# Assignment 5

Jatin Tarachandani-CS20BTECH11021

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https://github.com/jatin−tarachandani/AI1103/blob/main /Assignment_5/main.tex

## 1 Problem Statement

Let $X$, $Y$, have the joint discrete distribution such that $X|Y = y \sim$ Binomial($y$, 0.5) and $Y \sim$ Poisson($\lambda$), $\lambda > 0$, where $\lambda$ is an unknown parameter. Let $T = T(X, Y)$ be any unbiased estimator of $\lambda$. Then

1. $Var(T) \leq Var(Y)$ for all $\lambda$
2. $Var(T) \geq Var(Y)$ for all $\lambda$
3. $Var(T) \geq \lambda$ for all $\lambda$
4. $Var(T) = Var(Y)$ for all $\lambda$

## 2 Solution

If an estimator of a certain parameter of a sample of data has the expected value exactly equal to the parameter it estimates, then it is said to be an unbiased estimator.
Suppose we have an estimator $T(X, Y)$, acting on a set of 2 RVs $X, Y$; which estimates a parameter $\theta$, then if

$$E\left(T(X, Y)\right) = \theta \tag{2.0.1}$$

the estimator is unbiased.
We know that:

$$Var(Y) = \lambda \tag{2.0.2}$$

as Y has a Poisson distribution.

From Bayes Theorem,

$$Pr\left(X = x, Y = y\right) = Pr\left(Y = y\right) \cdot Pr\left(X = x|Y = y\right) \tag{2.0.3}$$

$$= \frac{\lambda^y}{y!}e^{-\lambda} \cdot \binom{y}{x}\frac{1}{2^y} \tag{2.0.4}$$

Let us represent $Pr\left(X = x, Y = y\right)$ as a function of $x, y$, and $\lambda$: $f(x, y; \lambda)$.
$T(X, Y)$ is an unbiased estimator of $\lambda$. The Cramer-Rao bound for an unbiased estimator T of a parameter $\lambda$ states that:

$$Var\left(T\left(X, Y\right)\right) \geq \frac{-1}{E(\frac{\partial^2 ln(f(x,y;\lambda))}{\partial\lambda^2})} \tag{2.0.5}$$

From the above definition of Cramer-Rao bound,

$$Var(T(X, Y)) \geq \frac{-1}{E(\frac{\partial^2 ln(f(x,y;\lambda))}{\partial\lambda^2})} \tag{2.0.6}$$

$$\geq \frac{-1}{E\left(\frac{-y}{\lambda^2}\right)} \tag{2.0.7}$$

$$\geq \frac{\lambda^2}{E\left(y\right)} \tag{2.0.8}$$

$$\geq \lambda \tag{2.0.9}$$

This is because the expectation value of a Poisson distribution with parameter $\lambda$ is $\lambda$ itself.
The correct options are options (2) and (3), since by the application of Cramer-Rao bound, we see that the variance of $T$ is $\geq \lambda = Var(Y)$ from (2.0.2). Hence, (1) and (4) are incorrect as they would only be true in the case where Var(T) = $\lambda$ for some specific lambda.