

Generalized distance domination problems and their complexity on graphs of bounded mim-width

Lars Jaffke^{*1}, O-joung Kwon^{†2}, Torstein J.F. Strømme¹, and Jan Arne Telle¹

¹Department of Informatics, University of Bergen, Norway.

{lars.jaffke, torstein.stromme, jan.arne.telle}@uib.no

²Department of Mathematics, Incheon National University, South Korea.

ojoungkwon@gmail.com

July 6, 2018

Abstract

We generalize the family of (σ, ρ) -problems and locally checkable vertex partition problems to their distance versions, which naturally captures well-known problems such as distance- r dominating set and distance- r independent set. We show that these distance problems are XP parameterized by the structural parameter *mim-width*, and hence polynomial on graph classes where mim-width is bounded and quickly computable, such as k -trapezoid graphs, Dilworth k -graphs, (circular) permutation graphs, interval graphs and their complements, convex graphs and their complements, k -polygon graphs, circular arc graphs, complements of d -degenerate graphs, and H -graphs if given an H -representation. To supplement these findings, we show that many classes of (distance) (σ, ρ) -problems are W[1]-hard parameterized by mim-width + solution size.

1 Introduction

Telle and Proskurowski [20] defined the (σ, ρ) -domination problems, and the more general *locally checkable vertex partitioning* problems (LCVP). In (σ, ρ) -domination problems, feasible solutions are vertex sets with constraints on how many neighbours each vertex of the graph has in the set. The framework generalizes important and well-studied problems such as MAXIMUM INDEPENDENT SET and MINIMUM DOMINATING SET, as well as PERFECT CODE, MINIMUM SUBGRAPH WITH MINIMUM DEGREE d and a multitude of other problems. See Table 1. Bui-Xuan, Telle and Vatshelle [6] showed that (σ, ρ) -domination and locally checkable vertex partitioning problems can be solved in time XP parameterized by *mim-width*, if we are given a corresponding decomposition tree. Roughly speaking, the structural parameter mim-width measures how easy it is to decompose a graph along vertex cuts inducing a bipartite graph with small maximum induced matching size [21].

^{*}Supported by the Bergen Research Foundation (BFS).

[†]Partially supported by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (ERC consolidator grant DISTRUCT, agreement No. 648527).

In this paper, we consider distance versions of problems related to independence and domination, like DISTANCE- r INDEPENDENT SET and DISTANCE- r DOMINATING SET. The DISTANCE- r INDEPENDENT SET problem, also studied under the names r -SCATTERED SET and r -DISPERSION (see e.g. [2] and the references therein), asks to find a set of at least k vertices whose vertices have pairwise distance strictly longer than r . Agnarsson et al. [1] pointed out that it is identical to the original INDEPENDENT SET problem on the r -th power graph G^r of the input graph G , and also showed that for fixed r , it can be solved in linear time for interval graphs, and circular arc graphs. The DISTANCE- r DOMINATING SET problem was introduced by Slater [19] and Henning et al. [13]. They also discussed that it is identical to solve the original DOMINATING SET problem on the r -th power graph. Slater presented a linear-time algorithm to solve DISTANCE- r DOMINATING SET problem on forests.

We generalize all of the (σ, ρ) -domination and LCVp problems to their distance versions, which naturally captures DISTANCE- r INDEPENDENT SET and DISTANCE- r DOMINATING SET. Where the original problems put constraints on the size of the immediate neighborhood of a vertex, we consider the constraints to be applied to the ball of radius r around it. Consider for instance the MINIMUM SUBGRAPH WITH MINIMUM DEGREE d problem; where the original problem is asking for the smallest (number of vertices) subgraph of minimum degree d , we are instead looking for the smallest subgraph such that for each vertex there are at least d vertices at distance at least 1 and at most r . In the PERFECT CODE problem, the target is to choose a subset of vertices such that each vertex has exactly one chosen vertex in its closed neighbourhood. In the distance- r version of the problem, we replace the closed neighbourhood by the closed r -neighbourhood. This problem is known as PERFECT r -CODE, and was introduced by Biggs [4] in 1973. Similarly, for every problem in Table 1 its distance- r generalization either introduces a new problem or is already well-known.

We show that all these distance problems are XP parameterized by mim-width if a decomposition tree is given. The main result of the paper is of structural nature, namely that for any positive integer r the mim-width of a graph power G^r is at most twice the mim-width of G . It follows that we can reduce the distance- r version of a (σ, ρ) -domination problem to its non-distance variant by taking the graph power G^r , whilst preserving small mim-width.

The downside to showing results using the parameter mim-width, is that we do not know an XP algorithm computing mim-width. Computing a decomposition tree with optimal mim-width is NP-complete in general and W[1]-hard parameterized by itself. Determining the optimal mim-width is not in APX unless $\text{NP} = \text{ZPP}$, making it unlikely to have a polynomial-time constant-factor approximation algorithm [18], but saying nothing about an XP algorithm.

However, for several graph classes we are able to find a decomposition tree of constant mim-width in polynomial time, using the results of Belmonte and Vatshelle [3]. These include; permutation graphs, convex graphs and their complements, interval graphs and their complements (all of which have *linear* mim-width 1); (circular k -) trapezoid graphs, circular permutation graphs, Dilworth- k graphs, k -polygon graphs, circular arc graphs and complements of d -degenerate graphs. Fomin, Golovach and Raymond [12] show that we can find linear decomposition trees of constant mim-width for the very general class of H -graphs, see Definition 10, in polynomial time, *given*¹ an H -representation of the input graph. For all of the above graph classes, our results imply that the distance- r (σ, ρ) -domination and LCVp problems become polynomial time solvable.

Graphs represented by intersections of objects in some model are often closed under taking powers.

¹We would like to remark that it is NP-complete to decide whether a graph is an H -graph whenever H is not a cactus [7].

For instance, interval graphs, and generally d -trapezoid graphs [11, 1], circular arc graphs [17, 1], and leaf power graphs (by definition) are such graphs. We refer to [5, Chapter 10.6] for a survey of such results. For these classes, we already know that the distance- r version of a (σ, ρ) -domination problem can be solved in polynomial time. However, this closure property does not always hold; for instance, permutation graphs are not closed under taking powers. Our result provides that to obtain such algorithmic results, we do not need to know that these classes are closed under taking powers; it is sufficient to know that classes have bounded mim-width. To the best of our knowledge, for the most well-studied distance- r (σ, ρ) -domination problem, DISTANCE- r DOMINATING SET, we obtain the first polynomial time algorithms on Dilworth k -graphs, convex graphs and their complements, complements of interval graphs, k -polygon graphs, H -graphs (given an H -representation of the input graph), and complements of d -degenerate graphs.

The natural question to ask after obtaining an XP algorithm, is whether we can do better, e.g. can we show that for all fixed r , the distance- r (σ, ρ) -domination problems are in FPT. Fomin et al. [12] answered this in the negative by showing that (the standard, i.e. distance-1 variants of) MAXIMUM INDEPENDENT SET, MINIMUM DOMINATING SET and MINIMUM INDEPENDENT DOMINATING SET problems are W[1]-hard parameterized by (linear) mim-width + solution size. We modify their reductions to extend these results to several families of (σ, ρ) -domination problems, including the maximization variants of INDUCED MATCHING, INDUCED d -REGULAR SUBGRAPH and INDUCED SUBGRAPH OF MAX DEGREE $\leq d$, the minimization variants of TOTAL DOMINATING SET and d -DOMINATING SET and both the maximization and the minimization variant of DOMINATING INDUCED MATCHING.

The remainder of the paper is organized as follows. In Section 2 we introduce the (σ, ρ) problems and define their distance- r generalization. In Section 3 we introduce mim-width, and state previously known results. In Section 4 we show that the mim-width of a graph grows by at most a factor 2 when taking (arbitrary large) powers and give algorithmic consequences. We discuss LCVF problems, their distance- r versions and algorithmic consequences regarding them in Section 5 and in Section 6 we present the above mentioned lower bounds. Finally, we give some concluding remarks in Section 7. Some notational conventions are given in the appendix.

2 Distance- r (σ, ρ) -Domination Problems

Let σ and ρ be finite or co-finite subsets of the natural numbers $\sigma, \rho \subseteq \mathbb{N}$. For a graph G , a vertex set $S \subseteq V(G)$ is a (σ, ρ) -dominator if

- for each vertex $v \in S$ it holds that $|N(v) \cap S| \in \sigma$, and
- for each vertex $v \in V(G) \setminus S$ it holds that $|N(v) \cap S| \in \rho$.

For instance, a $(\{0\}, \mathbb{N})$ -set is an independent set as there are no edges inside of the set, and we do not care about adjacencies between S and $V(G) \setminus S$. For another example, a $(\mathbb{N}, \mathbb{N}^+)$ -set is a dominating set as we require that for each vertex in $V(G) \setminus S$, it has at least one neighbor in S .

There are 3 types of (σ, ρ) -domination problems; minimization, maximization and existence. We denote the problem of finding a minimum (σ, ρ) -dominator as the MIN- (σ, ρ) problem. Similarly, MAX- (σ, ρ) denotes the maximization problem, and \exists - (σ, ρ) denotes the existence problem. Many well-studied problems fit into this framework, see Table 1 for examples.

The d -value of a distance- r (σ, ρ) problem is a constant which will ultimately affect the runtime of the algorithm. For a set $\mu \subseteq \mathbb{N}$, the value $d(\mu)$ should be understood as the highest value in \mathbb{N} we

σ	ρ	d	Standard name
$\{0\}$	\mathbb{N}	1	Independent set *
\mathbb{N}	\mathbb{N}^+	1	Dominating set **
$\{0\}$	\mathbb{N}^+	1	Maximal Independent set **
\mathbb{N}^+	\mathbb{N}^+	1	Total Dominating set **
$\{0\}$	$\{0, 1\}$	2	Strong Stable set or 2-Packing
$\{0\}$	$\{1\}$	2	Perfect Code or Efficient Dom. set
$\{0, 1\}$	$\{0, 1\}$	2	Total Nearly Perfect set
$\{0, 1\}$	$\{1\}$	2	Weakly Perfect Dominating set
$\{1\}$	$\{1\}$	2	Total Perfect Dominating set
$\{1\}$	\mathbb{N}	2	Induced Matching *
$\{1\}$	\mathbb{N}^+	2	Dominating Induced Matching *, **
\mathbb{N}	$\{1\}$	2	Perfect Dominating set
\mathbb{N}	$\{d, d+1, \dots\}$	d	d -Dominating set **
$\{d\}$	\mathbb{N}	$d+1$	Induced d -Regular Subgraph *
$\{d, d+1, \dots\}$	\mathbb{N}	d	Subgraph of Min Degree $\geq d$
$\{0, 1, \dots, d\}$	\mathbb{N}	$d+1$	Induced Subg. of Max Degree $\leq d$ *

Table 1: Some vertex subset properties expressible as (σ, ρ) -sets, with $\mathbb{N} = \{0, 1, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$. Column d shows $d = \max(d(\sigma), d(\rho))$. For each problem, at least one of the minimization, the maximization and the existence problem is NP-complete. For problems marked with * (resp., **), W[1]-hardness of the maximization (resp., minimization) problem parameterized by mim-width + solution size is shown in the present paper. For problems marked with * (resp., **) the W[1]-hardness of maximization (resp., minimization) in the same parameterization was shown by Fomin et al. [12].

need to enumerate in order to describe μ . Hence, if μ is finite, it is simply the maximum value in μ , and if μ is co-finite, it is the maximum natural number *not* in μ (1 is added for technical reasons).

Definition 1 (d -value). Let $d(\mathbb{N}) = 0$. For every non-empty finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max\{x \mid x \in \mu\}, \max\{x \mid x \in \mathbb{N} \setminus \mu\})$.

For a given distance- r (σ, ρ) problem $\Pi_{\sigma, \rho}$, its d -value is defined as $d(\Pi_{\sigma, \rho}) := \max\{d(\sigma), d(\rho)\}$, see column d in Table 1.

3 Mim-width and Applications

Maximum induced matching width, or *mim-width* for short, was introduced in the Ph.D. thesis of Vatschelle [21], used implicitly by Belmonte and Vatschelle [3], and is a structural graph parameter described over *decomposition trees* (sometimes called *branch decompositions*), similar to graph parameters such as *rank-width* and *module-width*. Decomposition trees naturally appear in divide-and-conquer -style algorithms where one recursively partitions the pieces of a problem into two parts. When the algorithm is at the point where it combines solutions of its subproblems to form a full solution, the structure of the cuts are (unsurprisingly) important to the runtime; this is especially true of dynamic programming when one needs to store multiple sub-solutions at each intermediate node. We will briefly introduce the necessary machinery here, but for a more comprehensive introduction we refer the reader to [21].

A graph of maximum degree at most 3 is called *subcubic*. A *decomposition tree* for a graph G is a pair (T, δ) where T is a subcubic tree and $\delta : V(G) \rightarrow L(T)$ is a bijection between the vertices of G and the leaves of T . Each edge $e \in E(T)$ naturally splits the leaves of the tree in two groups depending on their connected component when e is removed. In this way, each edge $e \in E(T)$ also represent a partition of $V(G)$ into two partition classes A_e and $\overline{A_e}$. One way to measure the cut structure is by the *maximum induced matching* across a cut of (T, δ) . A set of edges M is called an *induced matching* if no pair of edges in M shares an endpoint and if the subgraph induced by the endpoints of M does not contain any additional edges.

Definition 2 (mim-width). Let G be a graph, and let (T, δ) be a decomposition tree for G . For each edge $e \in E(T)$ and corresponding partition of the vertices $A_e, \overline{A_e}$, we let $cutmim_G(A_e, \overline{A_e})$ denote the size of a maximum induced matching of the bipartite graph on the edges crossing the cut. Let the *mim-width of the decomposition tree* be

$$mimw_G(T, \delta) = \max_{e \in E(T)} \{cutmim(A_e, \overline{A_e})\}$$

The *mim-width of the graph* G , denoted $mimw(G)$, is the minimum value of $mimw_G(T, \delta)$ over all possible decompositions trees (T, δ) . The *linear mim-width of the graph* G is the minimum value of $mimw_G(T, \delta)$ over all possible decompositions trees (T, δ) where T is a caterpillar.

In previous work, Bui-Xuan et al. [6] and Belmonte and Vatshelle [3] showed that all (σ, ρ) problems can be solved in time $n^{\mathcal{O}(w)}$ where w denotes the mim-width of a decomposition tree that is provided as part of the input. More precisely, they show the following.²

Proposition 3 ([3, 6]). *There is an algorithm that given a graph G and a decomposition tree (T, δ) of G with $w := mimw_G(T, \delta)$ solves each (σ, ρ) problem Π with $d := d(\Pi)$*

- (i) *in time $\mathcal{O}(n^{4+2d \cdot w})$, if T is a caterpillar, and*
- (ii) *in time $\mathcal{O}(n^{4+3d \cdot w})$, otherwise.*

4 Mim-width on Graph Powers

Definition 4 (Graph power). Let $G = (V, E)$ be a graph. Then the k -th power of G , denoted G^k , is a graph on the same vertex set where there is an edge between two vertices if and only if the distance between them is at most k in G . Formally, $V(G^k) = V(G)$ and $E(G^k) = \{uv \mid \text{DIST}_G(u, v) \leq k\}$.

Theorem 5. *For any graph G and positive integer k , $mimw(G^k) \leq 2 \cdot mimw(G)$.*

Proof. Assume that there is a decomposition tree of mim-width w for the graph G . We show that the same decomposition tree has mim-width at most $2w$ for G^k .

We consider a cut A, \overline{A} of the decomposition tree. Let M be a maximum induced matching across the cut for G^k . To prove our claim, it suffices to construct an induced matching across the cut M' in G such that $|M'| \geq \frac{|M|}{2}$.

²We would like to remark that the original results in [6] are stated in terms of the number of d -neighborhood equivalence classes across the cuts in the decomposition tree ($nec_d(T, \delta)$) giving a runtime of $n^4 \cdot nec_d(T, \delta)^c$ (where $c = 2$ if the given decomposition is a caterpillar and $c = 3$ otherwise). In [3, Lemma 2], Belmonte and Vatshelle show that $nec_d(T, \delta) \leq n^{d \cdot mimw_G(T, \delta)}$.

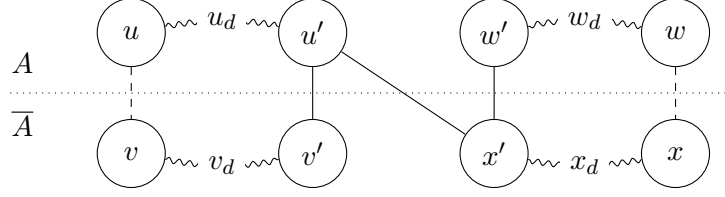


Figure 1: Structure of two paths P_{uv} and P_{wx} when the edge $u'x'$ exists in G . Dashed edges appear in G^k , solid edges appear in G , squiggly lines are (shortest) paths existing in G (possibly of length 0, and possibly crossing back and forth across the cut).

We begin by noticing that for an edge $uv \in M$, the distance between u and v is at most k in G . For each such edge $uv \in M$, we let P_{uv} denote some shortest path between u and v in G (including the endpoints u and v).

Claim 5.1. Let $uv, wx \in M$ be two distinct edges of the matching. Then P_{uv} and P_{wx} are vertex disjoint.

Proof. We may assume that $u, w \in A$ and $v, x \in \bar{A}$. Now assume for the sake of contradiction there exists a vertex $y \in P_{uv} \cap P_{wx}$. Because both paths have length at most k , we have that $\text{DIST}_G(u, y) + \text{DIST}_G(y, v) \leq k$, and $\text{DIST}_G(w, y) + \text{DIST}_G(y, x) \leq k$. Adding these together, we get

$$\text{DIST}_G(u, y) + \text{DIST}_G(y, v) + \text{DIST}_G(w, y) + \text{DIST}_G(y, x) \leq 2k.$$

Since uv and wx are both in M , there can not exist edges ux and wv in G^k . Hence, their distance in G is strictly greater than k , i.e. $\text{DIST}_G(u, y) + \text{DIST}_G(y, x) \geq \text{DIST}_G(u, x) > k$, and $\text{DIST}_G(w, y) + \text{DIST}_G(y, v) > k$. Putting these together, we obtain our contradiction:

$$\text{DIST}_G(u, y) + \text{DIST}_G(y, x) + \text{DIST}_G(w, y) + \text{DIST}_G(y, v) > 2k$$

This concludes the proof of the claim. \square

Our next observation is that for each $uv \in M$, the path P_{uv} starts (without loss of generality) in A , and ends in \bar{A} . There must hence exist at least one point at which the path cross from A to \bar{A} . For each $uv \in M$, we can thus safely let $u'v' \in E(P_{uv})$ denote an edge in G such that $u' \in A$ and $v' \in \bar{A}$.

We plan to construct our matching M' by picking a subset of such edges. However, we can not simply take all of them, since some pairs may be incompatible in the sense that they will not form an induced matching across the cut A, \bar{A} . We examine the structures that arise when two such edges $u'v'$ and $w'x'$ are incompatible, and can not both be included in the same induced matching across the cut. For easier readability, we let α_d be a shorthand notation for $\text{DIST}_G(\alpha, \alpha')$ for $\alpha \in \{u, v, w, x\}$.

Claim 5.2. Let $uv, wx \in M$ be two distinct edges of M and let $u'v'$ and $w'x'$ be edges on the shortest paths as defined above. If there is an edge $u'x' \in E(G)$, then all of the following hold. See Figure 1.

- (a) $u_d + x_d = k$
- (b) $u_d + v_d = w_d + x_d = k - 1$

(c) $w_d = u_d - 1$

Proof. (a) Since ux is not an edge in G^k , the distance between u and x must be at least $k + 1$ in G , and so $u_d + x_d$ must be at least k . It remains to show that $u_d + x_d \leq k$ for equality to hold. Similarly to the proof of Claim 5.1, we know that P_{uv} and P_{wx} both are of length at most k . We get

$$u_d + v_d + w_d + x_d \leq 2k - 2 \quad (1)$$

The -2 at the end is because we do not include the length contributed by edges $u'v'$ and $w'x'$ in our sum. Now assume for the sake of contradiction that $u_d + x_d \geq k + 1$. Then we get that

$$v_d + w_d \leq 2k - 2 - k - 1 = k - 3$$

Because $\text{DIST}_G(v', w') \leq 3$ (follow the edges $u'v' \rightarrow u'x' \rightarrow w'x'$), this implies that $\text{DIST}_G(v, w) \leq k$, and the edge vw would hence exist in G^k . This contradicts that uv and wx were both in the same induced matching M .

(b) Assume for the sake of contradiction that $u_d + v_d \leq k - 2$. Then, rather than Equation 1, we get the following bound

$$u_d + v_d + w_d + x_d \leq 2k - 3$$

By (a) we know that $u_d + x_d = k$, so by a similar argument as above we get that $v_d + x_d \leq k - 3$, obtaining a contradiction. An analogous argument holds for $w_d + x_d$.

(c) This follows immediately by substituting (a) into (b). \lrcorner

We will now construct our induced matching M' . We construct two candidates for M' , and we will pick the biggest one. First, we construct M'_0 by including $u'v'$ for each edge $uv \in M$ where $\text{DIST}_G(u, u')$ is even. Symetrically, M'_1 is constructed by including $u'v'$ if $\text{DIST}_G(u, u')$ is odd. Clearly, at least one of M'_0, M'_1 contains $\geq \frac{|M|}{2}$ edges. It remains to show that M' indeed forms an induced matching across the cut A, \bar{A} in G .

Consider two distinct edges $u'v'$ and $w'x'$ from M' . By Claim 5.1, the two edges are vertex disjoint. If there is an edge violating that $u'v'$ and $w'x'$ are both in the same induced matching, it must be either $u'x'$ or $v'w'$. Without loss of generality we may assume it is an edge of the type $u'x'$. By Claim 5.2 (c), we then have that the parities of $\text{DIST}_G(u, u')$ and $\text{DIST}_G(w, w')$ are different. But by how M' was constructed, this is not possible. This concludes the proof. \square

Observation 6. For a positive integer r , a graph G and a vertex $u \in V(G)$, the r -neighbourhood of u is equal to the neighbourhood of u in G^r , i.e. $N_G^r(u) = N_{G^r}(u)$.

The observation above shows that solving a distance- r (σ, ρ) problem on G is the same as solving the same standard distance-1 variation of the problem on G^r . Hence, we may reduce our problem to the standard version by simply computing the graph power. Combining Theorem 5 with the algorithms provided in Proposition 3, we have the following consequence.

Corollary 7. There is an algorithm that for all $r \in \mathbb{N}$, given a graph G and a decomposition tree (T, δ) of G with $w := \text{mimw}_G(T, \delta)$ solves each distance- r (σ, ρ) problem Π with $d := d(\Pi)$

- (i) in time $\mathcal{O}(n^{4+4d \cdot w})$, if T is a caterpillar, and
- (ii) in time $\mathcal{O}(n^{4+6d \cdot w})$, otherwise.

Proof. Let G be the input graph and (T, δ) the provided decomposition tree. We apply the following algorithm:

Step 1. Compute the graph G^r .

Step 2. Solve the standard (distance-1) version of the problem on G^r , providing (T, δ) as the decomposition tree.

Step 3. Return the answer of the algorithm ran in Step 2 without modification.

Computing G^r in Step 1 takes at most $\mathcal{O}(n^3)$ time using standard methods, Step 3 takes constant time. The worst time complexity is hence found in Step 2. By Theorem 5, the mim-width of (T, δ) on G^r is at most twice that of the same decomposition on G . The stated runtime then follows from Proposition 3. The correctness of this procedure follows immediately from Observation 6. \square

5 LCVP Problems

A generalization of (σ, ρ) problems are the *locally checkable vertex partitioning* (LCVP) problems. A *degree constraint matrix* D is a $q \times q$ matrix where each entry is a finite or co-finite subset of \mathbb{N} . For a graph G and a partition of its vertices $\mathcal{V} = \{V_1, V_2, \dots, V_q\}$, we say that it is a D -partition if and only if, for each $i, j \in [q]$ and each vertex $v \in V_i$, it holds that $|N(v) \cap V_j| \in D[i, j]$. Empty partition classes are allowed.

For instance, if a graph can be partitioned according to the 3×3 matrix whose diagonal entries are $\{0\}$ and the non-diagonal ones are \mathbb{N} , then the graph is 3-colorable. Typically, the natural algorithmic questions associated with LCVP properties are existential.³ Interesting problems which can be phrased in such terms include the H -COVERING and GRAPH H -HOMOMORPHISM problems where H is fixed, as well as q -COLORING, PERFECT MATCHING CUT and more. We refer to [20] for an overview.

We generalize LCVP properties to their distance- r version, by considering the ball of radius r around each vertex rather than just the immediate neighbourhood.

Definition 8 (Distance- r neighbourhood constraint matrix). A distance- r neighbourhood constraint matrix D is a $q \times q$ matrix where each entry is a finite or co-finite subset of \mathbb{N} . For a graph G and a partition of its vertices $\mathcal{V} = \{V_1, V_2, \dots, V_q\}$, we say that it is a D -distance- r -partition if and only if, for each $i, j \in [q]$ and each vertex $v \in V_i$, it holds that $|N^r(v) \cap V_j| \in D[i, j]$. Empty partition classes are allowed.

We say that an algorithmic problem is a *distance- r LCVP problem* if the property in question can be described by a distance- r neighbourhood constraint matrix. For example, the distance- r version of a problem such as q -COLORING can be interpreted as an assignment of at most q colours to vertices of a graph such that no two vertices are assigned the same colour if they are at distance r or closer.

For a given distance- r LCVP problem Π , its d -value $d(\Pi)$ is the maximum d -value over all the sets in the corresponding neighbourhood constraint matrix.

³Note however that each (σ, ρ) problem can be stated as an LCVP problem via the matrix $D_{(\sigma, \rho)} = \begin{bmatrix} \sigma & \mathbb{N} \\ \rho & \mathbb{N} \end{bmatrix}$, so maximization or minimization of some block of the partition can be natural as well.

As in the case of (σ, ρ) problems, combining Theorem 5 with Observation 6 and the works [3, 6] we have the following result.

Corollary 9. There is an algorithm that for all $r \in \mathbb{N}$, given a graph G and a decomposition tree (T, δ) of G with $w := \text{mimw}_G(T, \delta)$ solves each distance- r LCVP problem Π with $d := d(\Pi)$

- (i) in time $\mathcal{O}(n^{4+4qd \cdot w})$, if T is a caterpillar, and
- (ii) in time $\mathcal{O}(n^{4+6qd \cdot w})$, otherwise.

6 Lower Bounds

We show that several (σ, ρ) -problems are $\text{W}[1]$ -hard parameterized by linear mim-width plus solution size. Our reductions are based on two recent reductions due to Fomin, Golovach and Raymond [12] who showed that INDEPENDENT SET and DOMINATING SET are $\text{W}[1]$ -hard parameterized by linear mim-width plus solution size. In fact they show hardness for the above mentioned problems on *H-graphs* (the parameter being the number of edges in H plus solution size) which we now define formally.

Definition 10 (*H-Graph*). Let X be a set and \mathcal{S} a family of subsets of X . The *intersection graph* of \mathcal{S} is a graph with vertex set \mathcal{S} such that $S, T \in \mathcal{S}$ are adjacent if and only if $S \cap T \neq \emptyset$. Let H be a (multi-) graph. We say that G is an *H-graph* if there is a subdivision H' of H and a family of subsets $\mathcal{M} := \{M_v\}_{v \in V(G)}$ (called an *H-representation*) of $V(H')$ where $H'[M_v]$ is connected for all $v \in V(G)$, such that G is isomorphic to the intersection graph of \mathcal{M} .

All of the hardness results presented in this section are obtained via reductions to the respective problems on *H-graphs*, and the hardness for linear mim-width follows from the following proposition.

Proposition 11 (Theorem 1 in [12]). *Let G be an H-graph. Then, G has linear mim-width at most $2 \cdot ||H||$ and a corresponding decomposition tree can be computed in polynomial time given an H-representation of G .*

6.1 Maximization Problems

The first lower bound concerns several maximization problems that can be expressed in the (σ, ρ) framework. Recall that the INDEPENDENT SET problem can be formulated as $\text{MAX}-(\{0\}, \mathbb{N})$. The following result states that a class of problems that generalize the INDEPENDENT SET problem where each vertex in the solution is allowed to have at most some fixed number of d neighbors of the solution, and several variants thereof, is $\text{W}[1]$ -hard on *H-graphs* parameterized by $||H||$ plus solution size.

Theorem 12. *For any fixed $d \in \mathbb{N}$ and $x \leq d + 1$, the following holds. Let $\sigma^* \subseteq \mathbb{N}_{\leq d}$ with $d \in \sigma^*$. Then, $\text{MAX}-(\sigma^*, \mathbb{N}_{\geq x})$ DOMINATION is $\text{W}[1]$ -hard on *H-graphs* parameterized by the number of edges in H plus solution size, and the hardness holds even if an *H-representation* of the input graph is given.*

Proof. To prove the theorem, we provide a reduction from MULTICOLORED CLIQUE where given a graph G and a partition V_1, \dots, V_k of $V(G)$, the question is whether G contains a clique of size k using precisely one vertex from each V_i ($i \in [k]$). This problem is known to be $\text{W}[1]$ -complete [10, 15].

Let (G, V_1, \dots, V_k) be an instance of MULTICOLORED CLIQUE. We can assume that $k \geq 2$ and that $|V_i| = p$ for $i \in [k]$. If the second assumption does not hold, let $p := \max_{i \in [k]} |V_i|$ and add $p - |V_i|$ isolated vertices to V_i , for each $i \in [k]$. (Note that adding isolated vertices does not change the answer to the problem.) For $i \in [k]$, we denote by v_1^i, \dots, v_p^i the vertices of V_i . We first describe the reduction of Fomin et al. [12] and then explain how to modify it to prove the theorem.

The Construction of Fomin, Golovach and Raymond [12]. The graph H is obtained as follows.

1. Construct k nodes u_1, \dots, u_k .
2. For every $1 \leq i < j \leq k$, construct a node $w_{i,j}$ and two pairs of parallel edges $u_i w_{i,j}$ and $u_j w_{i,j}$.

We then construct the subdivision H' of H by first subdividing each edge p times. We denote the subdivision nodes for 4 edges of H constructed for each pair $1 \leq i < j \leq k$ in Step 2 by $x_1^{(i,j)}, \dots, x_p^{(i,j)}, y_1^{(i,j)}, \dots, y_p^{(i,j)}, x_1^{(j,i)}, \dots, x_p^{(j,i)}$, and $y_1^{(j,i)}, \dots, y_p^{(j,i)}$. To simplify notation, we assume that $u_i = x_0^{(i,j)} = y_0^{(i,j)}$, $u_j = x_0^{(j,i)} = y_0^{(j,i)}$ and $w_{i,j} = x_{p+1}^{(i,j)} = y_{p+1}^{(i,j)} = x_{p+1}^{(j,i)} = y_{p+1}^{(j,i)}$.

We now construct the H -graph G'' by defining its H -representation $\mathcal{M} = \{M_v\}_{v \in V(G'')}$ where each M_v is a connected subset of $V(H')$. (Recall that G denotes the graph of the MULTICOLORED CLIQUE instance.)

1. For each $i \in [k]$ and $s \in [p]$, construct a vertex z_s^i with model

$$M_{z_s^i} := \bigcup_{j \in [k], j \neq i} \left(\left\{ x_0^{(i,j)}, \dots, x_{s-1}^{(i,j)} \right\} \cup \left\{ y_0^{(i,j)}, \dots, y_{p-s}^{(i,j)} \right\} \right).$$

2. For each edge $v_s^i v_t^j \in E(G)$ for $s, t \in [p]$ and $1 \leq i < j \leq k$, construct a vertex $r_{s,t}^{(i,j)}$ with:

$$\begin{aligned} M_{r_{s,t}^{(i,j)}} := & \left\{ x_s^{(i,j)}, \dots, x_{p+1}^{(i,j)} \right\} \cup \left\{ y_{p-s+1}^{(i,j)}, \dots, y_{p+1}^{(i,j)} \right\} \\ & \cup \left\{ x_t^{(j,i)}, \dots, x_{p+1}^{(j,i)} \right\} \cup \left\{ y_{p-t+1}^{(j,i)}, \dots, y_{p+1}^{(j,i)} \right\}. \end{aligned}$$

Throughout the following, for $i \in [k]$ and $1 \leq i < j \leq k$, respectively, we use the notation

$$Z(i) := \bigcup_{s \in [p]} \{z_s^i\} \quad \text{and} \quad R(i, j) := \bigcup_{\substack{v_s^i v_t^j \in E(G), \\ s, t \in [p]}} \{r_{s,t}^{(i,j)}\}.$$

We now observe the crucial property of G'' .

Observation 12.1 (Claim 7 in [12]). For every $1 \leq i < j \leq k$, a vertex $z_h^i \in V(G')$ (a vertex $z_h^j \in V(G')$) is *not* adjacent to a vertex $r_{s,t}^{(i,j)} \in V(G')$ corresponding to the edge $v_s^i v_t^j \in E(G)$ if and only if $h = s$ ($h = t$, respectively).

The New Gadget. We now describe how to obtain from G'' a graph G' that will be the graph of the instance of $\text{MAX}-(\sigma^*, \mathbb{N}_{\geq x})$ DOMINATION. We do so by adding a gadget to each set $Z(i)$ and $R(i, j)$ (for all $1 \leq i < j \leq k$). We first describe the gadget and then explain how to modify H' to a new graph K' such that G' is a K -graph (where K denotes the graph obtained from K' by undoing the above described subdivisions that were made in H to obtain H'). Let X be any set of vertices of G'' . The gadget $\mathfrak{B}(X)$ is a complete bipartite graph on $2d - 1$ vertices and bipartition

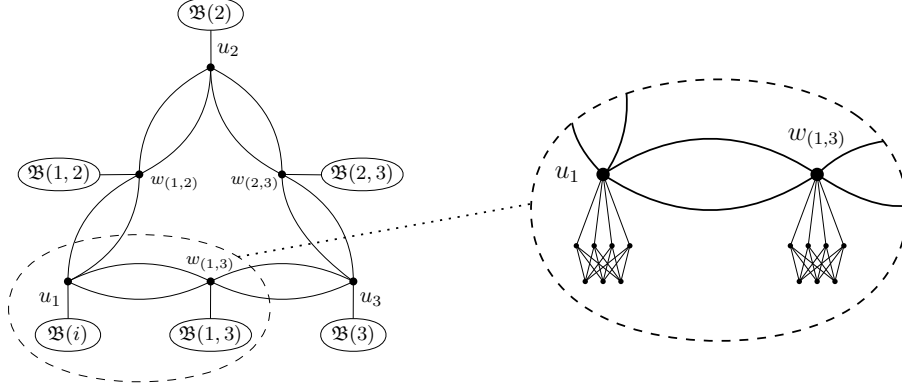


Figure 2: The graph K with respect to which the graph G' constructed in the proof of Theorem 12 is a K -graph. In this example, we have $k = 3$ and $d = 4$.

$(\{\beta_{1,1}, \dots, \beta_{1,d}\}, \{\beta_{2,1}, \dots, \beta_{2,d-1}\})$. such that for $h \in [d]$, each vertex $\beta_{1,h}$ is additionally adjacent to each vertex in X . For $1 \leq i < j \leq k$, we use the notation $\mathfrak{B}(i) := \mathfrak{B}(Z(i))$ and $\mathfrak{B}(i, j) := \mathfrak{B}(R(i, j))$ and we denote their vertices by $\beta_{\cdot, \cdot}^i$ and $\beta_{\cdot, \cdot}^{(i, j)}$, respectively.

We obtain K' by ‘hardcoding’ each gadget $\mathfrak{B}(\cdot)$ into H' . That is, for $i \in [k]$, we add the graph $\mathfrak{B}(i)$ and connect it to the remaining vertices via the edges $u_i \beta_{1,h}^i$ for $h \in [d]$. For $1 \leq i < j \leq k$, we proceed analogously in encoding $\mathfrak{B}(i, j)$ into H' . For an illustration of the graph K , see Figure 2. We observe that $|K| = 2d \left(k + \binom{k}{2} \right) = kd(k+1)$ and

$$||K|| = 4 \binom{k}{2} + \left(k + \binom{k}{2} \right) \cdot (d + d(d-1)) = \frac{1}{2}k (d^2(k+1) + 4(k-1)). \quad (2)$$

We subdivide all newly introduced edges, i.e. all edges in $E(K') \setminus E(H')$ and for an edge $xy \in E(K') \setminus E(H')$, we denote the resulting vertex by $s(x, y)$. We are now ready to describe (the K -representation of) G' .

1. For all $i \in [k]$ and $s \in [p]$, we add the vertices $s(u_i, \beta_{1,h}^i)$ (where $h \in [d]$) to the model of z_s^i . For all $1 \leq i < j \leq k$ and $s, t \in [p]$ with $v_s^i v_t^j \in E(G)$, we add the vertices $s(w_{(i,j)}, \beta_{1,h}^{(i,j)})$ for $h \in [d]$ to the model of $r_{s,t}^{(i,j)}$.
2. For all $i \in [k]$ and $h \in [d]$, we add a vertex $b_{1,h}^i$ with model $\{\beta_{1,h}^i, s(u_i, \beta_{1,h}^i)\} \cup \bigcup_{h' \in [d-1]} \{s(\beta_{1,h}^i, \beta_{2,h'}^i)\}$.
3. For all $i \in [k]$ and $h \in [d-1]$, we add a vertex $b_{2,h}^i$ with model $\{\beta_{2,h}^i\} \cup \bigcup_{h' \in [d]} \{s(\beta_{2,h}^i, \beta_{1,h'}^i)\}$.
4. For all $v_s^i v_t^j \in E(G)$ (where $1 \leq i < j \leq k$ and $s, t \in [p]$) and $h \in [d]$, we add a vertex $b_{1,h}^{(i,j)}$ with model $\{\beta_{1,h}^{(i,j)}, s(w_{(i,j)}, \beta_{1,h}^{(i,j)})\} \cup \bigcup_{h' \in [d-1]} \{s(\beta_{1,h}^{(i,j)}, \beta_{2,h'}^{(i,j)})\}$.
5. For all $v_s^i v_t^j \in E(G)$ (where $1 \leq i < j \leq k$ and $s, t \in [p]$) and $h \in [d-1]$, we add a vertex $b_{1,h}^{(i,j)}$ with model $\{\beta_{2,h}^{(i,j)}\} \cup \bigcup_{h' \in [d]} \{s(\beta_{2,h}^{(i,j)}, \beta_{1,h'}^{(i,j)})\}$.

One can verify that these five steps introduce the above described vertices to G' . For an illustration of G' , see Figure 3. We now turn to the correctness proof of the reduction.

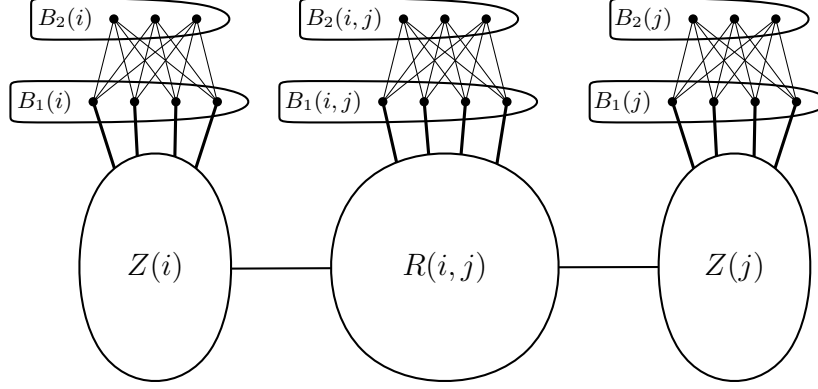


Figure 3: A part of the graph G' , where $1 \leq i < j \leq k$ and $d = 4$.

For $1 \leq i < j \leq k$, we let $B_1(i) := \{b_{1,1}^i, \dots, b_{1,d}^i\}$, $B_2(i) := \{b_{2,1}^i, \dots, b_{2,d-1}^i\}$, $B_1(i, j) := \{b_{1,1}^{(i,j)}, \dots, b_{1,d}^{(i,j)}\}$ and $B_2(i, j) := \{b_{2,1}^{(i,j)}, \dots, b_{2,d-1}^{(i,j)}\}$; furthermore $B(i) := B_1(i) \cup B_2(i)$, $B(i, j) := B_1(i, j) \cup B_2(i, j)$, and $B := \bigcup_{i \in [k]} B(i) \cup \bigcup_{1 \leq i < j \leq k} B(i, j)$. Note that $|B| = (2d - 1)(k + \binom{k}{2})$. We furthermore let $k' := 2d \cdot (k + \binom{k}{2})$ and throughout the following, we use the notation

$$Z_{+B}(i) := Z(i) \cup B(i) \text{ and } R_{+B}(i, j) := R(i, j) \cup B(i, j).$$

We now prove the first direction of the correctness of the reduction. Note that the following claim yields the forward direction of the correctness proof, since a $(\{d\}, \{d+1, \dots, d+k\})$ set is a $(\sigma^*, \mathbb{N}_{\geq x})$ set. (Recall that $d \in \rho^*$ and $x \leq d+1$.)

Claim 12.2. If G has a multicolored clique, then G' has a $(\{d\}, \{d+1, \dots, d+k\})$ set of size $k' = 2d \cdot (k + \binom{k}{2})$ (assuming $k \geq 3$).

Proof. Let $\{v_{h_1}^1, \dots, v_{h_k}^k\}$ be the vertex set in G that induces the multicolored clique. By Observation 12.1 we can verify that

$$I := \{z_{h_1}^1, \dots, z_{h_k}^k\} \cup \{r_{h_i, h_j}^{(i,j)} \mid 1 \leq i < j \leq k\} \quad (3)$$

is an independent set in G' . We let $S := I \cup B$ and observe that S is a $(\sigma^*, \mathbb{N}_{\geq x})$ set: By construction, there is no edge between any pair of distinct sets of $B(i)$, $B(i')$, $B(i, j)$, $B(i', j')$, for any choice of $1 \leq i < j \leq k$ and $1 \leq i' < j' \leq k$.

Consider any vertex $x \in S$ and suppose wlog.⁴ that $x \in Z_{+B}(i)$ for some $i \in [k]$. If $x = z_{h_i}^i$, then x is adjacent to the d vertices $b_{1,1}^i, \dots, b_{1,d}^i$, if $x = b_{1,\ell}^i$ for some $\ell \in [d]$, then x is adjacent to $z_{h_i}^i$ and the vertices $b_{2,1}^i, \dots, b_{2,d-1}^i$ and if $x = b_{2,\ell'}^i$ for some $\ell' \in [d-1]$, then it is adjacent to the vertices $b_{1,1}^i, \dots, b_{1,d}^i$. Hence, in all cases, x has precisely d neighbors in S .

Let $y \in V(G') \setminus S$ and note that $(V(G') \setminus S) \cap B = \emptyset$. If $y \in Z(i)$ for some $i \in [k]$, then $N(y) \cap S \supseteq \{z_{h_i}^i, b_{1,1}^i, \dots, b_{1,d}^i\}$, so $|N(y) \cap S| \geq d+1$. Since the only additional neighbors of y in S are in the set $R_i := \bigcup_{1 \leq j < i} R(j, i) \cup \bigcup_{i < j \leq k} R(i, j)$ and $R_i \cap S \subseteq I$, we can conclude that $|N(y) \cap (S \setminus B)| \leq k-1$, since I contains precisely one vertex from each set $R(i, j)$. We have argued that $d+1 \leq |N(y) \cap S| \leq d+k$. If $y \in R(i, j)$ for some $1 \leq i < j \leq k$, we can argue as before that

⁴The case when $x \in R_{+B}(i, j)$ can be argued for analogously.

$|N(y) \cap S| \geq d + 1$ and since all neighbors of y in $S \setminus B(i, j)$ are contained either in $Z(i)$ or $Z(j)$, we can conclude that $d + 1 \leq |N(y) \cap S| \leq d + 3 \leq d + k$.

It remains to count the size of S . Clearly, $|I| = k + \binom{k}{2}$ and as observed above, $|B| = (2d - 1)(k + \binom{k}{2})$, so

$$|S| = |I| + |B| = k + \binom{k}{2} + (2d - 1) \left(k + \binom{k}{2} \right) = 2d \left(k + \binom{k}{2} \right) = k',$$

as claimed. \square

We now prove the backward direction of the correctness of the reduction. We begin by making several observations about the structure of $(\sigma^*, \mathbb{N}_{\geq x})$ sets in the graph G' .

Claim 12.3. Let $1 \leq i < j \leq k$.

- (i) Any $(\sigma^*, \mathbb{N}_{\geq x})$ set in G' contains at most $d + 1$ vertices from each $Z(i) \cup B_1(i)$ or $R(i, j) \cup B_1(i, j)$.
- (ii) Any $(\sigma^*, \mathbb{N}_{\geq x})$ set contains at most $2d$ vertices from each $Z_{+B}(i)$ or $R_{+B}(i, j)$.
- (iii) If a $(\sigma^*, \mathbb{N}_{\geq x})$ set S contains $2d$ vertices from some $Z_{+B}(i)$ (resp., $R_{+B}(i, j)$), then it contains at least one vertex from $Z(i)$ (resp., $R(i, j)$) and each such vertex in $S \cap Z(i)$ (resp., $S \cap R(i, j)$) has at least d neighbors in $S \cap Z_{+B}(i)$ (resp., $S \cap R_{+B}(i, j)$).

Proof. (i) We prove the claim w.r.t. a set $Z(i) \cup B_1(i)$ and remark that a proof for $R(i, j) \cup B_1(i, j)$ works analogously. Suppose not and let $S \subseteq V(G')$ be such that it contains at least $d + 2$ vertices from some $Z(i) \cup B_1(i)$. Since $|B_1(i)| = d$, we know that S contains a vertex from $Z(i)$, say x . However, by construction, all vertices in $S \cap (Z(i) \cup B_1(i)) \setminus \{x\}$ are adjacent to x , implying that x has at least $d + 1$ neighbors in S , a contradiction with the fact that S is a $(\sigma^*, \mathbb{N}_{\geq x})$ set.

(ii) follows as a direct consequence, since $Z_{+B}(i) \setminus (Z(i) \cup B_1(i)) = B_2(i)$ and $|B_2(i)| = d - 1$. Similar for $R_{+B}(i, j)$.

For (iii), observe that if S contains $2d$ vertices from $Z_{+B}(i)$, then it contains $B_2(i)$ and $d + 1$ vertices from $Z(i) \cup B_1(i)$ by (i) and the fact that $Z_{+B}(i) \setminus (Z(i) \cup B_1(i)) = B_2(i)$ and $|B_2(i)| = d - 1$. Since $|B_1(i)| = d$, at least one vertex is in $S \cap Z(i)$. The claim now follows as any vertex in $Z(i)$ is adjacent to any other vertex in $Z(i)$ as well as any vertex in $B_1(i)$. Similar for $R_{+B}(i, j)$. \square

Claim 12.4. If G' contains a $(\sigma^*, \mathbb{N}_{\geq x})$ set S of size $k' = 2d(k + \binom{k}{2})$, then G contains a multicolored clique.

Proof. Let S be a $(\sigma^*, \mathbb{N}_{\geq x})$ set of size k' in G' . By Claim 12.3(ii), we can conclude that S contains precisely $2d$ vertices from each $Z_{+B}(i)$ and each $R_{+B}(i, j)$ (where $1 \leq i < j \leq k$). Consider any pair i, j with $1 \leq i < j \leq k$. By Claim 12.3(iii) we know that there are vertices

$$z_{s_i}^i \in Z(i) \cap S, \quad z_{s_j}^j \in Z(j) \cap S, \quad \text{and} \quad r_{t_i, t_j}^{(i, j)} \in R(i, j) \cap S.$$

Again by Claim 12.3(iii), we can conclude that $z_{s_i}^i r_{t_i, t_j}^{(i, j)} \notin E(G')$ and $z_{s_j}^j r_{t_i, t_j}^{(i, j)} \notin E(G')$: E.g., $z_{s_i}^i$ has d neighbors in $Z_{+B}(i) \cap S$, so if $z_{s_i}^i r_{t_i, t_j}^{(i, j)} \in E(G')$, then $z_{s_i}^i$ has $d + 1$ neighbors in S , a contradiction with the fact that S is a $(\sigma^*, \mathbb{N}_{\geq x})$ set. By Observation 12.1, we then have that $s_i = t_i$ and $s_j = t_j$. We can conclude that $v_{h_i}^i v_{h_j}^j \in E(G)$ and since the argument holds for any pair of indices i, j that G has a multicolored clique. \square

We would like to remark that by the proof of the previous claim, we have established that any $(\sigma^*, \mathbb{N}_{\geq x})$ set S in G' of size k' in fact contains all vertices from B and one vertex from each $Z(i)$ and from each $R(i, j)$. Since this is precisely the shape of the set constructed in the forward direction of the correctness proof, this shows that any $(\sigma^*, \mathbb{N}_{\geq x})$ set of size k' in G' is a $(\{d\}, \{d+1, \dots, d+k\})$ set (assuming $k \geq 3$).

Claims 12.2 and 12.4 establish the correctness of the reduction. We observe that $|V(G')| = \mathcal{O}(|V(G)| + d^2 \cdot k^2)$ and clearly, G' can be constructed from G in time polynomial in $|V(G)|$, d and k as well. Furthermore, by (2), $\|K\| = \mathcal{O}(d^2 \cdot k^2)$ and the theorem follows. \square

By Proposition 11, the previous theorem implies

Corollary 13. For any fixed $d \in \mathbb{N}$ and $x \leq d + 1$, the following holds. Let $\sigma^* \subseteq \mathbb{N}_{\leq d}$ with $d \in \sigma^*$. Then, $\text{MAX-}(\sigma^*, \mathbb{N}_{\geq x})$ DOMINATION is $\text{W}[1]$ -hard parameterized by linear mim-width plus solution size, and the hardness holds even if a corresponding decomposition tree is given.

6.2 Minimization Problems

In this section we prove hardness of minimization versions of several (σ, ρ) problems. We obtain our results by modifying a reduction from MULTICOLORED INDEPENDENT SET to DOMINATING SET due to Fomin et al. [12]. In the MULTICOLORED INDEPENDENT SET problem we are given a graph G and a partition V_1, \dots, V_k of its vertex set $V(G)$ and the question is whether there is an independent set $\{v_1, \dots, v_k\} \subseteq V(G)$ in G such that for each $i \in [k]$, $v_i \in V_i$. The $\text{W}[1]$ -hardness of this problem follows immediately from the $\text{W}[1]$ -hardness of the MULTICOLORED CLIQUE problem.

The Reduction of Fomin et al. [12]. Let G be an instance of MULTICOLORED INDEPENDENT SET with partition V_1, \dots, V_k of $V(G)$. Again we can assume that $k \geq 2$ and that $|V_i| = p$ for all $i \in [k]$. If the latter condition does not hold, let $p := \max_{i \in [k]} |V_i|$ and for each $i \in [k]$, add $p - |V_i|$ vertices to V_i that are adjacent to all vertices in each V_j where $j \neq i$. It is clear that the resulting instance has a multicolored independent set if and only if the original instance does.

The graph G' of the MINIMUM DOMINATING SET instance is obtained from the graph G'' as constructed in the proof of Theorem 12.5. The only difference is that for $i \in [k]$, a vertex b_i is added whose model is $\{u_i\}$, i.e. it is adjacent to all vertices in $Z(i)$ and nothing else. We argue that G has a multicolored independent set if and only if G' has a dominating set of size k .

For the forward direction, if G has a multicolored independent set $I := \{v_{h_1}^1, \dots, v_{h_k}^k\}$, then using Observation 12.1, one can verify that $D := \{z_{h_1}^1, \dots, z_{h_k}^k\}$ is a dominating set in G' . Clearly, for each $i \in [k]$, the vertices in $Z(i) \cup \{b_i\}$ are dominated by $z_{h_i}^i \in D$. Suppose there is a vertex $r_{s,t}^{(i,j)} \in R(i, j)$ that is not dominated by D , then in particular it is neither adjacent to $z_{h_i}^i$ nor to $z_{h_j}^j$. By Observation 12.1, this implies that G contains the edge $v_{h_i}^i v_{h_j}^j$, a contradiction with the fact that I is an independent set.

For the backward direction, suppose that G' has a dominating set D of size k . Due to the vertices b_i (for $i \in [k]$), we can conclude that for all $i \in [k]$, $D \cap (Z(i) \cup \{b_i\}) \neq \emptyset$. If D contains b_i for some $i \in [k]$, then we can replace b_i by any vertex in $Z(i)$ such that the resulting set is still a dominating set of D , so we can assume that $D = \{z_{h_1}^1, \dots, z_{h_k}^k\}$. We claim that $\{v_{h_1}^1, \dots, v_{h_k}^k\}$ is an independent set in G . Suppose that for $i, j \in [k]$, there is an edge $v_{h_i}^i v_{h_j}^j \in E(G)$. Observation 12.1 implies that $r_{h_i, h_j}^{(i,j)}$ is neither adjacent to $z_{h_i}^i$ nor to $z_{h_j}^j$, so $r_{h_i, h_j}^{(i,j)}$ is not dominated by D , a contradiction.

⁵See the paragraph ‘The Construction of Fomin, Golovach and Raymond’.

Remark 6.1. We would like to remark that the above reduction is to the $\text{MIN}-(\sigma^*, \rho^*)$ DOMINATION problem, for all $\sigma^* \subseteq \mathbb{N}$ with $0 \in \sigma^*$ and $\rho^* \subseteq \mathbb{N}^+$ with $\{1, 2\} \subseteq \rho^*$.

Adaption to Total Domination Problems. Recall that the (σ, ρ) -formulation for DOMINATING SET is $(\mathbb{N}, \mathbb{N}^+)$. We now explain how to modify the above reduction to obtain $W[1]$ -hardness for dominating set problems where each vertex in the solution has to have at least one neighbor in the solution. These problems include TOTAL DOMINATING SET and DOMINATING INDUCED MATCHING, which can be formulated as $(\mathbb{N}^+, \mathbb{N}^+)$ and $(\{1\}, \mathbb{N}^+)$, respectively. The minimization problem of either of them is known to be NP-complete.

Theorem 14. *For $\sigma^* \subseteq \mathbb{N}^+$ with $1 \in \sigma^*$ and $\rho^* \subseteq \mathbb{N}^+$ with $\{1, 2\} \subseteq \rho^*$, $\text{MIN}-(\sigma^*, \rho^*)$ DOMINATION is $W[1]$ -hard on H -graphs parameterized by the number of edges in H plus solution size, and the hardness holds even when an H -representation of the input graph is given.*

Proof. We modify the above reduction from MULTICOLORED INDEPENDENT SET as follows. For each $i \in [k]$, we add another vertex c_i to G' which is only adjacent to b_i . We let $B := \bigcup_{i \in [k]} \{b_i\}$ and $C := \bigcup_{i \in [k]} \{c_i\}$. Note that these new vertices can be hardcoded into H with the number of edges in H increasing only by k . To argue the correctness of the reduction, we now show that G has a multicolored independent set if and only if G' has a (σ^*, ρ^*) set of size $k' := 2k$.

For the forward direction, suppose that G has an independent set $\{v_{h_1}^1, \dots, v_{h_k}^k\}$. Then, $D' := \{z_{h_1}^1, \dots, z_{h_k}^k\}$ dominates all vertices in $V(G') \setminus C$ by the same argument as above and $D := D' \cup B$ dominates all vertices of G' . Furthermore, each $x \in D$ has precisely one neighbor in D : For each such x , either $x = z_{h_i}^i$ or $x = b_i$ for some $i \in [k]$. In the former case, $N(x) \cap D = \{b_i\}$ and in the latter case, $N(x) \cap D = \{z_{h_i}^i\}$. Now let $y \in V(G') \setminus D$. If $y \in Z(i) \cup \{c_i\}$ for $i \in [k]$, then $N(y) \cap D = \{z_{h_i}^i, b_i\}$. If $y \in R(i, j)$ for some $1 \leq i < j \leq k$, then y is either dominated by one of $z_{h_i}^i$ and $z_{h_j}^j$ or by both and it cannot have any other neighbors in D by construction. Since $1 \in \sigma^*$ and $\{1, 2\} \subseteq \rho^*$, D is a (σ^*, ρ^*) set and clearly, $|D| = 2k$.

For the backward direction, suppose that G' has a (σ^*, ρ^*) set D of size $2k$. Let $i \in [k]$. Since $0 \notin \sigma^*$ and $0 \notin \rho^*$, we have that either $c_i \in D$ or $b_i \in D$ (either c_i is dominating or it needs to be dominated). Since c_i does not dominate any vertex in G' other than b_i and b_i dominates c_i plus all vertices in $Z(i)$, we can always assume that $b_i \in D$ and hence that $B \subseteq D$. Since $0 \notin \sigma^*$, all vertices of B have a neighbor in D . For each $i \in [k]$, we can assume that this neighbor is some $z_{h_i}^i$ (rather than c_i , for similar reasoning as above). We have that $D = B \cup \{z_{h_1}^1, \dots, z_{h_k}^k\}$ and since D is a dominating set (in other words, $0 \notin \rho^*$), we can again argue using Observation 12.1 that $\{v_{h_1}^1, \dots, v_{h_k}^k\}$ is an independent set in G . \square

As a somewhat orthogonal result to Theorem 12, we now show hardness of several problems related to the d -DOMINATING SET problem, where each vertex that is not in the solution set has to be dominated by at least some fixed number of d neighbors in the solution.

Adaption to d -Domination Problems. We use a similar gadget constructed in the proof of Theorem 12 to prove $W[1]$ -hardness of several (σ, ρ) problems where each vertex has to be dominated by at least d vertices. In particular, we prove the following theorem. Note that the analogous statement of the following theorem for $d = 1$ is proved by the reduction explained in the beginning of this section, see Remark 6.1.

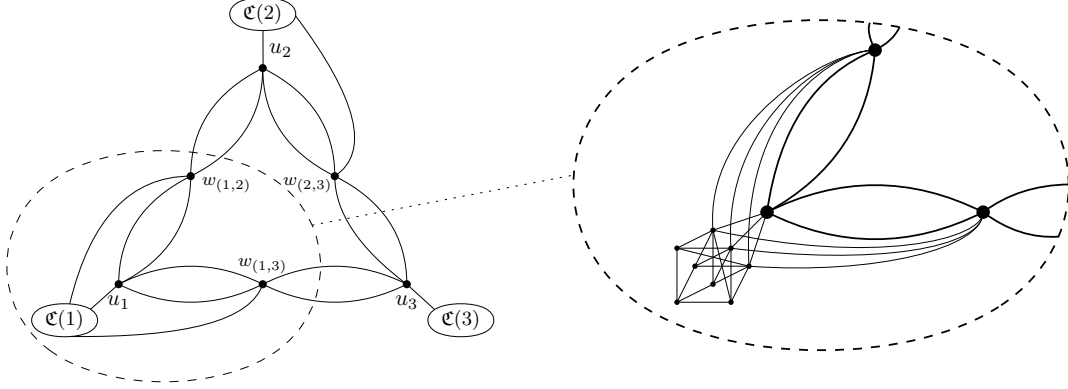


Figure 4: An example graph K w.r.t. which the graph G' constructed in the proof of Theorem 15 is a K -graph. In this example, $k = 3$.

Theorem 15. *For any fixed $d \in \mathbb{N}_{\geq 2}$,⁶ the following holds. Let $\sigma^* \subseteq \mathbb{N}$ with $\{0, 1, d-1\} \subseteq \sigma^*$ and $\rho^* \subseteq \mathbb{N}_{\geq d}$ with $\{d, d+1\} \subseteq \rho^*$. Then, $\text{MIN}-(\sigma^*, \rho^*)$ DOMINATION is $\text{W}[1]$ -hard on H -graphs parameterized by the number of edges in H plus solution size, and the hardness even holds when an H -representation of the input graph is given.*

Proof. We modify the above reduction from MULTICOLORED INDEPENDENT SET. Let G be a graph with vertex partition V_1, \dots, V_k and $|V_i| = p$ for all $i \in [k]$ and assume $k \geq 2$. We first describe the gadget we use and then how to construct the graph G' of the $\text{MIN}-(\sigma^*, \rho^*)$ DOMINATION instance.

The Gadget $\mathfrak{C}(i)$. Let $i \in [k]$. The gadget $\mathfrak{C}(i)$ is a complete bipartite graph on bipartition $(C_1(i), C_2(i))$ with $C_1(i) := \{c_{1,1}^i, \dots, c_{1,d}^i\}$ and $C_2(i) := \{c_{2,1}^i, \dots, c_{2,d}^i\}$ such that each vertex $c_{1,j}^i$ for $j \in [d-1]$ is additionally adjacent to all vertices in $Z(i)$ as well as to all vertices in $R(i, j)$ for $j > i$. (Note that $c_{1,d}^i$ does not have these additional adjacencies.) Throughout the following, we let $C(i) := C_1(i) \cup C_2(i)$ and $C := \bigcup_{i \in [k]} C(i)$.

The graph G' is now obtained by constructing the graph G'' as in the proof of Theorem 12 and then, for each $i \in [k]$, adding the gadget $\mathfrak{C}(i)$ and adding a ‘satellite vertex’ s_i , adjacent to all vertices in $Z(i) \cup C_1(i)$. G' is a K -graph for the graph $K \supseteq H$, obtained by ‘hardcoding’ each $\mathfrak{C}(i)$, for $i \in [k]$, into H . That is, for each $i \in [k]$, we add a complete bipartite graph with bipartition $(\{\gamma_{1,1}^i, \dots, \gamma_{1,d}^i\}, \{\gamma_{2,1}^i, \dots, \gamma_{2,d}^i\})$, and make all vertices $\gamma_{1,h}^i$, where $h \in [d-1]$, adjacent to u_i as well as to all vertices $w_{(i,j)}$ with $j > i$. For an illustration of K see Figure 4. Note that

$$||K|| = ||H|| + kd^2 + \sum_{i=1}^k (k-i)(d-1) = \mathcal{O}(k^2 \cdot d + k \cdot d^2) \quad (4)$$

One can now argue that G' is a K -graph. Since the construction is completely analogous to that explained in the proof of Theorem 12, we skip the details here. We illustrate the structure of the graph G' in Figure 5.

Claim 15.1. If G has a multicolored independent set, then G' has a (σ^*, ρ^*) set of size $k' := k \cdot (d+1)$.

⁶Note that the analogous statement for $d = 1$ follows from the reduction given in [12].

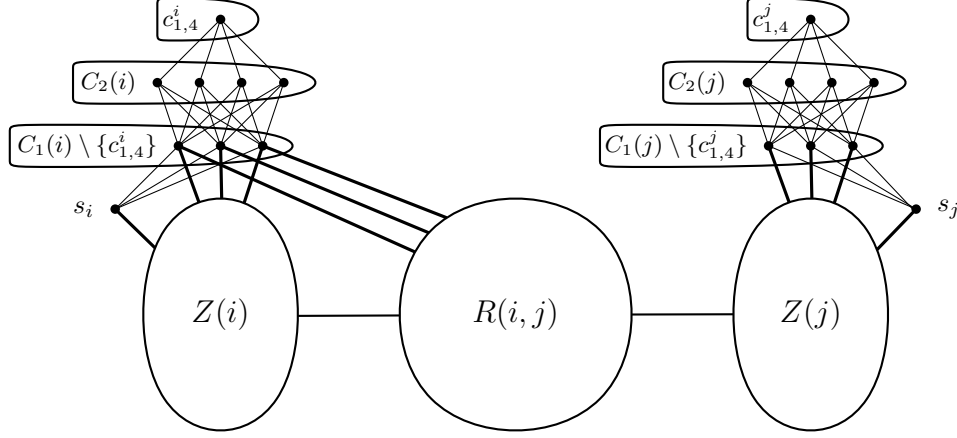


Figure 5: Illustration of a part of G' constructed in the proof of Theorem 15, where $1 \leq i < j \leq k$ and $d = 4$.

Proof. Let $\{v_{h_1}^1, \dots, v_{h_k}^k\}$ be the independent set in G . By the above reduction, $D' := \{z_{h_1}^1, \dots, z_{h_k}^k\}$ is a $(\{0\}, \{1, 2\})$ -set of $G' - C$ (see also Remark 6.1) of size k . Let $C_1 := \bigcup_{i \in [k]} C_1(i)$, $C_2 := C \setminus C_1$ and $D := D' \cup C_1$.

Since each vertex in $V(G') \setminus (D \cup C)$ is adjacent to precisely $d - 1$ vertices in C_1 and to either one or two vertices in D' (and $D' \cap C_1 = \emptyset$), we can conclude that each vertex in $V(G') \setminus (D \cup C)$ is adjacent to either d or $d + 1$ vertices in D . Since each $C(i)$ induces a $K_{d,d}$, we can conclude that all vertices in C_2 have d neighbors in D as well. Furthermore, $N(s_i) \cap D = (C_1(i) \setminus \{c_{1,d}^i\}) \cup \{z_{h_i}^i\}$, so we have that all vertices in G' that are not contained in D have either d or $d + 1$ neighbors in D .

Let $i \in [k]$. Then, $N(z_{h_i}^i) \cap D = \{c_{1,1}^i, \dots, c_{1,d-1}^i\}$, $N(c_{1,d}^i) \cap D = \emptyset$ and for $\ell \in [d - 1]$, $N(c_{1,\ell}^i) \cap D = \{z_{h_i}^i\}$. We can conclude that D is a $(\{0, 1, d - 1\}, \{d, d + 1\})$ -set in G' and clearly, $|D| = k + kd = k'$. \lrcorner

In what follows, the strategy is to argue that each (σ^*, ρ^*) set of size k' contains a set $\{z_{h_1}^1, \dots, z_{h_k}^k\}$ which will imply that $\{v_{h_1}^1, \dots, v_{h_k}^k\}$ is an independent set in G .

Claim 15.2. For all $i \in [k]$, any (σ^*, ρ^*) set D in G' contains at least d vertices from $C(i)$ and at least $d + 1$ vertices from $Z_+(i)$.

Proof. We first show that each such D contains at least d vertices from $C(i)$. Suppose not, then $|D \cap C(i)| \leq d - 1$ for some $i \in [k]$. If $c_{1,d}^i \notin D$, then $C_2(i) \subseteq D$, otherwise $c_{1,d}^i$ cannot have d or more neighbors in D . But $|C_2(i)| = d$, a contradiction. We can assume that $c_{1,d}^i \in D$. Furthermore, there is at least one vertex $c_{2,\ell}^i$ for $\ell \in [d]$ with $c_{2,\ell}^i \notin D$. To ensure that $c_{2,\ell}^i$ has at least d neighbors in D , we would have to include all remaining vertices from $C_1(i)$ in D , but then $|D \cap C(i)| \geq d$, a contradiction. The claim now follows since the vertex s_i only has neighbors in $Z_+(i)$ and at most $d - 1$ neighbors in D (namely $C_1(i) \setminus \{c_{1,d}^i\}$): Since D is a (σ^*, ρ^*) set, it either has to contain s_i or at least one additional neighbor of s_i . \lrcorner

Claim 15.3. For all $i \in [k]$, any (σ^*, ρ^*) set D of size at most $k' = k(d + 1)$ contains $C_1(i)$. We furthermore can assume that it additionally contains some $z_{h_i}^i \in Z(i)$, where $h_i \in [p]$.

Proof. By Claim 15.2 we have that D contains $d + 1$ vertices from each $Z_+(i')$, $i' \in [k]$, and no other vertices. Consider any vertex $z_s^i \in Z(i)$ (where $s \in [p]$) that is not contained in D . Recall that z_s^i has to have at least d neighbors in D . By Claim 15.2, z_s^i has precisely one neighbor in $(Z(i) \cup \{s_i\}) \cap D$ and since D does not contain any vertex from any $R(j, i)$ ($1 \leq j < i$) or $R(i, j')$ ($i < j' \leq k$), the only possible neighbors of z_s^i in D are $C_1(i) \setminus \{c_{1,d}^i\}$. We can conclude that $C_1(i) \subseteq D$. Now suppose that $s_i \in D$. Then, after swapping s_i with any vertex in $Z(i)$, the resulting set remains a (σ^*, ρ^*) set, and the claim follows. \lrcorner

We are now ready to conclude the correctness proof of the reduction.

Claim 15.4. If G' has a (σ^*, ρ^*) set of size $k' = k(d + 1)$, then G has a multicolored independent set.

Proof. Let D be a (σ^*, ρ^*) set of size k' . By Claim 15.3, we can assume that $D = C_1 \cup \{z_{h_1}^1, \dots, z_{h_k}^k\}$ for some $h_1, \dots, h_k \in [p]$. Now, since for each $1 \leq i < j \leq k$, all vertices in $R(i, j)$ have precisely $d - 1$ neighbors in C_1 , each of them has to have at least one of $z_{h_i}^i$ and $z_{h_j}^j$ as a neighbor. By Observation 12.1, this allows us to conclude that $\{v_{h_1}^1, \dots, v_{h_k}^k\}$ is an independent set in G . \lrcorner

Claims 15.1 and 15.4 establish the correctness of the reduction. Clearly, $|V(G')| = \mathcal{O}(|V(G)| + d^2 \cdot k)$ (and G' can be constructed in polynomial time) and by (4), $\|K\| = \mathcal{O}(k^2 \cdot d + k \cdot d^2)$. The theorem follows. \square

Similarly to above, a combination of the previous two theorems with Proposition 11 yields the following hardness results for (σ, ρ) minimization problems on graphs of bounded linear mim-width.

Corollary 16. Let $\sigma^* \subseteq \mathbb{N}$ and $\rho^* \subseteq \mathbb{N}$. Then, MIN- (σ^*, ρ^*) DOMINATION is W[1]-hard parameterized by linear mim-width plus solution size, if one of the following holds.

- (i) $\sigma^* \subseteq \mathbb{N}^+$ with $1 \in \sigma^*$ and $\rho^* \subseteq \mathbb{N}^+$ with $\{1, 2\} \subseteq \rho^*$.
- (ii) For some fixed $d \in \mathbb{N}_{\geq 2}$, $\{0, 1, d - 1\} \subseteq \sigma^*$ and $\rho^* \subseteq \mathbb{N}_{\geq d}$ with $\{d, d + 1\} \subseteq \rho^*$.

Furthermore, the hardness holds even if a corresponding decomposition tree is given.

7 Concluding Remarks

We have introduced the class of distance- r (σ, ρ) and LCVP problems. This generalizes well-known graph distance problems like distance- r domination, distance- r independence, distance- r coloring and perfect r -codes. It also introduces many new distance problems for which the standard distance-1 version naturally captures a well-known graph property.

Using the graph parameter mim-width, we showed that all these problems are solvable in polynomial time for many interesting graph classes. These meta-algorithms will have runtimes which can likely be improved significantly for a particular problem on a particular graph class. For instance, blindly applying our results to solve DISTANCE- r DOMINATING SET on permutation graphs yields an algorithm that runs in time $\mathcal{O}(n^8)$: Permutation graphs have linear mim-width 1 (with a corresponding decomposition tree that can be computed in linear time) [3, Lemmas 2 and 5], so we can apply Corollary 7(i). However, there is an algorithm that solves DISTANCE- r DOMINATING SET on permutation graphs in time $\mathcal{O}(n^2)$ [16]; a much faster runtime. A concrete example of improving a mim-width based algorithm on a specific graph class has recently been provided by Chiarelli et al. [8] who gave algorithms for the (TOTAL) k -DOMINATING SET problems that run in time $\mathcal{O}(n^{3k})$.

on proper interval graphs. The fastest previously known algorithm runs in time $\mathcal{O}(n^{4+6k})$ [3, 6], the generic mim-width based algorithm (cf. Proposition 3 and [3, Lemmas 2 and 3])

We would like to draw attention to the most important and previously stated [14, 18, 21] open question regarding the mim-width parameter: Is there an XP approximation algorithm for computing mim-width? An important first step could be to devise a polynomial-time algorithm deciding if a graph has mim-width 1, or even linear mim-width 1.

Regarding lower bounds, we expanded on the previous results by Fomin et al. [12] and showed that many (σ, ρ) problems are W[1]-hard parameterized by mim-width. However, it remains open whether there exists a problem which is NP-hard in general, yet FPT by mim-width. In particular, there are currently no hardness results when σ and ρ are both finite. Even so, we conjecture that every NP-hard (distance) (σ, ρ) problem is W[1]-hard parameterized by mim-width.

References

- [1] Geir Agnarsson, Peter Damaschke, and Magnús M. Halldórsson. Powers of geometric intersection graphs and dispersion algorithms. *Discrete Appl. Math.*, 132(1-3):3–16, 2003.
- [2] Gábor Bacsó, Dániel Marx, and Zsolt Tuza. H-free graphs, independent sets, and subexponential-time algorithms. In *Proc. IPEC 2016*, pages 3:1–3:12, 2017.
- [3] Rémy Belmonte and Martin Vatshelle. Graph classes with structured neighborhoods and algorithmic applications. *Theor. Comput. Sci.*, 511:54 – 65, 2013.
- [4] Norman Biggs. Perfect codes in graphs. *J. Combin. Theory, Ser. B*, 15(3):289 – 296, 1973.
- [5] A. Brandstädt, V. Le, and J. Spinrad. *Graph Classes: A Survey*. SIAM, 1999.
- [6] Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theor. Comput. Sci.*, 511:66 – 76, 2013.
- [7] Steven Chaplick, Martin Töpfer, Jan Voborník, and Peter Zeman. On H-topological intersection graphs. In *Proc. WG 2017*, pages 167–179. Springer, 2017.
- [8] Nina Chiarelli, Tatiana Romina Hartinger, Valeria Alejandra Leoni, Maria Inés Lopez Pujato, and Martin Milanič. New algorithms for weighted k -domination and total k -domination problems in proper interval graphs. *arXiv preprint*, 2018. arXiv:1803.04327, to appear at ISCO ’18.
- [9] Marek Cygan, Fedor V Fomin, Lukasz Kowalik, Daniel Lokshtanov, Dániel Marx, Marcin Pilipczuk, Michał Pilipczuk, and Saket Saurabh. *Parameterized algorithms*. Springer, 2015.
- [10] Michael R Fellows, Danny Hermelin, Frances Rosamond, and Stéphane Vialette. On the parameterized complexity of multiple-interval graph problems. *Theor. Comput. Sci.*, 410(1):53–61, 2009.
- [11] Carsten Flotow. On powers of m-trapezoid graphs. *Discrete Appl. Math.*, 63(2):187 – 192, 1995.

- [12] Fedor V. Fomin, Petr A. Golovach, and Jean-Florent Raymond. On the tractability of optimization problems on H-graphs. *arXiv preprints*, 2017. arXiv:1709.09737, to appear at ESA '18.
- [13] M. A. Henning, Ortrud R. Oellermann, and Henda C. Swart. Bounds on distance domination parameters. *J. Combin. Inform. Syst. Sci.*, 16(1):11–18, 1991.
- [14] Lars Jaffke, O joungh Kwon, and Jan Arne Telle. A unified polynomial-time algorithm for feedback vertex set on graphs of bounded mim-width. In *Proc. STACS 2018*, pages 42:1–42:14, 2018.
- [15] Krzysztof Pietrzak. On the parameterized complexity of the fixed alphabet shortest common supersequence and longest common subsequence problems. *J. Comput. Syst. Sci.*, 67(4):757–771, 2003.
- [16] Akul Rana, Anita Pal, and Madhumangal Pal. An efficient algorithm to solve the distance k -domination problem on permutation graphs. *J. Discrete Math. Sci. Cryptography*, 19(2):241–255, 2016.
- [17] Arundhati Raychaudhuri. On powers of strongly chordal and circular arc graphs. *Ars Combin.*, 34:147–160, 1992.
- [18] Sigve Hortemo Sæther and Martin Vatshelle. Hardness of computing width parameters based on branch decompositions over the vertex set. *Theor. Comput. Sci.*, 615:120–125, 2016.
- [19] Peter J. Slater. R -domination in graphs. *J. ACM*, 23(3):446–450, 1976.
- [20] Jan Arne Telle and Andrzej Proskurowski. Algorithms for vertex partitioning problems on partial k -trees. *SIAM J. Discrete Math.*, 10(4):529–550, 1997.
- [21] Martin Vatshelle. New width parameters of graphs. 2012.

A Basic definitions and notation

We let the set of natural numbers be $\mathbb{N} = \{0, 1, 2, \dots\}$, and the positive natural numbers be $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. For a set S and a given property ψ , we denote by S_ψ the biggest subset of S where ψ is satisfied for all elements. For instance, $\mathbb{N}_{\leq k}^+$ denotes the set $\{1, 2, \dots, k\}$. For this particular property, we also use the shorthand $[k] = \mathbb{N}_{\leq k}^+$.

A set $A \subseteq \mathbb{N}$ is finite if it has finite cardinality, and it is co-finite if $\mathbb{N} \setminus A$ has finite cardinality.

Graphs. For a graph G and a vertex $u \in V(G)$, its *neighborhood* $N(u)$ is the set of all vertices adjacent to u . The *closed neighborhood* of u is denoted $N[u] = N(u) \cup \{u\}$. The *degree* of a vertex is the number of vertices adjacent to v in the graph, $\deg(v) = |N(v)|$. A vertex of degree 1 is called a leaf, and the set of all leaves of a graph G is denoted $L(G)$. In cases where it would otherwise be unclear which graph is being referred to, a subscript is added, e.g. $N_{G'}(u)$ denotes the neighborhood of u in the graph G' .

For two vertices $u, v \in V(G)$, the *distance* between them $\text{DIST}(u, v)$ is the shortest possible length of a path with u and v as its endpoints, or ∞ if no path exist. A graph is *connected* if $\text{DIST}(u, v) < \infty$ for all vertices $u, v \in V(G)$. For a positive integer r , the *r -neighbourhood* of a vertex u is the set of vertices at distance r or less from u , denoted $N^r(u) = \{v \in V(G) \setminus \{u\} \mid \text{DIST}(u, v) \leq r\}$.

A *connected component* is a vertex maximal induced subgraph which is connected. A *tree* T is a connected graph which contains no cycles. A *caterpillar* is a graph which consists of a path, and for each non-endpoint vertex of the path there is an additional leaf attached to that vertex.

We denote by $n = |G| = |V(G)|$ the number of vertices, and by $m = ||G|| = |E(G)|$ the number of edges of a graph G .

Parameterized Complexity Theory. A *parameterized problem* is a problem where the input instances come along with a non-negative integer k , the *parameter*. Formally, it is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a fixed, finite alphabet. The parameter k is sometimes given implicitly.

A parameterized problem is in the class FPT if there exists an algorithm which correctly decides every instance (x, k) of the problem in time $f(k) \cdot |x|^c$ for some constant c . Such an algorithm is called an FPT algorithm.

A parameterized problem is in the class XP if for every instance (x, k) there exists an algorithm which solves it in time $f(k) \cdot |x|^{g(k)}$.

For both FPT and XP algorithms, the runtime becomes polynomial when the parameter is bounded. But while it is clear that $\text{FPT} \subseteq \text{XP}$, the converse is not true under basic complexity assumptions, and there is a hierarchy of complexity classes between them called the W hierarchy: $\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \dots \subseteq \text{XP}$. We say that a problem is $\text{W}[1]$ -hard if every problem in $\text{W}[1]$ can be reduced to that problem by a parameter-preserving reduction. For a more thorough introduction, we refer the reader to [9].