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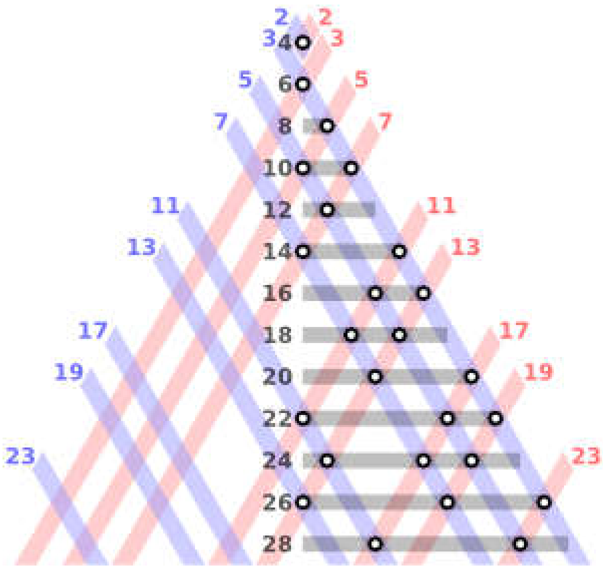
Goldbach's conjecture

Goldbach's conjecture is one of the oldest and best-known unsolved problems in number theory and all of mathematics. It states:

> Every even integer greater than 2 can be expressed as the sum of two primes.[1]

The conjecture has been shown to hold for all integers less than 4×10^{18},[2] but remains unproven despite considerable effort.



The even integers from 4 to 28 as sums of two primes: Even integers correspond to horizontal lines. For each prime, there are two oblique lines, one red and one blue. The sums of two primes are the intersections of one red and one blue line, marked by a circle. Thus the circles on a given horizontal line give all partitions of the corresponding even integer into the sum of two primes.

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Goldbach number

A Goldbach number is a positive integer that can be expressed as the sum of two odd primes.[4] Since four is the only even number greater than two that requires the even prime 2 in order to be written as the sum of two primes, another form of the statement of Goldbach's conjecture is that all even integers greater than 4 are Goldbach numbers.

The expression of a given even number as a sum of two primes is called a Goldbach partition of that number. The following are examples of Goldbach partitions for some even numbers:

> 6 = 3 + 3
> 8 = 3 + 5
> 10 = 3 + 7 = 5 + 5
> 12 = 7 + 5
> ...
> 100 = 3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53
> ...

The number of ways in which $2n$ can be written as the sum of two primes (for n starting at 1) is:

> 0, 1, 1, 1, 2, 1, 2, 2, 2, 2, 3, 3, 3, 2, 3, 2, 4, 4, 2, 3, 4, 3, 4, 5, 4, 3, 5, 3, 4, 6, 3, 5, 6, 2, 5, 6, 5, 5, 7, 4, 5, 8, 5, 4, 9, 4, 5, 7, 3, 6, 8, 5, 6, 8, 6, 7, 10, 6, 6, 12, 4, 5, 10, 3, ... (sequence A045917 in the OEIS).

Origins

On 7 June 1742, the German mathematician Christian Goldbach wrote a letter to Leonhard Euler (letter XLIII)[6] in which he proposed the following conjecture:

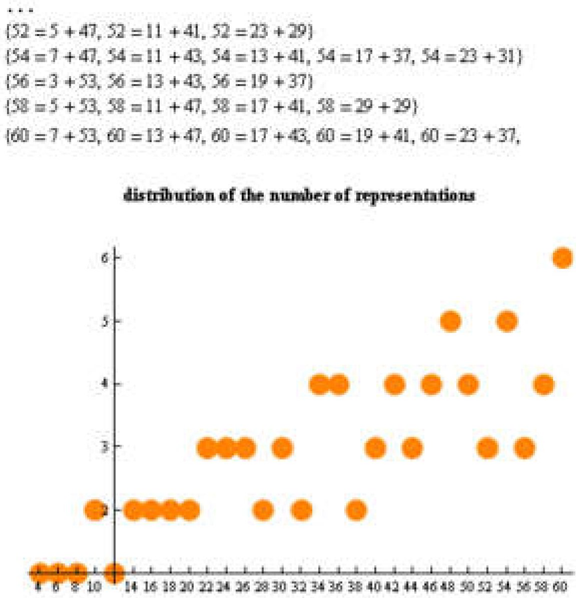
> Every integer which can be written as the sum of two primes, can also be written as the sum of as many primes as one wishes, until all terms are units.

He then proposed a second conjecture in the margin of his letter:

> Every integer greater than 2 can be written as the sum of three primes.

He considered 1 to be a prime number, a convention subsequently abandoned.[1] The two conjectures are now known to be equivalent, but this did not seem to be an issue at the time. A modern version of Goldbach's marginal conjecture is:

> **Every integer greater than 5 can be written as the sum of three primes**.



The number of ways an even number can be represented as the sum of two primes.[3]

Euler replied in a letter dated 30 June 1742, and reminded Goldbach of an earlier conversation they had ("...so Ew vormals mit mir communicirt haben..."), in which Goldbach remarked his original (and not marginal) conjecture followed from the following statement

> **Every even integer greater than 2 can be written as the sum of two primes**,


Letter from Goldbach to Euler dated on 7. June 1742 (Latin-German).[5]

which is, thus, also a conjecture of Goldbach. In the letter dated 30 June 1742, Euler stated:

> "Dass ... ein jeder numerus par eine summa duorum primorum sey, halte ich für ein ganz gewisses theorema, ungeachtet ich dasselbe nicht demonstriren kann." ("That ... every even integer is a sum of two primes, I regard as a completely certain theorem, although I cannot prove it.")[7][8]

Goldbach's third version (equivalent to the two other versions) is the form in which the conjecture is usually expressed today. It is also known as the "strong", "even", or "binary" Goldbach conjecture, to distinguish it from a weaker conjecture, known today variously as the **Goldbach's weak conjecture**, the "odd" Goldbach conjecture, or the "ternary" Goldbach conjecture. This weak conjecture asserts that *all odd numbers greater than 7 are the sum of three odd primes*, and appears to have been proved in 2013.[9][10] The weak conjecture is a corollary of the strong conjecture, as, if $n - 3$ is a sum of two primes, then n is a sum of three primes. The converse implication, and the strong Goldbach conjecture remain unproven.

Verified results