Assignment 5

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Download latex codes from

https://github.com/jatin−tarachandani/AI1103/blob/main/Assignment_5/main.tex

1 Problem Statement

Let X, Y, have the joint discrete distribution such that $X|Y = y \sim \text{Binomial}(y, 0.5)$ and $Y \sim \text{Poisson}(\lambda)$, $\lambda > 0$, where λ is an unknown parameter. Let $T = T(X, Y)$ be any unbiased estimator of λ. Then

1. $Var(T) \leq Var(Y)$ for all λ
2. $Var(T) \geq Var(Y)$ for all λ
3. $Var(T) \geq \lambda$ for all λ
4. $Var(T) = Var(Y)$ for all λ

2 Solution

Definition 2.1. The pmf for a binomial distribution with parameters n and p is

$$\Pr\left(X = x; n, p\right) = {}^{n}\mathrm{C}_{x} p^{x} \left(1 - p\right)^{n-x} \tag{2.0.1}$$

Definition 2.2. The pmf for a Poisson distribution with parameter λ is:

$$\Pr\left(Y = y; \lambda\right) = \frac{\lambda^{y}}{y!} e^{-\lambda} \tag{2.0.2}$$

Lemma 2.1. Since Y has a Poisson distribution, we know that:

$$Var(Y) = \lambda \tag{2.0.3}$$

Definition 2.3. Suppose we have an estimator $T(X, Y)$, acting on a set of 2 RVs X, Y; which estimates a parameter θ, then if

$$E\left(T(X, Y)\right) = \theta \tag{2.0.4}$$

the estimator is unbiased.

We define $p_{XY}\left(X, Y\right)$ as the joint discrete distribution of X and Y. From Bayes Theorem,

$$p_{XY}\left(X, Y\right) = \Pr\left(Y = y\right) \Pr\left(X = x|Y = y\right) \tag{2.0.5}$$

$$= {}^{y}\mathrm{C}_{x} \frac{1}{2^{y}} \frac{\lambda^{y}}{y!} e^{-\lambda} \tag{2.0.6}$$

Definition 2.4. If $T(X, Y)$ is an unbiased estimator of a parameter λ, the Cramer-Rao bound states that:

$$Var\left(T\left(X, Y\right)\right) \geq -\frac{1}{E\left(\frac{\partial^{2} \ln(p_{XY}(x,y)))}{\partial \lambda^{2}}\right)} \tag{2.0.7}$$

Lemma 2.2.

$$\frac{\partial^{2} \ln(p_{XY}\left(X, Y\right))}{\partial \lambda^{2}} = -\frac{Y}{\lambda^{2}} \tag{2.0.8}$$

Proof.

$$p_{XY}\left(X, Y\right) = {}^{Y}\mathrm{C}_{X} \frac{1}{2^{Y}} \frac{\lambda^{Y}}{Y!} e^{-\lambda} \tag{2.0.9}$$

$$\frac{\partial^{2} \ln(p_{XY}\left(X, Y\right))}{\partial \lambda^{2}} = \frac{\partial^{2} \begin{pmatrix} \ln {}^{Y}\mathrm{C}_{X} - Y \ln 2^{Y} \\ + Y \ln \lambda - \ln Y! - \lambda \end{pmatrix}}{\partial \lambda^{2}} \tag{2.0.10}$$

$$= \frac{\partial \left(\frac{Y}{\lambda} - 1\right)}{\partial \lambda} \tag{2.0.11}$$

$$= -\frac{Y}{\lambda^{2}} \tag{2.0.12}$$

which is the required result. □

From using (2.0.7) on $T(X, Y)$,

$$Var(T(X, Y)) \geq -\frac{1}{E\left(\frac{\partial^{2} \ln(p_{XY}(X,Y))}{\partial \lambda^{2}}\right)} \tag{2.0.13}$$

(2.0.14)

From *Lemma* 2.2,

$$Var(T(X,Y)) \geq -\frac{1}{E\left(-\frac{Y}{\lambda^2}\right)} \tag{2.0.15}$$

$$\geq \frac{\lambda^2}{E(Y)} \tag{2.0.16}$$

$$\geq \lambda \tag{2.0.17}$$

because the expectation value of a Poisson distribution with parameter λ is λ.
The correct options are options (2) and (3), since by (2.0.17), we see that

$$Var(T) \geq \lambda = Var(Y)$$

(from (2.0.3)). (1) and (4) do not hold for all λ.