

Assignment 5

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https://github.com/jatin-tarachandani/AI1103/blob/main/Assignment_5/main.tex

mates a parameter θ , then if

$$E(T(X, Y)) = \theta \quad (2.0.4)$$

the estimator is unbiased.

We define $p_{XY}(x, y)$ as the joint discrete distribution of X and Y . From Bayes Theorem,

$$p_{XY}(x, y) = \Pr(Y = y) \Pr(X = x|Y = y) \quad (2.0.5)$$

$$= {}^yC_x \frac{1}{2^y} \frac{\lambda^y}{y!} e^{-\lambda} \quad (2.0.6)$$

Definition 2.4. If $T(X, Y)$ is an unbiased estimator of a parameter λ , the Cramer-Rao bound states that:

$$\text{Var}(T(X, Y)) \geq -\frac{1}{E\left(\frac{\partial^2 \ln(p_{XY}(x, y))}{\partial \lambda^2}\right)} \quad (2.0.7)$$

Lemma 2.2.

$$\frac{\partial^2 \ln(p_{XY}(x, y))}{\partial \lambda^2} = -\frac{y}{\lambda^2} \quad (2.0.8)$$

1 Problem Statement

Let X, Y , have the joint discrete distribution such that $X|Y = y \sim \text{Binomial}(y, 0.5)$ and $Y \sim \text{Poisson}(\lambda)$, $\lambda > 0$, where λ is an unknown parameter. Let $T = T(X, Y)$ be any unbiased estimator of λ . Then

1. $\text{Var}(T) \leq \text{Var}(Y)$ for all λ
2. $\text{Var}(T) \geq \text{Var}(Y)$ for all λ
3. $\text{Var}(T) \geq \lambda$ for all λ
4. $\text{Var}(T) = \text{Var}(Y)$ for all λ

2 Solution

Definition 2.1. The pmf for a binomial distribution with parameters n and p is

$$\Pr(X = x; n, p) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \quad (2.0.1)$$

Definition 2.2. The pmf for a Poisson distribution with parameter λ is:

$$\Pr(Y = y; \lambda) = \frac{\lambda^y}{y!} e^{-\lambda} \quad (2.0.2)$$

Lemma 2.1. Since Y has a Poisson distribution, we know that:

$$\text{Var}(Y) = \lambda \quad (2.0.3)$$

Definition 2.3. Suppose we have an estimator $T(X, Y)$, acting on a set of 2 RVs X, Y ; which esti-

Proof.

$$p_{XY}(x, y) = {}^yC_x \frac{1}{2^y} \frac{\lambda^y}{y!} e^{-\lambda} \quad (2.0.9)$$

$$\frac{\partial^2 \ln(p_{XY}(x, y))}{\partial \lambda^2} = \frac{\partial^2 \left(\ln {}^yC_x - y \ln 2^y + y \ln \lambda - \ln y! - \lambda \right)}{\partial \lambda^2} \quad (2.0.10)$$

$$= \frac{\partial \left(\frac{y}{\lambda} - 1 \right)}{\partial \lambda} \quad (2.0.11)$$

$$= -\frac{y}{\lambda^2} \quad (2.0.12)$$

which is the required result. \square

From using (2.0.7) on $T(X, Y)$,

$$\text{Var}(T(X, Y)) \geq -\frac{1}{E\left(\frac{\partial^2 \ln(p_{XY}(x, y))}{\partial \lambda^2}\right)} \quad (2.0.13)$$

$$(2.0.14)$$

From *Lemma 2.2*,

$$\text{Var}(T(X, Y)) \geq -\frac{1}{E\left(-\frac{y}{\lambda^2}\right)} \quad (2.0.15)$$

$$\geq \frac{\lambda^2}{E(y)} \quad (2.0.16)$$

$$\geq \lambda \quad (2.0.17)$$

because the expectation value of a Poisson distribution with parameter λ is λ .

The correct options are options (2) and (3), since by (2.0.17), we see that

$$\text{Var}(T) \geq \lambda = \text{Var}(Y)$$

(from (2.0.3)). (1) and (4) do not hold for all λ .