Resolving complexity of matrix surjectivity

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Abstract. Degree refinement matrices have tight connections to graph homomorphisms that locally, on the neighborhoods of a vertex and its image, are constrained to three types: bijective, injective or surjective. If a graph G has a homomorphism of a given type to a graph H then we say that the degree refinement matrix $\mathrm{drm}(G)$ of G is smaller than $\mathrm{drm}(H)$. This way we obtain three partial orders. Computing $\mathrm{drm}(G)$ is easy, so an algorithm deciding comparability of two matrices in one of these partial orders would be a heuristic for deciding if G has a homomorphism of given type to H. Both for the locally bijective and injective constraints there exists such an algorithm deciding comparability, and for the locally bijective constraint this gives a well-known heuristic for the special case of graph isomorphism. For the locally surjective constraint the existence of an algorithm deciding comparability has been posed as an open problem in [13]. We resolve this open problem by showing that it belongs to the complexity class NP.

Keywords: degree matrix, graph homomorphism, order, local constraints

1 Introduction

Graph homomorphisms, originally obtained as a generalization of graph coloring, have a great deal of applications in computer science and other fields. Beyond these computational aspects they impose an interesting structure on the class of graphs, with many important categorical properties, see e.g. the recent monograph [15]. We focus our attention on *locally constrained* graph homomorphisms [10], where for any vertex u the mapping f induces a function from the neighborhood of u to the neighborhood of $f(u)$ that is required to be either *bijective* [1, 16], *injective* [9, 10], or *surjective* [17, 12].

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Locally bijective homomorphisms (also known as local isomorphisms or full covers) have important applications, for example in distributed computing [5], in recognizing graphs by networks of processors [2, 3], or in constructing highly transitive regular graphs [4]. Locally injective homomorphisms (local epimorphisms or partial covers) are used in distance constrained labelings of graphs [11] with applications to frequency assignment, and as indicators of the existence of homomorphisms of derivate graphs (line graphs) [19].

Locally surjective homomorphisms, which are the main focus of this paper, play a role in distributed computing as well: in a model of distributed computing, where a node changes state based on its own state and the state of one of its neighbors, the naming problem can be solved if and only if the network topology graph G does not allow a locally surjective homomorphism to a smaller graph [7]. Furthermore, in the literature on social network theory, locally surjective homomorphisms are known as *role assignments*. Here, the target graph models roles and their relationships, and for a given society we can ask if its individuals can be assigned roles such that the relationships are preserved: each person playing a particular role has among its neighbors all necessary roles as prescribed by the model [8]. In the paper [21] the complexity of the case of two roles (target graph on two vertices) is studied and later generalized in [12]. A greedy algorithm for deciding two-role assignability of a triangulated graph is given in [22], while structural results are given in [20].

Just as in a graph isomorphism, a locally bijective homomorphism maintains vertex degrees and degrees of neighbors and degrees of neighbors of neighbors and so on. The existence of such a mapping from G to H therefore implies equality of their so-called degree refinement matrices $\mathrm{drm}(G)$ and $\mathrm{drm}(H)$. Since these are easy to compute, the test "is $\mathrm{drm}(G) = \mathrm{drm}(H)$?" gives a necessary condition for the existence of a locally bijective homomorphism from G to H. Indeed, this test is a well-known heuristic for the graph isomorphism problem (cf. [18]). Since locally injective and surjective homomorphisms do not maintain vertex degrees the existence of similar tests giving necessary conditions are in these cases not obvious. Nevertheless, in related work [14] we have shown the existence of partial orders $\overset{I}{\leqslant}$ and $\overset{S}{\leqslant}$ on degree refinement matrices such that the tests "is $\mathrm{drm}(G) \overset{I}{\leqslant} \mathrm{drm}(H)$?" and "is $\mathrm{drm}(G) \overset{S}{\leqslant} \mathrm{drm}(H)$?" give similar necessary conditions for the existence of a locally injective and surjective homomorphism, respectively, from G to H. But are these tests decidable at all?

For local injectivity we have shown that the answer is positive [13]. We did this by showing that for given matrices M and N there is an upper bound on the size of the smallest pair of witness graphs G and H with $\mathrm{drm}(G) = M$ and $\mathrm{drm}(H) = N$ such that G has a locally injective homomorphism to H. Since $M \overset{I}{\leqslant} N$ if and only if such graphs G and H exist we could use this upper bound to prove decidability. For local surjectivity this proof technique is not sufficient, and in the paper [13] we left the question of existence of an algorithm deciding if $M \overset{S}{\leqslant} N$ as a "major open problem".

In the current paper we give an affirmative answer also for the surjective case. To achieve this we introduce graph partitions reflecting the conditions of local surjectivity. As the core theorem of this paper we show that instead of searching for two witness graphs G and H it suffices for local surjectivity to search for a single witness graph that allows a suitable partition with respect to the given matrices M and N. Using this stronger result we are able to apply the techniques developed in [13] to resolve the open problem and provide an NP algorithm.

2 Preliminaries

If not stated otherwise graphs considered in this paper are finite and *simple*, i.e., without loops and multiple edges. For graph terminology not defined below we refer to [6]. For a function $f : V(G) \to V(H)$ and a set $S \subseteq V(G)$ we use the shorthand notation $f(S)$ to denote the image set of S under f, i.e., $f(S) = \{f(u) \mid u \in S\}$. For a vertex $u \in V(G)$ we denote its *neighborhood* by $N_G(u) = \{v \mid (u, v) \in E(G)\}$. The *degree* $\deg(u)$ of a vertex u is the number of edges incident with it, or equivalently the size of its neighborhood. In particular, $u \in N(u)$ if and only if u is incident with a loop, and a loop increases the degree of the associated vertex by exactly one. We write $G(A, B)$ to denote that a graph G is bipartite with classes A and B. The complete graph on n vertices is denoted by K_n.

A *graph homomorphism* from a graph G to a graph H is a vertex mapping $f : V(G) \to V(H)$ satisfying the property that for any edge (u, v) in $E(G)$, we have $(f(u), f(v))$ in $E(H)$ as well, i.e., $f(N_G(u)) \subseteq N_H(f(u))$ for all $u \in V(G)$. A graph homomorphism from G to H satisfying $f(N_G(u)) = N_H(f(u))$ for all $u \in V(G)$ is called *locally surjective*. A homomorphism that induces a one-to-one mapping on the neighborhood of every vertex is called *locally bijective*, i.e., for all $u \in V(G)$ it satisfies $f(N_G(u)) = N_H(f(u))$ and $|N_G(u)| = |N_H(f(u))|$. A homomorphism that satisfies $|N_G(u)| = |N_H(f(u))|$ for all $u \in V(G)$ is called *locally injective*.

A *degree partition* of a graph G is a partition of the vertex set $V(G)$ into *blocks* $B_1, \ldots, B_k$ such that whenever two vertices u and v belong to the same block B_i, then for any $j \in \{1, \ldots, k\}$ we have $|N_G(u) \cap B_j| = |N_G(v) \cap B_j| = m_{i,j}$. The $k \times k$ matrix M such that $(M)_{i,j} = m_{i,j}$ is a *degree matrix*. Observe that a graph G allows several degree matrices, with an adjacency matrix itself being the largest one. There is also a unique matrix corresponding to the partition in the minimum number of blocks, called the *degree refinement matrix* $\mathrm{drm}(G)$, see e.g. [10]. Observe that for any degree partition it holds that $m_{i,j}|B_i| = m_{j,i}|B_j|$ for all $1 \leq i < j \leq k$.

Definition 1. *Let M be a degree matrix of order k, and let G be an arbitrary graph.*

We write $G \xrightarrow{S} M$ if there is a partition of $V(G)$ into blocks $B_1, \dots, B_k$ that for every i and $u \in B_i$ satisfies:

$$\forall j : |N(u) \cap B_j| \begin{cases} = 0 & \text{if } m_{i,j} = 0 \\ \geq m_{i,j} & \text{if } m_{i,j} > 0. \end{cases} \tag{1}$$

We write $G \xrightarrow{B} M$ if the partition for every i and $u \in B_i$ satisfies

$$\forall j : |N(u) \cap B_j| = m_{i,j}. \tag{2}$$

Note that $G \xrightarrow{B} M$ implies $G \xrightarrow{S} M$. To explain this notation observe that for any adjacency matrix $\mathrm{adj}(H)$ of a graph H we have $G \xrightarrow{B} \mathrm{adj}(H)$ if and only if there exists a locally bijective homomorphism from G to H. With a slight abuse of notation we will in this case write $G \xrightarrow{B} H$. Similarly, $G \xrightarrow{S} \mathrm{adj}(H)$ if and only if there exists a locally surjective homomorphism from G to H, in which case we write $G \xrightarrow{S} H$. Moreover, for any matrix M we have $G \xrightarrow{B} M$ if and only if G allows degree matrix M. We define a binary relation $\overset{S}{\leqslant}$ on degree matrices as follows:

Definition 2. *For degree matrices M and N we write $M \overset{S}{\leqslant} N$ if and only if there exist graphs G with $G \xrightarrow{B} M$ and H with $H \xrightarrow{B} N$ such that $G \xrightarrow{S} H$.*

It is easy to show that the $\overset{S}{\leqslant}$ relation on degree matrices is a quasiorder. In the paper [14] it has been shown that the $\overset{S}{\leqslant}$ relation is a partial order on degree refinement matrices. Consider the following decision problem:

MATRIX SURJECTIVITY
Instance: Degree matrices M and N.
Question: Does $M \overset{S}{\leqslant} N$ hold?

In [13] the question of decidability of the MATRIX SURJECTIVITY problem restricted on degree refinement matrices was left as an open problem. In this paper we show that the MATRIX SURJECTIVITY problem belongs to the complexity class NP.

3 Graph reconstruction

In the following two lemmas we consider some cases in which the target matrix N is relatively simple. These cases will be the basic cases for the graph reconstruction in our main theorem.

Lemma 1. *Let N be a degree matrix of order two with zeros on the diagonal. Let G be a graph with $G \xrightarrow{S} N$. Then for any graph H with $H \xrightarrow{B} N$ there exists a graph G^* such that $G^* \xrightarrow{B} G$ and $G^* \xrightarrow{S} H$.*

$$\begin{tikzcd}
G^* \arrow[r, "B"] \arrow[dr, "S"] & G \arrow[dr, "S"] & \\
& H \arrow[r, "B"] & N
\end{tikzcd}$$

Proof. Without loss of generality we may assume that N is not a zero matrix, since otherwise G, H contain no edges and G^* can be an arbitrary edgeless graph.

Since $G \xrightarrow{S} N$ we have a partition (V_1, V_2) of $V(G)$ satisfying equation (1). Let H be a graph with $H \xrightarrow{B} N$ witnessed by a partition (W_1, W_2) of $V(H)$ satisfying equation (2).

First take the graph G' as the disjoint union of $|E(H)| = |W_1|n_{1,2} = |W_2|n_{2,1}$ copies of the graph G. Note that $G' \xrightarrow{B} G$. We construct the graph G^* as demanded by the Lemma by a series of edge swappings in G'. The mapping of $V(G^*) = V(G')$ into $V(G)$ witnessing $G^* \xrightarrow{B} G$ is the projection where all copies of $v \in V(G)$ are sent to v. Throughout, the copy of G corresponding to the edge $e \in E(H)$ will be denoted by $G(e)$ and its copy of the partition (V_1, V_2) by $(V_1(e), V_2(e))$.

We define a homomorphism $f : G' \to H$ such that for every edge $e = (x, y) \in E(H)$ with $x \in W_1$ and $y \in W_2$ all vertices in $V_1(e)$ are mapped on x and similarly $f(V_2(e)) = \{y\}$.

To make this homomorphism f locally surjective we perform appropriate edge swappings. As $G \xrightarrow{S} N$ by our assumptions, any u in V_1 has at least $n_{1,2}$ neighbors in V_2. Out of $N_G(u)$ we choose $n_{1,2}$ different neighbors $v_1, v_2, \ldots, v_{n_{1,2}}$. We denote the isomorphic copy of u in $V_1(e)$ by $u(e)$ and for $1 \le i \le n_{1,2}$ we denote the isomorphic copy of v_i in $V_2(e)$ by $v_i(e)$. Then, for any vertex x in W_1, we act as follows. Let $N_H(x) = \{y_1, \ldots, y_{n_{1,2}}\}$. We denote the corresponding edges in H by $e_h = (x, y_h)$. Consider the $n_{1,2}$ copies $G(e_h)$ in which $f(V_1(e_h)) = \{x\}$ and $f(V_2(e_h)) = \{y_h\}$. We swap suitable edges such that in the resulting graph G'' obtained from G' every vertex mapped onto x has at least one neighbor mapped onto y_h for $1 \le h \le n_{1,2}$ in such a way that we still have $G'' \xrightarrow{B} G$ (see also Figure 1):

- Delete edges $(u(e_h), v_i(e_h))$ for all $1 \le i, h \le n_{1,2}$.
- Add edges $(u(e_h), v_{h+i}(e_i))$, for all $1 \le i, h \le n_{1,2}$, here $h + i$ is taken modulo $n_{1,2}$.

It is clear that after performing appropriate edge swappings for all copies of all $u \in V_1$ the resulting graph G'' still allows a locally bijective homomorphism to G via the projection $\pi : u(e) \to u$ and that the homomorphism f is surjective on the neighborhood of every vertex in any $V_1(e_h)$.

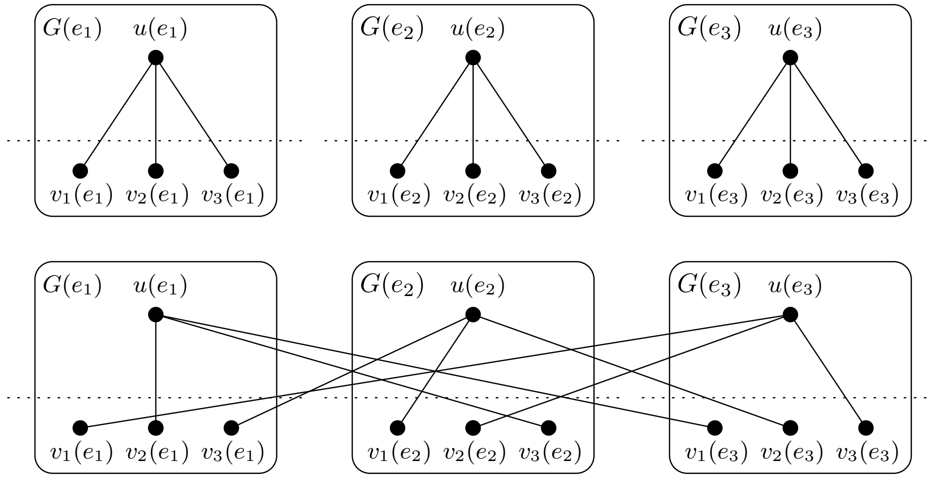

Fig. 1. Swapping edges for V_1.

We now swap edges in G'' to construct the graph G^* where also the neighbors of any vertex in $V_2(e_h)$ will receive their desired images, while ensuring $G^* \xrightarrow{B} G$.

From our previous edge swappings it is clear that G'' is the disjoint union of $|W_1|$ disjoint bipartite graphs $F_i(A_i, B_i)$ that are all *isomorphic* to each other. Here, each set A_i is chosen such that all its vertices are mapped onto x_i.

For every vertex $y \in W_2$ we act as follows. For simplicity we assume that the graphs F_i are numbered in such a way that $N_H(y) = \{x_1, \ldots, x_{n_{2,1}}\}$. We denote the corresponding edges in H by $e_j = (x_j, y)$. So $V_2(e_j)$ is in graph F_j for $1 \leq j \leq n_{2,1}$.

Recall that $f(V_2(e_j)) = \{y\}$ and that all neighbors of vertices from $V_2(e_j)$ are mapped onto x_j. Because $G \xrightarrow{S} N$, the number of neighbors of any vertex in any $V_2(e_j)$ is at least $n_{2,1}$. Because the graphs $F_i(A_i, B_i)$ are all isomorphic, any isomorphic copy $v(e_i)$ of any vertex $v \in V_2$ with $p = |N_G(v)|$ neighbors $u_1, \ldots, u_p$ is adjacent to copies $u_1((x_i, y_{j_1})), u_2((x_i, y_{j_2})), \ldots, u_p((x_i, y_{j_p}))$ for some $j_1, j_2, \ldots, j_p$ (with possibly $j_s = j_t$ for some $1 \leq s, t \leq p$). Then it is clear that, just as before, we can choose $n_{2,1}$ neighbors of v and perform appropriate edge swappings in G'' such that afterwards

- for $1 \leq i \leq n_{2,1}$ the neighbors of any vertex in B_i are mapped into the desired vertices of H;
- for $1 \leq i \leq n_{2,1}$ the neighbors of any vertex in A_i still maintain their right images;
- the resulting graph maps locally bijectively to G via the projection π.

See Figure 2 for an example of two subgraphs F_1 and F_2 in which the copies $v(e_1)$ and $v(e_2)$ of a vertex v with $|N_G(v)| = 4$ and $n_{2,1} = 2$ are displayed together with their neighbors. We swap in the same way as for V_1. After performing appropriate edge swappings for all copies of all $v \in V_2$ we obtain our desired graph G^*. $\square$

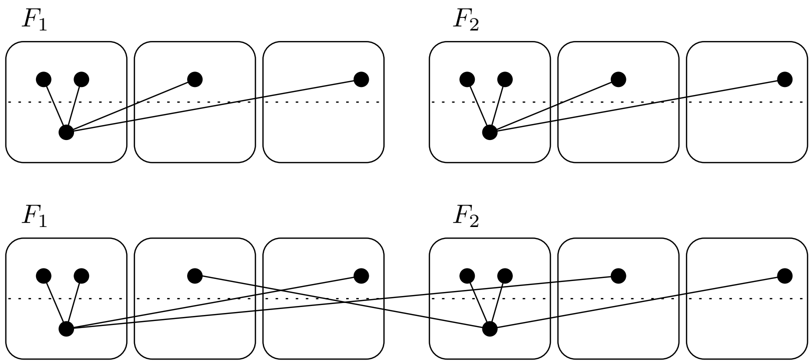

Fig. 2. Swapping edges for V_2.

The case of matrices of order one cannot be treated directly as in the above case. The reason is that the construction heavily depends on the bipartition of the graph H, which cannot be assumed in this new setting. We present here a useful trick (motivated by [10]) that allows us to focus on bipartite graphs.

Lemma 2. *Let N be a degree matrix of order one. Let G be a graph with $G \xrightarrow{S} N$. Then for any graph H with $H \xrightarrow{B} N$ there exists a graph G^* such that $G^* \xrightarrow{B} G$ and $G^* \xrightarrow{S} H$.*

Proof. Let us first recall the notion of Kronecker double cover $G \times K_2$ of a graph G. For vertices we take twice the vertex set of G, i.e., $V(G \times K_2) = V(G) \times \{1, 2\}$ and define the edges as $E(G \times K_2) = \{((u, i), (v, j)) \mid (u, v) \in E(G), i \neq j\}$. If the graph G is bipartite then its Kronecker double cover consists of two disjoint copies of G. Otherwise the resulting graph is connected and bipartite. In both cases it allows a locally bijective homomorphism $\pi : G \times K_2 \xrightarrow{B} G$ by the projection to the first coordinate: $\pi(u, i) = u$.

For the proof of the lemma take $G' = G \times K_2$, $H' = H \times K_2$ and matrix $N' = \begin{pmatrix} 0 & n_{1,1} \\ n_{1,1} & 0 \end{pmatrix}$. Then $H' \xrightarrow{B} N'$, and we can apply Lemma 1 for N', G' and H'. The resulting graph G^* satisfies $G^* \xrightarrow{B} G' \xrightarrow{B} G$ and $G^* \xrightarrow{S} H' \xrightarrow{B} H$, which proves the statement. $\square$

By definition every degree matrix contains only nonnegative integers and its zero entries are placed symmetrically around the diagonal. Before proving our main theorem we introduce some definitions that allow us to use a result on degree matrices from [13].

The *quotient graph* F_N of a degree matrix N of order l is a weighted directed graph defined as follows. Its vertex set $V(F_N)$ consists of vertices $\{1,\ldots,l\}$. There is an arc or a loop from i to j with weight $n_{i,j}$ if and only if $n_{i,j} \geq 1$. We say that a cycle $(1,2,\ldots,c,1)$ in a quotient graph F_N has the *cycle product identity* if

$$1 = \left(\prod_{i=1}^{c-1} \frac{n_{i,i+1}}{n_{i+1,i}}\right)\frac{n_{c,1}}{n_{1,c}}$$

In other words, a cycle has the cycle product identity if the product of arc weights going clockwise around the cycle is the same as the product counter-clockwise.

Proposition 1 ([13]). *For any degree matrix N, all cycles of the quotient graph F_N have the cycle product identity.*

Theorem 1. *Let M and N be degree matrices of size k and l, respectively. The following statements are equivalent.*

(i) $M \overset{S}{\leqslant} N$.
(ii) *There exists G such that $G \xrightarrow{B} M$ and $G \xrightarrow{S} N$.*
(iii) *For any H such that $H \xrightarrow{B} N$ there exists G^* such that $G^* \xrightarrow{B} M$ and $G^* \xrightarrow{S} H$.*

Proof. $(iii) \Rightarrow (i)$ This is trivially true since $M \overset{S}{\leqslant} N$ requires the existence of only a single pair G and H with $G \xrightarrow{B} M, H \xrightarrow{B} N$, and $G \xrightarrow{S} H$.

$(i) \Rightarrow (ii)$ Since $M \overset{S}{\leqslant} N$ there exist a graph G and a graph H such that $G \xrightarrow{B} M, H \xrightarrow{B} N$ and $G \xrightarrow{S} H$. The composition of $G \xrightarrow{S} H$ and $H \xrightarrow{B} N$ gives $G \xrightarrow{S} N$.

$(ii) \Rightarrow (iii)$ This is the core implication of the proof. Let G be the graph with $G \xrightarrow{B} M$ and $G \xrightarrow{S} N$. Since $G \xrightarrow{S} N$ we have a partition $\{V_1,\ldots,V_l\}$ of $V(G)$ satisfying equation (1). Let H be an arbitrary graph with $H \xrightarrow{B} N$ witnessed by a partition $\{W_1,\ldots,W_l\}$ of $V(H)$ satisfying equation (2). We will construct a graph G^* such that $G^* \xrightarrow{B} M$ and $G^* \xrightarrow{S} H$.

Let $V(F_N) = \{1,2,\ldots,l\}$, where vertex i corresponds to the i-th row and column of N. Let i,j with $i < j$ be a pair of distinct adjacent vertices in F_N. Let $H^{(i,j)}$ be the bipartite subgraph of H on vertices $W_i \cup W_j$, with all edges of H connecting sets W_i and W_j. Similarly, let $G^{(i,j)}$ be the bipartite subgraph of G with edges between V_i and V_j. Construct a graph $G^{(i,j)*}$ as in the proof of Lemma 1. Recall that we take $|E(H^{(i,j)})|$ copies $G^{(i,j)}(e)$ of $G^{(i,j)}$ that correspond to edges $e \in E(H^{(i,j)})$, and perform edge swappings in such a way that we have $f^{(i,j)} : G^{(i,j)*} \xrightarrow{S} H^{(i,j)}$. Let $V(G^{(i,j)*}) = V_i^{(i,j)*} \cup V_j^{(i,j)*}$, where all vertices in $V_i^{(i,j)*}$ are mapped into vertices of the block W_i and all vertices in $V_j^{(i,j)*}$ into vertices of W_j.

Observe that, by the construction of each G^*_{ij}, the preimage of every vertex from W_i has the same size, i.e., for all $u \in W_i$ we have that $|(f^{(i,j)})^{-1}(u)| = \frac{|V_i^{(i,j)*}|}{|W_i|}$, and vice versa for vertices from the block W_j.

According to the proof of Lemma 2 we construct a graph $G^{(i,i)*}$ for each loop (i,i) in F_N such that $G^{(i,i)*} \xrightarrow{B} G^{(i,i)}$ and $G^{(i,i)*}$ allows a mapping $f^{(i,i)} : G^{(i,i)*} \xrightarrow{S} H^{(i,i)}$. As above each vertex has the same size of the preimage, i.e., $|(f^{(i,i)})^{-1}(u)| = \frac{|V_i^{(i,i)*}|}{|W_i|}$ holds for each vertex u from W_i.

At this moment we have constructed graphs that will provide connections between blocks V^*_i and V^*_j of the final graph G^*. The graph G^* will be formed by a series of unifications of vertices from sufficiently many copies of graphs G^{e*}, where $e = (i,j)$ is taken over all arcs and loops of the quotient graph F_N with $i \leq j$. For any i and all $e \in E(F_N)$ such that $i \in e$ it must hold that the number of copies x^e of the graph G^{e*} provide the same number of vertices for the block V^*_i, i.e., $|V^*_i| = x^e \cdot |V^{e*}_i|$ whenever $i \in e$.

So far any vertex of V_i appears with the same number in repetitions in the x^e copies of V^{e*}_i for an arbitrary $e : i \in e$. By a closer look, we can partition vertices of these copies of V^{e*}_i into $|V_i| \cdot |W_i|$ groups of the same size. Vertices in the same group are mapped onto the same target vertex in W_i and appear as a copy of the same vertex from V_i. For a fixed i and each $e : i \in e$ we take one vertex from some copy of V^{e*}_i such that the chosen vertices belong to the corresponding groups and merge them into a single vertex. We repeat this $|V_i|$ times until we obtain the block V_i. We perform such a series of unification for all $i = 1, \ldots, l$ and get the desired graph G^*.

The mapping $G^* \xrightarrow{B} G$ is the projection to the original vertices of G as in Lemma 1 and Lemma 2. The locally surjective homomorphism $f : G^* \to H$ follows from the partial mappings $f^{(i,j)}$ and $f^{(i,i)}$.

The only thing that remains to verify is whether we can find nontrivial integers x^e that determine the number of copies of the building blocks G^{e*}. If we fix some $x^{(i,j)} > 0$, the sizes of sets V^*_i and V^*_j are uniquely determined. Then also the values of all x^e for all arcs e of F_N where $j \in e$ are determined and all are positive. The following equation expresses the size of V^*_i in terms of the block sizes of the original graphs G and H. (We let $x^{(i,j)} = x^{(j,i)}$ for $i > j$.)

$$|V^*_i| = x^{(i,j)} \cdot |V_i^{(i,j)*}| = x^{(i,j)} \cdot |V_i| \cdot |E(H^{(i,j)})| = x^{(i,j)} \cdot |V_i| \cdot |W_i| \cdot n_{i,j} \tag{3}$$

Assume without loss of generality that F_N contains a cycle $(1, \ldots, c, 1)$. Then the size of V^*_c can be expressed in two ways as

$$|V^*_1| \cdot \frac{|V_c|}{|V_1|} \cdot \frac{|W_c|}{|W_1|} \cdot \frac{n_{c,1}}{n_{1,c}} = |V^*_c| = |V^*_1| \cdot \frac{|V_c|}{|V_1|} \cdot \frac{|W_c|}{|W_1|} \cdot \prod_{j=1}^{c-1} \frac{n_{j+1,j}}{n_{j,j+1}}.$$

Here in the first case we have considered only the arc $(1, c)$ while in the other we have iterated (3) along the path $(1, 2, \ldots, c)$.

As each cycle of F_N satisfies the cycle product identity due to Proposition 1, the two expressions above cause no conflict. Hence, values of $\mathbf{x}$ can be derived from a single entry $x^{(i,j)}$ of each connected component of F_N, regardless which paths were used during the computation. Since all coefficients in the system of linear equations determining $\mathbf{x}$ are integers, a nontrivial integer solution exists as well. $\square$

4 Computational complexity

We are now ready to show decidability of the Matrix Surjectivity problem, i.e., deciding if $M \overset{S}{\leqslant} N$ for two degree matrices M and N. We use case (ii) of Theorem 1 and show that the existence of a suitable G can be verified in polynomial time. For a degree matrix M define $m^* = 2 + \max\{m_{i,j} \mid 1 \leq i, j \leq k\}$.

Theorem 2. *Let M, N be degree matrices of order k and l. If $M \overset{S}{\leqslant} N$, then there exists a graph G of size $(klm^*)^{O(k^2l^2)}$ such that $G \xrightarrow{S} N$ and $G \xrightarrow{B} M$.*

Proof. We first explore properties of such a hypothetic graph G. Since $G \xrightarrow{B} M$ we have a partition $\{U_1, \ldots, U_k\}$ of $V(G)$ satisfying equation (2) and since $G \xrightarrow{S} N$ we have a partition $\{V_1, \ldots, V_l\}$ of $V(G)$ satisfying equation (1). Then, for each pair of indices r and s, we define the set $W_{r,s} = \{v \mid v \in U_r \cap V_s\}$.

For vertices from $W_{r,s}$ we determine all feasible vectors describing the distribution of neighbors in the classes $W_{1,1}, \ldots, W_{k,l}$. These are vectors $\mathbf{p}^{r,s}$ of length kl whose entries $\mathbf{p}^{r,s}_{i,j}$ describe the number of neighbors a vertex in the class $W_{r,s}$ has in $W_{i,j}$. We call such a vector $\mathbf{p}^{r,s}$ a *surjective distribution row for indices r and s.* It is clear that the entries of a surjective distribution row $\mathbf{p}^{r,s}$ are positive integers and satisfy

$$
\begin{array}{lcll}
 & & \displaystyle\sum_{j=1}^{l} p^{r,s}_{i,j} = m_{r,i} & \text{for all } 1 \leq i \leq k, \\
n_{s,j} > 0 & \Rightarrow & \displaystyle\sum_{i=1}^{k} p^{r,s}_{i,j} \geq n_{s,j} & \text{for all } 1 \leq j \leq l. \\
n_{s,j} = 0 & \Rightarrow & \displaystyle\sum_{i=1}^{k} p^{r,s}_{i,j} = 0 & \text{for all } 1 \leq j \leq l.
\end{array}
$$

Obviously, the set of all different surjective distribution rows for r and s is finite and we write it as $\{\mathbf{p}^{r,s(1)}, \ldots, \mathbf{p}^{r,s(t(r,s))}\}$. We denote the total number of distribution rows by t_0. Now consider a set of t_0 variables $w^{r,s(t)}$ for all feasible r, s and all

$1 \leq t \leq t(r,s)$. The existence of a nontrivial *nonnegative* solution of the following homogeneous system of k^2l^2 equations in t_0 variables:

$$\sum_{t=1}^{t(r,s)} p_{i,j}^{r,s(t)} w^{r,s(t)} = \sum_{t'=1}^{t(i,j)} p_{r,s}^{i,j(t')} w^{i,j(t')} \qquad 1 \leq i,r \leq k,\ 1 \leq j,s \leq l \tag{4}$$

is a necessary and sufficient condition for the existence of the desired graph G. Showing necessity is easy. Sufficiency can be proved in exactly the same way as it has been done in [13] for *locally injective distribution vectors.* If a nontrivial solution of the above system exists, this solution can be transformed to some solution with $k^2l^2 + 1$ nonzero integer entries, each bounded by $(klm^*)^{O(k^2l^2)}$ (see Lemma 1 of [13]). Afterwards, a desired graph G that is within the given size bound can be constructed analogously to the construction of the graph in [13] for the locally injective homomorphisms. □

Corollary 1. *The* Matrix Surjectivity *problem belongs to the complexity class* NP*.*

Proof. For the $M \overset{S}{\leqslant} N$ comparison it is essential to check whether the system of equations (4) allow a nontrivial solution $\mathbf{w}$, which is a necessary and sufficient condition. The certificate for membership in NP consists of the $k^2l^2 + 1$ nonzero entries of the vector $\mathbf{w}$ together with the corresponding surjective distribution rows. The size of this certificate is $O(k^4l^4 \log(klm^*))$, which is polynomial in the size of both matrices M and N. It can be tested in linear time (with respect to the length of the certificate) whether all distribution rows are valid. The test whether the vector $\mathbf{w}$ satisfies (4) can also be performed in polynomial time.

5 Conclusion

We have resolved the decidability of the Matrix Surjectivity problem "is $\mathrm{drm}(G) \overset{S}{\leqslant} \mathrm{drm}(H)$?" that was left as an open problem in [13]. A nice corollary of our proof technique is that an affirmative answer implies not only the existence of a single pair of witness graphs G', H', but also the existence of an infinite set of graphs G' for which $\mathrm{drm}(G') = \mathrm{drm}(G)$ and $G' \xrightarrow{S} H$ (this follows easily from case (iii) of Theorem 1.) We arrive at an NP algorithm which means that for the many input graphs G having $\mathrm{drm}(G)$ of size at most logarithmic in G we get a polynomial-time heuristic for the H-Role Assignment problem (the problem that asks whether $G \xrightarrow{S} H$ for a fixed graph H, which is NP-complete whenever H has at least three vertices.) We note that it is not obvious whether the Matrix Surjectivity problem is NP-complete. A set of witness pairs G, H for $M \overset{S}{\leqslant} N$ can be of infinite cardinality, but the comparison $M \overset{S}{\leqslant} N$ itself provides little indication how any witness pair G, H can be useful in an NP-hardness reduction.

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