10

First-Order Logic

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Based on slides by: Jacques Fleuriot, Michael Rovatsos, Michael Herrmann, Vaishak Belle

10.a

A case for FOL

PROPOSITIONAL LOGIC AS A LANGUAGE

COMPARED TO...

LANGUAGES IN COMPUTER SCIENCE

Serves as a basis for declarative languages

Allows partial/disjunctive/negated information

unlike most data structures and databases

Is compositional

unlike some instances of concurrent programming

e.g. the meaning of $B_{1,1} \wedge P_{1,2}$ is derived from that of $B_{1,1}$ and of $P_{1,2}$

PROPOSITIONAL LOGIC AS A LANGUAGE

COMPARED TO...

NATURAL LANGUAGES

Meaning is context-independent

unlike natural languages, where meaning depends on context

Propositional logic has very limited expressive power

e.g. we can say *pits cause breezes in adjacent squares* only by writing one sentence for **each** square

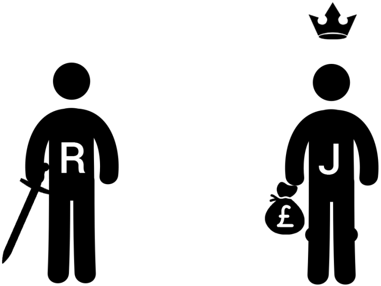
FIRST-ORDER LOGIC

Propositional logic deals with atomic facts (i.e. atomic, non-structured propositional symbols; usually finitely many).

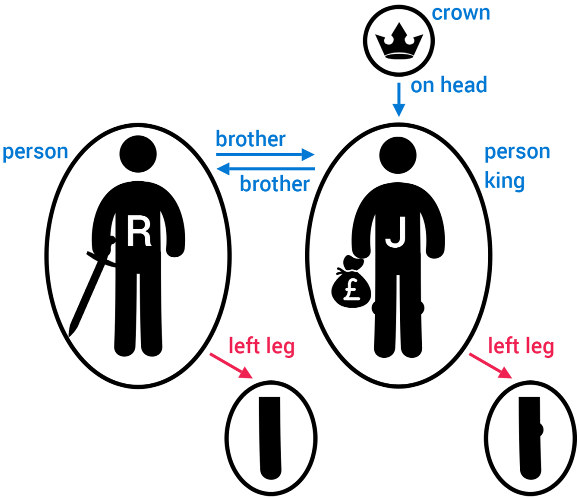
FOL brings structure to facts, which can be built from:

Objects	people, houses, numbers, colours, football games
Functions	father of, best friend, one more than, plus
Relations	red, round, prime, brother of, bigger than, part of

EXAMPLE



EXAMPLE · OF BROTHERS AND KINGS



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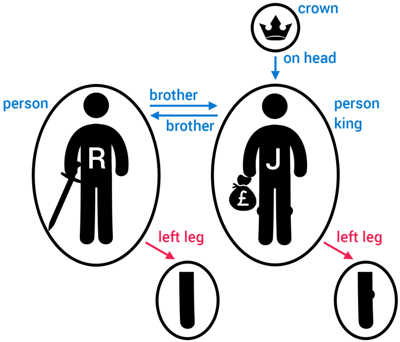

$\text{Brother}(\text{KingJohn}, \text{RichardTheLionheart})$

$\text{Length}(\text{LeftLegOf}(\text{Richard})) > \text{Length}(\text{LeftLegOf}(\text{John}))$

10.b

Defining FOL

A first-order **signature** is a pair (F, P)

- F – indexed family $(F_n)_{n \in \mathbb{N}}$ of sets of **function** symbols (operations)
- P – indexed family $(P_n)_{n \in \mathbb{N}}$ of sets of **relation** symbols (predicates)

For $\sigma \in F_n$ and $\pi \in P_n$, n is called **arity**.

Constant symbols are function symbols with arity zero.

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EXAMPLE

functions	$F_0 = \{\text{Richard, John}\}, F_1 = \{\text{LeftLegOf}\}$
predicates	$P_1 = \{\text{Crown, King, Person}\}$
	$P_2 = \{\text{Brother, OnHead}\}$

Terms Least set T_F such that $\sigma(t_1, \dots, t_n) \in T_F$ for every $\sigma \in F_n$ and $t_1, \dots, t_n \in T_F$.

In particular, T_F contains all constants.

Variables Every set of (F, P)-variables X determines an an extended signature $(F \cup X, P)$ with the variables in X added to F_0 as **new constants**.

Sentences over a signature (F, P) are defined by the grammar

$$\begin{aligned}\varphi ::= {} & \pi(t_1, \ldots, t_n) \mid t = t' && \text{atoms}\\ & \mid \neg\varphi \mid \varphi \wedge \varphi' \mid \varphi \vee \varphi' \mid \varphi \to \varphi' \mid \varphi \leftrightarrow \varphi' && \text{connectives}\\ & \mid \forall X.\varphi \mid \exists X.\varphi && \text{quantifiers}\end{aligned}$$

where $\pi \in P_n$ is a predicate symbol, $t, t', t_1, \ldots, t_n$ are terms, and X is a set of variables.

Precedence $\forall X, \exists X, \neg, \wedge, \vee, \to, \leftrightarrow$

SYNTAX · SENTENCES

Sentences over a signature (F, P) are defined by the grammar

$$\begin{aligned} \varphi ::= {} & \pi(t_1, \ldots, t_n) \mid t = t' && \text{atoms} \\ & \mid \neg\varphi \mid \varphi \land \varphi' \mid \varphi \lor \varphi' \mid \varphi \to \varphi' \mid \varphi \leftrightarrow \varphi' && \text{connectives} \\ & \mid \forall X.\varphi \mid \exists X.\varphi && \text{quantifiers} \end{aligned}$$

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EXAMPLE

$\mathrm{Brother}(\mathrm{John}, \mathrm{Richard})$

$\mathrm{Brother}(\mathrm{John}, \mathrm{Richard}) \land \mathrm{Brother}(\mathrm{Richard}, \mathrm{John})$

$\neg\mathrm{Brother}(\mathrm{LeftLegOf}(\mathrm{Richard}), \mathrm{John})$

$\neg\mathrm{King}(\mathrm{Richard}) \to \mathrm{King}(\mathrm{John})$

$\forall x.\mathrm{King}(x) \to \mathrm{Person}(x)$

Given a signature (F, P), a **model** M consists of

- a non-empty set $|M|$, called **the carrier set** (**domain**) of M, whose elements are called **objects**
- a function $M_\sigma \colon |M|^n \to |M|$ for each operation symbol $\sigma \in F_n$
- a subset $M_\pi \subseteq |M|^n$ for each relation symbol $\pi \in P_n$

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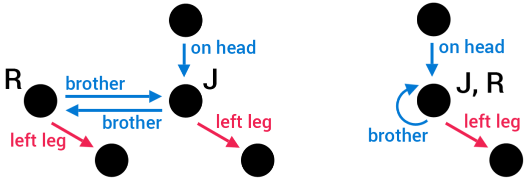
EXAMPLE



SATISFACTION RELATION

We read $M \models \varphi$ as "M satisfies φ", for M a model and φ a sentence, both for the same signature (F, P).

To make (F, P) explicit, we sometimes write $M \models_{(F,P)} \varphi$.

The satisfaction relation is defined according to the structure of sentences, based on the evaluation of terms in models.

EVALUATION OF TERMS

M_t denotes the interpretation of a term t in a model M.

$$M_{\sigma(t_1,\ldots,t_n)} = M_\sigma(M_{t_1}, \ldots, M_{t_n})$$

e.g. $M_{\mathsf{LeftLegOf}(\mathsf{John})} = M_{\mathsf{LeftLegOf}}(M_{\mathsf{John}})$

$= M_{\mathsf{LeftLegOf}}($  $) =$ 

SATISFACTION RELATION · $M \models \varphi$

ATOMS

$$M \models t = t' \quad \text{iff} \quad M_t = M_{t'}$$

$$M \models \pi(t_1, \dots, t_n) \quad \text{iff} \quad (M_{t_1}, \dots, M_{t_n}) \in M_\pi$$

BOOLEAN CONNECTIVES

$$M \models \neg\varphi \quad \text{iff} \quad M \not\models \varphi$$

$$M \models \varphi_1 \wedge \varphi_2 \quad \text{iff} \quad M \models \varphi_1 \text{ and } M \models \varphi_2$$

$$M \models \varphi_1 \vee \varphi_2 \quad \text{iff} \quad M \models \varphi_1 \text{ or } M \models \varphi_2$$

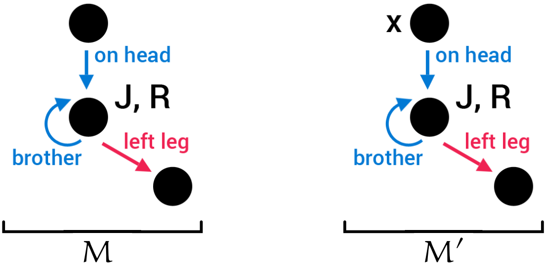
$$M \models \varphi_1 \rightarrow \varphi_2 \quad \text{iff} \quad M \models \varphi_2 \text{ whenever } M \models \varphi_1$$

$$M \models \varphi_1 \leftrightarrow \varphi_2 \quad \text{iff} \quad M \models \varphi_1 \rightarrow \varphi_2 \text{ and } M \models \varphi_2 \rightarrow \varphi_1$$

QUANTIFIERS

A model $\mathcal{M}'$ for $(F \cup X, P)$ is called an **expansion** of a model $\mathcal{M}$ for (F, P) if it interprets all symbols in F and in P the same as $\mathcal{M}$. Expansions formalize assignments of elements from $\mathcal{M}$ to the variables in X.

EXAMPLE



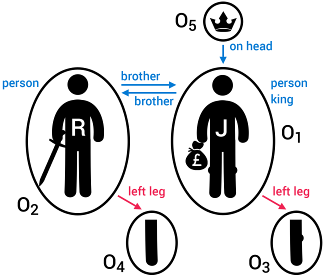
QUANTIFIERS

A model M' for $(F \cup X, P)$ is called an **expansion** of a model M for (F, P) if it interprets all symbols in F and in P the same as M. Expansions formalize assignments of elements from M to the variables in X.

$M \models_{(F,P)} \forall X.\varphi$ iff $M' \models_{(F \cup X,P)} \varphi$
for all expansions M' along the inclusion $(F, P) \subseteq (F \cup X, P)$

$M \models_{(F,P)} \exists X.\varphi$ iff there exists an expansion M' along the inclusion $(F, P) \subseteq (F \cup X, P)$ such that $M' \models_{(F \cup X,P)} \varphi$

SATISFACTION RELATION · EXAMPLE



True or False?

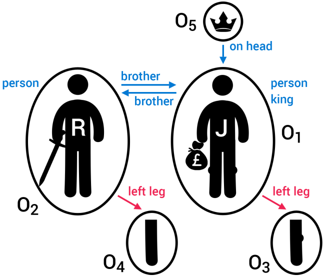
$\text{Brother}(\text{John}, \text{Richard}) \land \text{Brother}(\text{Richard}, \text{John})$

$\neg\text{Brother}(\text{LeftLegOf}(\text{Richard}), \text{John})$

$\neg\text{King}(\text{Richard}) \to \text{King}(\text{John})$

$\forall x. \text{King}(x) \to \text{Person}(x)$

SATISFACTION RELATION · EXAMPLE



True or False?

$\forall x.\text{King}(x) \rightarrow \text{Person}(x)$

$x \mapsto O_1$ (i.e. $M'_x = O_1$) O_1 (John) is a king $\rightarrow O_1$ is a person.

$x \mapsto O_2$ O_2 (Richard) is a king $\rightarrow O_2$ is a person.

$x \mapsto O_3$ O_3 (John's left leg) is a king $\rightarrow O_3$ is a person.

$x \mapsto O_4$ O_4 (Richard's left leg) is a king $\rightarrow O_4$ is a person.

$x \mapsto O_5$ O_5 (crown) is a king $\rightarrow O_5$ is a person.

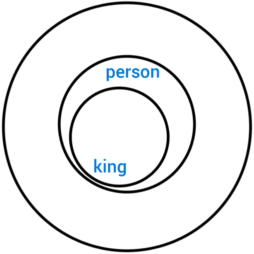
10.c

Expressivity

EXPRESSIVITY · QUANTIFIERS

$\forall x.\text{King}(x) \rightarrow \text{Person}(x)$

$$\forall x.\mathsf{King}(x) \to \mathsf{Person}(x)$$



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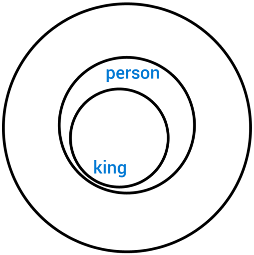
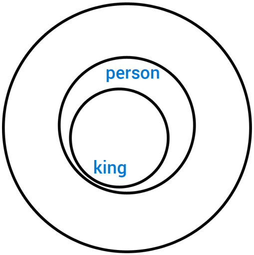


$\forall x.\mathrm{King}(x) \wedge \mathrm{Person}(x)$

EXPRESSIVITY · QUANTIFIERS

$$\forall x.\text{King}(x) \rightarrow \text{Person}(x)$$



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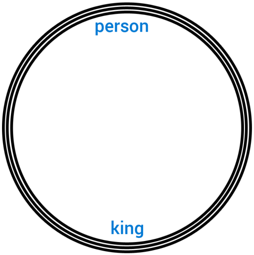
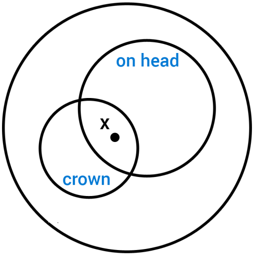


$$\exists x.\mathsf{Crown}(x) \wedge \mathsf{OnHead}(x, \mathsf{John})$$

$\exists x.\text{Crown}(x) \wedge \text{OnHead}(x, \text{John})$



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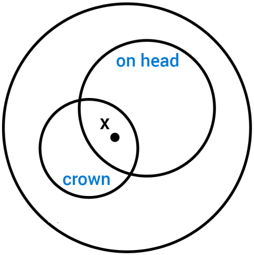
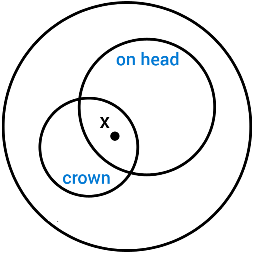


$\exists x.\text{Crown}(x) \rightarrow \text{OnHead}(x, \text{John})$

$\exists x.\text{Crown}(x) \land \text{OnHead}(x, \text{John})$



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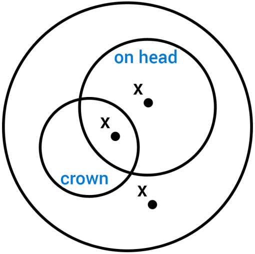

THE ORDER OF QUANTIFIERS

$\exists X.\forall Y.\varphi$ is not the same thing as $\forall Y.\exists X.\varphi$

$\exists x.\forall y.\text{Loves}(x, y)$
There is a person who loves everyone in the world.

$\forall y.\exists x.\text{Loves}(x, y)$
Everyone in the world is loved by someone.

DUALITY

$\varphi \wedge \varphi' \equiv \neg(\neg\varphi \vee \neg\varphi')$ and $\varphi \vee \varphi' \equiv \neg(\neg\varphi \wedge \neg\varphi')$

$\forall X.\varphi \equiv \neg\exists X.\neg\varphi$

$\forall x.\text{Likes}(x, \text{IceCream}) \equiv \neg\exists x.\neg\text{Likes}(x, \text{IceCream})$

$\exists X.\varphi \equiv \neg\forall X.\neg\varphi$

$\exists x.\text{Likes}(x, \text{Broccoli}) \equiv \neg\forall x.\neg\text{Likes}(x, \text{Broccoli})$

AXIOMS · definitions, theorems

One's mother is one's female parent.

$\forall m, c. m = \text{Mother}(c) \leftrightarrow (\text{Female}(m) \land \text{Parent}(m, c))$

Parent and child are inverse relations.

$\forall p, c. \text{Parent}(p, c) \leftrightarrow \text{Child}(c, p)$

A sibling is another child of one's parents.

$\forall x, y. \text{Sibling}(x, y) \leftrightarrow x \neq y \land \exists p. \text{Parent}(p, x) \land \text{Parent}(p, y)$

Brothers are siblings.

$\forall x, y. \text{Brother}(x, y) \rightarrow \text{Sibling}(x, y)$

The sibling relation is symmetric.

$\forall x, y. \text{Sibling}(x, y) \leftrightarrow \text{Sibling}(y, x)$

INTERACTING WITH FOL KBS

TELL/ASK INTERFACE

Assertions

$\text{Tell}(\text{KB}, \text{King}(\text{John}))$
$\text{Tell}(\text{KB}, \text{Person}(\text{Richard}))$
$\text{Tell}(\text{KB}, \forall x.\text{King}(x) \rightarrow \text{Person}(x))$

Queries (goals)

$\text{Ask}(\text{KB}, \text{Person}(\text{John}))$	*true*
$\text{Ask}(\text{KB}, \exists x.\text{Person}(x))$	*true*
$\text{Ask}(\text{KB}, \text{Person}(x))$	$\{x/\text{John}\}, \{x/\text{Richard}\}$

IDEA

$\text{Ask}(\text{KB}, \varphi)$ returns all **substitutions** θ such that $\text{KB} \models \theta(\varphi)$.

EXAMPLE · WUMPUS WORLD

Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at $t = 5$.

$\textsc{Tell}(\text{KB}, \text{Percept}([\text{Smell}, \text{Breeze}, \text{None}], 5))$

Does the KB entail some best action at $t = 5$?

$\textsc{Ask}(\text{KB}, \exists a.\text{BestAction}(a, 5))$

Answer: *true*, $\{a/\text{Shoot}\}$

PERCEPTION

$\forall t, s, b.\text{Percept}([s, b, \text{Glitter}], t) \to \text{Glitter}(t)$

REFLEX

$\forall t.\text{Glitter}(t) \to \text{BestAction}(\text{Grab}, t)$

THE ENVIRONMENT

$$\forall x, y, a, b.\mathsf{Adjacent}([x, y]) \leftrightarrow [a, b] \in \{[x+1, y], [x-1, y], [x, y+1], [x, y-1]\}$$

$$\forall s, t.\mathsf{At}(\mathsf{Agent}, s, t) \land \mathsf{Breeze}(t) \rightarrow \mathsf{Breezy}(s)$$

Squares are breezy near a pit.

Diagnostic rule – infer cause from effect

$$\forall s.\mathsf{Breezy}(s) \rightarrow \exists r.\mathsf{Adjacent}(r, s) \land \mathsf{Pit}(r)$$

Causal rule – infer effect from cause

$$\forall r.\mathsf{Pit}(r) \rightarrow (\forall s.\mathsf{Adjacent}(r, s) \rightarrow \mathsf{Breezy}(s))$$

FOL IN SHORT

Objects and relations are semantic primitives.

Syntax: constants, functions, predicates, quantifiers.

Increased expressive power – sufficient to define the Wumpus world.