Sample 1

Chapter 2

Mathematics and Logic

Before you get started, make sure you've read Chapter 1, which sets the tone for the work we will begin doing here.

2.1 A Taste of Number Theory

In this section, we will work with the set of integers, $\mathbb{Z} = \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$. The purpose of this section is to get started with proving some theorems about numbers and study the properties of $\mathbb{Z}$. Because you are so familiar with properties of the integers, one of the issues that we will bump into knowing which facts about the integers we can take for granted. As a general rule of thumb, you should attempt to use the definitions provided without relying too much on your prior knowledge. We will likely need to discuss this further as issues arise.

It is important to note that we are diving in head first here. There are going to be some subtle issues that you will bump into and our goal will be to see what those issues are, and then we will take a step back and start again. See what you can do!

Recall that we use the symbol "$\in$" as an abbreviation for the phrase "is an element of" or sometimes simply "in." For example, the mathematical expression "$n \in \mathbb{Z}$" means "n is an element of the integers."

Definition 2.1. An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$.

Definition 2.2. An integer n is **odd** if $n = 2k+1$ for some $k \in \mathbb{Z}$.

Notice that we did not define "even" as being divisible by 2. When tackling the next few theorems and problems, you should use the formal definition of even as opposed to the well-known divisibility condition. For the remainder of this section, you may assume that every integer is either even or odd but never both.

Theorem 2.3. The sum of two consecutive integers is odd.

Theorem 2.4. If n is an even integer, then n^2 is an even integer.

Problem 2.5. Prove or provide a counterexample: The sum of an even integer and an odd integer is odd.

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Question 2.6. Did Theorem 2.3 need to come before Problem 2.5? Could we have used Problem 2.5 to prove Theorem 2.3? If so, outline how this alternate proof would go. Perhaps your original proof utilized the approach I'm hinting at. If this is true, can you think of a proof that does not rely directly on Problem 2.5? Is one approach better than the other?

Problem 2.7. Prove or provide a counterexample: The product of an odd integer and an even integer is odd.

Problem 2.8. Prove or provide a counterexample: The product of an odd integer and an odd integer is odd.

Problem 2.9. Prove or provide a counterexample: The product of two even integers is even.

Definition 2.10. An integer n **divides** the integer m, written $n|m$, if and only if there exists $k \in \mathbb{Z}$ such that $m = nk$. In the same context, we may also write that m **is divisible by** n.

Question 2.11. For integers n and m, how are following mathematical expressions similar and how are they different?

(a) $m|n$

(b) $\dfrac{m}{n}$

(c) m/n

In this section on number theory, we allow addition, subtraction, and multiplication of integers. In general, division is not allowed since an integer divided by an integer may result in a number that is not an integer. The upshot: don't write $\frac{m}{n}$. When you feel the urge to divide, switch to an equivalent formulation using multiplication. This will make your life much easier when proving statements involving divisibility.

Problem 2.12. Let $n \in \mathbb{Z}$. Prove or provide a counterexample: If 6 divides n, then 3 divides n.

Problem 2.13. Let $n \in \mathbb{Z}$. Prove or provide a counterexample: If 6 divides n, then 4 divides n.

Theorem 2.14. Assume $n, m, a \in \mathbb{Z}$. If $a|n$, then $a|mn$.

A theorem that follows almost immediately from another theorem is called a **corollary** (see Appendix B). See if you can prove the next result quickly using the previous theorem. Be sure to cite the theorem in your proof.

Corollary 2.15. Assume $n, a \in \mathbb{Z}$. If a divides n, then a divides n^2.

Problem 2.16. Assume $n, a \in \mathbb{Z}$. Prove or provide a counterexample: If a divides n^2, then a divides n.

Sample 2

Example Generation: Equations, Unknowns, and Intersections

Write a system of linear equations and the row reduced echelon form (RREF) of the corresponding augmented matrix that meets the requirements described in the table. If no such system exists, state this and explain why.

	No intersection	Intersects in a point	Intersects in a line	Intersects in a plane
2 equations & 2 unknowns				
2 equations & 3 unknowns				
3 equations & 2 unknowns				
3 equations & 3 unknowns				

Write at least 2 generalizations that can be made from these examples and the strategies you used to create them.

Sample 3

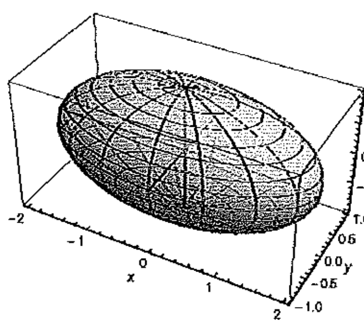
Group Work §10.6 Quadric surfaces

Instructions for the instructor. Have the students form groups of 3-4 people and assign each group a number between one and four.

Acknowledgments. I am grateful to Julie Barnes for introducing me to the idea of using PlayDoh in Calculus 3 and to Anthony Rizzie for sharing his PlayDoh activities with me.

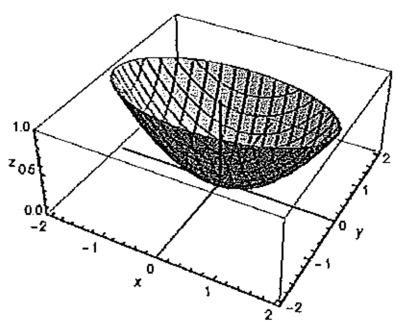
1. Four families of surfaces

Molding models. The number assigned to your group corresponds to one of the four families of quadric surfaces listed below. You will be focusing on that family and the corresponding general equation. The equation has three constants a, b and c. Each row of the table below gives a set of values for those constants. Plugging those values into the general equation gives a specific equation which is satisfied by one surface in the family. For each of these surfaces, use Play-Doh to build a model of a solid that it bounds. Use a consistent scale and orientation for your xyz-axes for all four of your models.



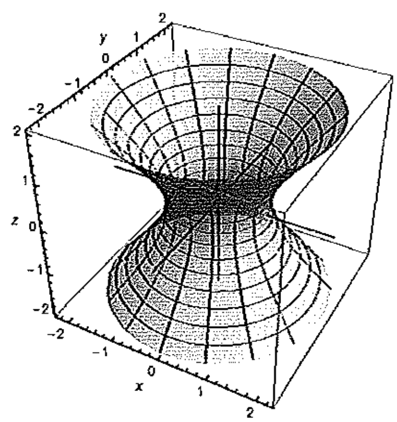
I. Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



II. Elliptic paraboliod

$$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

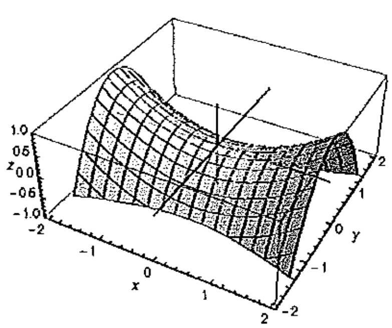


III. Hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

IV. Hyperbolic Paraboloid

$$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$



Values of Constants

a	b	c
1	1	1
0.5	1	-1
1	2	1
2	1	0.5

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Show and tell. Each member of your group should be prepared to explain how the structure of the general equation corresponds to the general shape of the surfaces in your family. You should also be able to explain how the values of the constants effect the shape of the surfaces in your family.

Once all of the groups have created their models and prepared their explanations, you will take turns sending group members off to tour the room and discover what the other groups have learned. At least one group member must stay behind at all times to provide explanations of your groups work to touring classmates.

2. One important parameter

Consider the general degree two equation below. We are interested in how the values of the constant d effect the shape of the corresponding surface. In your group, mold four new models corresponding to the four values of d listed below.

$$x^2 + y^2 - z^2 = d \text{ where } d = -1, 0, 1, 4$$

Questions.

1.) Are any of these shapes familiar? If so what are their names?

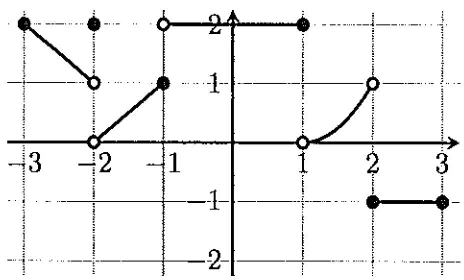
2.) How does the value of d effect the shape? What happens at $d \to \infty$? What about as $d \to -\infty$?

Sample 4

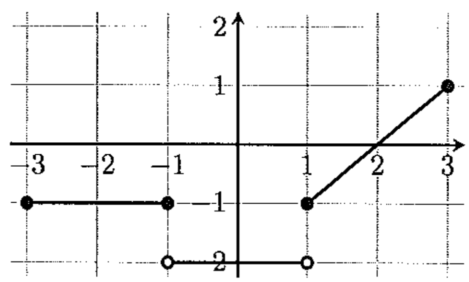
Wacky Limits Name: ______________

These limits are wacky. Help me understand the key. All I have is the answers and not the reasons why the answers are what they are. Do this by providing the correct mathematical reasons/work explaining how one gets the correct answer.

Graph of f



Graph of g



1. $\lim_{x\to 0}(f(x)+g(x)) = 0$

2. $\lim_{x\to 2^-}\dfrac{g(x)}{f(x)} = \lim_{x\to 2^+}\dfrac{g(x)}{f(x)} = \lim_{x\to 2}\dfrac{g(x)}{f(x)} = 0$

3. $\lim_{x\to -1}(f(x)+g(x)) = 0$

4. $\lim_{x\to -1}\dfrac{f(x)}{g(x)} = -1$

5. $\lim_{x\to 2}(f(x)g(x)) = 0$

6. $\lim_{x\to 3^-} f(g(x)) = 2$

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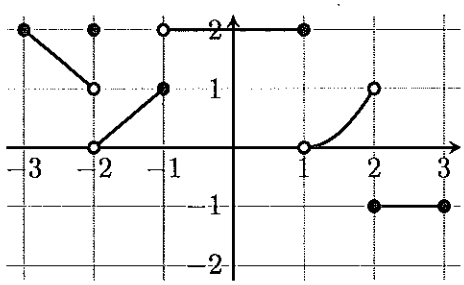
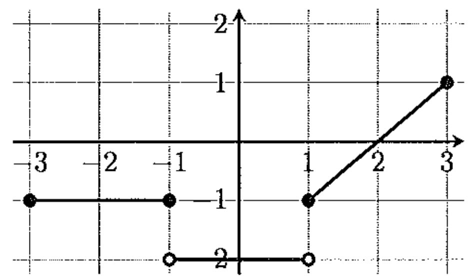
Graph of f



Graph of g



7. $\lim\limits_{x\to 1^+} f(g(x)) = 2$

8. $\lim\limits_{x\to -2^-} g(f(x)) = -1$ (and NOT -2)

9. $\lim\limits_{x\to 1^-} f(g(x)) = 2$ (and NOT 1)

10. $\lim\limits_{x\to 2^-} \dfrac{f(x)}{g(x)} = -\infty$

11. $\lim\limits_{x\to 2^+} \dfrac{f(x)}{g(x)} = -\infty$

12. $\lim\limits_{x\to 2} \dfrac{f(x)}{g(x)} = -\infty$