01/12/22

Unit - 1
Matrix

Matrix- An arrangement of mn number (real or complex) in m rows & n columns is called matrix of type $m \times n$ denoted by A, B, C, ...

Ex- $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 4 & 5 \\ 11 & 10 & 3 \end{bmatrix}$

$C = \begin{bmatrix} i & 3 & 5 \\ 3 & 7 & i \end{bmatrix}$ $D = \begin{bmatrix} 3+i & -1 & 0 \\ i-2 & \omega & -1 \end{bmatrix}$

Types of Matrix

1) **Rectangular Matrix**- Any $m \times n$ matrix is called a rectangular matrix, if $m \neq n$.

Ex- $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$

2) **Square Matrix**- Any $n \times n$ matrix is called square matrix of order n.

Ex- $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & 3 & -4 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

3) **Row Matrix**- Any $1 \times n$ matrix is called a row matrix.

Ex- $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

4) Column Matrix- Any $m \times 1$ matrix is called a column matrix.

Ex- $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$

5) Diagonal Matrix- A square matrix $A = [a_{ij}]$ is said to be diagonal matrix if $a_{ij} = 0$ for every i & j such that $i \neq j$. Evidently, if $A = [a_{ij}]$ is a diagonal matrix of order n, then A must be of the form.

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

6) Scalar Matrix- Any diagonal matrix in which all its diagonal elements are equal is called a scalar matrix.

Ex- $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

7) Identity matrix (or unit matrix)

Any diagonal matrix is called as identity matrix, if each of its diagonal elements is unity.

Ex-
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8) Null matrix or Zero matrix

Any $m \times n$ matrix is called a null matrix if each of its elements is zero & is denoted by $O_{m\times n}$ or simply O.

Ex-
$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

9) Triangular Matrix

If every element above or below the leading diagonal of a square matrix is zero, the matrix is called a triangular matrix.

a) Upper Triangular Matrix- A square matrix in which all the elements below the leading diagonal are zero i.e. $a_{ij} = 0$ for $i > j$ called an upper triangular matrix.

Ex-
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

b) Lower Triangular Matrix - A square matrix in which all the elements above the leading diagonal are zero i.e. $a_{ij}=0$ for $i<j$ is called a lower triangular matrix.

Ex- $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

c) Strictly Triangular Matrix - A triangular matrix in which all leading diagonal elements are zero i.e. $a_{ij}=0$ for $i \geq j$ or $i \leq j$ is called strictly triangular matrix.

Ex- $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix}$ & $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

Elementary Transformation

Any operation (row or column) may be called elementary transformation or operation.

1) Interchanging between two rows (columns)

R_{ij} or $R_i \leftrightarrow R_j$

C_{ij} or $C_i \leftrightarrow C_j$

2) By constant multiplication with any row or column

$$R_i \rightarrow KR_i \qquad (K \neq 0)$$
$$C_i \rightarrow KC_i$$

3) Replacement of any row by constant multiplication of other row & adding it to previous row.

$$R_i \rightarrow R_i + KR_j \qquad (K \neq 0)$$

Elementary Matrix – Any matrix form from unit matrix by using just one elementary row or column operation.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

R_{23}

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Gauss Jordan Method

To find A^{-1} by using row elementary transformation.

Q1- Find inverse of following:

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

Solution- $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix}$

$A \sim I_3$

$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_1 \rightarrow -2R_1 + R_3$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$R_3 \rightarrow R_3 - 5R_1$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 10 & 0 & -4 \end{bmatrix}$

$R_3 \rightarrow \frac{R_3}{2}$

$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & -2 \end{bmatrix}$

$R_2 \rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 2 \\ 5 & 0 & -2 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$R_1 \rightarrow R_1 + R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

$$I \sim B$$

$$B = A^{-1} = \begin{bmatrix} 8 & -1 & -3 \\ -5 & 1 & 2 \\ 10 & -1 & -4 \end{bmatrix}$$

(ii) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$A \sim I_3$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\quad$ $R_3 \to R_3 + 2R_1$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1 & \phi & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_2 \to \frac{R_2}{2}$ $\quad$ $R_3 \to \frac{R_3}{2}$

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -3 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ 1 & 0 & 1/2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$ $\quad$ $R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 1/2 \end{bmatrix}$$

$R_3 \to \frac{-1}{2} R_3$

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$R_2 \to R_2 + 3R_3$ $\qquad$ $R_1 \to R_1 - 6R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

$$I \sim B$$

$$B = A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Rank of matrix

A rank of matrix $A = [a_{ij}]_{mn}$ is said to be r, if

(i) it has atleast one non-zero minor of order

(ii) all the minors of higher than order r are zero.

it is denoted by $\rho(R_0)$.

$$\rho(A) = rank(A) = r$$

Note-

(i) $\rho(A) \leq \min(m, n)$

(ii) If $A = I_n$, then $\rho(A) = n$

(iii) If $A = [a_{ij}]_{n \times n}$ & $|A| = 0$
$\rho(A) < n$

(iv) The rank of matrix is no. of non-zero rows in echleon form of matrix.

Q1- Find the rank of following

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$$

Solution = $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$ $\quad R_3 \to R_3 - 3R_1$ $\quad R_4 \to R_4 - 6R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_2 \to -\frac{1}{4} R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 11/4 & -5/4 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_3 \to R_3 + 4R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 11/4 & -5/4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 11/4 & -5/4 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 3$

Normal form – By using elementary transformation (row & column) any matrix $A = [a_{ij}]_{mn}$ can be reduced into following four forms.

(i) $[I_r]$ (ii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

(iii) $[I_r \quad 0]$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Where I is unit matrix
fourth form is known as first canonical form. hence find r is rank of matrix.

Q1 – Reduce the following matrix to the canonical form.

(i) $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$

Solution – Let $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 3 & 4 & 1 & 2 \end{bmatrix}$

$R_1 \leftrightarrow R_4$

$\sim \begin{bmatrix} 0 & 3 & 4 & 1 & 2 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$C_1 \leftrightarrow C_4$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 & 2 \\ 3 & 1 & 2 & 0 & 4 \\ 4 & 2 & 3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1$ $\quad R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 3 & 4 & 0 & 2 \\ 0 & -8 & -10 & 0 & -2 \\ 0 & -10 & -13 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \to C_2 - 3C_1$, $\quad C_3 \to C_3 - 4C_1$ $\quad C_5 \to C_5 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -8 & -10 & 0 & -2 \\ 0 & -10 & -13 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to -\frac{1}{2} R_2$, $\quad R_3 \to -R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & 5 & 0 & 1 \\ 0 & 10 & 13 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_2 \leftrightarrow C_5$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 4 \\ 0 & 7 & 13 & 0 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 7R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 4 \\ 0 & 0 & -22 & 0 & -18 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow C_3 - 5C_2$ $\quad$ $C_5 \rightarrow C_5 - 4C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -22 & 0 & -18 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \rightarrow \frac{-1}{22} C_3$ $\quad$ $C_5 \rightarrow \frac{-1}{18} C_5$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_5 \rightarrow C_5 - C_3$

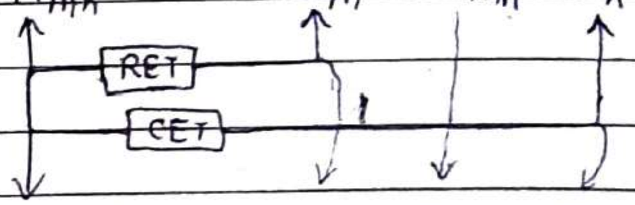
$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

Q2- To find two non-singular matrices P & Q for a given matrix $A = [a_{ij}]$ such that PAQ is in normal form. Hence find A^{-1} if exist.

Solution-

$$A_{mn} = I_m \; A_{mn} \; I_n$$



Normal form = $P \; A \; Q$

Now if A^{-1} exist, then $m=n$, $|A| \neq 0$

$$PAQ = \text{Normal form} = I_n$$

$$\cancel{(P^{-1}P)(AQ) =}$$

$$P^{-1}(PAQ) = P^{-1}I$$

$$(P^{-1}P)(AQ) = P^{-1}$$

$$A = P^{-1}Q^{-1}$$

$$A = (QP)^{-1}$$

$$\boxed{A^{-1} = QP}$$

Q3- Find non-singular P & Q such that PAQ is in normal form for the matrix.

(i) $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Hence A^{-1} if exist.

Solution- (i) Let $A = I A I$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$ $R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & -1 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) $A = I A I$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_1 \to C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & -1 \\ 5 & 1 & 0 \\ 3 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 5R_1 \qquad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_1 + C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 5R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -15 & 6 & -5 \\ 2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$I_3 = P\,A\,Q$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -15 & 6 & -5 \\ 2 & -1 & 1 \end{bmatrix} \qquad Q = \cancel{\begin{bmatrix} 1 & 0 & -1 \\ -15 & 6 & -5 \\ 2 & -1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = Q\,P$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -15 & 6 & -5 \\ 2 & -1 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & 0 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} //$$

Solution of system of linear eq.

Let the matrix form is given by

$$AX = B \quad — (1)$$

Now construct augmented matrix,

$$C = [A \ B]$$

Now find rank of C.

1) If $\rho(C) = \rho(A)$
Given system of linear eq. are consistent
We have ~~unique~~ solution.

(i) If $\rho(C) = \rho(A) = n$ (i.e. no of unknown)
We have ~~infinite~~ unique solution.

(ii) If $\rho(C) = \rho(A) < n$. (i.e. of unknowns)
We have infinite solution.

2) If $\rho(C) \neq \rho(A)$
No solution exists.

Q1- Solve with the help of matrices the simultaneou[s] eq.,

$x+y+z=3$, $x+2y+3z=4$
$x+4y+9z=6$.

Solution-

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$AX = B$$

Let $C = [A\ B]$

$$C = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}$$

$R_2 \to R_2 - R_1$ $\qquad$ $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$\rho(C) = 3$, $\rho(A) = 3$

Here $\rho(C) = \rho(A) = 3 =$ no. of unknowns.
So we have unique solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

$$x + y + z = 3$$
$$y + 2z = 1$$
$$2z = 0$$

$z = 0$, $y = 1$, $x = 2$

Q2- Solve
$$3x + 3y + 2z = 1$$
$$x + 2y = 4$$
$$10y + 3z = -2$$
$$2x - 3y - z = 5$$

Solution- The matrix form is

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

Let $C = [A \; B]$

$$= \begin{bmatrix} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 3R_1$ $\qquad$ $R_4 \rightarrow R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{bmatrix}$$

$R_3 \to R_3 - 10R_2$ $\qquad$ $R_4 \to R_4 + 7R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & -87 & 348 \\ 0 & 0 & 62 & -248 \end{bmatrix}$$

$R_3 \to \frac{-1}{87} R_3$, $R_4 \to \frac{1}{62} R_4$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 1 & 9 & -35 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(C) = 3$, $\rho(A) = 3$

Here $\rho(C) = \rho(A) = 3 =$ no. of unknowns
So we have unique solution.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -35 \\ -4 \\ 0 \end{bmatrix}$$

$$x + 2y = 4$$
$$y + 9z = -35$$
$$\boxed{z = -4} \qquad \boxed{x = 2}$$

$$y - 36 = -35$$
$$\boxed{y = 1}$$

Q3- Check ~~for~~ consistancy

$$x + y + z = -3$$
$$3x + y - 2z = -2$$
$$2x + 4y + 7z = 7$$

Solution- The matrix eq. is

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix}$$

$$AX = B \quad (\text{say})$$

Let $C = [A\ B]$

$$= \begin{bmatrix} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{bmatrix}$$

$R_2 \to R_2 - 3R_1 \qquad R_3 \Rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{bmatrix}$$

~~$R_2 \rightarrow R_2 + R_3$~~ ~~$C_3$~~ $R_3 \rightarrow R_2 + R_3$

$$C \sim \begin{bmatrix} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 14 \end{bmatrix}$$

$\rho(C) = 3$ $\rho(A) = 2$

As $\rho(C) \neq \rho(A)$

The system is inconsistant.

no solution exist

Q4- Solve:

$$4x - 2y + 6z = 8$$
$$x + y - 3z = -1$$
$$15x - 3y + 9z = 21$$

Solution- The matrix is

$$\begin{bmatrix} 4 & -2 & 6 \\ 1 & 1 & -3 \\ 15 & -3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 21 \end{bmatrix}$$

$AX = B$ (say)

Let $C = [A \ B]$

$$= \begin{bmatrix} 4 & -2 & 6 & 8 \\ 1 & 1 & -3 & -1 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 4 & -2 & 6 & 8 \\ 15 & -3 & 9 & 21 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$ $\qquad$ $R_3 \to R_3 - 15R_1$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & -6 & 18 & 12 \\ 0 & -18 & 54 & 36 \end{bmatrix}$$

$R_2 \to \frac{-1}{6} R_2$ $\qquad$ $R_3 \to \frac{-1}{18} R_3$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 1 & -3 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence $\rho(C) = 2$, $\rho(A) = 2$

$\rho(C) = \rho(A) = 2 < 3$ (no. of unknowns)
so system is consistence & we have infinite solution.

$$\begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

$$x + y - 3z = -1$$
$$y - 3z = -2$$

let $z = t$, $y = 3t - 2$

$$x = 3z - y - 1$$
$$= 3t - (3t - 2) - 1$$
$$\Rightarrow x = 1$$

$x = 1$, $y = 3t - 2$, $z = t$

Q5- Investigate for what values of λ & μ do the system of eq. $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have

(i) no solution
(ii) unique solution
(iii) infinite solution

Solution- The matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX = B \text{ (say)}$$

Let $C = [A \; B]$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1$ $\qquad$ $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & (\lambda-1) & (\mu-6) \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & (\lambda-3) & (\mu-10) \end{bmatrix}$$

(i) For no solution $\rho(C) \neq \rho(A)$

$\rho(C) = 3$ i.e. $\mu - 10 \neq 0$ or $\mu \neq 10$

$\rho(A) = 2$ i.e. $\lambda - 3 = 0$ or $\lambda = 3$

(ii) For unique solution $\rho(C) = \rho(A) = 3$

$\rho(A) = 3$ i.e. $\lambda - 3 \neq 0$ or $\lambda \neq 3$

Now for $\rho(C) = 3$, μ taken any value or $\mu \in R$.

(iii) Infinite solution

$\rho(C) = \rho(A) = 2 <$ no. of function

$\lambda - 3 = 0$ & $\mu - 10 = 0$

$\lambda = 3$ $\qquad$ $\mu = 10$

Q6- Show that the eq.

$$-2x + y + z = a$$
$$x - 2y + z = b$$
$$x + y - 2z = c$$

have no solution unless $a+b+c=0$, In which case they have infinite solution? Find these solutions when $a=1$, $b=1$, $c=-2$

Solution- The matrix is

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$AX = B \text{ (say)}$$

$$C = [A \; B]$$

$$= \begin{bmatrix} -2 & 1 & 1 & a \\ 1 & -2 & 1 & b \\ 1 & 1 & -2 & c \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ -2 & 1 & 1 & a \\ 1 & 1 & -2 & c \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$ $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & -3 & 3 & a+2b \\ 0 & 3 & -3 & c-b \end{bmatrix}$$

$R_2 \to -\frac{1}{3} R_2$ $\qquad$ $R_3 \to \frac{1}{3} R_3$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 1 & -1 & -\frac{(a+2b)}{3} \\ 0 & 1 & -1 & \frac{c-b}{3} \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 1 & -1 & -\frac{(a+2b)}{3} \\ 0 & 0 & 0 & \frac{c-b+a+2b}{3} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 1 & b \\ 0 & 1 & -1 & -\frac{(a+2b)}{3} \\ 0 & 0 & 0 & \frac{a+b+c}{3} \end{bmatrix}$$

$\rho(A) = 2$ $\qquad$ $\rho(C) = 3$

For infinite solution $\rho(A) = \rho(C) < 3$
which is possible when $a+b+c = 0$
then $\rho(C) = 2 = \rho(A) < 3$.

n $a=1, b=1, c=-2$

$$\sim \begin{bmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(A) = 2$ $\qquad$ $\rho(C) = 2$

$\rho(A) = \rho(C) = 2 < 3$ (no of unknowns)

So System is consistence & we have infinite solution.

Cayley - Hamiltion theorem

Every square matrix satisfy its own characteristic eq.

Ex- Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$

then characteristic eq. is

$$|A - \lambda I| = 0$$

where λ is non-zero scalar

$$[A - \lambda I] = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{bmatrix}$$

So characteristic eq.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(2-\lambda) = 0$$

$$2 - \lambda - 2\lambda + \lambda^2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

the matrix eq. can be written as,

$$X^2 - 3X + 2I = 0$$

by cayley - hamilton theorem, above matrix eq. satisfied by $X = A$

$$A^2 - 3A + 2I = 0$$

Now LHS $= A^2 - 3A + 2I$

$$= \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 9 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 9 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We have

$$A^2 - 3A + 2I = 0$$

$$A - 3I + 2A^{-1} = 0$$

$$2A^{-1} = 3I - A$$

$$= 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

$$A^{-1} = \cancel{4}\ \frac{1}{2}\begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}$$

$\lambda^3 - (\text{trace} A)\lambda^2 + (M_{11} + M_{22} + M_{33})\lambda - |A| = 0$

Q1- Find the characteristic eq. of the matrix

$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ & hence compute A^{-1} also find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Solution- characteristic eq.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)(2-\lambda)] + 1(\lambda - 1) = 0$$
$$(2-\lambda)(2-\lambda-2\lambda+\lambda^2) + \lambda - 1 = 0$$
$$(2-\lambda)(\lambda^2 - 3\lambda + 2) + \lambda - 1 = 0$$
$$2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda + \lambda - 1 = 0$$
$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$
$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

the matrix eq. can be written as

$$X^3 - 5X^2 + 7X - 3I = 0$$

by cayley - hamilton theorem, above matrix eq. satisfied by $X = A$.

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^2 - 5A + 7I - 3A^{-1} = 0$$
$$A^2 - 5A + 7I = 3A^{-1}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} - 5\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - \begin{bmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} = 3A^{-1}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix} = 3A^{-1}$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix} //$$

$$\begin{array}{r} A^5 + A \\ A^3 - 5A^2 + 7A - 3I \,\big)\, A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 \\ + 8A^2 - 2A + I \\ A^8 - 5A^7 + 7A^6 - 3A^5 \\ - \quad + \quad - \quad + \\ \hline A^4 - 5A^3 + 8A^2 - 2A \\ A^4 - 5A^3 + 7A^2 - 3A \\ - \quad + \quad - \quad + \\ \hline A^2 + A + I \end{array}$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I)$$
$$+ A^2 + A + I$$

$$= A^5(0) + A(0) + A^2 + A + I$$
$$= A^2 + A + I$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix} //$$

(ii) $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}$

$$A^6 - 6A^5 + 9A^4 - 2A^3 + 12A^2 + 23A - 9I$$

<u>Solution</u>- The characteristic eq.

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\left[(2-\lambda)^2 - 2\right] + 1\left[\lambda - 2 + 1\right] + 1\left[2 - (2-\lambda)\right] = 0$$
$$(2-\lambda)^3 - 2(2-\lambda) + \lambda - 1 + \cancel{2} - \cancel{2} + \lambda = 0$$
$$8 - \lambda^3 - 6\lambda(2-\lambda) - 4 + 2\lambda + \lambda - 1 + \lambda = 0$$
$$3 - \lambda^3 - 12\lambda + 6\lambda^2 + 2\lambda + \lambda + \lambda = 0$$
$$3 - \lambda^3 - 8\lambda + 6\lambda^2 = 0$$
$$\lambda^3 - 6\lambda^2 + 8\lambda - 3 = 0$$

$$X^3 - 6X^2 + 8X - 3I = 0$$

$$A^3 - 6A^2 + 8A - 3I = 0$$

$$A^2 - 6A + 8I - 3A^{-1} = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -2 & 2 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 3A^{-1}$$

$$\begin{bmatrix} 6 & -6 & 5 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -12 & 12 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 3A^{-1}$$

$$\frac{1}{3}\begin{bmatrix} 2 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 3 & 3 \end{bmatrix} = A^{-1}$$

$$A^3 + A + 7$$

$$A^3 - 6A^2 + 8A - 3I \,\big)\, \cancel{A^6} - \cancel{6A^5} + 9A^4 - 2A^3 + 12A^2 + 23A - 9I$$

$$\cancel{A^6} - \cancel{6A^5} + 8A^4 - 3A^3$$

$$- \quad + \quad - \quad +$$

$$\cancel{A^4} + A^3 + 12A^2 + 23A$$

$$\cancel{A^4} - 6A^3 + 8A^2 - 3A$$

$$- \quad + \quad - \quad +$$

$$\cancel{7A^3} + 4A^2 + 26A - 9I$$

$$\cancel{7A^3} - 42A^2 + 56A - 21I$$

$$- \quad + \quad - \quad +$$

$$46A^2 + 82A - 12I$$

$$= (A^3 - 6A^2 + 8A - 3I)(A^3 + A + 7) + 46A^2 + 82A - 12I$$

$$= 46A^2 + 82A - 12I$$

$$= 46\begin{bmatrix} 6 & -6 & 5 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix} + 82\begin{bmatrix} 6 & -6 & 5 \\ -5 & 7 & -5 \\ 6 & -9 & 7 \end{bmatrix} - 12\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 756 & -768 & 640 \\ -640 & 884 & -640 \\ 768 & -1152 & 884 \end{bmatrix}$$

System of Homogeneous Linear ~~e~~ equations

In matrix eq. of ~~sq~~ system of linear eq. $AX = B$ if $B = 0$ (zero matrix) then this ~~Sys~~ is called homogeneous linear eq.

$$AX = 0$$
$$C = [A \; 0]$$

In homogeneous case
$$\rho(C) = \rho(A)$$
so this system is always consistant

(i) If $\rho(A) =$ no. of unknowns then we have unique solution called zero solution or trivial solution.

$\boxed{\text{for zero sol. } |A| \neq 0}$

(ii) if $\rho(A) <$ no. of unknowns.
then we have infinite sol. called non zero or non-trivial solution.

$\boxed{\text{for non-trivial sol. } |A| = 0}$

Q1- Solve eq. using matrix method.

$$x_1 + 3x_2 + 2x_3 = 0$$
$$2x_1 - x_2 + 3x_3 = 0$$
$$3x_1 - 5x_2 + 4x_3 = 0$$
$$x_1 + 17x_2 + 4x_3 = 0$$

Solution- Let the matrix form is

$$AX = 0 \quad - (1)$$

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & -1 & 3 \\ 3 & -5 & 4 \\ 1 & 17 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1 \qquad R_3 \to R_3 - 3R_1 \quad R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & -14 & -2 \\ 0 & 14 & 2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2 \;, \quad R_4 \to R_4 + 2R_2$

$$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (2) \qquad \rho(A) = 2 \; (< 3 \text{ no. of unknowns})$$

from ②,

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -7 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 3x_2 + 2x_3 = 0$$
$$-7x_2 - x_3 = 0$$

let $x_2 = t$, $x_3 = -7t$

$$x_1 = 11t$$

$$\boxed{x_1 = 11t, \quad x_2 = t, \quad x_3 = -7t} \quad t \in R$$

Q2- Find the values of k for which the system of eq.

$$(3k-8)x + 3y + 3z = 0$$
$$3x + (3k-8)y + 3z = 0$$
$$3x + 3y + (3k-8)z = 0$$

Solution- Let the matrix form is

$$AX = 0$$

$$A = \begin{bmatrix} (3k-8) & 3 & 3 \\ 3 & (3k-8) & 3 \\ 3 & 3 & (3k-8) \end{bmatrix}$$

For non-trivial sol.

$$|A| = 0$$

~~(3k-8)~~

$$\begin{vmatrix} 3k-8 & 3 & 3 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$R_1 \to R_1 + R_2 + R_3$

$$\begin{vmatrix} 3k-2 & 3k-2 & 3k-2 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$$(3k-2)\begin{vmatrix} 1 & 1 & 1 \\ 3 & 3k-8 & 3 \\ 3 & 3 & 3k-8 \end{vmatrix} = 0$$

$C_2 \to C_2 - C_1 \qquad C_3 \to C_3 - C_1$

$$(3k-2)\begin{vmatrix} 1 & 0 & 0 \\ 3 & 3k-11 & 0 \\ 3 & 0 & 3k-11 \end{vmatrix} = 0$$

$$(3k-2)(3k-11)(3k-11) = 0$$

$$k = \frac{2}{3}, \frac{11}{3}$$

Q3- For what values of k the equations $x+y+z=1$, $2x+y+4z=k$, $4x+y+10z=k^2$ have a solution & solve them completely in each case.

Solution- Let the matrix form is

$$AX = B \quad — (1)$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 4 & 1 & 10 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 6 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = [A \ B]$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & K \\ 4 & 1 & 10 & k^2 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1 \quad , \quad R_3 \to R_3 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & -3 & 6 & k^2-4 \end{bmatrix}$$

$$R_3 \to R_3 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & k-2 \\ 0 & 0 & 0 & k^2-3k+2 \end{bmatrix}$$

For sol. $\rho(A) = \rho(C)$

Here $\rho(A) = 2$

So the $\rho(C)$ is also 2

$$k^2 - 3k + 2 = 0$$
$$k^2 - 2k - k + 2 = 0$$
$$k(k-2) - 1(k-2) = 0$$
$$(k-1)(k-2) = 0$$
$$\boxed{k = 1, 2}$$

Case I - When $k=1$, from eq ①

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$x + y + z = 1$$
$$-y + 2z = -1$$

Let $z = t$, $y = 2t + 1$

$$x = 1 - y - z$$
$$= 1 - (2t+1) - t$$
$$= 1 - 2t - 1 - t$$
$$x = -3t$$
$$y = 2t + 1$$
$$z = t$$

Case II - When $k = 2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$x + y + z = 1$$
$$-y + 2z = 2$$

Let $z = t$

$$y = 2t - 2$$
$$x = 3 - 3t$$

Eigen values & eigen vectors:

Eigen values - For a given square matrix A, the characteristic equation is given by

$$|A-\lambda I| = 0$$

The roots above eq is called eigen values, characteristic roots, latent roots.

Note - (i) A & A' have same set of eigen values

(ii) If eigen values of A are $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ then the eigen values of

(a) KA are $K\lambda_1, K\lambda_2, \ldots\ldots, K\lambda_n$.

(b) A^m are $\lambda_1^m, \lambda_2^m, \ldots\ldots, \lambda_n^m$

(c) A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots\ldots, \frac{1}{\lambda_n}$

(iii) The product of all eighen values is equal to the determine of A matrix.

(iv) The sum of all eighen values is equal to trace of matrix i.e. sum of elements of principle diagonal.

Q1- Find the eigen values of following

(i) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 2 & 0 \\ 3 & 1 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 & 7 \\ 5 & 3 & 1 \\ 7 & 0 & 2 \end{bmatrix}$

Solution- The characteristic eq. is

$$\begin{bmatrix} 1-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 3 & 1 & -3-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(-3-\lambda)\right] + 4\left[-3(2-\lambda)\right] = 0$$

$$(1-\lambda)\left[(\lambda-2)(3+\lambda)\right] + 12(\lambda-2) = 0$$

$$(1-\lambda)\left[3\lambda+\lambda^2-6 \mp 3\lambda\right] + 12\lambda - 24 = 0$$

$$(1-\lambda)(+\lambda^2+\lambda-6) + 12\lambda - 24 = 0$$

$$\cancel{+\lambda^2} + \lambda - 6 - \lambda^3 - \cancel{\lambda^2} + 6\lambda + 12\lambda - 24 = 0$$

$$-\lambda^3 + 19\lambda - 30 = 0$$

$$\lambda^3 - 19\lambda + 30 = 0$$

$$\lambda = -5, 2, 3$$

(ii) the characteristic eq

$$|A - I\lambda| = 0$$

$$\begin{bmatrix} 2-\lambda & 5 & 7 \\ 5 & 3-\lambda & 1 \\ 7 & 0 & 2-\lambda \end{bmatrix} = 0$$

$$(2-\lambda)\left[(3-\lambda)(2-\lambda)\right] - 5\left[10-5\lambda-7\right] + 7\left[-7(3-\lambda)\right] = 0$$

$$(2-\lambda)\left[6-3\lambda-2\lambda+\lambda^2\right] - 5\left[3-5\lambda\right] + 7\left[7\lambda-21\right] = 0$$

$$(2-\lambda)\left[\lambda^2-5\lambda+6\right] - 15 + 25\lambda + 49\lambda - 147 = 0$$

$$2\lambda^2 - 10\lambda + 12 - \lambda^3 + 5\lambda^2 - 6\lambda + 74\lambda - 162 = 0$$

$$-\lambda^3 + 7\lambda^2 + 58\lambda - 150 = 0$$

$$\lambda^3 - 7\lambda^2 - 58\lambda + 150 = 0$$

$$\lambda = 11.02,\ 2.18,\ -6.21$$

Eigen vector – The non zero ~~ve~~ vector X is said to be eigen vector if $[A-\lambda I]X = 0$

Q1 – Find the eigen vectors of following

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$$

Solution – the characteristic eq. is

$$|A - I\lambda| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left[(2-\lambda)(2-\lambda) - 2\right] - 1\left[2 - 4 + 2\lambda\right] = 0$$

$$(1-\lambda)\left[4 - 2\lambda - 2\lambda + \lambda^2 - 2\right] + 2 - 2\lambda = 0$$

$$(1-\lambda)\left[\lambda^2 - 4\lambda + 2\right) + 2 - 2\lambda = 0$$

$$\lambda^2 - 4\lambda + 2 - \lambda^3 + 4\lambda^2 - 2\lambda + 2 - 2\lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$\lambda^2 - 4\lambda + 4 = 0$

$\lambda = \frac{4 \pm \sqrt{16 - 4\times1\times4}}{2\times1}$

$= \frac{4 \pm}{2}$

$= 2, 2$

$$\lambda = 1, 2, 2$$

Case I – When $\lambda = 1$, let the eighen vector is

$$X_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A-\lambda I]X_1 = 0$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2b + 2c = 0$$
$$b + c = 0$$
$$-a + 2b + c = 0$$

let $c = t$, $b = -t$, $a = -t$

$$X_1 = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Case II - When $\lambda = 2$, let the eigen vector is

$$X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$[A-\lambda I]X_2 = 0$$

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x + 2y + 2z = 0$ $\qquad$ $y = t$

$\boxed{z = 0}$ $\qquad$ $x = 2t$

$-x + 2y = 0$ $\qquad$ $z = 0$

(-3, -1, 5)   (0, 3, 15)   (2, 1, 8)

$$X_2 = \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Q2- Find the eigen values & eigen vectors

(i) $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(i) Characteristic eq.

$$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$$(-2-\lambda)\left[(1-\lambda)(-\lambda) + 2(-6)\right] - 2\left[-2\lambda - 6\right]$$

$$-3\left[-4 + 1(1-\lambda)\right] = 0$$

$$(-2-\lambda)\left[-\lambda + \lambda^2 - 12\right) + 4\lambda + 12 + 9 + 3\lambda = 0$$

$$(-2-\lambda)(\lambda^2 - \lambda - 12) + 7\lambda + 21 = 0$$

$$-2\lambda^2 + 2\lambda + 24 - \lambda^3 + \lambda^2 + 12\lambda + 7\lambda + 21 = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$$

$$\lambda = -3, 5$$

Case I – When $\lambda = -3$,

$$X_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$[A - \lambda I] X_1 = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a + 2b - 3c = 0$$
$$2a + 4b - 6c = 0$$
$$-a - 2b + 3c = 0$$

Let $c = t$,

(ii) Let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

the characteristic eq

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2[24 - 2(7-\lambda)] = 0$$

$$(8-\lambda)[21 - 7\lambda - 3\lambda + \lambda^2 - 16] + 6[-18 + 6\lambda + 8] + 2[24 - 14 + 2\lambda] = 0$$

$$(8-\lambda)\left[\lambda^2-10\lambda+5\right]+6(6\lambda-10)+2(2\lambda+10)=0$$

$$8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20=0$$

$$-\lambda^3+18\lambda^2-45\lambda=0$$

$$\lambda^3-18\lambda^2+45\lambda=0$$

$$\lambda(\lambda^2-18\lambda+45)=0$$

$$\lambda=0,3,15$$

Case I - when $\lambda=0$, let $X_1=\begin{bmatrix}a\\b\\c\end{bmatrix}$

then $[A-\lambda I]X_1=0$

$$\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

R_{13}

$$\sim\begin{bmatrix}2&-4&3\\-6&7&-4\\8&-6&2\end{bmatrix}$$

$R_2\to R_2+3R_1$, $R_3\to R_3-4R_1$

$$\sim\begin{bmatrix}2&-4&3\\0&-5&5\\0&10&-10\end{bmatrix}$$

$R_3\to R_3+2R_2$

$$\sim\begin{bmatrix}2&-4&3\\0&-5&5\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2a - 4b + 3c = 0$$
$$-5b + 5c = 0$$
or $$-b + c = 0$$

let $c = t$, $b = t$, $a = \frac{t}{2}$

$$X_1 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} t/2 \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$$

Case II - When $\lambda = 3$ let $X_2 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$

then $[A - \lambda I] X_2 = 0$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}

$$\sim \begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}$$

$R_1 \rightarrow \frac{1}{2} R_1$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}$$

$R_2 \rightarrow R_2 + 6R_1 \quad , \quad R_3 \rightarrow R_3 - 5R_1$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & 4 & 2 \end{bmatrix}$$

$R_3 \rightarrow 2R_3 + R_2$

$$\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 0 & -8 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$l - 2m = 0$$
$$-8m - 4n = 0$$
$$2m + n = 0$$

let $m = k \quad , \quad n = -2k \quad , \quad l = 2k$

$$X_2 = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ -2k \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \,//$$

Case III - When $\lambda = 15$, let $X_3 = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$

$$[A - \lambda I] X = 0$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

R_{13}

$$\sim \begin{bmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{bmatrix}$$

$R_1 \to \frac{1}{2} R_1 \qquad R_2 \to \frac{1}{2} R_2$

$$\sim \begin{bmatrix} 1 & -2 & -6 \\ -3 & -4 & -2 \\ -7 & -6 & 2 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1 \qquad R_3 \to R_3 + 7R_1$

$$\sim \begin{bmatrix} 1 & -2 & -6 \\ 0 & -10 & -20 \\ 0 & -20 & -40 \end{bmatrix}$$

~~$R_2 \to -\frac{1}{10} R_2 \qquad R_3 \to -\frac{1}{20} R_3$~~

$$\sim \begin{bmatrix} 1 & -2 & -6 \\ 0 & 1 & 2 \\ 0 & 5 & 17 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\sim \begin{bmatrix} 1 & -2 & -6 \\ 0 & -10 & -20 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 \\ 0 & -10 & -20 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$r - 2s - 6t = 0$$
$$-10s - 20t = 0$$

$t = e$, $s = -2e$, $r = 2s + 6t$
$= -4e + 6e$
$= 2e$

$$X_3 = \begin{bmatrix} 2e \\ -2e \\ e \end{bmatrix} = e \begin{bmatrix} 2 \\ -2 \\ +1 \end{bmatrix}$$

(iii) Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

the characteristic eq.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)\left[(3-\lambda)^2-1\right]+2\left[-6+2\lambda+2\right]+2\left[2-6+2\lambda\right]=0$$

$$(6\cancel{\lambda}-\lambda)\left[9+\lambda^2-6\lambda-1\right]+2(2\lambda-4)+2(2\lambda-4)=0$$

$$\cancel{48\ +6\lambda^2-36\lambda+4\lambda-8+4\lambda-8=0}$$

$$(6-\lambda)(\lambda^2-6\lambda+8)+4\lambda-8+4\lambda-8=0$$

$$6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2\cancel{-8\lambda}\cancel{+8\lambda}-16=0$$

$$-\lambda^3+12\lambda^2-36\lambda+32=0$$

$$\lambda^3-12\lambda^2+36\lambda-32=0$$

$$\lambda=2,2,8$$

Case I - When $\lambda=2$, let $X_1=\begin{bmatrix}a\\b\\c\end{bmatrix}$

then $[A-\lambda I]X_1=0$

$$\cancel{\begin{bmatrix}8&-6&2\\-6&7&-4\\2&-4&3\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}\\ \\ \end{bmatrix}}$$

$$\begin{bmatrix}4&-2&2\\-2&1&-1\\2&-1&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1\to R_1+2R_2 \qquad R_3\to R_3+R_2$

$$\sim\begin{bmatrix}0&0&0\\-2&1&-1\\0&0&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-2a+b-c=0$$

linear dependence & independence of vectors

Vector $X_1, X_2, X_3, \ldots\ldots, X_n$ are said to be linearly dependent if,

(i) They are of same type.

(ii) There exist n scalars (not all zero) such that $\lambda_1 X_1 + \lambda_2 X_2 + \ldots\ldots + \lambda_n X_n = 0$ otherwise set of vectors are linearly independent.

Def.— Row & column matrix are called vectors

Ex- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \longrightarrow$ Column vector

$[0 \ -1 \ 3] \longrightarrow$ Row vector

Q1- Prove that vectors $X_1 = (1,2,3)$, $X_2 = (3,-2,1)$ & $X_3 = (1,-6,-5)$ form a linearity dependent system. Also, find the relation between them

Solution- Let $\lambda_1, \lambda_2, \lambda_3$ (not all zero) such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0 \quad -① $$

$$\lambda_1(1,2,3) + \lambda_2(3,-2,1) + \lambda_3(1,-6,-5) = (0,0,0)$$

$$(\lambda_1, 2\lambda_1, 3\lambda_1) + (3\lambda_2, -2\lambda_2, \lambda_2) + (\lambda_3, -6\lambda_3, -5\lambda_3) = (0,0,0)$$

$$(\lambda_1 + 3\lambda_2 + \lambda_3,\ 2\lambda_1 - 2\lambda_2 - 6\lambda_3,\ 3\lambda_1 + \lambda_2 - 5\lambda_3) = (0, 0, 0)$$

$$\lambda_1 + 3\lambda_2 + \lambda_3 = 0$$
$$2\lambda_1 - 2\lambda_2 - 6\lambda_3 = 0$$
$$3\lambda_1 + \lambda_2 - 5\lambda_3 = 0$$

Here

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -2 & -6 \\ 3 & 1 & -5 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & -8 & -8 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & -8 & -8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_1 + 3\lambda_2 + \lambda_3 = 0$$
$$-8\lambda_2 - 8\lambda_3 = 0$$
$$\lambda_2 = -\lambda_3$$

Let $\lambda_3 = t$, $\lambda_2 = -t$, $\lambda_1 = 2t$

from ①

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0$$
$$2t X_1 - t X_2 + t X_3 = 0$$

$t \neq 0$, $\boxed{2X_1 - X_2 + X_3 = 0}$

Q2- Prove that vectors $X_1 = (1,2,4)$, $X_2 = (2,-1,3)$, $X_3 = (0,1,2)$; $X_4 = (-3,7,2)$ form a linearity dependent system. Also, find the relation between them.

Solution- Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ (not all zero) such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 + \lambda_4 X_4 = 0 \quad - ①$$

$$\lambda_1 + 2\lambda_2 - 3\lambda_4 = 0$$
$$2\lambda_1 - \lambda_2 + \lambda_3 + 7\lambda_4 = 0$$
$$4\lambda_1 + 3\lambda_2 + 2\lambda_3 + 2\lambda_4 = 0$$

$$A = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix}.$$

$R_2 \to R_2 - 2R_1$ $\qquad$ $R_3 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$d_1 + 2d_2 - 3d_4 = 0$$
$$-5d_2 + d_3 + 13d_4 = 0$$
$$d_3 + d_4 = 0$$

$d_4 = t \quad , \quad d_3 = -t$

$d_2 = \frac{12}{5}t \quad , \quad d_1 = \frac{-9}{5}t$

from ①

$$d_1 X_1 + d_2 X_2 + d_3 X_3 + d_4 X_4 = 0$$

$$\frac{-9}{5}t X_1 + \frac{12}{5}t X_2 - t X_3 + t X_4 = 0$$

$t \neq 0 \quad , \quad \frac{-9}{5}X_1 + \frac{12}{5}X_2 - X_3 + X_4 = 0$

$$\boxed{-9X_1 + 12X_2 - 5X_3 + 5X_4 = 0}$$

Q3- Prove that vectors $X_1 = (3, 2, 4)$, $X_2 = (1, 0, 2)$, $X_3 = (1, -1, -1)$ form a linearity dependent system. Also find the relation between them.

Solution- Let $\lambda_1, \lambda_2, \lambda_3$ (not all zero) such that

$$\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3 = 0 \quad - (1)$$

$$3\lambda_1 + \lambda_2 + \lambda_3 = 0$$
$$2\lambda_1 \cancel{+} - \lambda_3 = 0$$
$$4\lambda_1 + 2\lambda_2 - \lambda_3 = 0$$

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \\ 4 & 2 & -1 \end{bmatrix}$$

$C_1 \longleftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 2 & 4 & -1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & -3 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\lambda_1 + 3\lambda_2 + \lambda_3 = 0$

$2\lambda_2 - \lambda_3 = 0$

$-4\lambda_3 = 0$

$\lambda_3 = 0$ $\quad$ $\lambda_2 = 0$ $\quad$ $\lambda_1 = 0$

$\because$ $\lambda_1 = \lambda_2 = \lambda_3 = 0$, therefore the given vectors $X_1, X_2, \& X_3$ are linearly independent & there exist no relationship.

Complex Matrix-

If atleast one entry in a matrix is a complex no. then the matrix is called complex matrix.

Ex- $A = \begin{bmatrix} 1 & i & 0 \\ 3 & 1+i & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -i \\ 0 & 3-i \end{bmatrix}$

The θ - matrix or $*$ - matrix, the conjugate transpose of a matrix is called θ - matrix.

$$A^\theta = (\bar{A})' = (\overline{A'})$$

$$\bar{B} = \begin{bmatrix} 1 & i \\ 0 & 3+i \end{bmatrix}$$

$$B^\theta = (\bar{B})' = \begin{bmatrix} 1 & 0 \\ i & 3+i \end{bmatrix}$$

Hermitian Matrix- A square complex matrix $A = [a_{ij}]$ is said to be Hermitian if $a_{ij} = \overline{a_{ji}}$

or $A^\theta = A$.

Note- In hermitian matrix,

$$a_{ii} = \overline{a_{ii}}$$

$$x + iy = x - iy$$

$$2iy = 0$$

$$y = 0$$

In hermitian matrix the principle diagonal is real.

Ex- $A = \begin{bmatrix} 0 & 1+i & 3 \\ 1-i & 1 & i \\ 3 & -i & 0 \end{bmatrix}$

Skew Hermitian Matrix- A square complex matrix $A = [a_{ij}]$ is said to be skew - Hermitian if

$$a_{ij} = -\overline{a_{ji}}$$

or $$A^{\theta} = -A$$

Note- Here $a_{ij} = -\overline{a_{ij}}$

$$x + iy = -(x - iy)$$

$$\boxed{x = 0}$$

In skew - hermitian matrix element of principle diagonal is pure imaginary or zero.

Ex- $A = \begin{bmatrix} i & 1+i & i \\ -1+i & 0 & 3-i \\ i & -3-i & 2i \end{bmatrix}$

~~Unu~~

Unitary Matrix- A square complex matrix $A = [a_{ij}]$ is said to be unitary iff

$$A^{\theta}A = I = AA^{\theta}$$

Q1- Show that every complex matrix can be written as sum of two matrices one is hermitian & another is skew-hermitian.

Solution- Let A is given complex matrix.

Now $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$ — ①

Or $A = P + Q$

where $P = \frac{1}{2}(A + A^{\theta})$ & $Q = \frac{1}{2}(A - A^{\theta})$

$$P^{\theta} = \frac{1}{2}(A + A^{\theta})^{\theta} = \frac{1}{2}(A^{\theta} + A^{\theta\theta}) = \frac{1}{2}(A^{\theta} + A) = P$$

This show that P is hermitian matrix.

$$\& \quad Q^{\theta} = \frac{1}{2}(A - A^{\theta})^{\theta} = \frac{1}{2}(A^{\theta} - A^{\theta\theta}) = \frac{1}{2}(A^{\theta} - A)$$

$$= -\frac{1}{2}(A - A^{\theta}) = -Q$$

This show that Q is skew-hermitian.

Q2- Express the matrix

(i) $\begin{bmatrix} i & 2-3i & 4+5i \\ 6+i & 0 & 4-5i \\ -i & 2-i & 2+i \end{bmatrix}$ (ii) $\begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$

as a sum of two matrices one is hermitian & another is skew-hermitian

Solution-(i) We know that

$$A = P + Q = \left\{\frac{1}{2}(A + A^{\theta})\right\} + \left\{\frac{1}{2}(A - A^{\theta})\right\} \quad \text{—(1)}$$

Here P is Hermitian & Q is skew hermitian

$$A^{\theta} = (\bar{A})' = \begin{bmatrix} -i & 6-i & i \\ 2+3i & 0 & 2+i \\ 4-5i & 4+5i & 2-i \end{bmatrix}$$

$$P = \frac{1}{2}(A + A^{\theta})$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 8-4i & 4+6i \\ 8+4i & 0 & 6-4i \\ 4-6i & 6-4i & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 4-2i & 2+3i \\ 4+2i & 0 & 3-2i \\ 2-3i & 3-2i & 2 \end{bmatrix}$$

$P^{\theta} = P$

P is hermitian matrix

$$Q = \frac{1}{2}(A - A^{\theta})$$

$$= \frac{1}{2}\begin{bmatrix} 2i & -4-2i & 4+4i \\ 4-2i & 0 & 2-6i \\ -4+4i & -2-6i & 2i \end{bmatrix}$$

$$= \begin{bmatrix} i & -2-i & 2+2i \\ 2-i & 0 & 1-3i \\ -2+2i & -1-3i & i \end{bmatrix}$$

$$Q^{\theta} = -Q$$

Q is skew hermitian matrix.

(ii) We know that

$$A = P+Q = \left\{\frac{1}{2}(A+A^{\theta})\right\} + \left\{\frac{1}{2}(A-A^{\theta})\right\} \quad \text{--- (1)}$$

$$A^{\theta} = \begin{bmatrix} 1-i & -2i & -1-i \\ 2 & 2-i & -4 \\ 5+5i & 4-2i & 7 \end{bmatrix}$$

$$P = \frac{1}{2}(A+A^{\theta})$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 2-2i & 4-6i \\ 2+2i & 4 & 2i \\ 4+6i & -2i & 14 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1-i & 2-3i \\ 1+i & 2 & i \\ 2+3i & -i & 7 \end{bmatrix}$$

$P^{\theta} = P$

P is hermitian matrix

$$Q = \frac{1}{2}(A - A^{\theta})$$

$$= \frac{1}{2}\begin{bmatrix} 2i & 2+2i & 6 \\ 2i-2 & 2i & 8+2i \\ -6-4i & -8+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & 1+i & 3 \\ i-1 & i & 4+i \\ -3-2i & -4+i & 0 \end{bmatrix}$$

$Q^{\theta} = -Q$

Q is skew hermitian matrix

Q3- If $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$, verify that $A^{\theta}A$ is a hermitian matrix.

Solution- Given $A = \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$

$$A^{\theta} = \begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix}$$

$\cancel{A^\theta}$ $A^\theta A$

$$\begin{bmatrix} 2-i & -5 \\ 3 & -i \\ -1-3i & 4+2i \end{bmatrix} \begin{bmatrix} 2+i & 3 & -1+3i \\ -5 & i & 4-2i \end{bmatrix}$$

$$\begin{bmatrix} 30 & 6-8i & -19+17i \\ 6+8i & 10 & -5+5i \\ -19-17i & -5-5i & 30 \end{bmatrix}$$

In above matrix $a_{ij} = \bar{a}_{ji}$ & elements diagonal are real number therefore $A^\theta A$ is hermitian matrix.

Q4- Show that the matrix

$$\begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$$ is unitary matrix

if $\alpha^2+\beta^2+\gamma^2+\delta^2 = 1$.

Solution- Let $A = \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix}$

$$A^\theta = \begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix}$$

$$A^\theta A = I$$

$$\begin{bmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{bmatrix} \begin{bmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\cancel{\begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & (\alpha+i\gamma)(-\beta+i\delta)+(\beta-i\delta)(\alpha-i\gamma) \\ (-\beta+i\delta)(\alpha+i\gamma)+(\alpha-i\gamma)(\beta+i\delta) & \end{bmatrix}}$$

$$\begin{bmatrix} \alpha^2+\gamma^2+\beta^2+\delta^2 & (\alpha-i\gamma)(-\beta+i\delta)+(\beta-i\delta)(\alpha-i\gamma) \\ (-\beta-i\delta)(\alpha+i\gamma)+(\alpha+i\gamma)(\beta+i\delta) & \beta^2+\delta^2+\alpha^2+\gamma^2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & \cancel{-\alpha\beta}+\cancel{\alpha\delta i}+\cancel{\beta\gamma i}+\cancel{\gamma\delta}+\cancel{\alpha\beta}-\cancel{i\beta\gamma}-\cancel{i\alpha\delta} \\ \cancel{-\alpha\beta}-\cancel{i\beta\gamma}-\cancel{i\alpha\delta}+\cancel{\delta\gamma}+\cancel{\alpha\beta}+\cancel{\alpha\delta i}+\cancel{i\beta\gamma}-\cancel{\gamma\delta} & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix}$$

$$\begin{bmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{bmatrix}$$

is unitary matrix

if $\alpha^2+\beta^2+\gamma^2+\delta^2=1$ //

Q1- Prove that

(i) $\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ (ii) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

(iii) $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ where ω is cube root of unity is a unitary matrix.

Solution Let $A = \frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$

$$A^\theta = \frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$A^\theta \cdot A = I = AA^\theta$$

$$\frac{1}{2}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix} = I = \frac{1}{4}\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}\begin{bmatrix} 1-i & 1-i \\ -1-i & 1+i \end{bmatrix}$$

$$\frac{1}{4}\begin{bmatrix} 2+2 & 0 \\ 0 & 2+2 \end{bmatrix} = I = \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

$1+i^2+2j$

$2j$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence A is unitary matrix.

(iii) Let $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$

$$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$A^\theta A = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

again

$$A A^\theta = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ //

(ii) $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$A^{\theta} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$AA^{\theta} = \frac{1}{3} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$= \frac{1}{3} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ //

Hence A is unitary matrix //

Rank & Nullity theorem

If A is a matrix then

$\boxed{\text{rank of } A + \text{Nullity of } A = \text{No. of columns}}$

$\because$ = since
$\therefore$ = therefore

Theorem – The characteristic roots of a unitary matrix are of unit matrix

Solution – Let A be a unitary matrix,
so $A^\theta A = I = AA^\theta$ — (1)
Now let λ be the eigen value of A & corresponding eigen vector is X.

$$[A-\lambda I]X = 0$$
$$AX = \lambda X \quad \text{— (2)}$$

from (2)

$$(AX)^\theta = (\lambda X)^\theta$$
$$X^\theta A^\theta = \bar{\lambda} X^\theta \quad \text{— (3)}$$

from (3) & (2)

$$(X^\theta A^\theta)(AX) = (\bar{\lambda} X^\theta)(\lambda X)$$
$$X^\theta (A^\theta A) X = \lambda\bar{\lambda} X^\theta X$$
$$X^\theta I X = \lambda\bar{\lambda} X^\theta X$$
$$X^\theta X = \lambda\bar{\lambda} X^\theta X$$
$$X^\theta X - \lambda\bar{\lambda} X^\theta X = 0$$
$$(1-\lambda\bar{\lambda}) X^\theta X = 0$$

$\because$ $X \neq 0$

$\therefore$ $1 - \lambda\bar{\lambda} = 0$

i.e. $|\lambda|^2 = 1$

$$\boxed{|\lambda| = 1}$$