# Local Envy Free Allocation in Network Graph Setting

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- We also assume that agents have a limited information, i.e, they can get the info for only those agents who are directly connected to them.
- Our aim in this presentation is to find the fair allocation for this setting.

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# Literature Review

- We focus on the problem DEC-LEF: to find out whether the local-envy free allocation exists or not?
- It is known that even for sparse graphs, and for regular graphs of degree n-3 the problem of determining the existence of local envy-free allocations is NP-hard.

- We focus on the problem DEC-LEF: to find out whether the local-envy free allocation exists or not?
- It is known that even for sparse graphs, and for regular graphs of degree n-3 the problem of determining the existence of local envy-free allocations is NP-hard.
- We showed that if we restrict the agents domain to Single Peak and even less strict domain like local-Single Peak there is an efficient procedure to determine the existence of a local envy-free allocation.

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- DEC-LEF is polynomial for regular graphs with degree atleast n-2.
- DEC-LEF is NP-hard even for regular graphs with degree n-3. The proof of this result is through reduction from 3(2B)-SAT.

# Definitions

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- **Single Peaked over $\triangleright$:** An ordering $\triangleright$ over the objects O is single peaked over N if $\exists o^* \in O$ such that $\forall i \in N$, we have
  (1) $a \triangleright b \triangleright o^* \rightarrow b \succ_i a$
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- **Locally single peaked:** The problem $e =< N, O, \succ, G = (N, E) >$ is locally single peaked if $\forall i \in N$, there exists an ordering $\triangleright_i$ over the objects for which the set $N_i$ is single peaked.

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- **Locally single peaked with fixed ordering:** The problem $e$ is locally single peaked with fixed ordering if there exists an ordering $\triangleright$ over the objects O such that $\forall i \in N$, the set $N_i$ is single peaked with respect to $\triangleright$.

- **Problem:** Given a regular graph G = (N,E) with each vertex degree = n-k and the preference profile is single peaked, then we have to return the local envy free allocation(if exists), otherwise return no, i.e. no such allocation exists.

# Problem Statement

- **Problem:** Given a regular graph G = (N,E) with each vertex degree = n-k and the preference profile is single peaked, then we have to return the local envy free allocation(if exists), otherwise return no, i.e. no such allocation exists.
- We will describe the algorithm that will solve this problem in $O(n)$ time where $n = |N|$.

# Observations

- Since the degree of each vertex is n-k, then an agent is **local envy free** only if he gets an object from his **top k preferences**. Therefore, we will focus only on the top k preferred objects of each agent.

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- Since the profile is **single peaked**, lets say over $\triangleright$. Then the top k preferred objects of each agent forms an interval in the ordering $\triangleright$ over the objects.

# Model

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- Let $p_i$ denotes the position of the leftmost positioned object among the $r_i(1), r_i(2), r_i(3), ..., r_i(k)$ objects in the ordering $\triangleright$. So, objects at $p_i, p_i + 1, p_i + 2, ..., p_i + k - 1$ will denote the positions of top-k preferred objects.

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- Let $a_1, a_2, a_3, ...a_n$ denotes the agents. We define $A(i)$ agents at position i, as the set of agents such that $\{a_j | p_j = i\}$. Therefore, the number of agents at position i is $|A(i)|$.

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- We denote **stage i** as the point when agents in $A(1) \cup A(2) \cup A(3).. \cup A(i)$ gets the envy free allocation.

# Inferences

- if $|A(i)| > k$, then there doesn't exist any envy free allocation. Since, all the agents in A(i) needs the objects from $o_i, o_{i+1}, o_{i+2}, ..o_{i+k-1}$, and by pigeon hole principal at least one agent will not receive the object.

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- If we find the envy free allocation for the agents in $A(1) \cup A(2) \cup A(3) \cup ... \cup A(i)$, then the agents in $A(1) \cup A(2) \cup ..A(i-k+1)$ are always local envy-free because for any extension of this allocation all the other agents get the object of lower preference than their own.

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- The agents $A(i)$ can't be envy to other agent for the objects at position $1, 2, ..., i-1$. Hence, agents in $A(i+1) \cup A(i+2).. \cup A(n)$ will care only for the objects $o_i$ onwards.

# Inferences

## Continued ..

- From previous 2 arguments, we can say that given the allocation of agents $A(i-k+2), A(i-k+3), ..A(i-1), A(i)$ all the possible local envy free allocation of $A(1) \cup A(2) \cup A(3) \cup ... \cup A(i)$ comes in the same equivalence class. Hence, no of equivalence class $< k^{k-1}$ at each stage.

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- This is the crux of our incremental algorithm that number of equivalent class at each stage is independent of n.

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- Total time complexity is the time to reach the stage-n starting from stage-2 and repeated using the previous step. Hence, overall time complexity $< nk^{k-1}k!$.

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- We can also extend this result for the graphs whose minimum degree is n-k when the agents profile is single peaked.
- We can also extend this result for the graphs when the minimum degree is n-k but the agents profile preference is local single peaked(less stricter than single peaked).