* **First Fundamental Theorem of asset pricing** :

A risk-neutral probability measure exists iff the market exhibits no arbitrage principle

i.e., NAP $\Leftrightarrow$ $\exists$ RNPM

Pf : - NAP $\Rightarrow$ $\exists$ RNPM

we will be using linear program & duality theorem

$\rightarrow$ $\min \; c^t x$

subject to $Ax \leq b$

x^*

$x \geq 0$

$A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $c \in \mathbb{R}^n$

x : unknown

Dual : $\max \; b^t \omega$

s.to $\underset{n \times m}{A^t} \; \underset{m \times 1}{\omega} \geq \underset{n \times 1}{c}$

ω^*

$\omega \geq 0$

$\omega \in \mathbb{R}^m$: dual variable

$$\left.\begin{aligned} \underset{1 \times m}{\omega^{*t}} \; \underset{m \times 1}{(b - Ax^*)} &= 0 \\ x^{*t}(A^t \omega^* - c) &= 0 \end{aligned}\right\} \text{ complementary slackness theorem}$$

* **Goldman Tucker theorem** :

$\exists$ optimal pair (x^*, ω^*) solution of the two problems such that

$$\omega^* + (b - Ax^*) > 0$$

$$x^* + (A^t \omega^* - c) > 0$$

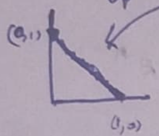
$$0 = \omega^{*t} \cdot (b - Ax^*)$$

$$= \underbrace{\sum_{i=1}^{m} \underbrace{\omega_i^*}_{\geq 0} \underbrace{(b - Ax^*)_i}_{\geq 0}}_{\geq 0}$$

$$w_i^*(b-Ax^*)_i = 0 \quad \forall i$$

$w_i^* = 0$	$w_i^* > 0$	$w_i^* = 0$
$(b-Ax^*)_i > 0$	$(b-Ax^*)_i = 0$	$(b-Ax^*)_i = 0$
✓	✓	✗

Eg:



optimal sol^{ns}

$$\begin{cases} \min & x_1 + x_2 \\ \text{s. to} & x_1 + x_2 = 1 \leftarrow y_1 \\ & x_1, x_2 \geq 0 \end{cases}$$

$A = [1 \quad 1]$

$A^t = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Dual:

$$\begin{cases} \max & y_1 \\ \text{s.to} & y_1 \leq 1 \\ & y_1 \leq 1 \\ & y_1 : \text{unrestricted} \end{cases}$$

$y_1^* = 1$

$x^* = (1, 0)$

$y_1^*(x_1^* + x_2^* - 1) = 0$

$y_1^* + (x_1^* + x_2^* - 1) = 1 > 0$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 - y_1^* \\ 1 - y_1^* \end{pmatrix}$

$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \not> 0$

Let $x^* = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$y_1^* = 1$

Goldman Tucker holds

Proof: NAP $\Rightarrow$ RNPM exists

market exhibits some finite no. of states.

We will be using few notations —

$S_0^k(\omega_j)$: price of k^{th} asset at time $t = 0$ when market is in state ω_j

$k = 0, \underbrace{1, 2, \ldots, n}_{\text{risky assets}}$

$j = 1, 2, \ldots, m$

$k = 0$ is a risk free asset

$S_T^k(\omega_j)$: price of k^{th} asset at time $t = T$ when market is in state ω_j

Let us construct a portfolio $P = (x_0, x_1, \ldots, x_n)^t$ investing x_k in k^{th} asset at $t = 0$.

$$V_P(t=0) = \sum_{k=0}^{n} \underbrace{x_k}_{\text{unknown}} S_0^k(\omega_j)$$

Construct a linear program

(LP)

NAP

$$\begin{cases} \min \sum_{k=0}^{n} (x_k S_0^k) \geq 0 \\ \quad = 0 \\ \text{s. to } \quad 0 = \sum_{k=0}^{n} x_k S_T^k(\omega_j) \geq 0 \quad \forall j = 1, 2, \ldots, m \\ \quad x_k : \text{unrestricted} \\ \quad \text{linear in } x_k \end{cases}$$

(Dual):

$$\max \sum_{j=1}^{m} 0 \cdot p_j = 0$$

$$\text{s. to} \quad \sum_{j=1}^{m} S_T^k(\omega_j) p_j = S_0^k, \quad k = 0, 1, 2, \ldots, n$$

$$p_j \geq 0 \quad \forall j = 1, \ldots, m$$

$\exists\ x^*$ a soln of (LP) & p^* a soln of dual such that

$$p_j^* + \underbrace{\sum_{k=0}^{n} x_k^* S_T^k(\omega_j)}_{\text{under NAP } = 0} > 0 \implies p_j^* > 0, \ \forall j$$

[Under NAP, value of P at $t = T$, = value of P at $t = 0$

$\implies V_P(0) = V_P(T) = 0$]

Also, $\sum_{j=1}^{m} S_T^k(\omega_j) p_j^* = S_0^k, \ \forall k = 0, 1, \ldots, n$

In particular $k = 0$ (risk free asset)

$$\sum_{j=1}^{m} S_T^{0}(\omega_j)\, p_j^{*} = S_0^{0} = A(0) \quad \forall \omega_j$$

S_T^0 = growth in $A(0)$ at T
$= R\,A(0)$

$$\Rightarrow R\,A(0) \sum_{j=1}^{m} p_j^{t} = A(0)$$

$$\Rightarrow \sum_{j=1}^{m} \hat{p}_j = 1 \qquad \hat{p}_j = R\, p_j^{*} > 0 \quad \forall j$$

$$\Rightarrow \hat{p}_j \in (0,1) \quad \forall j$$

$\hat{p}_j$: like a probability

Also,

$$\sum_{j=1}^{m} S_T^{k}(\omega_j)\, p_j^{*} = S_0^{k}, \quad \forall\, k = 1, \dots, n$$

$$\frac{1}{R} \sum_{j=1}^{m} S_T^{k}(\omega_j)\, \hat{p}_j = S_0^{k} \quad \forall\, k = 1, \dots, n$$

$$E_{\hat{p}}(S_T^{k}) = \sum_{j=1}^{m} S_T^{k}(\omega_j)\, \hat{p}_j$$

$$\boxed{\frac{1}{R} E_{\hat{p}}(S_T^{k}) = S_0^{k}} \quad \forall\, K = 1, \dots, n.$$

$S_0^k \xrightarrow{\hat{p}_1} \omega_1 \rightarrow S_T^k(\omega_1)$

$S_0^k \xrightarrow{\hat{p}_2} \omega_2 \rightarrow S_T^k(\omega_2)$

$S_0^k \xrightarrow{\hat{p}_m} \omega_m \; S_T^k(\omega_m)$

T

discount it back

* Black Schole's Formula :

① CRR model [Cox Rubinstein Ross] → discrete model

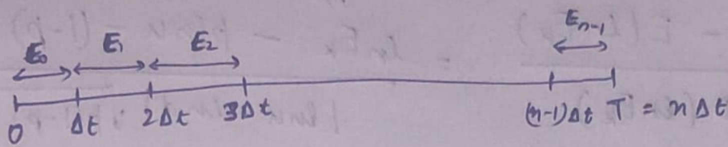
② Ito Calculus

③ Change of measure

(② and ③) — Stochastic calculus

↳ Stochastic Process

* CRR model :



$$E_K = \begin{cases} u & \text{with prob } p \\ d & \text{with prob } 1-p \end{cases} \qquad K = 0, 1, 2, \ldots, n-1$$

where $p \in (0, 1)$

$$\underbrace{S(T)}_{\text{random}} = \underbrace{S(0)}_{\text{deterministic}} \underbrace{E_0 E_1 E_2 \ldots E_{n-1}}_{\text{random}} = S(0)e^H$$

$$e^H = E_0 E_1 \ldots E_{n-1}$$

$$\Rightarrow H = \ln(E_0 E_1 \ldots E_{n-1})$$

$$= \sum_{k=0}^{n-1} \ln E_K$$

CRR model : Every risky stock has two constants attached with it
— drift : $\mu \in \mathbb{R}$
— volatility : $\sigma > 0$ (smoothness in a short interval)

These two parameters μ, σ are defined by the model as—

$$\mu \Delta t = E(\ln E_K)$$

$$\sigma^2 \Delta t = Var(\ln E_K)$$

$$\ln(E_K) = \begin{cases} \ln u & \text{with } p \\ \ln d & \text{with } 1-p \end{cases}$$

$$\mu \Delta t = E(\ln E_K) = p \ln u + (1-p)\ln d$$

$$\sigma^2 \Delta t = Var(\ln E_K) = (\ln u - \ln d)^2 p(1-p)$$

Let us define -

$$X_k = \frac{\ln E_k - E(\ln E_k)}{SD(\ln E_k)} = \frac{\ln E_k - p\ln u - (1-p)\ln d}{|\ln u - \ln d|\sqrt{p(1-p)}}$$

$$= \begin{cases} \dfrac{1-p}{\sqrt{p(1-p)}} & \text{with } p \\ \dfrac{-p}{\sqrt{p(1-p)}} & \text{with } 1-p \end{cases}$$ Bernoulli r.v

$E(X_k) = 0$

$Var(X_k) = 1$, $\forall k$

$$\therefore \ln E_k = E(\ln E_k) + X_k(SD \ln(E_k))$$

$$= \mu \Delta t + \sigma\sqrt{\Delta t}\, X_k$$

$$\sum_{k=0}^{n-1} \ln E_k = \sum_{k=0}^{n-1} \mu \Delta t + \sum_{k=0}^{n-1} \sigma\sqrt{\Delta t}\, X_k$$

$$= \mu T + \sigma\sqrt{\Delta t} \sum_{k=0}^{n-1} X_k$$

$$\Rightarrow S(T) = S(0)\, e^{\mu T + \sigma\sqrt{\Delta t} \sum_{k=0}^{n-1} X_k}$$

$$= S(0)\, e^{\mu T + \sigma\sqrt{\Delta t}\, Y}$$

simple random walk

randomness