Assignment 5

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https://github.com/jatin−tarachandani/AI1103/blob/main/Assignment_5/main.tex

1 Problem Statement

Let X, Y, have the joint discrete distribution such that $X|Y = y \sim \text{Binomial}(y, 0.5)$ and $Y \sim \text{Poisson}(\lambda)$, $\lambda > 0$, where λ is an unknown parameter. Let $T = T(X, Y)$ be any unbiased estimator of λ. Then

1. $Var(T) \leq Var(Y)$ for all λ
2. $Var(T) \geq Var(Y)$ for all λ
3. $Var(T) \geq \lambda$ for all λ
4. $Var(T) = Var(Y)$ for all λ

2 Solution

Definition 2.1. Suppose we have an estimator $T(X, Y)$, acting on a set of 2 RVs X, Y; which estimates a parameter θ, then if

$$E\left(T(X, Y)\right) = \theta \tag{2.0.1}$$

the estimator is unbiased.

Lemma 2.1. Since Y has a Poisson distribution, we know that:

$$Var(Y) = \lambda \tag{2.0.2}$$

We define $p_{XY}(x, y)$ as the joint discrete distribution of X and Y. From Bayes Theorem,

$$\begin{aligned} p_{XY}(x, y) &= \Pr(Y = y)\Pr(X = x|Y = y) && (2.0.3) \\ &= {}^{y}\mathrm{C}_{x}\frac{1}{2^{y}}\frac{\lambda^{y}}{y!}e^{-\lambda} && (2.0.4) \end{aligned}$$

Definition 2.2. If $T(X, Y)$ is an unbiased estimator of a parameter λ, the Cramer-Rao bound states that:

$$Var\left(T\left(X, Y\right)\right) \geq -\frac{1}{E\left(\frac{\partial^2 \ln(p_{XY}(x,y)))}{\partial\lambda^2}\right)} \tag{2.0.5}$$

Lemma 2.2.

$$\frac{\partial^2 \ln(p_{XY}(x, y))}{\partial\lambda^2} = -\frac{y}{\lambda^2} \tag{2.0.6}$$

Proof.

$$\begin{aligned} p_{XY}(x, y) &= {}^{y}\mathrm{C}_{x}\frac{1}{2^{y}}\frac{\lambda^{y}}{y!}e^{-\lambda} && (2.0.7) \\ \frac{\partial^2 \ln(p_{XY}(x, y))}{\partial\lambda^2} &= \frac{\partial^2\begin{pmatrix}\ln {}^{y}\mathrm{C}_{x} - y\ln 2^{y} \\ + y\ln\lambda - \ln y! - \lambda\end{pmatrix}}{\partial\lambda^2} && (2.0.8) \\ &= \frac{\partial\left(\frac{y}{\lambda} - 1\right)}{\partial\lambda} && (2.0.9) \\ &= -\frac{y}{\lambda^2} && (2.0.10) \end{aligned}$$

which is the required result. □

From using (2.0.5) on $T(X, Y)$,

$$\begin{aligned} Var(T(X, Y)) &\geq -\frac{1}{E\left(\frac{\partial^2 \ln(p_{XY}(x,y))}{\partial\lambda^2}\right)} && (2.0.11) \\ & && (2.0.12) \end{aligned}$$

From *Lemma* 2.2,

$$\begin{aligned} Var(T(X, Y)) &\geq -\frac{1}{E\left(-\frac{y}{\lambda^2}\right)} && (2.0.13) \\ &\geq \frac{\lambda^2}{E\left(y\right)} && (2.0.14) \\ &\geq \lambda && (2.0.15) \end{aligned}$$

because the expectation value of a Poisson distribution with parameter λ is λ.

The correct options are options (2) and (3), since by (2.0.15), we see that

$$Var(T) \geq \lambda = Var(Y)$$

(from (2.0.2)). (1) and (4) do not hold for all λ.