Generalized Master Theorem

Jatin Shah

Yale University

Lemma 1. *Given a polynomial $p(x)$ such that $p(0) = 0$ then for any $0 < t < 1$, $p(tx) \leq tp(x)$.*

Proof. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x$.

Now,

$$\begin{aligned} p(tx) &= a_n(tx)^n + a_{n-1}(tx)^{n-1} + \ldots + a_1(tx) \\ &= a_n t^n x^n + a_{n-1}t^{n-1}x^{n-1} + \ldots + a_1(tx) \\ &\leq a_n t x^n + a_{n-1} t x^{n-1} + \ldots + a_1(tx) \qquad (\because \quad t^n \leq t \ \ \forall n \geq 1) \\ &= t(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x) = tp(x) \end{aligned}$$

□

Theorem 1. *Given a recursive function $T(x)$ defined as*

$$\begin{array}{ll} T(x) = \sum_{i=1}^{m} T(x_i) + f(x) & \textit{if } x > K \\ T(x) = b & \textit{if } 1 \leq x \leq K \end{array}$$

where x and x_i's are positive integers, there exists functions $g_i(\cdot)$ such that $x_i = g_i(x)$ for all $i = 1, \ldots, m$; and K, b and m are positive integer constants.

If $f(x)$ is a polynomial defined on positive integers such that $f(0) \geq 0$ and $f(x) \geq d > 0$ for all $1 \leq x \leq K$, and $x > \sum_{i=1}^{m} x_i$ then there exists a constant $c \geq 1$ such that $T(x) \leq cxf(x)$ for all $x \geq 1$.

Proof. Choose $c = max\{1, b/d\}$. We shall prove by induction.

Base case: When $1 \leq x \leq K$, $T(x) = b = (b/d)d \leq cf(x) \leq cxf(x)$.

Induction case: When $x > K$,

$$\begin{aligned} T(x) &= \sum_{i=1}^{m} T(x_i) + f(x) \\ &\leq \sum_{i=1}^{m} cx_i f(x_i) + f(x) \\ &\qquad \text{(Using Induction Hypothesis on } x_i < x\text{: } T(x_i) \leq cx_i f(x_i)) \\ &= c\sum_{i=1}^{m} x\left(\frac{x_i}{x}\right) f\left(x\frac{x_i}{x}\right) + f(x) \\ &\leq c\sum_{i=1}^{m} \left(\frac{x_i}{x}\right) xf(x) + f(x) \\ &\qquad \text{(Let } t = \left(\tfrac{x_i}{x}\right) < 1 \text{ and } p(x_i) = x_i f(x_i) \text{ in Lemma 1)} \\ &= cxf(x)\sum_{i=1}^{m} \left(\frac{x_i}{x}\right) + f(x) \\ &= xf(x)\left(c\sum_{i=1}^{m} \frac{x_i}{x} + \frac{1}{x}\right) \\ &\leq cxf(x) \qquad (\because x > \textstyle\sum_{i=1}^{m} x_i) \end{aligned}$$

□