Sim-width and induced minors (Full Version)

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Abstract

We introduce a new graph width parameter, called *special induced matching width*, shortly *sim-width*, which does not increase when taking the induced minor operation. For a vertex partition (A, B) of a graph G, this parameter is based on the maximum size of an induced matching $\{a_1b_1, \ldots, a_mb_m\}$ in G where $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$. Classes of graphs of bounded sim-width are much wider than classes of bounded tree-width, rank-width, or mim-width. As examples, we show that chordal graphs and co-comparability graphs have sim-width at most 1, while they have unbounded value for the other three parameters. In this paper, we obtain general algorithmic results on graphs of bounded sim-width by further excluding certain graphs as induced minors.

A *t-matching complete graph* is a graph that consists of two vertex sets $\{v_1, \ldots, v_t\}$ and $\{w_1, \ldots, w_t\}$ where between the two sets, $\{v_1w_1, \ldots, v_tw_t\}$ is an induced matching, and for two distinct $i, j \in \{1, \ldots, t\}$, either v_i is adjacent to v_j or w_i is adjacent to w_j. We prove that for positive integers w and t, a large class of domination and partitioning problems, including the Minimum Dominating Set problem, can be solved in time $n^{\mathcal{O}(wt^2)}$ on n-vertex graphs of sim-width at most w and having no induced minor isomorphic to a t-matching complete graph, when the decomposition tree is given. As far as we know, these are the first infinite non-trivial classes of graphs that are closed under induced minors, but not under minors, and have general algorithmic applications. For chordal graphs and co-comparability graphs, we provide polynomial-time algorithms to obtain their decomposition trees certifying sim-width at most 1. Note that Minimum Dominating Set is NP-complete on chordal graphs.

Keywords and phrases sim-width, induced minor, chordal graphs, co-comparability graphs

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1 Introduction

It is well known that a graph has tree-width at most k if and only if it is a subgraph of a chordal graph with maximum clique size $k + 1$. The algorithmic usefulness of tree-width came from the property of chordal graphs that minimal separators are cliques and form a tree-like structure. Since the tree-width of a graph restricts the number of vertices in each minimal separator of its minimal chordal supergraphs, we can naturally design efficient (FPT) dynamic programming algorithms for many problems on graphs of bounded tree-width. On the other hand, tree-width is unbounded on some very simple classes of graphs, such as complete graphs and complete bipartite graphs. To deal with such classes, several other width parameters based on restricting the number of neighborhood types have been introduced, and they are mostly equivalent to clique-width [17, 15, 5]. Nowadays,

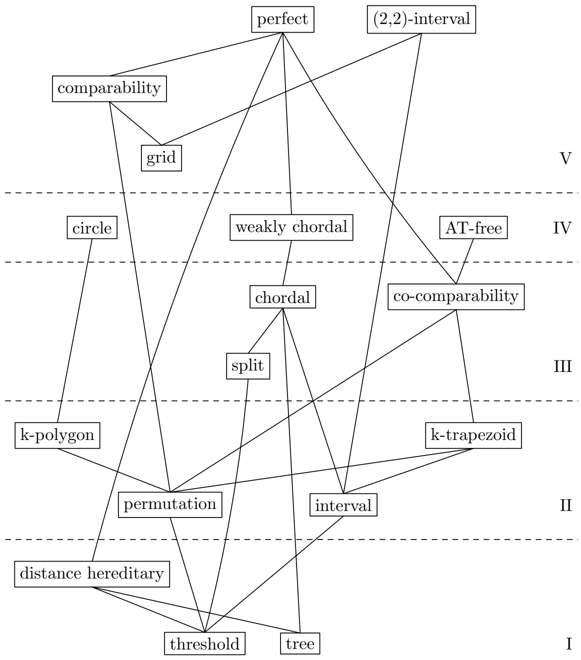

Figure 1 Inclusion diagram of some well-known graph classes. (I) Classes where clique-width and rank-width are constant. (II) Classes where mim-width is constant. (III) Classes where sim-width is constant. (IV) Classes where it is unknown if sim-width is constant. (V) Classes where sim-width is unbounded.

most people believe tree-width and clique-width are the best width parameters for FPT algorithms in this jungle of width parameters.

In this paper we focus on XP algorithms and introduce a new graph width parameter using the maximum size of an induced matching between parts of a vertex partition. For a graph G and its vertex subset A, we denote by $\text{simval}_G(A)$ the maximum size of an induced matching $\{a_1b_1, a_2b_2, \ldots, a_mb_m\}$ where $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in V(G) \setminus A$. We take a sim-decomposition (T, L) of a graph where T is a subcubic tree and L is a bijection from the vertex set of G to the leaves of T. For each edge e in T inducing the vertex partition (A_e, B_e) of G, the width of e is defined as $\text{simval}_G(A_e)$. As usual, the width of (T, L) is the maximum width among all edges in T, and the *sim-width* of the graph is defined as the minimum width among all sim-decompositions of it. The linear variant of sim-width will be called *linear sim-width*. We will define these more carefully in Section 2. An important point is that when we consider the induced matching, we also care about edges in A_e and B_e. If one ignores the edges in A_e and B_e, then it creates another parameter called *mim-width*, introduced formally by Vatshelle in [22], but implicitly used in a paper by Belmonte and Vatshelle [3] to give a common explanation for the existence of efficient XP algorithms for many well-known graph classes of unbounded clique-width and constant mim-width. We show that the modelling power of sim-width is strictly stronger than mim-width. See Figure 1 for an inclusion diagram of some well-known graph classes.

▶ **Theorem 1.** ***1.*** *Chordal graphs have sim-width at most* 1 *but unbounded mim-width, and a sim-decomposition of width at most* 1 *can be found in polynomial time.*

2. *Co-comparability graphs have linear sim-width at most* 1 *but unbounded mim-width, and a linear sim-decomposition of width at most* 1 *can be found in polynomial time.*

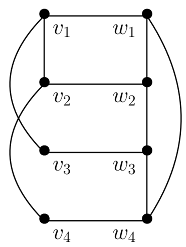

Figure 2 A 4-matching complete graph.

We conjecture that circle graphs also have constant sim-width. Note that for problems like Minimum Dominating Set which are NP-complete on chordal graphs [4], we cannot expect an XP algorithm parameterized by sim-width, i.e. with runtime $|V(G)|^{f(\mathrm{simw}(G))}$, even if we are given a sim-decomposition. In this paper, we nevertheless obtain a general algorithmic result of this type for a large class of such problems, on graphs of bounded sim-width, by further excluding certain graphs as induced minors.

A *t-matching complete graph* is a graph that consists of two vertex sets $\{v_1, \dots, v_t\}$ and $\{w_1, \dots, w_t\}$ where between the two sets, $\{v_1w_1, \dots, v_tw_t\}$ is an induced matching, and for two distinct $i, j \in \{1, \dots, t\}$, either v_i is adjacent to v_j or w_i is adjacent to w_j. See Figure 2 for an example. The class of *Locally Checkable Vertex Subset and Vertex Partitioning problems*, shortly LC-VSVP problems, is a subclass of MSO_1 problems that generalize Maximum Independent Set, Minimum Dominating Set, and q-Coloring problems [21]. We define this class of problems in Section 6. We show the following.

▶ **Theorem 2.** *Given an n-vertex graph having no t-matching complete graph as an induced minor and its sim-decomposition of width w, every fixed LC-VSVP problem on G can be solved in time $n^{\mathcal{O}(wt^2)}$. For instance, the* Minimum Dominating Set *and q-*Coloring *problems can be solved in time $\mathcal{O}(n^{6(2w+1)t^2+4})$ and $\mathcal{O}(qn^{6q(2w+1)t^2+4})$, respectively.*

We explain why both conditions in Theorem 2, on t-matching complete graphs and on sim-width, are necessary. First, the class of graphs having no 5-matching complete graph as an induced minor contains all planar graphs, because if a graph contains a 5-matching complete graph as an induced minor then it contains K_5 as a minor and is thus not planar. Moreover, it is known that the Maximum Independent Set problem is NP-complete on planar graphs [13]. Second, we show in Theorem 1 that chordal graphs admit a sim-decomposition of width at most 1, and moreover the Minimum Dominating Set problem is NP-complete on chordal graphs [4]. Therefore, if we remove one of the two conditions, we cannot hope to obtain the general result in Theorem 2.

For chordal and co-comparability graphs, Theorem 1 provides a polynomial-time algorithm to compute a sim-decomposition of width at most 1. One might ask if testing whether a given graph has a t-matching complete graph as an induced minor can be done efficiently, because of two reasons; testing H-induced minor for a fixed graph H is NP-complete in general [9], and the number of possible t-matching complete graphs are exponential in t. It has been recently shown that for fixed graph H, testing H-induced minor can be solved in polynomial time on chordal graphs [2], and AT-free graphs [14] which contain all co-comparability graphs. We can use those algorithms. Furthermore, in Section 7, we prove that there is a unique t-matching complete graph that is a chordal graph, and there are only t possible t-matching complete graphs that are co-comparability graphs. Therefore, we can efficiently check the conditions of Theorem 2 for chordal graphs and co-comparability graphs.

However, we do not know whether H-induced minor testing can be solved in polynomial time on graphs of sim-width at most w in general.

The paper is organized as follows. Section 2 contains the necessary notions required for our results, including other width parameters tree-width, rank-width, and mim-width. In Section 3, we show that the class of graphs of sim-width at most w is closed under taking induced minors. In Section 4, we prove that chordal graphs have sim-width at most 1, and co-comparability graphs have linear sim-width at most 1, and provide polynomial-time algorithms to find such decompositions. We also provide lower bounds on mim-width of those classes, which were not previously known. Towards our main algorithmic result, we show in Section 5 that every graph of sim-width at most w and having no t-matching complete graph as an induced minor has mim-width at most $(4w+2)t^2$ (Proposition 11). Then in Section 6 we introduce LC-VSVP problems and obtain the general algorithmic results (Corollary 15). We give the results on t-matching complete graphs in chordal graphs and co-comparability graphs in Section 7. We list some questions on sim-width in Section 8.

2 Preliminaries

We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. We denote by $N_G(v)$ the set of neighbors of a vertex v in G, and let $N_G[v] := N_G(v) \cup \{v\}$. For two graphs G_1 and G_2 on disjoint vertex sets, the *union* of G_1 and G_2 is the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$. For $v \in V(G)$ and $X \subseteq V(G)$, we denote by $G - v$ the graph obtained from G by removing v, and denote by $G - X$ the graph obtained from G by removing all vertices in X. For $e \in E(G)$, we denote by $G - e$ the graph obtained from G by removing e, and denote by G/e the graph obtained from G by contracting e. For a vertex v of G with exactly two neighbors v_1 and v_2 that are non-adjacent, the operation of removing v and adding the edge between its neighbors is called *smoothing* a vertex v. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced on X. A *clique* is a set of vertices that are pairwise adjacent, and an *independent set* is a set of vertices that are pairwise non-adjacent. A set of edges $\{v_1w_1, v_2w_2, \ldots, v_mw_m\}$ of G is called an *induced matching* in G if there are no other edges in $G[\{v_1, \ldots, v_m, w_1, \ldots, w_m\}]$. For a vertex partition (A, B) of G and an induced matching $\{v_1w_1, v_2w_2, \ldots, v_mw_m\}$ in G where $v_1, \ldots, v_m \in A$ and $w_1, \ldots, w_m \in B$, we say that it is an induced matching in G between A and B.

For two graphs H and G, H is a *subgraph* of G if H can be obtained from G by removing some vertices and edges, and H is an *induced subgraph* of G if $H = G[X]$ for some $X \subseteq V(G)$, and H is an *induced minor* of G if H can be obtained from G by a sequence of removing vertices and contracting edges, and H is a *minor* of G if H can be obtained from G by a sequence of removing vertices, removing edges, and contracting edges. We note that it is not allowed to remove an edge in the induced minor relation; for instance, the complete graph on 4 vertices cannot contain the cycle of length 4 as an induced minor.

A pair of vertex subsets (A, B) of a graph G is called a *vertex partition* if $A \cap B = \emptyset$ and $A \cup B = V(G)$. For a vertex partition (A, B) of a graph G, we denote by $G[A, B]$ the bipartite graph on the bipartition (A, B) where for $a \in A, b \in B$, a and b are adjacent in $G[A, B]$ if and only if they are adjacent in G. For a bipartite graph G with a bipartition (A, B), we say that a matrix M is a *bipartitie-adjacency* matrix of G, if the rows of M are indexed by A, the columns of M are indexed by B, and for $a \in A, b \in B$, $M_{a,b} = 1$ if a is adjacent to b in G, and $M_{a,b} = 0$ otherwise.

A tree is called *subcubic* if every internal node has exactly 3 neighbors. A tree T is called a *caterpillar* if contains a path P where for every vertex in T either it is in P or has a neighbor on P. A graph is called *chordal* if it contains no induced subgraph isomorphic to a cycle of length 4 or more. For a graph G, an ordering $v_1, \dots, v_n$ of the vertex set of G is called a *co-comparability ordering* if for every triple i, j, k with $i < j < k$, v_j has a neighbor in each path from v_i to v_k avoiding v_j. A graph is called a *co-comparability graph* if it admits a co-comparability ordering. The complete graph on n vertices will be denoted by K_n.

2.1 Sim-width

For a graph G, let $\mathrm{simval}_G : 2^{V(G)} \to \mathbb{N}$ be the function such that for $A \subseteq V(G)$, $\mathrm{simval}_G(A)$ is the maximum size of an induced matching $\{a_1b_1, a_2b_2, \dots, a_mb_m\}$ in G where $a_1, \dots, a_m \in A$ and $b_1, \dots, b_m \in V(G) \setminus A$. For a graph G, a pair (T, L) of a subcubic tree T and a function L from $V(G)$ to the set of leaves of T is called a *branch-decomposition*. For each edge e of T, let (A_1^e, A_2^e) be the vertex partition of G where T_1^e, T_2^e are the two connected components of $T - e$, and for each $i \in \{1, 2\}$, A_i^e is the set of all vertices in G mapped to leaves contained in T_i^e. We call it the vertex partition of G associated with e. For a branch-decomposition (T, L) of a graph G and an edge e in T, the *width* of e with respect to the simval_G function, denote by $\mathrm{simval}_{(T,L)}(e)$, is define as $\mathrm{simval}_G(A_1^e)$ where (A_1^e, A_2^e) is the vertex partition associated with e. The *width* of (T, L) with respect to the simval_G function is the maximum width over all edges in T. The *sim-width* of a graph G is the minimum width over all its branch-decompositions, and we denote it by $\mathrm{simw}(G)$. If T is a subcubic caterpillar tree, then (T, L) is called a *linear branch-decomposition*. The *linear sim-width* of a graph G is the minimum width over all its linear branch-decompositions, and we denote it by $\mathrm{lsimw}(G)$. When we consider sim-width, branch-decompositions are also called as sim-decompositions.

2.2 Other width parameters

A *tree-decomposition* of a graph G is a pair $(T, \mathcal{B} = \{B_t\}_{t \in V(T)})$ such that (1) $\bigcup_{t \in V(T)} B_t = V(G)$, (2) for every edge in G, there exists B_t containing both end vertices, and (3) for $t_1, t_2, t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever t_2 is on the path from t_1 to t_3. Each vertex subset B_t is called a *bag* of the tree-decomposition. The *width* of a tree-decomposition is $w - 1$ where w is the maximum size of bags in the decomposition, and the *tree-width* of a graph is the minimum width over all tree-decompositions of the graph.

For a graph G, we define two functions $\mathrm{mimval}_G : 2^{V(G)} \to \mathbb{N}$ and $\mathrm{cutrk}_G : 2^{V(G)} \to \mathbb{N}$ such that $\mathrm{mimval}_G(A)$ is the maximum size of an induced matching of $G[A, V(G) \setminus A]$, and $\mathrm{cutrk}_G(A)$ is the rank of the bipartite-adjacency matrix of $G[A, V(G) \setminus A]$ where the rank is computed over the binary field. For a branch-decomposition (T, L) of a graph G and $e \in E(T)$ and the vertex partition (A, B) of G associated with e, we define $\mathrm{cutrk}_{(T,L)}(e) := \mathrm{cutrk}_G(A)$, and $\mathrm{mimval}_{(T,L)}(e) := \mathrm{mimval}_G(A)$. The *rank-width*, and *mim-width* of a graph are defined in the same way as sim-width, with cutrk_G and mimval_G functions, respectively. The tree-width, rank-width, mim-width of a graph G are denoted by $\mathrm{tw}(G)$, $\mathrm{rw}(G)$, $\mathrm{mimw}(G)$, respectively.

▶ **Lemma 3.** *For a graph G, we have* $\mathrm{simw}(G) \le \mathrm{mimw}(G) \le \mathrm{rw}(G) \le \mathrm{tw}(G) + 1$.

Proof. Oum [18] proved that $\mathrm{rw}(G) \le \mathrm{tw}(G) + 1$. To show $\mathrm{mimw}(G) \le \mathrm{rw}(G)$, it is enough to show that for every branch-decomposition (T, L) of G and $e \in E(T)$, $\mathrm{mimval}_{(T,L)}(e) \le \mathrm{cutrk}_{(T,L)}(e)$. This holds since the size of an induced matching in $G[A_1, A_2]$ gives a lower bound on the rank of the bipartite-adjacency matrix of $G[A_1, A_2]$ where (A_1, A_2) is the vertex partition associated with e. By definition $\mathrm{simw}(G) \le \mathrm{mimw}(G)$ is clear. ◀

2.3 t-matching complete graphs

A *t-matching complete graph* is a graph that consists of two vertex sets $\{v_1, \ldots, v_t\}$ and $\{w_1, \ldots, w_t\}$ where between the two sets, $\{v_1w_1, \ldots, v_tw_t\}$ is an induced matching, and for two distinct $i, j \in \{1, \ldots, t\}$, either v_i is adjacent to v_j or w_i is adjacent to w_j. We say that $(\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\})$ is the *canonical bipartition* of the t-matching complete graph. Notice that every t-matching complete graph contains K_t as a minor. We will use this graph to arrive at an algorithmically useful subclass of graphs of bounded sim-width, still of unbounded rank-width and still closed under induced minors. The following observation will be useful.

▶ **Lemma 4.** *Let G be a t-matching complete graph with the canonical bipartition (A, B). Then either $G[A]$ is connected or $G[B]$ is connected.*

Proof. By the definition of a t-matching complete graph, $G[B]$ should contain a subgraph isomorphic to the complement of $G[A]$. The result follows from the fact that the complement of a disconnected graph is connected. ◀

3 Sim-width and contraction

Let us start by showing that the sim-width of a graph does not increase when taking an induced minor. This is one of the main motivations to consider this parameter.

▶ **Lemma 5.** *The sim-width of a graph does not increase when taking an induced minor.*

Proof. Clearly, the sim-width of a graph does not increase when removing a vertex. We prove for contractions.

Let G be a graph, $v_1v_2 \in E(G)$, and let (T, L) be a sim-decomposition of G of width w. For convenience, let the contracted vertex in G/v_1v_2 be called v_1. We claim that G/v_1v_2 admits a sim-decomposition of G of width at most w. We may assume that G has at least 3 vertices. For G/v_1v_2, we obtain a sim-decomposition (T', L') as follows:

- Let T' be the tree obtained from T by removing $L(v_2)$, and smoothing its neighbor. This neighbor of $L(v_2)$ has degree 3 in T because T is a subcubic tree and G has at least 3 vertices.
- Let L' be the function from $V(G/v_1v_2)$ to the set of leaves of T' such that $L'(w) = L(w)$ for $w \in V(G/v_1v_2) \setminus \{v_1\}$ and $L'(v_1) = L(v_1)$.

Let e_1 and e_2 be the two edges of T incident with the neighbor of $L(v_2)$, but not incident with $L(v_2)$. Let e_{cont} be the edge of T' obtained by smoothing.

▶ Claim 1. For each $e \in E(T')$, $\text{simval}_{(T',L')}(e) \le \text{simval}_{(T,L)}(e)$ if $e \in E(T) \setminus \{e_1, e_2\}$, and $\text{simval}_{(T',L')}(e_{cont}) \le \min(\text{simval}_{(T,L)}(e_1), \text{simval}_{(T,L)}(e_2))$.

Proof. Let $e \in E(T')$, and first assume that $e \in E(T) \setminus \{e_1, e_2\}$. Let (A, B) be the vertex partition of G/v_1v_2 associated with e. Without loss of generality, we may assume that $v_1 \in A$. Suppose there exists an induced matching $\{a_1b_1, \ldots, a_mb_m\}$ in G/v_1v_2 with $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$. Let (A', B') be the vertex partition of G associated with e. We will show that there is also an induced matching in G of same size between A' and B'.

We have either $A \cup \{v_2\} = A'$ and $B = B'$, or $A = A'$ and $B \cup \{v_2\} = B'$. If $v_1 \notin \{a_1, \ldots, a_m\}$, then $\{a_1b_1, \ldots, a_mb_m\}$ is also an induced matching between A' and B' in G. Without loss of generality, we may assume that $v_1 = a_1$.

Case 1. $A \cup \{v_2\} = A'$ **and** $B = B'$.

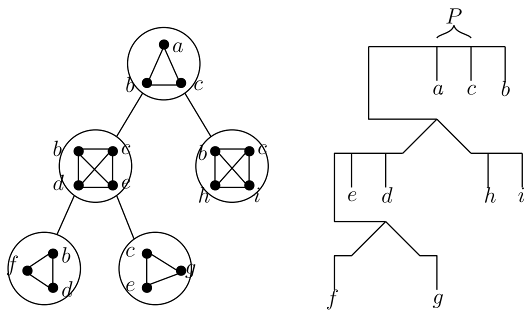

Figure 3 Constructing a sim-decomposition (T, L) of a chordal graph G of width at most 1 from its tree-decomposition. We denote the path P where the order of the assigned vertices in this part can be freely changed.

Proof. Note that in G, one of v_1 and v_2, say v', is adjacent to b_2. And also, v_1 and v_2 are not adjacent to any of $\{a_2, \ldots, a_m, b_1, \ldots, b_m\}$. Therefore, $\{v'b_1, a_2b_2, \ldots, a_mb_m\}$ is an induced matching in G between A' and B', as required. ◀

Case 2. $A = A'$ **and** $B \cup \{v_2\} = B'$**.**

Proof. If v_1 is adjacent to b_1 in G, then we have the same induced matching in G between A' and B'. Howver, v_1 is not necessary adjacent to b_1 in G. In this case, v_2 should be adjacent to b_1 in G. We now assume that v_1 is not adjacent to b_1 in G. In this case, $\{v_1v_2, a_2b_2, \ldots, a_mb_m\}$ is an induced matching between B_1 and B_2, because v_1 and v_2 are not adjacent to any of $\{a_2, \ldots, a_m, b_2, \ldots, b_m\}$. ◀

It shows that $\mathrm{simval}_{(T',L')}(e) \leq \mathrm{simval}_{(T,L)}(e)$ if $e \in E(T) \setminus \{e_1, e_2\}$. We can follow the same procedure to show that the same holds for e_{cont} as well. ◀

Claim **??** implies that the width of (T', L') is at most the width of (T, L). We conclude that $\mathrm{simw}(G/v_1v_2) \leq \mathrm{simw}(G)$. ◀

4 Sim-width of chordal graphs and co-comparability graphs

In this section, we show that *chordal graphs* and *co-comparability graphs* have sim-width at most 1, but have unbounded mim-width. Belmonte and Vatshelle [3] showed that chordal graphs either do not have constant mim-width or it is NP-complete to find such a decomposition. We strengthen their result.

4.1 Chordal graphs

For chordal graphs, we recursively construct a sim-decomposition of width at most 1. We use the fact that chordal graphs admit a tree-decomposition whose bags are maximal cliques.

▶ **Proposition 6.** Chordal graphs have sim-width at most 1. Given a chordal graph, one can output a sim-decomposition of width at most 1 in polynomial time.

Proof. Let G be a chordal graph. We may assume that G is connected. We compute a tree-decomposition $(F, \mathcal{B} = \{B_t\}_{t \in V(F)})$ of G where every bag induces a maximal clique of G. It is known that such a decomposition can be computed in polynomial time; for instance, see [**?**]. Let us choose a root node r of F, and for each node t, let F_t be the subtree of F induced on the union of all descendant nodes of t, and let G_t be the subgraph of G induced by the union of bags $B_{t'}$ where $t' \in V(F_t)$. We remark that for each node t, $(F_t, \{B_x\}_{x\in V(F_t)})$ can be regarded as a rooted tree-decomposition of G_t with root node t.

We recursively compute a sim-decomposition (T_t, L_t) of G_t of width at most 1 satisfying the following property:

- There are two internal nodes p and q of T_t where the path P from p to q in T_t has no branch nodes, and q is a node incident with two leaves,
- all vertices in B_t are assigned to leaves that are attached in a linear way on P, and
- any reordering of vertices of B_t also gives a sim-decomposition of G_t of width at most 1.

We describe such a sim-decomposition in Figure 3. If t is a leaf node, then G_t is a complete graph, and thus, we can take any linear sim-decomposition of G_t. We may assume that t is not a leaf node.

Let $t_1, \dots, t_m$ be the set of children of t in T. Note that for each $i \in \{1, \dots, m\}$, $V(B_{t_i}) \cap V(B_t) \neq \emptyset$ as G is connected.

By induction hypothesis, for each $i \in \{1, \dots, m\}$, there exists a sim-decomposition (T_i, L_i) of G_{t_i} of width at most 1 satisfying the required property. We reorder the vertices of B_{t_i} so that the vertices in $B_{t_i} \cap B_t$ appear at the last part of the ordering of B_{t_i}. We obtain a new tree T_i' by contracting the minimal subtree connecting vertices of $B_{t_i} \cap B_t$ into one vertex x_i, and call it a modified tree.

Now, we obtain a sim-decomposition (T, L) as follows.

1. Let $Q := q_1 q_2 \cdots q_{m+|V(B_t)|}$ be a path. Let T be the tree obtained from Q by adding a leaf r_i to q_i for each $i \in \{1, \dots, |V(B_t)|\}$, and adding a modified tree T_i' and identifying x_i with $q_{|B_t|+i}$ for each $i \in \{1, \dots, m\}$, and smoothing degree 2 nodes q_1 and $q_{m+|V(B_t)|}$.
2. Let L be the function from the set of leaves of T to $V(G_t)$ such that $L|_{B_t}$ is a bijective function from $\{r_1, \dots, r_{|B_t|}\}$ to B_t, and $L(v) = L_i(v)$ for all vertices v in T_i'.

We can easily check that (T, L) has width 1 as the root bag induces a clique, and thus it forbid having two induced matchings. The second statement also holds as we take any ordering when constructing the decomposition. Note that we can update the sim-decomposition (T, L) of G_t in linear time. Therefore, we can construct a sim-decomposition of width at most 1 for a chordal graph in polynomial time. ◀

We now prove the lower bound on the mim-width of chordal graphs. We in fact show this for the class of split graphs that is a subclass of chordal graphs. A *split* graph is a graph that can be partitioned into two vertex sets C and I where C is a clique and I is an independent set. The Sauer-Shelah lemma [19, 20] is essential in the proof.

▶ **Theorem 7** (Sauer-Shelah lemma [19, 20]). *Let t be a positive integer and let M be an $X \times Y$ $(0,1)$-matrix such that any two row vectors of M are distinct. If $|X| \geq |Y|^t$, then there are $X' \subseteq X$, $Y' \subseteq Y$ such that $|X'| = 2^t$, $|Y'| = t$, and all possible row vectors of length t appear in $M[X', Y']$.*

▶ **Proposition 8.** For every large enough n, there is a split graph on n vertices having mim-width at least $\sqrt{\log_2 \frac{n}{2}}$.

Proof. Let $m \geq 10000$ be an integer and let $n := m + (2^m - 1)$. Let G be a split graph on the vertex partition (C, I) where C is a clique of size m, I is an independent set of size $2^m - 1$, and all vertices in I have pairwise distinct non-empty neighborhoods on C. We claim that every branch-decomposition of G has width at least $\sqrt{\log_2 \frac{n}{2}}$ with respect to the mimval$_G$ function.

Let (T, L) be a branch-decomposition of G. It is well known that there is an edge of T inducing a balanced vertex partition, but we add a short proof for it. We subdivide an edge of T, and regard the new vertex as a root node. For each node $t \in V(T)$, let $\mu(t)$ be the number of leaves of T that are descendants of t. Now, we choose a node t that is farthest from the root node such that $\mu(t) > \frac{n}{3}$. By the choice of t, for each child t' of t, $\mu(t') \leq \frac{n}{3}$. Therefore, $\frac{n}{3} < \mu(t) \leq \frac{2n}{3}$. Let e be the edge connecting the node t and its parent. Then clearly, the vertex partition (A_1, A_2) of G induced by the edge e satisfies that for each $i \in \{1, 2\}$, $\frac{n}{3} < |A_i| \leq \frac{2n}{3}$. Without loss of generality, we may assume that $|A_1 \cap C| \geq |A_2 \cap C|$, and thus we have $\frac{m}{2} \leq |A_1 \cap C| \leq m$.

Note that $|A_2 \cap I| > \frac{n}{3} - m \geq \frac{2^m - 2m - 1}{3} \geq 2^{m-3}$. Since $|A_2 \cap C| < \frac{m}{2}$ and $m \geq 8$, there are at least

$$\frac{2^{m-3}}{2^{\frac{m}{2}}} \geq 2^{\frac{m}{2}-3}$$

vertices in $A_2 \cap I$ that have pairwise distinct neighbors on $A_1 \cap C$. Let $I' \subseteq A_2 \cap I$ be the set of such vertices.

Now, by the Sauer-Shelah lemma, if $|I'| \geq |A_1 \cap C|^k$ for some positive integer k, then there will be an induced matching of size k between $A_1 \cap C$ and I' in $G[A_1, A_2]$. We choose $k := \sqrt{m}$. As $m \geq 10000$, we can deduce that $\frac{m}{2} - 3 \geq \sqrt{m} \log_2 m$. Therefore, we have

$$|I'| \geq 2^{\frac{m}{2}-3} \geq m^{\sqrt{m}} \geq |A_1 \cap C|^{\sqrt{m}},$$

and there is an induced matching of size $\sqrt{m}$ between $A_1 \cap C$ and I' in $G[A_1, A_2]$. It implies that $\mathrm{mimval}_{(T,L)}(e) \geq \sqrt{m}$. As (T, L) was chosen arbitrary, the mim-width of G is at least $\sqrt{m} \geq \sqrt{\log_2 \frac{n}{2}}$. ◀

4.2 Co-comparability graphs

We observe the same properties for co-comparability graphs. We recall that co-comparability graphs are exactly graphs that admit a co-comparability ordering.

▶ **Theorem 9** (McConnell and Spinrad [16]). *Given a co-comparability graph G, one can output a co-comparability ordering in polynomial time.*

▶ **Proposition 10.** Co-comparability graphs have linear sim-width at most 1. Given a co-comparability graph, one can output a linear sim-decomposition of width at most 1 in polynomial time.

Proof. Let G be a co-comparability graph. Using Theorem 8, we can obtain its co-comparability ordering $v_1, \ldots, v_n$. From this, we take a linear branch-decomposition (T, L) following the sequence. We claim that for each $i \in \{2, \ldots, n-1\}$, there is no induced matching of size 2 between $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$. Suppose there are $i_1, i_2 \in \{1, \ldots, i\}$ and $j_1, j_2 \in \{i+1, \ldots, n\}$ such that $\{v_{i_1}v_{j_1}, v_{i_2}v_{j_2}\}$ is an induced matching. Without loss of generality we may assume that $i_1 < i_2$. Then we have $i_1 < i_2 < j_1$, and thus by the definition of the co-comparability ordering, v_{i_2} should be adjacent to one of v_{i_1} and v_{j_1}, which contradicts to our assumption. Therefore, there is no induced matching of size 2. It implies that (T, L) has width at most 1. ◀

To show that co-comparability graphs have unbounded mim-width, we provide a grid-like structure. For positive integers p, q, the $(p \times q)$ *column-clique grid* is the graph on the vertex set $\{v_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ where

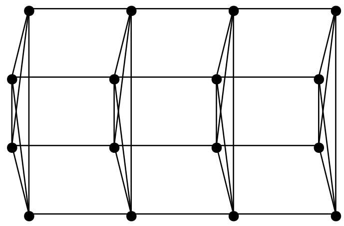

Figure 4 The (4×4) column-clique grid.

- for every $i \in \{1, \dots, q\}$, $\{v_{1,j}, \dots, v_{p,j}\}$ is a clique,
- for every $i \in \{1, \dots, p\}$ and $j_1, j_2 \in \{1, \dots, q\}$, v_{i,j_1} is adjacent to v_{i,j_2} if and only if $|j_2 - j_1| = 1$,
- for $i_1, i_2 \in \{1, \dots, p\}$ and $j_1, j_2 \in \{1, \dots, q\}$, v_{i_1,j_1} is not adjacent to v_{i_2,j_2} if neither $i_1 \neq i_2$ nor $j_1 \neq j_2$.

We depict an example in Figure ??. For each $1 \leq i \leq p$, we call $\{v_{i,1}, \dots, v_{i,h}\}$ *the i-th row* of G, and define its columns similarly.

▶ **Lemma 11.** *For integers $p, q \geq 12$, the $(p \times q)$ column-clique grid has mim-width at least* $\min(\frac{p}{4}, \frac{q}{3})$.

Proof. Let G be the $(p \times q)$ column-clique grid. Suppose that G has a branch-decomposition of width at most d with respect to the mimval_G function, for some positive integer d. It is enough to show that $d \geq \min(\frac{p}{4}, \frac{q}{3})$.

Firstly, assume that for each row R of G, $R \cap A \neq \emptyset$ and $R \cap B \neq \emptyset$. Then there is an edge between $R \cap A$ and $R \cap B$, as $G[R]$ is connected. For each i-th row R_i, we choose a pair of vertices $v_{i,a_i} \in R \cap A$ and $v_{i,b_i} \in R \cap B$ that are adjacent. We know that there is an index subset $X \subseteq \{1, \dots, p\}$ such that $|X| \geq \frac{p}{2}$ and every pair (v_{i,a_i}, v_{i,b_i}) in $\{(v_{i,a_i}, v_{i,b_i}) : i \in X\}$ satisfies that $a_i + 1 = b_i$. By taking the same parity of a_i's, we know there is an index subset $Y \subseteq \{1, \dots, p\}$ such that $|Y| \geq \frac{p}{4}$, all integers in $\{a_i : i \in Y\}$ have the same parity, and every pair (v_{i,a_i}, v_{i,b_i}) in $\{(v_{i,a_i}, v_{i,b_i}) : i \in Y\}$ satisfies that $a_i + 1 = b_i$.

We observe that $\{v_{i,a_i} v_{i,b_i} : i \in Y\}$ is an induced matching in $G[A, B]$. If it is not, then there are distinct integers $y, z \in Y$ such that either v_{y,a_y} is adjacent to v_{z,b_z}, or v_{y,b_y} is adjacent to v_{z,a_z}. But this is not possible; for instance, if v_{y,a_y} is adjacent to v_{z,b_z}, then $a_y = b_z$, and we have $a_y = a_z + 1$ as $z \in Y$. However, it contradicts to our assumption that all integers in $\{a_i : i \in Y\}$ have the same parity. Therefore, we conclude that $G[A, B]$ contains an induced matching of size at least $\frac{p}{4}$.

Now, we assume that there exists a row R such that R is fully contained in one of A and B. Without loss of generality, we may assume that R is contained in A. Since $|B| > \frac{|V(G)|}{3}$, we can choose an index set $X \subseteq \{1, \dots, q\}$ such that $|X| > \frac{q}{3}$ and for each $i \in X$, the i-th column contains a vertex of B. For each i-th column where $i \in X$, we choose a vertex $v_{a_i,i}$ in B. It is not hard to verify that the edges between $\{v_{a_i,i} : i \in X\}$ and the rows in R form an induced matching of size $\frac{q}{3}$ in $G[A, B]$.

Therefore, we have $d \geq \min(\frac{p}{4}, \frac{q}{3})$. ◀

▶ **Corollary 12.** *For every large enough n, there is a co-comparability graph on n vertices having mim-width at least $\sqrt{\frac{n}{12}}$.*

Proof. Let $p \geq 4$ be an integer, and let $n := 12p^2$. Let G be the $(4p \times 3p)$ clique-grid graph. It is not hard to see that

$$v_{1,1}, v_{2,1}, \dots, v_{4p,1}, v_{1,2}, v_{2,2}, \dots, v_{4p-1,3p}, v_{4p,3p}$$

is a co-comparability ordering. Thus, G is a co-comparability graph. By Lemma **??**, $\text{mimw}(G) \geq p = \sqrt{\frac{n}{12}}$. ◀

5 Excluding t-matching complete graphs

In the previous section, we proved that graphs of sim-width at most 1 contain all chordal graphs. A classical result on chordal graphs is that the problem of finding a minimum dominating set in a chordal graph is NP-complete [4]. So, even for this kind of locally-checkable problem, we cannot expect efficient algorithms on graphs of sim-width at most w. Therefore, to obtain a meta-algorithm for graphs of bounded sim-width encompassing many locally-checkable problems, we must impose some restrictions. We approach this problem in a way analogous to what has previously been done in the realm of rank-width [11].

It is well known that complete graphs have rank-width at most 1, but they have unbounded tree-width. Fomin, Oum, and Thilikos [11] showed that if G is K_r-minor free, then the tree-width of a graph is bounded by $c \cdot \text{rw}(G)$ where c is a constant depending on r. This can be utilized algorithmically, to get a result for graphs of bounded rank-width when excluding a fixed minor, as the class of problems solvable in FPT time is strictly larger when parameterized by tree-width than rank-width [15].

We will do something similar by focusing on the distinction between mim-width and sim-width. However, K_r-minor free graphs are too strong, as one can show that on K_r-minor free graphs, the tree-width of a graph is also bounded by some constant factor of its sim-width. To see this, one can use Lemma 5 and the result on contraction obstructions for graphs of bounded tree-width [10].

Vatshelle [22], in his Ph.D Thesis, developed a way of obtaining XP-algorithms on graphs of bounded mim-width for locally checkable problems such as the MINIMUM DOMINATING SET problem. We will recall such problems formally. In the same spirit of work as Fomin, Oum, and Thilikos [11], we try to bound the mim-width of a graph in terms of a function of sim-width by excluding a certain configuration. On the other hand, we do not want to destroy, by excluding this certain configuration, the property that the class is closed under taking induced minors. Based on this idea, t-matching complete graphs naturally came up.

▶ **Proposition** 13. Every graph with sim-width w and no induced minor isomorphic to a t-matching complete graph has mim-width at most $(4w + 2)t^2$.

We use the following result. Notice that the optimal bound of Theorem 12 has been slightly improved by Fox [12], and then by Balogh and Kostochka [1].

▶ **Theorem 14** (Duchet and Meyniel [8]). *For positive integers k and n, every n-vertex graph contains either an independent set of size k or a K_t-minor where $t \geq \frac{n}{2k-1}$.*

Proof of Proposition 11. Let G be a graph with sim-width w and no induced minor isomorphic to a t-matching complete graph. Let (T, L) be a branch-decomposition of G of width w with respect to the simval_G function. We claim that for each edge e of T, $\text{mimval}_{(T,L)}(e) \leq 2(2w+1)t^2 - t - 1$. It implies that G has mim-width at most $2(2w+1)t^2 - t - 1 \leq (4w+2)t^2$.

Let $e \in E(T)$, and let (A, B) be the vertex partition of G associated with e. Suppose for contradiction that there is an induced matching $\{v_1w_1, \ldots, v_mw_m\}$ in $G[A, B]$ where $v_1, \ldots, v_m \in A$, $w_1, \ldots, w_m \in B$, and $m \geq 2(2w + 1)t^2 - t$. Let f be the function from $\{v_1, \ldots, v_m\}$ to $\{w_1, \ldots, w_m\}$ such that $f(v_i) = w_i$ for each $i \in \{1, \ldots, m\}$. As $m \geq 2(2w+1)t^2 - t$, by Theorem 12, the subgraph $G[\{v_1, \ldots, v_m\}]$ contains either an independent set of size $(2w + 1)t$, or a K_t-minor.

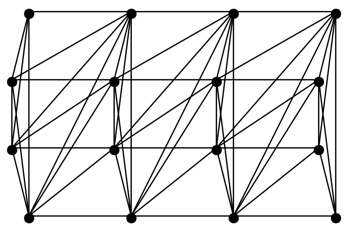

Figure 5 The (4×4) Hsu-clique chain graph.

Suppose that $G[\{v_1, \dots, v_m\}]$ contains a K_t-minor. Thus, there exist pairwise disjoint subsets $S_1, \dots, S_t$ of $\{v_1, \dots, v_m\}$ such that

- for each $i \in \{1, \dots, t\}$, $G[S_i]$ is connected,
- for two distinct integers $i, j \in \{1, \dots, t\}$, there is an edge between S_i and S_j.

From this, we can obtain a t-matching complete graph by contracting each set S_i to a vertex, and taking one vertex among vertices in each set $f(S_i)$. It contradicts to the assumption that G contains no t-matching complete graphs as an induced minor. Thus, $G[\{v_1, \dots, v_m\}]$ contains an independent set of size $(2w+1)t$. Let $\{v_{i_1}, v_{i_2}, \dots, v_{i_{(2w+1)t}}\}$ be an independent set in $G[\{v_1, \dots, v_m\}]$.

Now, by applying Theorem 12 again, $G[\{w_{i_1}, w_{i_2}, \dots, w_{i_{(2w+1)t}}\}]$ contains an independent set of size $w+1$ or a K_t-minor. If it has an independent set of size $w+1$, then we have $\mathrm{simval}_{(T,L)}(e) \geq w+1$ which contradicts to our assumption. If it contains a K_t-minor, then by the same argument in the previous paragraph, we can find a t-matching complete graph as an induced minor, which also contradicts to our assumption. Thus, we conclude that $\mathrm{mimval}_{(T,L)}(e) \leq 2(2w+1)t^2 - t - 1$, as required. ◀

One might wonder whether the class of graphs of sim-width at most k and having no induced minor isomorphic to a t-matching complete graph falls into a class of graphs of bounded rank-width or tree-width. We confirm that this is not true, by showing that Hsu-clique chain graphs in Figure 4 are chordal, but do not contain any 3-matching complete graph as an induced minor. Belmonte and Vatshelle showed that a $(p \times q)$ Hsu-clique chain graph has rank-width at least $\frac{p}{3}$ [3, Lemma 16] when $q = 3p+1$. So, our algorithmic applications based on Proposition 11 are beyond algorithmic applications of graphs of bounded tree-width or rank-width.

We formally define Hsu-clique chain graphs. For positive integers p, q, the $(p \times q)$ *Hsu-clique chain grid* is the graph on the vertex set $\{v_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ where

- for every $i \in \{1, \dots, q\}$, $\{v_{1,j}, \dots, v_{p,j}\}$ is a clique
- for every $i_1, i_2 \in \{1, \dots, p\}$ and $j \in \{1, \dots, q-1\}$, $v_{i_1,j}$ is adjacent to $v_{i_2,j+1}$ if and only if $i_1 \leq i_2$,
- for $i_1, i_2 \in \{1, \dots, p\}$ and $j_1, j_2 \in \{1, \dots, q\}$, v_{i_1,j_1} is not adjacent to v_{i_2,j_2} if $|j_1 - j_2| > 1$.

► **Proposition 15.** The class of graphs of sim-width 1 and having no induced minor isomorphic to a 3-matching complete graph has unbounded rank-width.

Proof. Let p be a positive integer and $q := 3p+1$. Let G be a $(p \times q)$ Hsu-clique chain graph. Belmonte and Vatshelle showed that a $(p \times q)$ Hsu-clique chain graph has rank-width at least $\frac{p}{3}$ [3, Lemma 16]. It is not hard to see that this graph is chordal, and thus it has sim-width at most 1 by Proposition 6. Now, we claim that G has no induced minor isomorphic to a 3-matching complete graph. Let H be a 3-matching complete graph with canonical bipartition $(\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\})$ where $\{v_1w_1, v_2w_2, v_3w_3\}$ is the induced matching in

$H[\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\}]$. For contradiction, suppose that G contains an induced minor isomorphic to H.

We observe that the class of chordal graphs is closed under taking induced minors. And later, we will show in Proposition 16 that every chordal t-matching complete graph is a t-matching complete graph with the canonical bipartition (A, B) where A is a clique and B is an independent set. Therefore, without loss of generality, we may assume that $\{v_1, v_2, v_3\}$ is a clique and $\{w_1, w_2, w_2\}$ is an independent set in H.

Since G contains H as an induced minor, there is a mapping μ from $V(H)$ to $2^{V(G)}$ where

- $\{\mu(v) : v \in V(H)\}$ are pairwise disjoint vertex subsets of G, and each set in $\{\mu(v) : v \in V(H)\}$ induces a connected subgraph of G,
- for two distinct vertices $v, w \in V(H)$, $vw \in E(H)$ if and only if there is an edge between $\mu(v)$ and $\mu(w)$.

For each $v \in V(H)$, let $I_v = \{i : v_{i,j} \in \mu(v)\}$. For convenience, we say that a set I of integers is called an interval, if all integers in I consecutively appear; for instance, $\{5, 6, 7, 8\}$ is an interval. For an non-empty interval I, $\mathbb{Z} \setminus I$ is divided into two parts of consecutive integers. Let us denote by $L(I)$ the set of all integers that are smaller than integers in I, and let us denote by $R(I)$ the set of all integers that are larger than integers in I. Notice that for two distinct integers $i_1, i_2 \in \{1, 2, 3\}$, $\mu(v_{i_1}) \cup \mu(v_{i_2})$ is an interval, as there is an edge between $\mu(v_{i_1})$ and $\mu(v_{i_2})$.

We claim that one of intervals $I_{v_1}, I_{v_2}, I_{v_3}$ is contained in the union of two others. Suppose that this is not true. If one of I_{v_1} and I_{v_2} is contained in the other, then we have a contradiction. We may assume that I_{v_2} contains an integer in exactly one part of $L(I_{v_1})$ and $R(I_{v_1})$. Without loss of generality we may assume that I_{v_2} contains an integer in $R(I_{v_1})$, but does not contain an integer in $L(I_{v_1})$. Clearly I_{v_1} does not contain an integer in $R(I_{v_2})$.

Let $I := I_{v_1} \cup I_{v_2}$. Since I_{v_3} is not contained in I, I_{v_3} should contain an integer in $L(I)$ or $R(I)$, and it should contain an integer from exactly one of them, otherwise, I will be contained in I_{v_3}, which is not possible by assumption. If I_{v_3} contains an integer in $L(I)$, then I_{v_1} is contained in $I_{v_2} \cup I_{v_3}$. If I_{v_3} contains an integer in $R(I)$, then I_{v_2} is contained in $I_{v_1} \cup I_{v_3}$. Both cases lead a contradiction. So, we conclude the claim.

Now, without loss of generality, we assume that $I_{v_2} \subseteq I_{v_1} \cup I_{v_3}$. Let $I := I_{v_1} \cup I_{v_3}$. Let x be the smallest integer in I, and y be the largest integer in I. Let $z \in \{v_1, v_2, v_3\}$ be the vertex where $\mu(z)$ contains a vertex in the x-th column, and there are no other vertex in $\{v_1, v_2, v_3\} \setminus \{z\}$ containing a higher vertex in the column. Similarly, let $z' \in \{v_1, v_2, v_3\}$ be the vertex where $\mu(z')$ contains a vertex in the y-th column, and there are no other vertex in $\{v_1, v_2, v_3\} \setminus \{z'\}$ containing a lower vertex in the column. It is not hard to see that the possible positions of $\mu(w_1), \mu(w_2), \mu(w_3)$ is left-hand side of $\mu(z)$ or right-hand side of $\mu(z')$, otherwise, it will be adjacent to a vertex that are not matched with the selected vertex in $\{w_1, w_2, w_3\}$.

It shows that G has no induced minor isomorphic to a 3-matching complete graph. ◀

6 Algorithms for LC-VSVP problems

In this section, we describe algorithmic applications for graphs of bounded sim-width having no t-matching complete graph as an induced minor. Telle and Proskurowski [21] classified a class of problems called *Locally Checkable Vertex Subset and Vertex Partitioning problems*, which is a subclass of MSO_1 problems. These problems generalize problems like Maximum Independent Set, Minimum Dominating Set, q-Coloring etc.

Let σ, ρ be finite or co-finite subsets of natural numbers. For a graph G and $S \subseteq V(G)$, we call S a (σ, ρ)*-set* of G if

- for every $v \in S$, $|N_G(v) \cap S| \in \sigma$, and
- for every $v \in V(G) \setminus S$, $|N_G(v) \cap S| \in \rho$.

For instance, a $(0, \mathbb{N})$-set is an independent set as there are no edges inside of the set, and we do not care about adjacency between S and $V(G) \setminus S$. Another example is that a $(\mathbb{N}, \mathbb{N}^+)$-set is a dominating set as we require that for each vertex in $V(G) \setminus S$, it has at least one neighbor in S. The class of *locally checkable vertex subset problems* consist of finding a minimum or maximum (σ, ρ)-set in an input graph G, and possibly on vertex-weighted graphs.

For a positive integer q, a $(q \times q)$-matrix D_q is called a *degree constraint* matrix if each element is either a finite or co-finite subset of natural numbers. A partition $\{V_1, V_2, \dots, V_q\}$ of the vertex set of a graph G is called a D_q*-partition* if

- for every $i, j \in \{1, \dots, q\}$ and $v \in V_i$, $|N_G(v) \cap V_j| \in D_q[i, j]$.

For instance, if we take a matrix D_q where all diagonal entries are 0, and all other entries are $\mathbb{N}$, then a D_q-partition is a partition into q independent sets, which corresponds to a q-coloring of the graph. The class of *locally checkable vertex partitioning problems* consist of deciding if G admits a D_q-partition.

All these problems will be called *Locally Checkable Vertex Subset and Vertex Partitioning problems*, shortly LC-VSVP problems. As shown in [6] the runtime solving an LC-VSVP problem by dynamic programming relates to the finite or co-finite subsets of natural numbers used in its definition. The following function d is central.

1. Let $d(\mathbb{N}) = 0$.
2. For every finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max\{x \in \mathbb{N} : x \in \mu\}, \max\{x \in \mathbb{N} : x \notin \mu\})$.

▶ **Theorem 16** (Belmonte and Vatshelle [3] and Bui-Xuan, Telle, and Vatshelle [6])**.** *Given an n-vertex graph and its branch-decomposition (T, L) of mim-width w we solve*

- *any (σ, ρ)-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $\mathcal{O}(n^{3dw+4})$,*
- *any D_q-vertex partitioning problem with $d = \max_{i,j} d(D_q[i, j])$ in time $\mathcal{O}(qn^{3dwq+4})$.*

Combining Theorem 14 with Proposition 11 we get the following.

▶ **Corollary 17.** *Given an n-vertex graph having no t-matching complete graph as an induced minor and its branch-decomposition (T, L) of sim-width w, we solve*

- *any (σ, ρ)-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $\mathcal{O}(n^{6d(2w+1)t^2+4})$,*
- *any D_q-vertex partitioning problem with $d = \max_{i,j} d(D_q[i, j])$ in time $\mathcal{O}(qn^{6d(2w+1)qt^2+4})$.*

For example, for Minimum Dominating Set and q-Coloring we plug in $d = 1$ since $\max(d(\mathbb{N}), d(\mathbb{N}^+)) = 1$ and $\max(d(0), d(\mathbb{N})) = 1$.

7 Finding t-matching complete graphs in chordal graphs and co-comparability graphs

We observe structural properties of t-matching complete graphs in chordal graphs or co-comparability graphs. We say that a t-matching complete graph with canonical bipartition (A, B) is of *type 1* if A is a clique and B is an independent set, and it is of *type 2* if A is a clique and B is one clique or the disjoint union of two cliques.

▶ **Proposition** 18. Let t be a positive integer.

1. Every chordal t-matching complete graph is a t-matching complete graph of type 1.

2. Every co-comparability t-matching complete graph is a t-matching complete graph of type 2.

Proof. (1) Let G be a chordal graph that is a t-matching complete graph with canonical bipartition $(\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\})$ where $\{v_1w_1, \ldots, v_tw_t\}$ is an induced matching. By Lemma 4, we know that one of vertex sets $\{v_1, \ldots, v_t\}$ and $\{w_1, \ldots, w_t\}$ induces a connected graph. Without loss of generality, we may assume that $G[\{v_1, \ldots, v_t\}]$ is connected. We claim that $G[\{w_1, \ldots, w_t\}]$ has no edges.

Suppose there is an edge w_iw_j for some $i, j \in \{1, \ldots, t\}$. We choose a shortest path P from v_i to v_j in $G[\{v_1, \ldots, v_t\}]$. Since $G[\{v_1, \ldots, v_t\}]$ is connected, we can find such a path. As w_i and w_j are not adjacent to vertices in $\{v_1, \ldots, v_t\}$ other than v_i and v_j, $G[V(P) \cup \{w_i, w_j\}]$ is an induced cycle of length at least 4. This contradicts to the fact that G is chordal. Therefore, $\{w_1, \ldots, w_t\}$ is an independent set, and it implies that $\{v_1, \ldots, v_t\}$ is a clique.

(2) Let G be a co-comparability t-matching complete graph with canonical bipartition $(\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\})$ where $\{v_1w_1, \ldots, v_tw_t\}$ is an induced matching. Let $V_1 := \{v_1, \ldots, v_t\}$ and $V_2 := \{w_1, \ldots, w_t\}$. We claim that for each $i \in \{1, 2\}$, V_i has no independent set of size 3, and it also has no induced path of length 2. It will imply that each V_i consists of at most two cliques.

Let us fix $i \in \{1, 2\}$. Suppose for contradiction that V_i has an independent set of size 3. Then they are matched with three vertices on the other part that form a clique by the definition of t-matching complete graphs. However, in this case, the independent set is an asteroidal triple. So, it is contradiction.

Suppose that V_i has an induced path of length 3. Then two end vetices of this path will be matched with two vertices on the other part that are adjacent. Therefore, it creates an induced cycle of length 5. However, co-comparability graphs have no induced subgraph isomorphic to C_5.

So, we can conclude that each V_i consists of at most two cliques. By Lemma 4, we know that one of vertex sets $\{v_1, \ldots, v_t\}$ and $\{w_1, \ldots, w_t\}$ induces a connected graph. Therefore, one part is a clique, and the other part consists of at most 2 cliques, as required. ◀

Corollary 15 assumed that the given graph has no t-matching complete graph as an induced minor, and also its sim-decomposition of width at most w is given. For chordal graphs and co-comparability graphs, we can produce a sim-decomposition of width at most 1 in polynomial time using Propositions 6 and 9. To test whether a chordal or co-comparability graph contains a t-matching complete graph as an induced minor, we can use known XP algorithms [2, 14]. Notice that for fixed graph H, testing H-induced minor in general graphs is known to be NP-complete [9]. By Proposition 16, there are few candidates for t-matching complete graphs; there is a unique chordal t-matching complete graph, and there are t pairwise non-isomorphic co-comparability t-matching complete graphs. Thus, the runtime will not be exponentially multiplied from the number of possible t-matching complete graphs.

8 Concluding remarks

In this paper, we show that every graph with sim-width at most w and having no induced minor isomorphic to a t-matching complete graph has mim-width at most $(4w+2)t^2$, and every LC-VSVP problem can be solved in time $n^{\mathcal{O}(wt^2)}$ on such n-vertex graphs, when its branch-decomposition is given. Also, polynomial-time algorithms to find branch-decompositions certifying sim-width at most 1 for chordal graphs and co-comparability graphs are provided.

It is worth noting that there remains a number of interesting open problems about this new parameter. We would like to find more classes that have constant sim-width, but unbounded value for tree-width, rank-width, or mim-width. We propose some possible classes, that are also presented in Figure 1.

▶ Question 1. Do weakly chordal graphs, AT-free graphs, or circle graphs have constant sim-width?

In the line of this research on sim-width and mim-width, one of main problems is to find an efficient algorithm to find a branch-decomposition of relevant width. As we see, we obtain general XP algorithms, and thus it make sense to ask whether there is an XP-algorithm to find a branch-decomposition. We remark that simval_G is not a submodular function.

▶ Question 2. Is there a function f such that, given a graph G, we can in XP time parameterized by $k = \mathrm{simw}(G)$ (or $k = \mathrm{mimw}(G)$) compute a branch-decomposition of width at most $f(k)$ with respect to the simval_G function (or the mimval_G function)?

Probably, most interesting property of sim-width is that the class of graphs of sim-width at most w is closed under taking induced minors. We could try to characterize classes of graphs of bounded sim-width in terms of forbidden induced minors. We remark that interval graphs are not well-quasi-ordered by the induced minor operation [7], and thus, graphs of bounded sim-width are not well-quasi-ordered by the induced minor operation in general. So, we even do not know whether there is a finite list of obstructions for graphs of bounded sim-width at most 1.

▶ Question 3. What are the induced minor obstructions for the class of graphs of sim-width at most 1? For fixed ℓ, what are the induced minor obstructions for the class of graphs of sim-width at most 1 and having no ℓ-matching complete graphs as an induced minor?

The last question is about finding a certain induced minor in a graph of sufficiently large sim-width. For the minor operation, planar graphs satisfy that for fixed planar graph, every graph of sufficiently large tree-width contains it as a minor. It would be interesting to know whether we can get some grid-like structure from graphs of sufficiently large sim-width as an induced minor.

▶ Question 4. Is there any non-trivial graph class $\mathcal{C}$ such that every graph of sufficiently large sim-width contain a graph in $\mathcal{C}$ of certain size as an induced minor? In particular, can $\mathcal{C}$ be the class of planar graphs?

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