Assignment 5

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Download latex codes from

https://github.com/jatin−tarachandani/AI1103/blob/main/Assignment_5/main.tex

1 Problem Statement

Let X, Y, have the joint discrete distribution such that $X|Y = y \sim$ Binomial(y, 0.5) and $Y \sim$ Poisson(λ), $\lambda > 0$, where λ is an unknown parameter. Let $T = T(X, Y)$ be any unbiased estimator of λ. Then

1. $Var(T) \leq Var(Y)$ for all λ
2. $Var(T) \geq Var(Y)$ for all λ
3. $Var(T) \geq \lambda$ for all λ
4. $Var(T) = Var(Y)$ for all λ

2 Solution

Definition 2.1. Suppose we have an estimator $T(X, Y)$, acting on a set of 2 RVs X, Y; which estimates a parameter θ, then if

$$E\left(T(X, Y)\right) = \theta \tag{2.0.1}$$

the estimator is unbiased.

Lemma 2.1. Since Y has a Poisson distribution, we know that:

$$Var(Y) = \lambda \tag{2.0.2}$$

From Bayes Theorem,

$$Pr\left(X = x, Y = y\right) = Pr\left(Y = y\right) Pr\left(X = x|Y = y\right) \tag{2.0.3}$$
$$= {}^{y}\mathrm{C}_{x} \frac{1}{2^{y}} \frac{\lambda^{y}}{y!} e^{-\lambda} \tag{2.0.4}$$

Let us represent $Pr\left(X = x, Y = y\right)$ as a function of x, y, and λ: $f(x, y; \lambda)$.

Definition 2.2. If $T(X, Y)$ is an unbiased estimator of a parameter λ, the Cramer-Rao bound states that:

$$Var\left(T\left(X, Y\right)\right) \geq \frac{-1}{E\left(\frac{\partial^2 ln(f(x,y;\lambda))}{\partial\lambda^2}\right)} \tag{2.0.5}$$

From applying Definition 2.2 on $T(X, Y)$,

$$Var(T(X, Y)) \geq \frac{-1}{E\left(\frac{\partial^2 ln(f(x,y;\lambda))}{\partial\lambda^2}\right)} \tag{2.0.6}$$
$$\geq \frac{-1}{E\left(\frac{-y}{\lambda^2}\right)} \tag{2.0.7}$$
$$\geq \frac{\lambda^2}{E\left(y\right)} \tag{2.0.8}$$
$$\geq \lambda \tag{2.0.9}$$

because the expectation value of a Poisson distribution with parameter λ is λ.
The correct options are options (2) and (3), since by (2.0.9), we see that the variance of T is $\geq \lambda = Var(Y)$ (from (2.0.2)). (1) and (4) do not hold for all λ.