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Some Definitions

Definition

The pmf for a binomial distribution with parameters n and p is

$$\Pr\left(X = x; n, p\right) = {}^{n}\mathrm{C}_{x} p^{x} \left(1 - p\right)^{n-x} \tag{1}$$

Definition

The pmf for a Poisson distribution with parameter λ is:

$$\Pr\left(Y = y; \lambda\right) = \frac{\lambda^{y}}{y!} e^{-\lambda} \tag{2}$$

Unbiased Estimators

Definition

Suppose we have an estimator $T(X, Y)$, acting on a set of 2 RVs X, Y; which estimates a parameter θ, then if

$$E\left(T(X, Y)\right) = \theta \tag{3}$$

the estimator is unbiased.

Problem statement

Let X, Y, have the joint discrete distribution such that $X|Y = y \sim$ Binomial$(y, 0.5)$ and $Y \sim$ Poisson(λ), $\lambda > 0$, where λ is an unknown parameter. Let $T = T(X, Y)$ be any unbiased estimator of λ. Then

1. $Var(T) \leq Var(Y)$ for all λ
2. $Var(T) \geq Var(Y)$ for all λ
3. $Var(T) \geq \lambda$ for all λ
4. $Var(T) = Var(Y)$ for all λ

Solution

We know that since Y has a Poisson distribution with parameter λ,

$$Var(Y) = \lambda \tag{4}$$

Since X, Y are discrete, we define $p_{XY}(X, Y)$ as the joint discrete distribution of X and Y. From Bayes Theorem,

$$p_{XY}(X, Y) = \Pr(X = x | Y = y) \Pr(Y = y) \tag{5}$$

$$= {}^{Y}\mathrm{C}_{X} \frac{1}{2^{Y}} \frac{\lambda^{Y}}{Y!} e^{-\lambda} \tag{6}$$

Cramer Rao lower bound

Definition

If $T(X,Y)$ is an unbiased estimator of a parameter λ, the Cramer-Rao bound states that:

$$Var\left(T\left(X,Y\right)\right) \geq -\frac{1}{E\left(\frac{\partial^2 \ln(p_{XY}(X,Y)))}{\partial \lambda^2}\right)} \tag{7}$$

Since λ is a continuous parameter, we can apply this inequality to the logarithm of the joint discrete distribution seen in (6).

Lemma

$$\frac{\partial^2 \ln(p_{XY}(x, y))}{\partial \lambda^2} = -\frac{y}{\lambda^2} \tag{8}$$

Proof.

$$p_{XY}(X, Y) = {}^{Y}\mathrm{C}_X \frac{1}{2^Y} \frac{\lambda^Y}{Y!} e^{-\lambda} \tag{9}$$

$$\frac{\partial^2 \ln(p_{XY}(X, Y))}{\partial \lambda^2} = \frac{\partial^2 \begin{pmatrix} \ln {}^{Y}\mathrm{C}_X - Y \ln 2^Y \\ + Y \ln \lambda - \ln Y! - \lambda \end{pmatrix}}{\partial \lambda^2} \tag{10}$$

$$= \frac{\partial \left(\frac{Y}{\lambda} - 1\right)}{\partial \lambda} \tag{11}$$

$$= -\frac{Y}{\lambda^2} \tag{12}$$

which is the required result. □

From using (7) on $T(X, Y)$,

$$Var(T(X, Y)) \geq -\frac{1}{E\left(\frac{\partial^2 \ln(p_{XY}(X,Y))}{\partial \lambda^2}\right)} \tag{13}$$

From (8),

$$\begin{aligned} Var(T(X, Y)) &\geq -\frac{1}{E\left(-\frac{Y}{\lambda^2}\right)} && (14) \\ &\geq \frac{\lambda^2}{E(Y)} && (15) \\ Var(T(X, Y)) &\geq \lambda && (16) \end{aligned}$$

because the expectation value of a Poisson R.V with parameter λ is λ.

The correct options are options (2) and (3), since by (16), we see that

$$Var(T) \geq \lambda = Var(Y)$$

(from (4)). (1) and (4) do not hold for all λ.