14

Resolution II

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14.a

Example

RESOLUTION ONE MORE TIME

Negate query α.

Convert everything to **CNF**.

Repeat: Choose clauses and resolve (based on unification).

If resolution results in empty clause, α is proved.

Return all substitutions (or Fail).

RESOLUTION IN IMPLICATION FORM

Ground binary resolution

$$\frac{C \vee P \qquad D \vee \neg P}{C \vee D}$$

Set $C = \neg A$.

$$\frac{A \to P \qquad P \to D}{A \to D}$$

Some students like all memes.

$$F_1 : \quad \exists x. S(x) \wedge \forall y. M(y) \rightarrow \mathsf{Likes}(x, y)$$

No student likes any theorem.

$$F_2 : \quad \forall x, y. S(x) \wedge \mathsf{T}(y) \rightarrow \neg \mathsf{Likes}(x, y)$$

Show: No meme is a theorem.

$$F : \quad \forall x. M(x) \rightarrow \neg \mathsf{T}(x)$$


Memes that are not about themselves
A meme about memes that are not about themselves

CNF · Eliminating implications

$$F_1: \quad \exists x.S(x) \land \forall y.M(y) \to \mathsf{Likes}(x,y)$$
$$\exists x.S(x) \land \forall y.\neg M(y) \lor \mathsf{Likes}(x,y)$$

$$F_2: \quad \forall x,y.S(x) \land \mathsf{T}(y) \to \neg\mathsf{Likes}(x,y)$$
$$\forall x,y.\neg S(x) \lor \neg\mathsf{T}(y) \lor \neg\mathsf{Likes}(x,y)$$

$$F: \quad \forall x.M(x) \to \neg\mathsf{T}(x)$$
$$\forall x.\neg M(x) \lor \neg\mathsf{T}(x)$$

CNF · Standardising variables apart, skolemising, dropping universal quantifiers

$F_1 : \quad \exists x. S(x) \land \forall y. \neg M(y) \lor \mathsf{Likes}(x, y)$

$\quad S(G) \land (\neg M(y) \lor \mathsf{Likes}(G, y))$

$F_2 : \quad \forall x, y. \neg S(x) \lor \neg T(y) \lor \neg \mathsf{Likes}(x, y)$

$\quad \neg S(w) \lor \neg T(z) \lor \neg \mathsf{Likes}(w, z)$

$F : \quad \forall x. \neg M(x) \lor \neg T(x)$

$\quad \neg M(x) \lor \neg T(x)$

Unification

$$F_1 : \quad S(G) \wedge (\neg M(y) \vee \mathsf{Likes}(G, y))$$

$$F_2 : \quad \neg S(w) \vee \neg \mathsf{T}(z) \vee \neg \mathsf{Likes}(w, z)$$

$$w/G : \ \neg S(G) \vee \neg \mathsf{T}(z) \vee \neg \mathsf{Likes}(G, z)$$

Negation of proof goal

$$\neg(\neg M(x) \vee \neg \mathsf{T}(x)) \equiv M(x) \wedge \mathsf{T}(x)$$

EXAMPLE · MEMES & THEOREMS

$S(G) \wedge (\neg M(y) \vee \text{Likes}(G, y))$
$\neg S(G) \vee \neg T(z) \vee \neg \text{Likes}(G, z)$
$M(x) \wedge T(x)$

Clauses $S(G), M(x), T(x), \neg M(y) \vee \text{Likes}(G, y),$
$\neg S(G) \vee \neg T(z) \vee \neg \text{Likes}(G, z)$

$$\frac{S(G) \qquad \neg S(G) \vee \neg T(z) \vee \neg \text{Likes}(G, z)}{\neg T(z) \vee \neg \text{Likes}(G, z)}$$

$$\frac{\neg M(y) \vee \text{Likes}(G, y) \qquad \neg T(z) \vee \neg \text{Likes}(G, z)}{\neg M(z) \vee \neg T(z)}$$

Substitute z/x

$$\frac{\neg M(x) \vee \neg T(x) \qquad M(x)}{\neg T(x)} \quad \text{and} \quad \frac{\neg T(x) \qquad T(x)}{\square}$$

Therefore, $\neg M(x) \vee \neg T(x)$, i.e. $M(x) \to \neg T(x)$.

EXAMPLE · MEMES & THEOREMS 2.0

Some students like all memes.

$$F_1 : \quad \exists x.S(x) \wedge \forall y.M(y) \to \mathsf{Likes}(x, y)$$

No student likes any theorem.

$$F_2 : \quad \forall x.S(x) \to \forall y.T(y) \to \neg\mathsf{Likes}(x, y)$$

Show: No meme is a theorem.

$$F : \quad \forall x.M(x) \to \neg T(x)$$

CNF · Eliminating implications

$$F_1: \quad \exists x.S(x) \land \forall y.M(y) \rightarrow \mathsf{Likes}(x, y)$$

$$\exists x.S(x) \land \forall y.\neg M(y) \lor \mathsf{Likes}(x, y)$$

$$F_2: \quad \forall x.S(x) \rightarrow \forall y.T(y) \rightarrow \neg\mathsf{Likes}(x, y)$$

$$\forall x.\neg S(x) \lor \forall y.\neg T(y) \lor \neg\mathsf{Likes}(x, y)$$

$$F: \quad \forall x.M(x) \rightarrow \neg T(x)$$

$$\forall x.\neg M(x) \lor \neg T(x)$$

CNF · Standardising variables apart, skolemising, dropping universal quantifiers

$$F_1 : \quad \exists x.S(x) \wedge \forall y.\neg M(y) \vee \text{Likes}(x, y)$$

$$S(G) \wedge (\neg M(y) \vee \text{Likes}(G, y))$$

$$F_2 : \quad \forall x.\neg S(x) \vee \forall y.\neg T(y) \vee \neg\text{Likes}(x, y)$$

$$\neg S(w) \vee (\neg T(z) \vee \neg\text{Likes}(w, z))$$

$$F : \quad \forall x.\neg M(x) \vee \neg T(x)$$

$$\neg M(x) \vee \neg T(x)$$

14.b

Completeness

SOUNDNESS AND COMPLETENESS

Resolution is *sound and complete*.
A set of clauses S is unsatisfiable if and only if one can derive the empty clause (false) from S.

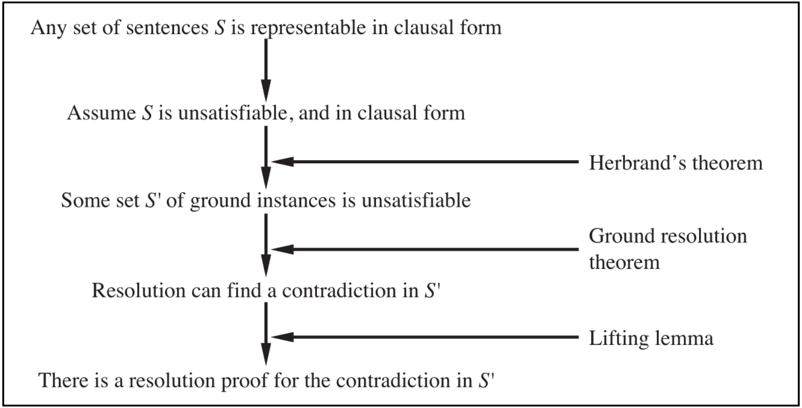
Soundness – derivability of empty clause implies unsatisfiability.

Can be proved by noticing that every model that satisfies the premises of resolution also satisfies its conclusion.

Completeness – every unsatisfiable clause can be refuted by resolution.

Can be proved using completeness of propositional resolution and *lifting* (as in the following slides; the full proof is beyond the scope of this course).

COMPLETENESS PROOF



COMPLETENESS PROOF · STEP 1

For a set of clauses S, we call the **Herbrand universe of** S the set H_S of all ground terms that can be constructed from the function symbols in S.

EXAMPLE

For $S = \{\neg P(x, F(x, A)) \vee \neg Q(x, A) \vee R(x, B)\}$ we have

$H_S = \{A, B, F(A, A), F(A, B), F(B, A), F(B, B), F(A, F(A, A)), \ldots\}$

COMPLETENESS PROOF · STEP 1

For a set of clauses S and P a set of ground terms,
$P(S)$, **the saturation of S with respect to P**, is the set of all ground clauses obtained by applying all possible consistent substitutions of variables in S with ground terms from P.

The saturation of a set S with respect to its Herbrand universe is called the **Herbrand base** of S and denoted $H_S(S)$.

EXAMPLE

$$
\begin{aligned}
H_S(S) = \{&\neg P(A, F(A, A)) \vee \neg Q(A, A) \vee R(A, B), \\
&\neg P(B, F(B, A)) \vee \neg Q(B, A) \vee R(B, B), \\
&\neg P(F(A, A), F(F(A, A), A)) \vee \neg Q(F(A, A), A) \vee R(F(A, A), B), \\
&\neg P(F(A, B), F(F(A, B), A)) \vee \neg Q(F(A, B), A) \vee R(F(A, B), B), \ldots\}
\end{aligned}
$$

Herbrand's theorem (1930)

If a set S of clauses is unsatisfiable, then there exists a finite subset of $H_S(S)$ that is also unsatisfiable.

COMPLETENESS PROOF · STEP 2

Let S' be that finite unsatisfiable subset of ground sentences.

Running propositional resolution to completion on S' will derive a contradiction.

COMPLETENESS PROOF · STEP 3

Lifting lemma

Let C_1 and C_2 be two clauses with no shared variables, and let C_1' and C_2' ground instances of C_1 and C_2.

If C' is a resolvent of C_1' and C_2', then there exists a clause C such that:

- C is a resolvent of C_1 and C_2
- C' is a ground instance of C.

$$\begin{array}{ccc} C_1, C_2 & \xrightarrow{\text{instantiation}} & C_1', C_2' \\ \downarrow & & \downarrow \\ \dfrac{C_1 \quad C_2}{C} & \xleftarrow{\text{lifting}} & \dfrac{C_1' \quad C_2'}{C'} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\text{instantiation}} & C' \end{array}$$

COMPLETENESS PROOF · STEP 3

EXAMPLE

$$C_1 = \neg P(x, F(x, A)) \lor \neg Q(x, A) \lor R(x, B)$$
$$C_2 = \neg N(G(y), z) \lor P(H(y), z)$$

$$C_1' = \neg P(H(B), F(H(B), A)) \lor \neg Q(H(B), A) \lor R(H(B), B)$$
$$C_2' = \neg N(G(B), F(H(B), A)) \lor P(H(B), F(H(B), A))$$

$$C' = \neg N(G(B), F(H(B), A)) \lor \neg Q(H(B), A) \lor R(H(B), B)$$
$$C = \neg N(G(y), F(H(y), A)) \lor \neg Q(H(y), A) \lor R(H(y), B)$$

EFFICIENT ALGORITHMS FOR RESOLUTION

Heuristics to make resolution more efficient:

Unit preference	prefer clauses with only one symbol.
Pure clauses	a pure clause contains symbol A which does not occur in any other clause. Can't lead to contradiction.
Tautology	clauses containing A and $\neg A$.
Set of support	identify *useful* clauses and ignore the rest.
Input resolution	intermediately generated clauses can only be combined with original input clauses.
Subsumption	if a clause contains another one, use only the shorter clause. Prune unnecessary facts from KB.

Including heuristics, resolution is more efficient than DPLL.