Assignment 2 - GATE problem 49

Jatin Tarachandani-CS20BTECH11021

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Download the python codes from

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https://github.com/jatin−tarachandani/AI1103/
    blob/main/Assignment%202/codes/
    Assignment_2.py
```

and latex codes from

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https://github.com/jatin−tarachandani/AI1103/
    blob/main/Assignment%202/main.tex
```

1 Problem Statement

A fair coin is tossed 3 times in succession. If the first toss is a head, then the probability of getting exactly two heads in three tosses is?

2 Solution

Let $K \in \{0, 1, 2, 3\}$ be the random variable denoting the possible numbers of heads we obtain in three consecutive tosses of the coin. Let $L \in \{0, 1\}$ be the random variable denoting the result of the first flip, with 0 representing a result of tails. We see that K represents the probabilities of getting a certain number of successes in 3 distinct Bernoulli trials, so we can find the probabilities for K to take its possible values via a binomial distribution b(n, p). We know that for a binomial distribution, $Pr(K = r) = \binom{n}{r} p^r (1-p)^{n-r}$. Here, $n = 3, p = 0.5$.

$$Pr(K = 0) = \binom{3}{0} 0.5^0 0.5^{3-0} = 0.125 \tag{1}$$

$$Pr(K = 1) = \binom{3}{1} 0.5^1 0.5^{3-1} = 0.375 \tag{2}$$

$$Pr(K = 2) = \binom{3}{2} 0.5^2 0.5^{3-2} = 0.375 \tag{3}$$

$$Pr(K = 3) = \binom{3}{3} 0.5^3 0.5^{3-3} = 0.125 \tag{4}$$

$$Pr(L = 1|K = 0) = 0 \tag{5}$$

$$Pr(L = 1|K = 1) = \frac{1}{3} \equiv 0.33 \tag{6}$$

$$Pr(L = 1|K = 2) = \frac{2}{3} \equiv 0.67 \tag{7}$$

$$Pr(L = 1|K = 3) = 1 \tag{8}$$

Using Bayes theorem, we get:

$$\begin{aligned} Pr(K = 2|L = 1) &= \frac{Pr(L = 1|K = 2) \cdot Pr(K = 2)}{\sum_{i=0}^{3} Pr(L = 1|K = i) \cdot Pr(K = i)} \\ &= \frac{\frac{2}{3} \cdot \binom{3}{2} p^2 (1-p)^{3-2}}{\begin{array}{l} 0 \cdot \binom{3}{0} p^0 (1-p)^3 + \frac{1}{3} \cdot \binom{3}{1} p^1 (1-p)^2 \\ + \frac{2}{3} \cdot \binom{3}{2} p^2 (1-p)^1 + 1 \cdot \binom{3}{3} p^3 (1-p)^{3-3} \end{array}} \end{aligned}$$

Since $p = 0.5 = 1 - p$, we can replace $1 - p$ by p.

$$\begin{aligned} Pr(K = 2|L = 1) &= \frac{0.67 \cdot 3 \cdot p^3}{\begin{array}{l} 0 \cdot p^3 + 0.33 \cdot 3 \cdot p^3 \\ + 0.67 \cdot 3 \cdot p^3 + 1 \cdot p^3 \end{array}} \\ &= \frac{2 \cdot p^3}{4 \cdot p^3} \\ &= 0.5. \end{aligned}$$

The probability of getting exactly 2 heads in 3 tosses, if the first toss is a head, is 0.5.