1 3-SAT Reduction

$$
\begin{array}{rcl}
\text{variable} & : & \text{type} \\
\text{literal} & : & \text{type} \\
\text{clause} & : & \text{type} \\
\text{term} & : & \text{type} \\
\\
\text{pos} & : & \text{variable} \to \text{literal} \\
\text{neg} & : & \text{variable} \to \text{literal} \\
\\
\text{3disjunct} & : & \text{literal} \to \text{literal} \to \text{literal} \to \text{type} \\
\text{ndisjunct} & : & \text{clause} \to \text{literal} \to \text{type} \\
\\
\text{convert} & : & \text{ndisjunct}\ C\ L_1 \multimap \text{ndisjunct}\ C\ L_2 \multimap \text{ndisjunct}\ C\ L_3 \multimap \text{ndisjunct}\ C\ L_4 \\
 & & \multimap \{\exists u : \text{variable.3disjunct}\ L_1\ L_2\ (\text{pos}\ u) \otimes \text{ndisjunct}\ C\ (\text{neg}\ u) \otimes \\
 & & \text{ndisjunct}\ C\ L_3 \otimes \text{ndisjunct}\ C\ L_4\} \\
\text{terminate} & : & (\text{term} \to \{\top\}) \multimap \text{cnfreduction} \\
\text{term1} & : & \text{term} \to \text{ndisjunct}\ C\ L_1 \multimap \text{ndisjunct}\ C\ L_2 \multimap \text{ndisjunct}\ C\ L_3 \\
 & & \multimap \{\text{3disjunct}\ L_1\ L_2\ L_3\} \\
\text{term2} & : & \text{term} \to \text{ndisjunct}\ C\ L_1 \multimap \text{ndisjunct}\ C\ L_2 \\
 & & \multimap \{\exists u : \text{variable3disjunct}\ L_1\ L_2\ (\text{pos}\ u) \otimes \text{3disjunct}\ L_1\ L_2\ (\text{neg}\ u)\} \\
\text{term3} & : & \text{term} \to \text{ndisjunct}\ C\ L_1 \multimap \{\exists u_1 : \text{variable}\ u_2 : \text{variable} \\
 & & \ \text{3disjunct}\ L_1\ (\text{pos}\ u_1)\ (\text{pos}\ u_2) \otimes \\
 & & \ \text{3disjunct}\ L_1\ (\text{pos}\ u_1)\ (\text{neg}\ u_2) \otimes \\
 & & \ \text{3disjunct}\ L_1\ (\text{neg}\ u_1)\ (\text{pos}\ u_2) \otimes \\
 & & \ \text{3disjunct}\ L_1\ (\text{neg}\ u_1)\ (\text{neg}\ u_2)\}
\end{array}
$$

2 Satisfiability

$$
\begin{array}{rcl}
\text{boolean} & : & \text{type} \\
\text{true} & : & \text{boolean} \\
\text{false} & : & \text{boolean} \\
\text{assign} & : & \text{variable} \to \text{boolean} \to \text{type} \\
\\
\text{3cnf1} & : & \text{assign } V \text{ true} \to \text{3disjunct (pos } V\text{) } L_2\ L_3 \multimap \{1\} \\
\text{3cnf2} & : & \text{assign } V \text{ true} \to \text{3disjunct } L_1 \text{ (pos } V\text{) } L_3 \multimap \{1\} \\
\text{3cnf3} & : & \text{assign } V \text{ true} \to \text{3disjunct } L_1\ L_2 \text{ (pos } V\text{) } \multimap \{1\} \\
\text{3cnf4} & : & \text{assign } V \text{ false} \to \text{3disjunct (neg } V\text{) } L_2\ L_3 \multimap \{1\} \\
\text{3cnf5} & : & \text{assign } V \text{ false} \to \text{3disjunct } L_1 \text{ (neg } V\text{) } L_3 \multimap \{1\} \\
\text{3cnf6} & : & \text{assign } V \text{ false} \to \text{3disjunct } L_1\ L_2 \text{ (neg } V\text{) } \multimap \{1\} \\
\\
\text{sat} & : & \text{clause} \to \text{type} \\
\\
\text{cnf1} & : & \text{sat } C \to \text{ndisjunct } C \text{ (pos } V\text{)} \multimap \{1\} \\
\text{cnf2} & : & \text{sat } C \to \text{ndisjunct } C \text{ (neg } V\text{)} \multimap \{1\} \\
\\
\text{cnf3} & : & \text{assign } V \text{ true} \to \text{ndisjunct } C \text{ (pos } V\text{)} \multimap \{!\text{sat } C\} \\
\text{cnf4} & : & \text{assign } V \text{ false} \to \text{ndisjunct } C \text{ (neg } V\text{)} \multimap \{!\text{sat } C\}
\end{array}
$$

3 Proofs

Let $\Gamma \in (u : \text{variable})^*$, $\Sigma \in (\text{assign } u \text{ true}|\text{assign } u \text{ false})^*$, $\Lambda \in (s : \text{sat } c)^*$, $\Phi \in (c : \text{clause})^*$, $\Omega \in (t : \text{term})$, $\Delta \in (l : \text{ndisjunct } c\ x)^*$, $\Theta \in (k : \text{3disjunct } x_1\ x_2\ x_3)^*$

Lemma 3.1 *If* $\Gamma, \Phi; \Delta, \Theta; [u, k \otimes l' \otimes l_3 \otimes l_4] \vdash E \leftarrow \text{cnfreduction}$ *where* $c : \text{clause} \in \Phi$, $u : \text{variable}$, $l' : \text{ndisjunct } c \text{ (neg } u)$, $l_3 : \text{ndisjunct } c\ x_3$, *and* $l_4 : \text{ndisjunct } c\ x_4$ *where* $x_3, x_4 : \text{literal}$, *then there exists a derivation* $\mathcal{D}$ *such that*

$$
\begin{array}{c}
\dfrac{\Gamma, u : \text{variable}, \Phi; \Delta, l', l_3, l_4, \Theta, k \vdash E \leftarrow \text{cnfreduction}}{\Gamma, u : \text{variable}, \Phi; \Delta, l', l_3, l_4, \Theta, k; \cdot \vdash E \leftarrow \text{cnfreduction}} \leftarrow\leftarrow \\
\vdots\ \mathcal{D} \\
\Gamma, \Phi; \Delta, \Theta; [u, k \otimes l' \otimes l_3 \otimes l_4] \vdash E \leftarrow \text{cnfreduction}
\end{array}
$$

.

Proof: The derivation $\mathcal{D}$ is given by (*modulo commutativity of these rules*):

$$\begin{array}{c l}
\Gamma, u : \mathsf{variable}, \Phi; \Delta, l', l_3, l_4\Theta, k \vdash E \leftarrow \mathsf{cnfreduction} & \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, l', l_3, l_4\Theta, k; \cdot \vdash E \leftarrow \mathsf{cnfreduction} & \leftarrow\leftarrow \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, l', l_3, \Theta, k; l_4 \vdash E \leftarrow \mathsf{cnfreduction} & A\mathbf{L} \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, l', \Theta, k; l_3, l_4 \vdash E \leftarrow \mathsf{cnfreduction} & A\mathbf{L} \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, \Theta, k; l', l_3, l_4 \vdash E \leftarrow \mathsf{cnfreduction} & A\mathbf{L} \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, \Theta; k, l', l_3, l_4 \vdash E \leftarrow \mathsf{cnfreduction} & A\mathbf{L} \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, \Theta; k, l', l_3 \otimes l_4 \vdash E \leftarrow \mathsf{cnfreduction} & \otimes\mathbf{L} \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, \Theta; k, l' \otimes l_3 \otimes l_4 \vdash E \leftarrow \mathsf{cnfreduction} & \otimes\mathbf{L} \\ \hline
\Gamma, u : \mathsf{variable}, \Phi; \Delta, \Theta; k \otimes l' \otimes l_3 \otimes l_4 \vdash E \leftarrow \mathsf{cnfreduction} & \otimes\mathbf{L} \\ \hline
\Gamma, \Phi; \Delta, \Theta; [u, k \otimes l' \otimes l_3 \otimes l_4] \vdash E \leftarrow \mathsf{cnfreduction} & \exists\mathbf{L}
\end{array}$$

□

Lemma 3.2 *If*

$$\begin{array}{c}
\Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\
\vdots\ \mathcal{S}_1 \\
\Gamma, \Phi, \Sigma; \Delta, l_1, l_2, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} E_{s_1} \leftarrow 1
\end{array}$$

where $c : \mathsf{clause} \in \Phi$, $l_i : \mathsf{ndisjunct}\ c\ x_i$ *and* $x_i : \mathsf{literal}$, *then there exists a* Σ' *and a derivation* $\mathcal{S}_2$ *such that*

$$\begin{array}{c}
\Gamma, \Gamma', \Phi, \Sigma, \Sigma', \Lambda'; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\
\vdots\ \mathcal{S}_2 \\
\Gamma, u : \mathsf{variable}, \Phi, \Sigma, \Sigma'; \Delta, l', l_3, l_4, \Theta, k \vdash_{\mathsf{SAT}} E_{s_2} \leftarrow 1
\end{array}$$

where $l' : \mathsf{ndisjunct}\ c\ (\mathsf{neg}\ u)$ *and* $k : \mathsf{3disjunct}\ x_1\ x_2\ (\mathsf{pos}\ u)$.

Proof: Let us assume $x_1 = \mathsf{pos}\ u_1$ and $x_2 = \mathsf{pos}\ u_2$. The other three cases are similar.

The derivation $\mathcal{S}_1$ can be in one of the following forms:

Case: Let $\Sigma = \Sigma_1, a_1 : \mathsf{assign}\ u_1\ \mathsf{true} = \Sigma_2, a_2 : \mathsf{assign}\ u_2\ \mathsf{true}$.

$$\dfrac{\Gamma, \Phi, \Sigma; \cdot \vdash_{\mathsf{SAT}} \mathsf{cnf3}\ a_1^{\wedge} l_1 \Rightarrow \{\exists s_1 : \mathsf{sat}\ c.1\} \quad \begin{array}{c}
\dfrac{\Gamma, \Phi, \Sigma; \cdot \vdash_{\mathsf{SAT}} \mathsf{cnf3}\ a_2^{\wedge} l_2 \Rightarrow \{\exists s_2 : \mathsf{sat}\ c.1\} \quad \begin{array}{c}
\Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\
\vdots\ \mathcal{S}_1' \\
\Gamma, \Phi, \Sigma, s_1 : \mathsf{sat}\ c, s_2 : \mathsf{sat}\ c; \Delta, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} E_{s_1}'' \leftarrow 1 \\
\vdots\ \mathcal{D}_2 \\
\Gamma, \Phi, \Sigma, s_1 : \mathsf{sat}\ c; \Delta, l_3, l_4, \Theta; [s_2, 1] \vdash_{\mathsf{SAT}} \mathsf{in}\ E_{s_1}'' \leftarrow 1
\end{array}}{\Gamma, \Phi, \Sigma_2, a_2 : \mathsf{assign}\ u_2\ \mathsf{true}, s_1 : \mathsf{sat}\ c; \Delta, l_2, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} \mathsf{let}\ [s_2, 1] = \mathsf{cnf3}\ a_2^{\wedge} l_2 \mathsf{in}\ E_{s_1}'' \leftarrow 1}\ \{\}\mathbf{E} \\
\vdots\ \mathcal{D}_1 \\
\Gamma, \Phi, \Sigma_1, a_1 : \mathsf{assign}\ u_1\ \mathsf{true}; \Delta, l_2, l_3, l_4, \Theta; [s_1, 1] \vdash_{\mathsf{SAT}} E_{s_1}' \leftarrow 1
\end{array}}{\Gamma, \Phi, \Sigma, a_1 : \mathsf{assign}\ u_1\ \mathsf{true}; \Delta, l_1, l_2, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} \mathsf{let}\ [s_1, 1] = \mathsf{cnf3}\ a_1^{\wedge} l_1 \mathsf{in}\ E_{s_1}' \leftarrow 1}\ \{\}\mathbf{E}$$

In this case $E_{s_1} = (\mathsf{let}\ [s_1, 1] = \mathsf{cnf3}\ a_1^{\wedge} l_1 \mathsf{in}\ E_{s_1}')$ and $E_{s_1}' = (\mathsf{let}\ [s_2, 1] = \mathsf{cnf3}\ a_2^{\wedge} l_2 \mathsf{in}\ E_{s_1}'')$.

Let $\Sigma' = a' : \mathsf{assign}\ u\ \mathsf{false}$. Now, the derivation $\mathcal{S}_2$ is given by

$$
\cfrac{\Gamma, \ldots ; \cdot \vdash_{\mathsf{SAT}} \mathsf{3cnf1} a_1^{\wedge} k \ldots \qquad \begin{array}{c} \cfrac{\Gamma, \ldots \qquad \begin{array}{c} \Gamma, u, \Phi, \Sigma, \Sigma', s' : \mathsf{sat}\ c; \Delta, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} E''_{s_1} \leftarrow 1 \\ \vdots \end{array}}{\Gamma, u : \mathsf{variable}, \Phi, \Sigma, \Sigma'; \Delta, l', l_3, l_4, \Theta \vdash_{\mathsf{SAT}} \mathsf{let}\ [s', 1] = \mathsf{cnf4}\ a'^{\wedge} l' \mathsf{in}\ E''_{s_1} \leftarrow 1} \\ \vdots \end{array}}{\Gamma, u : \mathsf{variable}, \Phi, \Sigma, \Sigma'; \Delta, l', l_3, l_4, \Theta, k \vdash_{\mathsf{SAT}} \mathsf{let}\ \{1\} = \mathsf{3cnf1}\ a_1^{\wedge} k\ \mathsf{in}\ E'_{s_2} \leftarrow 1}\ \{\}\mathbf{E}
$$

We have a proof of $\Gamma, u, \Phi, \Sigma, \Sigma', s' : \mathsf{sat}\ c; \Delta, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} E''_{s_1} \leftarrow 1$ due to $\mathcal{S}'_1$ and *weakening/contraction* of contexts.

Case: Let $\Sigma = \Sigma_1, a_1 : \mathsf{assign}\ u_1\ \mathsf{true} = \Sigma_2, a_2 : \mathsf{assign}\ u_2\ \mathsf{false}$.

$$
\cfrac{\Gamma, \Phi, \Sigma; \cdot \vdash_{\mathsf{SAT}} \mathsf{cnf3}\ a_1^{\wedge} l_1 \Rightarrow \{\exists s_1 : \mathsf{sat}\ c.1\} \qquad \begin{array}{c} \cfrac{\Gamma, \Phi, \Sigma; \cdot \vdash_{\mathsf{SAT}} \mathsf{cnf1}\ s_1^{\wedge} l_2 \Rightarrow \{1\} \qquad \begin{array}{c} \Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}'_1 \\ \Gamma, \Phi, \Sigma, s_1 : \mathsf{sat}\ c; \Delta, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} E''_{s_1} \leftarrow 1 \\ \vdots\ \mathcal{D}_2 \\ \Gamma, \Phi, \Sigma, s_1 : \mathsf{sat}\ c; \Delta, l_3, l_4, \Theta; \{1\} \vdash_{\mathsf{SAT}} \mathsf{in}\ E''_{s_1} \leftarrow 1 \end{array}}{\Gamma, \Phi, \Sigma, s_1 : \mathsf{sat}\ c; \Delta, l_2, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} \mathsf{let}\ \{1\} = \mathsf{cnf1}\ s_1^{\wedge} l_2 \mathsf{in}\ E''_{s_1} \leftarrow 1}\ \{\}\mathbf{E} \\ \vdots\ \mathcal{D}_1 \\ \Gamma, \Phi, \Sigma_1, a_1 : \mathsf{assign}\ u_1\ \mathsf{true}; \Delta, l_2, l_3, l_4, \Theta; [s_1, 1] \vdash_{\mathsf{SAT}} E'_{s_1} \leftarrow 1 \end{array}}{\Gamma, \Phi, \Sigma, a_1 : \mathsf{assign}\ u_1\ \mathsf{true}; \Delta, l_1, l_2, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} \mathsf{let}\ [s_1, 1] = \mathsf{cnf3}\ a_1^{\wedge} l_1 \mathsf{in}\ E'_{s_1} \leftarrow 1}\ \{\}\mathbf{E}
$$

Let $\Sigma' = a' : \mathsf{assign}\ u\ \mathsf{false}$. The rest of the proof same as above.

Case: Let $\Sigma = \Sigma_1, a_1 : \mathsf{assign}\ u_1\ \mathsf{false} = \Sigma_2, a_2 : \mathsf{assign}\ u_2\ \mathsf{true}$.

Same as above.

Case: Let $\Sigma = \Sigma_1, a_1 : \mathsf{assign}\ u_1\ \mathsf{false} = \Sigma_2, a_2 : \mathsf{assign}\ u_2\ \mathsf{false}$. Let $\Sigma' = a' : \mathsf{assign}\ u\ \mathsf{true}$.

Using Lemma 4.3, there exists $\mathcal{S}'_1$ given by

$$
\begin{array}{c} \Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}'_1 \\ \Gamma, \Phi, \Sigma; \Delta, l_3, l_4, \Theta \vdash_{\mathsf{SAT}} E''_{s_1} \leftarrow 1 \end{array}
$$

and S''_1 given by

$$
\begin{array}{c} \Gamma, u, \Phi, \Sigma, \Sigma', \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}'_1 \\ \Gamma, u : \mathsf{variable}, \Phi, \Sigma, \Sigma'; \Delta, l_3, l_4, l', \Theta \vdash_{\mathsf{SAT}}\ E'''_{s_1} \leftarrow 1 \end{array}
$$

Now, $\mathcal{S}_2$ is given by

$$
\cfrac{\begin{array}{c} \Gamma, u, \Phi, \Sigma, \Sigma', \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}'_1 \\ \Gamma, u, \Phi, \Sigma, \Sigma'; \Delta, l_3, l_4, l', \Theta \vdash_{\mathsf{SAT}} \mathsf{let}\ E'''_{s_1} \leftarrow 1 \end{array}}{\Gamma, u, \Phi, \Sigma, \Sigma'; \Delta, l_3, l_4, l', \Theta, k \vdash_{\mathsf{SAT}} \mathsf{let}\{1\} = \mathsf{3cnf3}\ a'^{\wedge} k\ \mathsf{in}\ E'''_{s_1} \leftarrow 1}
$$

$\square$

Lemma 3.3

$$\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda';\cdot\vdash_{\mathsf{SAT}} 1 \Leftarrow 1\\ \vdots\ \mathcal{S}_1\\ \Gamma,\Phi,\Sigma,\Lambda;\Delta,l_1,\Theta\vdash_{\mathsf{SAT}} E_{s_1}\ \leftarrow 1\end{array}$$

where c : $\mathsf{clause} \in \Phi$, l_i : $\mathsf{ndisjunct}\ c\ x_i$ *and* x_i : $\mathsf{literal}$, $\Sigma = \Sigma_1, a_1$: $\mathsf{assign}\ u_1\ b$ *(*$b = \mathsf{false}$ *if* x_1 : $\mathsf{pos}\ u_1$ *and* $b = \mathsf{true}$ *if* x_1 : $\mathsf{neg}\ u_1$*), and* Λ *does not contain* s : $\mathsf{sat}\ c$*, if and only if there exists*

$$\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda';\cdot\vdash_{\mathsf{SAT}} 1 \Leftarrow 1\\ \vdots\ \mathcal{S}_1{}'\\ \Gamma,\Phi,\Sigma,\Lambda;\Delta,\Theta\vdash_{\mathsf{SAT}} E'_{s_1}\ \leftarrow 1\end{array}$$

.

Proof: ($\Rightarrow$) Let $l_2 = \mathsf{pos}\ u_2$. The other case is similar.

By induction on $\mathcal{S}_1$:

Case:

When l' : $\mathsf{ndisjunct}\ c'\ x'$ and $c' \neq c$.

$$\dfrac{\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda';\cdot\vdash_{\mathsf{SAT}} 1 \Leftarrow 1\\ \vdots\ \mathcal{S}_1{}''\\ \Gamma,\Phi,\Sigma,\Lambda,s:\mathsf{sat}\ c'?;\Delta',l_1,\Theta\vdash_{\mathsf{SAT}} E'_{s_1}\ \leftarrow 1\end{array}}{\Gamma,\Phi,\Sigma,\Lambda;\Delta',l',l_1,\Theta\vdash_{\mathsf{SAT}} \mathsf{let}\{\ldots\} = \mathsf{cnfi}\ (a|s)\hat{\ }l'\mathsf{in}\ E'_{s_1}\ \leftarrow 1}\ \{\}\mathbf{E}$$

Apply induction hypothesis to $\mathcal{S}''_1$.

Case: When l' : $\mathsf{ndisjunct}\ c\ x'$, x' : $\mathsf{pos}\ u'$ (the other neg case is similar) and $\Sigma = \Sigma', a'$: $\mathsf{assign}\ u'\ \mathsf{true}$ (The case $\Sigma = \Sigma', a'$: $\mathsf{assign}\ u'\ \mathsf{false}$ is not possible.).

$$\dfrac{\dfrac{\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda';\cdot\vdash_{\mathsf{SAT}} 1 \Leftarrow 1\\ \vdots\ \mathcal{S}_1{}''\\ \Gamma,\Phi,\Sigma,\Lambda,s';\Delta',\Theta\vdash_{\mathsf{SAT}} E''_{s_1}\end{array}}{\Gamma,\Phi,\Sigma,\Lambda,s';\Delta',l_1,\Theta\vdash_{\mathsf{SAT}} \mathsf{let}\ \{1\} = \mathsf{cnf1}\ s'\hat{\ }l'\mathsf{in}\ E''_{s_1}\ \leftarrow 1}\ \{\}\mathbf{E}}{\Gamma,\Phi,\Sigma,\Lambda;\Delta',l',l_1,\Theta\vdash_{\mathsf{SAT}} \mathsf{let}\ [s',1] = \mathsf{cnf3}\ a'\hat{\ }l'\mathsf{in}\ E'_{s_1}\ \leftarrow 1}\ \{\}\mathbf{E}$$

In this case, $\mathcal{S}'_1$ is given by:

$$\dfrac{\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda';\cdot\vdash_{\mathsf{SAT}} 1 \Leftarrow 1\\ \vdots\ \mathcal{S}_1{}''\\ \Gamma,\Phi,\Sigma,\Lambda,s';\Delta',\Theta\vdash_{\mathsf{SAT}} E''_{s_1}\end{array}}{\Gamma,\Phi,\Sigma,\Lambda;\Delta',l',\Theta\vdash_{\mathsf{SAT}} \mathsf{let}\ [s',1] = \mathsf{cnf3}\ a'\hat{\ }l'\mathsf{in}\ E'_{s_1}\ \leftarrow 1}\ \{\}\mathbf{E}$$

(Base) Case: When $\Delta' = \cdot$. The first case is not possible as the intuitionistic context does not contain s : $\mathsf{sat}\ c$. The second case is trivial.

($\Leftarrow$) By induction on $\mathcal{S}'_1$. The proof is similar to above. $\square$

Lemma 3.4 *If* $\Gamma, \Phi, \Omega; \Delta, \Theta; k_1 \vdash E \leftarrow \top$ *where* $c : \textsf{clause} \in \Phi$, $k_1 : \textsf{3disjunct}\ x_1\ x_2\ x_3$, *where* $x_1, x_2, x_3 : \textsf{literal}$, *then there exists a derivation* $\mathcal{D}$ *such that*

$$\cfrac{\dfrac{\Gamma, u : \textsf{variable}, \Phi; \Delta, \Theta, k_1 \vdash E \leftarrow \top}{\Gamma, u : \textsf{variable}, \Phi; \Delta, \Theta, k_1; \cdot \vdash E \leftarrow \top} \leftarrow\leftarrow}{\Gamma, \Phi; \Delta, \Theta; k_1 \vdash E \leftarrow \top}\ \mathcal{D}$$

.

Lemma 3.5 *If* $\Gamma, \Phi, \Omega; \Delta, \Theta; [u, k_1 \otimes k_2] \vdash E \leftarrow \top$ *where* $c : \textsf{clause} \in \Phi$, $u : \textsf{variable}$, $k_1 : \textsf{3disjunct (pos } u\textsf{)}\ x_1\ x_2$, *and* $k_2 : \textsf{3disjunct (neg } u\textsf{)}\ x_1\ x_2$ *where* $x_1, x_2 : \textsf{literal}$, *then there exists a derivation* $\mathcal{D}$ *such that*

$$\cfrac{\dfrac{\Gamma, u : \textsf{variable}, \Phi; \Delta, \Theta, k_1, k_2 \vdash E \leftarrow \top}{\Gamma, u : \textsf{variable}, \Phi; \Delta, \Theta, k_1, k_2; \cdot \vdash E \leftarrow \top} \leftarrow\leftarrow}{\Gamma, \Phi; \Delta, \Theta; [u, k_1 \otimes k_2] \vdash E \leftarrow \top}\ \mathcal{D}$$

.

Lemma 3.6 *If* $\Gamma, \Phi, \Omega; \Delta, \Theta; [u_1 u_2, k_1 \otimes k_2 \otimes k_3 \otimes k_4] \vdash E \leftarrow \top$ *where* $c : \textsf{clause} \in \Phi$, $u : \textsf{variable}$, $k_1 : \textsf{3disjunct (pos } u_1\textsf{) (pos } u_2\textsf{)}\ x_1$, $k_2 : \textsf{3disjunct (pos } u_1\textsf{) (neg } u_2\textsf{)}\ x_1$, $k_3 : \textsf{3disjunct (neg } u_1\textsf{) (pos } u_2\textsf{)}\ x_1$, $k_4 : \textsf{3disjunct (neg } u_1\textsf{) (neg } u_2\textsf{)}\ x_1$ *where* $x_1 : \textsf{literal}$, *then there exists a derivation* $\mathcal{D}$ *such that*

$$\cfrac{\dfrac{\Gamma, u_1 : \textsf{variable}, u_2 : \textsf{variable}, \Phi; \Delta, \Theta, k_1, k_2, k_3, k_4 \vdash E \leftarrow \top}{\Gamma, u_1 : \textsf{variable}, u_2 : \textsf{variable}, \Phi; \Delta, \Theta, k_1, k_2, k_3, k_4; \cdot \vdash E \leftarrow \top} \leftarrow\leftarrow}{\Gamma, \Phi; \Delta, \Theta; [u_1 u_2, k_1 \otimes k_2 \otimes k_3 \otimes k_4] \vdash E \leftarrow \top}\ \mathcal{D}$$

.

Lemma 3.7 *If*

$$\cfrac{\Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\textsf{SAT}} 1 \Leftarrow 1}{\Gamma, \Phi, \Sigma; \Delta, l_1, l_2, l_3, \Theta \vdash_{\textsf{SAT}} E_{s_1} \leftarrow 1}\ \mathcal{S}_1$$

where $c : \textsf{clause} \in \Phi$, $l_i : \textsf{ndisjunct}\ c\ x_i$ *and* $x_i : \textsf{literal}$, *then there exists a derivation* $\mathcal{S}_2$ *such that*

$$\cfrac{\Gamma, \Gamma', \Phi, \Sigma, \Lambda'; \cdot \vdash_{\textsf{SAT}} 1 \Leftarrow 1}{\Gamma, \Phi, \Sigma; \Delta, \Theta, k_1 \vdash_{\textsf{SAT}} E_{s_2} \leftarrow 1}\ \mathcal{S}_2$$

where $k_1 : \textsf{3disjunct}\ x_1\ x_2\ x_3$.

Lemma 3.8 *If*

$$\cfrac{\Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\textsf{SAT}} 1 \Leftarrow 1}{\Gamma, \Phi, \Sigma; \Delta, l_1, l_2, \Theta \vdash_{\textsf{SAT}} E_{s_1} \leftarrow 1}\ \mathcal{S}_1$$

where c : clause $\in \Phi$, l_i : ndisjunct $c\ x_i$ *and* x_i : literal, *then there exists a* Σ' *and a derivation* $\mathcal{S}_2$ *such that*

$$\begin{array}{c} \Gamma, \Gamma', \Phi, \Sigma, \Sigma', \Lambda'; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}_2 \\ \Gamma, u : \mathsf{variable}, \Phi, \Sigma, \Sigma'; \Delta, \Theta, k_1, k_2 \vdash_{\mathsf{SAT}} E_{s_2} \leftarrow 1 \end{array}$$

where k_1 : 3disjunct (pos u) $x_1\ x_2$, *and* k_2 : 3disjunct (neg u) $x_1\ x_2$.

Lemma 3.9 *If*

$$\begin{array}{c} \Gamma, \Phi, \Sigma, \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}_1 \\ \Gamma, \Phi, \Sigma; \Delta, l_1, \Theta \vdash_{\mathsf{SAT}} E_{s_1} \leftarrow 1 \end{array}$$

where c : clause $\in \Phi$, l_i : ndisjunct $c\ x_i$ *and* x_i : literal, *then there exists a* Σ' *and a derivation* $\mathcal{S}_2$ *such that*

$$\begin{array}{c} \Gamma, \Gamma', \Phi, \Sigma, \Sigma', \Lambda'; \cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}_2 \\ \Gamma, u_1 : \mathsf{variable}, u_2 : \mathsf{variable}, \Phi, \Sigma, \Sigma'; \Delta, \Theta, k_1, k_2, k_3, k_4 \vdash_{\mathsf{SAT}} E_{s_2} \leftarrow 1 \end{array}$$

where k_1 : 3disjunct (pos u_1) (pos u_2) x_1, k_2 : 3disjunct (pos u_1) (neg u_2) x_1, k_3 : 3disjunct (neg u_1) (pos u_2) x_1, *and* k_4 : 3disjunct (neg u_1) (neg u_2) x_1

Lemma 3.10 (Termination) *If*

$$\begin{array}{c} \dfrac{\Gamma, \Phi, \Gamma', \Omega; \Theta, \Theta' \vdash \langle\rangle \Leftarrow \top}{\Gamma, \Phi, \Gamma', \Omega; \Theta, \Theta' \vdash \langle\rangle \leftarrow \top} \Leftarrow\leftarrow \\ \vdots\ \mathcal{R} \\ \Gamma, \Phi, \Omega; \Delta, \Theta \vdash E_r \leftarrow \top \end{array}$$

where only expression checking and pattern expansion rules are permitted within $\mathcal{R}$ *and*

$$\begin{array}{c} \Gamma, \Phi, \Sigma, \Omega, \Lambda; \cdot \vdash_{\mathsf{SAT}} 1 : 1 \\ \vdots\ \mathcal{S}_1 \\ \Gamma, \Phi, \Omega, \Sigma; \Delta, \Theta \vdash_{\mathsf{SAT}} E_{s_1} \Leftarrow \{1\} \end{array}$$

then there exists a Σ' *such that*

$$\begin{array}{c} \Gamma, \Gamma', \Phi, \Omega, \Sigma, \Sigma', \Lambda'; \cdot \vdash_{\mathsf{SAT}} 1 : 1 \\ \vdots\ \mathcal{S}_2 \\ \Gamma, \Gamma', \Phi, \Omega, \Sigma, \Sigma'; \Theta, \Theta' \vdash_{\mathsf{SAT}} E_{s_2} \Leftarrow \{1\}. \end{array}$$

Proof: By induction on the derivation $\mathcal{R}$:

Case:

$$\begin{array}{c} \dfrac{\Gamma, \Phi, \Gamma'; \Delta', \Theta, \Theta' \vdash E'_r \Leftarrow \top}{\Gamma, \Phi, \Gamma'; \Delta', \Theta, \Theta' \vdash E'_r \leftarrow \top} \Leftarrow\leftarrow \\ \vdots\ \mathcal{R}' \\ \dfrac{\Gamma, \Phi; \Delta, \Theta \vdash E_r \Leftarrow \top}{\Gamma, \Phi; \Delta, \Theta \vdash E_r \leftarrow \top} \Leftarrow\leftarrow \end{array}$$

Base Case. Here $E_r = E'_r$, $\Delta = \Delta'$ and $\Theta' = \cdot$. Let $\Sigma' = \cdot$ and the lemma follows.

Case:

$$\dfrac{\overset{\mathcal{C}}{\Gamma,\Phi,\Omega;\cdot \vdash \textsf{term1}\ t^\wedge l_1^\wedge l_2^\wedge l_3 \Rightarrow \{\ldots\}} \qquad \begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}'' \\ \Gamma,\Phi,\Omega;\Delta,\Theta;k_1 \vdash E_r \leftarrow \top\end{array}}{\Gamma,\Phi,\Omega;\Delta,l_1,l_2,l_3,\Theta \vdash \textsf{let}\ \{k_1\} = \textsf{term1}\ t^\wedge l_1^\wedge l_2^\wedge l_3\ \textsf{in}\ E_r \leftarrow \top}\{\}\mathbf{E}$$

where $l_i : \textsf{ndisjunct}\ c\ x_i$, $k : \textsf{3disjunct}\ x_1\ x_2\ x_3$ and $t : \textsf{term} \in \Omega$.

By Lemma 4.4, there exists a derivation $\mathcal{D}$ such that

$$\begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}'' \\ \Gamma,\Phi,\Omega;\Delta,\Theta,k_1 \vdash E_r \leftarrow \top \\ \vdots\ \mathcal{D} \\ \Gamma,\Phi,\Omega;\Delta,\Theta;k_1 \vdash E_r \leftarrow \top\end{array}$$

Apply Lemma 4.7 and induction hypothesis on $\mathcal{R}''$.

Case:

$$\dfrac{\overset{\mathcal{C}}{\Gamma,\Phi,\Omega;\cdot \vdash \textsf{term2}\ t^\wedge l_1^\wedge l_2 \Rightarrow \{\ldots\}} \qquad \begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}'' \\ \Gamma,\Phi,\Omega;\Delta,\Theta;[u,k_1 \otimes k_2] \vdash E_r \leftarrow \top\end{array}}{\Gamma,\Phi,\Omega;\Delta,l_1,l_2,\Theta \vdash \textsf{let}\ [u,k_1 \otimes k_2] = \textsf{term2}\ t^\wedge l_1^\wedge l_2\ \textsf{in}\ E_r \leftarrow \top}\{\}\mathbf{E}$$

where $l_i : \textsf{ndisjunct}\ c\ x_i, k_1 : \textsf{3disjunct}\ (\textsf{pos}\ u)\ x_1\ x_2, k_2 : \textsf{3disjunct}\ (\textsf{neg}\ u)\ x_1\ x_2$ and $t : \textsf{term} \in \Omega$.

Apply Lemma 4.5 and Lemma 4.8 and induction hypothesis on $\mathcal{R}''$.

Case:

$$\dfrac{\overset{\mathcal{C}}{\Gamma,\Phi,\Omega;\cdot \vdash \textsf{term3}\ t^\wedge l_1 \Rightarrow \{\ldots\}} \qquad \begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Delta',\Theta,\Theta' \vdash E_r' \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}'' \\ \Gamma,\Phi,\Omega;\Delta,\Theta;[u_1u_2,k_1 \otimes k_2 \otimes k_3 \otimes k_4] \vdash E_r \leftarrow \top\end{array}}{\Gamma,\Phi,\Omega;\Delta,l_1,\Theta \vdash \textsf{let}\ [u_1u_2,k_1 \otimes k_2 \otimes k_3 \otimes k_4] = \textsf{term3}\ t^\wedge l_1\ \textsf{in}\ E_r \leftarrow \top}\{\}\mathbf{E}$$

where $l_i : \textsf{ndisjunct}\ c\ x_i, k_1 : \textsf{3disjunct}\ (\textsf{pos}\ u_1)\ (\textsf{pos}\ u_2)\ x_1, k_2 : \textsf{3disjunct}\ (\textsf{pos}\ u_1)\ (\textsf{neg}\ u_2)\ x_1$, $k_1 : \textsf{3disjunct}\ (\textsf{neg}\ u_1)\ (\textsf{pos}\ u_2)\ x_1$, $k_1 : \textsf{3disjunct}\ (\textsf{neg}\ u_1)\ (\textsf{neg}\ u_2)\ x_1$ and $t : \textsf{term} \in \Omega$.

Apply Lemma 4.6 and Lemma 4.9 and induction hypothesis on $\mathcal{R}''$.

$\square$

Theorem 3.11 (Conversion) *If*

$$\begin{array}{c}\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \Leftarrow \top \\ \vdots\ \mathcal{R} \\ \Gamma,\Phi;\Delta \vdash E_r \leftarrow \mathsf{cnfreduction}\end{array}$$

where only expression checking and pattern expansion rules are permitted within $\mathcal{R}$ and $\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda;\cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}_1 \\ \Gamma,\Phi,\Sigma;\Delta \vdash_{\mathsf{SAT}} E_{s_1} \Leftarrow \{1\}\end{array}$, *then there exists a Σ' such that* $\begin{array}{c}\Gamma,\Gamma',\Phi,\Omega,\Sigma,\Sigma',\Lambda';\cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}_2 \\ \Gamma,\Gamma',\Phi,\Omega,\Sigma,\Sigma';\Theta \vdash_{\mathsf{SAT}} E_{s_2} \Leftarrow \{1\}\end{array}$.

Proof: By induction on the derivation $\mathcal{R}$:

Case:

$$\begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}' \\ \dfrac{\overset{\mathcal{C}}{\Gamma,\Phi;\cdot \vdash \mathsf{convert}^\wedge l_1^\wedge l_2^\wedge l_3^\wedge l_4 \Rightarrow \{\ldots\}} \quad \Gamma,\Phi;\Delta,\Theta;[u,k_1\otimes l'\otimes l_3\otimes l_4] \vdash E_r \leftarrow \mathsf{cnfreduction}}{\Gamma,\Phi;\Delta,l_1,l_2,l_3,l_4,\Theta \vdash \mathsf{let}\ \{[u,k_1\otimes l'\otimes l_3\otimes l_4]\} = \mathsf{convert}^\wedge l_1^\wedge l_2^\wedge l_3^\wedge l_4\ \mathsf{in}\ E_r \leftarrow \mathsf{cnfreduction}}\{\}\mathbf{E}\end{array}$$

where l_i : ndisjunct $c\ x_i$, c : ndisjunct c (neg u), k : 3disjunct $x_1\ x_2\ x_3$

By Lemma 4.1, the derivation $\mathcal{R}'$ has a subderivation $\mathcal{D}$ such that

$$\begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}'' \\ \dfrac{\Gamma,u:\mathsf{variable},\Phi;\Delta,l',l_3,l_4,\Theta,k_1 \vdash E_r \leftarrow \mathsf{cnfreduction}}{\Gamma,u:\mathsf{variable},\Phi;\Delta,l',l_3,l_4,\Theta,k_1;\cdot \vdash E_r \leftarrow \mathsf{cnfreduction}}\leftarrow\leftarrow \\ \vdots\ \mathcal{D} \\ \Gamma,\Phi;\Delta,\Theta;[u,k_1\otimes l'\otimes l_3\otimes l_4] \vdash E_r \leftarrow \mathsf{cnfreduction}\end{array}$$

Using Lemma 4.2, there exists a derivation $\mathcal{S}_1''$ such that

$$\begin{array}{c}\Gamma,u,\Phi,\Sigma,\Lambda;\cdot \vdash_{\mathsf{SAT}} 1 \Leftarrow 1 \\ \vdots\ \mathcal{S}_1'' \\ \Gamma,u,\Phi,\Sigma;\Delta,l',l_3,l_4,\Theta,k_1 \vdash_{\mathsf{SAT}} \{E''_{s_1}\} \Leftarrow \{1\}\end{array}$$

Apply induction hypothesis to $\mathcal{R}''$.

Case:

$$\begin{array}{c}\dfrac{\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \Leftarrow \top}{\Gamma,\Phi,\Gamma',\Omega;\Theta \vdash \langle\rangle \leftarrow \top}\Leftarrow\leftarrow \\ \vdots\ \mathcal{R}' \\ \dfrac{\Gamma,\Phi;\Delta,\Theta \vdash E_r \Leftarrow \mathsf{cnfreduction}}{\Gamma,\Phi;\Delta,\Theta \vdash E_r \leftarrow \mathsf{cnfreduction}}\Leftarrow\leftarrow\end{array}$$

Base Case. Here $E_r = \mathsf{terminate}^\wedge(\lambda t : \mathsf{term}.E_r')$. Apply Theorem 4.12.

□

Theorem 3.12 (Termination) *If*

$$\begin{array}{c}\Gamma,\Phi,\Gamma',\Omega;\Theta,\Theta'\vdash\langle\rangle\Leftarrow\top\\ \vdots\ \mathcal{R}\\ \Gamma,\Phi;\Delta,\Theta\vdash \mathsf{terminate}^{\wedge}(\lambda t:\mathsf{term}.E_r)\Leftarrow\{\mathsf{cnfreduction}\}\end{array}$$

where only expression checking and pattern expansion rules are permitted within $\mathcal{R}$ *and*
$\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda;\cdot\vdash_{\mathsf{SAT}}1\Leftarrow 1\\ \vdots\ \mathcal{S}_1\\ \Gamma,\Phi,\Sigma;\Delta,\Theta\vdash_{\mathsf{SAT}}E_{s_1}\Leftarrow\{1\}\end{array}$, *then there exists a* Σ' *such that* $\begin{array}{c}\Gamma,\Gamma',\Phi,\Omega,\Sigma,\Sigma',\Lambda';\cdot\vdash_{\mathsf{SAT}}1\Leftarrow 1\\ \vdots\ \mathcal{S}_2\\ \Gamma,\Gamma',\Phi,\Omega,\Sigma,\Sigma';\Theta,\Theta'\vdash_{\mathsf{SAT}}E_{s_2}\Leftarrow\{1\}\end{array}$.

Proof: The reduction $\mathcal{R}$ can be written as shown below:

$$\dfrac{\Gamma,\Phi;\Delta,\Theta\vdash\Rightarrow\ldots\quad \dfrac{\dfrac{\begin{array}{c}\Gamma,\Phi,\Gamma',\Omega;\Theta,\Theta'\vdash\langle\rangle\Leftarrow\top\\ \vdots\ \mathcal{R}'\\ \Gamma,\Phi,t:\mathsf{term};\Delta,\Theta\vdash E_r\Leftarrow\top\end{array}}{\Gamma,\Phi,t:\mathsf{term};\Delta,\Theta\vdash E_r\Leftarrow\{\top\}}\ \{\}\mathbf{I}}{\Gamma,\Phi;\Delta,\Theta\vdash\lambda t:\mathsf{term}.E_r\Leftarrow\mathsf{term}\to\{\top\}}\ \Pi\mathbf{I}}{\Gamma,\Phi;\Delta,\Theta\vdash\mathsf{terminate}^{\wedge}(\lambda t:\mathsf{term}.E_r)\Leftarrow\{\mathsf{cnfreduction}\}}\ \multimap\mathbf{E}$$

Apply Lemma 4.10. □

Theorem 3.13 (Main Theorem) *If*

$$\begin{array}{c}\Gamma,\Phi,\Gamma',\Omega;\Theta\vdash\langle\rangle\Leftarrow\top\\ \vdots\ \mathcal{R}\\ \Gamma,\Phi;\Delta\vdash E_r\Leftarrow\{\mathsf{cnfreduction}\}\end{array}$$

where only expression checking and pattern expansion rules are permitted within $\mathcal{R}$ *and*
$\begin{array}{c}\Gamma,\Phi,\Sigma,\Lambda;\cdot\vdash_{\mathsf{SAT}}1\Leftarrow 1\\ \vdots\ \mathcal{S}_1\\ \Gamma,\Phi,\Sigma;\Delta\vdash_{\mathsf{SAT}}E_{s_1}\Leftarrow\{1\}\end{array}$, *then there exists a* Σ' *such that* $\begin{array}{c}\Gamma,\Gamma',\Phi,\Omega,\Sigma,\Sigma',\Lambda';\cdot\vdash_{\mathsf{SAT}}1\Leftarrow 1\\ \vdots\ \mathcal{S}_2\\ \Gamma,\Gamma',\Phi,\Omega,\Sigma,\Sigma';\Theta\vdash_{\mathsf{SAT}}E_{s_2}\Leftarrow\{1\}\end{array}$.

Proof: The reduction $\mathcal{R}$ can be written as shown below:

$$\dfrac{\begin{array}{c}\Gamma,\Phi,\Gamma',\Omega;\Theta\vdash\langle\rangle\Leftarrow\top\\ \vdots\ \mathcal{R}'\\ \Gamma,\Phi;\Delta\vdash E_r\leftarrow\mathsf{cnfreduction}\end{array}}{\Gamma,\Phi;\Delta\vdash E_r\Leftarrow\{\mathsf{cnfreduction}\}}\ \{\}\mathbf{I}$$

Apply Theorem 4.11.

□