neq 2l + k$ follows directly from the fact that $deg_H(a) \neq deg_H(c)$. □

Before we proceed to the proof of Theorem 1 we partition the class of 2-block graphs to several subclasses and show the complexity separately for each subclass. In Figure 3 we show this partition and the computational complexity of H-Cover for graphs H in corresponding sets.

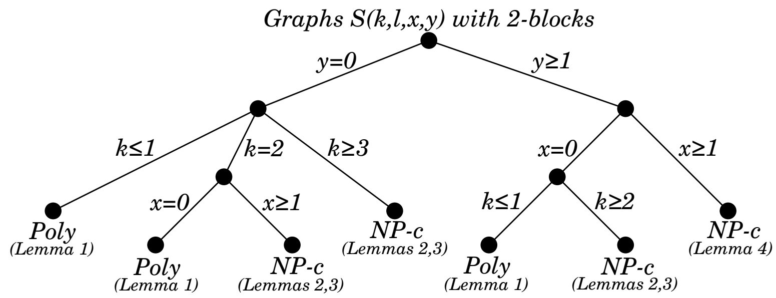

Fig. 3. Partition of 2-block graphs. Leaf vertices indicate the computational complexity of H-Cover for the corresponding graphs H.

Lemma 1. *Let H be a 2-block graph on three vertices. If H is isomorphic to $S(k', l, x, 0)$, $S(k', l, 0, y)$ or $S(2, l, 0, 0)$ for some $k' \in \{0, 1\}$ and $l, x, y \geq 0$ then H-Cover is polynomially solvable.*

Proof. Let G be the input of H-Cover and let AB, resp. C be the block of G that corresponds to block $\{a, b\}$, resp. $\{c\}$ of H.

At first suppose that H is isomorphic to $S(k', l, x, 0)$ or $S(k', l, 0, y)$. We will construct a conjunctive normal form boolean formula φ_G with clauses of size 2, such that φ_G is satisfiable if and only if G covers H.

Let variables of φ_G be $\{x_u | u \in AB\}$ and for each $u, v \in AB$ we add to φ_G the following clauses:

- $(x_u \vee x_v)$ and $(\neg x_u \vee \neg x_v)$, if $u \neq v$ and u, v share a common neighbor in C
- $(x_u \vee \neg x_v)$ and $(\neg x_u \vee x_v)$, if $uv \in E(G)$ and $H = S(k', l, x, 0)$
- $(x_u \vee x_v)$ and $(\neg x_u \vee \neg x_v)$, if $uv \in E(G)$ and $H = S(k', l, 0, y)$

Suppose that φ_G is satisfiable and fix one satisfying evaluation of variables. Define a mapping $f \colon V(G) \to V(H)$ by:

- $f(u) = a$, if $u \in AB$ and x_u is positive
- $f(u) = b$, if $u \in AB$ and x_u is negative
- $f(u) = c$, if $u \in C$

It is a routine check to show that f is a covering projection from G to H. On the other side if $f\colon G \to H$ is a covering projection then we can define an evaluation of φ_G such that x_u is positive if and only if $f(u) = a$. Such evaluation satisfies formula φ_G since there is exactly one positive literal in every clause. The facts that size of φ_G is polynomial in size of G and 2-SAT is polynomially solvable imply that H-Cover is polynomially solvable.

In the rest of the proof suppose that $H = S(2, l, 0, 0)$. Then graph G covers H if and only if we can color the vertices of AB by two colors, say black and white, in such a way, that for each $u \in C$ exactly two out of four vertices from $N_G(u) \cap AB$ are black.

We construct an auxiliary 4-regular graph G'. Let $V(G') = C$ and the edge multiplicities are $m_{G'}(u_1, u_2) = |\{u \in AB : uu_1, uu_2 \in E(G)\}|$ for all $u_1, u_2 \in C$. Note that G' can generally contain loops and multi-edges. Also note that in G' every edge corresponds to some vertex from AB.

Then coloring of vertices of AB in G corresponds to the coloring of edges in G' such that the black edges induce a 2-factor of G'. The problem of deciding existence of 2-factor in 4-regular graph can be solved in polynomial time (in fact such 2-factor always exists). That concludes the proof. □

In Lemma 2 we show NP-hardness by reduction from the following problem.

Problem: m-in-$2m$-SAT$_q$
Input: A formula φ in CNF where every clause contains exactly $2m$ variables without negation and every variable occurs in φ exactly q times.
Task: Does there exist an evaluation of variables of φ such that every clause contains exactly m positively valued variables?

Kratochvíl [15, Corollary 1] shows that this problem is NP-complete for every $q \geq 3$ and $m \geq 2$. If formula φ is a positive instance of m-in-$2m$-SAT$_q$ we simply say that φ is m-in-$2m$ *satisfiable*.

For the purpose of our NP-hardness reductions in Lemma 2 we will build a specific gadget according to the following needs:

Definition 1 (Variable gadget). *Let $H = S(k, l, x, y)$ and let F be a graph with $2q$ specified vertices $S = \{s_1, \dots, s_q\}$ and $S' = \{s'_1, \dots, s'_q\}$ of degree one. Let V, resp. V' be the set of neighbors of vertices in S, resp. S' in F. Suppose that whenever F is an induced subgraph of G with $\delta_G(F) \subseteq S \cup S'$ and $f : G \to H$ is a covering projection then $f(S \cup S') = c$ and one of the following occurs:*

i) $f(V) = a$ and $f(V') = b$
ii) $f(V) = b$ and $f(V') = a$
iii) $f(V \cup V') = a$
iv) $f(V \cup V') = b$

Furthermore suppose that any mapping $f\colon S \cup S' \cup V \cup V' \to V(H)$ such that $f(S \cup S') = c$ and satisfying i) or ii) can be extended to $V(F)$ in such a way that for each $u \in V(F) \setminus (S \cup S')$ the restriction of f to $N_F(u)$ is a bijection to $N_H(f(u))$.

We denote such F by $VG_H(q)$ and we call it a variable gadget *of size q.*

Next lemma shows how we use variable gadgets while Lemma 3 proves that $VG_H(q)$ exists for some graphs $S(k,l,x,0)$, $S(2,l,x,0)$, and $S(k,l,0,y)$. Note that in Definition 1 and Lemma 2 we do not use the fact that H is a 2-block graph. We also use them later in Section 4.

Lemma 2. *Let $k \geq 2$ and let $H = S(k,l,x,y)$. If for some $q \geq 3$ there exists a variable gadget $VG_H(q)$ then H-*Cover *is NP-complete.*

Proof. We reduce NP-hardness of H-Cover from k-in-$2k$-SAT_q. Let φ be an instance of k-in-$2k$-SAT_q. Let $x_1, x_2, \ldots, x_n$, resp. $C_1, C_2, \ldots, C_m$ be variables, resp. clauses of φ. For every clause C_i denote the variables in C_i by $l_i^1, \ldots, l_i^{2k}$ (recall that all variables have positive appearance in φ). We construct a graph G_φ such that G_φ covers H if and only if φ is k-in-$2k$ satisfiable.

We start the construction of G_φ by taking vertices $c_1, \ldots, c_m, c'_1, \ldots, c'_m$ (vertices correspond to clauses) and we add l loops to each of them. For every variable x_i we take a copy $VG^i(q)$ of variable gadget $VG_H(q)$. Denote copy of S, S', V, resp. V' in $VG^i(q)$ simply by S^i, S'^i, V^i, reps. V'^i. For every occurrence of x_i in clause C_j we identify one vertex from S^i, resp. S'^i with c_j, resp. c'_j. We do it in such a way that every vertex from $S^i \cup S'^i$ is identified exactly once, see Figure 4.

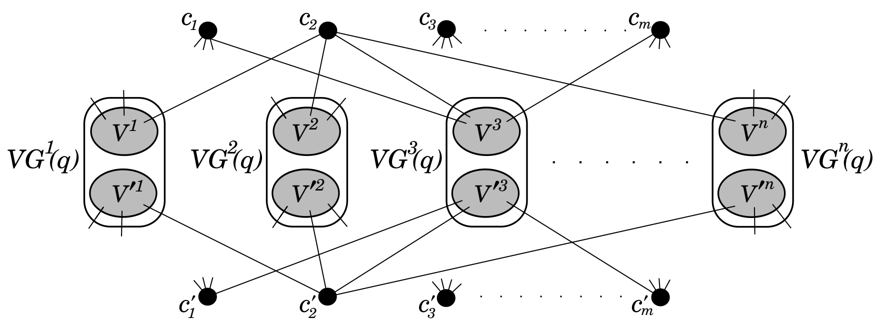

Fig. 4. Construction of graph G_φ for $k = 2$ and $q = 3$. In this example φ contains a clause $C_2 = (x_1 \wedge x_2 \wedge x_3 \wedge x_n)$ and variable x_3 appears in clauses C_1, C_2 and C_m.

We claim that G_φ covers H if and only if φ is k-in-$2k$ satisfiable.

Suppose that there exists a covering projection $f\colon G_\varphi \to H$. We define an evaluation of variables of φ such that x_i is true if and only if $f(V^i) = a$.

From the properties of variable gadget $VG_H(q)$ we know that $f(c_j) = c$ for every $j = 1, \ldots, m$. Then $|N_{G_\varphi}(c_j) \cap f^{-1}(a)| = |N_{G_\varphi}(c_j) \cap f^{-1}(b)| = k$. It means that in every clause of φ there are exactly k positive as well as negative variables.

For the opposite implication we fix one satisfying evaluation of φ. We define a mapping $f\colon G_\varphi \to H$ in a following way:

- $f(c_j) = f(c'_j) = c$, for all $j = 1, \ldots, m$

– $f(V^i) = a$ and $f(V'^i) = b$, if x_i is a positive variable
– $f(V^i) = b$ and $f(V'^i) = a$, if x_i is a negative variable

Then for each $i = 1, \dots, n : f(S^i) = c$ and $f(V^i) \neq f(V'^i)$. By the definition of variable gadget we know, that f can be extended to $VG^i(q)$ in such a way that for each $u \in V(VG^i(q)) \setminus (S^i \cup S'^i)$: the restriction of f to $N_{G_\varphi}(u)$ is a bijection to $N_H(f(u))$. It is a routine check to show that such mapping f is a covering projection from G_φ to H. □

Lemma 3. *If a* 2*-block graph* H *is one of the following:*

a) $S(k, l, x, 0)$*, where* $k \geq 3$*,* $l \geq 0$ *and* $x \geq 0$
b) $S(2, l, x, 0)$*, where* $l \geq 0$ *and* $x \geq 1$
c) $S(k, l, 0, y)$*, where* $k \geq 2$*,* $l \geq 0$ *and* $y \geq 1$

then there exists a variable gadget $VG_H(q)$ *for some* $q \geq 3$*.*

Proof. Depending onto which of $a), b)$ and $c)$ is graph H equal to, we define $VG_H(q)$ and corresponding sets S and S' as is depicted on Figure 5. Note that in case $a), b)$, resp. $c)$ is q equal to $k, 4$, resp. $2k$.

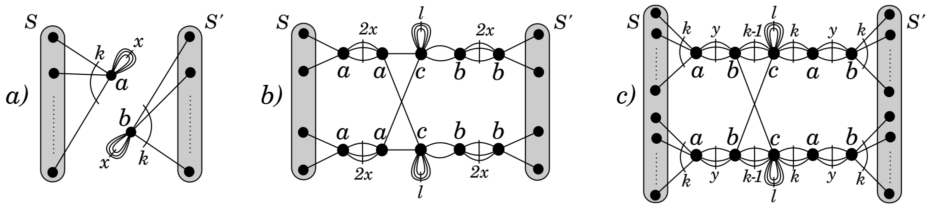

Fig. 5. Variable gadgets for cases $a), b)$ and $c)$.

The fact that depicted graphs are really variable gadgets follows from the case analyzes. Figure 5 also shows how one particular mapping $f\colon S \cup S' \cup V \cup V' \to H$ (where V, resp. V' are neighbors of S, resp. S') can be extended to all vertices of $VG_H(q)$. Other conditions from the definition of $VG_H(q)$ follows from the fact that H has 2 blocks. □

Lemma 4. *Let* $H = S(k, l, x, y)$ *be a* 2*-block graph where* $k, l \geq 0$ *and* $x, y \geq 1$*. Then* H*-*Cover *is NP-complete.*

Proof. Kratochvíl et al. [17, Theorem 11] proved that if H' is a graph on two vertices L and R such that $x = m_{H'}(L, L) = m_{H'}(R, R) \geq 1$ and $y = m_{H'}(L, R) \geq 1$, then H'-Cover is NP-complete.

We reduce NP-hardness of H-Cover from H'-Cover. Let G' be an instance of H'-Cover. We will construct a graph G such that G covers H if and only if G' covers H'.

We start the construction of G by taking two copies G^1 and G^2 of G'. Denote the copy of vertex $v \in V(G')$ in G^1, resp. G^2 by v^1, resp. v^2. For every $v \in V(G')$ we add to G a new vertex u_v with l loops and k-edges v^1u_v and v^2u_v.

Suppose that $f : G \to H$ is a covering projection. Then $f(u_v) = c$ for every $v \in V(G')$ and f restricted to G^1 is a covering projection to H'. It means that G' covers H'.

For the opposite implication suppose that $f' : G' \to H'$ is a covering projection. We define a mapping $f : V(G) \to V(H)$ in the following way:
- $f(u_v) = c$
- $f(v^1) = f'(v)$
- $f(v^2) = a$ if $f(v^1) = b$, and $f(v^2) = b$ otherwise

for every $v \in V(G')$. It is a routine check to show that f is a covering projection from G to H. □

Now we can proceed to the proof of Theorem 1.

Proof (of Theorem 1). The polynomial cases are settled by Lemma 1. Cases where $x, y \geq 1$ follow from Lemma 4. All other cases follow from Lemmata 2 and 3 (see Figure 3). □

4 Complexity for 1-block graphs on three vertices

In this section we focus on 1-block graphs H, i.e. regular graphs. We provide several definitions and lemmata that help us prove Theorem 2. The following lemma settles the polynomial part.

Lemma 5. *Let H be a t-regular graph on three vertices. If H is disconnected or $t \leq 2$, then H-Cover is polynomially solvable.*

Proof. Let G be a t-regular graph. At first suppose that H is disconnected. Without loss of generality suppose that $m_H(a, c) = m_H(b, c) = 0$. We define mapping $f : V(G) \to V(H)$ by $f(u) = c$ for every $u \in V(G)$. Then mapping f is a covering projection from G to H by the definition.

If H is connected and $t \leq 2$, then H is a triangle. Graph G covers a triangle if and only if G consists of disjoint cycles of lengths divisible by 3. This condition can be easily verified in linear time. □

We prove the NP-hardness part of Theorem 2 using a reduction from a problem we call H-Cover*. To define H-Cover* we need the following definitions.

Definition 2. *Let G be a graph on $3n$ vertices and let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a partition of its vertices into n sets of size 3. Then we say that $\mathcal{A}$, resp. the pair $(G, \mathcal{A})$ is a* 3-partition*, resp.* graph 3-partition*. Moreover, if $f : V(G) \to \{a, b, c\}$ is a mapping such that $f(A_i) = \{a, b, c\}$ for every $A_i \in \mathcal{A}$, then we say that f* respects *the 3-partition $\mathcal{A}$.*

Definition 3. *We say that a graph* 3*-partition* $(G, \mathcal{A})$ *covers* a graph* H *if there exists a covering projection* $f^*: G \to H$ *that respects the* 3*-partition* $\mathcal{A}$*. We denote such a mapping by "*$\to^*$*" and call it a* covering projection* *or simply a* cover*.

Definition 4. *Let* $(G, \mathcal{A})$ *be a graph* 3*-partition and* H *be a graph. If the existence of a covering projection* $f: G \to H$ *implies the existence of a covering projection** $f^*: (G, \mathcal{A}) \to^* H$*, then we say that* $(G, \mathcal{A})$ *is* nice *for* H*.*

Note that by the definition if G does not cover H then any graph 3-partition $(G, \mathcal{A})$ is nice for H.

Problem: H-Cover*
Parameter: A fixed graph H on three vertices.
Input: A graph 3-partition $(G, \mathcal{A})$ that is nice for H.
Task: Does there exist a covering projection* $f: (G, \mathcal{A}) \to^* H$?

Similarly as H-Cover also H-Cover* problem belongs to the class NP. The following observation allows us to prove the NP-hardness part of Theorem 2 by proving NP-hardness of H-Cover* for connected t-regular graphs on three vertices and $t \geq 4$ (note there are no t-regular graphs on three vertices with $t = 3$).

Observation 2 *Let* H *be a graph. Then* H-Cover* *is polynomially reducible to* H-Cover.

Proof. Suppose that $(G, \mathcal{A})$ is an instance of H-Cover*. Since $(G, \mathcal{A})$ is nice for H we know that $(G, \mathcal{A})$ covers* H if and only if G covers H, which concludes the proof. □

Theorem 3. *Let* H *be a connected* t*-regular graph on three vertices and* $t \geq 4$*. Then* H-Cover* *is NP-complete.*

The remainder of this section is devoted to a proof of Theorem 3. The proof will be by mathematical induction on t. The advantage of H-Cover* in comparing to H-Cover is that we can use a stronger induction hypothesis. The first three lemmata that we prove, Lemmata 6, 7 and 8, will settle the base case $t = 4$, but along the way it will also settle some other important classes of graphs. In the rest of the paper we assume that H is a connected t-regular graph and $t \geq 4$.

We first show NP-hardness of H-Cover* for the graph $H = S(1,1,1,1)$, see Figure 2, by reduction from the 3-edge coloring problem which was shown NP-complete for cubic graphs by Holyer [14]. Denote this problem by 3-ECol.

Lemma 6. *Let* $H = S(1,1,1,1)$*. Then* H-Cover* *is NP-complete.*

Proof. We reduce NP-hardness of H-Cover* from 3-ECol problem. For every simple cubic graph F we construct a nice graph 3-partition $(G_F, \mathcal{A})$ for H (of size polynomial to size of F) such that $(G_F, \mathcal{A})$ covers* H if and only if F is 3-edge colorable.

For every vertex $u \in V(F)$ we insert to G_F vertices u_1, u_2, u_3 and we add 1-edges u_1u_2, u_2u_3 and u_3u_1. For every edge $uv \in E(F)$ we choose vertices u_i and v_j and we add 2-edge u_iv_j to G_F. We choose indices i and j in such a way that the final graph G_F is 4-regular. We define 3-partition $\mathcal{A} = \bigcup_{u \in V(F)} \{\{u_1, u_2, u_3\}\}$.

We prove that $(G_F, \mathcal{A})$ is nice for H. Let $f : G_F \to H$ be a covering projection. Clearly all 2-edges of G_F must be mapped by f onto loops of H. It implies that for every $u \in V(F)$ we have $f(u_1, u_2, u_3) = \{a, b, c\}$ and so f respects $\mathcal{A}$.

Suppose that $f\colon (G_F, \mathcal{A}) \to^* H$ is a covering projection*. We know that every 2-edge u_iv_j corresponds to an edge uv of F and $f(u_i) = f(v_j)$. We define coloring $col\colon E(F) \to V(H)$ by $c(uv) = f(u_i)$. The fact that f respects 3-partition $\mathcal{A}$ implies that col is a proper 3-edge coloring of F.

In the rest of the proof suppose that $col\colon E(F) \to V(H)$ is a proper 3-edge coloring of F. We show that there exists a covering projection* $f\colon (G_F, \mathcal{A}) \to^* H$. For every 2-edge u_iv_j of G_F we define $f(u_i) = f(v_j) = col(uv)$. Since col is a proper 3-edge coloring of F we have $f(u_1, u_2, u_3) = \{a, b, c\}$ for every $u \in V(F)$. It means that f respects the 3-partition $\mathcal{A}$. It is a routine check to show that f is a covering and consequently a covering projection*. □

We continue with the graph $H = S(k, 0, x, y)$ where $k = 2x + y \geq 2$, see Figure 2.

Lemma 7. *Let $H = S(k, 0, x, y)$ where $k = 2x + y \geq 2$. Then H-*Cover* *is NP-complete.*

Proof. At first suppose that $x \geq 1$. We claim that gadget F on the left side of Figure 6 is a variable gadget $VG_H(2k)$ (see Definition 1). Suppose that F is an induced subgraph of G with $\delta_G(F) \subseteq S \cup S'$ and $f\colon G \to H$ is a covering projection. If $f(u_1) = a$ then without loss of generality $f(u_2) = c$ and $f(u_5) = a$ (we use the fact that $k > x \geq 1$). Moreover $y = 0$, because no neighbor of u_1 is mapped to b. It means that $k = 2x$ and vertices u_6 and u_7 must be mapped to c. That is not possible since $m_F(u_6, u_7) > 0 = m_H(c, c)$ and so $f(u_1) \neq a$. Analogously we have $f(u_1) \neq b$.

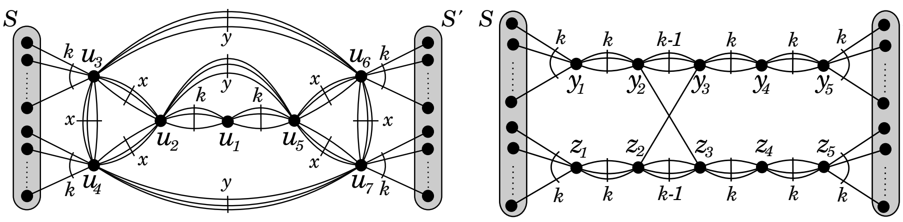

Fig. 6. Gadgets F, resp. F' for $S(2x + y, 0, x, y)$ where $x \geq 1$, resp. $S(y, 0, 0, y)$.

If $f(u_1) = c$ then by case analysis we get either $f(u_2, u_3, u_4) = a$, $f(u_5, u_6, u_7) = b$; or $f(u_2, u_3, u_4) = b$, $f(u_5, u_6, u_7) = a$. It is a routine check to verify that all other conditions from the definition of $VG_H(2k)$ are fulfilled.

The existence of $VG_H(2k)$ and Lemma 2 imply that problem H-Cover is NP-complete. Let G_φ be graph constructed in the proof of Lemma 2. To prove that H-Cover* is NP-complete it is enough to show, that for every G_φ there exists a graph 3-partition $(G_\varphi, \mathcal{A})$ that is nice for H.

Let $f\colon G_\varphi \to H$ be any covering projection. From the properties of variable gadget F we know that vertices $f^{-1}(c)$ are exactly copies of vertices from $S \cup S' \cup \{u_1\}$ from every copy of F in G_φ. By case analysis we have $f(u_2, u_5) = f(u_3, u_6) = f(u_4, u_7) = \{a, b\}$ in every copy of F. Using the fact that $|f^{-1}(a)| = |f^{-1}(b)| = |f^{-1}(c)|$ we can easily find 3-partition $\mathcal{A}$ of $V(G_\varphi)$ (not depending on f) such that every covering projection $f' : G_\varphi \to H$ respects $\mathcal{A}$.

In the rest of the proof suppose that $x = 0$, i.e. $k = y \geq 2$. Let F' be the right gadget from Figure 6 and let $V = \{y_1, z_1\}$ and $V' = \{y_5, z_5\}$.

We show that F' has some properties similar to variable gadget. More precisely suppose that F' is an induced subgraph of G with $\delta_G(F') \subseteq S \cup S'$ and that $f\colon G \to H$ is a covering projection. Suppose that $f(y_1) = a$ and $f(y_2) = b$. Then $f(y_3) = f(z_3) = c$, $f(y_4) = f(z_4) = a$, $f(y_5) = f(z_5) = b$, $f(z_2) = b$, and $f(z_1) = a$. It means that $f(S) = f(S') = c$, $f(V) = a$ and $f(V') = b$. Since H is symmetric, we have that for any covering $f\colon G \to H$ and any copy of F': $|f(S)| = |f(S')| = |f(V)| = |f(V')| = 1$, $f(S) = f(S')$ and $\{f(S), f(V), f(V')\} = \{a, b, c\}$.

We prove that H-Cover is NP-complete by reducing it to k-in-$2k$-SAT$_{2k}$ in an analogous way as we did in the proof of Lemma 2. For every instance φ of k-in-$2k$-SAT$_{2k}$ we construct graph G_φ in the same way as in the proof of Lemma 2, but we use gadget F' instead of variable gadget. We will show that G_φ covers H if and only if φ is k-in-$2k$ satisfiable.

In fact if φ is satisfiable, then we can construct covering $f\colon G_\varphi \to H$ in the same way as we did in the proof of Lemma 2. On the other side if there exists a covering projection $f\colon G_\varphi \to H$ then without loss of generality we can suppose that in every connected component of G_φ covering f maps all copies of S to vertex c (here we use the fact that composition of covering projection and an automorphism is a covering projection). Then we can define k-in-$2k$ satisfying evaluation of variables of φ in the same way as we did in the proof of Lemma 2.

To conclude the proof we need to show that also H-Cover* is NP-complete. We will find a 3-partition $\mathcal{A}$ of $V(G_\varphi)$ such that if G_φ covers H then $(G_\varphi, \mathcal{A})$ covers* H. Suppose that $f\colon G_\varphi \to H$ is a covering projection. Without lost of generality we can suppose that $f(S \cup S' \cup \{y_3, z_3\}) = c$ for every copy of F'. Using the fact that $f(y_1, y_2) = f(z_1, z_2) = f(y_4, y_5) = f(z_4, z_5) = \{a, b\}$, we can easily construct 3-partition $\mathcal{A}$ (without knowing f) such that any cover f respects $\mathcal{A}$, what means that $(G_\varphi, \mathcal{A})$ is nice for H. $\square$

We continue with the graph $H = S(k, l, 0, k + 2l)$ where $k \geq 1$ and $l \geq 1$, see Figure 2.

Lemma 8. *Let $H = S(k, l, 0, k + 2l)$ where $k \geq 1$ and $l \geq 1$. Then H-Cover* is NP-complete.*

Proof. We reduce NP-hardness of H-COVER* from 3-ECOL problem. The idea of the reduction is similar to the idea used in the proof of Lemma 6. For every simple cubic graph F we construct a nice graph 3-partition $(G_F, \mathcal{A})$ for H such that $(G_F, \mathcal{A})$ covers* H if and only if F is 3-edge colorable.

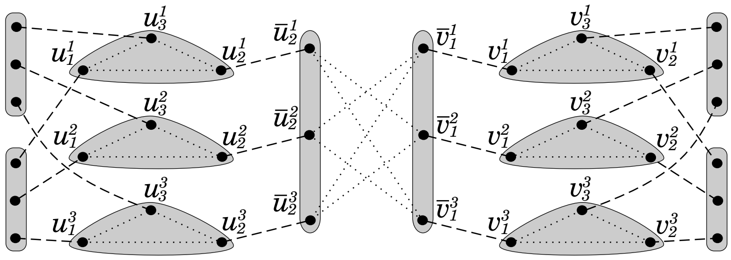

Fig. 7. Replacement of edge uv in the proof of Lemma 8. In this example we choose indices $i = 2$ and $j = 1$. We depict k-edges, resp. $2l$-edges by dotted, resp. dashed lines.

For every vertex $u \in V(F)$ and $i, t = 1, 2, 3$ we insert to G_F new vertices u_i^t and $\bar{u}_i^t$. We add k-edges $u_1^t u_2^t, u_2^t u_3^t, u_3^t u_1^t$ and $2l$-edges $u_1^t \bar{u}_1^t, u_2^t \bar{u}_2^t, u_3^t \bar{u}_3^t$ for all $t = 1, 2, 3$, see Figure 7. For every edge uv of F we choose indices $i, j = 1, 2, 3$ and add k-edges $\bar{u}_i^1 \bar{v}_j^2$, $\bar{v}_j^2 \bar{u}_i^3$, $\bar{u}_i^3 \bar{v}_j^1$, $\bar{v}_j^1 \bar{u}_i^2$, $\bar{u}_i^2 \bar{v}_j^3$, and $\bar{v}_j^3 \bar{u}_i^1$. We choose indices i and j in such a way that every vertex of G_F has degree $2k + 2l$.

We define 3-partition $\mathcal{A}$ of G_F as union of triples $\{u_1^t, u_2^t, u_3^t\}$ and $\{\bar{u}_i^1, \bar{u}_i^2, \bar{u}_i^3\}$ for all $u \in V(F)$ and $i, t = 1, 2, 3$ (see gray areas on Figure 7).

At first we will prove that $(G_F, \mathcal{A})$ is nice for H. Let $f : G_F \to H$ be a covering projection. We prove that f respects $\mathcal{A}$. Let $T = \{u_1^t, u_2^t, u_3^t\} \in \mathcal{A}$ and for contrary suppose that $f(T) \neq V(H)$. It means that at least two vertices from T, say u_1^t and u_2^t are mapped to c (because there are no loops neither at vertex a nor b). Vertex u_3^t also has to be mapped to c since at least $2k > k$ of its neighbors are mapped to c. It means that no neighbor of u_3^t is mapped to a or b, a contradiction.

Now take $T = \{\bar{u}_i^1, \bar{u}_i^2, \bar{u}_i^3\} \in \mathcal{A}$ and suppose that index i, resp. j corresponds to vertex u, resp. v and an edge $uv \in E(F)$. For contrary suppose that $f(T) \neq V(H)$. Without loss of generality $f(\bar{u}_i^1) = f(\bar{u}_i^2)$. If $f(\bar{u}_i^1) = c$ then by the same argument as in the previous case we have $f(\bar{v}_j^3) = c$ and either a or b is missing in $f(N_{G_F}(\bar{v}_j^3))$, a contradiction. If $f(\bar{u}_i^1) = a$ (analogously b) then we have $f(\bar{v}_j^3) = b$ and $f(v_j^3) = c$. Then at least two out of vertices v_1^3, v_2^3, v_3^3 must be mapped to c, a contradiction.

In the next step we will show that from the existence of a covering projection* $f : (G_F, \mathcal{A}) \to^* H$ follows that F is 3-edge colorable. Let i, resp. j be the index corresponding to vertex u, resp. v and edge $uv \in E(F)$. By case analysis and using the fact that f respects $\mathcal{A}$ we have $f(\bar{u}_i^1) = f(\bar{v}_j^1)$ for every edge $uv \in E(F)$. Moreover from $f(u_1^1, u_2^1, u_3^1) = V(H)$ we have $f(\bar{u}_1^1, \bar{u}_2^1, \bar{u}_3^1) = V(H)$ for all vertices $u \in V(F)$. These two facts directly imply that coloring defined by $col(uv) = f(\bar{u}_i^1)$ is a proper 3-edge coloring of F.

In the rest of the proof suppose that $col\colon E(F) \to V(H)$ is a proper 3-edge coloring of F. We show that $(G_F, \mathcal{A})$ covers* H. Let $col_1, col_2, col_3\colon E(F) \to V(H)$ be three proper 3-edge colorings of F such that $\{col_1(uv), col_2(uv), col_3(uv)\} = V(H)$ for every $uv \in E(F)$. Such colorings can be easily made from col by "shifting" colors. We define mapping $f\colon V(G_F) \to V(H)$ in the following way:

- $f(\bar{u}_i^t) = col_t(uv)$ for every $\bar{u}_i^t \in V(G_F)$ and $uv \in E(F)$, where i is the index corresponding to vertex u and edge uv in G_F
- $f(u_i^t) = a$, if $f(\bar{u}_i^t) = b$, for every $u_i^t \in V(G_F)$
- $f(u_i^t) = b$, if $f(\bar{u}_i^t) = a$, for every $u_i^t \in V(G_F)$
- $f(u_i^t) = c$, if $f(\bar{u}_i^t) = c$, for every $u_i^t \in V(G_F)$

Properties of col_1, col_2, col_3 imply that for every $T \in \mathcal{A}$ is $f(T) = V(H)$ and so f respects $\mathcal{A}$. Furthermore it is a routine check to show that f is a covering projection from G_F to H, which concludes the proof. ◻

Corollary 3. *Let H be a connected 4-regular graph on three vertices. Then H-*Cover* *is NP-complete.*

Proof. If H does not contain triangle as an subgraph, then H is isomorphic to $S(2,0,1,0)$ and NP-completeness of H-Cover* follows from Lemma 7.

Furthermore suppose that H contains a triangle. By an easy case analysis we get that H is isomorphic to $S(2,0,0,2)$, $S(1,1,1,1)$, or $S(1,1,0,3)$. Then NP-completeness of H-Cover* is covered by Lemmata 7, 6, and 8 (in this order). ◻

Notice that Lemmata 7 and 8 show NP-hardness of H-Cover* for connected t-regular graphs $H = S(k,l,x,y)$ with $t \geq 4$ having at least one vertex with no loop, i.e. $l = 0$ or $x = 0$. It means that to prove NP-hardness of H-Cover* for all $H = S(k,l,x,y)$ we can assume that $l, x \geq 1$. Moreover, using Lemma 6 we can assume that such graphs H are at least 6-regular. We prove that H-Cover* is NP-hard for such graphs by mathematical induction using the next lemma as a key instrument.

Lemma 9 (Adding loops). *Let H be a t-regular graph on three vertices with at least one loop at every vertex and $t \geq 6$. Let H' be the graph constructed from H by removing a loop from every vertex. Then H'-*Cover* *can be reduced to H-*Cover**.*

Proof. Let $(G', \mathcal{A})$ be an instance of H'-Cover*. We construct graph G by adding one loop to every vertex of G'. Then it is enough to show that $(G, \mathcal{A})$ is nice for H and $(G', \mathcal{A})$ covers* H' if and only if $(G, \mathcal{A})$ covers* H.

Suppose that $f\colon (G, \mathcal{A}) \to^* H$ is a covering projection*. From definition for every $v \in V(G) = V(G')$ and $x \in V(H) = V(H')$ we have $|N_G(v) \cap f^{-1}(x)| = m_H(f(v), x)$. If we map vertices of G' by f to H' then if $x \neq f(v)$: $|N_{G'}(v) \cap$

$f^{-1}(x)| = m_H(f(v),x) = m_{H'}(f(v),x)$ and if $x = f(v)$: $|N_{G'}(v) \cap f^{-1}(x)| = |N_G(v) \cap f^{-1}(x)| - 2 = m_H(f(v),x) - 2 = m_{H'}(f(v),x)$. It means that f is a covering projection from G' to H'. Moreover f respects $\mathcal{A}$ and so f is a covering projection* from $(G',\mathcal{A})$ to H'.

For the opposite implication suppose that $f\colon (G',\mathcal{A}) \to^* H'$ and we prove that $(G,\mathcal{A})$ covers* H. For every $v \in V(G) = V(G')$ and every $x \in V(H) = V(H')$, $x \neq f(v)$ we have $|N_G(v) \cap f^{-1}(x)| = |N_{G'}(v) \cap f^{-1}(x)| = m_{H'}(f(v),x) = m_H(f(v),x)$ and for $x = f(v)$ we have $|N_G(v) \cap f^{-1}(x)| = |N_{G'}(v) \cap f^{-1}(x)| + 2 = m_{H'}(f(v),x) + 2 = m_H(f(v),x)$. Together with the fact that f respects $\mathcal{A}$ it means that f is a covering projection* from $(G,\mathcal{A})$ to H.

To conclude the proof we need to verify that $(G,\mathcal{A})$ is nice for H. Let $f\colon G \to H$ be a covering projection. From above we know that f is also covering projection from G' to H'. Since $(G',\mathcal{A})$ is nice for H' we know that there exists a covering projection* $f'\colon G' \to H'$. Mapping f' is clearly also covering projection* from G to H what implies that $(G,\mathcal{A})$ is nice for H. □

Corollary 4. *Let $H = S(k,l,x,y)$ be a connected t-regular graph. If $t \geq 4$ then H-*Cover* is NP-complete.*

Proof. The proof proceeds by mathematical induction on t:

- If $t = 4$ then NP-completeness follows from Corollary 3
- If $t > 4$ and at least one vertex of H does not contain a loop, i.e. $l = 0$ or $x = 0$, then NP-completeness follows from Lemmata 7 and 8
- If $t > 4$ and there is a loop at every vertex of H, then NP-completeness of H-Cover* follows from Lemma 9. Here we use the fact that H'-Cover* is NP-complete by an induction hypothesis. □

One can observe that every regular graph H on three vertices that allows non-trivial automorphism is isomorphic to some $S(k,l,x,y)$. It means, that in the rest of this section we can restrict to connected t-regular graphs H that allow only trivial automorphisms and $t \geq 6$ (Corollary 3). Moreover we can assume that such graphs contain at least one vertex with no loop. Otherwise NP-completeness of H-Cover* can be proved by mathematical induction using Lemma 9.

It is a routine check to show that every such graph is isomorphic to some $H(m,n,z)$ for $m > n > 0$, see Figure 2.

Before we prove NP-completeness of $H(m,n,z)$-Cover* for $m > n > 0$ we need to define new gadgets and auxiliary lemmata. From this point further we count all indices *mod 6*.

Definition 5 (of $I(m,n,z)$). *Let m,n,z be integers such that $m > n > 0$ and $z \geq 0$. Define $I(m,n,z)$ to be a graph on vertex set $\{u_1,u_2,v_1,\dots,v_6\}$ with edges (see Figure 8):*

- $m(v_1,v_3) = m(v_3,v_5) = m(v_5,v_1) = m(v_2,v_4) = m(v_4,v_6) = m(v_6,v_2) = z$
- $m(v_1,v_2) = m(v_2,v_3) = m(v_4,v_5) = m(v_6,v_1) = 2n$
- $m(v_3,v_4) = m(v_5,v_6) = 2m$
- $m(v_1,u_1) = m(v_2,u_2) = 2(m-n)$

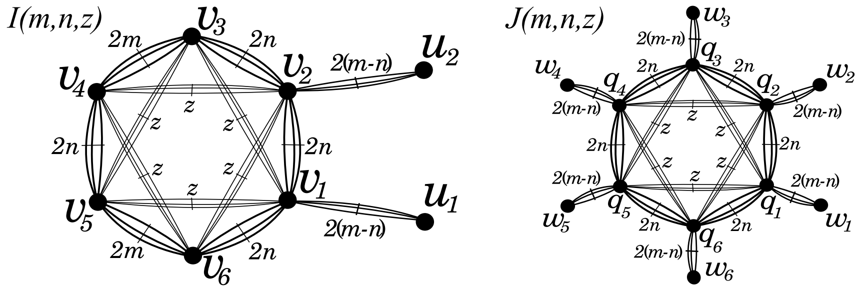

Fig. 8. Gadgets $I(m,n,z)$ and $J(m,n,z)$.

Lemma 10. *Let m, n, z be integers such that $m > n > 0$ and $z \geq 0$. Let $I(m,n,z)$ be an induced subgraph of graph G with $\delta_G(I(m,n,z)) = \{u_1, u_2\}$. Suppose that $f\colon G \to H(m,n,z)$ is a covering projection. Then $f(u_1) = f(v_2)$, $f(u_2) = f(v_1)$, $f(v_1, v_3, v_5) = f(v_2, v_4, v_6) = \{a, b, c\}$ and one of the following holds:*

i) $f(u_1) = b$ and $f(u_2) = c$
ii) $f(u_1) = c$ and $f(u_2) = b$
iii) $f(u_1) = f(u_2) = a$

Proof. Denote $I(m,n,z)$, resp. $H(m,n,z)$ simply by I, resp. H. At first suppose that $z > 0$. Let $f\colon G \to H$ be a covering projection. We will show that exactly one of the vertices v_1, v_3, and v_5 is mapped by f to c. Since there is no loop at c in H at most one of those vertices could be mapped to c. For contrary suppose that none of them is mapped to c. If $f(v_4) \neq c$ then $f(v_6) = f(v_2) = c$ since v_4 needs more than z neighbors to be mapped to c. That is a contradiction with the fact that f is a covering projection (since $m_I(v_2, v_6) > 0$) and so $f(v_4) = c$. Analogously we get $f(v_6) = c = f(v_4)$, a contradiction. It means that exactly one of vertices v_1, v_3, v_5 is mapped by f to c. By symmetry the same holds for vertices v_2, v_4, v_6.

Suppose that $v_i \neq v_j$ are the two vertices mapped to c. Then necessarily $j = i + 3\ (mod\ 6)$. If $f(v_1) = f(v_4) = c$ then $f(v_3) = b$ since b is the only vertex of H with at least $2m+z$ edges going to c. From the fact that $f(v_5) \neq c$ we know that $f(N_I(v_5)) = \{a, b, c\}$ what implies $f(v_6) = a$ and consequently $f(v_5) = a$ as well. By a routine check we also get $f(v_2, u_1) = b$ and $f(v_1, u_2) = c$. Now we can see that f fulfills all conditions from the lemma.

If $f(v_2) = f(v_5) = c$ then the situation is symmetric.

If $f(v_3) = f(v_6) = c$ then $f(v_4) = f(v_5) = b$ since b is the only vertex of H with at least $2m + z$ edges going to c. By a routine check we also have $f(v_1, v_2, u_1, u_2) = a$ and f again fulfills all conditions of the lemma.

If $z = 0$ then by routine case analysis we get:

- if $f(v_3) = a$ then $f(v_3, v_4) = a$, $f(v_1, v_6, u_2) = b$, and $f(v_2, v_5, u_1) = c$
- if $f(v_3) = b$ then $f(v_5, v_6) = a$, $f(v_2, v_3, u_1) = b$, and $f(v_1, v_4, u_2) = c$
- if $f(v_3) = c$ then $f(v_1, v_2, u_1, u_2) = a$, $f(v_4, v_5) = b$, and $f(v_3, v_6) = c$

All conditions of lemma are fulfilled in every case, that concludes the proof. □

Observation 5 *Let* $f\colon \{u_1, u_2, v_1, v_2\} \to V(H(m,n,z))$ *be a mapping such that* $f(u_1) = f(v_2)$, $f(u_2) = f(v_1)$ *and either* $f(u_1, u_2) = \{a\}$ *or* $f(u_1, u_2) = \{b, c\}$. *Then* f *can be extended to* $V(I(m,n,z))$ *in such a way that the restriction of* f *to* $N_{I(m,n,z)}(v_i)$ *is a bijection to* $N_{H(m,n,z)}(f(v_i))$ *for each* $v_i \in V(I(m,n,z))$ *and* $f(v_1, v_3, v_5) = f(v_2, v_4, v_6) = \{a, b, c\}$.

The proof of Observation 5 follows directly from the case analyzes used in the proof of Lemma 10.

Definition 6 (of $J(m,n,z)$**).** *Let* m, n, z *be integers such that* $m > n > 0$ *and* $z \geq 0$. *Define* $J(m,n,z)$ *to be a graph on vertex set* $\{q_1, \dots, q_6, w_1, \dots, w_6\}$ *with edges (see Figure 8):*

- $m(q_i, q_{i+1}) = 2n$, *for every* $i = 1, \dots, 6$
- $m(q_i, q_{i+2}) = z$, *for every* $i = 1, \dots, 6$
- $m(q_i, w_i) = 2(m-n)$, *for every* $i = 1, \dots, 6$

In the following definition we use the copies I^j, resp. J^j of gadget $I(m,n,z)$, resp. $J(m,n,z)$. Intuitively, the symbol x_i^j will denote the vertex of I^j, resp. J^j corresponding to x_i in $I(m,n,z)$, resp. $J(m,n,z)$.

Definition 7 (of K(m,n,z)). *Let* m, n, z *be integers such that* $m > n > 0$ *and* $z \geq 0$. *Let* I^1, I^2, I^3 *be disjoint copies of* $I(m,n,z)$. *Let* J^1, *resp.* J^2 *be copy of* $J(m,n,z) \setminus \{w_1, \dots, w_6\}$, *resp.* $J(m,n,z)$. *We proceed by the set of following unification (see Figure 9):*

- $u_2^j = q_{2j}^1$, *for every* $j = 1, 2, 3$
- $w_{2j-1}^2 = q_{2j-1}^1$, *for every* $j = 1, 2, 3$

Denote vertices u_1^j, *resp.* w_{2j}^2 *by* r_j, *resp.* s_j. *Denote this gadget by* $K(m,n,z)$.

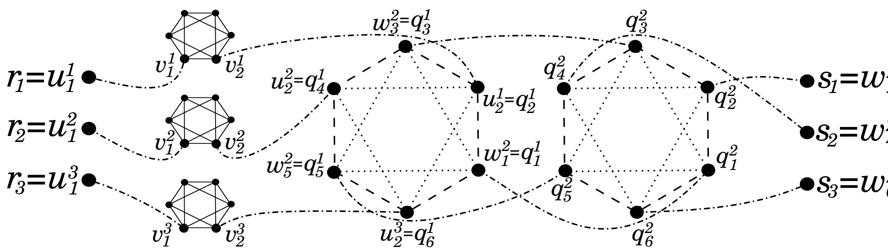

Fig. 9. Gadget $K(m,n,z)$. Some z-edges, $2n$-edges, resp. $2(m-n)$-edges are depicted by dotted, dashed, resp. dash-dotted curves.

Gadget $K(m,n,z)$ is symmetric in the following way (proof of Observation 6 follows directly from definition of $K(m,n,z)$):

Observation 6 *For every permutation π on the set $\{1,2,3\}$, there exists an automorphism ψ of $K(m,n,z)$ such that $\psi(r_i) = r_{\pi(i)}$ and $\psi(s_i) = s_{\pi(i)}$ for every $i = 1,2,3$.*

Lemma 11. *Let m, n, z be integers such that $m > n > 0$ and $z \geq 0$. Let $K(m,n,z)$ be an induced subgraph of a graph G such that $\delta_G(K(m,n,z)) = \{r_1, r_2, r_3, s_1, s_2, s_3\}$. Suppose that $f\colon G \to H(m,n,z)$ is a covering projection. Then*
$f(r_1, r_2, r_3) = \{a, b, c\}$.

Proof. Denote the set $\{q_2^1, q_4^1, q_6^1\}$ by Q and $H(m,n,z)$, resp. $K(m,n,z)$ simply by H, resp. K. From Lemma 10 follows that it is enough to show that $f(Q) = \{a,b,c\}$. We will prove that no two vertices of Q are mapped to the same vertex of H. At first we suppose that $z = 0$.

For contradiction suppose that $|f^{-1}(a) \cap Q| \geq 2$. From Observation 6 follows that without lost of generality $f(q_2^1) = f(q_4^1) = a$, see Figure 10. Then necessarily $f(q_3^1) = a$ since $m_H(a,b) < m_H(a,c) < 4n = m_K(q_3^1, q_2^1) + m_K(q_3^1, q_4^1)$ (recall that $m > n > 0$). From Lemma 10 we know that $f(v_2^1) = f(v_2^2) = a$. It means that vertices q_1^1 and q_5^1 must be mapped to c, because at least one neighbor of q_2^1, resp. q_4^1 has to be mapped to c. Then q_6^1 has to be mapped to b since $m_H(c,c) < m_H(a,c) < 4n = m_K(q_6^1, q_1^1) + m_K(q_6^1, q_5^1)$. Using Lemma 10 we see that all neighbors of q_6^1 are mapped to c, a contradiction.

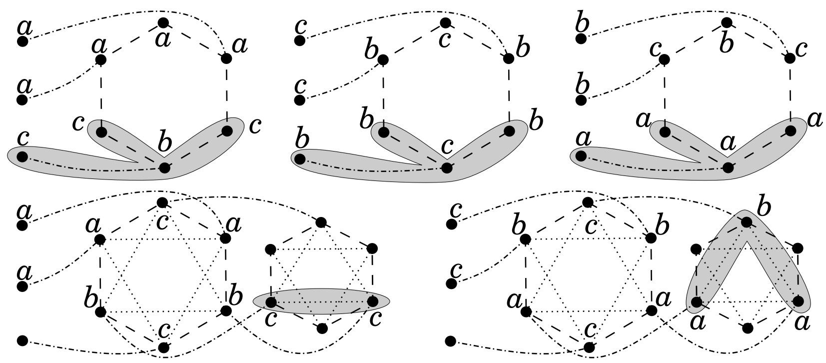

Fig. 10. All contradictions used in case analysis in the proof of Lemma 11 are depicted in gray area. At the top are cases with $z = 0$ while at the bottom are cases with $z > 0$.

If $|f^{-1}(b) \cap Q| \geq 2$ then without loss of generality suppose $f(q_2^1) = f(q_4^1) = b$. Since $m_H(a,b) < m_H(b,b) < 4n = m_K(q_3^1, q_2^1) + m_K(q_3^1, q_4^1)$ necessarily $f(q_3^1) = c$. By Lemma 10 and the fact that at least one neighbor of vertices mapped to b must be mapped to b as well we have $f(q_1^1) = f(q_5^1) = b$. Vertex q_6^1 only can be mapped to c since $m_H(a,b) < m_H(b,b) < 4n = m_K(q_6^1, q_1^1) + m_K(q_6^1, q_5^1)$. Using Lemma 10 we again see that all neighbors of q_6^1 are mapped to b, a contradiction.

Finally suppose that $|f^{-1}(c) \cap Q| \geq 2$ and that $f(q_2^1) = f(q_4^1) = c$. Since $m_H(c,c) < m_H(a,c) < 4n = m_K(q_3^1, q_2^1) + m_K(q_3^1, q_4^1)$ we have $f(q_3^1) = b$. By Lemma 10 and the fact that at least one neighbor of every vertex mapped to c must be mapped to a we have $f(q_1^1) = f(q_5^1) = a$. Vertex q_6^1 has to be mapped to a since $m_H(a,b) < m_H(a,c) < 4n = m_K(q_6^1, q_1^1) + m_K(q_6^1, q_5^1)$. From Lemma 10 follows that all neighbors of q_6^1 are mapped to a, a contradiction.

For the rest of the proof we suppose that $z > 0$.

Suppose that $|f^{-1}(a) \cap Q| \geq 2$ and that $f(q_2^1) = f(q_4^1) = a$. From the fact that $m_H(a,c) > \max\{z, 2n\}$ and Lemma 10 follow that at least two of vertices q_1^1, q_3^1, q_6^1, resp. q_3^1, q_5^1, q_6^1 must be mapped to c. This is only possible if $f(q_3^1) = f(q_6^1) = c$, see Figure 10. Since at least one neighbor of q_2^1, resp. q_4^1 has to be mapped to b we have $f(q_1^1) = f(q_5^1) = b$. Because of $m_H(b,c) = 2m + z > 2n + z = m_K(q_1^1, q_3^1) + m_K(q_1^1, q_6^1) = m_K(q_5^1, q_3^1) + m_K(q_5^1, q_6^1)$, we have $f(q_1^2) = c = f(q_5^2)$. This is in contrary with the facts that $m_H(c,c) = 0$ and $m_K(q_1^2, q_5^2) = z > 0$.

If $|f^{-1}(b) \cap Q| \geq 2$, then we suppose that $f(q_2^1) = f(q_4^1) = b$. From the fact that $m_H(b,c) = 2m + z > 2(m-n) + \max\{z, 2n\}$ and Lemma 10 we know that at least two of vertices q_1^1, q_3^1, q_6^1, resp. q_3^1, q_5^1, q_6^1 have to be mapped to c. This is only possible if $f(q_3^1) = f(q_6^1) = c$. Since at least one neighbor of q_2^1, resp. q_4^1 has to be mapped to a we have $f(q_1^1) = f(q_5^1) = a$.

From the mapping of vertices q_1^1, q_3^1 and q_5^1 we have that $2z \leq m_H(a,c) = z + 2n < z + 2m$ and so $z < 2m$. It means that vertex q_1^1, resp. q_5^1 needs more neighbors to be mapped to a and we have $f(q_1^2) = f(q_5^2) = a$. Moreover in this case necessarily $z = 2n$, since $m_H(a,b) = m_K(q_1^1, q_2^1)$. Finally vertex q_3^2 has to be mapped to b, since $4n < z + 2m = m_H(b,c)$. From the mapping of vertices q_1^2, q_3^2 and q_5^2 we have $z = m_H(a,b) \geq m_K(q_3^2, q_1^2) + m_K(q_3^2, q_5^2) = 2z$, a contradiction.

We conclude the proof by the fact that the case $|f^{-1}(c) \cap Q| \geq 2$ is not possible since vertices of $f^{-1}(c)$ form a stable set in G. $\square$

Observation 7 *For every bijection $\pi\colon \{1,2,3\} \to V(H(m,n,z))$ there exists a mapping $f\colon V(K(m,n,z)) \to H(m,n,z)$ such that the restriction of f to $N_{K(m,n,z)}(v)$ is a bijection to $N_{H(m,n,z)}(f(v))$ for every $v \in V(K(m,n,z)) \setminus \{r_1, r_2, r_3, s_1, s_2, s_3\}$ and for every $i = 1,2,3$ and $j = 1,2$ the following claims hold:*

- $f(r_i) = \pi(i)$
- $f(r_i, s_i)$ *is either* $\{a,b\}$ *or* $\{c\}$
- $f(r_i, v_1^i)$ *is either* $\{a\}$ *or* $\{b,c\}$
- $f(s_i, q_{2i}^2)$ *is either* $\{a\}$ *or* $\{b,c\}$
- $f(v_1^i, v_3^i, v_5^i) = f(v_2^i, v_4^i, v_6^i) = \{a,b,c\}$
- $f(q_1^j, q_3^j, q_5^j) = f(q_2^j, q_4^j, q_6^j) = \{a,b,c\}$

Proof. Using Observation 6 it is enough to show that such mapping f exists for at least one permutation π. The example of such mapping f is depicted on the Figure 11. Note that vertices in copies of I^1, I^2 and I^3 can be properly mapped according to Observation 5. $\square$

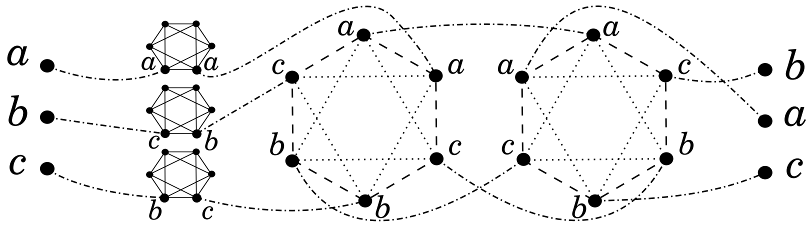

Fig. 11. Possible mapping of $K(m,n,z)$ to $H(m,n,z)$ such that the restriction to $N_{K(m,n,z)}(v)$ is a bijection to $N_{H(m,n,z)}(f(v))$ for every $v \in V(K(m,n,z)) \setminus \{r_1, r_2, r_3, s_1, s_2, s_3\}$

Lemma 12 (Adding loops and 2-edges). *Let m, n, z be integers such that $m > n > 0$ and $z \geq 0$. Then $H(m,n,z)$-Cover* is NP-complete.*

Proof. From definition we know that $H(n,n,z) = S(2n+z, 0, n, z)$. We also know that $S(2n+z, 0, n, z)$-Cover* is NP-complete (Lemma 7). We reduce NP-hardness of $H(m,n,z)$-Cover* from this problem. For every instance $(G', \mathcal{A}')$ of $H(n,n,z)$-Cover* we construct a nice graph 3-partition $(G, \mathcal{A})$ such that $(G, \mathcal{A})$ covers* $H(m,n,z)$ if and only if $(G', \mathcal{A}')$ covers* $H(n,n,z)$.

We begin the construction of G by taking two disjoint copies G'^1 and G'^2 of G'. Denote the copy of vertex v in G'^1, resp. G'^2 by v^1, resp. v^2. For every triple $\{a_1, a_2, a_3\} \in \mathcal{A}'$ we take a copy of gadget $K(m,n,z)$ and identify vertices a_i^1, resp. a_i^2 with r_i, resp. s_i for all $i = 1, 2, 3$. Denote this graph by G.

For every $\{a_1, a_2, a_3\} \in \mathcal{A}'$ we add to $\mathcal{A}$ triples $\{a_1^1, a_2^1, a_3^1\}$ and $\{a_1^2, a_2^2, a_3^2\}$. Furthermore for every copy of $K(m,n,z)$ we add to $\mathcal{A}$ the following triples:

- $\{v_1^j, v_3^j, v_5^j\}$ and $\{v_2^j, v_4^j, v_6^j\}$, for all $j = 1, 2, 3$
- $\{q_1^j, q_3^j, q_5^j\}$ and $\{q_2^j, q_4^j, q_6^j\}$, for all $j = 1, 2$

Suppose that $f\colon (G, \mathcal{A}) \to^* H(m,n,z)$ is a covering projection*. Let $\mathcal{A}'^1$ be 3-partition corresponding to G'^1. We prove that the restriction of f to $(G'^1, \mathcal{A}'^1)$ is a covering projection* to $H(n,n,z)$.

Let $v \in V(G'^1)$ and let $x \in V(H(m,n,z))$. If $\{f(v), x\}$ is not equal to $\{a\}$ nor $\{b, c\}$ then by Lemma 10 we have $|N_{G'^1}(u) \cap f^{-1}(x)| = |N_G(u) \cap f^{-1}(x)| = m_{H(m,n,z)}(f(v), x) = m_{H(n,n,z)}(f(v), x)$. Otherwise $|N_{G'^1}(u) \cap f^{-1}(x)| = |N_G(u) \cap f^{-1}(x)| - 2(m-n) = m_{H(m,n,z)}(f(v), x) - 2(m-n) = m_{H(n,n,z)}(f(v), x)$. It means that $f|_{G'^1}\colon G'^1 \to H(n,n,z)$ is a covering projection. From Lemma 11 and construction of G follows that $f|_{G'^1}$ respects 3-partition $\mathcal{A}'$. It means that $(G', \mathcal{A}')$ covers* $H(n,n,z)$.

For the opposite implication suppose that $f'\colon (G', \mathcal{A}') \to^* H(n,n,z)$ is a covering projection*. Define mapping $f''\colon V(G') \to \{a, b, c\}$ by $f''(f'^{-1}(a)) = b$, $f''(f'^{-1}(b)) = a$, and $f''(f'^{-1}(c)) = c$. From the symmetry of $H(n,n,z)$ follows that f'' is also a covering projection* from $(G', \mathcal{A}')$ to $H(n,n,z)$.

We construct covering* $f : (G, \mathcal{A}) \to H(m,n,z)$ in the following way: for each $v \in V(G')$ we define $f(v^1) = f'(v)$ and $f(v^2) = f''(v)$. We extend this mapping to every copy of $K(m,n,z)$ using Observation 7 where for every corresponding triple $A \in \mathcal{A}'$ we use permutation $\pi = f|_A$. It is a routine check to show that f is a covering projection. The fact that f respects 3-partition $\mathcal{A}$ follows from Lemma 11 and Observation 7.

To conclude the proof we need to show that $(G, \mathcal{A})$ is nice for $H(m, n, z)$. Suppose that $f : G \to H(m, n, z)$ is a covering projection. We will show that there exists a covering projection* $f^* \colon (G, A) \to^* H(m, n, z)$. By Lemma 11 we know that for every triple $(a_1, a_2, a_3) \in \mathcal{A}'$ is $f(a_1^1, a_2^1, a_3^1) = \{a, b, c\}$. It means that $f|_{G'^1}$ is a covering projection* from $(G'^1, \mathcal{A}'^1)$ to $H(n, n, z)$ (by Lemma 10). We already proved that the existence of such a mapping implies the existence of f^*, that concludes the proof. □

Note that we are done with the proof of Theorem 3.

Proof (of Theorem 3). This follows directly from Corollary 4, Lemma 9, Lemma 12 and the fact that every connected t-regular H with $t \geq 4$ and at least one vertex with no loop is isomorphic to $S(k, l, x, y)$ or $H(m, n, z)$. □

This settles Theorem 2 that handles the complexity of H-Cover for all 1-block graphs H on three vertices.

Proof (of Theorem 2). Lemma 5 covers all polynomial cases while Theorem 3 with Observation 2 covers the NP-complete cases. □

5 Conclusion

We have settled the computational complexity of H-cover for all multigraphs on three vertices. Not surprisingly, the characterization is substantially more involved than the 2-vertex case. This constitutes an important step towards the goal of a full dichotomy for complexity of H-cover of simple graphs, a goal that requires a full dichotomy also for colored mixed multigraphs, as shown in [17], and in particular the multigraphs handled in this paper.

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