Computer Organization and Architecture

Module 6

Computer Arithmetic

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Computer Arithmetic

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Introduction

- Computers are built using tiny electronic switches.
    - Typically made up of MOS transistors.
    - The state of the switches are typically expressed in binary (ON/OFF).
- To design arithmetic circuits for use in computers, we need to work with *binary numbers*.
    - How to carry out various arithmetic operations in binary?
    - How to implement them efficiently in hardware?

Addition / Subtraction

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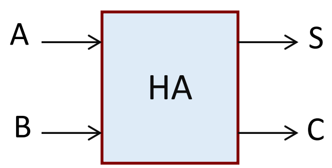
Addition of Two Binary Digits (Bits)

- When two bits *A* and *B* are added, a sum (*S*) and carry (*C*) are generated as per the following truth table:

Inputs		Outputs	
A	B	S	C
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

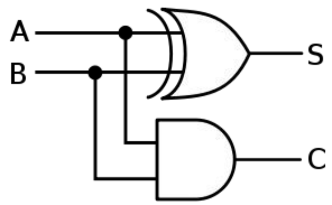
0 + 0 = 00
0 + 1 = 01
1 + 0 = 01
1 + 1 = 10



HALF ADDER

$S = A'.B + A.B' = A \oplus B$

$C = A.B$



Addition of Multi-bit Binary Numbers

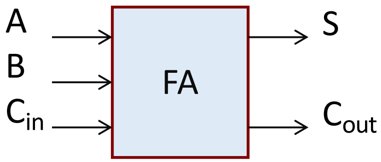
0 0 1 0 1 1 0	Carry		1 1 1 1 1 1 0	Carry
0 1 0 1 0 1 1	Number A		0 1 1 1 1 1 1	Number A
+ 0 0 0 1 0 0 1	Number B		+ 0 0 0 0 0 0 1	Number B
0 1 1 0 1 0 0	Sum S		1 0 0 0 0 0 0	Sum S

- At every bit position (stage), we require to add 3 bits:
  - 1 bit for number A
  - 1 bit for number B
  - 1 carry bit coming from the previous stage

WE NEED A FULL ADDER

Full Adder

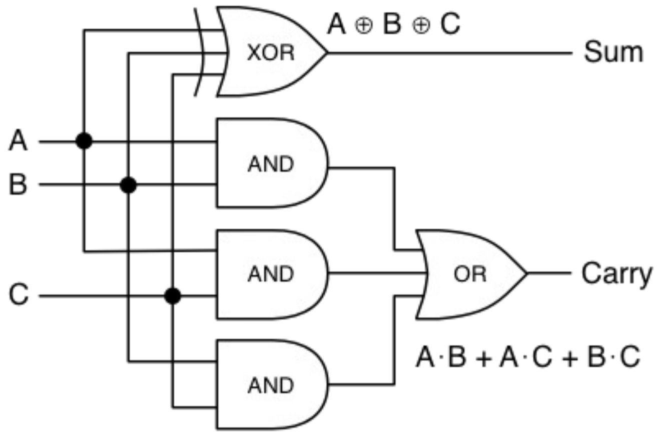
Inputs			Outputs	
A	B	C_{in}	S	C_{out}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

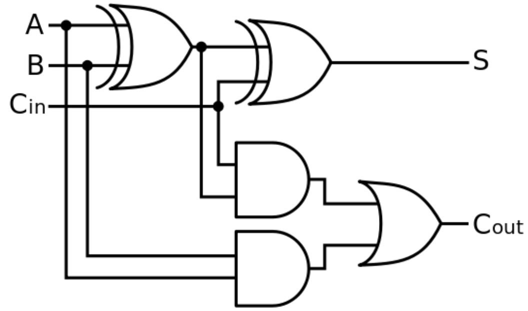


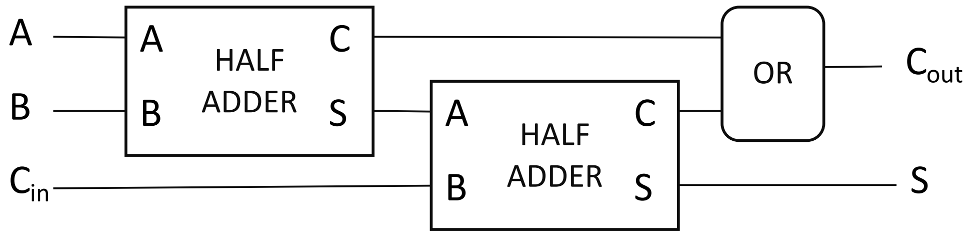
$$S = A'.B'.C_{in} + A'.B.C_{in}' + A.B'C_{in}' + A.B.C$$
$$= A \oplus B \oplus C_{in}$$

$$C_{out} = B.C_{in} + A.C_{in} + A.B + A.B.C_{in}$$
$$= A.B + B.C_{in} + A.C_{in}$$

Various Implementations of Full Adder

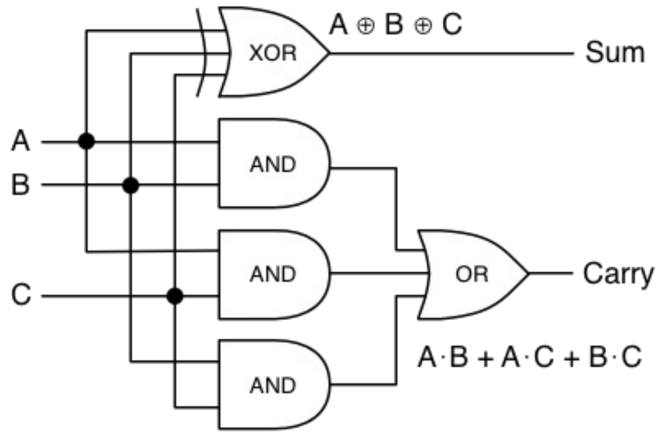






- **Delay of a full adder:**
  - Assume that the delay of all basic gates (AND, OR, NAND, NOR, NOT) is δ
  - Delay for Carry = 2δ
  - Delay for Sum = 3δ

    (AND-OR delay plus one inverter delay)



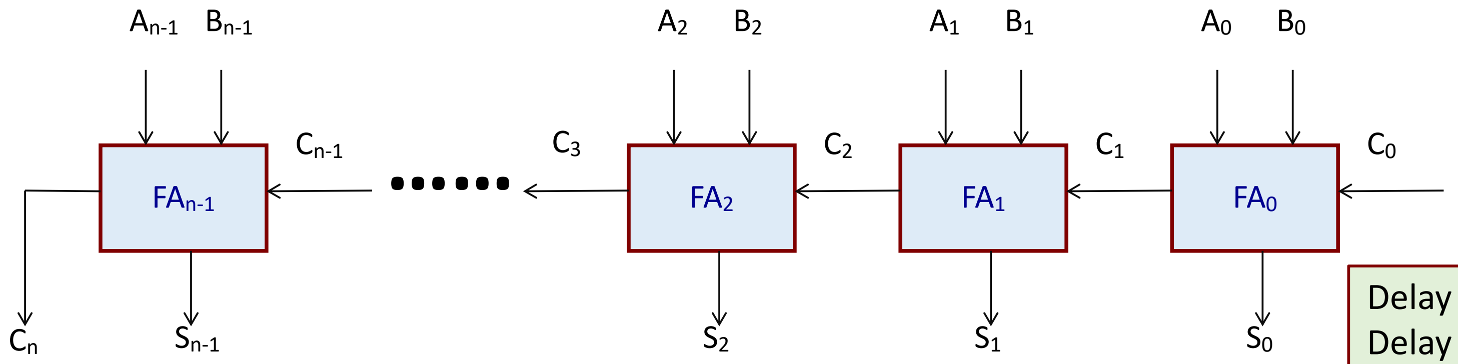
Parallel Adder Design

- We shall look at the various designs of *n*-bit parallel adder.
  a) Ripple carry adder
  b) Carry look-ahead adder
  c) Carry save adder
  d) Carry select adder

Ripple Carry Adder

- Cascade *n* full adders to create a *n*-bit parallel adder.
- Carry output from stage-*i* propagates as the carry input to stage-(*i*+1).
- In the worst-case, carry ripples through all the stages.

	1 1 1 1 1 1 0	Carry
	0 1 1 1 1 1 1	Number A
+	0 0 0 0 0 0 1	Number B
	1 0 0 0 0 0 0	Sum S



Delay for C_1 $= 2\delta$
Delay for C_2 $= 4\delta$
Delay for C_{n-1} $= 2(n-1)\delta$
Delay for C_n $= 2n\delta$

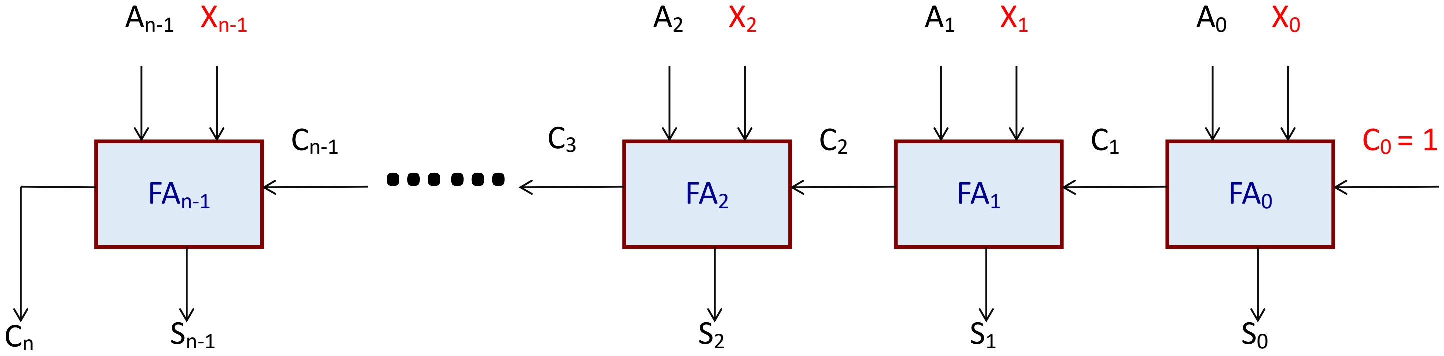
Two numbers:	$A_{n-1}\ldots A_2A_1A_0$ and $B_{n-1}\ldots B_2B_1B_0$
Input carry:	C_0
Sum:	$S_{n-1}\ldots S_2S_1S_0$
Output carry:	C_n

Delay for S_0 $= 3\delta$
Delay for S_1 $= 2\delta + 3\delta = 5\delta$
Delay for S_2 $= 4\delta + 3\delta = 7\delta$
Delay for S_{n-1} $= 2(n-1)\delta + 3\delta = (2n+1)\,\delta$

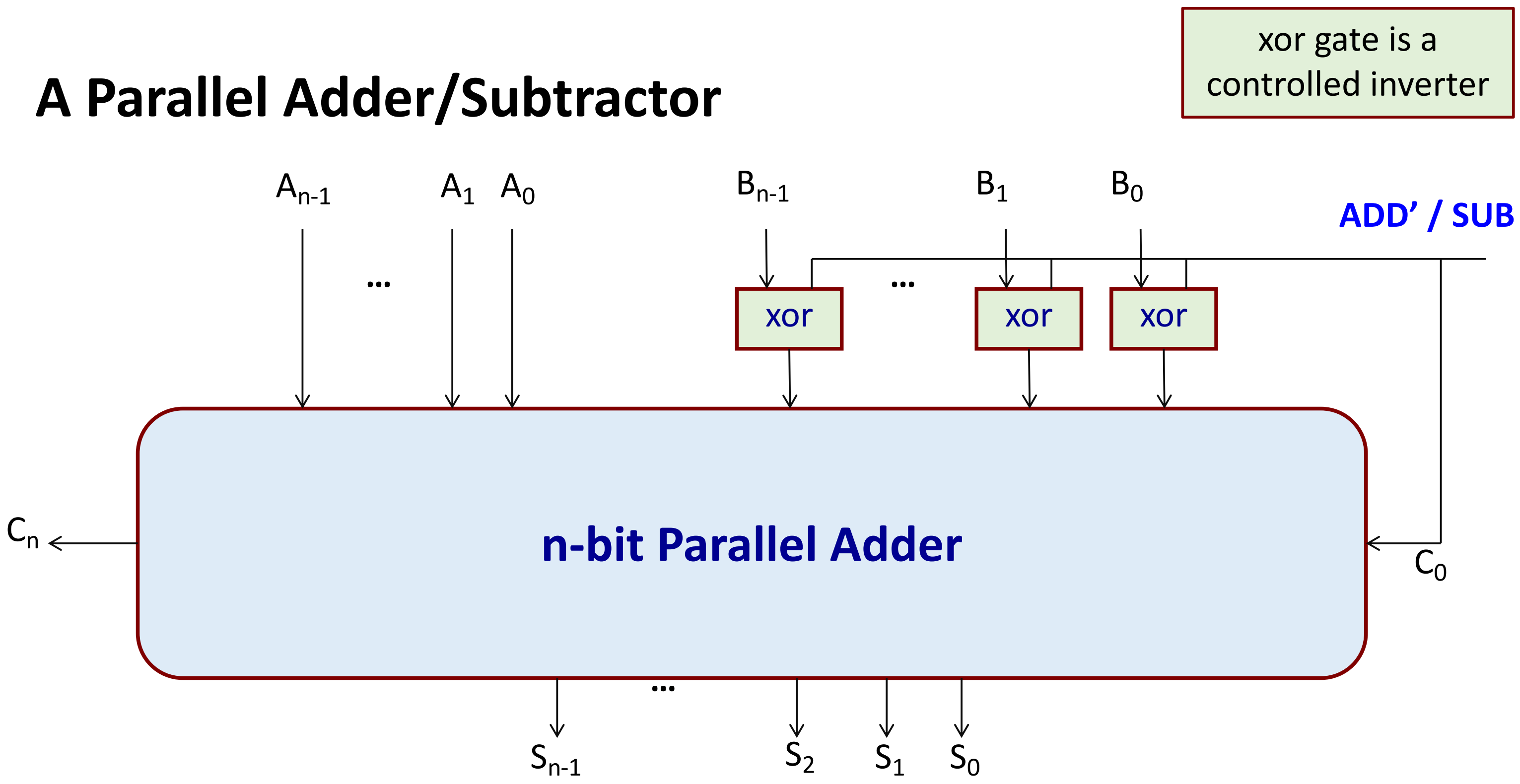
Delay is proportional to n

How to Design a Parallel Subtractor?

- **Observation:**
  - Computing *A - B* is the same as adding the 2’s complement of *B* to *A*.
  - 2’s complement is equal to 1’s complement plus 1.
  - Let $X_i = B_i'$.



A Parallel Adder/Subtractor



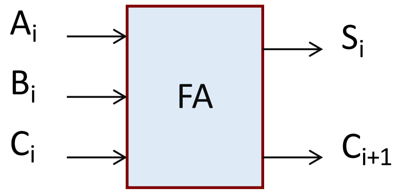
Carry Look-ahead Adder

- The propagation delay of an *n*-bit ripple carry order has been seen to be proportional to *n*.
  - Due to the rippling effect of carry sequentially from one stage to the next.
- One possible way to speedup the addition.
  - Generate the carry signals for the various stages in parallel.
  - Time complexity reduces from O(n) to O(1).
  - However, hardware complexity increases rapidly with *n*.

- Consider the *i*-th stage in the addition process.
- We define the *carry generate* and *carry propagate* functions as:

  $G_i = A_i.B_i$

  $P_i = A_i \oplus B_i$



- $G_i = 1$ represents the condition when a carry is generated in stage-*i* independent of the other stages.
- $P_i = 1$ represents the condition when an input carry C_i will be propagated to the output carry C_{i+1}.

$$\boxed{C_{i+1} = G_i + P_i.C_i}$$

Unrolling the Recurrence

$C_{i+1} = G_i + P_iC_i = G_i + P_i(G_{i-1} + P_{i-1}C_{i-1}) = G_i + P_iG_{i-1} + P_iP_{i-1}C_{i-1}$

$= G_i + P_iG_{i-1} + P_iP_{i-1}(G_{i-2} + P_{i-2}C_{i-2})$

$= G_i + P_iG_{i-1} + P_iP_{i-1}G_{i-2} + P_iP_{i-1}P_{i-2}C_{i-2} = \ldots..$

$$\boxed{C_{i+1} = G_i + \sum_{k=0}^{i-1} G_k \prod_{j=k+1}^{i} P_j + C_0 \prod_{j=0}^{i} P_j}$$

Design of 4-bit CLA Adder

$C_4 = G_3 + G_2P_3 + G_1P_2P_3 + G_0P_1P_2P_3 + C_0P_0P_1P_2P_3$

$C_3 = G_2 + G_1P_2 + G_0P_1P_2 + C_0P_0P_1P_2$

$C_2 = G_1 + G_0P_1 + C_0P_0P_1$

$C_1 = G_0 + C_0P_0$

4 AND2 gates
3 AND3 gates
2 AND4 gates
1 AND5 gate
1 OR2, 1 OR3, 1 OR4 and
1 OR5 gate

$S_0 = A_0 \oplus B_0 \oplus C_0 = P_0 \oplus C_0$

$S_1 = P_1 \oplus C_1$

$S_2 = P_2 \oplus C_2$

$S_3 = P_3 \oplus C_3$

4 XOR2 gates

Design of 4-bit CLA Adder

$C_4 = G_3 + G_2P_3 + G_1P_2P_3 + G_0P_1P_2P_3 + C_0P_0P_1P_2P_3$

$C_3 = \boxed{G_2 + G_1P_2 + G_0P_1P_2 + C_0P_0P_1P_2}$

$C_2 = G_1 + G_0P_1 + C_0P_0P_1$

$C_1 = G_0 + C_0P_0$

4 AND2 gates
3 AND3 gates
2 AND4 gates
1 AND5 gate
1 OR2, 1 OR3, 1 OR4 and
1 OR5 gate

$S_0 = A_0 \oplus B_0 \oplus C_0 = P_0 \oplus C_0$

$S_1 = P_1 \oplus C_1$

$S_2 = P_2 \oplus C_2$

$S_3 = P_3 \oplus C_3$

4 XOR2 gates

Design of 4-bit CLA Adder

$C_4 = G_3 + \mathbf{C_3}P_3$

$C_3 = G_2 + G_1P_2 + G_0P_1P_2 + C_0P_0P_1P_2$

$C_2 = G_1 + G_0P_1 + C_0P_0P_1$

$C_1 = G_0 + C_0P_0$

4 AND2 gates
2 AND3 gates
1 AND4 gates
~~1 AND5 gate~~
1 OR2, 1 OR3, 1 OR4 and
1 **OR2** gate

$S_0 = A_0 \oplus B_0 \oplus C_0 = P_0 \oplus C_0$

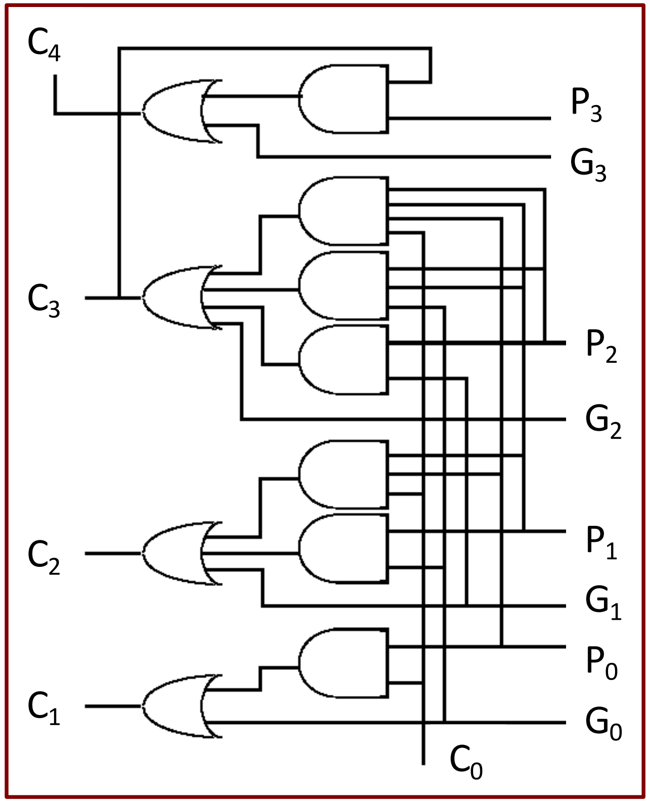
$S_1 = P_1 \oplus C_1$

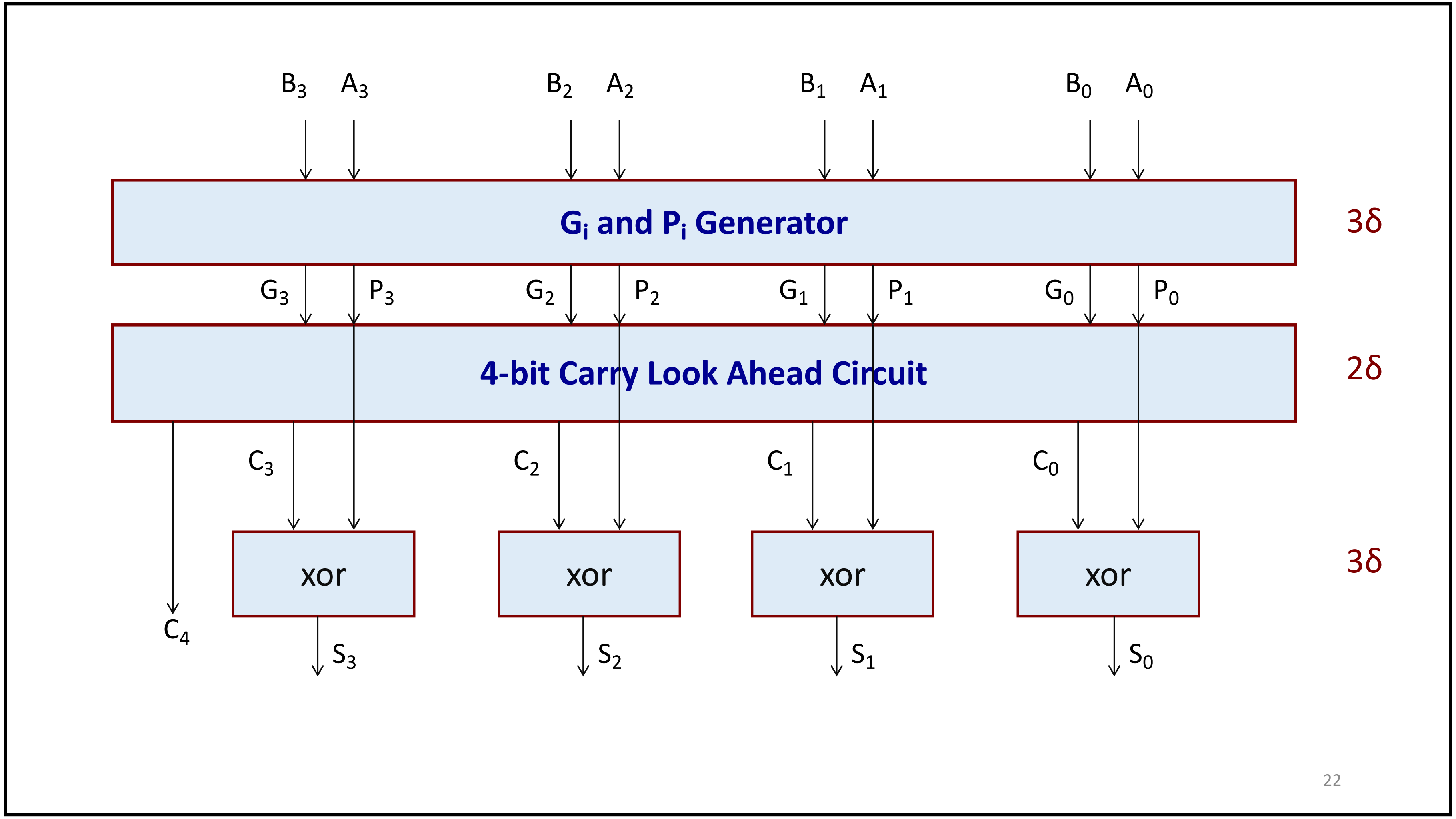
$S_2 = P_2 \oplus C_2$

$S_3 = P_3 \oplus C_3$

4 XOR2 gates

The 4-bit CLA Circuit




B_3 A_3 B_2 A_2 B_1 A_1 B_0 A_0
G_i and P_i Generator
3δ
G_3 P_3 G_2 P_2 G_1 P_1 G_0 P_0
4-bit Carry Look Ahead Circuit
2δ
C_3 C_2 C_1 C_0
xor
xor
xor
xor
3δ
C_4
S_3 S_2 S_1 S_0
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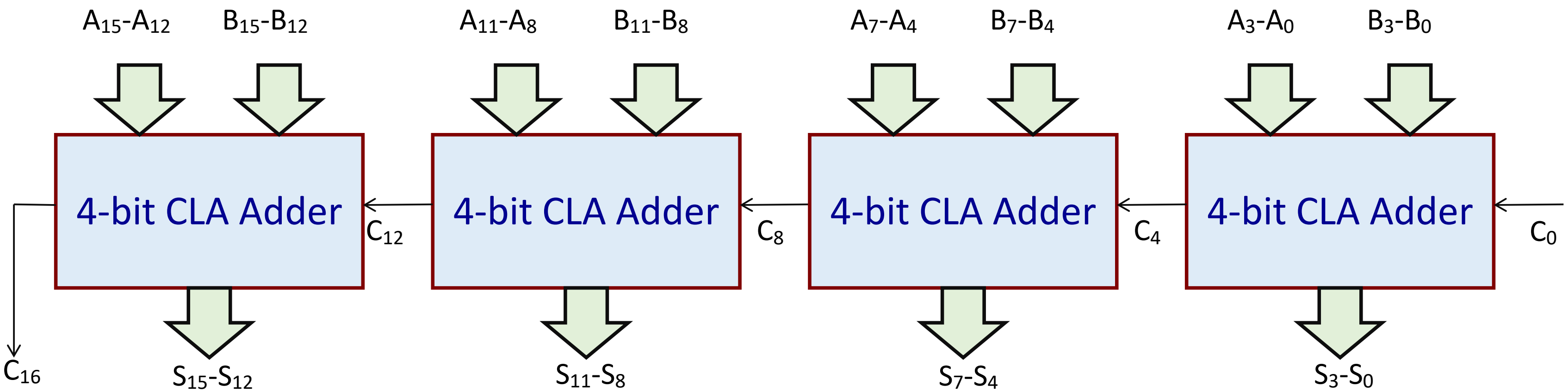
16-bit Adder Using 4-bit CLA Modules



Problem: Carry propagation between modules still slows down the adder

- **Solution:**
  - Use a second level of carry look-ahead mechanism to generate the input carries to the CLA blocks in parallel.
  - The second level of CLA generates C4, C8, C12 and C16 in parallel with two gate delays (2δ).
  - For larger values of n, more CLA levels can be added.
- Delay calculation of a 16-bit adder:
  a) For original single-level CLA: 14δ
  b) For modified two-level CLA: 10δ

Delay of a k-bit Adder

n	T_{CLA}	T_{RCA}
4	8δ	9δ
16	10δ	33δ
32	12δ	65δ
64	12δ	129δ
128	14δ	257δ
256	14δ	513δ

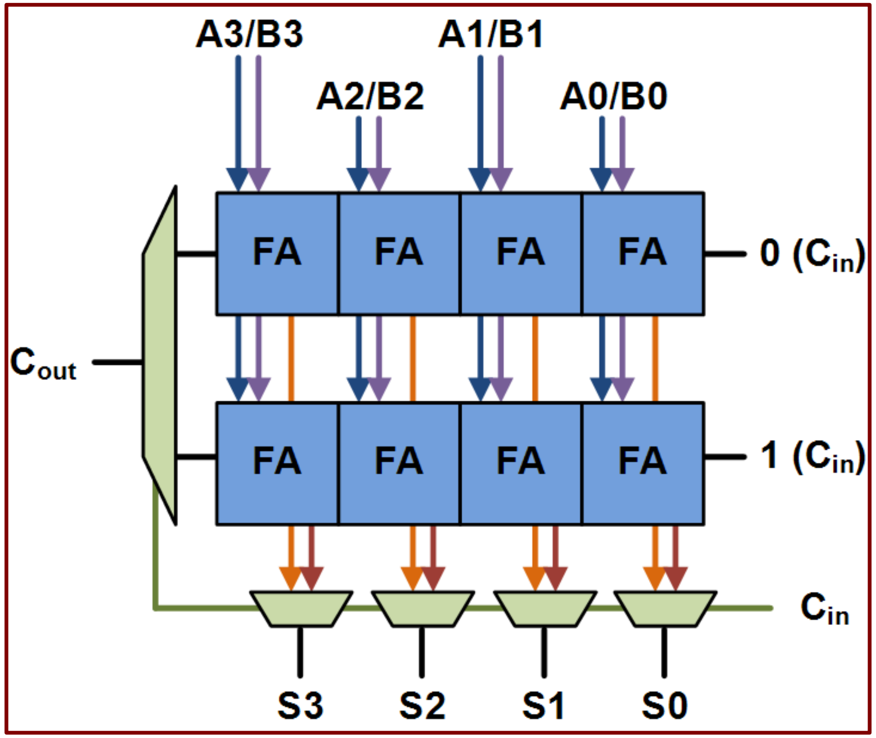
$$\begin{aligned} T_{CLA} &= (6 + 2\lceil \log_4 n \rceil)\,\delta \\ T_{RCA} &= (2n+1)\,\delta \end{aligned}$$

Carry Select Adder

- Basically consists of two parallel adders (say, ripple-carry adder) and a multiplexer.
- For two given numbers *A* and *B*, we carry out addition twice:
  - With carry-in as 0
  - With carry-in as 1
- Once the correct carry-in is known, the correct sum is selected by a multiplexer.

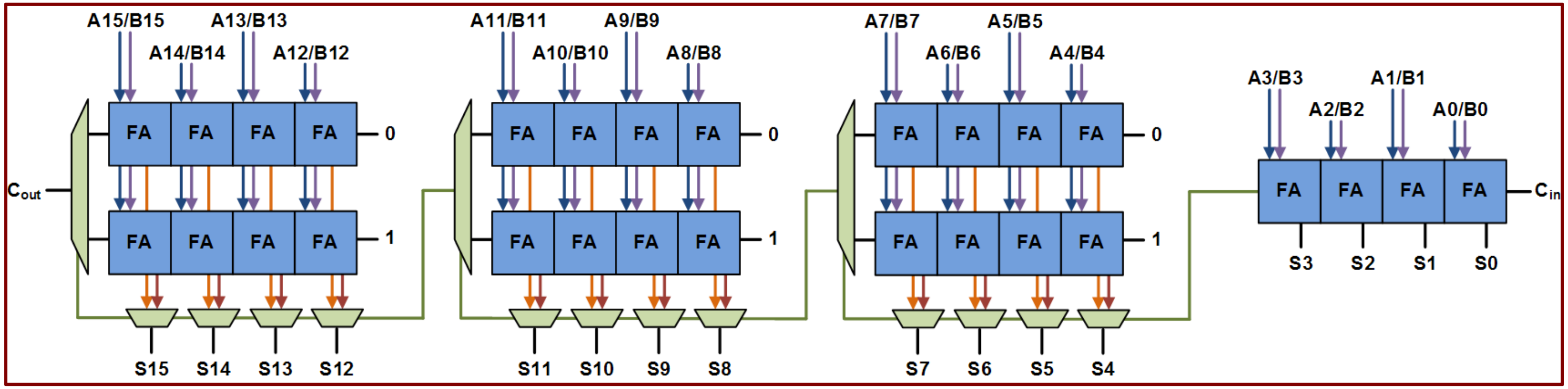
Basic building block of a carry-select adder, with block size of 4.

- For a multi-bit adder, the number of bits in each carry select block can be either uniform or variable.



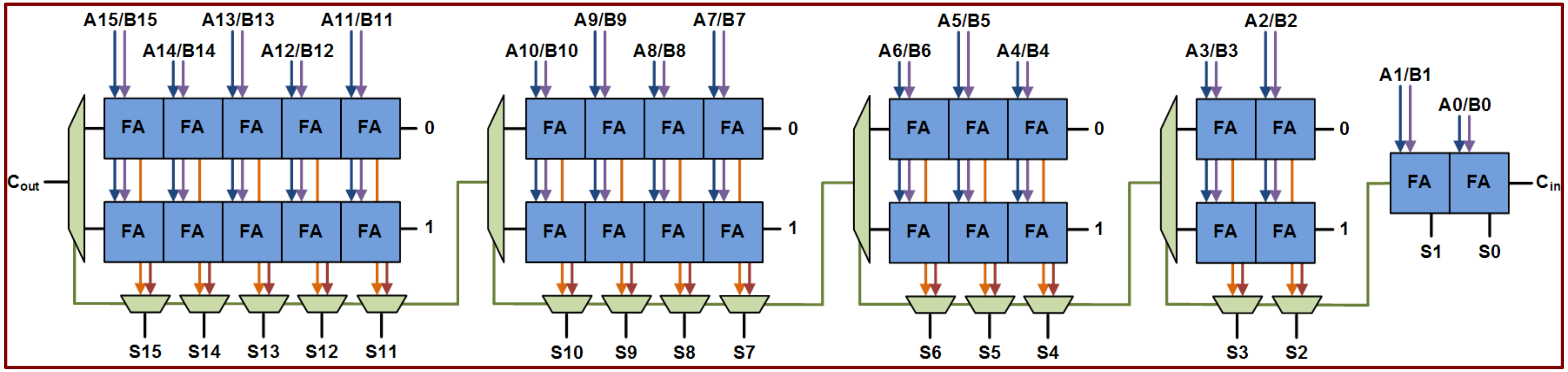
Uniform sized adder

- A 16-bit carry select adder with a uniform block size of 4 is shown.
- The least significant block needs a single adder (since the carry-in is known).
- Total delay is 4 full adder delays, plus 3 MUX delays.



Variable-sized adder

- A 16-bit carry select adder with variable block sizes of 2-2-3-4-5 is shown.
- Total delay is 2 full adder delays, plus 4 MUX delays.



Carry Save Adder

- Here we add three operands (say, *X*, *Y* and *Z*) together.
- For adding multiple numbers, we have to construct a tree of carry save adders.
  - Used in combinational multiplier design.
- Each carry save adder is simply an independent full adder without carry propagation.
- A parallel adder is required only at the last stage.

- **An illustrative example:**

```
X:      1 0 0 1 1
Y:    + 1 1 0 0 1
Z:    + 0 1 0 1 1
-----------------
C:    1 1 0 1 1
```

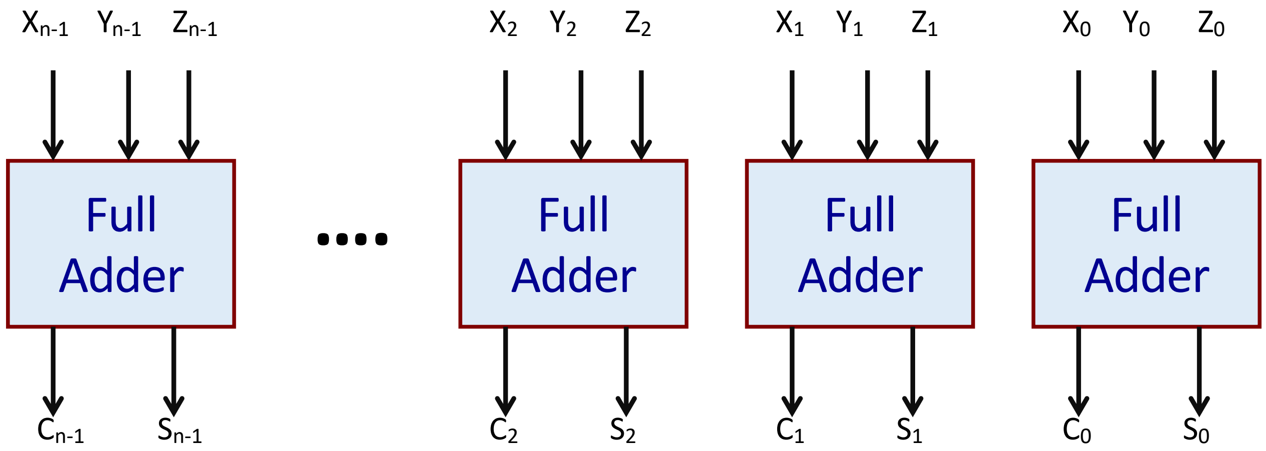
```
X:      1 0 0 1 1
Y:    + 1 1 0 0 1
Z:    + 0 1 0 1 1
-----------------
S:      0 0 0 0 1
```

A set of full adders generate carry and sum bits in parallel

```
X:        1 0 0 1 1
Y:      + 1 1 0 0 1
Z:      + 0 1 0 1 1
-------------------
S:        0 0 0 0 1
C:      1 1 0 1 1
-------------------
Sum:    1 1 0 1 1 1
```

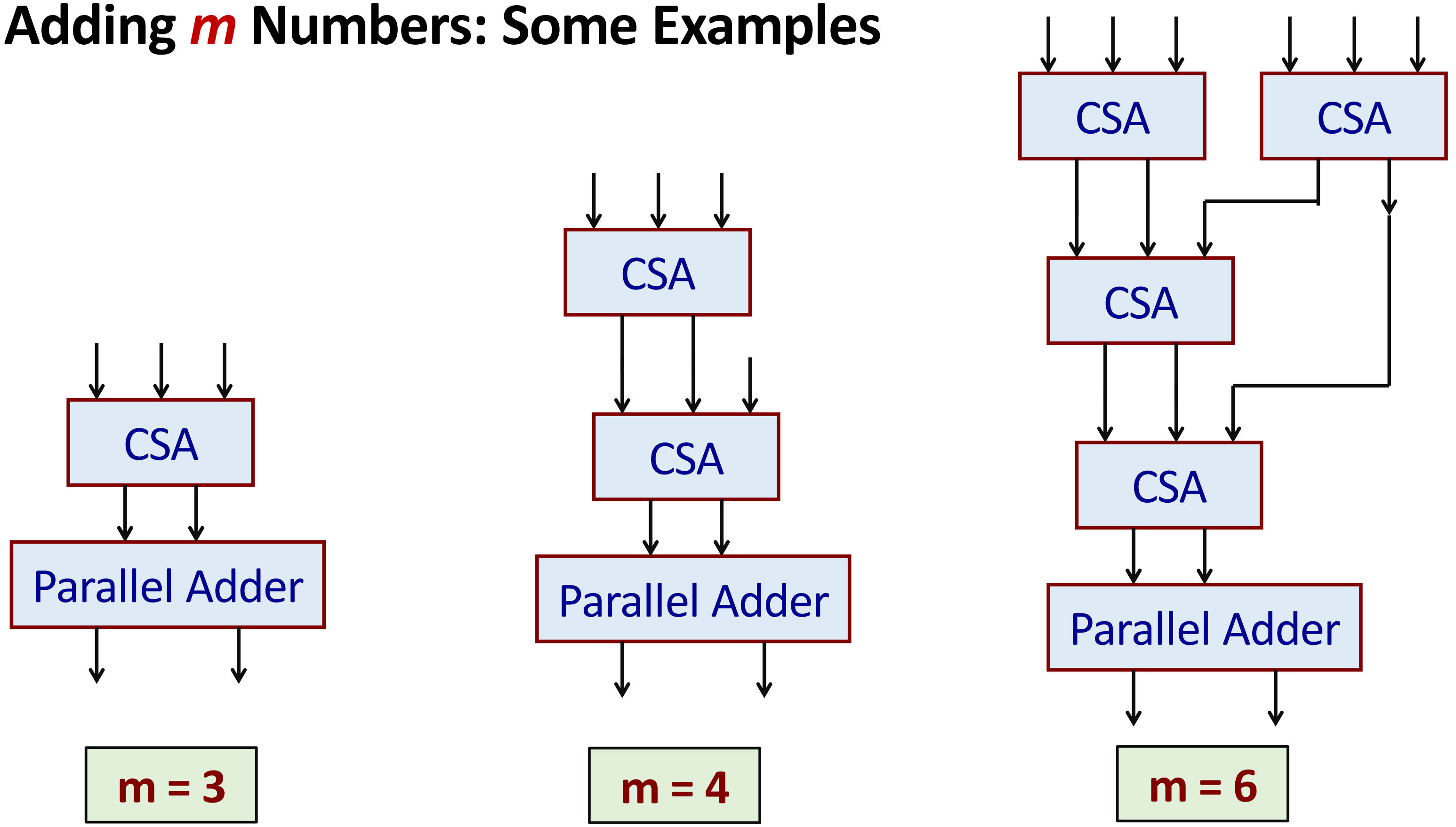
The sum and carry vectors are added later (with proper shifting)

An n-bit Carry Save Adder



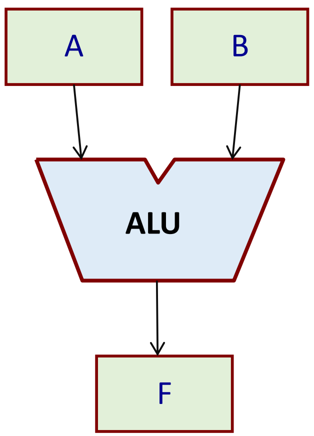
The carry input of the full adder is used as the third input

Adding *m* Numbers: Some Examples

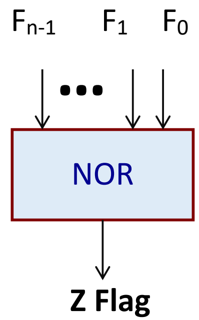


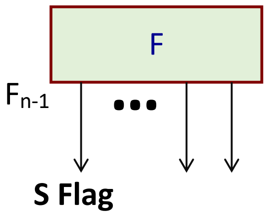
Generating the Status Flags

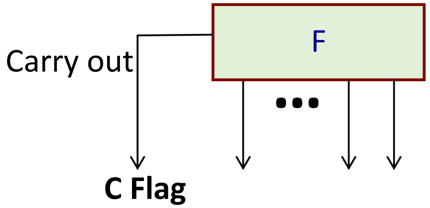
- Many contemporary processors have a *flag register* that contains the status of the last arithmetic / logic operation.
  - **Zero (Z)**: tells whether the result is zero.
    - Can be used for both arithmetic and logic operations.
  - **Sign (S)**: tells whether the result is positive (=0) or negative (=1).
    - Can be used for both arithmetic and logic operations.
  - **Carry (C)**: tells whether there has been a carry out of the most significant stage.
    - Used only for arithmetic operations.
  - **Overflow (V)**: tells whether the result is too large to fit in the target register.
    - Used only for arithmetic operations (addition and subtraction).



Assume *A*, B and *F* are *n*-bit registers







- Overflow can occur during addition when the sign of the two operands are the same.
    - Sign of the result becomes different from the sign of the operand(s).

$$V = A_{n-1}.B_{n-1}.F_{n-1}' + A_{n-1}'.B_{n-1}'.F_{n-1}$$

$$V = F_{n-1} \oplus Carry\_out$$

- The MIPS32 processor does not have any status flags.
- Why?
  - MIPS ISA is designed for efficient pipeline implementation.
  - Several instructions can be in various stages of execution in the pipeline.
  - Flag registers result in *side effects* among instructions.
- MIPS stores information about the flags temporarily in a GPR.

```
slt  $t0,$s1,$s2
beq  $t0,$zero,Label
```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